Deformation theory of Cayley submanifolds

Kimberley Moore

Trinity Hall

University of Cambridge

This dissertation is submitted for the degree of

Doctor of Philosophy

February 2017

Summary

Cayley submanifolds are naturally arising volume minimising submanifolds of $Spin(7)$-manifolds. In the special case that the ambient manifold is a four-dimensional Calabi–Yau manifold, a Cayley submanifold might be a complex surface, a special Lagrangian submanifold or neither. In this thesis, we study the deformation theory of Cayley submanifolds from two different perspectives.

Firstly, we seek special Lagrangian submanifolds by deforming a Lagrangian submanifold in the direction given by its mean curvature vector. It is expected that a Lagrangian submanifold evolved under Lagrangian mean curvature flow will develop singular points in finite time. We will prove that if a compact Lagrangian submanifold of $\mathbb{C}^m$ develops finitely many singular points under Lagrangian mean curvature flow, each asymptotic to a non-area-minimising pair of transversely intersecting Lagrangian planes, then the flow can be continued (in a weak sense) smoothly beyond the formation of these singularities.

Secondly, motivated by the question of whether a complex surface can be deformed into a Cayley submanifold that is not complex, we study deformations of compact and conically singular Cayley submanifolds and complex surfaces. We will show that the expected dimension of the moduli space of Cayley deformations of a compact or conically singular Cayley submanifold is given by the index of a linear elliptic partial differential operator. In particular, we will prove directly that complex and Cayley deformations of a compact or conically singular complex surface are the same.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

The material of Chapter 2 below is based on a joint work of the author with Tom Begley [5].

Acknowledgements

First and foremost I would like to thank Jason Lotay for giving up so much of his time to guide me through this thesis with such patience and kindness. I would also like to thank Alexei Kovalev for his many helpful comments on this thesis as well as many significantly more primitive versions over the past few years. Thank you also to Felix Schulze for useful conversations on Lagrangian mean curvature flow.

I would like to thank everyone in the CCA (third cohort in particular) for being such good company over the last few years. Special thanks go to my officemates, Henry Jackson and Tom Holding, for the fun distractions from work and for tolerating (and sometimes helping!) my office decorating. Thank you to Tom Begley, for being an awesome collaborator and great company during our trips to London (and Disneyland!); Alex Bastounis for being one of the least socially inept mathematicians that I know; James Mathews for being a great lunch buddy; Karen Habermann for infiltrating our lunchtimes and becoming a great friend; and Rob Hocking for brightening up my day with his quite frankly ridiculous (and hilarious) stories.

I would never even have considered studying maths at university if it hadn't been for my wonderful school maths teachers, Cath Wise and Alyson Coates, and I would certainly have been too daunted to apply to a PhD program without the support and encouragement of Gregory Sankaran at the University of Bath.

I'd like to thank all of my friends and family, most of all Martin Taylor, for their love, support and most importantly distraction from work.

Contents

Chapter 1

Preliminaries

The theory of *calibrated geometry* was proposed by Harvey and Lawson [15] in 1982. It can be deduced from *Wirtinger's inequality* that complex submanifolds of a Kähler manifold are homologically volume minimising. Given a manifold with a *calibration*, this manifold can have *calibrated submanifolds* which by definition satisfy a Wirtinger-type inequality, and as a result are volume minimising in their homology class. Calibrated submanifolds exist naturally in manifolds with *special holonomy*, which are of special interest to physicists working in certain branches of string theory.

We will be interested in three particular related types of calibrated submanifolds in this thesis: *Cayley* submanifolds, which are four-dimensional submanifolds of manifolds with holonomy contained in $Spin(7)$, two-dimensional complex surfaces inside manifolds with holonomy contained in $SU(4)$, which in this thesis are four-dimensional *Calabi–Yau* manifolds and minimal Lagrangian submanifolds of Calabi–Yau manifolds known as *special Lagrangian* submanifolds. Since $SU(4) \subseteq Spin(7)$, Cayley submanifolds can exist in four-dimensional Calabi–Yau manifolds. In fact, a two-dimensional complex surface and a real four-dimensional special Lagrangian submanifold are Cayley submanifolds, however a Cayley submanifold may be neither complex nor special Lagrangian in general.

In this thesis, we will consider two different methods for finding examples of calibrated submanifolds. Firstly, in Chapter 2, we talk about finding special Lagrangian submanifolds of a Calabi–Yau manifold by flowing a Lagrangian submanifold of the Calabi–Yau under *mean curvature flow*. Despite Lagrangian mean curvature flow having several nice properties, it is known that finite time singularities of the flow are unavoidable. We will present a joint work of the author with Tom Begley which says that a smooth solution to Lagrangian mean curvature flow exists for short time when the initial condition is a Lagrangian submanifold with a certain type of singularity.

The second method that we will consider for finding examples of calibrated submanifolds is by studying their *deformation theory*. Since a lot of work has already been done on the deformation theory of special Lagrangian submanifolds, we will focus on the deformation theory of Cayley submanifolds and complex surfaces in this thesis.

Given a calibrated submanifold Y of a manifold X, we can *deform* Y as a submanifold of X. An interesting question to ask is whether we can characterise the deformations of Y that are themselves calibrated submanifolds of X. The first study of the deformation theory of calibrated submanifolds can be attributed to Kodaira [31], who, some twenty years before the conception of calibrated geometry, studied complex deformations of compact complex submanifolds of complex manifolds using methods from algebraic geometry. Analogues of Kodaira's result for certain other examples of compact calibrated submanifolds were proved by McLean [43], this time using methods from differential geometry. Roughly, these results say that if an *obstruction space* vanishes, then the *moduli space of calibrated deformations* of Y in X is locally isomorphic to the kernel of a linear partial differential operator.

A question that we will be aiming to answer in this thesis is the following.

Question 1. *Given a two-dimensional complex submanifold N of a four-dimensional Calabi–Yau manifold M, are the complex and Cayley deformations of N in M the same?*

When N is compact, one may deduce somewhat indirectly from the work of Harvey and Lawson that the answer to Question 1 is yes. In Chapter 3 of this thesis, we will give an argument in the style of McLean on the complex and Cayley deformation theory of a compact complex surface in a Calabi–Yau four-fold, which will enable us to see directly that the answer to Question 1 is yes. In fact, we will see that the *expected dimension* of the moduli space of Cayley deformations of N in M can be identified with the index of the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)), \tag{1.0.1}$$

whereas the dimension of the moduli space of complex deformations of N in M is isomorphic to the kernels of

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)), \tag{1.0.2}$$

$$\bar{\partial}^* : C^\infty(\Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)), \tag{1.0.3}$$

which are in fact isomorphic to one another. Therefore, Cayley and complex deformations of the compact complex surface N are the same because the images of (1.0.2) and (1.0.3) are orthogonal on a compact manifold.

Having answered Question 1 when N is compact, we would like to compare Cayley and complex deformations of a noncompact complex surface $\hat{N}$ in a Calabi–Yau manifold M. This adds an extra layer of difficulty to the deformation problem, which is almost entirely due to the failure of elliptic operators to be Fredholm on even the simplest of noncompact manifolds. In Chapter 4 we will review the results of Lockhart and McOwen [35] on Fredholm theory for elliptic operators on manifolds with a *cylindrical end*. These results can be applied to elliptic operators on manifolds with *conical singularities* N, whose nonsingular part $\hat{N}$ is a noncompact manifold. This involves introducing weighted norms on spaces of sections, which, roughly speaking, force sections and their derivatives to decay at a rate proportional to a power μ of their distance from the singular point. As an example, we will see that if $\hat{N}$ is a two-dimensional complex submanifold of a Calabi–Yau four-fold M then for $\mu \in \mathbb{R} \backslash \mathcal{D}$, where $\mathcal{D}$ is a discrete set of 'bad weights', the elliptic operator

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\mu-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \qquad (1.0.4)$$

is Fredholm. Moreover, we can write down a formula for how the index of (1.0.4) changes as μ varies.

In Chapter 5 we will consider Cayley deformations of a conically singular Cayley submanifold Y inside a manifold X with holonomy $Spin(7)$, and complex and Cayley deformations of a conically singular complex surface N inside a Calabi–Yau four-fold M. Since we are restricted by the Fredholm theory we have available to us for elliptic operators on Y and N, we will consider deformations of Y and N that are themselves conically singular. This time, the expected dimension of the moduli space of conically singular Cayley deformations of N in M is equal to the index of (1.0.4) (for a certain value of μ) whereas the moduli space of conically singular complex deformations of N in M is isomorphic to the kernels of

$$\bar{\partial} : C^\infty_\mu(\nu^{1,0}_M(N)) \to C^\infty_{\mu-1}(\Lambda^{0,1}N \otimes \nu^{1,0}_M(N)), \qquad (1.0.5)$$

$$\bar{\partial}^* : C^\infty_\mu(\Lambda^{0,2}N \otimes \nu^{1,0}_M(N)) \to C^\infty_{\mu-1}(\Lambda^{0,2}N \otimes \nu^{1,0}_M(N)), \qquad (1.0.6)$$

where $C^\infty_\mu(E)$ denotes smooth sections σ of a vector bundle E with $|\nabla^j\sigma(x)|$ proportional to the $(\mu - j)$th power of the distance between x and the singular point $\hat{x}$ when x is close to $\hat{x}$. It can again be shown that the images of (1.0.5) and (1.0.6) are orthogonal for the values of μ that we consider, and so the answer to Question 1 is again yes when N is conically singular.

In Chapter 6 we will perform some calculations. We first characterise the set $\mathcal{D}$ for which the operator (1.0.4) is not Fredholm. The set $\mathcal{D}$ can be described by an eigenvalue problem for certain operators on the *complex link* of a complex cone. We will apply the Atiyah–Patodi–Singer index theorem [3] to the operator (1.0.1) on a conically singular manifold, which will allow us to give an expression for the index of (1.0.4) in terms of topological invariants of N and the set $\mathcal{D}$. In particular, we will calculate the η-invariant for the operator (1.0.1) acting on some two-dimensional complex cones in $\mathbb{C}^4$. We will apply the analysis of Chapter 5 to two-dimensional complex cones in $\mathbb{C}^4$. We will consider *conical* Cayley and complex deformations of these cones, which is equivalent to deforming the links of these cones as associative submanifolds of S^7, a problem studied by Kawai [30]. In particular, we will see explicitly that the answer to Question 1 is 'not necessarily', by taking N to be a complex cone.

1.1 Introduction

This chapter is dedicated to preliminary information that we will require to study the problems described above. It is intended to be an introduction to the fundamental concepts in this thesis for the unfamiliar, or as a point of reference for definitions taken to be standard in Chapters 2, 3, 4, 5 and 6 for the more experienced reader. In Section 1.1.1, we will define notation taken to be standard throughout the rest of this thesis.

Section 1.2.1 focusses on Calabi–Yau manifolds. In Section 1.2.2, we will discuss the possible holonomy groups of a manifold, with an emphasis on the special holonomy groups. In particular, we will define $Spin(7)$-manifolds and state some facts about them which will be useful in the sequel. We will introduce calibrations in Section 1.2.3, before going into more detail about special Lagrangian and Cayley submanifolds in Sections 1.2.4 and 1.2.5 respectively.

1.1.1 Notation and conventions

Before we begin we will explain the notation in this thesis that is taken to be standard throughout.

Euclidean space

We denote by $B_r(x)$ the open ball of radius r in Euclidean space. We denote by $A(r, R) := B_R(0)\setminus\overline{B_r(0)}$ the annulus centred at zero.

The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$.

Manifolds

Manifolds will not have a boundary unless clearly stated. Submanifolds will be embedded. When talking about the dimension of a complex manifold, we mean the complex dimension unless otherwise stated.

By a cone in $\mathbb{R}^n$ we mean a subspace C of $\mathbb{R}^n$ so that for any $\lambda > 0$ and any $x \in C$ we have that $\lambda x \in C$. Define the *link* of C to be $L := C \cap S^{n-1}$.

Complex projective space of dimension n will be denoted $\mathbb{C}P^n$.

We will denote the second fundamental form of a submanifold by A, and the mean curvature vector by $\vec{H}$.

Smooth functions $M \to \mathbb{R}$ will be denoted by $C^\infty(M)$, with $C^k(M)$ denoting k-times differentiable functions $M \to \mathbb{R}$ with continuous kth derivative.

Vector bundles

Let X be a real manifold. We will denote by $\Lambda^p X$ the vector bundle whose fibre at $x \in X$ is $\Lambda^p T_x^* X$, the exterior algebra of the cotangent space to X at x. The vector bundle $T_s^q X = TX^{\otimes s} \otimes T^* X^{\otimes q}$ denotes the bundle of (s, q)-tensors on X. The positive and negative spinor bundles over X will be denoted by $\mathbb{S}_+$ and $\mathbb{S}_-$ respectively.

If Y is a submanifold of X then $\nu_X(Y)$ denotes the normal bundle of Y in X. The conormal bundle of Y in X is denoted by $\nu_X^*(Y)$. The normal space to Y at $y \in Y$ will be denoted by $\nu_y(Y)$ or $\nu_{y,X}(Y)$ if there is possible ambiguity about the ambient space.

Let (M, J) be a complex manifold. Write $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ and $T^*M \otimes \mathbb{C} = T^{*1,0}M \oplus T^{*0,1}M$ for the decomposition of the complexified (co)tangent space to X into i and $-i$ eigenspaces of the complex structure J respectively. Write $\Lambda^{p,q}M =$

$\Lambda^p T^{*1,0}M \otimes \Lambda^q T^{*0,1}M$. If $\dim_{\mathbb{C}} M = m$ write $K_M := \Lambda^{m,0}M$ for the canonical bundle of M.

Let N be a complex submanifold of M. Then $\nu_M(N) \otimes \mathbb{C} = \nu_M^{1,0}(N) \oplus \nu_M^{0,1}(N)$ and $\nu_M^*(N) \otimes \mathbb{C} = \nu_M^{*1,0}(N) \oplus \nu^{*0,1}(N)$ under the decomposition of the complexified (co)normal bundle of N in M into the i and $-i$ eigenspaces of J respectively. Call $\nu_M^{1,0}(N)$ the holomorphic normal bundle of N in M.

We will denote by $\mathcal{O}_{\mathbb{C}P^n}(-1)$ the tautological line bundle over $\mathbb{C}P^n$ and its dual bundle by $\mathcal{O}_{\mathbb{C}P^n}(1)$. If $k > 0$ then $\mathcal{O}_{\mathbb{C}P^n}(k) := \mathcal{O}_{\mathbb{C}P^n}(1)^{\otimes k}$ and if $k < 0$ then $\mathcal{O}_{\mathbb{C}P^n}(k) := \mathcal{O}_{\mathbb{C}P^n}(-1)^{\otimes -k}$. The bundle $\mathcal{O}_{\mathbb{C}P^n}(0)$ is the trivial line bundle over $\mathbb{C}P^n$.

Sections

Let F be a vector bundle over a manifold X. Denote by $C^\infty(F)$ the smooth sections of F and by $C_0^\infty(F)$ smooth sections of F with compact support. If $U \subseteq F$ is an open set in F, define

$$C^\infty(U) := \{\sigma \in C^\infty(F) \,|\, \sigma(x) \in U \text{ for all } x \in X\}.$$

Denote by $\Omega^{p,q}(F)$ the bundle of F-valued (p,q)-forms. If F has an inner product and a connection ∇, we can define a norm on sections of F by

$$\|\sigma\|_{C^k} := \sum_{j=0}^{k} \sup_{x \in X} |\nabla^j \sigma|.$$

For $1 < p < \infty$ and $k \in \mathbb{N}$ denote by $L_k^p(F)$ the Sobolev space with norm

$$\|\sigma\|_{p,k} = \left(\sum_{j=0}^{k} \int_X |\nabla^j \sigma|^p \, \mathrm{vol}_X \right)^{1/p}.$$

Say that $\sigma \in L_{k,\mathrm{loc}}^p(F)$ if $\psi\,\sigma \in L_k^p(F)$ for all smooth functions ψ on X with compact support.

Differential operators

Say that a bounded linear operator

$$A : X \to Y,$$

between Banach spaces is *Fredholm* if it has finite-dimensional kernel and cokernel (Coker $A := Y/\mathrm{Im}\ A$) and closed image. The cokernel of a Fredholm operator A is isomorphic to the kernel of its adjoint $A^* : Y^* \to X^*$. Define the *index* of A to be

$$\mathrm{ind}\ A = \dim \mathrm{Ker}\ A - \dim \mathrm{Coker} A = \dim \mathrm{Ker}\ A - \dim \mathrm{Ker}\ A^*.$$

If X is a compact manifold and A is a linear elliptic differential operator

$$A : C^\infty(F_1) \to C^\infty(F_2),$$

for vector bundles F_1, F_2 over X, then we take

$$\mathrm{ind}\ A := \dim \mathrm{Ker}\ A - \dim \mathrm{Ker}\ A^*, \tag{1.1.1}$$

where

$$A^* : C^\infty(F_2) \to C^\infty(F_1),$$

is the formal adjoint of A. If A acts on sections of complex vector bundles, we can take the real or complex dimension in (1.1.1) and refer to the real or complex index of A.

Cohomology groups

If V is a complex variety and S is a sheaf on V denote by $H^j(X, S)$ the jth sheaf cohomology group of S.

If M is a complex manifold and F is a holomorphic vector bundle over M denote by $H^{p,q}_{\bar\partial}(M, F)$ the cohomology of the complex

$$\dots \xrightarrow{\bar\partial} C^\infty(\Lambda^{p,q-1}M \otimes F) \xrightarrow{\bar\partial} C^\infty(\Lambda^{p,q}M \otimes F) \xrightarrow{\bar\partial} C^\infty(\Lambda^{p,q+1}M \otimes F) \xrightarrow{\bar\partial} \dots$$

Operations

Let X be a manifold. If $\alpha \in C^\infty(\Lambda^p X)$ and $v \in C^\infty(TX)$, write $v \lrcorner \alpha := \alpha(v, \cdot, \ldots, \cdot)$. Suppose X has a metric g. We denote by $*_X : \Lambda^p X \to \Lambda^{n-p} X$ the Hodge star on X, where $n = \dim X$. Denote the musical isomorphisms on X by $\flat : TX \to T^*X$ and $\sharp : T^*X \to TX$.

1.2 Preliminaries

Here we will give some background results that will be useful in this thesis. References are given for specific results, however there are some general references that the material is based on. Joyce's book [25] is an excellent reference for anyone interested in studying manifolds with special holonomy and calibrated submanifolds and is highly recommended for the reader who would like to fill in the gaps of what is written subsequently. Huybrechts' book [17] on complex geometry is again highly recommended for those less familiar with complex geometry, as is Griffiths and Harris's classic textbook [14] on algebraic geometry.

1.2.1 Calabi–Yau manifolds

Let (M, J, ω) be a compact Kähler manifold with complex structure J and Kähler form ω. The Kähler form defines a cohomology class on M, $[\omega]$, called the *Kähler class*. A natural question that arises when considering cohomology classes is whether there is a 'preferred' representative of each class. For example, the Hodge theorem [21, Thm 1.1.4] tells us that each de Rham cohomology class on a compact oriented manifold has a unique harmonic representative.

The question of whether the Kähler class on a manifold (M, J, ω) has a preferred representative was studied by Calabi [10]. He conjectured that if the underlying complex manifold (M, J) satisfies a certain topological condition, then there exists $\omega' \in [\omega]$

that induces a Ricci-flat metric on M, and proved that if such an ω' exists, then it is unique. The proof of the so-called Calabi conjecture was completed by Yau [58] some twenty years later, resulting in the following theorem, referred to by some authors as Yau's theorem.

Theorem 1.2.1 (Calabi–Yau)**.** *Let M be a compact Kähler manifold with complex structure J and Kähler form ω' and suppose that M satisfies $c_1(M) = 0$, where $c_1(M)$ is the first Chern class of M. Then there exists a unique Ricci-flat Kähler metric ω satisfying $[\omega'] = [\omega]$.*

Motivated by Theorem 1.2.1, we make the following definition.

Definition 1.2.1. We say that a compact Kähler manifold M of complex dimension m is a *Calabi–Yau* manifold if the canonical bundle of M, $K_M := \Lambda^{m,0}M$, is holomorphically trivial, that is, if there exists an $(m, 0)$-form α that satisfies $\alpha(x) \neq 0$ for all $x \in M$ and $\bar{\partial}\alpha = 0$.

Remark. A Calabi–Yau manifold M in the sense of Definition 1.2.1 satisfies $c_1(M) = 0$, and so we can apply the Calabi–Yau theorem 1.2.1 to M. However, a Kähler manifold X satisfying $c_1(X) = 0$ is not, in general, a Calabi–Yau manifold in the sense of Definition 1.2.1.

Later, when we have a Calabi–Yau manifold M we will require a particular choice of nonvanishing holomorphic section of K_M. Let M be a Calabi–Yau manifold with Ricci-flat metric ω. Notice that any nonvanishing section α of K_M satisfies

$$\frac{\omega^m}{m!} = \psi_\alpha\, \alpha \wedge \bar{\alpha}, \tag{1.2.1}$$

for some $\psi_\alpha : M \to (0, \infty)$, where $m = \dim_{\mathbb{C}} M$. The following result about Ricci-flat Kähler manifolds will allow us to deduce that if α is holomorphic then ψ_α is constant.

Proposition 1.2.2 ([25, Prop.7.1.5])**.** *Suppose X is a compact Kähler Ricci-flat manifold, and let ξ be a smooth $(p, 0)$-form on X. Then $\nabla\xi = 0$ iff $d\xi = 0$ iff $\bar{\partial}\xi = 0$, where ∇ is the Levi-Civita connection of the metric on X.*

Suppose that α in Equation (1.2.1) is holomorphic. Proposition 1.2.2 ensures that α is parallel. Differentiating both sides of Equation (1.2.1) with respect to ∇ allows us to deduce that

$$\nabla\psi_\alpha = 0,$$

where we note that $d\omega = 0$ implies that $\nabla\omega = 0$ on a Hermitian manifold [25, Prop 5.4.2]. The following definition allows us to choose a particular holomorphic section of K_M by taking ψ_α in Equation (1.2.1) to be a particular choice of constant. This choice of constant will be important in Section 1.2.3.

Definition 1.2.2. Let M^m be a Calabi–Yau manifold with Ricci-flat Kähler metric ω. We say that a holomorphic nowhere vanishing section Ω of K_M satisfying

$$\frac{\omega^m}{m!} = \left(\frac{i}{2}\right)^m (-1)^{m(m-1)/2}\Omega \wedge \overline{\Omega}. \tag{1.2.2}$$

is a *holomorphic volume form* of M. A holomorphic volume form on a Calabi–Yau manifold is unique up to multiplication by a unit complex number.

We will end this section with some examples of Calabi–Yau manifolds.

Example. A model example of a Calabi–Yau manifold is $\mathbb{C}^m$ with the Euclidean metric and standard complex structure. This is clearly not a compact Kähler manifold, however our reason for asking for compactness in Definition 1.2.1 is so that we may apply the Calabi–Yau theorem 1.2.1 to find a Ricci-flat Kähler metric on the manifold. The Euclidean metric is flat, and so already Ricci-flat, so we do not require the Calabi–Yau theorem here.

Define the Euclidean Kähler form $\omega_0 \in C^\infty(\Lambda^{1,1}\mathbb{C}^m)$ and the Euclidean holomorphic volume form $\Omega_0 \in C^\infty(\Lambda^{m,0}\mathbb{C}^m)$ to be

$$\begin{aligned} \omega_0 &= \frac{i}{2}\left(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m\right), \qquad (1.2.3)\\ \Omega_0 &= dz_1 \wedge \cdots \wedge dz_m. \end{aligned}$$

It is easy to check that Ω_0 is a holomorphic volume form of $\mathbb{C}^m$ in the sense of Definition 1.2.2.

Given a point $x \in M$, a Calabi–Yau manifold of dimension m with Ricci-flat Kähler form ω and holomorphic volume form Ω we can choose a neighbourhood V of x and a biholomorphism $\chi : U \subseteq \mathbb{C}^m \to V \subseteq M$ such that $\chi(0) = x$ and

$$\chi^*\omega = \omega_0 + O(|z|^2), \tag{1.2.4}$$

where ω_0 is as in (1.2.3) by [14, pg 107]. Notice that with this choice of coordinates, using Equation (1.2.2)

$$\chi^*(\Omega \wedge \overline{\Omega}) = \pm 2^m i \frac{\omega_0^m}{m!} + O(|z|^2) = \Omega_0 \wedge \overline{\Omega}_0 + O(|z|^2),$$

where Ω_0 was defined in (1.2.3) and so we must also have that in these coordinates,

$$\chi^*\Omega = e^{i\theta}\Omega_0 + O(|z|^2), \tag{1.2.5}$$

where θ is constant. We can always choose a holomorphic volume form on M so that $\theta = 0$.

Example. The compact manifold $\mathbb{C}P^m$ with the Fubini–Study metric [17, Ex 3.1.9 i)] is a Kähler manifold. It is well known that a smooth hypersurface M of degree $m+1$ in $\mathbb{C}P^m$ has trivial canonical bundle [17, Cor 2.4.9]. Since M is also Kähler with respect to the metric induced by the Fubini–Study metric, Definition 1.2.1 tells us that M is a Calabi–Yau manifold. Note that although the Calabi–Yau theorem 1.2.1 says that a Ricci-flat Kähler metric exists on M, it is very hard to find an explicit expression for this metric (even though we know what cohomology class it lies in). Finding an explicit expression for the Ricci-flat metric on a Calabi–Yau manifold is a difficult problem in general.

1.2.2 $Spin(7)$-manifolds

So-called $Spin(7)$-manifolds are of interest because of their unusual, or *exceptional* holonomy. We will therefore introduce the Riemannian holonomy group of a Riemannian manifold. The overview given here is based on the material in [25, §2.2].

Let (X, g) be a Riemannian manifold of dimension n and denote by ∇ the Levi-Civita connection of g. Consider a loop γ in X based at $x \in X$, that is, a piecewise smooth map $[0,1] \to X$ with $\gamma(0) = \gamma(1) = x$. Given $v_0 \in T_x X$ we can find a unique section v of TX defined along γ satisfying

$$\nabla_{\dot{\gamma}(t)} v(\gamma(t)) = 0,$$

and $v(0) = v_0$. This defines a map

$$P_\gamma : T_x X \to T_x X,$$

with $P_\gamma(v_0) = v(\gamma(1))$. Consider the set

$$H_x := \{P_\gamma : T_x X \to T_x X \,|\, \gamma \text{ is a loop in } X \text{ based at } x\}. \tag{1.2.6}$$

It can be shown that $(H_x, \circ)$ is a subgroup of $Gl(n, \mathbb{R})$ and moreover that $(H_x, \circ)$ and $(H_{x'}, \circ)$ are the same subgroup of $Gl(n, \mathbb{R})$ up to conjugation, and so we can write $H = H_x$. We will call the group $(H, \circ)$ the *Riemannian holonomy group* of (X, g) denoted $\mathrm{Hol}(g)$. It can be shown that on a simply connected manifold, the Riemannian holonomy group is a connected Lie subgroup of $Gl(n, \mathbb{R})$.

The Riemannian holonomy group of a metric allows us to deduce properties of the manifold. For example, if the Riemannian holonomy group of (X, g) is contained in $U(n/2)$, then (X, g) is Kähler. The possible Riemannian holonomy groups of a simply connected manifold (X, g) of dimension n which is nonsymmetric and irreducible were characterised by Berger [6, Thm 3], who showed that $\mathrm{Hol}(g)$ must be equal to one of the following groups:

(i) $SO(n)$,

(ii) $U(m)$ in $SO(2m)$ $(n = 2m)$,

(iii) $SU(m)$ in $SO(2m)$ $(n = 2m)$,

(iv) $Sp(m)$ in $SO(4m)$ $(n = 4m)$,

(v) $Sp(m)Sp(1)$ in $SO(4m)$ $(n = 4m)$,

(vi) G_2 in $SO(7)$ $(n = 7)$,

(vii) $Spin(7)$ in $SO(8)$ $(n = 8)$.

Around the time that this result was proved, examples of metrics with holonomy groups given by (i)-(v) were already well-known – for example as we mentioned above a Kähler manifold (X, g) has $\mathrm{Hol}(g) \subseteq U(m)$, whereas a Calabi–Yau manifold (X, g) has $\mathrm{Hol}(g) \subseteq SU(m)$ (yielding yet another possible way of defining a Calabi–Yau manifold – this is actually equivalent [25, Prop 7.1.4] to Definition 1.2.1). However, the inclusion of the groups G_2 and $Spin(7)$ on this list was initially considered anomalous (Berger's original list included the group $Spin(9)$ in $SO(16)$. It was later shown that metrics with holonomy $Spin(9)$ are symmetric), and metrics with holonomy equal to G_2 and $Spin(7)$ were not expected to exist. It was discovered, however, some thirty years later by Bryant [9] that such metrics did indeed exist, with explicit examples constructed soon after by Bryant and Salamon [8]. Finally, examples of metrics with holonomy G_2 and $Spin(7)$ on compact manifolds were constructed by Joyce, who gives a succinct overview of how this is done in his book [25, Ch 11].

Manifolds with Riemannian holonomy equal to G_2 or $Spin(7)$ are known as manifolds with *exceptional holonomy*. Their discovery has attracted the attention of physicists, who are interested in manifolds with special holonomy (that is, holonomy G_2, $Spin(7)$ or $SU(n)$) as models in certain branches of string theory such as M-theory. In particular, physicists would like mathematicians to come up with more examples of manifolds with holonomy G_2 and $Spin(7)$ with certain additional properties, for example isolated conical singularities, to support their theories.

We will now give the definition of a $Spin(7)$-manifold that we will use throughout this exposition. One can perhaps infer from the overview above that a $Spin(7)$-manifold might be a manifold with a metric whose holonomy group is equal to $Spin(7)$, however, this is not the only way to define a $Spin(7)$-manifold. The following definition has been chosen because of the context in which we will need $Spin(7)$-manifolds, not because we are interested in these manifolds themselves, but in naturally occurring volume minimising submanifolds of these manifolds (which we will discuss in Sections 1.2.3

and 1.2.5 later). The following definition is based on the one given in Joyce's book [25, Defn 11.4.2].

Definition 1.2.3. Let $(x_1, \ldots, x_8)$ be coordinates on $\mathbb{R}^8$ with the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_8^2$. Define a four-form on $\mathbb{R}^8$ by

$$\begin{aligned}\Phi_0 :=& dx_{1234} - dx_{1256} - dx_{1278} - dx_{1357} + dx_{1368} - dx_{1458} - dx_{1467} \\ & - dx_{2358} - dx_{2367} + dx_{2457} - dx_{2468} - dx_{3456} - dx_{3478} + dx_{5678},\end{aligned} \tag{1.2.7}$$

where $dx_{ijkl} := dx_i \wedge dx_j \wedge dx_k \wedge dx_l$.

Let M be an eight-dimensional oriented manifold. Define for each $p \in M$ the subset $\mathcal{A}_p M \subseteq \Lambda^4 T_p^* M$ to be those four-forms Φ for which there exists an oriented isomorphism $T_p M \to \mathbb{R}^8$ identifying Φ and Φ_0 given in (1.2.7), and define the vector bundle $\mathcal{A}M$ to be the vector bundle with fibre $\mathcal{A}_p M$.

A four-form Φ on M satisfying $\Phi|_p \in \mathcal{A}_p M$ for all $p \in M$ defines a metric g on M, using the fact that each tangent space to M is identified with $\mathbb{R}^8$ with the Euclidean metric. We call (Φ, g) a *Spin*(7)*-structure* on M. Let ∇ denote the Levi-Civita connection of g. Say that (Φ, g) is a *torsion-free* *Spin*(7)-structure on M if $\nabla\Phi = 0$.

We say that (M, Φ, g) is a *Spin*(7)*-manifold* if M is an eight-dimensional oriented manifold and (Φ, g) is a torsion-free *Spin*(7)-structure on M.

Remark. We may actually use Φ_0 to define the group $Spin(7)$ by taking it to be the subgroup of $Gl(8, \mathbb{R})$ that preserves Φ_0 in the sense that $A \in Spin(7)$ if, and only if,

$$\Phi_0(Ax, Ay, Az, Aw) = \Phi_0(x, y, z, w),$$

for all $x, y, z, w \in \mathbb{R}^8$.

The link between Definition 1.2.3 and manifolds with holonomy contained in $Spin(7)$ is made rigorous in the following proposition. It is taken from Joyce's book [25, Prop 11.4.3].

Proposition 1.2.3. *Let M be an oriented eight-dimensional manifold and (Φ, g) a $Spin(7)$-structure on M. Then the following are equivalent:*

(i) (Φ, g) *is torsion-free,*

(ii) $\mathrm{Hol}(g) \subseteq Spin(7)$,

(iii) $\nabla\Phi = 0$ *on* M, *where* ∇ *is the Levi-Civita connection of* g, *and*

(iv) $d\Phi = 0$ *on* M.

Metrics with holonomy contained in $Spin(7)$ are Ricci-flat [25, Prop 11.4.5], so we can deduce that a $Spin(7)$-manifold is Ricci-flat.

Since the holonomy group of a $Spin(7)$-manifold is contained in $Spin(7)$, and not necessarily equal to $Spin(7)$, we are allowed to have the following special example of a $Spin(7)$-manifold.

Example. We will consider the special type of $Spin(7)$-manifold frequently in this exposition which is a four-dimensional Calabi–Yau manifold (note that $SU(4) \subseteq Spin(7)$). Let M be a four-dimensional Calabi–Yau manifold as in Definition 1.2.1 with Ricci-flat Kähler metric ω. Choose a holomorphic volume form Ω so that in local coordinates on M

$$\omega = \omega_0 + O(|z|^2), \quad \Omega = \Omega_0 + O(|z|^2),$$

where ω_0 and Ω_0 were defined in (1.2.3), as we described in (1.2.4)-(1.2.5).

Consider the naturally occurring four-form on M given by

$$\Phi := \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\ \Omega.$$

Then given any $p \in M$ we can choose coordinates so that

$$\Phi|_p = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}\ \Omega_0. \tag{1.2.8}$$

Identifying $\mathbb{C}^4$ with $\mathbb{R}^8$ using $(z_1, z_2, z_3, z_4) = (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7, x_4 + ix_8)$ and comparing Equations (1.2.7) and (1.2.8), it follows that Φ defines a $Spin(7)$-structure on M. Clearly $d\Phi = 0$ and therefore by Proposition 1.2.3, (M, ω, Ω) is a $Spin(7)$-manifold.

Properties of $Spin(7)$-manifolds

A $Spin(7)$-structure on a manifold M induces a natural three-fold *cross product* on TM. The following lemma follows from [29, Lem 4.4.2].

Lemma 1.2.4. *Let M^8 be an oriented manifold with $Spin(7)$-structure (Φ, g). Then there is a natural alternating, trilinear map*

$$\kappa : TM \times TM \times TM \to TM,$$

called the triple cross product *on M defined by*

$$\Phi(x, y, z, w) = g(x, \kappa(y, z, w)),$$

satisfying

$$|\kappa(y, z, w)| = |y \wedge z \wedge w|.$$

Remark. In general, an m-fold cross product on an n-dimensional Riemannian manifold (X, g) is an alternating, m-linear map from m copies of TX to TX. It must satisfy

$$|v_1 \times \cdots \times v_m| = |v_1 \wedge \cdots \wedge v_m|,$$
$$g(v_1 \times \ldots v_m, v_i) = 0, \quad i = 1, \ldots, m,$$

for all vector fields $v_1, \ldots, v_m$ on X. Lemma 1.2.4 ensures that the map κ is a cross product in this sense.

We can decompose bundles of forms on $Spin(7)$-manifolds into irreducible representations of $Spin(7)$. The vector bundle Λ^2_7 defined below will appear frequently in this exposition, making its first appearance in Section 1.2.5. The following proposition can be found in Joyce's book [25, Prop 11.4.4].

Proposition 1.2.5. *Let M be a $Spin(7)$-manifold. Then the bundles of two-forms and self-dual four-forms on M admit the following decompositions into irreducible representations of $Spin(7)$:*

$$\Lambda^2 M \cong \Lambda^2_7 \oplus \Lambda^2_{21},$$
$$\Lambda^4_+ M \cong \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27},$$

where $\Lambda^4_+ M$ denotes the self-dual four-forms on M and Λ^k_l denotes the irreducible representation of $Spin(7)$ on k-forms of dimension l.

Further, there is a canonical isomorphism $\Lambda^2_7 \cong \Lambda^4_7$.

Remark. Given an orthonormal frame for M $\{e_1, \dots, e_8\}$ with coframe $\{e^1, \dots, e^8\}$, we can explicitly define Λ^2_7 and Λ^4_7. The following expressions are taken from [51, Thm 9.5] and [29, Eqn 4.17] respectively. We have that

$$\Lambda^2_7 = \{e^i \wedge e^j - (e_i \lrcorner (e_j \lrcorner \Phi)) \,|\, 1 \le i < j \le 8\}, \tag{1.2.9}$$

$$\Lambda^4_7 = \{\alpha^j_i e^i \wedge (e_j \lrcorner \Phi) - \alpha^i_j e^j \wedge (e_i \lrcorner \Phi) \,|\, \alpha_{ij} e^i \wedge e^j \in \Lambda^2_7\}, \tag{1.2.10}$$

where we use the summation convention in (1.2.10).

1.2.3 Calibrated submanifolds

It was first noticed by Federer [12] that complex submanifolds of Kähler manifolds are volume minimising in their homology class. We can see this from *Wirtinger's inequality* for Kähler manifolds [17, Ex 1.2.9]: Let (X, ω) be a Kähler manifold of complex dimension n. Then for any oriented real $2p$-dimensional submanifold Y of X,

$$\left.\frac{\omega^p}{p!}\right|_Y \le \mathrm{vol}_Y, \tag{1.2.11}$$

with equality if, and only if, Y is a complex submanifold of X. So we see that if Y is a p-dimensional compact complex submanifold of (X, ω) and Y' is any real $2p$-dimensional submanifold of X homologous to Y, then

$$\mathrm{vol}(Y) = \int_Y \mathrm{vol}_Y = \int_Y \frac{\omega^p}{p!} = \int_{Y'} \frac{\omega^p}{p!} \le \int_{Y'} \mathrm{vol}_{Y'} = \mathrm{vol}(Y'), \tag{1.2.12}$$

by Wirtinger's inequality (1.2.11). Further, equality holds in (1.2.12) if, and only if, Y' is a complex submanifold of X too.

Harvey and Lawson [15] exploited the property (1.2.11) that makes a complex submanifold homologically mass minimising in making the following definition.

Definition 1.2.4. Let (X, g) be a Riemannian manifold and let α be a p-form on X. If $d\alpha = 0$ and, for any $x \in X$ and any oriented p-dimensional subspace $V \subseteq T_x X$,

$$\alpha|_V \leq \mathrm{vol}_V,$$

then we say that α is a *calibration* on X.

We say that an oriented p-dimensional submanifold Y of X is a *calibrated submanifold* of X if $\alpha|_Y = \mathrm{vol}_Y$.

Example. Let (X, ω) be a Kähler manifold of complex dimension n. Then for any integer $1 \leq p \leq n$ it follows from Wirtinger's inequality (1.2.11) that

$$\frac{\omega^p}{p!},$$

is a calibration on X, and the calibrated submanifolds of X are the complex p-submanifolds of X.

Replacing the complex submanifold calibration in Equation (1.2.12) by an arbitrary calibration leads us straight to the following result.

Proposition 1.2.6 ([15, II.4 Thm 4.2])**.** *Let X be a Riemannian manifold with calibration α and let Y be a compact α-calibrated submanifold. Let Y' be any other compact submanifold of X homologous to Y. Then*

$$\int_Y \mathrm{vol}_Y \leq \int_{Y'} \mathrm{vol}_{Y'},$$

with equality if, and only if, Y' is also α-calibrated.

1.2.4 Special Lagrangian submanifolds

We will now define the first type of calibrated submanifold that will be studied in this thesis.

Definition 1.2.5. Let M be a Calabi–Yau manifold with Ricci-flat Kähler metric ω and holomorphic volume form Ω. Then

$$e^{-i\theta}\mathrm{Re}\ \Omega,$$

for any constant $\theta \in [0, 2\pi)$ is a calibration on M. The calibrated submanifolds of M in this case are called *special Lagrangian* submanifolds of M with phase θ.

Remark. Our choice of constant relating ω and Ω in Definition 1.2.2 was made to ensure that Re Ω is a calibration.

Harvey and Lawson found the following equivalent condition for a real m-dimensional submanifold of a Calabi–Yau manifold (of dimension m) to be special Lagrangian.

Proposition 1.2.7 ([15, Cor III.1.11])**.** *A real m-dimensional submanifold L of an m-dimensional Calabi–Yau manifold (M, ω, Ω) is a special Lagrangian submanifold of M with phase θ if, and only if,*

$$\omega|_L \equiv 0,$$

and

$$\mathrm{Im}(e^{-i\theta}\Omega)|_L = (\cos\theta \,\mathrm{Im}\ \Omega - \sin\theta \,\mathrm{Re}\ \Omega)|_L \equiv 0.$$

In particular, this confirms that a special Lagrangian submanifold is indeed a Lagrangian submanifold.

1.2.5 Cayley submanifolds

We will now introduce the protagonist of this thesis. Recall the material of Section 1.2.2, where we talked about manifolds with special holonomy and, in particular, $Spin(7)$-manifolds. Taking Definition 1.2.3 to be our definition of a $Spin(7)$-manifold (and supposing (correctly) that there exists a similar definition of a G_2-manifold) shows us that manifolds with exceptional holonomy are examples of manifolds equipped with a naturally arising calibration.

Definition 1.2.6. Let (X, Φ, g) be a $Spin(7)$-manifold. Then Φ is a calibration on X, called the *Cayley* calibration, and submanifolds of X calibrated by Φ are called *Cayley* submanifolds of X.

Remark. We saw in Section 1.2.2 that a four-dimensional Calabi–Yau manifold M is a $Spin(7)$-manifold, and in this case the Cayley calibration is given by

$$\Phi = \text{Re}\ \Omega + \frac{1}{2}\omega \wedge \omega,$$

where ω is the Ricci-flat Kähler metric and Ω is the holomorphic volume form on M. From this expression it can be seen that complex surfaces (calibrated by $\frac{1}{2}\omega \wedge \omega$) and special Lagrangian submanifolds (with phase zero, calibrated by Re Ω) of M are both examples of Cayley submanifolds. We are particularly interested in the relationship between two-dimensional complex submanifolds and Cayley submanifolds that are not complex submanifolds inside a four-dimensional Calabi–Yau manifold in Chapters 3–6 of this thesis.

It is easy to see then that Cayley submanifolds exist by taking any two-dimensional complex submanifold of a degree six hypersurface in $\mathbb{C}P^5$. It is interesting to note that Cayley submanifolds exist that are not complex or special Lagrangian submanifolds of a Calabi–Yau four-fold. These can be linear subspaces of $\mathbb{R}^8$, but more interesting Cayley submanifolds of $\mathbb{R}^8$ were constructed by Lotay [36]. Joyce [21] has constructed Cayley submanifolds of compact $Spin(7)$-manifolds.

Properties of Cayley submanifolds

Let (M, g, Φ) be a $Spin(7)$-manifold. We can identify the tangent space to M at any point with $\mathbb{O}$, the *octonions* or *Cayley numbers*. This is because the automorphism group of $\mathbb{O}$ is $Spin(7)$. In *Calibrated geometries* [15], the theory of Cayley submanifolds unfolds mainly in terms of linear subspaces of $\mathbb{O}$. The following characterisation of Cayley subspaces of $\mathbb{O}$ is particularly interesting.

Proposition 1.2.8 ([15, IV.1.C Cor 1.29])**.** *Let ζ be an oriented four-dimensional linear subspace of $\mathbb{O}$. Then ζ is a Cayley subspace of $\mathbb{O}$ if, and only if, for any basis $\{u, v, w, x\}$ of ζ*

$$\text{Im}\ (u \times v \times w \times x) \equiv 0,$$

where $u \times v \times w \times x$ is a four-fold cross product naturally arising on $\mathbb{O}$.

We can interpret the four-fold octonian cross product as a Λ^2_7-valued four-form on a $Spin(7)$-manifold. The next result, which can be considered as a generalisation of Proposition 1.2.8, allows us to characterise Cayley submanifolds of a $Spin(7)$-manifold (X, Φ, g) in terms of a four-form that vanishes exactly when restricted to a Cayley submanifold of X.

Proposition 1.2.9 ([51, Lem 10.3])**.** *Let X be a real eight-dimensional manifold with $Spin(7)$-structure (Φ, g). Let κ denote the triple cross product on X induced by Φ given in Lemma 1.2.4. Let Y be an oriented real four-dimensional submanifold of X. Then Y is a Cayley submanifold of X if, and only if, $\tau|_Y \equiv 0$, where $\tau \in C^\infty(\Lambda^4 X \otimes \Lambda^2_7)$ is defined by, for any vector fields x, u, v, w on X*

$$\begin{aligned}\tau(x,u,v,w) = \frac{1}{4}\big(&\pi_7(\kappa(u,v,w)^\flat \wedge x^\flat) - \pi_7(\kappa(v,w,x)^\flat \wedge u^\flat)\\ &+\pi_7(\kappa(w,x,u)^\flat \wedge v^\flat) - \pi_7(\kappa(x,u,v)^\flat \wedge w^\flat)\big),\end{aligned}$$

where $\pi_7 : \Lambda^2 X \to \Lambda^2_7$ is the projection map given by $\pi_7(x^\flat \wedge y^\flat) = \frac{1}{2}(x^\flat \wedge y^\flat + \Phi(x, y, \cdot, \cdot))$ and $\flat$ denotes the musical isomorphism $TX \to T^*X$.

Moreover, if x, u, v, w are orthogonal then

$$\tau(x,u,v,w) = \pi_7(\kappa(u,v,w)^\flat \wedge x^\flat).$$

If $\{e^1, \ldots, e^8\}$ is an orthonormal coframe for T^*X so that

$$\begin{aligned}\Phi =& e^{1234} - e^{1256} - e^{1278} - e^{1357} + e^{1368} - e^{1458} - e^{1467}\\ &-e^{2358} - e^{2367} + e^{2457} - e^{2468} - e^{3456} - e^{3478} + e^{5678},\end{aligned}$$

then τ takes the form

$$\begin{aligned}\tau =&(e^{1358} + e^{1367} - e^{1457} + e^{1468}\\ &-e^{2357} + e^{2368} - e^{2458} - e^{2467}) \otimes \frac{1}{2}(e^{12} + e^{34} - e^{56} - e^{78})\\ &+(-e^{1258} - e^{1267} + e^{1456} + e^{1478}\\ &+e^{2356} + e^{2378} - e^{3458} - e^{3467}) \otimes \frac{1}{2}(e^{13} - e^{24} - e^{57} + e^{68})\end{aligned}$$

$$
\begin{aligned}
&+(e^{1257}-e^{1268}-e^{1356}-e^{1378}\\
&+e^{2456}+e^{2478}+e^{3457}-e^{3468})\otimes\frac{1}{2}(e^{14}+e^{23}-e^{58}-e^{67})\\
&+(e^{1238}-e^{1247}+e^{1346}-e^{1678}\\
&-e^{2345}+e^{2578}-e^{3568}+e^{4567})\otimes\frac{1}{2}(e^{15}+e^{26}+e^{37}+e^{48})\\
&+(e^{1237}+e^{1248}-e^{1345}+e^{1578}\\
&-e^{2346}+e^{2678}-e^{3567}-e^{4568})\otimes\frac{1}{2}(e^{16}-e^{25}-e^{38}+e^{47})\\
&+(-e^{1236}+e^{1245}+e^{1348}-e^{1568}\\
&-e^{2347}+e^{2567}+e^{3678}-e^{4578})\otimes\frac{1}{2}(e^{17}+e^{28}-e^{35}-e^{46})\\
&+(-e^{1235}-e^{1246}-e^{1347}+e^{1567}\\
&-e^{2348}+e^{2568}+e^{3578}+e^{4678})\otimes\frac{1}{2}(e^{18}-e^{27}+e^{36}-e^{45}).
\end{aligned}
\tag{1.2.13}
$$

This expression looks fairly nasty, however we can equivalently write this as

$$
\tau=\frac{1}{4}\sum_{i<j\in\{1,\dots,8\}}(e^j\wedge(e_i\lrcorner\,\Phi)-e^i\wedge(e_i\lrcorner\,\Phi))\otimes\pi_7(e^i\wedge e^j),
\tag{1.2.14}
$$

which is slightly less intimidating.

Chapter 2

Lagrangian mean curvature flow

Much of what follows in this thesis is on deformation theory of Cayley submanifolds with a particular emphasis on complex surfaces inside Calabi–Yau four-folds. As we have seen, the other subclass of Cayley submanifolds in Calabi–Yau four-folds is *special Lagrangian* submanifolds. The deformation theory of special Lagrangian submanifolds has been studied extensively, firstly by McLean [43, §3] who studied deformations of compact special Lagrangians, and later by Joyce [22], who in a five paper series studied the deformation theory of compact special Lagrangians with isolated conical singularities, asymptotically conical special Lagrangians, as well as related problems such as desingularisation. The Cayley deformations of special Lagrangians were studied by Ohst [48, §5.4]. In this chapter, we will discuss a different method for finding special Lagrangian submanifolds and present a joint work of the author with Tom Begley on Lagrangian mean curvature flow [5].

2.1 Introduction

An important open question in the theory of special Lagrangian submanifolds is whether given a homology or Hamiltonian isotopy class in a suitable ambient manifold (Calabi–Yau, Kähler–Einstein or symplectic) there exists a special (or minimal) Lagrangian representative of that class. This problem, whilst reasonably simple to state, is very subtle and fraught with difficulty. For example, Schoen–Wolfson [52] studied a constrained minimisation problem for the area functional, minimising over Lagrangian submanifolds of a four-dimensional symplectic manifold in a given homology class. They found that such a minimiser exists, but is not guaranteed to be a minimal surface (that is, a special Lagrangian) or even smooth. Later, Wolfson [57] went on to explicitly construct a Lagrangian sphere inside a K3-surface whose homology class has a Lagrangian representative with least volume, but that isn't a special Lagrangian, and moreover the minimiser among all submanifolds in this homology class is not Lagrangian.

A different approach to studying this problem is to evolve a Lagrangian submanifold under mean curvature flow in the hope of reaching a special Lagrangian. In this chapter, we will give a brief review of mean curvature flow in Section 2.2 before describing some of the special properties of Lagrangian mean curvature flow in Section 2.2.1. In Section 2.2.2, we will talk about singularities of mean curvature flow and describe two special types of solution to Lagrangian mean curvature flow. In Section 2.3, we will motivate and present the following theorem (Theorem 2.3.1).

Theorem. *Suppose that $L \subset \mathbb{C}^m$ is a compact Lagrangian submanifold of $\mathbb{C}^m$, satisfying some additional properties, with finitely many singularities each asymptotic to a pair of transversally intersecting planes which is not area minimising. Then we can find a solution to mean curvature flow starting from L which exists for time $T > 0$.*

The first step towards proving this theorem is to construct a family of compact smooth Lagrangian submanifolds L^s by removing a neighbourhood of the singular point of L and carefully gluing in something smooth. This construction is presented in detail in Section 2.3.1. A solution to Lagrangian mean curvature flow with initial condition L^s exists for short time T_s, and so as long as $\inf_{s>0} T_s > 0$, we can apply a compactness result to find a limiting flow which exists for short time, which is what we require.

We will give an overview of this part of the proof in Section 2.3.2, followed by some concluding remarks in Section 2.3.3.

2.2 Mean curvature flow

Given an m-dimensional manifold M and an embedding $F_0 : M \to \mathbb{R}^{m+k}$, we say that M evolves under *mean curvature flow* if there exist a $T > 0$ and a family of immersions $F : M \times [0, T) \to \mathbb{R}^{m+k}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t), \tag{2.2.1}$$

$$F(\,\cdot\,, 0) = F_0(\,\cdot\,), \tag{2.2.2}$$

where $\vec{H}(p,t)$ is the mean curvature vector of the immersed submanifold $M_t := F(M,t)$ at the point $x(p,t) = F(p,t)$. In what follows, we will frequently denote $F(p,t)$ by x, suppressing the arguments where there is no chance of ambiguity. It is hoped that if the flow exists for all time (i.e., $T = \infty$) then M_t will converge (as $t \to \infty$) to a *stationary* submanifold of $\mathbb{R}^{m+k}$ (i.e., with $\vec{H} = 0$), and that this submanifold is a *minimal submanifold*. It is well known that if M is a closed manifold then the initial value problem (2.2.1)–(2.2.2) has a unique solution up to time $T \in (0, \infty]$. This is known as *short-time existence* for mean curvature flow.

There are several important tools for studying properties of mean curvature flow, which we will introduce here. We will abuse notation slightly by omitting the family of immersions $F(\,\cdot\,, t)$ and say that a family $(M_t)_{0 \le t < T}$ is a mean curvature flow if there exists a family of immersions F with $F(M_0, t) = M_t$ satisfying (2.2.1)–(2.2.2).

Definition 2.2.1. For any $(x_0, t_0) \in \mathbb{R}^{m+k} \times \mathbb{R}$ define the *backwards heat kernel* $\rho_{(x_0,t_0)}$ to be

$$\rho_{(x_0,t_0)}(x,t) := \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \tag{2.2.3}$$

for $x \in \mathbb{R}^{m+k}$ and $t < t_0$.

A simple, yet important result in mean curvature flow is the *monotonicity formula* proved by Huisken [16, Thm 3.1].

Theorem 2.2.1 (Monotonicity formula). *Let $(M_t)_{0\leq t<t_0}$ be a mean curvature flow. Then*

$$\frac{d}{dt}\int_{M_t}\rho_{(x_0,t_0)}(x,t)\,d\mathcal{H}^m(x) = -\int_{M_t}\left|\vec{H}-\frac{(x_0-x)^\perp}{2(t_0-t)}\right|^2\rho_{(x_0,t_0)}(x,t)\,d\mathcal{H}^m(x). \quad (2.2.4)$$

Definition 2.2.2. Let $(M_t)_{0\leq t<T}$ be a mean curvature flow. Then for $0<t_0\leq T$, $0<r\leq\sqrt{t_0}$ and any $x_0\in\mathbb{R}^{m+k}$ we define the *Gaussian density ratio* centred at (x_0,t_0) and at scale r by

$$\begin{aligned}\Theta(x_0,t_0,r) &:= \int_{M_{t_0-r^2}}\rho_{(x_0,t_0)}(x,t_0-r^2)\,d\mathcal{H}^m(x),\\ &= \int_{M_{t_0-r^2}}\frac{1}{(4\pi r^2)^{m/2}}\exp\left(-\frac{|x-x_0|^2}{4r^2}\right)\,d\mathcal{H}^m(x).\end{aligned}$$

Define the *Gaussian density* to be

$$\Theta(x_0,t_0) := \lim_{r\searrow 0}\Theta(x_0,t_0,r).$$

Theorem 2.2.1 tells us that this limit exists.

It is known that $\Theta(x_0,t_0)=1$ if, and only if, (x_0,t_0) is a regular point of the flow. The following local regularity theorem of White [56] says that it is enough to show that the Gaussian density ratios are close to one in a ball to find a priori estimates on the curvature of the manifolds in the flow inside a smaller ball.

Theorem 2.2.2 (Local regularity). *Let $(M_t)_{0\leq t<T}$ be a mean curvature flow and let $\tau>0$. There are constants $\epsilon_0(m,k)>0$ and $C_0(m,k,\tau)<\infty$ such that if $\partial M_t\cap B_{2r}=\emptyset$ for $t\in[0,r^2)$ and*

$$\Theta(x,t,\rho)\leq 1+\epsilon_0,$$

for $\rho\leq\tau\sqrt{t}, x\in B_{2r}(x_0), t\in[0,r^2)$, then

$$|A|(x,t)\leq\frac{C_0}{\sqrt{t}},$$

for $x\in M_t\cap B_r(x_0), t\in[0,r^2)$, where $A(x,t)$ is the second fundamental form of M_t at the point x.

Finally, we state what it means for two manifolds to be ϵ-close in $C^{1,\alpha}$.

Definition 2.2.3. Let $U \subseteq \mathbb{R}^{m+k}$ be an open set and let Σ and L be two m-dimensional submanifolds of $\mathbb{R}^{m+k}$ that are defined in U. We say that Σ and L are 1-close in $C^{1,\alpha}(W)$ for any $W \subseteq U$ with $\mathrm{dist}(W, \partial U) \geq 1$ if for all $x \in W$ we have that $B_1(x) \cap \Sigma$ and $B_1(x) \cap L$ can be written as graphs u and v over the same m-dimensional plane with $\|u - v\|_{1,\alpha} \leq 1$. We say that Σ and L are ϵ-close in W if after rescaling by a factor of $1/\epsilon$, Σ and L are 1-close in $\epsilon^{-1}W$ for any W with $\mathrm{dist}(\epsilon^{-1}W, \epsilon^{-1}\partial U) \geq 1$.

2.2.1 Lagrangian mean curvature flow

Suppose that $(L_t)_{0 \leq t < T}$ is a mean curvature flow in $\mathbb{C}^m$ with L_0 a Lagrangian submanifold of $\mathbb{C}^m$. It was shown by Smoczyk [53, Thm 1.9] that L_t remains a Lagrangian submanifold of $\mathbb{C}^m$ for all $t \in [0, T)$. (In fact this result holds as long as the ambient manifold is Kähler–Einstein.) This is known as *Lagrangian mean curvature flow*.

We will now gather some useful preliminaries specific to Lagrangian mean curvature flow.

Definition 2.2.4. Let L be a Lagrangian submanifold of a Calabi–Yau manifold and let Ω be a holomorphic volume form on M. Then

$$\Omega|_L = e^{i\theta_L}\mathrm{vol}_L, \tag{2.2.5}$$

where θ_L is a multi-valued function on L called the *Lagrangian angle* of L. If $\theta_L : L \to \mathbb{R}$ is a well-defined function, then we call L *zero-Maslov* or *graded*.

Remark. A zero-Maslov Lagrangian remains zero-Maslov under mean curvature flow [53, Thm 2.9].

We have the following remarkable relationship between the Lagrangian angle and the mean curvature vector on a Lagrangian submanifold (see [55, Lem 2.1] for a proof)

$$\vec{H} = J\nabla\theta_L, \tag{2.2.6}$$

where J is the complex structure of the ambient manifold. Since a special Lagrangian has constant Lagrangian angle, this tells us that stationary Lagrangians are special

Lagrangians and therefore minimal. Since a special Lagrangian is zero-Maslov, and the property of being zero-Maslov is preserved along the flow, if we hope to evolve a Lagrangian under mean curvature flow to a special Lagrangian, we should take the initial Lagrangian itself to be zero-Maslov.

Definition 2.2.5. Consider $\mathbb{C}^m$ with coordinates $(x_1, ...x_m, y_1, ..., y_m)$ and complex structure J_0 acting as follows

$$J_0\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J_0\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

with the standard Kähler form ω_0 and holomorphic volume form Ω_0. Define the *Liouville form* on $\mathbb{C}^m$ to be

$$\lambda := \sum_{j=1}^{m} x_j dy_j - y_j dx_j. \tag{2.2.7}$$

Notice that $d\lambda = 2\omega_0$. Say that a Lagrangian submanifold L of $\mathbb{C}^m$ is *exact* if $\lambda|_L = d\beta_L$, where $\beta_L \in C^\infty(L)$.

Remark. Since $\lambda|_L$ is a closed one-form on any Lagrangian L, it follows that all Lagrangian submanifolds of $\mathbb{C}^m$ are locally exact.

2.2.2 A special type of solution to mean curvature flow

We are motivated in this chapter by the occurrence of singularities in mean curvature flow.

Definition 2.2.6. Say that a mean curvature flow $(M_t)_{0\leq t<T}$ has a *finite time singularity* at time $t = T$ if the flow cannot be smoothly extended to $T + \epsilon$ for any $\epsilon > 0$.

Remark. It is well known that if the maximal existence time T of a mean curvature flow $(M_t)_{0\leq t<T}$ is finite then

$$\limsup_{t\to T} \sup_{x\in M_t} |A(x,t)|^2 = \infty,$$

where $A(x,t)$ is the second fundamental form of M_t at x (see [54, Prop 3.11] for more details). So gaining a priori control of the curvature is an important tool for showing existence of a mean curvature flow.

In this section we will introduce two special types of solution to mean curvature flow.

Definition 2.2.7. Call an m-dimensional manifold Σ' in $\mathbb{R}^{m+k}$ a *self-shrinker* if its mean curvature vector satisfies the elliptic equation

$$\vec{H}(x) = -x^{\perp}, \tag{2.2.8}$$

where $x \in \Sigma' \subseteq \mathbb{R}^{m+k}$. In this case, $(\Sigma'_t)_{-\infty<t<0}$ with $\Sigma'_t := \sqrt{-2t}\Sigma'$ is a homothetically shrinking solution to mean curvature flow.

Remark. Self-shrinkers model the formation of certain types of singularity in mean curvature flow. However, there are no nontrivial zero-Maslov Lagrangian self-shrinkers in $\mathbb{C}^m$ [46, Cor 3.5(i)].

Definition 2.2.8. An m-dimensional manifold Σ in $\mathbb{R}^{m+k}$ is a *self-expander* if its mean curvature vector satisfies the elliptic equation

$$\vec{H}(x) = x^{\perp}, \tag{2.2.9}$$

where $x \in \Sigma \subseteq \mathbb{R}^{m+k}$. In this case, $(\Sigma_t)_{0<t<\infty}$, where $\Sigma_t := \sqrt{2t}\Sigma$ is a homothetically expanding solution to mean curvature flow.

Remark. It is known [46, Cor 3.5 ii)] that if $(L_t)_{t>0}$ is an exact, smooth, zero-Maslov Lagrangian mean curvature flow with area ratios bounded below such that L_{ϵ_i} converges in the sense of varifolds to a cone L_0 as $\epsilon_i \to 0$, then L_1 is a self-expander. Thus we can think of self-expanders as providing us with a solution to Lagrangian mean curvature flow starting from a cone.

To prove Theorem 2.3.1, we seek a solution to Lagrangian mean curvature flow for a Lagrangian with singular points asymptotic to a particular cone. As a result, we are interested in self-expanders that are asymptotic to this cone. Such a family of Lagrangian self-expanders has been constructed explicitly by Joyce–Lee–Tsui.

Theorem 2.2.3 ([27, Thms C, D]). *Let P_1 and P_2 be transversally intersecting Lagrangian planes in $\mathbb{C}^m$ so that neither $P_1 + P_2$ or $P_1 - P_2$ are area minimising. Then there exists an exact, zero-Maslov, Lagrangian self-expander Σ with bounded Lagrangian angle so that $\sqrt{2t}\Sigma$ converges to $P_1 + P_2$ in the sense of varifolds as $t \to 0$.*

A result fundamental to the proof of Theorem 2.3.1 is the uniqueness of the self-expander given in Theorem 2.2.3. The following result was proved by Lotay–Neves when the ambient manifold is $\mathbb{C}^2$ [40, Thm B] using analytic techniques. The result was proved by Imagi–Joyce–Oliveira dos Santos [19, Thm 1.2] in $\mathbb{C}^m$ for $m > 2$ using Fukaya categories.

Theorem 2.2.4 ([40, Thm B],[19, Thm 1.2]). *Let P_1 and P_2 be a pair of transversally intersecting Lagrangian planes in $\mathbb{C}^m$ such that neither $P_1 + P_2$ or $P_1 - P_2$ are area minimising. Suppose that Σ' is a zero-Maslov self-expander asymptotic to $P_1 + P_2$. Then $\Sigma' = \Sigma$, where Σ is the self-expander constructed in the proof of Theorem 2.2.3.*

2.3 Singularities of LMCF

We have now covered the preliminaries required to state the main theorem of this chapter, Theorem 2.3.1 below, as well as discuss the highlights of the proof. We will first discuss some results on Lagrangian mean curvature flow that motivate this result.

We first note that when the initial condition for Lagrangian mean curvature flow is zero-Maslov, the structure of finite time singularities that develop under the flow is relatively well understood. It was shown by Neves [45, Thm A] that if a Lagrangian mean curvature flow beginning from a zero-Maslov Lagrangian with bounded Lagrangian angle develops a finite time singularity, then the singular point must be asymptotic to a finite union of special Lagrangian cones. Recall that special Lagrangian cones in $\mathbb{C}^2$ are intersecting planes.

A later result of Neves [47, Thm 6.1] indicates that finite time singularities of Lagrangian mean curvature flow are unavoidable. Specifically, he showed that if M is a two-dimensional Calabi–Yau manifold and Σ is an embedded Lagrangian, then there is a Lagrangian L in the same Hamiltonian isotopy class as Σ that develops a finite time singularity when evolved under Lagrangian mean curvature flow. Noting that the Hamiltonian isotopy class of a zero-Maslov Lagrangian is preserved by mean curvature flow, this means that even if a Hamiltonian isotopy class contains a special Lagrangian, it is likely that under Lagrangian mean curvature flow, a Lagrangian in this class will develop singularities before converging to the special Lagrangian. Therefore in order for the theory to progress, new techniques must be developed in order to continue the flow beyond the occurrence of singularities.

It was conjectured by Joyce [26, Prob 3.14] that if the singularities developed by the flow were of a certain form then one would be able to continue the flow uniquely in some weak sense beyond the singular time. The following result of the author in collaboration with Tom Begley [5, Thm 6.1] is a proof of the existence part of the aforementioned conjecture.

Theorem 2.3.1. *Suppose that $L \subset \mathbb{C}^m$ is a compact zero-Maslov Lagrangian submanifold of $\mathbb{C}^m$ with a finite number of singularities, each of which is asymptotic to a pair of transversally intersecting planes $P_1 + P_2$ where neither $P_1 + P_2$ nor $P_1 - P_2$ are area minimising. Then there exist $T > 0$ and a Lagrangian mean curvature flow $(L_t)_{0<t<T}$ such that as $t \searrow 0$, $L_t \to L$ as varifolds and in C^∞_{loc} away from the singularities.*

Remark. While it has been assumed the singular Lagrangian L lies inside Euclidean space, the local nature of the analysis used to prove the result means that this result can quite easily be generalised to a singular Lagrangian inside a Calabi–Yau manifold. The assumption on the structure of the singular points of L may seem somewhat restrictive, however, as stated above, we know that singularities developed by the flow must be asymptotic to a finite union of special Lagrangian cones, and such a union in $\mathbb{C}^2$ is a union of Lagrangian planes.

The proof of this result is very similar to the proof of an analogous result for *network flows* of Ilmanen–Neves–Schulze [18, Thm 1.1]. The idea is very simple. We construct a family of smooth compact Lagrangians L^s for $0 < s \leq c$ by desingularising L. We will discuss how this is done in detail in Section 2.3.1 below. Then standard short time existence for mean curvature flow yields for each s a mean curvature flow $(L^s_t)_{0\leq t<T_s}$ for some time $T_s > 0$. We will show that $\inf_{s>0} T_s = T_0 > 0$, and so we may apply a compactness result that allows us to pass to a subsequential limit (in s) of flows, where the limiting flow $(L_t)_{0<t<T_0}$ will be smooth and exist for time $0 < t < T_0$. To do this we need to be able to control the Gaussian density ratios of the manifolds L^s_t uniformly in t and s. We will give an overview of this part of the proof in Section 2.3.2.

2.3.1 Construction of the approximating family

We consider a Lagrangian submanifold L of $\mathbb{C}^n$ with a singularity at the origin which is asymptotic to the pair of planes P considered in Theorem 2.3.1. We will approximate L by gluing in the unique self-expander Σ of Theorem 2.2.3 which is asymptotic to P at smaller and smaller scales in place of the singularity.

Proposition 2.3.2. *Let L be an exact, zero-Maslov Lagrangian submanifold of $\mathbb{C}^m$ with a singular point at the origin asymptotic to the pair of planes P described in Theorem 2.3.1. Let Σ denote the unique zero-Maslov self-expander asymptotic to P. Then there exists a family $(L^s)_{0<s\leq c}$ of smooth compact zero-Maslov Lagrangians, exact in the ball $B_4(0)$ satisfying the following conditions.*

(H1) *The area ratios are uniformly bounded, i.e. there exists a constant D_1 such that*

$$\mathcal{H}^m(L^s \cap B_r(x)) \leq D_1 r^m \qquad \forall r > 0,\ \forall s \in (0, c],\ \forall x.$$

(H2) *There is a constant D_2 such that for every s and $x \in L^s \cap B_4(0)$*

$$|\theta^s(x)| + |\beta^s(x)| \leq D_2(|x|^2 + 1),$$

where θ^s and β^s are, respectively, the Lagrangian angle of L^s and a primitive for the Liouville form on L^s.

(H3) *For any $\alpha \in (0,1)$, the rescaled manifolds $\tilde{L}^s := (2s)^{-1/2}L^s$ converge in $C^{1,\alpha}_{loc}$ to Σ. Moreover the second fundamental form of $\tilde{L}^s$ is bounded uniformly in s and without loss of generality we can assume that*

$$\lim_{s\to 0}(\tilde{\theta}^s + \tilde{\beta}^s) = 0$$

locally on $\tilde{L}^s$. (Note that $\tilde{L}^s$ is exact in the ball $B_{4(2s)^{-1/2}}(0)$ so we can make sense of $\tilde{\beta}^s$ in the limit.)

(H4) *The connected components of $P \cap A(r_0\sqrt{s}, 4)$ are in one to one correspondence with the connected components of $L^s \cap A(r_0\sqrt{s}, 4)$, and each component can be parametrised as a graph over the corresponding plane P_i*

$$L^s \cap A(r_0\sqrt{s}, 3) \subset \{x + u_s(x) | x \in P \cap A(r_0\sqrt{s}, 3)\} \subset L^s \cap A(r_0\sqrt{s}, 4),$$

where the function $u_s : P \cap A(r_0\sqrt{s}, 3) \to P^\perp$ is normal to P and satisfies the estimate

$$|u_s(x)| + |x|\,|\nabla u_s(x)| + |x|^2|\nabla^2 u_s(x)| \leq D_3\left(|x|^2 + \sqrt{2s}e^{-b|x|^2/2s}\right),$$

where ∇ denotes the covariant derivative on P, and $b > 0$.

Proof. Since L is conically singular we may write $L \cap B_4(0)$ as a graph over $P \cap B_4(0)$ (possibly rescaling L so that this is the case). We may further apply the Lagrangian neighbourhood theorem (its extension to cones was proved by Joyce, [23, Thm 4.3]), so that we may identify $L \cap B_4(0)$ with the graph of a one-form γ on P. Recall that the manifold corresponding to the graph of such a one-form is Lagrangian if and only if the one-form is closed.

Moreover, since we have assumed that L is exact inside $B_4(0)$, there exists $u \in C^\infty(P \cap B_4(0))$ such that $du = \gamma$. Since we know that γ must decay quadratically, we can choose a primitive for γ which has cubic decay, i.e.,

$$|\nabla^k u(x)| \leq C|x|^{3-k}. \tag{2.3.1}$$

We saw in Theorems 2.2.3 and 2.2.4 that there exists a unique, smooth zero-Maslov self-expander asymptotic to P. We may also identify the self-expander outside a ball of radius r_0 with the graph of a one-form over P and, since a zero-Maslov class Lagrangian self-expander is globally exact, there exists a function $v \in C^\infty(P \backslash B_{r_0}(0))$ such that the self-expander is described by the exact one-form $\psi = dv$ on $P \backslash B_{r_0}(0)$. Further, Lotay and Neves proved that [40, Thm 3.1]

$$\|v\|_{C^k(P \backslash B_r(0))} \leq C e^{-br^2} \quad \text{for all } r \geq r_0. \tag{2.3.2}$$

We will glue $\Sigma_s := \sqrt{2s}\Sigma$ into the initial condition L to resolve the singularity. Our new manifold, L^s, will be the rescaled self-expander Σ^s inside $B_{r_0\sqrt{2s}}(0)$, the manifold L outside $B_4(0)$ and will smoothly interpolate between the two on the annulus $A(r_0\sqrt{2s}, 4)$.

To do this, we will glue together the primitives of the one-forms corresponding to these manifolds, before taking the exterior derivative. This gives us a one-form that will describe L^s on the annulus $A(r_0\sqrt{2s}, 4)$, which ensures L^s is still Lagrangian and is exact in $B_4(0)$. We will then show that this family satisfies the properties (H1)-(H4).

Let $\varphi : \mathbb{R}_+ \to [0,1]$ be a smooth function satisfying $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$. Consider the one-form given by, for $r_0\sqrt{2s} \leq |x| \leq 4$, $0 < s \leq c$

$$\gamma_s(x) = dw_s(x) = d\left[\varphi(s^{-1/4}|x|) 2s v(x/\sqrt{2s}) + (1 - \varphi(s^{-1/4}|x|))u(x)\right], \tag{2.3.3}$$

where we have that $r_0\sqrt{2s} < s^{1/4} < 2s^{1/4} < 4$ holds for all $s \leq c$. Notice that in particular we must have $c < 1$. Then $\gamma_s(x) \equiv \psi_s(x) := \sqrt{2s}\psi(x/\sqrt{2s})$, the one-form corresponding to the rescaled self-expander Σ_s for $|x| < s^{1/4}$ and $\gamma_s \equiv \gamma$ for $|x| > 2s^{1/4}$. Notice that since γ_s is exact, it is closed and therefore its graph corresponds to an exact Lagrangian.

We define the smooth exact Lagrangian L^s by

- $L^s \cap B_{r_0\sqrt{2s}}(0) = \Sigma_s \cap B_{r_0\sqrt{2s}}(0)$,
- $L^s \cap A(r_0\sqrt{2s}, 4) = \text{graph } \gamma_s$,

- $L^s \backslash B_4(0) = L \backslash B_4(0)$.

We will now show that L^s satisfies (H1)-(H4).

For (H1), notice that both the self-expander and the initial condition individually satisfy (H1), and so for the rescaled self-expander, we have that for all x

$$\begin{aligned}\mathcal{H}^n(\Sigma_s \cap B_R(x)) = \mathcal{H}^n((\sqrt{2s}\Sigma) \cap B_R(x)) = (2s)^{n/2}\mathcal{H}^n(\Sigma \cap B_{R/\sqrt{2s}}(x))\\ \leq (2s)^{n/2} D_1 \left(\frac{R}{\sqrt{2s}}\right)^n = D_1 R^n.\end{aligned}$$

Since L^s interpolates between Σ_s and L on a compact region, L^s satisfies (H1).

We see that (H2) is satisfied because the Lagrangian angle of the initial condition L and the self-expander Σ are bounded, as is that of the rescaled self-expander Σ_s by the evolution equation [5, Lem 3.1(i)] and the maximum principle, since the Lagrangian angle of P is locally constant. When we interpolate between the two, we may consider the formula for the Lagrangian angle of a Lagrangian graph, as seen in [11, pg 5]. This tells us that a Lagrangian graph in $\mathbb{C}^n$ (over $\mathbb{R}^n$) given by $(x_1, ..., x_n, u_1(x), ..., u_n(x))$, where $u : \mathbb{R}^n \to \mathbb{R}, u_i := \frac{\partial u}{\partial x_i}$, has Lagrangian angle

$$\theta = \sum \arctan \lambda_i,$$

where the λ_i's are the eigenvalues of the Hessian of u. Since the eigenvalues of the Hessian of u are some nonlinear function of the second derivatives of u, if the C^2 norm of u is small we have that the Lagrangian angle of the graph is close to that of the Lagrangian angle of the plane that u is a graph over. So we can uniformly bound the Lagrangian angle of the graph. Since in our case, the Lagrangian angle of γ_s is given by the sum of arctangents of the eigenvalues of the Hessian of the function w_s, and, as we will show when we prove (H4), the C^2 norm of w_s is small, this means that we can uniformly bound the Lagrangian angle of the graph γ_s, and so the Lagrangian angle of L^s.

On the initial condition, since $\lambda = Jx$, we have that $d\beta_L = \lambda|_L = (Jx)^T$. Therefore, β_L is bounded quadratically, and so is the primitive for the Liouville form of $L^s \backslash B_{2s^{1/4}}(0)$.

On the self-expander, applying the maximum principle to the evolution equation for θ_t^s [5, Lem 3.1(i)], we have β_s (the primitive of $\lambda|_{\Sigma_s}$) is bounded by β_P, and so $|\beta^s(x)| \leq |\beta_P(x)| \leq C|x|^2$ for $|x| < s^{1/4}$. So it remains to check this still holds where we interpolate. We perform a calculation similar to that in the proof of [5, Lem 3.1(ii)]. We have that, for L_t^s the manifold described by the graph of the one-form tdw_s,

$$\frac{d}{dt}\lambda|_{L_t^s} =: \mathcal{L}_{J\nabla w_s}\lambda|_{L_t^s} = d(J\nabla w_s \lrcorner \lambda|_{L_t^s}) + J\nabla w_s \lrcorner d\lambda|_{L_t^s}.$$

Since $d\lambda = \omega$ and $J\nabla w_s \lrcorner \omega = dw_s$ and possibly adding constant to β_t^s dependent on s and t, we have that

$$\frac{d\beta_t^s}{dt} = -2w_s + \langle x, \nabla w_s\rangle|_{L_t^s},$$

where $d\beta_t^s$ is equal to the restriction of the Liouville form λ to graph of $t\gamma_s$. Integrating, we find that

$$\beta^s = \beta_P - 2w_s + \int_0^1 \langle x, \nabla w_s\rangle|_{L_t^s}\, dt,$$

where β_P is the primitive for λ on P. Now, w_s is bounded independently of s by $D(1+|x|^2)$, using (2.3.1) and (2.3.2), as is $\langle x, \nabla w_s\rangle$, using Cauchy-Schwarz and the estimates (2.3.1) and (2.3.2) so we find that β^s is bounded independently of s on the annulus $A(s^{1/4}, 2s^{1/4})$. Therefore, we have that

$$|\theta^s(x)| + |\beta^s(x)| \leq D_2(|x|^2 + 1).$$

and so (H2) is satisfied.

To show that (H3) is satisfied, recall that we define L^s as $L^s \cap B_{r_0\sqrt{2s}}(0) = \Sigma_s \cap B_{r_0\sqrt{2s}}(0)$, $L^s \backslash B_4(0) = L \backslash B_4(0)$ and we interpolate smoothly between the two, which exactly happens when $s^{1/4} \leq |x| \leq 2s^{1/4}$. Therefore when we rescale by $1/\sqrt{2s}$, we have that $\tilde{L}^s \cap B_{r_0}(0) \equiv \Sigma$. So it remains to check convergence outside this ball.

On the annulus $r_0 \leq |x| \leq 4/\sqrt{2s}$, $\tilde{L}^s$ is identified with the graph of the following one-form

$$\tilde{\gamma}_s(x) = d\left[\varphi(s^{1/4}|x|)v(x) + (1 - \varphi(s^{1/4}|x|))\frac{u(\sqrt{2s}x)}{2s}\right].$$

From this expression, noticing that

$$\frac{u(\sqrt{2s}x)}{2s} \leq C\frac{(2s)^{3/2}x^3}{2s} = C\sqrt{2s}x,$$

we see that as $s \to 0$, $\tilde{\gamma}_s \to dv = \psi$, the one-form whose graph is identified with Σ. This says that, outside $B_{r_0}(0)$, $\tilde{L}^s \to \Sigma$ as $s \to 0$ smoothly. Therefore we actually have stronger than the required $C^{1,\alpha}_{loc}$ convergence.

Finally, we check that the second fundamental form of $\tilde{L}^s$ is uniformly bounded in s. We have that the second fundamental form of Σ must be bounded, and if A is the second fundamental form of L, rescaling L by $1/\sqrt{2s}$ means that the second fundamental form scales by $\sqrt{2s}$. Since $\sqrt{2s} < 1$, we can uniformly bound both second fundamental forms so that $\tilde{L}^s$, which is a combination of both Σ and $1/\sqrt{2s}L$, has second fundamental form uniformly bounded in s.

To see (H4), first notice that since we can write $L^s \cap A(r_0\sqrt{2s}, 4)$ as a graph over $P \cap A(r_0\sqrt{2s}, 4)$, we have that L^s has the same number of connected components as P in the annulus $A(r_0\sqrt{2s}, 4)$.

We now must estimate γ_s. Firstly, note that we have

$$|\nabla^k(v(x/\sqrt{2s}))| \leq |(2s)^{-k/2}(\nabla^k v)(x/\sqrt{2s})| \leq C(2s)^{-k/2}e^{-b|x|^2/2s}, \tag{2.3.4}$$

where we have used (2.3.2).

We will need different estimates on $2s\nabla^2 v(x/\sqrt{2s})$ and $2s\nabla^3 v(x/\sqrt{2s})$, which we find as follows.

$$\begin{aligned}|2s\nabla^2 v(x/\sqrt{2s})| &\leq Ce^{-b|x|^2/2s} = C\frac{\sqrt{2s}}{|x|}\frac{|x|}{\sqrt{2s}}e^{-b|x|^2/2s} \\ &= C\frac{\sqrt{2s}}{|x|}e^{-\tilde{b}|x|^2/2s}\frac{|x|}{\sqrt{2s}}e^{-\tilde{b}|x|^2/2s} \leq \tilde{C}\frac{\sqrt{2s}}{|x|}e^{-\tilde{b}|x|^2/2s},\end{aligned} \tag{2.3.5}$$

where $\tilde{b} = b/2$ and $\tilde{C} = Ce^{-1/2}/\sqrt{b}$, since the function $y \mapsto ye^{-by^2/2}$ is bounded independently of y (by $e^{-1/2}/\sqrt{b}$) on $\mathbb{R}$, and so $\tilde{C}$ is independent of s.

A similar calculation, this time noticing the uniform boundedness of the function $y \mapsto ye^{-by/2}$ for $y > 0$ we can show that

$$|2s\nabla^3 v(x/\sqrt{2s})| \leq C\frac{\sqrt{2s}}{|x|^2}e^{-b|x|^2/2s}, \tag{2.3.6}$$

where we make C (which remains independent of s) larger if necessary and b smaller (which does not affect the previous estimates).

We have, using the definition in (2.3.3),

$$\begin{aligned}|\gamma_s| = |\nabla w_s| = |&\varphi'(s^{-1/4}|x|)2s^{3/4}v(x/\sqrt{2s}) + \varphi(s^{-1/4}|x|)2s\nabla[v(x/\sqrt{2s})] \\ &- s^{-1/4}\varphi'(s^{-1/4}|x|)u(x) + (1-\varphi(s^{-1/4}|x|))\nabla u(x)|,\end{aligned}$$

and, using that $s^{3/4} = \sqrt{s}s^{1/4} < \sqrt{s}$ since $s < 1$, (2.3.1) and (2.3.4) imply that

$$\begin{aligned}|\gamma_s| &\leq \sqrt{2s}Ce^{-b|x|^2/2s} + \sqrt{2s}Ce^{-b|x|^2/2s} + C|x|^{3-1} + C|x|^2 \\ &\leq C\left[\sqrt{2s}e^{-b|x|^2/2s} + |x|^2\right],\end{aligned} \tag{2.3.7}$$

where we have made C larger.

Now consider

$$\begin{aligned}|\nabla\gamma_s| = |\nabla^2 w_s| = |&\varphi''(s^{-1/4}|x|)2s^{1/2}v(x/\sqrt{2s}) + \varphi'(s^{-1/4}|x|)4s^{3/4}\nabla[v(x/\sqrt{2s})] \\ &+ \varphi(s^{-1/4}|x|)2s\nabla^2[v(x/\sqrt{2s})] - s^{-1/2}\varphi''(s^{-1/4}|x|)u(x) \\ &- 2s^{-1/4}\varphi'(s^{-1/4}|x|)\nabla u(x) + (1-\varphi(s^{-1/4}|x|))\nabla^2 u(x)|\end{aligned}$$

Using that on the support of φ' and φ'' we have ($s < 1$) $\sqrt{s} < s^{1/4} \leq \sqrt{2}\sqrt{2s}/|x|$, and applying the estimates (2.3.4) and (2.3.5)

$$\begin{aligned}|\nabla\gamma_s| &\leq C\left[\left(\frac{\sqrt{2s}}{|x|} + \frac{\sqrt{2s}}{|x|} + \frac{\sqrt{2s}}{|x|}\right)e^{-b|x|^2/2s} + |x|^{3-2} + |x|^{2-1} + |x|\right] \\ &\leq C\left[\frac{\sqrt{2s}}{|x|}e^{-b|x|^2/2s} + |x|\right].\end{aligned} \tag{2.3.8}$$

Finally, performing a similar computation to those above and combining (2.3.4), (2.3.5) and (2.3.6) we find that

$$|\nabla^2\gamma_s| \leq C\left[\frac{\sqrt{2s}}{|x|^2}e^{-b|x|^2/2s} + 1\right]. \tag{2.3.9}$$

Combining (2.3.7), (2.3.8) and (2.3.9), we have that

$$|\gamma_s| + |x||\nabla\gamma_s| + |x|^2|\nabla^2\gamma_s| \leq D_3 \left(|x|^2 + \sqrt{2s}e^{-b|x|^2/2s}\right),$$

where D_3 is a constant independent of s. Therefore (H4) is satisfied. □

2.3.2 Overview of the proof of Theorem 2.3.1

Having proved the existence of the approximating family $(L^s)_{0<s\leq c}$ in Proposition 2.3.2, we will now give an overview of the main ideas needed to complete the proof of Theorem 2.3.1. The following theorem, a rescaled version of [5, Thm 5.1], gives uniform control over a modified version of the Gaussian density ratios. From this result, the proof of Theorem 2.3.1 essentially follows from the local regularity theorem 2.2.2.

Theorem 2.3.3. *Let $(L^s)_{0<s\leq c}$ be the family of compact Lagrangians satisfying* (H1)–(H4) *constructed in Proposition 2.3.2. Denote by $(L^s_t)_{0\leq t<T_s}$ the smooth solution to Lagrangian mean curvature flow with initial condition L^s. Let $\epsilon_0 > 0$. Then there exist s_0, δ_0 and τ depending on D_1, D_2, D_3, Σ and r_0 so that if*

$$t \leq \delta_0, r^2 \leq \tau \text{ and } s \leq s_0,$$

then

$$\tilde{\Theta}^s_t(x_0, r) \leq 1 + \epsilon_0, \tag{2.3.10}$$

for all x_0 with $|x_0| \leq (2(s+t))^{-1/2}$. Here

$$\tilde{\Theta}^s_t(x_0, r) := \int_{\tilde{L}^s_t} \rho_{(x_0,0)}(x, -r^2)\, d\mathcal{H}^m(x) = \int_{\tilde{L}^s_t} \frac{1}{(4\pi r^2)^{m/2}} e^{-|x-x_0|^2/4r^2}\, d\mathcal{H}^m(x),$$

are the Gaussian density ratios for the rescaled manifolds

$$\tilde{L}^s_t = \frac{L^s_t}{\sqrt{2(s+t)}}.$$

The properties (H1)–(H4) of the family L^s allow us to obtain estimates of the form (2.3.10) everywhere. The most interesting part of the proof is showing that these estimates hold uniformly for $s \leq s_0$ and $t \leq \delta_0$ and x_0 in some compact set K. We will sketch this part of the proof. We first require the following stability result [5, Thm 4.2].

Theorem 2.3.4. *Fix $R, r, \tau > 0$, $\alpha, \epsilon_0 < 1$, and $C, M < \infty$. Let Σ be the unique smooth zero-Maslov Lagrangian self-expander asymptotic to $P_1 + P_2$ given by Theorem 2.2.4. Then for all $\epsilon > 0$ there exists $\tilde{R} \geq R, \eta, \nu > 0$ each dependent on $\epsilon_0, \epsilon, r, R, \tau, \alpha, C, M$ and $P_1 + P_2$ such that if L is a smooth Lagrangian submanifold which is zero-Maslov in $B_{\tilde{R}}(0)$ and*

(i) $|A| \leq M$ *on* $L \cap B_{\tilde{R}}(0)$,

(ii) $\int_L \rho_{(x,0)}(y, -r^2)\, d\mathcal{H}^m \leq 1 + \epsilon_0$ *for all* x *and* $0 < r \leq \tau$,

(iii) $\int_{L \cap B_{\tilde{R}}(0)} |\vec{H} - x^\perp|^2\, d\mathcal{H}^m \leq \eta$,

(iv) *The connected components of $L \cap A(r, \tilde{R})$ are in one to one correspondence with the connected components of $(P_1 + P_2) \cap A(r, \tilde{R})$ and*
$$\operatorname{dist}(x, P) \leq \nu + C \exp\left(-\frac{|x|^2}{C}\right),$$
for all $x \in L \cap A(r, \tilde{R})$;

then L is ϵ-close to Σ in $C^{1,\alpha}(\overline{B_{\tilde{R}}(0)})$ in the sense of Definition 2.2.3.

This theorem is proved by contradiction. If Theorem 2.3.4 does not hold then we can negate it to construct a sequence of Lagrangians (L^i), none of which are ϵ-close to Σ, satisfying properties which force a subsequence of (L^i) to converge to a smooth self-expander L^∞ asymptotic to $P_1 + P_2$. By the uniqueness theorem 2.2.4 we must have that $L^\infty = \Sigma$, which is a contradiction.

The second result that we require is the following monotonicity result, see [5, Lem 5.7] for this result in full generality.

Lemma 2.3.5. *Let $a > 1, \eta > 0$ and $R > 0$. Then there exists $\delta > 0$ such that if $s \leq T \leq \delta$, then*
$$\frac{1}{(a-1)T} \int_T^{aT} \int_{\tilde{L}_t^s \cap B_R(0)} |\vec{H} - x^\perp|^2\, d\mathcal{H}^m\, dt \leq \eta.$$

Lemma 2.3.5 follows roughly from the evolution equations for the Lagrangian angle and primitive for the Liouville form [5, Lem 3.1] and the monotonicity formula 2.2.1.

Sketch proof of Theorem 2.3.3. Let $\tau > 0$ be a carefully chosen real number and let K be a compact set. Since each L^s is a compact Lagrangian, standard short time existence for Lagrangian mean curvature flow tells us that

$$T_s := \sup\{ T \mid \tilde{\Theta}_t^s(x_0, r) \leq 1 + \epsilon_0 \; \forall r^2 \leq \tau, t \leq T, x_0 \in K\} > 0,$$

for each $s > 0$.

Fix $\epsilon > 0$, and let $\tilde{R}$, ν and η be as in Theorem 2.3.4. Choose $a > 1$. An application of Lemma 2.3.5 with a, $\tilde{R}$ and η yields a $\delta > 0$. It is claimed that $T_s > \delta$ for all $s > 0$.

Suppose not, that is, $T_{s_1} < \delta$ for some $s_1 > 0$. Let $T = T_{s_1}/a$. Then we may deduce from Lemma 2.3.5 that there exists $t_1 \in (T_{s_1}/a, T_{s_1})$ with

$$\int_{\tilde{L}_{t_1}^{s_1} \cap B_{\tilde{R}}(0)} |\vec{H} - x^\perp|^2 \, d\mathcal{H}^m < \eta.$$

So $\tilde{L}_{t_1}^{s_1}$ satisfies condition (iii) of Theorem 2.3.4. Further, by definition of T_{s_1}, $\tilde{L}_{t_1}^{s_1}$ must satisfy hypothesis (ii) of Theorem 2.3.4. With some extra work because of the rescaling, we may deduce that $\tilde{L}_{t_1}^{s_1}$ satisfies hypothesis (i) of Theorem 2.3.4 from the local regularity theorem 2.2.2. Finally, from condition (H4) and a little work it can be deduced that $\tilde{L}_{t_1}^{s_1}$ satisfies hypothesis (iv) of Theorem 2.3.4.

So Theorem 2.3.4 tells us that $\tilde{L}_{t_0}^{s}$ is close in $C^{1,\alpha}(\overline{B_{\tilde{R}}(0)})$ to the self-expander Σ. However, this gives us control over the Gaussian density ratios for a time interval exceeding T_{s_1} (see [5, Lem 8.2]), a contradiction. □

2.3.3 Concluding remarks

While Theorem 2.3.1 brings us a step closer to a method for extending Lagrangian mean curvature flow beyond a finite time singularity, there are still several issues that need to be overcome before this result can be made more general. For instance, while Theorem 2.3.1 establishes existence for Lagrangian mean curvature flow, it remains to see whether this solution is unique.

Moreover, the method used to prove Theorem 2.3.1 may be somewhat restrictive. Say that we wanted to prove short-time existence for Lagrangian mean curvature flow starting from any compact conically singular zero-Maslov Lagrangian, i.e., a Lagrangian with a singular point asymptotic to a cone C. In order to establish short-time existence for Lagrangian mean curvature flow starting from such a Lagrangian using the method described in this chapter, one requires the existence of a unique self-expander asymptotic to C. This may be a lot to ask, and so it could be beneficial to seek an alternative method to prove Theorem 2.3.1 that is more robust.

Chapter 3

Deformation theory of compact Cayley submanifolds

In this chapter, we consider Cayley deformations of compact Cayley submanifolds. We will pay special attention to the case where the ambient manifold is a four-dimensional Calabi–Yau manifold and the Cayley submanifold is a two-dimensional complex submanifold.

3.1 Introduction

Consider a complex surface N inside a Calabi–Yau four-fold M. As we have seen, N is not only a complex submanifold of M, but also a Cayley submanifold of M. A natural question to ask is whether we can deform N as a Cayley submanifold into a Cayley submanifold N' of M that is not a complex submanifold of M. We can actually see from the work of Harvey and Lawson [15] that this is not possible when N is compact. Recall the result of Harvey and Lawson quoted in Proposition 1.2.6. Let N be as above and let N' be a Cayley deformation of N. Then N' is certainly homologous to N, and since calibrated submanifolds are volume minimising in their homology class, we must have that

$$\int_N \mathrm{vol}_N = \int_{N'} \mathrm{vol}_{N'}.$$

But then Proposition 1.2.6 tells us that N' must also be a complex submanifold. The ultimate aim of this chapter is to recover this result directly and to understand geometrically why this is the case.

We begin this chapter with a literature review of some important works relevant to this problem. In particular, we will state Kodaira's theorem 3.1.1 on the deformation theory of compact complex submanifolds and McLean's theorem 3.1.3 on the deformation theory of compact Cayley submanifolds.

In Section 3.2 we give a proof of Theorem 3.2.6, stated below, on Cayley deformations of a compact Cayley submanifold inside a $Spin(7)$-manifold.

Theorem. *Let (X, g, Φ) be a $Spin(7)$-manifold and let Y be a compact Cayley submanifold of X. Let D denote the first order elliptic operator defined in (3.2.6). Then there exist a smooth manifold K_0, which is an open neighbourhood of 0 in $\mathrm{Ker}\ D$, and a smooth map $g_2 : K_0 \to \mathrm{Ker}\ D^*$ with $g_2(0) = 0$ so that an open neighbourhood of Y in the moduli space of Cayley deformations of Y in X is homeomorphic to an open neighbourhood of 0 in $\mathrm{Ker}\ g_2$.*

Moreover, the expected dimension of the moduli space of Cayley deformations of Y in X is given by

$$\mathrm{ind}\ D := \dim \mathrm{Ker}\ D - \dim \mathrm{Ker}\ D^*,$$

where

$$D^* : C^\infty(E) \to C^\infty(\nu_X(Y)),$$

is the formal adjoint of D. If $\mathrm{Ker}\ D^* = \{0\}$ then the moduli space of Cayley deformations of Y in X is a smooth manifold near Y of dimension

$$\dim \mathrm{Ker}\ D.$$

This theorem is proved by following a method developed by McLean [43] for studying the deformations theory of compact calibrated submanifolds.

In Section 3.3 we will consider Cayley deformations of a compact complex surface N in a Calabi–Yau four-fold M. We may apply the results of Section 3.2 to N, however since N is a complex surface we can exploit its complex structure to identify 'small' Cayley deformations of N with the kernel of a geometrically (and holomorphically) natural first order elliptic operator. Once we have made this identification, it will not be difficult to deduce the next main result of the chapter, Theorem 3.3.4.

Theorem. *Let N be a two-dimensional compact complex submanifold of a Calabi–Yau four-fold M. Then the expected (real) dimension of the moduli space of Cayley deformations of N in M is equal to the (complex) index of the operator*

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Moreover, if the kernel of

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)),$$

is $\{0\}$ then the moduli space of Cayley deformations of N in M is a smooth manifold near N of dimension

$$\dim \operatorname{Ker} \bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

This will allow us to deduce an expression for the expected dimension of this moduli space in terms of topological invariants on N in Theorem 3.3.5.

In Section 3.4 we will compare complex and Cayley deformations of N by applying a McLean-style argument to characterise the complex deformations of N in M. Recall that in Proposition 1.2.9 we defined a differential form τ on a $Spin(7)$-manifold that vanishes if, and only if restricted to a Cayley submanifold. Section 3.4.1 will be dedicated to finding a differential form, σ, say, on a Calabi–Yau manifold that vanishes if, and only if, it is restricted to a two-dimensional complex submanifold of that Calabi–Yau. This will allow us to identify complex deformations of a compact complex surface in a Calabi–Yau manifold with the kernel of a nonlinear operator. In Section 3.4.2, we will study the properties of this operator. The section will culminate in the proof of Theorem 3.4.7.

Theorem. *Let N be a compact complex surface inside a four-dimensional Calabi–Yau manifold M. Then the moduli space of Cayley deformations of N in M near N is isomorphic to the moduli space of complex deformations of N in M, which near N is a smooth manifold of dimension*

$$\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial} + \dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial}^* = 2\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial},$$

where

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$
$$\bar{\partial}^* : C^\infty(\Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Finally, in Section 3.5 we will compute the index of the operator $\bar{\partial} + \bar{\partial}^*$ for a family of examples.

3.1.1 Literature review

The problem that we study in the subsequent chapters of thesis, in various forms, can be broadly stated as follows. Let M be a four-dimensional Calabi–Yau manifold with two-dimensional complex submanifold N. As we noted in Section 1.2.5, N is also a Cayley submanifold of M. We can deform N as a real submanifold of M to some other submanifold N'. We would like to know when N' is still complex, and whether N' can still be Cayley but no longer complex. In this section, we will discuss two works that are related to this problem. The first, in Section 3.1.1, is Kodaira's theory of the deformation of compact complex submanifolds. The second, in Section 3.1.1 is McLean's study of the deformation of compact Cayley submanifolds inside a $Spin(7)$-manifold.

Kodaira's deformation theory of complex submanifolds

The deformation theory of the canonical example of a calibrated submanifold, complex submanifolds, was studied using techniques from algebraic geometry by Kodaira [31]. We will later be deforming complex submanifolds of a Calabi–Yau manifold, although our approach will be very different. It will be interesting to compare the results of these different approaches, and so we will quote Kodaira's theorem here.

Write $H^k(N, \nu_M^{1,0}(N))$ for the kth sheaf cohomology group of the sheaf of holomorphic sections of the holomorphic normal bundle of a complex submanifold N of a complex manifold M. Define the moduli space of complex deformations of N in M to be the set of complex submanifolds N' of M so that there exists a diffeomorphism $N \to N'$ isotopic to the identity.

Theorem 3.1.1 (Kodaira [31, Main Thm])**.** *Let M be a complex manifold with compact complex submanifold N. If $H^1(N, \nu_M^{1,0}(N)) = 0$, then the moduli space of complex deformations of N is isomorphic to $H^0(N, \nu_M^{1,0}(N))$.*

Remark. In the context of Theorem 3.1.1, we call $H^0(N, \nu_M^{1,0}(N))$ the *infinitesimal complex deformations* of N, and $H^1(N, \nu_M^{1,0}(N))$ the *obstruction space*.

Dolbeault's theorem [14, pg 45] allows us to identify the infinitesimal complex deformations of a compact complex submanifold N in M with the Dolbeault cohomology group $H_{\bar\partial}^{0,0}(N, \nu_M^{1,0}(N))$ and the obstruction space with $H_{\bar\partial}^{0,1}(N, \nu_M^{1,0}(N))$. Since N is compact we may deduce from the Hodge decomposition theorem [17, Thm 4.1.13] the following corollary to Kodaira's theorem.

Corollary 3.1.2. *Let M be a complex manifold with compact complex submanifold N. Then the space of infinitesimal complex deformations of N is isomorphic to the kernel of the operator*

$$\bar\partial : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

McLean's deformation theory of compact Cayley submanifolds

In the mid-nineties, McLean [43] proved analogous results to Kodaira's theorem 3.1.1 for the compact calibrated submanifolds that arise naturally in G_2- and $Spin(7)$-manifolds, including compact Cayley submanifolds of a $Spin(7)$-manifold. We quote his main result about compact Cayley submanifolds of $Spin(7)$-manifolds here.

Theorem 3.1.3 ([43, Thm 6-3])**.** *Let Y be a compact Cayley submanifold of a $Spin(7)$-manifold X and suppose that Y admits a spin structure. There exists a rank two complex vector bundle A over Y so that moduli space of Cayley deformations of Y in X is isomorphic to the kernel of the twisted Dirac operator*

$$\not{D} : C^\infty(\mathbb{S}_+ \otimes A) \to C^\infty(\mathbb{S}_- \otimes A), \tag{3.1.1}$$

as long as the kernel of the formal adjoint to (3.1.1) *vanishes.*

The method developed by McLean to prove this result is one that we will utilise extensively in this chapter.

3.2 Deformation theory of compact Cayley submanifolds inside $Spin(7)$-manifolds

To prove McLean's result Theorem 3.1.3, one identifies deformations of a compact Cayley submanifold with the zero set of a nonlinear partial differential operator

$$\not{F} : C^\infty(\mathbb{S}_+ \otimes A) \to C^\infty(\mathbb{S}_- \otimes A),$$

which has linear part

$$\not{D} : C^\infty(\mathbb{S}_+ \otimes A) \to C^\infty(\mathbb{S}_- \otimes A),$$

the twisted Dirac operator. In this section, following McLean's approach for proving Theorem 3.1.3, we will prove a more general version of McLean's result, Theorem 3.2.6. Theorem 3.2.6 is more general in the sense that it is not assumed that the compact Cayley submanifold being deformed has a spin structure, and moreover gives an expression for the expected dimension of the moduli space of Cayley deformations of a compact Cayley submanifold Y inside a $Spin(7)$-manifold X, defined formally in Definition 3.2.1 below, in terms of the index of a first order elliptic differential operator. In Proposition 3.2.3, we prove a result that we can consider isomorphic to Theorem 3.1.3 in the following sense. We will identify Cayley deformations of Y with the kernel of a nonlinear partial differential operator

$$F : C^\infty(\nu_X(Y) \otimes \mathbb{C}) \to C^\infty(E \otimes \mathbb{C}),$$

taking complexified normal vector fields to complexified sections of some rank four vector bundle E, which has linear part described by the elliptic operator

$$D : C^\infty(\nu_X(Y) \otimes \mathbb{C}) \to C^\infty(E \otimes \mathbb{C}).$$

If Y admits a spin structure, then we have that

$$\begin{tikzcd} C^\infty(\nu_X(Y) \otimes \mathbb{C}) \arrow[r, "F"] \arrow[d] & C^\infty(E \otimes \mathbb{C}) \arrow[d] \\ C^\infty(\mathbb{S}_+ \otimes A) \arrow[r, "\not{F}"] & C^\infty(\mathbb{S}_- \otimes A) \end{tikzcd}$$

commutes, with a similar commutative diagram for D and $\not{D}$.

The proof of Theorem 3.2.6 has been split into three main steps. Firstly, in Section 3.2.1 we describe how to identify deformations of Y with normal vector fields on Y. The main result of this section will be Proposition 3.2.2, identifying Cayley deformations of Y with the kernel of a nonlinear partial differential operator F. In Section 3.2.2, we will study the operator F. In particular, we compute the linear part of F in Proposition 3.2.3. Finally, in Section 3.2.3 the proof of Theorem 3.2.6 is completed by applying the Banach space implicit function theorem to F. This step relies on the observation that an elliptic operator on a compact manifold is Fredholm.

3.2.1 Deformations as normal vector fields

Let X be a manifold with a submanifold Y. We say that Y' is a deformation of Y in X if there exists a smooth family of embeddings $\iota_t : Y \to X$ such that $\iota_0(Y) = Y$ and $\iota_1(Y) = Y'$.

Definition 3.2.1. Let (X, g, Φ) be a $Spin(7)$-manifold, and let Y be a Cayley submanifold of X. Define the *moduli space of Cayley deformations of* Y, $\mathcal{M}_{\text{Cay}}(Y)$, to be the set of deformations Y' of Y that are Cayley submanifolds of (X, g, Φ).

The aim of this section is to study properties of $\mathcal{M}_{\text{Cay}}(Y)$ when Y is compact. To do this, we will identify nearby deformations of Y with small normal vector fields on Y. For this we require the tubular neighbourhood theorem. A proof of this result can be found in [33, IV, Thm 5.1].

Theorem 3.2.1 (Tubular neighbourhood theorem)**.** *Let X be a Riemannian manifold and Y be a closed embedded submanifold of X. Then there exists an open set $V \subseteq \nu_X(Y)$ containing the zero section and an open set $Y \subseteq T \subseteq X$ such that the exponential map*

$$\exp|_V : V \to T,$$

is a diffeomorphism.

Remark. Given a normal vector field v on Y taking values in V, we define $Y_v := \exp_v(Y) \subseteq T \subseteq X$. Then Y_v is a deformation of Y. We will denote by $\exp_v$ the diffeomorphism $Y \to Y_v$. Conversely, given another submanifold Y' of X so that $Y' \subseteq T$, we can use the inverse of $\exp|_V$ to define a normal vector field on Y.

Recall the alternative characterisation of Cayley submanifold given by Proposition 1.2.9. A submanifold Y' of a $Spin(7)$-manifold is Cayley if, and only if,

$$\tau|_{Y'} \equiv 0,$$

where τ is the Λ^2_7-valued four-form defined in Proposition 1.2.9. We will use this characterisation to construct a partial differential operator acting on normal vector fields on a compact Cayley submanifold Y whose kernel will be precisely the normal vector fields on Y that yield Cayley deformations of Y.

Proposition 3.2.2. *Let (X, g, Φ) be a $Spin(7)$-manifold with compact Cayley submanifold Y. Use the notation of the tubular neighbourhood theorem 3.2.1. The moduli space of Cayley deformations of Y in X is isomorphic near Y to the kernel of the following partial differential operator*

$$\begin{aligned} F : C^\infty(V) &\to C^\infty(E), \\ v &\mapsto \pi(*_Y \exp_v^*(\tau|_{Y_v})), \end{aligned} \tag{3.2.1}$$

where τ is defined in Proposition 1.2.9 and

$$\Lambda^2_7|_Y = \Lambda^2_+ Y \oplus E, \tag{3.2.2}$$

with $\pi : \Lambda^2_7|_Y \to E$ the projection map.

Proof. First note that we take smooth normal vector fields on Y to ensure that the deformation corresponding to v, Y_v, is a smooth manifold. Then Y_v is Cayley if, and only if, $\tau|_{Y_v} = 0$. Since $\exp_v$ is a diffeomorphism, $\tau|_{Y_v} = 0$ if, and only if, $\exp_v^*(\tau|_{Y_v}) = 0$, if, and only if $*_Y \exp_v^* \tau|_{Y_v} \equiv 0$. Finally, since τ, v and $\exp_v$ are smooth, we see that $*_Y \exp_v^* \tau|_{Y_v} \in C^\infty(\Lambda^2_7|_Y)$. It remains to show that $\pi(*_Y \exp_v^* \tau|_{Y_v}) = 0$ implies that $*_Y \exp_v^* \tau|_{Y_v} = 0$. For this we will employ a local argument.

Choose $y \in Y$. Then $T_yX|_Y = T_yY \oplus \nu_y(Y)$. Choose an orthonormal basis $\{e_1, \ldots, e_8\}$ for $T_yX|_Y$ so that

$$T_yY = \mathrm{span}\{e_1, e_2, e_3, e_4\}.$$

Let Y' be a small deformation of Y, and denote by f the diffeomorphism $Y \to Y'$. Then T_yX is naturally isometric to $T_{f(y)}X$. Denote an orthonormal basis of $T_{f(y)}X$ by $\{e'_1, \ldots, e'_8\}$ where e_i maps to e'_i under this isometry. We have that

$$T_{f(y)}Y' = \mathrm{span}\{v_1, v_2, v_3, v_4\},$$

where, without loss of generality since Y' is a small deformation of Y we may take

$$v_j = e'_j + \sum_{i=5}^{8} \lambda^i_j e'_i.$$

To prove the proposition we will suppose that

$$\pi(\tau_{f(y)}(v_1, v_2, v_3, v_4)) = 0, \tag{3.2.3}$$

and show that

$$\tau_{f(y)}(v_1, v_2, v_3, v_4) = 0. \tag{3.2.4}$$

Equation (3.2.3) gives us the following four equations

$$\begin{aligned}
\lambda_5^1 + \lambda_6^2 + \lambda_7^3 + \lambda_8^4 - &\sum_{6,7,8} \epsilon_{pqr}\lambda_p^2\lambda_q^3\lambda_r^4 - \sum_{5,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^3\lambda_r^4 \\
-&\sum_{5,6,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^4 - \sum_{5,6,7} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^3 = 0, \\
\lambda_6^1 - \lambda_5^2 - \lambda_8^3 + \lambda_7^4 + &\sum_{5,7,8} \epsilon_{pqr}\lambda_p^2\lambda_q^3\lambda_r^4 - \sum_{6,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^3\lambda_r^4 \\
-&\sum_{5,6,7} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^4 + \sum_{5,6,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^3 = 0, \\
\lambda_7^1 + \lambda_8^2 - \lambda_5^3 - \lambda_6^4 - &\sum_{5,6,8} \epsilon_{pqr}\lambda_p^2\lambda_q^3\lambda_r^4 - \sum_{5,6,7} \epsilon_{pqr}\lambda_p^1\lambda_q^3\lambda_r^4 \\
+&\sum_{6,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^4 + \sum_{5,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^3 = 0, \\
\lambda_8^1 - \lambda_7^2 + \lambda_6^3 - \lambda_5^4 + &\sum_{5,6,7} \epsilon_{pqr}\lambda_p^2\lambda_q^3\lambda_r^4 - \sum_{5,6,8} \epsilon_{pqr}\lambda_p^1\lambda_q^3\lambda_r^4 \\
+&\sum_{5,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^4 - \sum_{6,7,8} \epsilon_{pqr}\lambda_p^1\lambda_q^2\lambda_r^3 = 0,
\end{aligned} \tag{3.2.5}$$

where ϵ_{pqr} is skew-symmetric in p, q, r and $\epsilon_{pqr} = 1$ when $p < q < r$. Notice that if λ_j^i is a linear term, then there will be cubic terms of the form $\pm\lambda_p^l\lambda_q^m\lambda_r^n$, where $\{l, m, n\} \in \{1, 2, 3, 4\}\backslash\{i\}$ and $\{p, q, r\} \in \{5, 6, 7, 8\}\backslash\{j\}$.

Using your favourite equation solving software, we can solve for $\lambda_5^1, \lambda_6^1, \lambda_7^1$ and λ_8^1, which gives us four very complicated expressions which we will not give here. To show that Equation (3.2.4) is satisfied, it remains to show that

$$\sum_{\{i,j\}=\{5,7\},\{6,8\}} \epsilon_{ij}(\lambda_i^1\lambda_j^4 + \lambda_i^2\lambda_j^3) + \sum_{\{i,j\}=\{6,7\},\{5,8\}} \epsilon_{ij}(\lambda_i^1\lambda_j^3 - \lambda_i^2\lambda_j^4) = 0,$$
$$\sum_{\{i,j\}=\{5,6\},\{7,8\}} \epsilon_{ij}(\lambda_i^1\lambda_j^4 + \lambda_i^2\lambda_j^3) - \sum_{\{i,j\}=\{5,8\},\{6,7\}} \epsilon_{ij}(\lambda_i^1\lambda_j^2 + \lambda_i^3\lambda_j^4) = 0,$$
$$\sum_{\{i,j\}=\{5,6\},\{7,8\}} \epsilon_{ji}(\lambda_i^2\lambda_j^4 - \lambda_i^1\lambda_j^3) - \sum_{\{i,j\}=\{5,7\},\{6,8\}} \epsilon_{ij}(\lambda_i^1\lambda_j^2 + \lambda_i^3\lambda_j^4) = 0,$$

where $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{75} = \epsilon_{68} = \epsilon_{56} = \epsilon_{67} = \epsilon_{78} = \epsilon_{58} = 1$. Substituting in the values of $\lambda_5^1, \lambda_6^1, \lambda_7^1$ and λ_8^1 we found when we solved Equations (3.2.5), these three equations vanish. Therefore, $\tau_{f(y)}(v_1, v_2, v_3, v_4) = 0$ if, and only if, $\pi \circ \tau_{f(y)}(v_1, v_2, v_3, v_4) = 0$. Since $y \in Y$ and Y' were arbitrary, it follows that the kernel of $\pi(*_Y \exp_v^*(\tau|_{Y_v}))$ and $*_Y \exp_v^*(\tau|_{Y_v})$ are the same. □

Remark. Harvey and Lawson proved that the kernel of $*_Y \exp_v^*(\tau|_{Y_v})$ is the same as the kernel of $\pi(*_Y \exp_v^*(\tau|_{Y_v}))$ [15, IV.2.C Thm 2.20] in a significantly more clever way. For an arbitrary linear subspace of the octonions $\mathbb{O}$ written as $\mathbb{H} \oplus f(\mathbb{H})$ where $f : \mathbb{H} \to \mathbb{H}$ is a linear map between the quaternions, they showed that $\mathbb{H} \oplus f(\mathbb{H})$ is Cayley four plane of $\mathbb{O}$ if, and only if, $\mathbb{H} \oplus g(\mathbb{H})$ is, where $g : \mathbb{H} \to \mathbb{H}$ takes a simpler form. Applying a similar analysis to that in Proposition 3.2.2, phrased in terms of cross products on the octonions, they were left with much simpler looking equations than (3.2.5), which they could solve explicitly by hand. However, their proof was only valid when $\det f \neq 1$, a condition that the 'brute force' approach used above has removed.

3.2.2 Properties of the partial differential operator F

Proposition 3.2.2 tells us that to study the moduli space of Cayley deformations of Y in the $Spin(7)$-manifold X we must study the kernel of the operator F (3.2.1). Linearising the operator F will tell us which normal vector fields on Y are Cayley deformations to first order. We can think of this as the Zariski tangent space to $\mathcal{M}_{\mathrm{Cay}}(Y)$ at Y.

Proposition 3.2.3. *Let (X, g, Φ) be a Spin(7)-manifold and let Y be a compact Cayley submanifold of X. Let $\{e_1, e_2, e_3, e_4\}$ be a frame for TY with dual coframe $\{e^1, e^2, e^3, e^4\}$. Denote by F the operator (3.2.1). Then the linearisation of F at zero is given by the elliptic operator*

$$\begin{aligned} D : C^\infty(\nu_X(Y)) &\to C^\infty(E), \\ v &\mapsto \sum_{i=1}^{4} \pi_7(e^i \wedge (\nabla^\perp_{e_i} v)^\flat), \end{aligned} \tag{3.2.6}$$

where E is the rank four vector subbundle of $\Lambda^2_7|_Y$ in Equation (3.2.2), $\nabla^\perp : TY \otimes \nu_X(Y) \to \nu_X(Y)$ denotes the connection on $\nu_X(Y)$ induced by the Levi-Civita connection of X and π_7 denotes the projection of two-forms onto Λ^2_7 as in Proposition 1.2.9.

Remark. We call the vector fields in the kernel of D *infinitesimal Cayley deformations* of Y in X.

Proof. First note that the operator (3.2.6) is elliptic: its symbol is given by the map

$$\begin{aligned} T^*Y \otimes \nu_X(Y) &\to E, \\ \xi \otimes v &\mapsto \pi_7(\xi \wedge v^\flat), \end{aligned}$$

which for each nonzero ξ surjects, and therefore the operator D is elliptic. (Note here that we can explicitly define E in the given frame for Y as the span of $\pi_7(e^1 \wedge e^j)$ for $j = 5, \ldots, 8$.)

To see that (3.2.6) is the linearisation of the operator F in Equation (3.2.1), we make an explicit computation. By definition, we have that

$$dF|_0(v) = \frac{d}{dt} F(tv)|_{t=0} = *\mathcal{L}_v \tau|_Y,$$

by definition of the Lie derivative. We have that

$$*\mathcal{L}_v \tau|_Y = (\mathcal{L}_v \tau)(e_1, e_2, e_3, e_4),$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame for TY with $\mathrm{vol}_Y(e_1, e_2, e_3, e_4) = 1$, and so we may apply a formula linking the Lie derivative to the Levi-Civita connection such as [20, Eqn (4.3.26)] to find that

$$\begin{aligned}(\mathcal{L}_v\tau)(e_1, e_2, e_3, e_4) &= (\nabla_v\tau)(e_1, e_2, e_3, e_4) + \tau(\nabla_{e_1}v, e_2, e_3, e_4)\\ &\quad + \tau(e_1, \nabla_{e_2}v, e_3, e_4) + \tau(e_1, e_2, \nabla_{e_3}v, e_4) + \tau(e_1, e_2, e_3, \nabla_{e_4}v),\\ &= (\nabla_v\tau)(e_1, e_2, e_3, e_4) + \tau(\nabla_{e_1}v, e_2, e_3, e_4)\\ &\quad - \tau(\nabla_{e_2}v, e_1, e_3, e_4) + \tau(\nabla_{e_3}v, e_1, e_2, e_4) - \tau(\nabla_{e_4}v, e_1, e_2, e_3),\end{aligned}$$

since τ is a differential form. We can write the Levi-Civita connection on $TX|_Y$ as $\nabla = \nabla^T + \nabla^\perp$, where ∇^T is the projection of ∇ onto $T^*Y \otimes TY$ and $\nabla^\perp$ is the projection of ∇ onto $T^*Y \otimes \nu_X(Y)$. Then

$$\tau(\nabla^T_{e_i}v, e_j, e_k, e_l) = 0,$$

for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (since Y is Cayley), and therefore we have that

$$\begin{aligned}(\mathcal{L}_v\tau)(e_1, e_2, e_3, e_4) &= (\nabla_v\tau)(e_1, e_2, e_3, e_4) + \tau(\nabla^\perp_{e_1}v, e_2, e_3, e_4)\\ &\quad - \tau(\nabla^\perp_{e_2}v, e_1, e_3, e_4) + \tau(\nabla^\perp_{e_3}v, e_1, e_2, e_4) - \tau(\nabla^\perp_{e_4}v, e_1, e_2, e_3).\end{aligned}$$

Recalling the triple cross product κ defined in Lemma 1.2.4, we can calculate that, since $\mathrm{vol}_Y = e^1 \wedge e^2 \wedge e^3 \wedge e^4$,

$$\begin{aligned}\kappa(e_1, e_2, e_3) &= -e_4, \quad \kappa(e_1, e_2, e_4) = e_3,\\ \kappa(e_1, e_3, e_4) &= -e_2, \quad \kappa(e_2, e_3, e_4) = e_1.\end{aligned}$$

Therefore by definition of τ (see Proposition 1.2.9) we have that

$$\begin{aligned}(\mathcal{L}_v\tau)(e_1, e_2, e_3, e_4) &= (\nabla_v\tau)(e_1, e_2, e_3, e_4) + \pi_7(e^1 \wedge (\nabla^\perp_{e_1}v)^\flat) + \pi_7(e^2 \wedge (\nabla^\perp_{e_2}v)^\flat)\\ &\quad + \pi_7(e^3 \wedge (\nabla^\perp_{e_3}v)^\flat) + \pi_7(e^4 \wedge (\nabla^\perp_{e_4}v)^\flat).\end{aligned} \tag{3.2.7}$$

It remains to deal with the term

$$(\nabla_v\tau)(e_1, e_2, e_3, e_4).$$

We saw in (1.2.14) that we can write, extending $\{e_1, \ldots, e_4\}$ to an orthonormal frame $\{e_1, \ldots, e_8\}$ for $TM|_N$

$$\tau = \frac{1}{4} \sum_{i<j\in\{1,\ldots,8\}} (e^j \wedge (e_i \lrcorner \Phi) - e^i \wedge (e_j \lrcorner \Phi)) \otimes \pi_7(e^i \wedge e^j).$$

We calculate that

$$\begin{aligned}\nabla_v(e^j \wedge (e_i \lrcorner \Phi) \otimes \pi_7(e^i \wedge e^j)) = (\nabla_v e^j) \wedge (e_i \lrcorner \Phi) \otimes \pi_7(e^i \wedge e^j) \\ + e^j \wedge \nabla_v(e_i \lrcorner \Phi) \otimes \pi_7(e^i \wedge e^j) \\ + e^j \wedge (e_i \lrcorner \Phi) \otimes \nabla_v(\pi_7(e^i \wedge e^j)).\end{aligned} \tag{3.2.8}$$

The terms in $\nabla_v \tau$ of the form

$$(e^j \wedge (e_i \lrcorner \Phi) - e^i \wedge (e_j \lrcorner \Phi)) \otimes \nabla_v(\pi_7(e^i \wedge e^j)),$$

will clearly vanish when evaluated at e_1, e_2, e_3, e_4 since this is a frame for a Cayley submanifold.

Using a formula for the Levi-Civita connection of a differential form such as [20, Eqn (4.3.23)] we see that

$$\nabla_v(e_i \lrcorner \Phi) = (\nabla_v e_i) \lrcorner \Phi + e_i \lrcorner \nabla_v \Phi.$$

Since the $Spin(7)$-structure is torsion free, $\nabla_v \Phi = 0$, and so by Equation (3.2.8) we are left with terms in $\nabla_v \tau$ of the form

$$((\nabla_v e^j) \wedge (e_i \lrcorner \Phi) + e^j \wedge (\nabla_v e_i) \lrcorner \Phi) \otimes \pi_7(e^i \wedge e^j).$$

That is,

$$\begin{aligned}\nabla_v \tau = \frac{1}{4} \sum_{i<j\in\{1,\ldots,8\}} \big[(\nabla_v e^j) \wedge (e_i \lrcorner \Phi) + e^j \wedge (\nabla_v e_i) \lrcorner \Phi \\ - (\nabla_v e^i) \wedge (e_j \lrcorner \Phi) - e^i \wedge (\nabla_v e_j) \lrcorner \Phi\big] \otimes \pi_7(e^i \wedge e^j).\end{aligned}$$

But we can gather $\nabla_v \tau$ into a sum of terms of the form

$$((\nabla_v e^j) \wedge (e_i \lrcorner \Phi) - e^i \wedge (\nabla_v e_j) \lrcorner \Phi) \otimes \pi_7(e^i \wedge e^j),$$

which vanish when evaluated on the Cayley frame e_1, e_2, e_3, e_4 since $(\nabla_v e_j)^\flat = \nabla_v e^j = v_k e^k$, for some $k \in \{1, \dots, 8\}$ and functions v_k. Therefore $\nabla_v \tau = 0$ and so the proposition follows from Equation (3.2.7). $\square$

3.2.3 The moduli space of Cayley deformations

In order to prove Theorem 3.2.6 on the expected dimension of the moduli space of Cayley deformations of a compact Cayley submanifold we must first prove that we can extend the operator (3.2.1) to a smooth map of Banach spaces. The argument we use to prove Lemma 3.2.4 is reasonably standard, and is based on the arguments in [24, Prop 2.10] and [37, Prop 6.9]. The proof is presented here in the hope that a similar, but more complicated, result in Chapter 5 for *conically singular* Cayley submanifolds will be easier to follow.

Lemma 3.2.4. *Let* (X, g, Φ) *be a* Spin(7)*-manifold and let* Y *be a compact Cayley submanifold of* X*. Let* F *be the partial differential operator defined in Equation* (3.2.1)*. Then we can extend* F *to a smooth map of Banach spaces*

$$F : L^p_{k+1}(V) \to L^p_k(E), \tag{3.2.9}$$

for any $1 < p < \infty$ *and* $k \in \mathbb{N}$ *satisfying* $k > 1 + 4/p$*. Moreover, the normal vector fields in the kernel of* (3.2.9) *are smooth.*

Proof. At each point y of Y we have that $F(v)(y)$ relates to the tangent space of the deformation $Y_v := \exp_v(Y)$ and therefore depends on v and ∇v. We may write

$$F(v)(x) = Dv(x) + Q(x, v(x), \nabla v(x)), \tag{3.2.10}$$

where D is the linearisation of F defined in Proposition 3.2.3, and use Equation (3.2.10) to define Q to be a map

$$\{(x, y, z) \,|\, (x, y) \in V, z \in T^*_x Y \otimes \nu_x(Y)\} \to E,$$

so that $Q(v)(x) := Q(x, v(x), \nabla v(x))$ is a section of E. By definition of F, Q is smooth in x, y and z. Since we can think of Q as a map $\nu_x(Y) \otimes T^*_x Y \otimes \nu_x(Y) \to E_x$, we can make sense of a Taylor expansion of $Q(x,y,z)$ around $(x,0,0)$. Since by definition Q has no linear part at zero we deduce that

$$|Q(x,y,z)| \leq C_x(|y|+|z|)^2,$$

for each $x \in Y$. Since Y is compact, we may deduce that

$$\|Q(v)(x)\|_{C^0} \leq C\|v\|_{C^1}^2,$$

where C is independent of x. From this we see that

$$\begin{aligned}\left(\int_Y |Q(v)(x)|^p \operatorname{vol}_Y\right)^{1/p} &\leq C\left(\int_Y (|v|+|\nabla v|)^{2p} \operatorname{vol}_Y\right)^{1/p}\\ &\leq C\|v\|_{C^1}\left(\int_Y (|v|+|\nabla v|)^p \operatorname{vol}_Y\right)^{1/p}\\ &\leq C\|v\|_{C^1}\left(\int_Y |v|^p \operatorname{vol}_Y + \int_Y |\nabla v|^p \operatorname{vol}_Y\right)^{1/p},\end{aligned}$$

by Minkowski's inequality. So we see that Q maps $L^p_1(\nu_X(Y)) \cap C^1(\nu_X(Y)) \to L^p(E)$. We can take the derivative of the Taylor expansion of Q, and apply the chain rule to estimate $|\nabla Q|$ by a polynomial in $|v|, |\nabla v|$ and $|\nabla^2 v|$. A similar argument to the $k=0$ case given above shows that for each $k \in \mathbb{N}$ there exists $C_k > 0$ so that

$$\|Q(v)\|_{p,k} \leq C_k\|v\|_{C^1}\|v\|_{p,k+1}. \tag{3.2.11}$$

In particular, when $k > 4/p$, $L^p_{k+1}(\nu_X(Y))$ is continuously embedded in $C^1(\nu_X(Y))$ by [4, Thm 2.10], and so for $k > 4/p$ there exist $\tilde{C}_k > 0$ so that

$$\|Q(v)\|_{p,k} \leq \tilde{C}_k\|v\|_{p,k+1}^2. \tag{3.2.12}$$

Since D is linear, we see that F takes $L^p_{k+1}(\nu_X(Y))$ into $L^p_k(E)$.

Now we must show that (3.2.9) is a smooth map of Banach spaces. Firstly, since

$$v \mapsto Dv,$$

is linear, it is clearly smooth as a map

$$L^p_{k+1}(V) \to L^p_k(E).$$

To see that

$$v \mapsto (x \mapsto Q(x, v(x), \nabla v(x))),$$

is a smooth map

$$L^p_{k+1}(V) \to L^p_k(E),$$

we proceed as follows. To see that F is once differentiable at zero in this sense, notice that

$$\frac{\|F(v) - F(0) - Dv\|_{p,k}}{\|v\|_{p,k+1}} = \frac{\|Q(v)\|_{p,k}}{\|v\|_{p,k+1}} \to 0,$$

as $\|v\|_{p,k+1} \to 0$ by the estimate (3.2.12). Repeating this argument for the derivatives of Q, we can show that we can differentiate Q as many times as we like. We deduce that (3.2.9) is a smooth map of Banach spaces.

Finally, regularity of the kernel of (3.2.9) follows from a nonlinear elliptic regularity result, such as [4, Thm 3.56], which we may apply since $k > 1 + 4/p$ (which allows us to embed $L^p_{k+1}(V)$ in $C^2(V)$ by Sobolev embedding [4, Thm 2.10]). □

We will now deduce the main result of this section. For the reader's convenience, we will present the Banach space implicit function theorem here in the form that we will need it. See, for example, [32, Ch 6 Thm 2.1] for a proof.

Theorem 3.2.5 (Implicit function theorem)**.** *Let X and Y be Banach spaces and let $U \subseteq X$ be an open neighbourhood of zero. Let $\mathcal{F} : U \to Y$ be a C^k-map, with $k \geq 1$, such that $\mathcal{F}(0) = 0$. Suppose further that $d\mathcal{F}|_0 : X \to Y$ is surjective, with kernel K such that $X = K \oplus X'$ for some closed subspace X' of X.*

Then there exist open sets $K_0 \subseteq K$, $X'_0 \subseteq X'$ both containing zero and a C^k-map $g : K_0 \to X'_0$ such that $g(0) = 0$ and

$$\mathcal{F}^{-1}(0) \cap (K_0 \times X'_0) = \{(x, g(x)) \,|\, x \in K_0\}.$$

Theorem 3.2.6. *Let (X, g, Φ) be a Spin(7)-manifold and let Y be a compact Cayley submanifold of X. Let D denote the first order elliptic operator defined in* (3.2.6)*. Then there exist a smooth manifold K_0, which is an open neighbourhood of* 0 *in* Ker D*,* *and a smooth map* $g_2 : K_0 \to \text{Ker } D^*$ *with* $g(0) = 0$ *so that an open neighbourhood of* Y *in the moduli space of Cayley deformations of* Y *in* X *is homeomorphic to an open neighbourhood of* 0 *in* $\text{Ker } g_2$.

Moreover, the expected dimension of the moduli space of Cayley deformations of Y *in* X *is given by*

$$\text{ind } D := \dim \text{Ker } D - \dim \text{Ker } D^*,$$

where

$$D^* : C^\infty(E) \to C^\infty(\nu_X(Y)),$$

is the formal adjoint of D*. If* $\text{Ker } D^* = \{0\}$ *then the moduli space of Cayley deformations of* Y *in* X *is a smooth manifold near* Y *of dimension*

$$\dim \text{Ker } D.$$

Proof. By Proposition 3.2.2 we know that the moduli space of Cayley deformations of Y in X is isomorphic near Y to the kernel of

$$\begin{aligned} F : C^\infty(V) &\to C^\infty(E), \\ v &\mapsto \pi(*_Y \exp_v^*(\tau|_{Y_v})), \end{aligned}$$

where $V \subseteq \nu_X(Y)$ is the subset given in the tubular neighbourhood theorem 3.2.1, $Y_v := \exp_v(Y)$ $\pi : \Lambda^2_7|_Y \to E$ is the projection map of the splitting given in (3.2.2) and $\tau \in C^\infty(\Lambda^4 \otimes \Lambda^2_7)$ is the four-form defined in Proposition 1.2.9. By Lemma 3.2.4, without changing the kernel, F extends to a smooth map

$$L^p_{k+1}(V) \to L^p_k(E),$$

for any $1 < p < \infty$ and $k \in \mathbb{N}$ and the linearisation of F at zero is the elliptic operator D defined in Equation (3.2.6), which extends by density to a smooth map

$$D : L^p_{k+1}(\nu_X(Y)) \to L^p_k(E). \tag{3.2.13}$$

Since Y is compact and (3.2.13) is elliptic, the map (3.2.13) is Fredholm, and therefore (3.2.13) has finite-dimensional kernel and cokernel, and closed image. As a consequence, we can write

$$L^p_{k+1}(\nu_X(Y)) = K' \oplus X',$$

where K' is the kernel of D and X' is closed, and

$$L^p_k(E) = D(L^p_{k+1}(\nu_X(Y))) \oplus \mathcal{O},$$

where $\mathcal{O}$ is a finite-dimensional space that we'll call the *obstruction space*, and

$$\mathcal{O} \cong L^p_k(E)/D(L^p_{k+1}(\nu_X(Y))) =: \text{Coker } D.$$

Notice that if the obstruction space vanishes, i.e., $\mathcal{O} = \{0\}$, then it follows immediately from the implicit function theorem 3.2.5 that the moduli space of Cayley deformations of Y is a smooth manifold near Y of dimension dim Ker D. However, the obstruction space is nonempty in general, and so D is not surjective, thus we are not able to apply the implicit function theorem 3.2.5 to F. Instead define

$$\begin{aligned} \mathcal{F} : L^p_{k+1}(V) \times \mathcal{O} &\to L^p_k(E), \\ (v, w) &\mapsto F(v) + w. \end{aligned}$$

We see that

$$d\mathcal{F}|_{(0,0)}(v, w) = Dv + w,$$

which surjects, and therefore we may apply the implicit function theorem 3.2.5 to $\mathcal{F}$. Denoting the kernel of $d\mathcal{F}|_{(0,0)}$ by $K = K' \times \{0\}$, we can write

$$L^p_{k+1}(V) \times \mathcal{O} = K \oplus (X' \times \mathcal{O}).$$

The implicit function theorem 3.2.5 gives us open sets $K_0 \subseteq K$, $X'_0 \subseteq X'$ and $\mathcal{O}_0 \subseteq \mathcal{O}$ and a smooth map $g = (g_1, g_2) : K_0 \to X'_0 \times \mathcal{O}_0$ such that

$$\mathcal{F}^{-1}(0) \cap (K_0 \times X'_0 \times \mathcal{O}_0) = \{(x, g_1(x), g_2(x)) \,|\, x \in K_0\}.$$

Then for $x \in K_0$ we have that

$$\mathcal{F}(x, g_1(x), g_2(x)) = F(x, g_1(x)) + g_2(x) = 0.$$

Therefore we can identify the kernel of F with the kernel of the map $g_2 : K_0 \to \mathcal{O}_0$. These spaces are finite-dimensional since D is Fredholm. By Sard's theorem, we may deduce that the expected dimension of the kernel of g_2 is equal to the difference of the dimensions of K_0 and $\mathcal{O}_0$, and therefore the expected dimension of the moduli space of Cayley deformations of Y in X is

$$\dim \operatorname{Ker} D - \dim \operatorname{Coker} D,$$

where D is considered as a map $L^p_{k+1}(V) \to L^p_k(E)$. We have that the cokernel D is isomorphic to the kernel of the adjoint to D, D^*, since Y is compact. Elliptic regularity tells us that the kernels of D and D^* acting on $L^p_{k+1}(\nu_X(Y))$ and $(L^p_k(E))^*$ for any $1 < p < \infty$ and $k \in \mathbb{N}$ are exactly equal to the kernels of D and D^* acting on $C^\infty(\nu_X(Y))$ and $C^\infty(E)$ respectively. $\square$

3.3 Cayley deformations of a compact complex surface

In Section 3.2 we saw that the moduli space of Cayley deformations of a compact Cayley submanifold Y inside a $Spin(7)$-manifold (X, g, Φ) can be identified with the kernel of a first order nonlinear partial differential operator

$$F : C^\infty(V) \to C^\infty(E),$$

where V is an open subset of the normal bundle of Y in X and E is a rank four subbundle of $\Lambda^2_7|_Y$. The operator F linearises at zero to the elliptic operator

$$\begin{aligned} D : C^\infty(\nu_X(Y)) &\to C^\infty(E), \\ v &\mapsto \sum_{i=1}^4 \pi_7(e^i \wedge (\nabla^\perp_{e_i} v)^\flat), \end{aligned} \tag{3.3.1}$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal frame for TY with dual frame $\{e^1, \ldots, e^4\}$, $\nabla^\perp$ is the connection on $\nu_X(Y)$ induced by the Levi-Civita connection of M, $\pi_7 : \Lambda^2 X \to \Lambda^2_7$ is the projection map and $\flat : \nu_X(Y) \to \nu^*_X(Y)$ is the musical isomorphism.

As remarked at the beginning of Section 3.2, McLean showed that under the identifications of $\nu_X(Y) \otimes \mathbb{C}$ and $E \otimes \mathbb{C}$ with bundles of positive and negative twisted spinors respectively, (3.3.1) is the twisted Dirac operator. While we avoid the use of spin structures in our treatment of the Cayley deformation problem in this chapter, the special form that a spin structure on a Kähler manifold takes provides motivation for the vector bundle isomorphisms that we construct in Section 3.3.1.

As we have seen, if M is a four-dimensional Calabi–Yau manifold with two-dimensional compact complex submanifold N, we can view N as a Cayley submanifold of M. Recall that on a two-dimensional Kähler manifold N with a fixed spin structure, we can identify [13, pg 82]

$$\begin{aligned} \mathbb{S}_+ &\cong (\Lambda^{0,0}N \oplus \Lambda^{0,2}N) \otimes S_k, \\ \mathbb{S}_- &\cong \Lambda^{0,1}N \otimes S_k, \end{aligned}$$

where S_k is a holomorphic line bundle satisfying $S_k \otimes S_k = \Lambda^{2,0}N$, and in this case the Dirac operator is given by

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

In Section 3.3.1, we will explicitly construct isomorphisms between $\nu_M(N) \otimes \mathbb{C}$ and $E \otimes \mathbb{C}$ with bundles of twisted $(0,0)$- and $(0,2)$-forms and $(0,1)$-forms respectively. In Section 3.3.2, we will show that under these identifications, (3.3.1) becomes the operator $\bar{\partial} + \bar{\partial}^*$. Combining this with the analysis of Section 3.1.1 will allow us to prove Theorem 3.3.4, where we identify the expected dimension of the moduli space of Cayley deformations of N in M with the index of $\bar{\partial} + \bar{\partial}^*$, and further Theorem 3.3.5 which gives an expression for the expected dimension of this moduli space in terms of topological invariants of N.

3.3.1 Identifications of vector bundles

In this section we construct isomorphisms of vector bundles on a complex surface N in a Calabi–Yau four-fold M.

Proposition 3.3.1. *Let N be a two-dimensional complex submanifold of a Calabi–Yau four-fold M. Then*

$$\nu_M(N) \otimes \mathbb{C} \cong \nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N), \tag{3.3.2}$$

where $\nu_M(N)$ denotes the normal bundle of N in M and $\nu_M^{1,0}(N)$ denotes the holomorphic normal bundle of N in M.

Proof. Recall that on a complex submanifold we have the following splitting of the complexified normal bundle into holomorphic and antiholomorphic parts

$$\nu_M(N) \otimes \mathbb{C} \cong \nu_M^{1,0}(N) \oplus \nu_M^{0,1}(N).$$

Therefore to prove the proposition, it suffices to show that

$$\nu_M^{0,1}(N) \cong \Lambda^{0,2}N \otimes \nu_M^{1,0}(N).$$

The adjunction formula for complex submanifolds [17, Prop 2.2.17] says that

$$\Lambda^{2,0}N \cong \Lambda^{4,0}M|_N \otimes \Lambda^2\nu_M^{1,0}(N), \tag{3.3.3}$$

however, since M is Calabi–Yau, it has trivial canonical bundle. Tensoring both sides of Equation (3.3.3) with $\nu_M^{*1,0}(N)$, we have that

$$\Lambda^{2,0}N \otimes \nu_M^{*1,0}(N) \cong \nu_M^{1,0}(N).$$

The musical isomorphism $\sharp : \nu_M^{*1,0}(N) \to \nu_M^{0,1}(N)$ yields that

$$\Lambda^{2,0}N \otimes \nu_M^{0,1}(N) \cong \nu_M^{1,0}(N),$$

and finally taking the complex conjugate we conclude that

$$\nu_M^{0,1}(N) \cong \Lambda^{0,2}N \otimes \nu_M^{1,0}(N).$$

□

Remark. In fact, we can even write down explicitly the isomorphism

$$\nu_M^{0,1}(N) \cong \Lambda^{0,2}N \otimes \nu_M^{1,0}(N).$$

Let Ω be a holomorphic volume form on M. Then the isomorphism is given by

$$\begin{aligned} \nu_M^{0,1}(N) &\to \Lambda^{0,2}N \otimes \nu_M^{1,0}(N), \\ v &\mapsto \frac{1}{4}(v \lrcorner \overline{\Omega})^\sharp, \end{aligned}$$

where $\sharp$ denotes the musical isomorphism $\nu^{*0,1}(N) \to \nu^{1,0}(N)$. Its inverse is given by

$$\begin{aligned} \Lambda^{0,2}N \otimes \nu_M^{1,0}(N) &\to \nu_M^{0,1}(N), \\ \alpha \otimes v &\mapsto -\left[*_N(\alpha \wedge (v \lrcorner \Omega))\right]^\sharp, \end{aligned}$$

where $*_N$ is the real Hodge star on N and $\sharp : \nu_M^{*1,0}(N) \to \nu_M^{0,1}(N)$ is the musical isomorphism.

Proposition 3.3.2. *Let N be a two-dimensional complex submanifold of a Calabi–Yau four-fold M. Denote by E the rank four vector bundle in the splitting*

$$\Lambda_7^2|_N = \Lambda_+^2 N \oplus E,$$

where Λ_7^2 was defined in 1.2.5. Then we have that

$$E \otimes \mathbb{C} \cong \Lambda^{0,1}N \otimes \nu_M^{1,0}(N), \tag{3.3.4}$$

where $\nu_M^{1,0}(N)$ denotes the holomorphic normal bundle of N in M.

Proof. Since we have the musical isomorphism $\flat : \nu_M^{1,0}(N) \to \nu_M^{*0,1}(N)$, it suffices to show that

$$E \otimes \mathbb{C} \cong \Lambda^{0,1}N \otimes \nu_M^{*0,1}(N).$$

To see this we will show that the projection map

$$\pi_7 : \Lambda^2 M \to \Lambda_7^2,$$

given by

$$\pi_7(v \wedge w) = \frac{1}{2}\left[v \wedge w + \Phi(v^\sharp, w^\sharp, \cdot, \cdot)\right],$$

is a bijection

$$\Lambda^{0,1}N \otimes \nu_M^{*0,1}(N) \to E \otimes \mathbb{C}.$$

Let ω be the Ricci-flat Kähler metric on M and choose a holomorphic volume form Ω so that the Cayley form on M is given by

$$\Phi = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\,\Omega.$$

Let $v \otimes w \in \Lambda^{0,1}N \otimes \nu_M^{*0,1}(N)$. Then viewing this as a two-form on M, we have that

$$\pi_7(v \wedge w) = \frac{1}{2}\left[v \wedge w + \frac{1}{2}\omega \wedge \omega(v^\sharp, w^\sharp, \cdot, \cdot) + \frac{1}{2}(\Omega + \overline{\Omega})(v^\sharp, w^\sharp, \cdot, \cdot)\right].$$

First note that $v^\sharp$ and $w^\sharp$ are of type $(1,0)$, and so straight away we can eliminate the $\overline{\Omega}$ term. Further, since

$$\omega(a,b) = g(Ja, b),$$

for all vector fields a and b on M, we see that

$$\begin{aligned}
\frac{1}{2}\omega \wedge \omega(v^\sharp, w^\sharp, \cdot, \cdot) &= \frac{1}{2}\left[\omega(v^\sharp, w^\sharp) \wedge \omega + \omega \wedge \omega(v^\sharp, w^\sharp)\right. \\
&\quad \left. - \omega(v^\sharp, \cdot) \wedge \omega(w^\sharp, \cdot) + \omega(w^\sharp, \cdot) \wedge \omega(v^\sharp, \cdot)\right] \\
&= \frac{1}{2}\left[-g(Jv^\sharp, \cdot) \wedge g(Jw^\sharp, \cdot) + g(Jw^\sharp, \cdot) \wedge g(Jv^\sharp, \cdot)\right] \\
&= \frac{1}{2}\left[g(v^\sharp, \cdot) \wedge g(w^\sharp, \cdot) - g(w^\sharp, \cdot) \wedge g(v^\sharp, \cdot)\right] \\
&= \frac{1}{2}\left[v \wedge w - w \wedge v\right] \\
&= v \wedge w,
\end{aligned}$$

since $v^\sharp$ and $w^\sharp$ are of type $(1,0)$ and using the definition of the musical isomorphism. So we have shown that

$$\pi_7(v \wedge w) = v \wedge w + \frac{1}{4}\Omega(v^\sharp, w^\sharp, \cdot, \cdot),$$

where we notice that the second term lies in $\Lambda^{1,0}N \otimes \nu_M^{*1,0}(N)$ when restricted to N. It can be shown similarly that for $v \in \Lambda^{1,0}N$ and $w \in \nu_M^{*1,0}(N)$ that

$$\pi_7(v \wedge w) = v \wedge w + \frac{1}{4}\overline{\Omega}(v^\sharp, w^\sharp, \cdot, \cdot),$$

and so we see that if $\sigma \in \Lambda^{1,0}N \otimes \nu_M^{*1,0}(N)$ or $\Lambda^{0,1}N \otimes \nu_M^{*0,1}(N)$ then $\pi_7(\sigma) \in \Lambda^{1,0}N \otimes \nu_M^{*1,0}(N) \oplus \Lambda^{0,1}N \otimes \nu_M^{*0,1}(N)$. In particular, $\pi_7(\Lambda^{1,0}N \otimes \nu_M^{*1,0}(N)) = \pi_7(\Lambda^{0,1}N \otimes \nu_M^{*0,1}(N))$. A similar calculation yields that for all $\sigma_1 \in \Lambda^{1,0}N \otimes \nu_M^{*0,1}(N), \sigma_2 \in \Lambda^{0,1}N \otimes \nu_M^{*1,0}(N)$

$$\pi_7(\sigma_1) = 0 = \pi_7(\sigma_2),$$

and therefore to check that $E \otimes \mathbb{C} = \pi_7(\Lambda^{0,1}N \otimes \nu_M^{*0,1}(N))$ it suffices to check that $\pi_7(\Lambda^{0,2}N) + \pi_7(\Lambda^{2,0}N) + \pi_7(\Lambda^{1,1}N) = \Lambda^2_+ N$. But since if $v \wedge w$ is a unit element of $\Lambda^{0,2}N, \Lambda^{2,0}N$ or $\Lambda^{1,1}N$ then

$$\begin{aligned}
\pi_7(v \wedge w)|_N &= \frac{1}{2}\left[v \wedge w + \frac{1}{2}\omega \wedge \omega(v^\sharp, w^\sharp, \cdot, \cdot)|_N\right] \\
&= \frac{1}{2}\left[v \wedge w + \mathrm{vol}_N(v^\sharp, w^\sharp, \cdot, \cdot)\right] \\
&= \frac{1}{2}\left[v \wedge w + *_N(v \wedge w)\right],
\end{aligned}$$

this is clear. Therefore $E \otimes \mathbb{C} = \pi_7(\Lambda^{0,1}N \otimes \nu_M^{*0,1}(N))$. The inverse map to π_7 is given by the projection map

$$\pi_{0,1} : E \otimes \mathbb{C} \to \Lambda^{0,1}N \otimes \nu_M^{*0,1}(N).$$

□

3.3.2 The Cayley operator on a complex submanifold

Now that we have made our vector bundle identifications we can identify the moduli space of Cayley deformations of the complex surface N in the Calabi–Yau manifold M with the kernel of a partial differential operator acting between vector bundles whose linearisation takes a the form of a familiar first order elliptic operator.

Proposition 3.3.3. *Let N be a two-dimensional compact complex submanifold of a Calabi–Yau four-fold M. Then the moduli space of Cayley deformations of N in M can be identified with the kernel of the partial differential operator*

$$F^{\mathrm{cx}} : C^\infty(U) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$

where $U \subseteq \nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)$ is the image of $V \otimes \mathbb{C}$ from the tubular neighbourhood theorem 3.2.1 under the isomorphism given in Proposition 3.3.1, and F^{cx} is defined so that the following diagram commutes

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{F^{\mathrm{cx}}} & C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)) \\ \downarrow & & \downarrow \\ C^\infty(V \otimes \mathbb{C}) & \xrightarrow{F} & C^\infty(E \otimes \mathbb{C}) \end{array}$$

where F is the operator defined in Proposition 3.2.2 and we use the isomorphisms given in Propositions 3.3.1 and 3.3.2.

Moreover, the linearisation of F^{cx} at zero is given by the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)). \tag{3.3.5}$$

Proof. By Proposition 3.2.2 the moduli space of Cayley deformations of N in M can be identified with the kernel of the map F, which by definition of F^{cx} has the same kernel as F^{cx}.

Recall that the linearisation of F is given by the operator

$$Dv = \sum_{i=1}^{4} \pi_7(e^i \wedge (\nabla_{e_i}^{\perp} v)^\flat),$$

where $\{e_1, \dots, e_4\}$ is an orthonormal frame for TN with dual frame $\{e^1, \dots, e^4\}$ for T^*N, $\nabla^\perp$ is the connection on $\nu_M(N)$ induced by the Levi-Civita connection of M, π_7 is the projection of two-forms onto Λ_7^2 and $\flat : \nu_M(N) \to \nu_M^*(N)$ is the musical isomorphism. We will use this to find the linearisation of F^{cx}.

Write $v \in C^\infty(\nu_M(N) \otimes \mathbb{C})$ as $v_1 \oplus v_2$, where $v_1 \in C^\infty(\nu_M^{1,0}(N))$ and $v_2 \in C^\infty(\nu_M^{0,1}(N))$. If we can show that

$$\bar{\partial} v_1 + \bar{\partial}^* \frac{1}{4}(v_2 \lrcorner \overline{\Omega}) = \pi_{0,1} \circ D(v_1 \oplus v_2),$$

where $\pi_{0,1} : E \otimes \mathbb{C} \to \Lambda^{0,1}N \otimes \nu_M^{*0,1}(N)$ is the projection map as we saw in Proposition 3.3.2, then we are done by definition of F^{cx} and the isomorphisms given in Propositions 3.3.1 and 3.3.2.

We will first show that

$$\bar{\partial} v_1 = \pi_{0,1} \circ D v_1.$$

Since $e^i \wedge \nabla^\perp_{e_i} v_1$ is a one-form with values in the bundle $\nu^{1,0}(N)$, we may split it into the sum of a $(1,0)$- and $(0,1)$-form. We have that

$$\sum e^i \wedge \nabla^\perp_{e_i} v_1 = \partial v_1 + \bar{\partial} v_1,$$

where $\partial v_1 \in C^\infty(\Lambda^{1,0} N \otimes \nu^{1,0}_M(N))$ and $\bar{\partial} v_1 \in C^\infty(\Lambda^{0,1} N \otimes \nu^{1,0}_M(N))$. (We use this notation because this is exactly the definition of $\bar{\partial}$ on $\nu^{1,0}_M(N)$ since the Levi-Civita connection on a Kähler manifold is compatible with the holomorphic structure [17, Prop 4.A.8].) We saw in Proposition 3.3.2 that $\pi_{0,1} \circ \pi_7 = \mathrm{id}$ on forms in $\Lambda^{0,1} N \otimes \nu^{*0,1}_M(N)$, and so

$$\pi_{0,1} \circ \pi_7(\bar{\partial} v_1) = \bar{\partial} v_1,$$

where we have implicitly used the musical isomorphisms where appropriate. Therefore it remains to show that

$$\pi_{0,1} \circ \pi_7(\partial v_1) = 0.$$

If $w_1 \otimes w_2 \in \Lambda^{1,0} N \otimes \nu^{*0,1}_M(N)$, then for ω the Ricci-flat Kähler form on M and Ω the holomorphic volume form on M so that the Cayley form on M is given by

$$\Phi = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\,\Omega,$$

we have that

$$\pi_7(w_1 \wedge w_2) = \frac{1}{2}\left[w_1 \wedge w_2 + \frac{1}{2}\omega \wedge \omega(w_1^\sharp, w_2^\sharp, \cdot, \cdot) + \frac{1}{2}\Omega(w_1^\sharp, w_2^\sharp, \cdot, \cdot) + \frac{1}{2}\overline{\Omega}(w_1^\sharp, w_2^\sharp, \cdot, \cdot)\right].$$

Now since $w_1^\sharp$ is of type $(0,1)$, whereas $w_2^\sharp$ is of type $(1,0)$, we have that

$$\pi_7(w_1 \wedge w_2) = \frac{1}{2}\left[w_1 \wedge w_2 + \frac{1}{2}\omega \wedge \omega(w_1^\sharp, w_2^\sharp \cdot, \cdot)\right].$$

However, from the relationship between the metric and the Kähler form, we find that

$$\frac{1}{2}\omega \wedge \omega(w_1^\sharp, w_2^\sharp, \cdot, \cdot) = -w_1 \wedge w_2,$$

and so we are done.

We will now show that for $v_2 \in \nu_M^{0,1}(N)$,

$$\bar{\partial}^* \frac{1}{4}(v_2 \lrcorner \overline{\Omega}) = \pi_{0,1} \circ D v_2.$$

We first find that

$$\bar{\partial}^*(v_2 \lrcorner \overline{\Omega}) = -\sum_{i=1}^{4} e_i \lrcorner \nabla_{e_i}(v_2 \lrcorner \overline{\Omega}) = \sum_{i=1}^{4} \overline{\Omega}(e_i, \nabla^{\perp}_{e_i} v_2, \cdot, \cdot),$$

for an orthonormal frame $\{e_1, ..., e_4\}$ of TN, since Ω and therefore $\overline{\Omega}$ is parallel.

We compute that

$$\begin{aligned} D v_2 &= \sum_{i=1}^{4} \pi_7(e^i \wedge (\nabla^{\perp}_{e_i} v_2)^\flat), \\ &= \frac{1}{2} \sum_{i=1}^{4} \left[e^i \wedge (\nabla^{\perp}_{e_i} v_2)^\flat + \frac{1}{2}\omega \wedge \omega(e_i, \nabla^{\perp}_{e_i} v_2, \cdot, \cdot) + \frac{1}{2}(\Omega + \overline{\Omega})(e_i, \nabla^{\perp}_{e_i} v_2, \cdot, \cdot) \right]. \end{aligned} \tag{3.3.6}$$

Since N and M are Kähler, the Levi-Civita connection is compatible with the complex structure. So we have that $\nabla^{\perp}_{e_i} v_2 \in \nu_M^{0,1}(N)$ which allows us to eliminate the term in Equation (3.3.6) involving Ω. Projecting Equation (3.3.6) onto $\Lambda^{0,1} N \otimes \nu_M^{*0,1}(N)$ leaves us with only one term, however, and so we have that

$$\pi_{0,1} \circ D v_2 = \frac{1}{4} \sum_{i=1}^{4} \overline{\Omega}(e_i, \nabla^{\perp}_{e_i} v_2, \cdot, \cdot),$$

which proves the proposition. $\square$

Now that we have linearised F^{cx}, we may use Theorem 3.2.6 to write the expected dimension of the moduli space of Cayley deformations of a compact complex surface inside a Calabi–Yau four-fold in terms of the index of $\bar{\partial} + \bar{\partial}^*$.

Theorem 3.3.4. *Let N be a two-dimensional compact complex submanifold of a Calabi–Yau four-fold M. Then the expected (real) dimension of the moduli space of Cayley deformations of N in M is equal to the (complex) index of the operator*

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2} N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1} N \otimes \nu_M^{1,0}(N)). \tag{3.3.7}$$

Moreover, if the kernel of

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)),$$

is $\{0\}$ *then the moduli space of Cayley deformations of* N *in* M *is a smooth manifold near* N *of dimension*

$$\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Proof. This result follows from applying Theorem 3.2.6 to N and applying Proposition 3.3.3 to identify the operator (3.3.1) with the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$

using the isomorphisms in Proposition 3.3.1 and Proposition 3.3.2, and moreover the kernel and cokernel of (3.3.7) must be the complexification of the kernel and kernel of (3.3.1). Therefore the index of (3.3.1) and the (complex) index of (3.3.7) are the same (i.e., taking the complex dimension instead of the real dimension). Therefore the expected dimension of the moduli space of Cayley deformations of N in M is given by the index of (3.3.7) as claimed. □

Remark. Notice that we could have proved Theorem 3.3.4 by repeating the proof of Theorem 3.2.6 for the operator F^{cx}.

3.3.3 Index theory

Now that we have Theorem 3.3.4 on the expected dimension of the moduli space of Cayley deformations of a compact complex surface inside a Calabi–Yau four-fold we would like to be able to calculate this dimension. We will do this by applying the Hirzebruch–Riemann–Roch theorem, which we can think of as an application of the Atiyah–Singer index theorem to the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}M \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$

to find an expression for the expected dimension of this moduli space in terms of topological invariants of N.

Theorem 3.3.5. *Let N be a compact complex surface inside a four-dimensional Calabi–Yau manifold M. Consider the operator*

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}M \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Then the index of this operator is given by

$$\text{ind } \bar{\partial} + \bar{\partial}^* = \frac{1}{2}\text{sign}(N) + \frac{1}{2}\chi(N) - [N] \cdot [N], \tag{3.3.8}$$

where $\text{sign}(N)$ *is the signature of N, $\chi(N)$ is the Euler characteristic of N and* $[N] \cdot [N]$ *is the self-intersection number of N.*

Proof. Since N is compact, we can identify the kernel of $\bar{\partial} + \bar{\partial}^*$ and the kernel of its adjoint with Dolbeault cohomology groups. That is,

$$\begin{aligned}\dim_{\mathbb{C}}\text{Ker } (\bar{\partial} + \bar{\partial}^*) &= \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,0}(N, \nu_M^{1,0}(N)) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,2}(N, \nu_M^{1,0}(N)),\\ \dim_{\mathbb{C}}\text{Ker } (\bar{\partial} + \bar{\partial}^*)^* &= \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(N, \nu_M^{1,0}(N)).\end{aligned}$$

By Dolbeault's theorem, we can then identify the index of the operator with the dimensions of certain sheaf cohomology groups. We have that

$$\text{ind } \bar{\partial} + \bar{\partial}^* = \dim_{\mathbb{C}} H^0(N, \nu_M^{1,0}(N)) - \dim_{\mathbb{C}} H^1(N, \nu_M^{1,0}(N)) + \dim_{\mathbb{C}} H^2(N, \nu_M^{1,0}(N)).$$

Then by the Hirzebruch–Riemann–Roch theorem [17, Thm 5.1.1], we have that

$$\text{ind } \bar{\partial} + \bar{\partial}^* = \int_N \text{ch}(\nu_M^{1,0}(N))\text{td}(N),$$

where $\text{ch}(\nu_M^{1,0}(N))$ is the Chern character of $\nu_M^{1,0}(N)$ and $\text{td}(N)$ is the Todd class of N. We calculate that

$$\begin{aligned}\int_N \text{ch}(\nu_M^{1,0}(N))\text{td}(N) = \int_N & \frac{1}{6}(c_1^2(N) + c_2(N)) + \frac{1}{2}c_1(\nu_M^{1,0}(N))c_1(N)\\ &+ \frac{1}{2}(c_1^2(\nu_M^{1,0}(N)) - 2c_2(\nu_M^{1,0}(N))).\end{aligned}$$

Since M is a Calabi–Yau manifold, $c_1(M) = 0$, and therefore

$$0 = c_1(T^{1,0}M|_N) = c_1(T^{1,0}N \oplus \nu_M^{1,0}(N)) = c_1(T^{1,0}N) + c_1(\nu_M^{1,0}(N)),$$

which tells us that

$$\begin{aligned}\int_N \mathrm{ch}(\nu_M^{1,0}(N))\mathrm{td}(N) &= \frac{1}{6}(c_1^2(N) + c_2(N)) - c_2(\nu_M^{1,0}(N)) \\ &= \frac{1}{6}(c_1^2(N) - 2c_2(N)) + \frac{1}{2}c_2(N) - c_2(\nu_M^{1,0}(N)).\end{aligned}$$

Finally, since $c_i(\overline{E}) = (-1)^i c_i(E)$,

$$\begin{aligned}c_2(TN \otimes \mathbb{C}) &= c_2(T^{1,0}N \oplus T^{0,1}N) = c_2(N) + c_1(N)c_1(T^{0,1}N) + c_2(T^{0,1}N) \\ &= 2c_2(N) - c_1(N)^2,\end{aligned}$$

and so by definition of the Pontryagin class $p_1(N)$, we see that

$$\int_N \mathrm{ch}(\nu_M^{1,0}(N))\mathrm{td}(N) = \frac{1}{6}p_1(N) + \frac{1}{2}c_2(N) - c_2(\nu_M^{1,0}(N)),$$

and therefore applying the Hirzebruch signature theorem [17, Cor 5.1.4] we have that

$$\mathrm{ind}\ \bar{\partial} + \bar{\partial}^* = \frac{1}{2}\mathrm{sign}(N) + \frac{1}{2}\chi(N) - [N] \cdot [N],$$

as required. □

3.4 Complex deformations of a compact complex surface

In this section, we are interested in finding out when a Cayley deformation of a compact complex surface N in a Calabi–Yau four-fold M is a complex deformation.

If N' is a Cayley deformation of N, we see that

$$\mathrm{vol}_{N'} = \frac{1}{2}\omega \wedge \omega|_{N'} + \mathrm{Re}\ \Omega|_{N'},$$

where ω is the Ricci-flat Kähler form and Ω is the holomorphic volume form of M. It is easy to see that N' is a complex submanifold of M if, and only if,

$$\mathrm{Re}\ \Omega|_{N'} \equiv 0.$$

We can actually go further than this, however, and find a differential form on M that vanishes exactly when restricted to a two-dimensional complex submanifold without thinking about Cayley deformations at all. In Section 3.4.1 we will find such a form, which we will then use to construct a partial differential operator G on a compact complex surface N in a Calabi–Yau four-fold M whose kernel we can identify with the moduli space of complex deformations of N in M.

In Section 3.4.2 we will study properties of the operator G. In particular, we will compute its linearisation in Proposition 3.4.4, which will allow us to deduce that infinitesimal Cayley and complex deformations of N are the same.

Our main result will be Theorem 3.4.7, which gives a local argument to show that the complex deformation problem for N is unobstructed. We may deduce from this that the Cayley deformation problem for N is unobstructed also. From Theorem 3.4.7 we deduce that complex and Cayley deformations of a compact complex surface in a Calabi–Yau four-fold are the same, recovering Proposition 1.2.6 directly for complex submanifolds.

3.4.1 A form that vanishes on complex surfaces

Let M be a four-dimensional Calabi–Yau manifold with Ricci-flat Kähler form ω and holomorphic volume form Ω so that

$$\Phi = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\ \Omega,$$

is the Cayley form on M. Recall the triple cross product on M given by

$$\kappa(u, v, w) = \Phi(u, v, w, \cdot)^\sharp = \frac{1}{2}\omega \wedge \omega(u, v, w, \cdot)^\sharp + \mathrm{Re}\ \Omega(u, v, w, \cdot)^\sharp,$$

for any vector fields u, v, w on M, where $\sharp : T^*M \to TM$ is the musical isomorphism. In particular, when $M = \mathbb{C}^4$, we can see that for any three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{C}^4$ then

$$\mathrm{span}\{v_1, v_2, v_3, \kappa(v_1, v_2, v_3)\},$$

is a Cayley subspace of $\mathbb{C}^4$. Since the triple cross product is nondegenerate, to guarantee that this subspace is, in fact a complex subspace of $\mathbb{C}^4$ we must ask that $\mathrm{Re}\ \Omega(v_1, v_2, v_3, \cdot) = 0$.

From this we can see that a Cayley subspace of $\mathbb{C}^4$ is complex if, and only if, the one-form valued three-form defined by

$$\sigma(u, v, w) = \mathrm{Re}\ \Omega(u, v, w, \cdot),$$

vanishes on the Cayley subspace.

It is quite remarkable then that we can prove in Proposition 3.4.2 that $\sigma|_N \equiv 0$ if, and only if, N is a two-dimensional complex submanifold of M, that is, we do not require that N is Cayley before we check this condition. We will first require a result which follows from a lemma of Harvey and Lawson [15, II.6 Lem 6.13].

Lemma 3.4.1. *Let V be a four-dimensional oriented linear subspace of $\mathbb{C}^4$. Then there exist a unitary basis $e_1, Je_1, \ldots, e_4, Je_4$ for $\mathbb{C}^4$ and angles $0 \leq \theta_1 \leq \pi/2$, $\theta_1 \leq \theta_2 \leq \pi$ such that*

$$V = \mathrm{span}\{e_1, Je_1 \cos\theta_1 + e_2 \sin\theta_1, e_3, Je_3 \cos\theta_2 + e_4 \sin\theta_2\}.$$

Given this result, we can prove the following proposition.

Proposition 3.4.2. *Let X be an oriented real four-dimensional submanifold of a Calabi–Yau four-fold M. Then X is a complex submanifold of M if, and only if*

$$\sigma(u, v, w) \equiv 0,$$

for all vector fields u, v, w on X, where

$$\sigma(u, v, w) := \mathrm{Re}\ \Omega(u, v, w, \cdot), \tag{3.4.1}$$

where Ω is the holomorphic volume form of M.

Proof. Recall that for u, v, w vector fields on X we have that

$$\sigma(u, v, w) := \text{Re } \Omega(u, v, w, \cdot),$$

where Ω is a holomorphic volume form on M. If X is complex, then by the adjunction formula [17, Prop 2.2.17],

$$K_M|_X \cong K_X \otimes \Lambda^2 \nu_M^{*1,0}(X),$$

where K_M denotes the canonical bundle of M. Since Ω is a nowhere vanishing section of K_M, it is each to see that for any three vector fields u, v, w on X

$$\Omega(u, v, w, \cdot) = \overline{\Omega}(u, v, w, \cdot) = 0,$$

and so,

$$\text{Re } \Omega(u, v, w, \cdot) = 0.$$

It remains to show that $\sigma|_X \equiv 0$ implies that X is a complex manifold. We show the contrapositive, that is, if X is not complex, then we can find vector fields u, v, w on X so that $\sigma(u, v, w) \neq 0$. It suffices to show that for an arbitrary $x \in X$, we can find nonzero $v_1, v_2, v_3 \in T_xX$ so that $\sigma_x(v_1, v_2, v_3) \neq 0$.

Identifying (T_xM, ω_x) with $\mathbb{C}^4$ with Euclidean Kähler form and holomorphic volume form as defined in (1.2.3), we can view T_xX as an oriented four-dimensional linear subspace V of $\mathbb{C}^4$. Apply Lemma 3.4.1 to choose a unitary basis $\{e_1, Je_1, \ldots, e_4, Je_4\}$ for $\mathbb{C}^4$ so that for some $0 \leq \theta_1 \leq \pi/2$, $\theta_1 \leq \theta_2 \leq \pi$

$$V = \text{span}\{e_1, Je_1 \cos\theta_1 + e_2 \sin\theta_1, e_3, Je_3 \cos\theta_2 + e_4 \sin\theta_2\}.$$

Since V is not a complex subspace of $\mathbb{C}^4$, suppose without loss of generality that $0 < \theta_2 < \pi$. The holomorphic volume form on $\mathbb{C}^4$ takes the form

$$\Omega = (e^1 - iJe^1) \wedge (e^2 - iJe^2) \wedge (e^3 - iJe^3) \wedge (e^4 - iJe^4),$$

where $e^i = g(e_i, \cdot)$. Notice that $Je^1 = -g(Je_1, \cdot)$. Then we have that

$$\begin{aligned}\sigma(e_1, e_3, Je_3\cos\theta_2 + e_4\sin\theta_2) &= \frac{1}{2}(\Omega + \overline{\Omega})(e_1, e_3, Je_3\cos\theta_2 + e_4\sin\theta_2, \cdot)\\ &= \frac{1}{2}(\sin\theta_2(e^2 - iJe^2) + \sin\theta_2(e^2 + iJe^2))\\ &= \sin\theta_2\, e^2,\end{aligned}$$

which doesn't vanish since we assumed that $0 < \theta_2 < \pi$. $\square$

Remark. Given this result, it is natural to wonder whether four-dimensional special Lagrangian submanifolds of a Calabi–Yau four-fold M (calibrated by Re Ω, where Ω is a holomorphic volume form on M) could be characterised by the vanishing of the T^*M valued three-form σ' defined by

$$\sigma'(u, v, w) := \frac{1}{2}\omega \wedge \omega(u, v, w, \cdot),$$

for u, v, w vector fields on M. However, unlike complex submanifolds, we cannot define special Lagrangian submanifolds using only the complex structure on M. We can, however, define Lagrangian submanifolds using only the complex structure, and therefore a similar argument to Proposition 3.4.2 will in fact show that σ' vanishing on a four-dimensional submanifold X of M is equivalent to X being Lagrangian. A local argument similar to Lemma 3.4.6 below shows that this definition is equivalent to the standard definition of Lagrangian which is that ω must vanish on X.

In the style of Proposition 3.2.2 we can now identify the moduli space of complex deformations of a compact complex surface in a Calabi–Yau four-fold with the kernel of a partial differential operator.

Proposition 3.4.3. *Let N be a compact complex surface inside a four-dimensional Calabi–Yau manifold M. Let $V \subseteq \nu_M(N)$ be the subset defined in the tubular neighbourhood theorem 3.2.1, and for $v \in C^\infty(V)$ define $N_v := \exp_v(N)$. Then the moduli space of complex deformations of N is isomorphic near N to the kernel of*

$$\begin{aligned}G : C^\infty(V \otimes \mathbb{C}) &\to C^\infty(\Lambda^1 N \otimes T^*M|_N \otimes \mathbb{C}),\\ v &\mapsto *_N \exp_v^* \sigma|_{N_v},\end{aligned} \tag{3.4.2}$$

where σ was defined in Proposition 3.4.2.

Proof. The identification of the moduli space with this operator follows immediately from Proposition 3.4.2. □

3.4.2 Properties of the operator G

In this section we will look at the operator G defined in Proposition 3.4.2 in detail. We will compute the linear part of G in Proposition 3.4.4, which will allow us to compare infinitesimal Cayley deformations to infinitesimal complex deformations of a compact complex surface, before using a local argument in Lemma 3.4.6 to show that the kernel of G is equal to the kernel of its linear part.

Proposition 3.4.4. *Let N be a compact complex surface in a Calabi–Yau four-fold M. Let G be the partial differential operator defined in Proposition 3.4.2. Then the linearisation of G at zero is equal to the operator*

$$v \mapsto -\partial^*(v \lrcorner \Omega) - \bar{\partial}^*(v \lrcorner \overline{\Omega}),$$

where $v \in C^\infty(\nu_M(N) \otimes \mathbb{C})$. Therefore v is an infinitesimal complex deformation of N if, and only if,

$$\partial^*(v \lrcorner \Omega) = 0 = \bar{\partial}^*(v \lrcorner \overline{\Omega})$$

Moreover, we have that, if $v = v_1 \oplus v_2$ where $v_1 \in \nu_M^{1,0}(N)$ and $v_2 \in \nu_M^{0,1}(N)$

$$\partial^*(v_1 \lrcorner \Omega) = 0 \iff \bar{\partial} v_1 = 0.$$

Remark. We can see from this proposition that if $v \in C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2} N \otimes \nu_M^{1,0}(N))$ is an infinitesimal Cayley deformation of N, that is,

$$(\bar{\partial} + \bar{\partial}^*)v = 0,$$

then for v to be an infinitesimal complex deformation of N we must have that

$$\bar{\partial} v = \bar{\partial}^* v = 0.$$

Since N is compact, this is already true, and so we see that infinitesimal complex and Cayley deformations of a compact complex surface inside a Calabi–Yau four-fold are the same.

Proof. By definition, we have that

$$dG|_0(V) = \frac{d}{dt} G(tv)|_{t=0} = \frac{d}{dt} *_N \exp^*_{tv}(\sigma|_{N_{tv}}) = *_N(\mathcal{L}_v \sigma).$$

Notice that if $\{e_i, e_j, e_k, e_l\}$ is an orthonormal frame for TN so that $\mathrm{vol}_N(e_i, e_j, e_k, e_l) = \pm 1$ then

$$\pm(*_N \mathcal{L}_v \sigma)(e_l) = (\mathcal{L}_v \sigma)(e_i, e_j, e_k).$$

Recalling that σ takes values in T^*M, we can compute using a formula such as [20, Eqn (4.3.26)] that for any vector field u on M, we have that

$$\begin{aligned}(\mathcal{L}_v \sigma)(e_i, e_j, e_k)(u) &= (\nabla_v \sigma)(e_i, e_j, e_k)(u) + \sigma(\nabla^\perp_{e_i} v, e_j, e_k)(u) \\ &\quad + \sigma(e_i, \nabla^\perp_{e_j} v, e_k)(u) + \sigma(e_i, e_j, \nabla^\perp_{e_k} v)(u) + \sigma(e_i, e_j, e_k)(\nabla_u v),\end{aligned}$$

where we write $\nabla_{e_p} v = \nabla^T_{e_p} v + \nabla^\perp_{e_p} v$ for $\nabla^T_{e_p} v \in TN$ and $\nabla^\perp_{e_p} v \in \nu_M(N)$, and use that σ vanishes when evaluated on three tangent vectors to N. We see immediately that since σ is defined using the parallel form Ω that the first term on the right hand side vanishes, while the last term on the right hand side vanishes since e_i, e_j and e_k are tangent to N.

Without loss of generality, suppose that $Je_i = e_j$. Then since $e^i \wedge e^j \in \Lambda^{1,1}N$, we have that $\Omega(e_i, e_j, \cdot\,, \cdot\,) = \overline{\Omega}(e_i, e_j, \cdot\,, \cdot\,) = 0$, since by the adjunction formula,

$$(\Omega + \overline{\Omega})|_N \in \Lambda^{2,0}N \otimes \Lambda^2 \nu^{*1,0}_M(N) \oplus \Lambda^{0,2}N \otimes \Lambda^2 \nu^{*0,1}_M(N).$$

So we may deduce that

$$\begin{aligned}(\mathcal{L}_v \sigma)(e_i, e_j, e_k) &= \sigma(\nabla^\perp_{e_i} v, e_j, e_k) + \sigma(e_i, \nabla^\perp_{e_j} v, e_k) \\ &= \frac{1}{2}\sigma((\nabla^\perp_{e_i} + i\nabla^\perp_{e_j})v, e_j + ie_i, e_k) + \frac{1}{2}\sigma((\nabla^\perp_{e_i} - i\nabla^\perp_{e_j})v, e_j - ie_i, e_k) \\ &= \frac{1}{4}\Omega((\nabla^\perp_{e_i} + i\nabla^\perp_{e_j})v, e_j + ie_i, e_k, \cdot) + \frac{1}{4}\overline{\Omega}((\nabla^\perp_{e_i} - i\nabla^\perp_{e_j})v, e_j - ie_i, e_k, \cdot),\end{aligned}$$

where we use the definition of σ and the fact that $e_j + ie_i$ is of type $(1,0)$ and $e_j - ie_i$ is of type $(0,1)$. Let us examine the term

$$\frac{1}{4}\Omega((\nabla^{\perp}_{e_i} + i\nabla^{\perp}_{e_j})v, e_j + ie_i, e_k, \,\cdot\,).$$

This is equal to

$$\frac{1}{4}\Omega((\nabla^{\perp}_{e_i} + i\nabla^{\perp}_{e_j})v, e_j + ie_i, e_k, \,\cdot\,) - \frac{1}{4}\Omega((\nabla^{\perp}_{e_i} - i\nabla^{\perp}_{e_j})v, e_j - ie_i, e_k, \,\cdot\,),$$

since $e_j - ie_i$ is of type $(0,1)$ and so we have added zero. By linearity, this is equal to

$$\frac{i}{2}\Omega(\nabla^{\perp}_{e_i}v, e_i, e_k, \,\cdot\,) + \frac{i}{2}\Omega(\nabla^{\perp}_{e_j}v, e_j, e_k, \,\cdot\,).$$

A similar computation for the $\overline{\Omega}$ term leads us to deduce that

$$\begin{aligned}\pm e_l \lrcorner *_N \mathcal{L}_v\sigma = (\mathcal{L}_v\sigma)(e_i, e_j, e_k) &= \frac{i}{2}\Omega(\nabla^{\perp}_{e_i}v, e_i, e_k, \,\cdot\,) \\ &+ \frac{i}{2}\Omega(\nabla^{\perp}_{e_j}v, e_j, e_k, \,\cdot\,) - \frac{i}{2}\overline{\Omega}(\nabla^{\perp}_{e_i}v, e_i, e_k, \,\cdot\,) - \frac{i}{2}\overline{\Omega}(\nabla^{\perp}_{e_j}v, e_j, e_k, \,\cdot\,).\end{aligned} \tag{3.4.3}$$

Suppose now that $e_l = Je_k$. Then we select plus in Equation (3.4.3). We have that $e_k - ie_l$ is of type $(1,0)$, while $e_k + ie_l$ is of type $(0,1)$. Therefore

$$(e_k + ie_l)\lrcorner\,\Omega = 0,$$
$$(e_k - ie_l)\lrcorner\,\overline{\Omega} = 0.$$

Rearranging these equations, we see that

$$e_k\lrcorner\,\Omega = -ie_l\lrcorner\,\Omega,$$
$$e_k\lrcorner\,\overline{\Omega} = ie_l\lrcorner\,\overline{\Omega}.$$

Using these identities, Equation (3.4.3) becomes

$$\begin{aligned}e_l \lrcorner *_N \mathcal{L}_v\sigma = (\mathcal{L}_v\sigma)(e_i, e_j, e_k) &= \frac{1}{2}\Omega(\nabla^{\perp}_{e_i}v, e_i, e_l, \,\cdot\,) \\ &+ \frac{1}{2}\Omega(\nabla^{\perp}_{e_j}v, e_j, e_l, \,\cdot\,) + \frac{1}{2}\overline{\Omega}(\nabla^{\perp}_{e_i}v, e_i, e_l, \,\cdot\,) + \frac{1}{2}\overline{\Omega}(\nabla^{\perp}_{e_j}v, e_j, e_l, \,\cdot\,).\end{aligned} \tag{3.4.4}$$

If $Je_k = -e_l$, a similar calculation yields the same expression. Therefore summing over terms of the form (3.4.4), we find that

$$\begin{aligned}
*_N \mathcal{L}_v \sigma &= \sum_{i=1}^{4} \Omega(\nabla^{\perp}_{e_i} v, e_i, \cdot, \cdot) + \overline{\Omega}(\nabla^{\perp}_{e_i} v, e_i, \cdot, \cdot) \\
&= \sum_{i=1}^{4} e_i \lrcorner ((\nabla^{\perp}_{e_i} v) \lrcorner \Omega) + e_i \lrcorner ((\nabla^{\perp}_{e_i} v) \lrcorner \overline{\Omega}) \\
&= \sum_{i=1}^{4} e_i \lrcorner \nabla_{e_i}(v \lrcorner \Omega) + e_i \lrcorner \nabla_{e_i}(v \lrcorner \overline{\Omega}) \\
&= -\partial^*(v \lrcorner \Omega) - \bar{\partial}^*(v \lrcorner \overline{\Omega}).
\end{aligned}$$

It remains to show that if $v_1 \in \nu^{1,0}_M(N)$ then

$$\bar{\partial} v_1 = 0 \iff \partial^*(v_1 \lrcorner \Omega) = 0.$$

Let $\{e_1, e_2, e_3, e_4\}$ be a frame for TN and suppose that $Je_1 = e_3$ and $Je_2 = e_4$. Then we have that

$$\begin{aligned}
\bar{\partial} v_1 = 0 &\iff (e^1 - ie^3) \wedge (\nabla^{\perp}_{e_1} + i\nabla^{\perp}_{e_3}) v_1 + (e^2 - ie^4) \wedge (\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_4}) v_1 = 0 \\
&\iff (\nabla^{\perp}_{e_1} + i\nabla^{\perp}_{e_3}) v_1 = (\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_4}) v_1 = 0 \\
&\iff [(\nabla^{\perp}_{e_1} + i\nabla^{\perp}_{e_3}) v_1] \lrcorner \Omega = [(\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_4}) v_1] \lrcorner \Omega = 0 \\
&\iff \Omega((\nabla^{\perp}_{e_1} + i\nabla^{\perp}_{e_3}) v_1, e_1 - ie_3, \cdot, \cdot) + \Omega((\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_4}) v_1, e_2 - ie_4, \cdot, \cdot) = 0 \\
&\iff 2 \sum_{i=1}^{4} \Omega(\nabla^{\perp}_{e_i} v_1, e_i, \cdot, \cdot) = 0 \\
&\iff 2 \sum_{i=1}^{4} e_i \lrcorner \nabla_{e_i}(v_1 \lrcorner \Omega) = 0 \\
&\iff -2\partial^*(v_1 \lrcorner \Omega) = 0,
\end{aligned}$$

where we have exploited the property that Ω never vanishes, that $(\nabla^{\perp}_{e_i} v)$ is of type $(1, 0)$ and used similar tricks to the proof of the first part of the proposition. $\square$

Similarly to Proposition 3.4.4 we may identify the kernels of the operators $\bar{\partial}$ and $\bar{\partial}^*$. This will be helpful when we compare the results of this chapter to Kodaira's theorem 3.1.1.

Corollary 3.4.5. *Let N be a complex surface inside a four-dimensional Calabi–Yau manifold M. Consider the operators*

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$
$$\bar{\partial}^* : C^\infty(\Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Then there is an isomorphism

$$\nu_M^{1,0}(N) \to \Lambda^{0,2}N \otimes \nu_M^{1,0}(N),$$

that induces an isomorphism

$$\operatorname{Ker} \bar{\partial} \to \operatorname{Ker} \bar{\partial}^*.$$

Proof. Let Ω be a holomorphic volume form on M. Then the isomorphism

$$\nu_M^{1,0}(N) \to \Lambda^{0,2}N \otimes \nu_M^{1,0}(N),$$

is given by

$$v \mapsto (\bar{v} \lrcorner \overline{\Omega})^\sharp,$$

where $\sharp : \nu_M^{*0,1}(N) \to \nu_M^{1,0}(N)$ is the musical isomorphism. It follows from Proposition 3.3.1 that this is an isomorphism. We proved in Proposition 3.4.4 that

$$\bar{\partial} v = 0 \iff \partial^*(v \lrcorner \Omega) = 0.$$

Since

$$\bar{\partial}^*(\bar{v} \lrcorner \overline{\Omega}) = \overline{\partial^*(v \lrcorner \Omega)},$$

the result follows. □

The following lemma allows us to see that the kernel of G defined in Proposition 3.4.2 is equal to the kernel of the linear part of G computed in Proposition 3.4.4.

Lemma 3.4.6. *Let N be a two-dimensional compact complex submanifold of a Calabi–Yau four-fold M. Let $v \in C^\infty(\nu_M^{1,0}(N) \otimes \mathbb{C})$ satisfy*

$$\partial^*(v \lrcorner \Omega) = 0 = \bar{\partial}^*(v \lrcorner \overline{\Omega}).$$

Let G be the operator defined in Proposition 3.4.2. Then $G(v) = 0$.

Proof. The argument here is similar to the argument of Proposition 3.2.2. We will write the tangent space to a deformation of N as a graph over the tangent space of N, identified with a complex subspace of $\mathbb{C}^4$ and write down the condition equivalent to $G(v) = 0$.

Choose $p \in N$. Then $T_pM = T_pN \oplus \nu_p(N)$. Choose an orthonormal basis $\{e_1, \ldots, e_8\}$ for T_pM with $Je_i = e_{i+4}$ for $i = 1, \ldots 4$ so that

$$T_pN = \mathrm{span}\{e_1, e_2, Je_1, Je_2\}.$$

Let N' be a small deformation of N with diffeomorphism $f : N \to N'$. Then there is a natural isometry $T_pM \to T_{f(p)}M$ preserving the complex structures J and J' on these spaces. Denote by $\{e'_1, \ldots, e'_8\}$ the orthonormal basis of $T_{f(p)}M$ where e_i maps to e'_i under this isometry, with $J'e'_i = e'_{i+4}$. Then

$$T_{f(p)}N' = \mathrm{span}\{v_1, v_2, v_5, v_6\},$$

where without loss of generality since N' is a small deformation of N we may take

$$v_j = e'_j + \sum_{i=3,4,7,8} \lambda^j_i e'_i,$$

for $\lambda^j_i \in \mathbb{R}$.

We can then evaluate

$$\sigma_{f(p)}(v_i, v_j, v_k) := \mathrm{Re}\ \Omega_{f(p)}(v_i, v_j, v_k, \cdot) = 0,$$

where $\{i, j, k\} \subseteq \{1, 2, 5, 6\}$. We have that

$$\mathrm{Re}\ \Omega_{f(p)} = e'^{1234} - e'^{1278} + e'^{1368} - e'^{1467} - e'^{2358} + e'^{2457} - e'^{3456} + e'^{5678},$$

where $e'^j(e'_k) = \delta_{jk}$ and $e'^{ijkl} := e'^i \wedge e'^j \wedge e'^k \wedge e'^l$.

We evaluate $\sigma(v_i, v_j, v_k) = 0$ for $\{i, j, k\} = \{1, 2, 5\}, \{1, 2, 6\}, \{1, 5, 6\}$ and $\{2, 5, 6\}$. Eliminating duplicate equations, we find that the λ^i_j must satisfy the following linear equations

$$\begin{aligned}
\lambda_3^1 - \lambda_7^5 &= 0,\\
\lambda_4^1 - \lambda_8^5 &= 0,\\
\lambda_7^1 + \lambda_3^5 &= 0,\\
\lambda_8^1 + \lambda_4^5 &= 0,\\
\lambda_3^2 - \lambda_7^6 &= 0,\\
\lambda_4^2 - \lambda_8^6 &= 0,\\
\lambda_7^2 + \lambda_3^6 &= 0,\\
\lambda_8^2 + \lambda_4^6 &= 0,
\end{aligned}$$

and the following nonlinear equations

$$\begin{aligned}
\lambda_3^1\lambda_4^5 - \lambda_4^1\lambda_3^5 - \lambda_7^1\lambda_8^5 + \lambda_8^1\lambda_7^5 &= 0,\\
\lambda_3^1\lambda_8^5 - \lambda_4^1\lambda_7^5 + \lambda_7^1\lambda_4^5 - \lambda_8^1\lambda_3^5 &= 0,\\
\lambda_3^2\lambda_4^6 - \lambda_4^2\lambda_3^6 - \lambda_7^2\lambda_8^6 + \lambda_8^2\lambda_7^6 &= 0,\\
\lambda_4^2\lambda_7^6 - \lambda_7^2\lambda_4^6 + \lambda_8^2\lambda_3^6 - \lambda_3^2\lambda_8^6 &= 0,\\
\lambda_3^1(\lambda_8^2 + \lambda_4^6) - \lambda_4^1(\lambda_7^2 + \lambda_3^6) + \lambda_7^1(\lambda_4^2 - \lambda_8^6) - \lambda_8^1(\lambda_3^2 - \lambda_7^6) &= 0,\\
\lambda_3^1(\lambda_4^2 - \lambda_8^6) - \lambda_4^1(\lambda_3^2 - \lambda_7^6) - \lambda_7^1(\lambda_8^2 + \lambda_4^6) + \lambda_8^1(\lambda_7^2 + \lambda_3^6) &= 0,\\
\lambda_3^2(\lambda_8^1 + \lambda_4^5) - \lambda_4^2(\lambda_7^1 + \lambda_3^5) + \lambda_7^2(\lambda_4^1 - \lambda_8^5) - \lambda_8^2(\lambda_3^1 - \lambda_7^5) &= 0,\\
\lambda_3^2(\lambda_4^1 - \lambda_8^5) - \lambda_4^2(\lambda_3^1 - \lambda_7^5) - \lambda_7^2(\lambda_8^1 + \lambda_4^5) + \lambda_8^2(\lambda_7^1 + \lambda_3^5) &= 0,\\
\lambda_3^5(\lambda_8^2 + \lambda_4^6) - \lambda_4^5(\lambda_7^2 + \lambda_3^6) + \lambda_7^5(\lambda_4^2 - \lambda_8^6) - \lambda_8^5(\lambda_3^2 - \lambda_7^6) &= 0,\\
\lambda_3^5(\lambda_4^2 - \lambda_8^6) - \lambda_4^5(\lambda_3^2 - \lambda_7^6) - \lambda_7^5(\lambda_8^2 + \lambda_4^6) + \lambda_8^5(\lambda_7^2 + \lambda_3^6) &= 0,\\
\lambda_3^6(\lambda_8^1 + \lambda_4^5) - \lambda_4^6(\lambda_7^1 + \lambda_3^5) + \lambda_7^6(\lambda_4^1 - \lambda_8^5) - \lambda_8^6(\lambda_3^1 - \lambda_7^5) &= 0,\\
\lambda_3^6(\lambda_4^1 - \lambda_8^5) - \lambda_4^6(\lambda_3^1 - \lambda_7^5) - \lambda_7^6(\lambda_8^1 + \lambda_4^5) + \lambda_8^6(\lambda_7^1 + \lambda_3^5) &= 0.
\end{aligned}$$

Since the first four equations may be rewritten as

$$\begin{aligned}
&\frac{1}{2}\left[(\lambda_8^1+\lambda_4^5)(\lambda_3^1+\lambda_7^5)-(\lambda_8^1-\lambda_4^5)(\lambda_3^1-\lambda_7^5)\right.\\
&\quad\left.+(\lambda_4^1-\lambda_8^5)(\lambda_7^1-\lambda_3^5)-(\lambda_4^1+\lambda_8^5)(\lambda_7^1+\lambda_3^5)\right]=0,\\
&\frac{1}{2}\left[(\lambda_3^1-\lambda_7^5)(\lambda_4^1+\lambda_8^5)-(\lambda_3^1+\lambda_7^5)(\lambda_4^1-\lambda_8^5)\right.\\
&\quad\left.-(\lambda_7^1+\lambda_3^5)(\lambda_8^1-\lambda_4^5)+(\lambda_7^1-\lambda_3^5)(\lambda_8^1+\lambda_4^5)\right]=0,\\
&\frac{1}{2}\left[(\lambda_3^2+\lambda_7^6)(\lambda_8^2+\lambda_4^6)-(\lambda_3^2-\lambda_7^6)(\lambda_8^2-\lambda_4^6)\right.\\
&\quad\left.+(\lambda_4^2-\lambda_8^6)(\lambda_7^2-\lambda_3^6)-(\lambda_4^2+\lambda_8^6)(\lambda_7^2+\lambda_3^6)\right]=0,\\
&\frac{1}{2}\left[(\lambda_4^2-\lambda_8^6)(\lambda_3^2+\lambda_7^6)-(\lambda_4^2+\lambda_8^6)(\lambda_3^2-\lambda_7^6)\right.\\
&\quad\left.+(\lambda_8^2-\lambda_4^6)(\lambda_7^2+\lambda_3^6)-(\lambda_8^2+\lambda_4^6)(\lambda_7^2-\lambda_3^6)\right]=0,
\end{aligned}$$

it is easy to see that if the linear equations are satisfied then all of the equations above are satisfied. Therefore an infinitesimal complex deformation is a full complex deformation. □

We will now prove the main theorem of the section.

Theorem 3.4.7. *Let N be a compact complex surface inside a four-dimensional Calabi–Yau manifold M. Then the moduli space of Cayley deformations of N in M near N is isomorphic to the moduli space of complex deformations of N in M, which near N is a smooth manifold of dimension*

$$\dim_{\mathbb{C}}\mathrm{Ker}\ \bar\partial+\dim_{\mathbb{C}}\mathrm{Ker}\ \bar\partial^*=2\dim_{\mathbb{C}}\mathrm{Ker}\ \bar\partial,$$

where

$$\begin{aligned}
\bar\partial &: C^\infty(\nu_M^{1,0}(N))\to C^\infty(\Lambda^{0,1}N\otimes\nu_M^{1,0}(N)),\\
\bar\partial^* &: C^\infty(\Lambda^{0,2}N\otimes\nu_M^{1,0}(N))\to C^\infty(\Lambda^{0,1}N\otimes\nu_M^{1,0}(N)).
\end{aligned}$$

Proof. The moduli space of complex deformations of N is isomorphic to the kernel of the operator G as we saw in Proposition 3.4.2. The kernel of the operator G is isomorphic to the kernel of its linear part by Lemma 3.4.6, which is isomorphic to the sum of the kernels of $\bar{\partial}$ and $\bar{\partial}^*$ by Proposition 3.4.4. The kernels of $\bar{\partial}$ and $\bar{\partial}^*$ are isomorphic by Corollary 3.4.5.

By Theorem 3.3.4, the moduli space of Cayley deformations of N, if it is smooth, has dimension at most equal to the dimension of the kernel of $\bar{\partial}+\bar{\partial}^*$, which is equal to the sum of the dimensions of the kernel of $\bar{\partial}$ and $\bar{\partial}^*$ since N is compact. Since the moduli space of Cayley deformations of N contains the moduli space of complex deformations of N, we see that they must have the same dimension, and therefore are the same. □

With this theorem, we achieve our aims of showing directly that complex and Cayley deformations of a compact complex surface inside a Calabi–Yau manifold are the same, as can be deduced from Proposition 1.2.6. Moreover, we have matched the result of Kodaira's theorem 3.1.1 that says that the infinitesimal complex deformations of N are isomorphic to the kernel of $\bar{\partial}$ (where we have counted every deformation twice by considering the complexified normal bundle of N in M).

3.5 Example

We now compute the index of the operator

$$\bar{\partial}+\bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)),$$

using the formula given in Theorem 3.3.5. In this example, the Calabi–Yau manifold M is a degree six hypersurface in $\mathbb{C}P^5$ and N is a complete intersection in M. Explicitly, take

$$M := \{[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}P^5 \mid f(z) = z_0^6 + z_1^6 + z_2^6 + z_3^6 + z_4^6 + z_5^6 = 0\},$$

and

$$N = \{z \in M \mid f_1(z) = f_2(z) = 0\},$$

where f_i are irreducible homogeneous polynomials of degree d_i. Notice that in order for N to be a smooth manifold we require the Jacobian of $g = (f, f_1, f_2)$ to have rank

3 at each point of N. We will compute the first and second Chern classes of M, N and $\nu_M^{1,0}(N)$.

First note that since M is Calabi–Yau, we have that $c_1(M) \equiv 0$. Since $\nu_{\mathbb{C}P^5}^{1,0}(M)$ is a rank one vector bundle, we have that $c_k(\nu_{\mathbb{C}P^5}^{1,0}(M)) = 0$ for $k > 1$. We can compute

$$\begin{aligned} c_2(T^{1,0}\mathbb{C}P^5|_M) &= c_2(T^{1,0}M \oplus \nu_{\mathbb{C}P^5}^{1,0}(M)) \\ &= c_2(M) + c_2(\nu_{\mathbb{C}P^5}^{1,0}(M)) + c_1(M)c_1(\nu_{\mathbb{C}P^5}^{1,0}(M)) \\ &= c_2(M). \end{aligned}$$

Therefore, for ω the Kähler form of $\mathbb{C}P^5$, we have that

$$c_2(M) = c_2(\mathbb{C}P^5)|_M = \binom{6}{2}\omega^2|_M = 15\omega^2|_M.$$

Denote by $\mathcal{O}_{\mathbb{C}P^5}(d)$ the $-d$th tensor power of the tautological line bundle over $\mathbb{C}P^5$ for $d < 0$ and the dth tensor power of the hyperplane bundle over $\mathbb{C}P^5$ when $d > 0$. Then $\nu_M^{1,0}(N) \cong \mathcal{O}_{\mathbb{C}P^5}(d_1)|_N \oplus \mathcal{O}_{\mathbb{C}P^5}(d_2)|_N$, and so

$$\begin{aligned} c_1(\nu_M^{1,0}(N)) &= (d_1 + d_2)\omega|_N, \\ c_2(\nu_M^{1,0}(N)) &= d_1 d_2 \omega^2|_N. \end{aligned}$$

So we calculate that

$$[N] \cdot [N] = \int_N d_1 d_2\, \omega^2 = 6d_1^2 d_2^2.$$

We have that

$$c_2(M)|_N = c_2(N) + c_2(\nu_M^{1,0}(N)) + c_1(N)c_1(\nu_M^{1,0}(N)),$$

and therefore

$$c_2(N) = \left(15 - d_1 d_2 + (d_1 + d_2)^2\right)\omega^2|_N,$$

so we find that

$$\chi(N) = \int_N (15 + d_1^2 + d_2^2 + d_1 d_2)\,\omega^2 = 90 d_1 d_2 + 6d_1^3 d_2 + 6d_2^3 d_1 + 6d_1^2 d_2^2.$$

Finally,

$$\begin{aligned}
p_1(N) = -c_2(TN \otimes \mathbb{C}) &= c_1(N)^2 - 2c_2(N) \\
&= [(d_1 + d_2)^2 - 2(15 + d_1^2 + d_2^2 + d_1 d_2)]\omega^2|_N \\
&= [-d_1^2 - d_2^2 - 30]\omega^2|_N.
\end{aligned}$$

Therefore

$$\operatorname{sign}(N) = \frac{1}{3}\int_N [-30 - d_1^2 - d_2^2]\,\omega^2 = -60 d_1 d_2 - 2d_1^3 d_2 - 2d_2^3 d_1.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\chi(N) + \frac{1}{2}\operatorname{sign}(N) - [N]\cdot[N] \\
&= 45 d_1 d_2 + 3d_1^3 d_2 + 3d_2^3 d_1 + 3d_1^2 d_2^2 - 30 d_1 d_2 - d_1^3 d_2 - d_2^3 d_1 - 6d_1^2 d_2^2 \\
&= 15 d_1 d_2 + 2d_1^3 d_2 + 2d_2^3 d_1 - 3d_1^2 d_2^2. \qquad (3.5.1)
\end{aligned}$$

Examining Equation (3.5.1), we see that this expression is strictly positive, unbounded and even for any $d_1, d_2 \in \mathbb{N}$.

Chapter 4

Fredholm theory on noncompact manifolds

Our main aim for the remainder of this thesis is to extend the results of Chapter 3 to noncompact manifolds. This chapter predominantly contains a literature review of some results for elliptic operators on noncompact manifolds that we will need in Chapters 5 and 6.

4.1 Introduction

Let (X, g, Φ) be a $Spin(7)$-manifold and let Y be a Cayley submanifold of X. In the proof of Theorem 3.2.6, there are two points at which we need Y to be compact. Firstly, for the tubular neighbourhood theorem to hold on Y so that we may identify deformations of Y with normal vector fields on Y. Secondly, we study a nonlinear elliptic partial differential operator, F, acting on normal vector fields. In order to apply the Banach space implicit function theorem to F, we require the linear part of F, D, to surject onto the target space of F. This does not happen in general, and so we must construct a new operator $\mathcal{F}$ that does satisfy the hypotheses of the implicit function theorem. This construction relies on the linear part of F being a Fredholm operator.

Asking when an elliptic operator on a noncompact manifold is Fredholm is a highly nontrivial problem, as this result can fail at even the simplest level, which we will see in the following well-known example.

Example. Consider the operator

$$\frac{d}{dt} : L^2_1(\mathbb{R}) \to L^2(\mathbb{R}), \tag{4.1.1}$$

where $L^p_k(\mathbb{R})$ denotes the Sobolev space of functions whose first k (weak) derivatives lie in $L^p(\mathbb{R})$. This operator is clearly elliptic, however, we will show that its image isn't closed, and therefore (4.1.1) is not Fredholm.

Consider the sequence of functions

$$u_n(t) := \begin{cases} t & 0 \leq t < 1, \\ t^{-1/2} & 1 \leq t < n, \\ \frac{1}{n^{1/2}}(n+1-t) & n \leq t \leq n+1, \\ 0 & \text{otherwise}, \end{cases}$$

in L^2_1 which have weak derivative

$$v_n(t) := \begin{cases} 1 & 0 \leq t < 1, \\ -\frac{1}{2}t^{-3/2} & 1 \leq t < n, \\ -\frac{1}{n^{1/2}} & n \leq t \leq n+1, \\ 0 & \text{otherwise}. \end{cases}$$

Then (v_n) is a sequence in $\operatorname{Im} \frac{d}{dt}$, and $v_n \to v$ in L^2, where

$$v = \begin{cases} 0 & t < 0, \\ 1 & 0 \leq t < 1, \\ -\frac{1}{2}t^{-3/2} & t \geq 1, \end{cases}$$

We can see that v is the weak derivative of u, where

$$u = \begin{cases} 0 & t < 0, \\ t & 0 \leq t < 1, \\ t^{-1/2} & t \geq 1, \end{cases}$$

but $u \notin L^2_1(\mathbb{R})$, and so $v \notin \operatorname{Im} \frac{d}{dt}$.

In the mid-eighties, a Fredholm theory for elliptic operators on manifolds with a cylindrical end was developed simultaneously by Lockhart and McOwen [35] and also Maz'ja and Plamenevskiĭ [42]. Following Lockhart and McOwen, we will quote the main results of this theory and discuss how these results can be applied to the type of noncompact manifold that we will consider in subsequent chapters.

We are interested in the index of Fredholm operators, and so in Section 4.4 we will discuss the Atiyah–Patodi–Singer index theorem for elliptic operators on compact manifolds with boundary. Most of the work in this chapter is a literature review, or an explanation of a reasonably common application of the theory in the literature review. The work in Section 4.4.3, which explains how to apply the Atiyah–Patodi–Singer index theorem to conically singular manifolds, is, to the author's knowledge, a new application of this result.

4.2 Elliptic operators on manifolds with cylindrical end

This section is a review of the main results of Lockhart and McOwen's paper [35].

In Section 4.2.1 we define the objects studied by Lockhart and McOwen, namely asymptotically translation invariant operators on manifolds with cylindrical end.

In Section 4.2.2 we define a weighted norm on sections that locally lie in a Sobolev space. This defines a Banach space, and we will see that for 'most' choices of weight, asymptotically translation invariant elliptic operators acting on sections in this space are Fredholm operators. We will quote this result in Section 4.2.3, where we will also explain how to find the 'bad weights' for which a translation invariant elliptic operator acting on the Sobolev space with weighted norms are not Fredholm.

4.2.1 Asymptotically translation invariant operators

Let X be a manifold with a cylindrical end, that is, a noncompact manifold without boundary that contains a compact manifold X_0 with $\partial X_0 = L$ (a closed manifold of dimension one less than X_0), satisfying

$$X \backslash X_0 = L \times \mathbb{R}_+ = \{(\omega, t) \,|\, l \in L,\, 0 < t < \infty\}.$$

Call $X \backslash X_0$ the cylindrical end of X. Let E and F be vector bundles of forms or (s, q)-tensors over X, and write $C^\infty(E)$ for the smooth sections of E and $C_0^\infty(E)$ for smooth sections of E with compact support. Let

$$A : C_0^\infty(E) \to C_0^\infty(F),$$

be an m^{th}-order linear differential operator with smooth coefficients.

The following definition is taken from Lockhart's paper [34, §2].

Definition 4.2.1. Let g be a metric on X that is asymptotic to a product metric $g_\infty = dt^2 + g_L$ on the cylindrical end $L \times (0, \infty)$ of X, in the sense that for all $j \in \mathbb{N}$

$$|\nabla_\infty^j (g - g_\infty)|_{g_\infty} = O(e^{-\delta t}), \tag{4.2.1}$$

as $t \to \infty$ for any $\delta > 0$, where ∇_∞ is the Levi-Civita connection of g_∞. Write

$$A = \sum_{j=0}^{m} a_j \cdot \nabla^j,$$

where $a_j \in C^\infty(E^* \otimes F \otimes (TX)^{\otimes j})$, '$\cdot$' denotes the tensor product followed by contraction and ∇ is the Levi-Civita connection of g. We say that A is *translation invariant* if it is invariant under the $\mathbb{R}_+$-action on the cylindrical end $L \times (0, \infty)$ of X.

Example. Let X be a manifold with cylindrical end $L \times (0, \infty)$, and let g be a metric on X that is asymptotic to the product metric $g_\infty = dt^2 + g_L$ on the cylindrical end of X. Then the exterior derivative

$$d : C^\infty(\Lambda^p X) \to C^\infty(\Lambda^{p+1} X),$$

is a translation invariant operator on X. This is easy to see. First recall that d does not depend on the choice of metric on X. Therefore, if $e_1, \ldots, e_n$ is an orthonormal frame for (L, g_L) with dual coframe $e^1, \ldots, e^n$ then we can take, for $\alpha \in C^\infty(\Lambda^m X)$

$$d\alpha = dt \wedge \frac{\partial \alpha}{\partial t} + \sum_{i=1}^{n} e^i \wedge \nabla_{e_i} \alpha,$$

where ∇ is the Levi-Civita connection of (L, g_L). Then since if $t' = t + t_0$ for some $t_0 \in (0, \infty)$,

$$dt = dt', \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'},$$

we see that d is translation invariant on $L \times (0, \infty)$.

Definition 4.2.2. Let g be a metric on X asymptotic to a product metric on the cylindrical end of X, and write

$$A = \sum_{j=0}^{m} a_j \cdot \nabla^j,$$
$$A_\infty = \sum_{j=0}^{m} a_j^\infty \cdot \nabla^j,$$

where A_∞ is translation invariant, $a_j, a_j^\infty \in C^\infty(E^* \otimes F \otimes (TX)^{\otimes j})$, '$\cdot$' denotes the tensor product followed by contraction and ∇ is the Levi-Civita connection of g.

We say that A is *asymptotically translation invariant* to A_∞ if on the cylindrical end $L \times (0, \infty)$ of X for $0 \leq j \leq m$ and all $k \in \mathbb{N}$

$$|\nabla^k (a_j - a_j^\infty)|_g = O(e^{-\delta t}),$$

as $t \to \infty$ for some $\delta > 0$.

Example. Let X be a manifold with cylindrical end and let g be a metric on X that is asymptotic to the product metric $g_\infty = dt^2 + g_L$ on the cylindrical end $L \times (0, \infty)$ of X. Then the operator

$$d^* : C^\infty(\Lambda^{m+1}X) \to C^\infty(\Lambda^m X),$$

is asymptotically translation invariant to the operator

$$d^*_\infty : C^\infty(\Lambda^{m+1}X) \to C^\infty(\Lambda^m X),$$

where d^* is computed using g and d^*_∞ is computed using g_∞. We can see that d^*_∞ is translation invariant by writing for $\beta \in C^\infty(\Lambda^{m+1}X)$

$$d^*_\infty \beta = -\frac{\partial}{\partial t} \lrcorner \frac{\partial \beta}{\partial t} - \sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} \beta,$$

where $e_1, \ldots, e_n$ is an orthonormal frame for (L, g_L) and ∇ is the Levi-Civita connection of g_L. Since the metric g is asymptotic to g_∞, it follows that we can write the operator d^* in terms of the Levi-Civita connection of g_∞ with coefficients that approach the coefficients of d^*_∞ as $t \to \infty$.

4.2.2 Weighted Sobolev spaces

For this definition we follow Lockhart [34, §3].

Definition 4.2.3. Let X be a manifold with a cylindrical end $L \times (0, \infty)$ as in Section 4.2.1. Let g be a metric on X asymptotic to a product metric on the cylindrical end of X. For a vector bundle E over X, define the weighted Sobolev spaces $W^p_{k,\delta}(E)$ to be the space of sections $\sigma \in L^p_{k,\mathrm{loc}}(E)$ so that

$$\|\sigma\|_{W^p_{k,\delta}} := \left(\sum_{j=0}^{k} \int_X |\rho^{-\delta} \nabla^j \sigma|^p \, \mathrm{vol}_g \right)^{1/p}, \tag{4.2.2}$$

is finite, where $\rho : X \to (0, 1]$ satisfies $ce^{-t} \le \rho(t) \le Ce^{-t}$ on $L \times (0, \infty)$ and is equal to one elsewhere.

Remark. By writing ρ here rather than e^{-t} Definition 4.2.3 will generalise to asymptotically cylindrical manifolds. It turns out that weighted norms (4.2.2) defined using different asymptotically invariant metrics on X are equivalent, and moreover the definition of the norm (4.2.2) is equivalent to the metric independent definition of this norm given in Lockhart and McOwen's original paper [35, Eqn 1.4].

The spaces $W^p_{k,\delta}(E)$ are Banach spaces. They contain $C^\infty_0(E)$ as a dense subset [34, Prop 3.9].

4.2.3 Results

We may now quote the main results of Lockhart and McOwen.

Theorem 4.2.1 ([35, Thm 6.2])**.** *Let X be a manifold with a cylindrical end, E and F vector bundles over X with the same rank and let*

$$A : C^\infty_0(E) \to C^\infty_0(F),$$

be a linear elliptic m^{th}-order differential operator with smooth coefficients which is asymptotically translation invariant to

$$A_\infty : C^\infty_0(E) \to C^\infty_0(F).$$

Then A and A_∞ extend to bounded maps

$$A : W^p_{k+m,\delta}(E) \to W^p_{k,\delta}(F), \tag{4.2.3}$$

$$A_\infty : W^p_{k+m,\delta}(E) \to W^p_{k,\delta}(F). \tag{4.2.4}$$

There exists a discrete set $\mathcal{D}_{A_\infty} \subseteq \mathbb{R}$ such that the maps (4.2.3) *and* (4.2.4) *are Fredholm if, and only if, $\delta \in \mathbb{R}\backslash\mathcal{D}_{A_\infty}$. Moreover, the indices of* (4.2.3) *and* (4.2.4) *differ by a constant independent of δ.*

Remark. We call the set $\mathcal{D}_{A_\infty}$ the *exceptional weights* for A_∞.

Using the notation of Theorem 4.2.1, it will be useful later, in particular in Chapter 6, to describe the set $\mathcal{D}_{A_\infty}$ explicitly. This will also help us to state a result, Theorem 4.2.2 below, about how the indices of (4.2.3) and (4.2.4) change as the weight δ varies.

Let A_∞ be a translation invariant operator as in Theorem 4.2.1. Take the Fourier transform of the following equation on the cylindrical end $L \times (0, \infty)$ of X

$$A_\infty\left(l, \nabla_L, \frac{\partial}{\partial t}\right)\sigma(l,t) = 0, \tag{4.2.5}$$

and consider the eigenvalue problem on L: for $\lambda \in \mathbb{C}$

$$A_\infty(l, \nabla_L, i\lambda)\hat{\sigma}(l,\lambda) = 0. \tag{4.2.6}$$

Define $\mathcal{C}_{A_\infty}$ to be the set of $\lambda \in \mathbb{C}$ for which the eigenvalue problem (4.2.6) has a nontrivial solution. Then define

$$\mathcal{D}_{A_\infty} := \{\mathrm{Im}\,\lambda \,|\, \lambda \in \mathcal{C}_{A_\infty}\}. \tag{4.2.7}$$

For $\lambda \in \mathcal{D}_{A_\infty}$ define $d(\lambda)$ to be the dimension of the space of solutions to (4.2.5) of the form

$$e^{-\lambda t}p(\omega, t),$$

where p is a polynomial in t with coefficients in $C^\infty(E|_L)$. The final result of Lockhart and McOwen that we will need tells us how the index of an elliptic operator acting on weighted spaces changes as the weight varies.

Theorem 4.2.2 ([35, Thm 1.2]). *Use the notation of Theorem 4.2.1 and the discussion above. Denote the indices of the operators* (4.2.3) *and* (4.2.4) *by* $\mathrm{ind}_\delta A$ *and* $\mathrm{ind}_\delta A_\infty$ *respectively. Let* $\delta_1, \delta_2 \in \mathbb{R}\backslash\mathcal{D}_{A_\infty}$ *with* $\delta_1 < \delta_2$. *Then*

$$\mathrm{ind}_{\delta_1} A - \mathrm{ind}_{\delta_2} A = \mathrm{ind}_{\delta_1} A_\infty - \mathrm{ind}_{\delta_2} A_\infty = \sum_{\delta \in (\delta_1,\delta_2)\cap\mathcal{D}_{A_\infty}} d(\delta).$$

Remark. Lockhart and McOwen generalised these results to manifolds with finitely many cylindrical ends [35, Thm 8.1]. In this case, the weighted Sobolev spaces of Definition 4.2.3 come with an n-tuple of weights which allows sections to decay with different rates on each end. Given a linear elliptic operator on such a manifold acting on these weighted Sobolev spaces, the operator is Fredholm as long as no single weight of the n-tuple is a 'bad weight'. This leads to a slightly more complicated change of index formula. The work in this thesis will deal with manifolds with one end for brevity, but it is worth mentioning that generalising to manifolds with multiple ends is not difficult.

4.3 Elliptic operators on cones

In Chapters 5 and 6 the manifolds that we will be considering will not be manifolds with cylindrical ends, but manifolds with *conical singularities*, which we will define in Section 4.3.1. We will still be able to apply Theorems 4.2.1 and 4.2.2 in this case, as we will explain in Section 4.3.2.

A particularly useful reference for the material in this section is the paper of Lockhart [34], who wrote about how the results of his paper with McOwen [35] can be generalised to noncompact manifolds with 'admissible' metrics. This class of metrics includes metrics on conically singular manifolds defined below.

4.3.1 Conically singular manifolds

Heuristically speaking, a conically singular manifold can be thought of as a compact topological space that is a smooth Riemannian manifold away from a point. If the manifold near this point is diffeomorphic to a product $L\times(0,\epsilon)$, and the metric on the manifold is close to the cone metric on $L\times(0,\epsilon)$, then we call the manifold conically singular. This idea is made formal in the following definition, taken from [37, Defn 3.1].

Definition 4.3.1. Let M be a connected Hausdorff topological space and let $\hat{x}\in M$. Suppose that $\hat{M}:=M\backslash\{\hat{x}\}$ is a smooth Riemannian manifold with metric g. Then we say that M is conically singular at $\hat{x}$ with cone C and rate λ if there exist $\epsilon>0$, $\lambda>1$ and a closed Riemannian manifold (L,g_L) of dimension one less than M, an open set $\hat{x}\in U\subseteq M$ and a diffeomorphism

$$\Psi:(0,\epsilon)\times L\to U\backslash\{\hat{x}\},$$

such that

$$|\nabla_C^j(\Psi^*g-g_C)|_{g_C}=O(r^{\lambda-1-j})\quad\text{for } j\in\mathbb{N}\text{ as } r\to 0,\tag{4.3.1}$$

where r is the coordinate on $(0, \infty)$ on the cone $C = (0, \infty) \times L$, $g_C = dr^2 + r^2 g_L$ is the cone metric on C and ∇_C is the Levi-Civita connection of g_C.

4.3.2 Elliptic operators on conically singular manifolds

In this section we will explain how we can relate conically singular manifolds to manifolds with cylindrical ends, and explain how we can apply Theorems 4.2.1 and 4.2.2 to elliptic operators on conically singular manifolds.

Cylinders and cones

Cylinders and cones are conformally equivalent. Let L be a closed Riemannian manifold and consider $L \times (0, \infty)$ with the metrics

$$g_{\text{cone}} = dr^2 + r^2 g_L, \quad g_{\text{cyl}} = dt^2 + g_L,$$

where g_L is a metric on L and r and t are coordinates on $(0, \infty)$. Then we can write $r = e^{-t}$, which enables us to see that

$$g_{\text{cone}} = e^{-2t}(dt^2 + g_L) = e^{-2t} g_{\text{cyl}}.$$

We will show that a conically singular manifold is conformally equivalent to an *asymptotically cylindrical* manifold. That is, a manifold with a cylindrical end equipped with a metric g which is asymptotic to a product metric g_{cyl} (in the sense of Equation (4.2.1)).

Definition 4.3.2. Let M be a conically singular manifold at $\hat{x}$ with cone $(0, \infty) \times L$. Use the notation of Definition 4.3.1. We say that a smooth function $\rho : \hat{M} \to (0, 1]$ is a radius function for M if ρ is bounded below by a positive constant on $M \backslash U$, while on $U \backslash \{\hat{x}\}$ there exist constants $0 < c < 1$ and $C > 1$ such that

$$cr < \Psi^* \rho < Cr,$$

on $(0, \epsilon) \times L$.

Comparing Equations (4.3.1) and (4.2.1), it is easy to see that if g is a metric on a conically singular manifold with radius function ρ then $\rho^{-2}g$ is an asymptotically translation invariant metric.

Weighted Sobolev spaces on conically singular manifolds

We will now define weighted Sobolev spaces for conically singular manifolds. The definition given here may be deduced from [34, Defn 4.1].

Definition 4.3.3. Let M be an m-dimensional conically singular manifold at $\hat{x}$ with metric g on $\hat{M} := M\backslash\{\hat{x}\}$. Let ρ be a radius function for M. For a vector bundle E define the weighted Sobolev space $L^p_{k,\mu}(E)$ to be the set of sections $\sigma \in L^p_{k,\mathrm{loc}}(E)$ such that

$$\|\sigma\|_{L^p_{k,\mu}} := \left(\sum_{j=0}^{k} \int_{\hat{M}} |\rho^{j-\mu}\nabla^j\sigma|^p \rho^{-m}\,\mathrm{vol}_g\right)^{1/p}, \tag{4.3.2}$$

is finite.

The following lemma allows us to see when the weighted Sobolev spaces of Definitions 4.2.3 and 4.3.3 are isomorphic.

Lemma 4.3.1 ([34, Prop and Defn 4.4])**.** *Let M be a conically singular manifold at $\hat{x}$ of dimension m with metric g on $\hat{M} := M\backslash\{\hat{x}\}$. Let ρ be a radius function for M. Let $T^q_s\hat{M}$ be the vector bundle of (s,q)-tensors on $\hat{M}$. Denote by $W^p_{k,\delta}(T^q_s\hat{M})$ the weighted space of Definition 4.2.3 with metric $\rho^{-2}g$ and denote by $L^p_{k,\mu}(T^q_s\hat{M})$ the weighted space of Definition 4.3.3. Then these spaces are isomorphic, with isomorphism given by*

$$\begin{aligned} L^p_{k,\mu}(T^q_s\hat{M}) &\to W^p_{k,\delta}(T^q_s\hat{M}), \\ \sigma &\mapsto \rho^{\delta-\mu+s-q}\sigma. \end{aligned}$$

Elliptic operators on conically singular manifolds

We will give an analogous result to Theorem 4.2.1 for elliptic operators on conically singular manifolds. This result is essentially a corollary of Theorem 4.2.1 and Lemma

4.3.1.

Theorem 4.3.2. *Let M be a conically singular manifold at $\hat{x}$, ρ a radius function for M and $T^q_s\hat{M}$ be the vector bundle of (s,q)-tensors on $\hat{M} := M\backslash\{\hat{x}\}$. Let*

$$A : C^\infty_0(T^q_s\hat{M}) \to C^\infty_0(T^{q'}_{s'}\hat{M}),$$

be a linear m^{th}-order elliptic differential operator with smooth coefficients such that there exists $\lambda \in \mathbb{R}$ so that

$$\tilde{A} := \rho^{\lambda+s'-q'}A\rho^{q-s},$$

is asymptotically translation invariant. Then

$$A : L^p_{k+m,\mu}(T^q_s\hat{M}) \to L^p_{k,\mu-\lambda}(T^{q'}_{s'}\hat{M}), \tag{4.3.3}$$

is a bounded map and there exists a discrete set $\mathcal{D}_A \subseteq \mathbb{R}$ such that (4.3.3) *is Fredholm if, and only if, $\mu \in \mathbb{R}\backslash\mathcal{D}_A$.*

Proof. By Theorem 4.2.1, $\tilde{A}$ extends to a bounded map

$$\tilde{A} : W^p_{k+m,\mu}(T^q_s\hat{M}) \to W^p_{k,\mu}(T^{q'}_{s'}\hat{M}).$$

By Lemma 4.3.1, the following diagram commutes

$$\begin{array}{ccc} L^p_{k+m,\mu}(T^q_s\hat{M}) & \xrightarrow{A} & L^p_{k,\mu-\lambda}(T^{q'}_{s'}\hat{M}) \\ \downarrow & & \downarrow \\ W^p_{k+m,\mu}(T^q_s\hat{M}) & \xrightarrow{\tilde{A}} & W^p_{k,\mu}(T^{q'}_{s'}\hat{M}) \end{array}$$

where we use the isomorphisms described in Lemma 4.3.1 between the weighted spaces. So we see that A is Fredholm exactly when $\tilde{A}$ is, and moreover they have the same Fredholm index. Applying Theorem 4.2.1 to $\tilde{A}$ yields the result. □

4.4 Atiyah–Patodi–Singer index theorem

We will be interested later in the index of elliptic operators on conically singular manifolds. The celebrated Atiyah–Singer index theorem gives an expression for the index of an elliptic operator on a compact manifold in terms of topological invariants. This result was extended in collaboration with V. K. Patodi to the Atiyah–Patodi–Singer index theorem [3] for certain types of elliptic operators on a compact manifold with boundary. We will quote and discuss the Atiyah–Patodi–Singer index theorem in Section 4.4.1, before explaining how we can reinterpret the theorem as an index theorem for translation invariant operators on a manifold with a cylindrical end in Section 4.4.2. Finally, in Section 4.4.3 we will discuss how we can apply the Atiyah–Patodi–Singer theorem to elliptic operators on conically singular manifolds.

The material in this section is taken from the original paper of Atiyah, Patodi and Singer [3], except for the material of Section 4.4.3 which is a new application of this material to conically singular manifolds.

4.4.1 The APS index theorem

The discovery of the Atiyah–Singer index theorem could arguably be described as one of the most exciting events in recent mathematical history. Putting aside the remarkable fact that one can write an analytic quantity such as the index of an elliptic operator purely in terms of topological invariants, results important in their own right such as the Hirzebruch signature theorem, the Hirzebruch–Riemann–Roch theorem and the Gauss-Bonnet theorem can be considered as special cases of this one result.

In order to extend the Atiyah–Singer index theorem to manifolds with boundary, Atiyah, Patodi and Singer noticed that the index formula for an elliptic operator on a manifold with boundary should involve a spectral invariant known as the η-invariant. Suppose that X is a manifold with boundary that is isometric to a product in the neighbourhood of the boundary. Then we can talk about translation invariant differential operators A on X. Recall the set of exceptional weights $\mathcal{D}_A$ of Theorem 4.2.1. These are actually the eigenvalues for the eigenproblem (4.2.6). The η-invariant is defined in terms of the elements of $\mathcal{D}_A$.

We now quote the Atiyah–Patodi–Singer index theorem.

Theorem 4.4.1 ([3, Thm 3.10]). *Let X be a compact manifold with boundary Y, E and F vector bundles over X and let*

$$A : C^\infty(E) \to C^\infty(F),$$

be a linear first order elliptic differential operator on X. We assume that, in a neighbourhood $Y \times I$ of the boundary, A takes the special form

$$A = \sigma \left(\frac{\partial}{\partial u} + B \right),$$

where u is the inward normal coordinate, σ is a bundle isomorphism $E|_Y \to F|_Y$ and B is a self-adjoint elliptic operator on Y. Let $C^\infty(E; P)$ denote the space of sections f of E satisfying the boundary condition

$$Pf(\cdot, 0) = 0,$$

where P is the spectral projection of A corresponding to non-negative eigenvalues. Then

$$A : C^\infty(E; P) \to C^\infty(F),$$

has a finite index given by

$$\text{ind } A = \int_X \alpha_0(x)\, dx - \frac{h + \eta(0)}{2}, \tag{4.4.1}$$

where α_0, h and η are defined as follows:

(i) $\alpha_0(x)$ *is the constant term in the asymptotic expansion (as $t \to 0$) of*

$$\sum e^{-t\mu'} |\phi'_\mu(x)|^2 - \sum e^{-t\mu''} |\phi''_\mu(x)|^2,$$

where μ', ϕ'_μ denote the eigenvalues and eigenfunctions of A^*A on the double of X, and μ'', ϕ''_μ are the corresponding objects for AA^*.

(ii) $h = \dim \operatorname{Ker} B =$ *multiplicity of the* 0*-eigenvalue of* B.

(iii) $\eta(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda |\lambda|^{-s}$, *where λ runs over the eigenvalues of B.*

In (iii) the series converges absolutely for Re s *large and then* $\eta(s)$ *extends as a meromorphic function on the whole* s*-plane with a finite value at* $s = 0$*. Moreover, if the asymptotic expansion in (i) has no negative power of* t *then* $\eta(s)$ *is holomorphic for* Re $s > -1/2$.

Example. Perhaps the most intimidating term in Equation (4.4.1) is $\alpha_0(x)$. However, this term can be identified with topological invariants. As an example, let X be a complex manifold, E be some holomorphic vector bundle over X and consider the operator

$$A = \bar{\partial} + \bar{\partial}^* : \Omega^{0,\mathrm{even}}(E) \to \Omega^{0,\mathrm{odd}}(E),$$

acting between E-valued $(0, 2k)$- and $(0, 2k+1)$-forms. Then in this case [50, Thm 1.6]

$$\int_X \alpha_0(x)dx = \int_X \mathrm{ch}(E)\mathcal{T}(X),$$

where $\mathrm{ch}(E)$ is the Chern character of E and $\mathcal{T}(X)$ is the Todd class of X.

4.4.2 The APS index theorem for translation invariant operators

Let X be a manifold with boundary Y which is isometric to a product near the boundary. Let E and F be vector bundles over X. Let

$$A : C^\infty(E) \to C^\infty(F),$$

be a first order linear elliptic operator that takes the form

$$\sigma\left(\frac{\partial}{\partial u} + B\right),$$

in a neighbourhood of the boundary of X. Then we can attach an infinite half cylinder $Y \times \mathbb{R}_+$ to the boundary of X, making a manifold $\hat{X}$ with cylindrical end. We can extend A to a translation invariant operator on $\hat{X}$, the objects of study in Section 4.2. Following [3, §3], we will explain how we can transform Theorem 4.4.1 into a theorem for translation invariant operators on manifolds with cylindrical ends.

Definition 4.4.1. Let $\hat{X}$ be a manifold with a cylindrical end of the form $Y \times \mathbb{R}_+$ with the product metric. Let E be a vector bundle over $\hat{X}$. Let

$$A : C^\infty(E) \to C^\infty(F),$$

be a first order linear elliptic operator that takes the form

$$A = \sigma\left(\frac{\partial}{\partial u} + B\right),$$

where B is a self-adjoint elliptic operator. Call f an *extended L^2-section* of E if $f \in L^2_{\mathrm{loc}}(E)$ and on the cylindrical end of $\hat{X}$, for large t, f takes the form

$$f(y,t) = g(y,t) + f_\infty(y),$$

for $g \in L^2(E)$ and $f_\infty \in \mathrm{Ker}\, B$.

Proposition 4.4.2 ([3, Prop 3.11, Cor 3.14])**.** *Let X be a manifold with boundary Y isometric to a product near the boundary. Define $\hat{X}$ to be X with the cylinder $Y \times \mathbb{R}_+$ attached to the boundary. Extend the vector bundles E and F over X to $\hat{X}$ in the natural way. Consider the operator*

$$A : C^\infty(E; P) \to C^\infty(F),$$

from Theorem 4.4.1. Then we have that

(i) Ker A *is isomorphic to the space of L^2 solutions of $Af = 0$ on $\hat{X}$,*

(ii) Ker A^* *is isomorphic to the space of extended L^2 solutions of $A^*f = 0$ on $\hat{X}$.*

We can write

$$\mathrm{ind}\, A = h(E) - h(F) - h_\infty(F), \tag{4.4.2}$$

where $h(E)$ is the dimension of the space of L^2-solutions of $Af = 0$ in $\hat{X}$, $h(F)$ the corresponding dimension for A^* and $h_\infty(F)$ is the dimension of the subspace of Ker B *consisting of limiting values of extended L^2 sections f of F satisfying $A^*f = 0$.*

Remark. Note that [3, (3.25)]

$$h = h_\infty(E) + h_\infty(F). \tag{4.4.3}$$

Combining Theorem 4.4.1 and Proposition 4.4.2 we have the following index theorem for translation invariant first order elliptic differential operators on manifolds with a cylindrical end.

Corollary 4.4.3. *Let A be a linear elliptic first order translation invariant differential operator on a manifold with a cylindrical end $\hat{X}$. Use the notation of Theorem 4.4.1 and Proposition 4.4.2. Then*

$$\mathrm{ind}_{L^2} A = \int_{\hat{X}} \alpha_0(x)\, dx - \frac{h + \eta(0)}{2} + h_\infty(F). \tag{4.4.4}$$

4.4.3 The APS index theorem for elliptic operators on conically singular manifolds

We bring this chapter to its conclusion by explaining how we can apply the Atiyah–Patodi–Singer index theorem 4.4.1 to elliptic operators on conically singular manifolds, Proposition 4.4.5 below. This is similar to the application of the results of Section 4.2 to conically singular manifolds described in Section 4.3.2.

We first give a technical result that relates the adjoint of a differential operator on a conically singular manifold to the adjoint of the related asymptotically translation invariant operator acting on the conformally equivalent manifold with cylindrical end.

Lemma 4.4.4. *Let M be an m-dimensional conically singular manifold at $\hat{x}$ and let ρ be a radius function for M. Write $\hat{M} := M \backslash \{\hat{x}\}$, and g for the metric on $\hat{M}$. Let*

$$A : C_0^\infty(T_s^q \hat{M}) \to C_0^\infty(T_{s'}^{q'} \hat{M}),$$

be a linear first order differential operator on $\hat{M}$ and suppose there exists $\lambda \in \mathbb{R}$ so that

$$\tilde{A} := \rho^{\lambda + s' - q'} A \rho^{q-s},$$

is an asymptotically translation invariant operator. Then the formal adjoint of the operator $\tilde{A}$ (with respect to the metric $\rho^{-2} g$)

$$\tilde{A}^* : C_0^\infty(T_{s'}^{q'} \hat{M}) \to C_0^\infty(T_s^q \hat{M}),$$

is of the form

$$\tilde{A}^* = \rho^{s-q+m} A^* \rho^{\lambda-s'+q'-m},$$

where

$$A^* : C_0^\infty(T_{s'}^{q'}\hat{M}) \to C_0^\infty(T_s^q\hat{M}),$$

is the formal adjoint of A with respect to g.

Moreover, using the notation of Definitions 4.2.3 and 4.3.3, the kernel of

$$\tilde{A}^* : W_{k+1,\mu}^p(T_{s'}^{q'}\hat{M}) \to W_{k,\mu}^p(T_s^q\hat{M}), \tag{4.4.5}$$

is isomorphic to the kernel of

$$A^* : L_{k+1,\mu+\lambda-m}^p(T_{s'}^{q'}\hat{M}) \to L_{k,\mu-m}^p(T_s^q\hat{M}), \tag{4.4.6}$$

for any $\mu \in \mathbb{R}$, $k \in \mathbb{N}$ and $1 < p < \infty$.

Proof. Let $v \in C_0^\infty(T_s^q\hat{M})$ and $w \in C_0^\infty(T_{s'}^{q'}\hat{M})$. Then

$$\begin{aligned}
\int_{\hat{M}} \langle \tilde{A}v, w\rangle_{\rho^{-2}g} \operatorname{vol}_{\rho^{-2}g} &= \int_{\hat{M}} \langle \rho^{\lambda+s'-q'} A\rho^{q-s}v, w\rangle_{\rho^{-2}g} \operatorname{vol}_{\rho^{-2}g} \\
&= \int_{\hat{M}} \rho^{2q'-2s'} \langle \rho^{\lambda+s'-q'} A\rho^{q-s}v, w\rangle_g \,\rho^{-m}\operatorname{vol}_g \\
&= \int_{\hat{M}} \langle \rho^{\lambda-s'+q'} A\rho^{q-s}v, w\rangle_g \,\rho^{-m}\operatorname{vol}_g \\
&= \int_{\hat{M}} \langle A\rho^{q-s}v, \rho^{\lambda-m-s'+q'}w\rangle_g \operatorname{vol}_g \\
&= \int_{\hat{M}} \langle v, \rho^{q-s}A^*\rho^{\lambda-m-s'+q'}w\rangle_g \operatorname{vol}_g \\
&= \int_{\hat{M}} \rho^{2q-2s}\langle v, \rho^{s-q+m}A^*\rho^{\lambda-m-s'+q'}w\rangle_g \,\rho^{-m}\operatorname{vol}_g \\
&= \int_{\hat{M}} \langle v, \rho^{s-q+m}A^*\rho^{\lambda-m-s'+q'}w\rangle_{\rho^{-2}g} \operatorname{vol}_{\rho^{-2}g},
\end{aligned}$$

where we have used that A^* is the formal adjoint of A with respect to the metric g, which shows that

$$\tilde{A}^* := \rho^{s-q+m}A^*\rho^{\lambda-m-s'+q'},$$

is the formal adjoint of $\tilde{A}$ with respect to the metric $\rho^{-2}g$. Since by Lemma 4.3.1,

$$\rho^{\lambda-m-s'+q'} : W^p_{k+1,\mu}(T^{q'}_{s'}\hat{M}) \to L^p_{k+1,\mu+\lambda-m}(T^{q'}_{s'}\hat{M}),$$

is an isomorphism and so by definition of $\tilde{A}^*$ and A^* the kernels of (4.4.5) and (4.4.6) are isomorphic. □

We may now deduce the following proposition from Theorem 4.4.1 and Lemma 4.4.4 to give an index theorem for operators on conically singular submanifolds.

Proposition 4.4.5. *Let M be an m-dimensional conically singular manifold at $\hat{x}$ with radius function ρ. Let $T^q_s\hat{M}$ be the vector bundle of (s,q)-tensors on $\hat{M} := M\backslash\{\hat{x}\}$. Let*

$$A : C^\infty_0(T^q_s\hat{M}) \to C^\infty_0(T^{q'}_{s'}\hat{M}),$$

be a first order linear elliptic differential operator so that

$$\tilde{A} := \rho^{\lambda+s'-q'}A\rho^{q-s},$$

is asymptotically translation invariant to $\tilde{A}_\infty$ for some $\lambda \in \mathbb{R}$. Then for $\mu \in \mathbb{R}\backslash\mathcal{D}$, given in Theorem 4.3.2, the index of

$$A : L^2_{k+1,\mu}(T^q_s\hat{M}) \to L^2_{k,\mu-\lambda}(T^{q'}_{s'}\hat{M}), \tag{4.4.7}$$

differs by a constant from the index $\mathrm{ind}_\mu A_\infty$ of

$$A_\infty := r^{q'-s'-\lambda}\tilde{A}_\infty r^{s-q} : L^2_{k+1,\mu}(T^q_s\hat{M}) \to L^2_{k,\mu-\lambda}(T^{q'}_{s'}\hat{M}), \tag{4.4.8}$$

which satisfies

$$\mathrm{ind}_\epsilon A_\infty = \int_{\hat{M}} \alpha_0(x)dx - \frac{h+\eta(0)}{2}, \tag{4.4.9}$$

for $\epsilon > 0$ chosen so that $(0,\epsilon]\cap\mathcal{D} = \emptyset$ and we use the notation of Theorem 4.4.1 and Proposition 4.4.2 for the terms on the right hand side of (4.4.9) *(and these terms are defined for the translation invariant operator $\tilde{A}_\infty$).*

Proof. By Theorem 4.3.2, we know that A and $\tilde{A}$ have the same kernel and cokernel when acting on weighted Sobolev spaces, and moreover, the index of these operators differ from the index of $\tilde{A}_\infty$ by a constant independent of the weight.

Since $\tilde{A}_\infty$ is translation invariant, we can apply Theorem 4.4.1 and Corollary 4.4.3 to $\tilde{A}_\infty$. Let $\operatorname{Ker}_\mu \tilde{A}_\infty$ and $\operatorname{Ker}_\mu \tilde{A}^*_\infty$ denote the kernels of

$$\begin{aligned} \tilde{A}_\infty &: W^2_{k+1,\mu}(T^q_s\hat{M}) \to W^2_{k,\mu}(T^{q'}_{s'}\hat{M}), \\ \tilde{A}^*_\infty &: W^2_{k+1,\mu}(T^{q'}_{s'}\hat{M}) \to W^2_{k,\mu}(T^q_s\hat{M}), \end{aligned}$$

respectively, where $\tilde{A}^*_\infty$ is the formal adjoint of $\tilde{A}_\infty$ with respect to the metric $\rho^{-2}g$, where g is the metric on $\hat{M}$. Then Theorem 4.4.1 and Corollary 4.4.3 yield that

$$\dim \operatorname{Ker}_0 \tilde{A}_\infty - \dim \operatorname{Ker}_0 \tilde{A}^*_\infty = \int_{\hat{M}} \alpha_0(x)dx - \frac{h+\eta(0)}{2} + h_\infty(T^{q'}_{s'}\hat{M}). \tag{4.4.10}$$

By definition of $\tilde{A}_\infty$, $\operatorname{Ker}_0 \tilde{A}_\infty \cong \operatorname{Ker}_0 A_\infty$, where $\operatorname{Ker}_\mu A_\infty$ denotes the kernel of (4.4.8), and by Lemma 4.4.4, $\operatorname{Ker}_0 \tilde{A}^*_\infty \cong \operatorname{Ker}_{\lambda-m} A^*_\infty$, where A^*_∞ is the formal adjoint of A_∞ with respect to the metric g and $\operatorname{Ker}_\mu A^*_\infty$ denotes the kernel of

$$A^*_\infty : L^2_{k+1,\mu}(T^{q'}_{s'}\hat{M}) \to L^2_{k,\mu-\lambda}(T^q_s\hat{M}).$$

So we see that

$$\dim \operatorname{Ker}_0 A_\infty - \dim \operatorname{Ker}_{\lambda-m} A^*_\infty = \int_{\hat{M}} \alpha_0(x)dx - \frac{h+\eta(0)}{2} + h_\infty(T^{q'}_{s'}\hat{M}). \tag{4.4.11}$$

Denote by $\mathcal{D}$ the subset of $\mathbb{R}$ for which $\mu \in \mathcal{D}$ if, and only if, (4.4.8) is not Fredholm. Then we might have a problem equating

$$\dim \operatorname{Ker}_0 A_\infty - \dim \operatorname{Ker}_{\lambda-m} A^*_\infty = \operatorname{ind}_0 A_\infty,$$

since if $0 \in \mathcal{D}$ then $\operatorname{ind}_0 A_\infty$ may not be defined. Take $\epsilon > 0$ so that

$$(0, \epsilon] \cap \mathcal{D} = \emptyset.$$

Then $\operatorname{ind}_\epsilon A_\infty$ is well-defined. Since $\epsilon > 0$, we have that

$$\operatorname{Ker}_\epsilon A_\infty \subseteq \operatorname{Ker}_0 A_\infty,$$

where $\mathrm{Ker}_\mu A_\infty$ denotes the kernel of (4.4.8). It is claimed that

$$\mathrm{Ker}_\epsilon A_\infty = \mathrm{Ker}_0 A_\infty.$$

To see this, suppose that $\alpha \in \mathrm{Ker}_0 A_\infty$. Then by elliptic regularity, α is smooth, and by definition of weighted norm on $L^2_{k+1,0}(T^q_s \hat{M})$ α must decay to zero as $r \to 0$ and so we must have that $\alpha = \mathcal{O}(r^{\epsilon'})$ for some $\epsilon' > 0$. Taking ϵ' smaller if necessary we can guarantee that $\mathcal{D} \cap (0, \epsilon'] = \emptyset$. The rate of decay of α allows us to deduce that $\alpha \in L^2_{k+1,\epsilon''}(T^q_s \hat{M})$ where $0 < \epsilon'' < \epsilon'$. But then we are done, since there is no exceptional weight between ϵ and ϵ'', and so [35, Lem 7.1] says that $\mathrm{Ker}_\epsilon A_\infty = \mathrm{Ker}_{\epsilon'} A_\infty$. Notice that this tells us that the function $\mu \mapsto \dim \mathrm{Ker}_\mu A_\infty$ is upper semi-continuous at zero.

Since $\epsilon > 0$

$$\mathrm{Ker}_{\lambda - m} A^*_\infty \subseteq \mathrm{Ker}_{-\epsilon+\lambda-m} A^*_\infty.$$

A similar argument to the one given above shows that the function $\mu \mapsto \dim \mathrm{Ker}_\mu A^*_\infty$ is lower semi-continuous (in particular at $\mu = \lambda - m$) and so the set

$$\mathrm{Ker}_{-\epsilon+\lambda-m} A^*_\infty \backslash \mathrm{Ker}_{\lambda-m} A^*_\infty,$$

is nonempty, but its elements are exactly the limiting sections of the extended L^2-sections of $T^{q'}_{s'} \hat{M}$. Therefore

$$\dim \mathrm{Ker}_{\lambda-m} A^*_\infty \backslash \mathrm{Ker}_{-\epsilon+\lambda-m} A^*_\infty = h_\infty(T^{q'}_{s'} \hat{M}),$$

i.e., exactly the dimension of the space of limiting sections of extended L^2-sections of $T^{q'}_{s'} \hat{M}$. This allows us to deduce that

$$\mathrm{Ker}_0 A_\infty - \mathrm{Ker}_{\lambda-m} A_\infty - h_\infty(T^{q'}_{s'}(\hat{M})) = \mathrm{ind}_\epsilon A_\infty.$$

Applying this to (4.4.11) we find that

$$\mathrm{ind}_\epsilon A_\infty = \int_{\hat{M}} \alpha_0(x) dx - \frac{h + \eta(0)}{2}, \tag{4.4.12}$$

as claimed. $\square$

Chapter 5

Deformation theory of conically singular Cayley submanifolds

In this chapter we will extend the results of Chapter 3 to conically singular Cayley submanifolds and complex surfaces.

5.1 Introduction

In this chapter we will study the deformation theory of conically singular Cayley submanifolds inside $Spin(7)$-manifolds. We define conically singular submanifolds in Definition 5.2.2 below, however, a conically singular submanifold is, in particular, a conically singular manifold which we defined earlier in Definition 4.3.1. We have already proved results on the deformation theory of compact Cayley submanifolds in $Spin(7)$-manifolds in Chapter 3, and we know that the two barriers to extending these results to noncompact Cayley manifolds are the lack of a tubular neighbourhood theorem and the failure of elliptic operators on noncompact manifolds to be Fredholm. In Chapter 4, however, we saw that by introducing weighted norms on spaces of sections on conically singular manifolds that as long as we are careful about our choice of weight we may deduce that elliptic operators on conically singular manifolds are Fredholm.

In Section 5.2 we will define conically singular submanifolds and give an example of a conically singular complex submanifold of a Calabi–Yau manifold. We will also prove a tubular neighbourhood theorem for cones in $\mathbb{R}^n$ in Proposition 5.2.3, which we will use to prove a tubular neighbourhood theorem for conically singular submanifolds in Proposition 5.2.4. In Definition 5.2.5 we will define the moduli space of conically singular Cayley deformations of a conically singular Cayley submanifold Y in a $Spin(7)$-manifold X that we will be studying in this chapter. Heuristically, these are deformations of the nonsingular part of Y, $\hat{Y}$, that are Cayley and are themselves conically singular with the same rate and cone as Y. We will then prove Proposition 5.2.5, which identifies this moduli space near to Y with the kernel of a nonlinear differential operator $\hat{F}$ acting on smooth sections of the normal bundle of Y in X which have a certain rate of decay near to the singular point of $\hat{Y}$.

In Section 5.3 we will prove our first main result of this chapter, Theorem 5.3.3, on deformations of conically singular Cayley submanifolds.

Theorem. *Let Y be a CS Cayley submanifold at $\hat{x}$ with cone C and rate $\mu \in (1,2)\backslash\mathcal{D}$ of a $Spin(7)$-manifold X. Let D denote the first order elliptic differential operator defined in* (3.2.6)*. Then there exist a smooth manifold $\hat{K}_0$, which is an open neighbourhood of* 0 *in the kernel of* (5.1.1)*, and a smooth map $\hat{g}_2$ from $\hat{K}_0$ into the cokernel of* (5.1.1) *with $\hat{g}_2(0) = 0$ so that an open neighbourhood of Y in the moduli space of CS Cayley deformations of Y in X, $\hat{\mathcal{M}}_\mu(Y)$ from Definition 5.2.5, is homeomorphic to an open neighbourhood of* 0 *in* Ker $\hat{g}_2$*.*

Moreover, the expected dimension of $\hat{\mathcal{M}}_\mu(Y)$ is given by the index of the linear elliptic operator

$$D : L^p_{k+1,\mu}(\nu_X(\hat{Y})) \to L^p_{k,\mu-1}(E). \tag{5.1.1}$$

If the cokernel of (5.1.1) *is* $\{0\}$ *then $\hat{\mathcal{M}}_\mu(Y)$ is a smooth manifold near Y of the same dimension as the kernel of* (5.1.1)*. Here $\mathcal{D}$ is the set of weights $\mu \in \mathbb{R}$ for which*

(5.1.1) *is not Fredholm from Theorem 4.3.2.*

In Section 5.4 we will consider Cayley deformations of a two-dimensional conically singular complex submanifold N of a Calabi–Yau four-fold M. We will prove Theorem 5.4.4.

Theorem. *Let N be a CS complex surface at $\hat{x}$ with cone C and rate $\mu \in (1,2)\backslash\mathcal{D}$ of a Calabi–Yau four-fold M. Then the expected dimension of $\hat{\mathcal{M}}_\mu(N)$ is given by the index of the linear elliptic operator*

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\mu-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \qquad (5.1.2)$$

Moreover if the cokernel of (5.1.2) *is* $\{0\}$ *then $\hat{\mathcal{M}}_\mu(N)$ is a smooth manifold near N of the same dimension as the (complex) dimension of the kernel of* (5.1.2)*. Here $\mathcal{D}$ is the set of weights $\mu \in \mathbb{R}$ for which* (5.1.1) *is not Fredholm from Theorem 4.3.2.*

This combines Theorem 5.3.3 with the vector bundle isomorphisms proved in Section 3.3. This will allow us to identify the expected dimension of the moduli space of conically singular Cayley deformations of N in M with the index of the operator $\bar{\partial} + \bar{\partial}^*$ acting on weighted Sobolev spaces.

Finally, in Section 5.5 we will consider complex deformations of a conically singular complex surface N inside a Calabi–Yau four-fold N. We will prove Theorem 5.5.2.

Theorem. *Let N be a conically singular complex surface at $\hat{x}$ with rate $\mu \in (1,2)$ and cone C inside a Calabi–Yau four-fold M. The moduli space of CS complex deformations of N in M, $\hat{\mathcal{M}}^{\mathrm{cx}}_\mu(N)$ given in Definition 5.5.1, is a smooth manifold of dimension*

$$\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial} + \dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial}^* = 2\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial},$$

where

$$\begin{aligned} \bar{\partial} : C^\infty_\mu(\nu^{1,0}_M(\hat{N})) &\to C^\infty_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \\ \bar{\partial}^* : C^\infty_\mu(\Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) &\to C^\infty_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \end{aligned}$$

This will allow us to deduce Corollary 5.5.4.

Corollary. *Let N be a CS complex surface inside a Calabi–Yau four-fold M. Then the moduli space of CS Cayley deformations of N in M is isomorphic to the moduli space of CS complex deformations of N in M.*

Deformations of conically singular calibrated submanifolds have been studied by other authors, most notably by Joyce [22], who studied conically singular special Lagrangian submanifolds and Lotay [37], who studied conically singular coassociative submanifolds.

5.2 Conically singular Cayley submanifolds

5.2.1 Defining conically singular submanifolds

We begin with the definition of conically singular submanifold that we will use throughout this chapter.

Let (X, g, Φ) be a $Spin(7)$-manifold as in Definition 1.2.3. By definition, given any $x \in X$, there exists an oriented isomorphism $\zeta : \mathbb{R}^8 \to T_xX$ identifying $(\Phi|_x, g|_x)$ with the $Spin(7)$-structure on $\mathbb{R}^8$ (Φ_0, g_0), where Φ_0 was defined in Equation (1.2.7) and g_0 is the Euclidean metric. The following definition gives a preferred choice of coordinates around any given point of X. This definition is analogous to [23, Defn 3.6] and [37, Defn 3.3], which are coordinate systems for almost Calabi–Yau manifolds and G_2-manifolds respectively.

Definition 5.2.1. Let (X, g, Φ) be a $Spin(7)$-manifold. Then given $x \in X$, there exist $\eta > 0$, an open set $x \in V \subseteq X$, $\eta > 0$ and a diffeomorphism

$$\chi : B_\eta(0) \to V, \tag{5.2.1}$$

where $B_\eta(0)$ denotes the ball of radius η around zero in $\mathbb{R}^8$, with $\chi(0) = x$ and so that $d\chi|_0 : \mathbb{R}^8 \to T_xX$ is an isomorphism identifying $(\Phi|_x, g|_x)$ with (Φ_0, g_0). Call χ a *$Spin(7)$ coordinate system* for X around x.

Call two $Spin(7)$-coordinate systems $\chi, \tilde{\chi}$ for X around x *equivalent* if

$$d\chi|_0 = d\tilde{\chi}|_0,$$

as maps $\mathbb{R}^8 \to T_x M$.

In particular, when the $Spin(7)$-manifold X is a four-dimensional Calabi–Yau manifold, as we saw in (1.2.4) and (1.2.5), we can choose a holomorphic volume form Ω for X so that χ is a biholomorphism and $d\chi|_0$ identifies the Ricci-flat Kähler form ω with ω_0 and Ω with Ω_0, the Euclidean Kähler form and holomorphic volume form respectively.

We may now define conically singular submanifolds inside $Spin(7)$-manifolds. This definition is analogous to [23, Defn 3.6] and [37, Defn 3.4] for conically singular submanifolds of almost Calabi–Yau manifolds and G_2-manifolds respectively.

Definition 5.2.2. Let (X, g, Φ) be a $Spin(7)$-manifold and $Y \subseteq X$ compact and connected such that there exists $\hat{x} \in Y$ such that $\hat{Y} := Y \backslash \{\hat{x}\}$ is a smooth submanifold of X. Choose a $Spin(7)$-coordinate system χ for X around $\hat{x}$. We say that Y is *conically singular* (CS) at $\hat{x}$ with rate μ and cone C if there exist $1 < \mu < 2$, $0 < \epsilon < \eta$, a compact Riemannian submanifold (L, g_L) of S^7 of dimension one less than Y, an open set $\hat{x} \in U \subset X$ and a smooth map $\phi : (0, \epsilon) \times L \to B_\eta(0) \subseteq \mathbb{R}^8$ such that $\Psi = \chi \circ \phi : (0, \epsilon) \times L \to U \backslash \{\hat{x}\}$ is a diffeomorphism and ϕ satisfies

$$|\nabla^j(\phi - \iota)| = O(r^{\mu - j}) \text{ for } j \in \mathbb{N} \text{ as } r \to 0, \tag{5.2.2}$$

where $\iota : (0, \infty) \times L \to \mathbb{R}^8$ is the inclusion map given by $\iota(r, l) = rl$, ∇ is the Levi-Civita connection of the cone metric $g_C = dr^2 + r^2 g_L$ on C, and $|\cdot|$ is computed using g_C.

Remark. If the smooth, noncompact submanifold $\hat{Y}$ is a Cayley (complex) submanifold of the $Spin(7)$-manifold (Calabi–Yau four-fold) X then we say that Y is a CS Cayley (complex) submanifold of X.

Conically singular submanifolds come with a rate $1 < \mu < 2$. We must have that $\mu > 1$ to guarantee that a conically singular submanifold is a conically singular manifold (in the sense of Definition 4.3.1). The reason for asking that $\mu < 2$ is so that μ does not depend on the choice of equivalent $Spin(7)$-coordinate system around the singular point of the conically singular submanifold.

Lemma 5.2.1. *Let Y be a conically singular submanifold at $\hat{x}$ with rate μ and cone C of a $Spin(7)$-manifold (X, g, Φ) with $Spin(7)$-coordinate system χ around $\hat{x}$. Then Definition 5.2.2 is independent of choice of equivalent $Spin(7)$-coordinate system.*

Proof. Let $\tilde{\chi}$ be another $Spin(7)$-coordinate system for X around $\hat{x}$ equivalent to χ. Then χ and $\tilde{\chi}$ and their differentials agree at zero. Let $\phi : (0, \epsilon) \times L \to B_\eta(0)$ be the map from Definition 5.2.2. We will show that Y is conically singular in X with $Spin(7)$-coordinate system $\tilde{\chi}$ around $\hat{x}$. Taking $\tilde{\phi} := \tilde{\chi}^{-1} \circ \chi \circ \phi$, we have that

$$|\nabla^j(\tilde{\phi} - \iota)| = |\nabla^j(\tilde{\chi}^{-1} \circ \chi \circ \phi - \iota)| = |\nabla^j(\phi - \iota)| + O(r^{2-j}), \tag{5.2.3}$$

since $\tilde{\chi}^{-1} \circ \chi(x) = x + x^T Ax + \ldots$, and $\phi(r, l) = rl + O(r^\mu)$. So we see that Y is conically singular at $\hat{x}$ with cone C in (X, g, Φ) with $Spin(7)$-coordinate system $\tilde{\chi}$, but in order for Y to be CS with rate μ in this case, Equation (5.2.3) tells us that we must have that $\mu < 2$. □

The following definition is independent of choice of equivalent $Spin(7)$-coordinate system. It is analogous to [37, Defn 3.5].

Definition 5.2.3. Let Y be a conically singular submanifold at $\hat{x}$ with rate μ and cone C of a $Spin(7)$-manifold (X, g, Φ) with $Spin(7)$-coordinate system χ. Denote by $\zeta := d\chi|_0 : T_0\mathbb{R}^8 \to T_{\hat{x}}X$. Define the *tangent cone* of Y at $\hat{x}$ to be

$$\hat{C} := \zeta \circ \iota(C) \subseteq T_{\hat{x}}X,$$

where $\iota : C \to \mathbb{R}^8$ is the inclusion map given in Definition 5.2.2.

On a Calabi–Yau manifold M we are given a Ricci-flat metric ω that we often have no explicit expression for. The following lemma tells us that Definition 5.2.2 is independent of choice of Kähler metric on M.

Lemma 5.2.2. *Let M be a Calabi–Yau four-fold with Ricci-flat Kähler form ω and let N be a CS submanifold of M as in Definition 5.2.2. Then if ω' is any other Kähler form on M then N is still a conically singular submanifold of M with the same rate and tangent cone.*

Proof. Suppose that N is a CS submanifold of M with respect to ω at $\hat{x}$. Choose a $Spin(7)$-coordinate system for M around $\hat{x}$,

$$\chi : B_\eta(0) \to V \backslash \{\hat{x}\},$$

for some $\eta > 0$ and open $V \subseteq M$ containing $\hat{x}$, so that $\chi(0) = \hat{x}$ and

$$\chi^*\omega = \omega_0 + O(|z|^2),$$

where ω_0 is the standard Euclidean Kähler form on $\mathbb{C}^4$. Let ϕ, ϵ, $C = (0, \infty) \times L$, ι and μ be as in Definition 5.2.2.

Now given any other Kähler form ω' on M, we can find by [14, pg 107] $\eta' > 0$, an open set $x \in V' \subseteq M$ and a diffeomorphism

$$\chi' : B_{\eta'}(0) \to V' \backslash \{x\},$$

with $\chi'(0) = x$ and

$$\chi'^*\omega' = \omega_0 + O(|z|^2).$$

Since χ and χ' are diffeomorphisms, $d\chi|_0$ and $d\chi'|_0$ are isomorphisms $\mathbb{C}^4 \to T_{\hat{x}}M$. Then $A := (d\chi'|_0)^{-1} \circ d\chi|_0$ is an invertible linear map $\mathbb{C}^4 \to \mathbb{C}^4$. We will show that N is conically singular in (M, ω') (taking χ' to be the coordinate system) with cone $C' = A\iota(C)$ and rate μ.

Firstly note that since A is a linear map, $C' = A\iota(C) = \{Av \,|\, v \in \iota(C)\}$ is also a cone. Denote by L' the link of C' (considered as a Riemannian submanifold of S^7), and for any $\epsilon' > 0$ write $\iota' : L' \times (0, \epsilon') \to \mathbb{C}^4$ for the inclusion map $(r', l') \mapsto r'l'$.

Define $\phi' : (0, \epsilon') \times L' \to \mathbb{C}^4$ by $\phi' = \chi'^{-1} \circ \chi \circ \phi \circ A^{-1}$, where $\epsilon' = \epsilon \|A\|$. Then this map is well defined (taking ϵ' smaller if necessary) and moreover $\chi' \circ \phi'$ is a diffeomorphism onto its image. Moreover, by a similar argument to Lemma 5.2.1 we have that

$$|\nabla^j(\phi'(r',l') - \iota(r',l'))| = O(r'^{\mu}),$$

since $\mu < 2$.

Finally, we have that

$$\hat{C'} = d\chi'|_0(\iota'(C')) = d\chi|_0 \circ (d\chi|_0)^{-1} \circ d\chi'|_0(A\iota(C)) = d\chi|_0(A^{-1}A\iota(C)) = \hat{C},$$

and so the tangent cone to N at $\hat{x}$ is the same with respect to each metric. □

Remark. Note that the proof Lemma 5.2.2 also shows that if N is conically singular with respect to one $Spin(7)$-coordinate system, it is conically singular with respect to any other $Spin(7)$-coordinate system, although with a different cone in general, but the same tangent cone.

We can now construct an example of a conically singular complex surface inside a Calabi–Yau four-fold.

Example. We will model our conically singular complex surface on the following complex cone in $\mathbb{C}^4$. Define C to be the set of $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ satisfying

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0,$$
$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0.$$

Clearly, if $z \in C$, then also $\lambda z \in C$ for any $\lambda \in \mathbb{R}\backslash\{0\}$, and so C is a cone.

Checking the rank of the matrix

$$\begin{pmatrix} 4z_1^3 & 4z_2^3 & 4z_3^3 & 4z_4^3 \\ 3z_1^2 & 3z_2^2 & 3z_3^2 & 3z_4^2 \end{pmatrix},$$

at each point of C, we see that the only singular point of C is zero.

As we will discuss in more detail in Chapter 6, a complex cone C in $\mathbb{C}^4$ has both a real link $L := S^7 \cap C$, and a complex link $\Sigma := \pi(L)$, where $\pi : S^7 \to \mathbb{C}P^3$ is the Hopf fibration. We can view the real link of a complex cone as a circle bundle over the complex link of the cone.

In this case, the complex link Σ of C is the Riemannian surface in $\mathbb{C}P^3$ is given by $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3$ satisfying

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0,$$
$$z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0.$$

We can apply the adjunction formula to find that the canonical bundle of Σ is given by

$$K_\Sigma = K_{\mathbb{C}P^3}|_\Sigma \otimes \mathcal{O}_{\mathbb{C}P^3}(4)|_\Sigma \otimes \mathcal{O}_{\mathbb{C}P^3}(3)|_\Sigma = \mathcal{O}_{\mathbb{C}P^3}(4+3-3-1)|_\Sigma = \mathcal{O}_{\mathbb{C}P^3}(3)|_\Sigma,$$

where $\mathcal{O}_{\mathbb{C}P^3}(k)$ denotes the $-k^{\text{th}}$ (tensor) power of the tautological line bundle over $\mathbb{C}P^3$ if k is a negative integer, the k^{th} power of the dual of the tautological line bundle if k is a positive integer, and the trivial line bundle if $k = 0$. Then it follows from the Hirzebruch–Riemann–Roch theorem [17, Thm 5.1.1] that the genus of Σ is

$$g = \frac{2 + \deg \mathcal{O}_{\mathbb{C}P^3}(3)|_\Sigma}{2} = \frac{2 + 3 \times \deg(\Sigma)}{2} = (2 + 3 \times 4 \times 3)/2 = 19.$$

Now consider the Calabi–Yau four-fold M defined by

$$\{[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}P^5 \,|\, z_0^6 + z_1^6 + z_2^6 + z_3^6 + z_4^6 + z_5^6 = 0\}.$$

Consider the singular submanifold N of M defined to be the set of all $[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \in \mathbb{C}P^5$ satisfying

$$z_0^6 + z_1^6 + z_2^6 + z_3^6 + z_4^6 + z_5^6 = 0,$$
$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0,$$
$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0.$$

The complex Jacobian matrix of the defining equations of N is given by

$$\begin{pmatrix} 6z_0^5 & 6z_1^5 & 6z_2^5 & 6z_3^5 & 6z_4^5 & 6z_5^5 \\ 0 & 4z_1^3 & 4z_2^3 & 4z_3^3 & 4z_4^3 & 0 \\ 0 & 3z_1^2 & 3z_2^2 & 3z_3^2 & 3z_4^2 & 0 \end{pmatrix}.$$

It can be calculated that there are six singular points on N of the form $[\omega : 0 : 0 : 0 : 0 : 1]$, where ω is a 6^{th} root of -1.

We will now prove that N satisfies Definition 5.2.2. We will exploit Lemma 5.2.2 and check the definition using the metric on M induced from the Fubini–Study metric on $\mathbb{C}P^5$, denoted by ω.

Denote the singular points of N by $\{p_1, \ldots, p_6\}$, where $p_k = [\omega_k : 0 : 0 : 0 : 0 : 1]$ for $\omega_k := e^{i(2k-1)\pi/6}$. We must construct maps χ_k so that there exist $\eta_k > 0$ and open sets $p_k \in V_k \subseteq M$ and diffeomorphisms

$$\chi_k : B_{\eta_k}(0) \to V_k,$$

with $\chi_k(0) = p_k$ and so that

$$\chi_k^* \omega = \omega_0 + O(|z|^2),$$

for $k = 1, \ldots, 6$.

For $k = 1, \ldots 6$, define $\chi_k : B_{\eta_k}(0) \to M$ by

$$\chi_k(w_1, w_2, w_3, w_4) = [\omega_k : \sqrt{2}w_1 : \sqrt{2}w_2 : \sqrt{2}w_3 : \sqrt{2}w_4 : (1 - 8(w_1^6 + w_2^6 + w_3^6 + w_4^6))^{1/6}], \tag{5.2.4}$$

where if $a = re^{i\theta}$ for $r > 0$ and $-\pi < \theta \leq \pi$, we define $a^{1/6} := r^{1/6}e^{i\theta/6}$. It is clear that (5.2.4) is a diffeomorphism onto its image. The induced Fubini–Study metric on M pulls back under χ_k to the Euclidean metric on $\mathbb{C}^4$ at each $p_k = [\omega_k : 0 : 0 : 0 : 0 : 1]$. Taking $\phi = \iota$, where $\iota : C \to \mathbb{C}^4$ is the inclusion map, we see that $\phi \circ \chi$ is a diffeomorphism C to N, and so the definition of conically singular is trivially satisfied.

Remark. In Chapter 6 we will consider three complex cones in $\mathbb{C}^4$ and perform some calculations based on the subsequent work in the current chapter. The reader might wonder why the above example was not constructed to be conically singular with one of these cones. Firstly, smooth varieties are trivially conically singular at every point with cone $C_1 := \mathbb{C}^2 \subseteq \mathbb{C}^4$, and so our example in this case would not actually be singular at any point. Applying the above method to the other cones considered in Chapter 6 that we consider yield examples with too many singularities, and so removing the conically singular points does not leave us with a smooth manifold. Conversely, the calculations performed in Chapter 6 rely on a result that is only available for cones with complex link given a constant curvature $\mathbb{C}P^1$ in $\mathbb{C}P^3$ and so we will not at this time be able to perform analogous calculations for the above example.

5.2.2 Tubular neighbourhood theorems

In this section we will prove a tubular neighbourhood theorem for conically singular submanifolds so that we can identify deformations of conically singular submanifolds with normal vector fields. We will do this in two steps. Firstly, in Proposition 5.2.3 we will construct a tubular neighbourhood of a cone in $\mathbb{R}^n$ using the compact tubular neighbourhood theorem 3.2.1. We will use this to construct a tubular neighbourhood of a conically singular submanifold in Proposition 5.2.4. Propositions 5.2.3 and 5.2.4 use ideas of similar results proved by Joyce [23, Thm 4.6] for special Lagrangian cones and Lotay [37, Prop 6.4] for CS coassociative submanifolds.

Proposition 5.2.3 (Tubular neighbourhood theorem for cones). *Let C be a cone in $\mathbb{R}^n$ with link L and let g be a Riemannian metric on $\mathbb{R}^n$ (not necessarily the Euclidean metric). There exists an action of $\mathbb{R}_+$ on $\nu_{\mathbb{R}^n}(C)$*

$$t : \nu_{\mathbb{R}^n}(C) \to \nu_{\mathbb{R}^n}(C),$$

so that

$$|t \cdot v| = t|v|. \tag{5.2.5}$$

We can construct open sets $V_C \subseteq \nu_{\mathbb{R}^n}(C)$, invariant under (5.2.5), containing the zero section and $T_C \subseteq \mathbb{R}^n$, invariant under multiplication by positive scalars, containing C that grow like r and a dilation equivariant diffeomorphism

$$\Xi_C : V_C \to T_C,$$

in the sense that $\Xi_C(t \cdot v) = t\,\Xi_C(v)$ for all $v \in \nu_{\mathbb{R}^n}(C)$. Moreover, Ξ_C maps the zero section of $\nu_{\mathbb{R}^n}(C)$ to C.

Proof. We will first address the claim that there exists an $\mathbb{R}_+$-action on $\nu_{\mathbb{R}^n}(C)$ so that (5.2.5) holds. First note that points in $\nu_{\mathbb{R}^n}(C)$ take the form

$$(r, l, v(r, l)),$$

where $r \in \mathbb{R}_+$, $l \in L$ and $v(r, l) \in \nu_{r,l}(C)$. Since any finite-dimensional inner product spaces of the same dimension are isometric, given any $r' \in \mathbb{R}_+$ we can think of $(r', l, v(r, l))$ as another point in $\nu_{\mathbb{R}^n}(C)$ with $|v(r, l)|_{r,l} = |v(r, l)|_{r',l}$, where $|\cdot|_{r,l}$ denotes the norm on $\nu_{r,l}(C)$ induced from g. Define an action of $\mathbb{R}_+$ on $\nu_{\mathbb{R}}(C)$ by

$$\begin{aligned} t : \nu_{\mathbb{R}^n}(C) &\to \nu_{\mathbb{R}^n}(C), \\ (r, l, v(r, l)) &\mapsto (tr, l, tv(r, l)). \end{aligned} \tag{5.2.6}$$

Then $|t \cdot v(r, l)|_{tr,l} = |tv(r, l)|_{r,l} = t|v(r, l)|_{r,l}$ as claimed. Notice that $t \cdot (t' \cdot v) = (tt') \cdot v$ and so (5.2.6) is a group action in the usual sense.

To prove the tubular neighbourhood part of this proposition, we first apply the tubular neighbourhood theorem 3.2.1 to the compact submanifold L of S^{n-1}. (Recall that we need a metric on S^{n-1} to define the exponential map. We take this to be the standard round metric on S^{n-1}.) This gives us an open set $V_L \subseteq \nu_{S^{n-1}}(L)$ containing the zero section and an open set $T_L \subseteq S^7$ containing L and a diffeomorphism

$$\Xi_L : V_L \to T_L,$$

so that Ξ_L maps the zero section of $\nu_{S^{n-1}}(L)$ to L. Again write points in $\nu_{\mathbb{R}^n}(C)$ as $(r, l, v(r, l))$, where $v \in \nu_{r,l}(C)$, and similarly points in $\nu_{S^{n-1}}(L)$ as $(l, v(l))$ where $v \in \nu_l(L) \cong \nu_{r,l}(C)$. Then define

$$V_C := \left\{(r, l, v(r, l)) \in \nu_{\mathbb{R}^n}(C) \;\middle|\; \left(l, r^{-1}v(r, l)\right) \in V_L\right\}.$$

It is clear that V_C is invariant under the $\mathbb{R}_+$-action (5.2.6) by construction of V_C and the $\mathbb{R}_+$-action. We see that V_C grows like r in the sense that if $v = (r, l, v(r, l)) \in V_C$ then

$$|v(r, l)|_{rl} \leq r|V|,$$

where V is the diameter of the set V. Now define

$$T_C := \{\lambda t \,|\, t \in T_L,\, \lambda \in \mathbb{R}_+\}.$$

Then it is clear that T_C is dilation invariant, in the sense that it is clearly invariant under multiplication by positive scalars, and that $C \subseteq T_C$. We see that T_C grows like r in the sense that if $t \in T$, $l \in L$ and $r \in \mathbb{R}_+$ then

$$|rt - rl| \leq r|T|,$$

where $|T|$ is the diameter of the set T. Define

$$\begin{aligned} \Xi_C : V_C &\to T_C, \\ (r, l, v(r, l)) &\mapsto r\, \Xi_L(l, r^{-1}v(r, l)). \end{aligned}$$

It is clear that Ξ_C is well-defined, bijective and smooth. It is also clear that

$$\Xi_C(t \cdot (r, l, v(r, l)) = t\, \Xi_C(r, l, v(r, l)).$$

Finally we have that

$$\Xi_C(r, l, 0) = r\, \Xi_L(l, 0) = rl,$$

by definition of Ξ_L and so Ξ_C maps the zero section of $\nu_{\mathbb{R}^n}(C)$ to C. □

We can use this result to prove a tubular neighbourhood theorem for a conically singular submanifold.

Proposition 5.2.4. *Let Y be a conically singular submanifold of X at $\hat{x}$ with cone C and rate μ. Write $\hat{Y} := Y \backslash \{\hat{x}\}$. Then there exist open sets $\hat{V} \subseteq \nu_X(\hat{Y})$ containing the zero section and $\hat{T} \subseteq X$ containing $\hat{Y}$ and a diffeomorphism*

$$\hat{\Xi} : \hat{V} \to \hat{T},$$

that takes the zero section of $\nu_X(\hat{Y})$ to $\hat{Y}$. Moreover, we can choose $\hat{V}$ and $\hat{T}$ to grow like ρ as $\rho \to 0$.

Proof. Notice that $K := Y\backslash U$ is a compact submanifold of X. So by the compact tubular neighbourhood theorem 3.2.1 we can find open sets $\hat{V}_1 \subseteq \nu_X(K)$ containing the zero section and $\hat{T}_1 \subseteq X$ containing K and a diffeomorphism

$$\hat{\Xi}_1 : \hat{V}_1 \to \hat{T}_1.$$

We will construct a tubular neighbourhood for $\hat{Y}$ near $\hat{x}$. Denote $C_\epsilon := C \cap B_\epsilon(0)$. Use the notation of Definition 5.2.2. Choose $\phi : C_\epsilon \to \mathbb{R}^n$ uniquely by asking that

$$\phi(r,l) - \iota(r,l) \in (T_{rl}\iota(C))^\perp.$$

Then since

$$|\phi - \iota| = O(r^\mu),$$

for $1 < \mu < 2$ as $r \to 0$, making ϵ smaller if necessarily, we can guarantee that $\phi(r,l)$ lies in the tubular neighbourhood of C given by Proposition 5.2.3. We can therefore identify $\phi(C_\epsilon)$ with a normal vector field v_ϕ on C.

Applying Proposition 5.2.3 gives us $V_C \subseteq \nu_{\mathbb{R}^n}(C)$, $T_C \subseteq \mathbb{R}^n$ and a diffeomorphism

$$\Xi_C : V_C \to T_C.$$

Denote by V_{C_ϵ} the restriction of V_C to C_ϵ, and define

$$V_\phi := \{v \in \nu_{B_\epsilon(0)}(C_\epsilon) \,|\, v + v_\phi \in V_{C_\epsilon}\},$$

with

$$\Xi_\phi(v) := \Xi_C(v + v_\phi),$$

for $v \in V_\phi$ and

$$T_\phi := \Xi_C(V_\phi).$$

Then $\Xi_C : V_\phi \to T_\phi$ is a diffeomorphism by construction.

Write $\hat{U} := U\backslash\{\hat{x}\}$. Define $\hat{V}_2 := F(V_\phi) \subseteq \nu_X(\hat{U})$, where F is the isomorphism $\nu_{B_\epsilon(0)}(C_\epsilon) \to \nu_X(\hat{U})$ induced from Ψ and ι and $\hat{T}_2 := \chi(T_\phi)$. By definition, these sets grow with order ρ as $\rho \to 0$. Then

$$\chi \circ \Xi_\phi \circ F^{-1} : \hat{V}_2 \to \hat{T}_2,$$

is a diffeomorphism taking the zero section of $\nu_X(\hat{U})$ to $\hat{U}$. Define $\hat{V}$, $\hat{T}$ and $\hat{\Xi}$ by interpolating smoothly between $\hat{V}_1$ and $\hat{V}_2$, $\hat{T}_1$ and $\hat{T}_2$ and $\hat{\Xi}_1$ and $\hat{\Xi}_2$. □

5.2.3 Deformation problem

The deformation problem that we will consider in this chapter has been chosen because of the Fredholm theory that we have available to us on a conically singular manifold. Let Y be a conically singular Cayley submanifold of a $Spin(7)$-manifold X. The tubular neighbourhood theorem for conically singular submanifolds 5.2.4 will allow us identify the moduli space of Cayley deformations of $\hat{Y}$ (Y with its singular point removed) with the kernel of a nonlinear partial differential operator

$$\hat{F} : C^\infty(\nu_X(\hat{Y})) \to C^\infty(E), \tag{5.2.7}$$

similarly to Proposition 3.2.2 for a compact Cayley manifold. However to deduce a theorem analogous to Theorem 3.2.6 about the moduli space of Cayley deformations of $\hat{Y}$ in X, we will need to extend $\hat{F}$ to a smooth map of Banach spaces. Moreover we will require the linear part of $\hat{F}$, which will be the elliptic first order differential operator D defined in Proposition 3.2.3 due to the local nature of this result, to be Fredholm as a map between these Banach spaces. Therefore we will need to extend $\hat{F}$ to a smooth map of the weighted Sobolev spaces we defined in Definition 4.3.3, so that we may apply the Fredholm theory of Theorem 4.3.2 to its linearisation. However, whereas we will still be able to use elliptic regularity to see that the kernel of $\hat{F}$ as a map between weighted Sobolev spaces contains smooth sections, these sections will have a certain decay as $\rho \to 0$, where ρ is a radius function on Y, and therefore will only be a subset of the kernel of the map (5.2.7). Restricting the map $\hat{F}$ to spaces of weighted smooth sections of the normal bundle of $\hat{Y}$ in X will not give all Cayley deformations of $\hat{Y}$, but will allow us to find the deformations of $\hat{Y}$ that are themselves conically singular. This moduli space will be defined in Definition 5.2.5 below, and this moduli space will be identified with the kernel of a nonlinear partial differential operator in Proposition

5.2.5. First we will define a weighted norm on spaces of differentiable sections of a vector bundle.

Weighted norms on spaces of differentiable sections

Let X will be an n-dimensional CS manifold with a radius function ρ, E a vector bundle over $\hat{X}$ (the nonsingular part of X) with a metric and connection.

Definition 5.2.4. Let $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$. Define the space $C^k_\lambda(E)$ to be the space of sections $\sigma \in C^k_{\mathrm{loc}}(E)$ satisfying

$$\|\sigma\|_{C^k_\lambda} := \sum_{j=0}^{k} \sup_{\hat{X}} |\rho^{j-\lambda} \nabla^j \sigma| < \infty.$$

We say that $\sigma \in C^\infty_\lambda(E)$ if $\sigma \in C^k_\lambda(E)$ for all $k \in \mathbb{N}$.

The space $C^k_\lambda(E)$ is a Banach space, but $C^\infty_\lambda(E)$ is not in general.

Moduli space

We will now formally define the moduli space of conically singular Cayley deformations of a Cayley submanifold that we will be studying in this chapter.

Definition 5.2.5. Let Y be a conically singular Cayley submanifold at $\hat{x}$ with cone C and rate μ of a $Spin(7)$-manifold (X, g, Φ) with respect to some $Spin(7)$-coordinate system χ, and denote the tangent cone of Y at $\hat{x}$ by $\hat{C}$. Write $\hat{Y} := Y \backslash \{\hat{x}\}$. Define the *moduli space of conically singular (CS) Cayley deformations* of Y in X, $\hat{\mathcal{M}}_\mu(Y)$, to be the set of CS Cayley submanifolds Y' at $\hat{x}$ with cone C, rate μ and tangent cone $\hat{C}$ of X so that there exists a continuous family of topological embeddings $\iota_t : Y \to X$ with $\iota_0(Y) = Y$ and $\iota_1(Y) = Y'$, so that $\iota_t(\hat{x}) = \hat{x}$ for all $t \in [0,1]$ and so that $\hat{\iota}_t := \iota_t|_{\hat{Y}}$ is a smooth family of embeddings $\hat{Y} \to X$ with $\hat{\iota}_0(\hat{Y}) = \hat{Y}$ and $\hat{\iota}_1(\hat{Y}) = \hat{Y}' := Y' \backslash \{\hat{x}\}$.

Remark. We will later be interested in Cayley and complex deformations of a CS complex surface in the $Spin(7)$-manifold that is a Calabi–Yau four-fold, analogous to the deformation problem that was the subject of study in Chapter 3. Replacing

'Cayley' and '$Spin(7)$-manifold' with 'complex' and 'Calabi–Yau four-fold' respectively in Definition 5.2.5 will allow us to define the moduli space of CS complex deformations of a complex surface inside a four-dimensional Calabi–Yau manifold.

We will now end this section by identifying the moduli space of Cayley CS deformations of a CS Cayley submanifold of a $Spin(7)$-manifold with the kernel of a nonlinear partial differential operator. This proposition is the analogy of Proposition 3.2.2 of Chapter 3 for the Cayley moduli space given in Definition 5.2.5.

Proposition 5.2.5. *Let Y be a CS Cayley submanifold at $\hat{x}$ with cone C and rate $\mu \in (1,2)$ of a $Spin(7)$-manifold (X, g, Φ). Let τ be the Λ^2_7-valued four-form defined in Proposition 1.2.9, $\pi : \Lambda^2_7 \to E$ be the projection defined in (3.2.2) and $\hat{V} \subseteq \nu_X(\hat{Y})$, $\hat{T} \subseteq M$ and $\hat{\Xi}$ be the open sets and diffeomorphism from the CS tubular neighbourhood theorem 5.2.4. For $v \in C^\infty(\nu_X(\hat{Y}))$ taking values in $\hat{V}$ write $\hat{\Xi}_v$ for the diffeomorphism $\hat{\Xi} \circ v : \hat{Y} \to \hat{Y}_v := \hat{\Xi}_v(\hat{Y})$.*

Then we can identify the moduli space of CS Cayley deformations of Y in X near Y with the kernel of the following differential operator

$$\begin{aligned} \hat{F} : C^\infty_\mu(\hat{V}) &\to C^\infty_{\mathrm{loc}}(E), \\ v &\mapsto \pi(*_{\hat{Y}} \hat{\Xi}_v^*(\tau|_{\hat{Y}_v})). \end{aligned} \tag{5.2.8}$$

Proof. The deformation $\hat{Y}_v$ is Cayley if, and only if, $\tau|_{\hat{Y}_v} \equiv 0$, which since $\hat{\Xi}_v$ is a diffeomorphism is equivalent to $v \in \operatorname{Ker} \hat{F}$ (similarly to the proof of Proposition 3.2.2), and since $v, \tau, \hat{\Xi}_v$ are all smooth, we see that $\hat{F}$ takes values in $C^\infty_{\mathrm{loc}}(E)$ at claimed.

It remains to show that $Y_v := \hat{Y}_v \cup \{\hat{x}\}$ is a CS submanifold of X at $\hat{x}$ with cone C and rate μ (with respect to the same $Spin(7)$-coordinate system as Y) if, and only if, $v \in C^\infty_\mu(\hat{V})$.

Let v be a smooth normal vector field on $\hat{Y}$, and let $\hat{Y}_v := \hat{\Xi}_v(\hat{Y})$. Use the notation of Definition 5.2.2. Choose $\phi : (0, \epsilon) \times L \to B_\epsilon(0)$ uniquely by requiring that

$$\phi(r, l) - \iota(r, l) \in (T_{rl}\iota(C))^\perp.$$

Now we can use Ψ and ι to identify $\nu_X(\hat{U})$ with $\nu_{B_\epsilon(0)}(\iota(C_\epsilon))$, where $\hat{U} := U\backslash\{\hat{x}\}$ and $C_\epsilon := (0, \epsilon) \times L$. Write v_C for the section of $\nu_{B_\epsilon(0)}(\iota(C_\epsilon))$ corresponding to v under this identification.

Making ϵ and U smaller if necessary, by the definition of the tubular neighbourhood map in Proposition 5.2.4, we can define a map $\phi_v : C_\epsilon \to B_\epsilon(0)$ by

$$\phi_v(r, l) = \Xi_\phi(v_C(r, l)),$$

where Ξ_ϕ was defined in the proof of Proposition 5.2.4, so that $\chi \circ \phi_v : C_\epsilon \to \Xi_v(\hat{U}) \subseteq \hat{Y}_v$ is a diffeomorphism. So we see that for Y_v to be a CS submanifold of X with rate μ and cone C we must have that

$$|\nabla^j(\phi_v(r, l) - \iota(r, l))| = O(r^{\mu-j}), \tag{5.2.9}$$

for all $j \in \mathbb{N}$ as $r \to 0$. Now we can write

$$|\nabla^j(\phi_v - \iota)| \leq |\nabla^j(\phi_v - \phi)| + |\nabla^j(\phi - \iota)|,$$

and so (5.2.9) holds if, and only if,

$$|\nabla^j(\phi_v - \phi)| = O(r^{\mu-j}),$$

for $j \in \mathbb{N}$ as $r \to 0$. But examining the definition of ϕ_v, we see that we can identify $\phi_v - \phi$ with the graph of v_C, and so (5.2.9) holds if, and only if,

$$|\nabla^j v_C| = O(r^{\mu-j}),$$

for $j \in \mathbb{N}$ as $r \to 0$. But then by definition of v_C this is equivalent to

$$|\nabla^j v| = O(\rho^{\mu-j}),$$

for $j \in \mathbb{N}$ as $\rho \to 0$, that is, $v \in C^j_\mu(\hat{V})$ for all $j \in \mathbb{N}$. So we see that the moduli space of CS Cayley deformations of Y in X can be identified with the kernel of (5.2.8). $\square$

5.3 Cayley deformations of a CS Cayley submanifold

In this section we prove Theorem 5.3.3 which gives the expected dimension of the moduli space of CS Cayley deformations of a conically singular Cayley submanifold Y in a $Spin(7)$-manifold X in terms of the index of a linear elliptic operator acting on weighted sections of the normal bundle of $\hat{Y}$ (the nonsingular part of Y) in X. Similar to the proof of Theorem 3.2.6, which was an analogous result for compact Cayley submanifolds, we first prove in Lemma 5.3.1 that the operator $\hat{F}$ defined in Proposition 5.2.5 is a smooth map of weighted Sobolev spaces. We then prove a weighted elliptic regularity result for the operator $\hat{F}$ in Proposition 5.3.2. Finally, these results combined with the Fredholm theory of Theorem 4.3.2 will allow us to prove Theorem 5.3.3.

The following lemma is similar to [23, Thm 5.1] and [37, Prop 6.9], and is an extension of Lemma 3.2.4 to conically singular Cayley submanifolds.

Lemma 5.3.1. *Let Y be a conically singular Cayley submanifold of a $Spin(7)$-manifold X. Let $\hat{F}$ be the operator defined in Proposition 5.2.5. Then we can write*

$$\hat{F}(v)(x) = Dv(x) + \hat{Q}(x, v(x), \nabla v(x)), \tag{5.3.1}$$

for $x \in \hat{Y}$, where

$$\hat{Q} : \{(x, y, z) \mid (x, y) \in \hat{V}, z \in \nu_x(\hat{Y}) \otimes T_x^*\hat{Y}\} \to E,$$

is smooth, D was defined in Proposition 3.2.3 and $\hat{Q}(v)(x) := \hat{Q}(x, v(x), \nabla v(x))$ is a section of E. Let $\mu > 1$. Then for each $k \in \mathbb{N}$, for $v \in C^{k+1}_{\mu}(\hat{V})$ with $\|v\|_{C^1_1}$ sufficiently small, there exist constants $C_k > 0$ so that

$$\|\hat{Q}(v)\|_{C^k_{2\mu-2}} \leq C_k \|v\|^2_{C^{k+1}_{\mu}}, \tag{5.3.2}$$

and if $v \in L^p_{k+1,\mu}(\hat{V})$ with $\|v\|_{C^1_1}$ sufficiently small, with $k > 1 + 4/p$, there exist constants $D_k > 0$ such that

$$\|\hat{Q}(v)\|_{p,k,2\mu-2} \leq D_k \|v\|^2_{p,k+1,\mu}. \tag{5.3.3}$$

Moreover, we may deduce that

$$\hat{F} : L^p_{k+1,\mu}(\hat{V}) \to L^p_{k,\mu-1}(E), \tag{5.3.4}$$

is a smooth map of Banach spaces for any $1 < p < \infty$ *and* $k \in \mathbb{N}$ *with* $k > 1 + 4/p$.

Proof. By definition of conically singular, we can split Y into a compact piece K, where we see similarly to Lemma 3.2.4 that the estimate (5.3.2) holds, and a piece diffeomorphic to a cone, which is where we must check how $\hat{F}$ behaves as $\rho \to 0$, where ρ is a radius function for $\hat{Y}$. Making the compact piece slightly larger, using the definition of $\hat{F}$, we may estimate $\hat{F}$ by estimating $\hat{F}_C$, the operator on the cone defined by

$$\hat{F}_C(v + v_\phi)(r, l) = \hat{F}(v)(\Psi(r, l)),$$

where v_ϕ is the normal vector field on C that describes $\phi(C)$ as described in the proof of Proposition 5.2.4, where we are using the notation of Definition 5.2.2. Define $\hat{Q}_C$ analogously by

$$\hat{F}_C(v + v_\phi)(r, l) = D(v + v_\phi)(r, l) + \hat{Q}_C(r, l, (v + v_\phi)(r, l), \nabla(v + v_\phi)(r, l)).$$

By definition of $\hat{Q}$ and $\hat{Q}_C$, we see that

$$\hat{Q}(r, l, v, \nabla v) = \hat{Q}_C(r, l, v + v_\phi, \nabla(v + v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi), \tag{5.3.5}$$

and so to estimate $\hat{Q}$ and its derivatives, it suffices to estimate the right hand side of Equation (5.3.5). Notice that since for each $(r, l) \in C$ we can think of $\hat{Q}_C$ as a map

$$\nu_{r,l}(C) \times \nu_{r,l}(C) \otimes T^*_{r,l}C \to E_{r,l},$$

and so we can make sense of a Taylor expansion of $\hat{Q}$ around points of the form (r, l, y, z), for $y \in \nu_{r,l}(C)$ and $z \in \nu_{r,l}(C) \otimes T^*_{r,l}C$. Abusing notation slightly, write

$$\frac{\partial \hat{Q}_C}{\partial y}(r, l, y, z),$$

for derivative of $\hat{Q}_C$ in the y direction at (r, l, y, z), and adopt similar notation for the derivative in the z direction and the higher derivatives. Then we have that

$$\begin{aligned}\hat{Q}_C(r, l, y + y_0, z + z_0) = \hat{Q}_C(r, l, y_0, z_0) &+ \frac{\partial \hat{Q}_C}{\partial y}(r, l, y_0, z_0)y + \frac{\partial \hat{Q}_C}{\partial z}(r, l, y_0, z_0)z \\ &+ \frac{1}{2}\frac{\partial^2 \hat{Q}_C}{\partial y^2}(r, l, ty + y_0, tz + z_0)(y, y) \\ &+ \frac{\partial^2 \hat{Q}_C}{\partial y \partial z}(r, l, ty + y_0, tz + z_0)(y, z) \\ &+ \frac{1}{2}\frac{\partial^2 \hat{Q}_C}{\partial z^2}(r, l, ty + y_0, tz + z_0)(z, z), \end{aligned} \tag{5.3.6}$$

for some $t \in [0, 1]$. We would like to estimate the derivatives of $\hat{Q}_C$. Since $\hat{Q}$ is smooth in all of its variables, this is possible as long as we restrict the domain of $\hat{Q}_C$ to a compact set. However, we are working on a cone (with its singular point removed) so this isn't possible. We may, however, fix $r = r_0$, for some $r_0 \in (0, \epsilon)$, perform our estimates, and use the definition of $\hat{F}_C$ and $\hat{Q}_C$ to study the behaviour of the estimates we find as we let r vary. Recall the action of $\mathbb{R}_+$ on $\nu_{\mathbb{R}^8}(C)$ that was defined in the proof of Proposition 5.2.3, and indeed the tubular neighbourhood map that we constructed in this proof, which forms part of the operator $\hat{F}_C$ that we are currently studying. By construction, we can see that

$$|\hat{F}_C(v(r, l))|_r = \left|\hat{F}_C\left(\frac{r_0}{r} \cdot v(r, l)\right)\right|_{r_0},$$

where $|\cdot|_r$ means that we are taking the norm at the point r. We may deduce that

$$|\nabla^k \hat{Q}_C(r, lv + v_\phi, \nabla(v + v_\phi))|_r = r^{-k}\left|\hat{Q}_C\left(r_0, l, \frac{r_0}{r} \cdot (v + v_\phi), \nabla \frac{r_0}{r} \cdot (v + v_\phi)\right)\right|_{r_0}.$$

We also have that by construction

$$\left|\nabla^k \frac{r_0}{r} \cdot v(r, l)\right|_{r_0} = \left(\frac{r}{r_0}\right)^{k-1} |\nabla^k v(r, l)|_r.$$

Since $\hat{Q}_C$ has no linear parts, we have that by equation (5.3.6),

$$\begin{aligned}\hat{Q}_C(r, l, v + v_\phi, \nabla(v + v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi) &\leq \frac{1}{2}\frac{\partial^2 \hat{Q}_C}{\partial y^2}(r, l, tv + v_\phi, \nabla(tv + v_\phi))(v, v) \\ + \frac{\partial^2 \hat{Q}_C}{\partial y \partial z}(r, l, tv + v_\phi, \nabla(tv + v_\phi))(v, \nabla v) &+ \frac{1}{2}\frac{\partial^2 \hat{Q}_C}{\partial z^2}(r, l, tv + v_\phi, \nabla(tv + v_\phi))(\nabla v, \nabla v),\end{aligned} \tag{5.3.7}$$

for some $t \in [0,1]$.

Consider

$$\frac{\partial^2 \hat{Q}_C}{\partial y^2}(r_0, l, (tv + v_\phi)(r_0, l), \nabla(tv + v_\phi)(r_0, l)).$$

Then by taking the supremum over the closed sets $l \in L$ and v with $|v(r_0,l)|_{r_0,l} + |\nabla v(r_0,l)|_{r_0,l} \leq \delta$, which is possible as long as we take δ sufficiently small, we may bound this expression, as well as the other coefficients of Equation (5.3.7). Using the scale equivariance properties of $\hat{Q}_C$ described above, we deduce that as long as $\|v\|_{C^1_1}$ is small, we have that

$$|\hat{Q}_C(r, l, v + v_\phi, \nabla(v + v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi)|_r \leq C_0(r^{-1}|v|_r + |\nabla v|_r)^2.$$

Therefore

$$\begin{aligned} r^{2-2\mu}|\hat{Q}_C(r, l, v + v_\phi, \nabla(v + v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi)|_r &\leq C_0 r^{2-2\mu}(r^{-1}|v|_r + |\nabla v|_r)^2 \\ &= C_0\|v\|^2_{C^1_\mu}. \end{aligned} \tag{5.3.8}$$

Finally, we can take k derivatives of Equation (5.3.7), which will give us a polynomial quadratic in v and its derivatives, whose coefficients depend on between two and $k+2$ derivatives of $\hat{Q}_C$, the C^1_1-norm of v_ϕ and δ as above. We can estimate these coefficients as we did above. We will find that

$$|\nabla^k(\hat{Q}_C(r, l, v+v_\phi, \nabla(v+v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi))| \leq C\left(\sum_{j_1, j_2} \frac{r^{j_1-1} r^{j_2-1}}{r^k}|\nabla^{j_1} v||\nabla^{j_2} v|\right),$$

and since

$$r^{k-(2\mu-2)} r^{j_1-1} r^{j_2-1} r^{-k} = r^{j_1-\mu} r^{j_2-\mu},$$

we may deduce that

$$r^{k-(2\mu-2)}|\nabla^k(\hat{Q}_C(r, l, v + v_\phi, \nabla(v + v_\phi)) - \hat{Q}_C(r, l, v_\phi, \nabla v_\phi))| \leq C_k\|v\|^2_{C^{k+1}_\mu},$$

and so we see that the estimate (5.3.2) holds. Finally, as long as $\mu > 1$, we have that $C^k_{2\mu-2}(E) \subseteq C^k_{\mu-1}(E)$. Similarly to the proof of Lemma 3.2.4, we can use (5.3.2) to deduce (5.3.3) and that (5.3.4) is a smooth map of Banach spaces, as $\hat{Q}$ is smooth. □

Now that we have described the behaviour of the operator $\hat{F}$ close to the singular point of the conically singular manifold $\hat{Y}$, we will prove a weighted elliptic regularity result for normal vector fields in the kernel of $\hat{F}$.

Proposition 5.3.2. *Let Y be a conically singular Cayley submanifold of a $Spin(7)$-manifold X. Let $\hat{F}$ be the map defined in Proposition 5.2.5. Then*

$$\{v \in C^\infty_\mu(\hat{V}) \,|\, \hat{F}(v) = 0\} \cong \{v \in L^p_{k+1,\mu}(\hat{V}) \,|\, \hat{F}(v) = 0\},$$

for any $\mu \in (1,2)\backslash\mathcal{D}$, $1 < p < \infty$ and $k \in \mathbb{N}$ satisfying $k > 1 + 4/p$. Here $\mathcal{D}$ is the set of exceptional weights given by applying Theorem 4.3.2 to the linear part of $\hat{F}$.

Proof. We will first show that if $v \in C^\infty_\mu(\hat{V})$ satisfying $\hat{F}(v) = 0$, then $v \in L^p_{k+1,\mu}$. This is a little trickier than it seems, since we have that for any $\epsilon > 0$, $C^\infty_\mu(\hat{V}) \subseteq L^p_{k,\mu-\epsilon}(\hat{V})$, which is weaker than what we require. We will show that if $v \in L^p_{k+1,\mu-\epsilon}(\hat{V})$, for $\epsilon > 0$ sufficiently small, satisfies $\hat{F}(v) = 0$, then we may deduce that $v \in L^p_{k+1,\mu}(\hat{V})$. Recall that in Lemma 5.3.1, we saw that we could write

$$\hat{F}(v) = Dv + \hat{Q}(v),$$

where D was defined in Proposition 3.2.3, and $\hat{Q}$ is nonlinear. It can be shown that D satisfies the hypotheses of Theorem 4.3.2 (we will see this explicitly in Section 6.2.2 below) and so there exists a discrete set $\mathcal{D}$ so that

$$D : L^p_{k+1,\lambda}(\nu_X(\hat{Y})) \to L^p_{k,\lambda-1}(E), \tag{5.3.9}$$

is Fredholm as long as $\lambda \notin \mathcal{D}$. Take $0 < \epsilon < (\mu-1)/2$ small enough so that $[\mu-\epsilon, \mu] \cap \mathcal{D} = \emptyset$. Let $v \in L^p_{k+1,\mu-\epsilon}(\hat{V})$ and suppose that $\hat{F}(v) = 0$. Since (5.3.9) is Fredholm when $\lambda = \mu - \epsilon$, we can write

$$L^p_{k,\mu-\epsilon-1}(E) = D(L^p_{k+1,\mu-\epsilon}(\nu_X(\hat{Y}))) \oplus \hat{\mathcal{O}}_{\mu-\epsilon},$$

where $\hat{\mathcal{O}}_{\mu-\epsilon}$ is finite-dimensional and

$$\hat{\mathcal{O}}_{\mu-\epsilon} \cong \mathrm{Coker}_{\mu-\epsilon} D,$$

where $\mathrm{Coker}_\lambda D$ denotes the cokernel of (5.3.9). Since $[\mu-\epsilon,\mu]\cap\mathcal{D}=\emptyset$, we know that (see [35, Lem 7.1])

$$\mathrm{Coker}_{\mu-\epsilon}D=\mathrm{Coker}_\mu D. \tag{5.3.10}$$

Now since $\hat{F}(v)=0$, we have that $Dv=-\hat{Q}(v)$, and so $\hat{Q}(v)$ is orthogonal to $\mathrm{Coker}_{\mu-\epsilon}D$. Also $\hat{Q}(v)\in L^p_{k,2\mu-2-2\epsilon}(E)\subseteq L^p_{k,\mu-1}(E)$ by Lemma 5.3.1 since $v\in L^p_{k+1,\mu-\epsilon}(\hat{V})$ and by our choice of ϵ. Therefore we have that $Dv=\hat{Q}(v)\in L^p_{k,\mu-1}(\hat{V})$, and it is orthogonal to $\mathrm{Coker}_\mu D$ by (5.3.10). Therefore there exists $\bar{v}\in L^p_{k+1,\mu}(\hat{V})$ with $Dv=D\bar{v}$. But then we must have that $v-\bar{v}\in\mathrm{Ker}_{\mu-\epsilon}D=\mathrm{Ker}_\mu D$, since $[\mu-\epsilon,\mu]\cap\mathcal{D}=\emptyset$, and so $v\in L^p_{k+1,\mu}(\hat{V})$, as required.

Conversely, let $v\in L^p_{k+1,\mu}(\hat{V})$ satisfy $\hat{F}(v)=0$. Here we perform a trick similar to that in [28, Prop 4.6]. Taylor expanding $\hat{F}(v)$ around zero we can write $\hat{F}(v)$ as a polynomial in v and ∇v. Differentiating and gathering terms we can write

$$\nabla\hat{F}(v)=L(x,v(x),\nabla v(x))\nabla^2 v+E(x,v(x),\nabla v(x)).$$

Consider the second order elliptic linear operator

$$\begin{aligned}L_v:\nu_X(\hat{Y})&\to E,\\ w&\mapsto L(x,v(x),\nabla v(x))\nabla^2 w.\end{aligned}$$

By Sobolev embedding, we know that $v\in C^l_\mu(\hat{V})$, for $l\geq 2$ by choice of p and k, and therefore the coefficients of the linear operator L_v lie in $C^{l-1}_{\mathrm{loc}}(\hat{V})$. Local regularity for linear elliptic operators with coefficients in Hölder spaces (a nice statement is given in [25, Thm 1.4.2], taken from [44, Thm 6.2.5]) tells us that $v\in C^{l+1}_{\mathrm{loc}}(\hat{V})$ which is an improvement on the regularity of v, and so bootstrapping we may deduce that $v\in C^\infty_{\mathrm{loc}}(\hat{V})$. (This is why we must differentiate $\hat{F}(v)$, to ensure that the coefficients of the linear operator have enough regularity to improve the regularity of v.) Therefore the coefficients of the operator L_v are smooth and so we may apply an estimate of Lockhart and McOwen [35, Eq. 2.4] in combination with a change of coordinates which tells us that

$$\|v\|_{p,k+2,\mu}\leq C(\|L_v v\|_{p,k,\mu-2}+\|v\|_{p,0,\mu}). \tag{5.3.11}$$

Since $\hat{F}(v) = 0 = \nabla\hat{F}(v)$, we have that

$$L_v v = -E(x, v(x), \nabla v(x)).$$

Since $E(x, v(x), \nabla v(x))$ is a polynomial in v and ∇v with coefficients that depend on the C^1_1-norm of v, and $v \in C^1_\mu(\hat{V})$ and $L^p_{k+1,\mu}(\hat{V})$, we have that $E(x, v(x), \nabla v(x)) \in L^p_{k,\mu-1}(E) \subseteq L^p_{k,\mu-2}(E)$. Therefore Equation (5.3.11) tells us that $v \in L^p_{k+2,\mu}(\hat{V})$, from which we may deduce that v is in fact in $C^\infty_\mu(\hat{V})$. $\square$

We may finally deduce the main theorem of this section, on the expected dimension of the moduli space of Cayley CS deformations of a CS Cayley submanifold of a $Spin(7)$-manifold X. This theorem is the analogy of Theorem 3.2.6 for this moduli space, and the proof is similar.

Theorem 5.3.3. *Let Y be a CS Cayley submanifold at $\hat{x}$ with cone C and rate $\mu \in (1,2)\backslash\mathcal{D}$ of a $Spin(7)$-manifold X. Let D denote the first order elliptic differential operator defined in* (3.2.6)*. Then there exist a smooth manifold $\hat{K}_0$, which is an open neighbourhood of* 0 *in the kernel of* (5.3.12)*, and a smooth map $\hat{g}_2$ from $\hat{K}_0$ into the cokernel of* (5.3.12) *with $\hat{g}_2(0) = 0$ so that an open neighbourhood of Y in the moduli space of CS Cayley deformations of Y in X, $\hat{\mathcal{M}}_\mu(Y)$ from Definition 5.2.5, is homeomorphic to an open neighbourhood of* 0 *in* Ker $\hat{g}_2$.

Moreover, the expected dimension of $\hat{\mathcal{M}}_\mu(Y)$ is given by the index of the linear elliptic operator

$$D : L^p_{k+1,\mu}(\nu_X(\hat{Y})) \to L^p_{k,\mu-1}(E). \tag{5.3.12}$$

If the cokernel of (5.3.12) *is* $\{0\}$ *then $\hat{\mathcal{M}}_\mu(Y)$ is a smooth manifold near Y of the same dimension as the kernel of* (5.3.12)*. Here $\mathcal{D}$ is the set of weights $\mu \in \mathbb{R}$ for which* (5.3.12) *is not Fredholm from Theorem 4.3.2.*

Proof. By Propositions 5.2.5 and 5.3.2, we can identify $\hat{\mathcal{M}}_\mu(Y)$ near Y with the kernel of the operator

$$\hat{F} : L^p_{k+1,\mu}(\hat{V}) \to L^p_{k,\mu-1}(E).$$

The linearisation of $\hat{F}$ at zero is the operator

$$D : L^p_{k+1,\mu}(\nu_X(\hat{Y})) \to L^p_{k,\mu-1}(E), \tag{5.3.13}$$

which is elliptic. Since $\mu \notin \mathcal{D}$, (5.3.13) is Fredholm. Therefore we may decompose

$$L^p_{k+1,\mu}(\nu_X(\hat{Y})) = \hat{K}' \oplus \hat{X}',$$

where $\hat{K}'$ is the kernel of (5.3.13) and $\hat{X}'$ is closed, and

$$L^p_{k,\mu-1}(E) = D(L^p_{k+1,\mu}(\nu_X(\hat{Y}))) \oplus \hat{\mathcal{O}}_\mu,$$

where $\hat{\mathcal{O}}_\mu$ is the finite-dimensional obstruction space and

$$\hat{\mathcal{O}}_\mu \cong L^p_{k,\mu-1}(E)/D(L^p_{k+1,\mu}(\nu_X(\hat{Y}))) =: \mathrm{Coker}_\mu D.$$

Then the map

$$\begin{aligned} \hat{\mathcal{F}} : L^p_{k+1,\mu}(\hat{V}) \times \hat{\mathcal{O}}_\mu &\to L^p_{k,\mu-1}(E), \\ (v, w) &\mapsto \hat{F}(v) + w, \end{aligned}$$

has

$$d\hat{\mathcal{F}}|_{(0,0)}(v, w) = Dv + w, \tag{5.3.14}$$

which is surjective. Write $\hat{K} = \hat{K} \times \{0\}$ for the kernel of (5.3.14). We then have that

$$L^p_{k+1,\mu}(\nu_X(\hat{Y})) \times \hat{\mathcal{O}}_\mu = \hat{K} \oplus (\hat{X}' \times \hat{\mathcal{O}}_\mu).$$

Now we may apply the implicit function theorem 3.2.5 to find $\hat{K}_0 \subseteq \hat{K}$ containing zero, $\hat{X}'_0 \subseteq \hat{X}'$, $\hat{\mathcal{O}}_0 \subseteq \hat{\mathcal{O}}_\mu$ and a smooth map $\hat{g} = (\hat{g}_1, \hat{g}_2) : \hat{K}_0 \to \hat{X}'_0 \times \hat{\mathcal{O}}_0$ so that

$$\hat{\mathcal{F}}^{-1}(0) \cap (\hat{K}_0 \times \hat{X}'_0 \times \hat{\mathcal{O}}_0) = \{(x, \hat{g}_1(x), \hat{g}_2(x)) \,|\, x \in \hat{K}_0\}.$$

So we may identify the kernel of $\hat{F}$, and therefore $\hat{M}_\mu(Y)$ with the kernel of $\hat{g}_2 : \hat{K}_0 \to \hat{\mathcal{O}}_0$, a smooth map between finite-dimensional spaces (since (5.3.13) is Fredholm). Sard's theorem tells us that the expected dimension of the kernel of $\hat{g}_2$ is given by the index of the operator (5.3.13). □

5.4 Cayley deformations of a CS complex surface

In this section we prove Theorem 5.4.4 which gives the expected dimension of the moduli space of CS Cayley deformations of a two-dimensional conically singular complex submanifold N of a Calabi–Yau four-fold M in terms of the index of the operator $\bar{\partial} + \bar{\partial}^*$ acting on weighted sections of a vector bundle over $\hat{N}$ (the nonsingular part of N). Similar to the proof of Theorem 3.3.4, which was an analogous result for compact complex surfaces, we have already seen in Proposition 5.2.5 that the moduli space of CS Cayley deformations of N in M can be identified with the kernel of a nonlinear partial differential operator $\hat{F}$. We can use the operator $\hat{F}$ to construct an operator $\hat{F}^{\mathrm{cx}}$ in Proposition 5.4.1, whose kernel is isomorphic to the kernel of $\hat{F}$, but whose linear part takes the form of the operator $\bar{\partial} + \bar{\partial}^*$. We will then give analytic results, Lemma 5.4.2 and Proposition 5.4.3, on extending the map $\hat{F}^{\mathrm{cx}}$ to act on weighted sections of Sobolev spaces and an elliptic regularity result for $\hat{F}^{\mathrm{cx}}$ respectively. These results follow immediately from Lemma 5.3.1 and Proposition 5.3.2, their counterparts for the operator $\hat{F}$. Finally, we prove Theorem 5.4.4, which can be proved in exactly the same way as Theorem 5.3.3 or simply by an application of this theorem.

5.4.1 Deformation problem

We would like to study the moduli space given in Definition 5.2.5 for the CS Cayley submanifold N that is a complex submanifold of a Calabi–Yau four-fold M. We will now identify this moduli space with the kernel of a nonlinear partial differential operator. The following proposition is the analogy of Proposition 3.3.3 for a CS complex surface.

Proposition 5.4.1. *Let N be a CS complex surface at $\hat{x}$ with cone C and rate $\mu \in (1, 2)$ inside a Calabi-Yau four-fold M. Write $\hat{N} := N \backslash \{\hat{x}\}$. Then the moduli space of CS Cayley deformations of N in M, $\hat{\mathcal{M}}_\mu(N)$, can be identified with the kernel of the* *operator*

$$\hat{F}^{\mathrm{cx}} : C^\infty_\mu(\hat{U}) \to C^\infty_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})),$$

where $\hat{U} \subseteq \nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})$ *is the image of* $\hat{V} \otimes \mathbb{C}$ *from the tubular neighbourhood theorem 5.2.4 under the isomorphism given in Proposition 3.3.1, and* $\hat{F}^{\mathrm{cx}}$ *is defined so that the following diagram commutes*

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{F^{\mathrm{cx}}} & C^\infty(\Lambda^{0,1}N \otimes \nu^{1,0}_M(N)) \\ \downarrow & & \downarrow \\ C^\infty(V \otimes \mathbb{C}) & \xrightarrow{F} & C^\infty(E \otimes \mathbb{C}) \end{array}$$

where $\hat{F}$ *is the operator defined in Proposition 5.2.5 and we use the isomorphisms given in Propositions 3.3.1 and 3.3.2.*

Moreover, the linearisation of $\hat{F}^{\mathrm{cx}}$ *at zero is the operator*

$$\bar{\partial} + \bar{\partial}^* : C^\infty_\mu(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to C^\infty_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \tag{5.4.1}$$

Proof. By Proposition 5.2.5 we can identify the moduli space of CS Cayley deformations of N in M with the kernel of $\hat{F}$, which is the same as the kernel of $\hat{F}^{\mathrm{cx}}$.

Since the linearisation of the operator of $\hat{F}$ is given by the operator D defined in Proposition 3.2.3, the local argument of Proposition 3.3.3 still holds, and so we see that the linearisation of $\hat{F}^{\mathrm{cx}}$ at zero is given by the operator (5.4.1) as claimed. □

5.4.2 Cayley deformations of a CS complex surface

In this section, we will give analogies of the results of Section 5.3, which were on analytic properties of the operator $\hat{F}$ defined in Proposition 5.2.5, for the operator $\hat{F}^{\mathrm{cx}}$ defined in Proposition 5.4.1. Due to the relation between the operators $\hat{F}$ and $\hat{F}^{\mathrm{cx}}$ noted in the proof of Proposition 5.4.1, these results follow immediately from their counterparts.

Lemma 5.4.2. *Let N be a conically singular complex surface inside a Calabi–Yau four-fold M. Let $\hat{F}^{\mathrm{cx}}$ be the operator defined in Proposition 5.4.1. Then we can write*

$$\hat{F}^{\mathrm{cx}}(w)(x) = (\bar{\partial} + \bar{\partial}^*)w(x) + \hat{Q}^{\mathrm{cx}}(x, w(x), \nabla w(x)), \tag{5.4.2}$$

for $x \in \hat{N}$, where

$$\begin{aligned}\hat{Q}^{\mathrm{cx}} : \{(x,y,z) \mid (x,y) \in \hat{U}, z \in \Big[\nu_x^{1,0}(\hat{N}) \oplus \Lambda_x^{0,2}\hat{N} \otimes \nu_x^{1,0}(\hat{N})\Big] \otimes T_x^*\hat{N}\} \\ \to \Lambda^{0,1}\hat{N} \otimes \nu_M^{1,0}(\hat{N}),\end{aligned}$$

is smooth and $\hat{Q}^{\mathrm{cx}}(w)(x) := \hat{Q}^{\mathrm{cx}}(x, w(x), \nabla w(x))$ is a section of $\Lambda^{0,1}\hat{N} \otimes \nu_M^{1,0}(\hat{N})$. Let $\mu > 1$. Then for each $k \in \mathbb{N}$, for $w \in C^{k+1}_\mu(\hat{U})$ with $\|w\|_{C^1_1}$ sufficiently small, there exist constants $C_k > 0$ so that

$$\|\hat{Q}^{\mathrm{cx}}(w)\|_{C^k_{2\mu-2}} \leq C_k \|w\|^2_{C^{k+1}_\mu}, \tag{5.4.3}$$

and if $w \in L^p_{k+1,\mu}(\hat{U})$ with $\|w\|_{C^1_1}$ sufficiently small, there exist constants $D_k > 0$ such that

$$\|\hat{Q}^{\mathrm{cx}}(w)\|_{p,k,2\mu-2} \leq D_k \|w\|^2_{p,k+1,\mu}. \tag{5.4.4}$$

Moreover, we may deduce that

$$\hat{F}^{\mathrm{cx}} : L^p_{k+1,\mu}(\hat{U}) \to L^p_{k,\mu-1}(E), \tag{5.4.5}$$

is a smooth map of Banach spaces for any $1 < p < \infty$ and $k \in \mathbb{N}$ with $k > 1 + 4/p$.

Proof. Since $\hat{F}^{\mathrm{cx}}$ is defined by composing the operator $\hat{F}$ defined in Proposition 5.2.5 with isomorphisms of vector bundles, the estimates (5.4.3) and (5.4.4) follow from the estimates (5.3.2) and (5.3.3) respectively since the isomorphisms defined in Propositions 3.3.1 and 3.3.2 are isometries.

Moreover, since these isomorphisms are smooth, the claim that (5.4.5) is a smooth map of Banach spaces follows from the corresponding fact for $\hat{F}$ from Lemma 5.3.1. □

We may now give a weighted elliptic regularity result for $\hat{F}^{\mathrm{cx}}$.

Proposition 5.4.3. *Let N be a conically singular complex surface inside a Calabi–Yau four-fold M. Let $\hat{F}^{\mathrm{cx}}$ be the map defined in Proposition 5.4.1. Then*

$$\{w \in C^{\infty}_{\mu}(\hat{U}) \,|\, \hat{F}^{\mathrm{cx}}(w) = 0\} \cong \{w \in L^p_{k+1,\mu}(\hat{U}) \,|\, \hat{F}^{\mathrm{cx}}(w) = 0\},$$

for any $\mu \in (1,2)\backslash\mathcal{D}$, $1 < p < \infty$ and $k \in \mathbb{N}$. Here $\mathcal{D}$ is the set of exceptional weights given by applying Theorem 4.3.2 to the linear part of $\hat{F}^{\mathrm{cx}}$.

Proof. This follows from Proposition 5.3.2 in combination with the fact that the kernels of $\hat{F}$, defined in Proposition 5.2.5, and $\hat{F}^{\mathrm{cx}}$ are isomorphic by definition, and the isomorphism given in Proposition 3.3.1 is an isometry. □

We deduce the following theorem on the moduli space of CS Cayley deformations of a CS complex surface inside a Calabi–Yau four-fold. This theorem can be proved by an identical argument to the proof of Theorem 5.3.3, but we will deduce it as a corollary of Theorem 5.3.3.

Theorem 5.4.4. *Let N be a CS complex surface at $\hat{x}$ with cone C and rate $\mu \in (1,2)\backslash\mathcal{D}$ of a Calabi–Yau four-fold M. Then the expected dimension of $\hat{\mathcal{M}}_{\mu}(N)$ is given by the index of the linear elliptic operator*

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\mu-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \tag{5.4.6}$$

Moreover if the cokernel of (5.4.6) *is $\{0\}$ then $\hat{\mathcal{M}}_{\mu}(N)$ is a smooth manifold near N of the same dimension as the (complex) dimension of the kernel of* (5.4.6). *Here $\mathcal{D}$ is the set of weights $\mu \in \mathbb{R}$ for which* (5.3.12) *is not Fredholm from Theorem 4.3.2.*

Proof. By Theorem 5.3.3, the expected dimension of $\hat{\mathcal{M}}_{\mu}(N)$ is given by the index of the operator (5.3.12). Since, by Proposition 3.3.3 we can consider the operator (5.4.6) as the composition of the operator (5.3.12) with the isomorphisms from Propositions 3.3.1 and 3.3.2, which are isometries, we may deduce that the index of (5.3.12) and (5.4.6) are equal. □

5.5 Complex deformations of a CS complex surface

In this section, we will compare the CS complex and Cayley deformations of a CS complex surface inside a four-dimensional Calabi–Yau manifold. In Definition 5.5.1 we formally define the moduli space of CS complex deformations of a conically singular complex surface N inside a Calabi–Yau four-fold M. We will then identify this moduli space with the kernel of a partial differential operator in Proposition 5.5.1. The local arguments of Section 3.4.2 still hold and so we may deduce, similarly to Theorem 3.4.7 on the moduli space of complex deformations of a compact complex surface in a Calabi–Yau four-fold, that the moduli space of CS complex deformations of N in M is a smooth manifold near N. A technical result in Proposition 5.5.3 will allow us to deduce that CS complex and Cayley deformations of N in M are the same.

Definition 5.5.1. Let N be a CS complex surface at $\hat{x}$ with rate μ and cone C inside a Calabi–Yau manifold M with respect to some $Spin(7)$-coordinate system χ, and denote by $\hat{C}$ the tangent cone of N. Write $\hat{N} := N \backslash \{\hat{x}\}$. Define the *moduli space of conically singular (CS) complex deformations of* N *in* M, $\hat{\mathcal{M}}^{\mathrm{cx}}_{\mu}(N)$, to be the set of CS complex surfaces N' at $\hat{x}$ with cone C, rate μ and tangent cone $\hat{C}$ of M so that there exists a continuous family of topological embeddings $\iota_t : N \to M$ with $\iota_0(N) = N$ and $\iota_1(N) = N'$, so that $\iota_t(\hat{x}) = \hat{x}$ for all $t \in [0, 1]$ and so that $\hat{\iota}_t := \iota_t|_{\hat{N}}$ is a smooth family of embeddings $\hat{N} \to X$ with $\hat{\iota}_0(\hat{N}) = \hat{N}$ and $\hat{\iota}_1(\hat{N}) = \hat{N}' := N' \backslash \{\hat{x}\}$.

We will now identify the moduli space of CS complex deformations of a CS complex surface in a Calabi–Yau manifold M with the kernel of a nonlinear partial differential operator.

Proposition 5.5.1. *Let N be a conically singular complex surface at $\hat{x}$ with rate μ and cone C inside a Calabi–Yau four-fold M. Write $\hat{N} := N \backslash \{\hat{x}\}$. Let $\hat{V} \subseteq \nu_M(\hat{N}) \otimes \mathbb{C}$ be the open set and $\hat{\Xi} : \hat{V} \to \hat{T}$ the diffeomorphism defined in the tubular neighbourhood theorem 5.2.4. For $v \in C^{\infty}_{\mathrm{loc}}(\hat{V})$ write $\Xi_v := \Xi \circ v$, and define $\hat{N}_v := \Xi_v(\hat{N})$. Then the moduli space of CS complex deformations of N in M, $\hat{\mathcal{M}}^{\mathrm{cx}}_{\mu}(N)$, is isomorphic near N*

to the kernel of

$$\hat{G} : C^{\infty}_{\mu}(\hat{V} \otimes \mathbb{C}) \to C^{\infty}_{\text{loc}}(\Lambda^1 \hat{N} \otimes T^*M|_{\hat{N}} \otimes \mathbb{C}),$$
$$v \mapsto *_{\hat{N}} \Xi^*_v(\sigma|_{\hat{N}_v}), \tag{5.5.1}$$

where σ was defined in Proposition 3.4.2. Moreover, the kernel of $\hat{G}$ is isomorphic to the kernel of its linear part given by the map

$$C^{\infty}_{\mu}(\nu_M(\hat{N}) \otimes \mathbb{C}) \to C^{\infty}_{\text{loc}}(\Lambda^{1,0}\hat{N} \otimes \nu^{*1,0}_M(\hat{N}) \oplus \Lambda^{0,1}\hat{N} \otimes \nu^{*0,1}_M(\hat{N})),$$
$$v \mapsto -\partial^*(v \lrcorner \Omega) - \bar{\partial}^*(v \lrcorner \overline{\Omega}). \tag{5.5.2}$$

The kernel of (5.5.2) *is isomorphic to*

$$\{v \in C^{\infty}_{\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \,|\, \bar{\partial} v = 0 = \bar{\partial}^* w\}. \tag{5.5.3}$$

Proof. By definition of σ we see that normal vector fields in the kernel of $\hat{G}$ correspond to complex deformations of $\hat{N}$, and a similar argument to Proposition 5.2.5 shows that weighted smooth sections of $\nu_M(\hat{N}) \otimes \mathbb{C}$ give conically singular deformations of $\hat{N}$ as required. The linear part of $\hat{G}$ follows from Proposition 3.4.4, which was a local argument, and similarly that the kernel of $\hat{G}$ is equal to the kernel of its linear part follows from the local argument of Lemma 3.4.6. Finally, that the kernel of (5.5.2) is equal to (5.5.3) follows from Proposition 3.4.4, where we proved that

$$\partial^*(v \lrcorner \Omega) = 0 \iff \bar{\partial}(\pi_{1,0}(v)) = 0,$$

where $\pi_{1,0} : \nu_M(\hat{N}) \otimes \mathbb{C} \to \nu^{1,0}_M(\hat{N})$ and the isomorphism of Proposition 3.3.1

$$\nu^{0,1}_M(\hat{N}) \cong \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}).$$

□

This proposition allows us to prove that the CS complex deformations of a conically singular complex surface are unobstructed. This theorem is a generalisation of Theorem 3.4.7 to conically singular submanifolds.

Theorem 5.5.2. *Let N be a conically singular complex surface at $\hat{x}$ with rate $\mu \in (1,2)$ and cone C inside a Calabi–Yau four-fold M. The moduli space of CS complex deformations of N in M, $\hat{\mathcal{M}}^{\mathrm{cx}}_{\mu}(N)$ given in Definition 5.5.1, is a smooth manifold of dimension*

$$\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial} + \dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial}^* = 2\dim_{\mathbb{C}}\mathrm{Ker}\ \bar{\partial}, \tag{5.5.4}$$

where

$$\bar{\partial} : C^{\infty}_{\mu}(\nu^{1,0}_M(\hat{N})) \to C^{\infty}_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \tag{5.5.5}$$

$$\bar{\partial}^* : C^{\infty}_{\mu}(\Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to C^{\infty}_{\mathrm{loc}}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \tag{5.5.6}$$

Proof. By Proposition 5.5.1 the moduli space of CS complex deformations of N in M can be identified with the kernels of the operators (5.5.5) and (5.5.6). Equation (5.5.4) follows from Corollary 3.4.5. □

To compare CS complex and Cayley deformations of a CS complex surface, we require the following result.

Proposition 5.5.3. *Let N be a CS complex surface at $\hat{x}$ with cone C and rate $\mu \in (1,2)$ in a Calabi–Yau four-fold M. Write $\hat{N} := N \backslash \{\hat{x}\}$. Then $w \in L^2_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}))$ is an infinitesimal CS Cayley deformation of $\hat{N}$ if, and only if, it is an infinitesimal complex deformation of $\hat{N}$. That is, $(\bar{\partial} + \bar{\partial}^*)w = 0$ if, and only if, $\bar{\partial}w = 0 = \bar{\partial}^* w$.*

Proof. Suppose that $w \in L^2_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}))$ satisfies $\bar{\partial}w = -\bar{\partial}^* w$ for $\mu \in (1,2)$. Then $\bar{\partial}^*\bar{\partial}w = 0$. We will check whether

$$\int_{\hat{N}} \langle \bar{\partial}u, v\rangle \ \mathrm{vol}_{\hat{N}} = \int_{\hat{N}} \langle u, \bar{\partial}^* v\rangle \ \mathrm{vol}_{\hat{N}},$$

holds for $u \in L^2_{1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}))$ and $v \in L^2_{1,\mu-1}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}))$, that is, whether the integrals on both sides converge. Let ρ be a radius function for N. We have that

$$\int_{\hat{N}} \langle \bar{\partial}u, v\rangle \ \mathrm{vol}_{\hat{N}} = \int_{\hat{N}} \langle \rho^{1-\mu-2}\bar{\partial}u, \rho^{\mu+3-2}v\rangle \ \mathrm{vol}_{\hat{N}} \leq \|\bar{\partial}u\|_{2,\mu-1}\|v\|_{2,-\mu-3},$$

by Hölder's inequality. This is finite since

$$|\rho^{\mu+3}v| \leq |\rho^{1-\mu}v|,$$

since $\mu \in (1,2)$. Similarly,

$$\int_{\hat{N}} \langle u, \bar{\partial}^* v\rangle \ \mathrm{vol}_{\hat{N}} = \int_{\hat{N}} \langle \rho^{-\mu-2}u, \rho^{\mu+4-2}\bar{\partial}^* v\rangle \ \mathrm{vol}_{\hat{N}} \leq \|u\|_{2,\mu}\|\bar{\partial}^* v\|_{2,-\mu-4},$$

which again is finite since

$$|\rho^{\mu+4}\bar{\partial}^* v| \leq |\rho^{2-\mu}\bar{\partial}^* v|,$$

for $\mu \in (1,2)$. Therefore

$$\|\bar{\partial}w\|^2_{L^2} = \int_{\hat{N}} \langle \bar{\partial}w, \bar{\partial}w\rangle \ \mathrm{vol}_{\hat{N}} = \int_{\hat{N}} \langle w, \bar{\partial}^*\bar{\partial}w\rangle \ \mathrm{vol}_{\hat{N}} = 0,$$

and so $\bar{\partial}w = 0$. □

This allows us to find that CS complex and Cayley deformations of a CS complex surface in a Calabi–Yau four-fold are the same.

Corollary 5.5.4. *Let N be a CS complex surface inside a Calabi–Yau four-fold M. Then the moduli space of CS Cayley deformations of N in M is isomorphic to the moduli space of CS complex deformations of N in M.*

Proof. There are no infinitesimal CS Cayley deformations of N by Proposition 5.5.3, i.e., no $w \in C^\infty_\mu(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N}))$ satisfying

$$\bar{\partial}w = -\bar{\partial}^* w,$$

where $\bar{\partial}w \neq 0$. Comparing the expected dimension of the moduli space of CS Cayley deformations of N in M, computed in Theorem 5.4.4, to the dimension of the moduli space of CS complex deformations of N in M, computed in Theorem 5.5.2 we see that these spaces must be the same, since any CS complex deformation of N is a Cayley deformation of N. □

Chapter 6

Index theory: Calculations and comparisons

In this chapter, we apply the theory of Chapter 4 to the first order linear elliptic operators D and $\bar{\partial}+\bar{\partial}^*$ featured in the analysis of Chapters 3 and 5. We also explicitly calculate the dimension of the space of infinitesimal complex and Cayley deformations of three two-dimensional complex cones in $\mathbb{C}^4$.

6.1 Introduction

Let Y be a CS Cayley submanifold of a $Spin(7)$-manifold X with nonsingular part $\hat{Y}$ and let N be a CS complex surface inside a Calabi–Yau manifold M with nonsingular part $\hat{N}$. In this chapter, we will be interested in the index of the operators

$$D : L^p_{k+1,\mu}(\nu_X(\hat{Y})) \to L^p_{k,\mu-1}(E), \tag{6.1.1}$$

defined in Proposition 3.2.3 on sections with compact support and extended by density to the above spaces, and

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\mu'}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\mu'-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})). \tag{6.1.2}$$

In Section 6.2 we will explain why we can apply Theorem 4.3.2 to (6.1.1) and (6.1.2) to see that they are Fredholm if, and only if, $\mu \in \mathbb{R}\backslash\mathcal{D}$ and $\mu' \in \mathbb{R}\backslash\mathcal{D}'$ respectively for some discrete sets $\mathcal{D}$ and $\mathcal{D}'$ of exceptional weights. In Propositions 6.2.1 and 6.2.2 we will describe the sets $\mathcal{D}$ and $\mathcal{D}'$ of exceptional weights in terms of eigenproblems on the links of Cayley and complex cones. Propositions 6.2.1 and 6.2.2 both take roughly the following form.

Proposition. *Suppose that Y is a CS Cayley submanifold of a $Spin(7)$-manifold with cone C and let $\mu \in \mathbb{R}$. Then $\mu \in \mathcal{D}$ if, and only if, there exists a nontrivial normal vector field v on the link L of C satisfying*

$$D_L v = -\mu v,$$

where D_L is a first order linear differential operator on L and $\mathcal{D}$ is the set of exceptional weights for the operator (6.1.1) *or* (6.1.2).

In Section 6.3, we will consider deformations of two-dimensional complex cones in $\mathbb{C}^4$, both as a Cayley submanifold and a complex submanifold of $\mathbb{C}^4$. In particular, we will consider Cayley deformations of the cone that are themselves cones. The (real) link of such a complex cone is an associative submanifold of S^7 with its nearly parallel G_2-structure inherited from the Euclidean $Spin(7)$-structure on $\mathbb{C}^4$, and so deforming the cone as a complex or Cayley cone in $\mathbb{C}^4$ is equivalent to deforming the link of the cone as an associative submanifold. Homogeneous associative submanifolds of S^7 were classified by Lotay [38], using the classification of homogeneous submanifolds of S^6 of Mashimo [41]. The deformation theory of these submanifolds was studied by Kawai [30], who explicitly calculated the dimension of the space of infinitesimal associative deformations of these explicit examples using techniques from representation theory. Motivated by these calculations, we will apply the analysis of Chapter 5 to compute the dimension of the space of infinitesimal Cayley conical deformations of the complex cones with these links, and check that these calculations match. We will be able to see explicitly which infinitesimal deformations correspond to complex deformations of the cone and which are Cayley but not complex deformations. In particular we will see that complex infinitesimal deformations and Cayley infinitesimal deformations of a two-dimensional complex submanifold of a Calabi–Yau four-fold are not the same in general.

In Section 6.4, we will apply the Atiyah–Patodi–Singer index theorem 4.4.1 to the operator (6.1.2). Using the version of the Atiyah–Patodi–Singer theorem deduced in Proposition 4.4.5, we will prove Theorem 6.4.1 which allows us to compare the dimension of the moduli space of CS complex deformations of a conically singular complex surface to what we will think of as the dimension of the moduli space of all complex deformations of a CS complex surface in a Calabi–Yau four-fold based on Kodaira's theorem 3.1.1. We will then proceed to perform some calculations of the quantities appearing in this theorem for some examples.

We will close this chapter, and this thesis, with some concluding remarks on the results in this thesis and some ideas for future research in Section 6.5.

6.2 Finding the exceptional weights for the operators D and $\bar{\partial} + \bar{\partial}^*$

In this section we will find the set $\mathcal{D}$ of exceptional weights for which the linear elliptic operators (6.1.1) and (6.1.2) that appeared in Chapter 5 are not Fredholm. To do this we will study these operators acting on cones in $\mathbb{R}^8$. We will see that the exceptional weights are actually eigenvalues for differential operators on the links of these cones.

6.2.1 Nearly parallel G_2 structure on S^7

We can consider $\mathbb{R}^8$ as a cone with link S^7. Let (Φ_0, g_0) be the Euclidean $Spin(7)$-structure (as given in Definition 1.2.3). Define a three-form φ on S^7 by the following relation:

$$\Phi_0|_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 * \varphi|_p. \tag{6.2.1}$$

Then (φ, g) is a G_2-structure on S^7 (here g is the standard round metric on S^7). Notice that this G_2-structure is not torsion-free, however, since Φ_0 is closed we have that

$$d\varphi = 4 * \varphi. \tag{6.2.2}$$

G_2-structures (φ, g) satisfying (6.2.2) are called *nearly parallel*.

6.2.2 Exceptional weights for the operator D

Let Y be a CS Cayley submanifold at $\hat{x}$ with rate μ and cone C of a $Spin(7)$-manifold X and write $\hat{Y} := Y \backslash \{\hat{x}\}$. Consider the linear elliptic operator on $\hat{Y}$ given by

$$\begin{aligned} D : C_0^\infty(\nu_X(\hat{Y})) &\to C_0^\infty(E), \\ v &\mapsto \sum_{i=1}^{4} \pi_7(e^i \wedge (\nabla^\perp_{e_i} v)^\flat), \end{aligned} \tag{6.2.3}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame for $T\hat{Y}$ with dual coframe $\{e^1, e^2, e^3, e^4\}$, Λ^2_7 is the seven-dimensional irreducible representation of $Spin(7)$ on two-forms with $\pi_7 : \Lambda^2 X \to \Lambda^2_7$ and $\Lambda^2_7|_{\hat{Y}} = \Lambda^2_+ \hat{Y} \oplus E$.

We will now describe the set of exceptional weights for D in terms of an eigenvalue problem on the link of C.

Proposition 6.2.1. *Let Y be a CS Cayley submanifold at $\hat{x}$ with cone C and rate μ of a $Spin(7)$-manifold X. Write $\hat{Y} := Y \backslash \{\hat{x}\}$. Let $\mathcal{D}_D$ denote the set of $\lambda \in \mathbb{R}$ for which*

$$D : L^p_{k+1,\lambda}(\nu_X(\hat{Y})) \to L^p_{k,\lambda-1}(E),$$

defined in (6.2.3) *is not Fredholm.*

Let $L := C \cap S^7$ be the link of the cone C, a submanifold of S^7. Then $\lambda \in \mathcal{D}_D$ if, and only if, there exists $v \in C^\infty(\nu_{S^7}(L))$ so that

$$D_L v = -\lambda v, \tag{6.2.4}$$

where for $\{e_1, e_2, e_3\}$ *an orthonormal frame for* TL *and* $\nabla^\perp$ *the connection on the normal bundle of* L *in* S^7 *induced by the Levi-Civita connection of the round metric on* S^7*,*

$$\begin{aligned} D_L : C^\infty(\nu_{S^7}(L)) &\to C^\infty(\nu_{S^7}(L)), \\ v &\mapsto \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} v, \end{aligned} \tag{6.2.5}$$

where $\times$ *is the cross product on* S^7 *induced from the nearly parallel* G_2*-structure* (φ, g) *defined by*

$$g(u \times v, w) = \varphi(u, v, w),$$

for any vector fields u, v, w *on* S^7*.*

Remark. The operator D_L can be defined on any associative submanifold of a G_2-manifold, that is, a manifold with torsion-free G_2-structure. Normal vector fields in its kernel correspond to infinitesimal associative deformations of the associative submanifold. This can be deduced from the work of McLean [43, Thm 5-2], however the operator first appears in this form in [1, Eqn 14]. Infinitesimal associative deformations of an associative submanifold of S^7 with its nearly parallel G_2-structure, however, satisfy (6.2.4) with $\lambda = 1$ as shown by Kawai [30, Lem 3.5]. Proposition 6.2.1 can be considered as a different proof of this fact.

Proof. We can apply Theorem 4.3.2 to the operator D. Suppose that ρ is a radius function for Y. Then since the given $Spin(7)$-structure on X approaches the Euclidean $Spin(7)$-structure as we move close to the singular point of Y,

$$\rho^{-1} D \rho^{-1}$$

is asymptotic to the translation invariant differential operator

$$r^{-1} D_0 r^{-1},$$

where D_0 is defined similarly to D but using the Euclidean $Spin(7)$-structure pulled back to X by a $Spin(7)$-coordinate system χ for X around $\hat{x}$ (see Definition 5.2.1).

From the discussion in Section 4.2.3, we see that $\lambda \in \mathcal{D}_D$ if, and only if, there exists a normal vector field $v \in C^\infty(\nu_L(S^7))$ satisfying

$$r^{-1} D_0(r^{\lambda-1} v) = 0,$$

where since $\nu_{rl,\mathbb{R}^8}(C) \cong \nu_{l,S^7}(L)$ for all $r > 0$ we can consider $(r, l) \mapsto (r, r^{\lambda-1} v(l))$ as a normal vector field on the cone. Note also that the induced Euclidean metric on the normal bundle of C in $\mathbb{R}^8$ takes the form $r^2 h$, where h is the metric on the normal bundle of L in S^7 induced from the round metric on S^7.

Let $\{e_1, e_2, e_3\}$ denote an orthonormal frame for TL with dual coframe $\{e^1, e^2, e^3\}$, and denote by Φ_0 the Euclidean Cayley form on $\mathbb{R}^8$ and φ the nearly parallel G_2-structure on S^7 defined in (6.2.1). We compute that

$$\begin{aligned} D_0(r^{\lambda-1} v) &= \pi_7\left(dr \wedge \left(\nabla^\perp_{\frac{\partial}{\partial r}} r^{\lambda-1} v\right)^\flat\right) + \sum_{i=1}^3 \pi_7\left(r e^i \wedge (\nabla^\perp_{\frac{e_i}{r}} r^{\lambda-1} v)^\flat\right) \\ &= \lambda r^{\lambda-2} dr \wedge v^\flat + \lambda r^{\lambda-2} \Phi_0\left(\frac{\partial}{\partial r}, v, \cdot, \cdot\right) \\ &\quad + \sum_{i=1}^3 \left(r^{\lambda-1} e^i \wedge \left(\nabla^\perp_{e_i} v\right)^\flat + r^{\lambda-3} \Phi_0(e_i, \nabla^\perp_{e_i} v, \cdot, \cdot)\right), \end{aligned}$$

since $\nabla^\perp_{\frac{\partial}{\partial r}} v = r^{-1} v$ as the metric on the normal bundle is of the form $r^2 h$. Using the definition of φ in (6.2.1), we find that

$$\begin{aligned} D_0(r^{\lambda-1} v) &= \lambda r^{\lambda-2} dr \wedge v^\flat + \lambda r^{\lambda+1} \varphi(v, \cdot, \cdot) \\ &\quad + \sum_{i=1}^3 \left(r^{\lambda-1} e^i \wedge (\nabla^\perp_{e_i} v)^\flat + r^\lambda dr \wedge \varphi(e_i, \nabla^\perp_{e_i} v, \cdot) + r^{\lambda+1} * \varphi(e_i, \nabla^\perp_{e_i} v, \cdot, \cdot)\right). \end{aligned}$$

Now we wish to replace the musical isomorphism $\flat : \nu_{\mathbb{R}^8}(C) \to \nu^*_{\mathbb{R}^8}(C)$ with the musical isomorphism $\flat_L : \nu_{S^7}(L) \to \nu^*_{S^7}(L)$. Since the metric on $\nu_{\mathbb{R}^8}(C)$ is of the form $r^2 h$, where h is a metric on $\nu_{S^7}(L)$, we find that

$$\begin{aligned} D_0(r^{\lambda-1} v) &= \lambda r^\lambda dr \wedge v^{\flat_L} + \lambda r^{\lambda+1} \varphi(v, \cdot, \cdot) \\ &\quad + \sum_{i=1}^3 \left(r^{\lambda+1} e^i \wedge (\nabla^\perp_{e_i} v)^{\flat_L} + r^\lambda dr \wedge \varphi(e_i, \nabla^\perp_{e_i} v, \cdot) + r^{\lambda+1} * \varphi(e_i, \nabla^\perp_{e_i} v, \cdot, \cdot)\right). \end{aligned}$$

Notice that $E \cong \nu_{S^7}(L)$ via the map

$$\alpha \mapsto \left(\frac{\partial}{\partial r} \lrcorner \alpha\right)^{\sharp_L},$$

where $\sharp_L : \nu^*_{S^7}(L) \to \nu_{S^7}(L)$ is the musical isomorphism, with inverse map

$$v \mapsto \pi_7(dr \wedge v^{\flat_L}).$$

Therefore we see that

$$r^{-1}D_0(r^{\lambda-1}v) = 0 \iff \left(\frac{\partial}{\partial r} \lrcorner r^{-1}D_0(r^{\lambda-1}v)\right)^{\sharp_L} = 0.$$

We find that

$$\left(\frac{\partial}{\partial r} \lrcorner r^{-1}D_0(r^{\lambda-1}v)\right)^{\sharp_L} = r^{\lambda-2}\left(\lambda v + \varphi(e_i, \nabla^{\perp}_{e_i}v, \cdot)^{\sharp}_{L}\right).$$

Since by definition,

$$D_L v = e_i \times \nabla^{\perp}_{e_i} v = \varphi(e_i, \nabla^{\perp}_{e_i} v, \cdot)^{\sharp_L},$$

we see that $\lambda \in \mathcal{D}_D$ if, and only if, there exists $v \in C^\infty(\nu_{S^7}(L))$ such that

$$D_L v = -\lambda v.$$

□

6.2.3 Exceptional weights for the operator $\bar{\partial} + \bar{\partial}^*$

Let N be a CS complex surface with rate μ and cone C inside a Calabi–Yau four-fold M, and write $\hat{N}$ for its nonsingular part. In order to prove an analogous result to Proposition 6.2.1 for the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty_0(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to C^\infty_0(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \tag{6.2.6}$$

we will need some preliminary facts about complex cones.

Definition 6.2.1. Let C be a complex cone in $\mathbb{C}^{n+1}$, with real link $L := C \cap S^{2n+1}$. Consider the Hopf projection $p : S^{2n+1} \to \mathbb{C}P^n$. Define the *complex link* Σ of C to be the image of L under the Hopf projection, i.e., $\Sigma := p(L) \subseteq \mathbb{C}P^n$.

The real link of a complex cone C is a circle bundle over the complex link of C. Thinking of L as $S^1 \times \Sigma$, we can find a globally defined vector field on L that we can think of as being tangent to S^1 in this product.

Definition 6.2.2. Let C be a complex cone in $\mathbb{C}^{n+1}$, and denote by J the complex structure on $\mathbb{C}^n$. The *Reeb* vector field is defined to be

$$\xi := J\left(r\frac{\partial}{\partial r}\right).$$

Notice that $|\xi|_L = 1$.

If $p|_L : L \to \Sigma$ is the restriction of the Hopf projection to L, then at each $l \in L$, ξ_l spans the kernel of $d\pi|_l : T_l L \to T_{p(l)}\Sigma$.

Definition 6.2.3. Let C be a complex cone in $\mathbb{C}^{n+1}$ with real link L. Let α be a p-form on L. We say that α is *horizontal* if $\xi \lrcorner \alpha = 0$, where ξ is the Reeb vector field. Denote by $\Lambda^p_h L$ the vector bundle of horizontal p-forms on L. Denote by d_h the projection of the exterior derivative onto horizontal forms.

By definition of the Reeb vector field, we see if J is the complex structure on $\mathbb{C}^{n+1}$ then $J(\Lambda^1_h L) \subseteq \Lambda^1_h L$. So we have a well-defined splitting $\Lambda^1_h L = \Lambda^{1,0}_h L \oplus \Lambda^{0,1}_h L$ of one-forms into the $\pm i$ eigenspaces of J. Define the operator $\bar{\partial}_h$ on functions to be the projection of d_h onto horizontal $(0,1)$-forms.

With these definitions, we may characterise the set of exceptional weights for the operator (6.2.6) in terms of an eigenproblem on the link of a cone.

Proposition 6.2.2. *Let N be a CS complex surface at $\hat{x}$ with rate μ and cone C inside a Calabi–Yau four-fold M. Write $\hat{N} := N \backslash \{\hat{x}\}$. Let $\mathcal{D}$ denote the set of $\lambda \in \mathbb{R}$ for which*

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\lambda}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\lambda-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \qquad (6.2.7)$$

is not Fredholm. Let L denote the real link of C. Then $\lambda \in \mathcal{D}$ if, and only if, there *exist* $v \in C^\infty(\nu^{1,0}_{S^7}(L))$ *and* $w \in C^\infty(\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L))$ *so that*

$$\bar{\partial}_h v = (\lambda + 2 - i\nabla_\xi)w, \tag{6.2.8}$$

$$\bar{\partial}^*_h w = \frac{1}{2}(\lambda + i\nabla_\xi)v, \tag{6.2.9}$$

where ξ *is the Reeb vector field on* L*. Here* ∇ *acts on* $\Lambda^{0,1}_h L$ *as the Levi-Civita connection of the metric on* L *and on* $\nu^{1,0}_{S^7}(L)$ *as the normal part of the Levi-Civita connection on* S^7*.*

Proof. Similarly to the proof of Proposition 6.2.1, if ρ is a radius function for N then we can see that

$$\bar{\partial} + \bar{\partial}^* \rho^2,$$

on $\hat{N}$ is asymptotically translation invariant to

$$\bar{\partial}_C + \bar{\partial}^*_C r^2,$$

on the cone C where this time we take a metric on $\nu_{\mathbb{C}^4}(C)$ that is independent of r. If $v \in C^\infty(\nu_{S^7}(L) \otimes \mathbb{C})$ we can think of $r^\mu v$ as a complexified normal vector field on C, and moreover the complex structure J on $\mathbb{C}^4$ induces a splitting

$$\nu_{S^7}(L) \otimes \mathbb{C} = \nu^{1,0}_{S^7}(L) \oplus \nu^{0,1}_{S^7}(L),$$

of the complexified normal bundle of L in S^7 into holomorphic and antiholomorphic parts (the i and $-i$ eigenspaces of J respectively). Also, by definition of the Reeb vector field, if we take $\theta \in C^\infty(\Lambda^1 L)$ to be the dual one-form to ξ we have that $dr - ir\theta$ is a $(0,1)$-form on C. Since $\Lambda^2 C \cong \Lambda^2 L \oplus dr \wedge \Lambda^1 L$, we can see that a $(0,2)$-form on C must be of the form

$$r^\mu (dr - ir\theta) \wedge w,$$

where $w \in C^\infty(\Lambda^{0,1}_h L)$. By the discussion in Section 4.2.3, we deduce that $\lambda \in \mathcal{D}$ if, and only if, there exists $v \in C^\infty(\nu_{S^7}(L))$ and $w \in C^\infty(\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L))$ so that

$$\bar{\partial}_C(r^\lambda v) + \bar{\partial}^*_C \left(r^{\lambda+2} \left(\frac{dr}{r} - i\theta \right) \wedge w \right) = 0,$$

where θ is dual to the Reeb vector field ξ. We can calculate that

$$d_C(r^\lambda v) = \lambda r^{\lambda-1} dr \otimes v + r^\lambda \theta \otimes \nabla_\xi v + r^\lambda d_h v,$$

and therefore

$$\bar{\partial}_C(r^\lambda v) = r^\lambda \frac{1}{2}\left(\frac{dr}{r} - i\theta\right) \otimes (\lambda + i\nabla_\xi)v + r^\lambda \bar{\partial}_h v. \tag{6.2.10}$$

We also have that

$$\begin{aligned}\bar{\partial}_C^*\left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) &= -\frac{\partial}{\partial r} \lrcorner \nabla_{\frac{\partial}{\partial r}}\left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) \\ &\quad - \frac{1}{r^2}\xi \lrcorner \nabla_\xi \left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) - r^\lambda \left(\frac{dr}{r} - i\theta\right)\bar{\partial}_h^* w,\end{aligned}$$

where since w is a horizontal $(0,1)$-form we see that any term gained from applying $\bar{\partial}_h^*$ to $r^{-1}dr - i\theta$ must be a multiple of w at each point and therefore will vanish under exterior product with w. We have that

$$-\frac{\partial}{\partial r} \lrcorner \nabla_{\frac{\partial}{\partial r}}\left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) = -(\lambda+1) r^\lambda w,$$

and

$$-\xi \lrcorner \nabla_\xi \left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) = -r^{\lambda+2} w + i r^{\mu+2} \nabla_\xi w,$$

since $\nabla_\xi dr = r\theta$ where ∇ is the Levi-Civita connection of the cone metric. We deduce that

$$\bar{\partial}_C^*\left(\left(\frac{dr}{r} - i\theta\right) \wedge r^{\lambda+2} w\right) = -r^\lambda(\lambda + 2 - i\nabla_\xi) w - r^\lambda \left(\frac{dr}{r} - i\theta\right)\bar{\partial}_h^* w. \tag{6.2.11}$$

Equating (6.2.10) and minus (6.2.11), we find that $\lambda \in \mathcal{D}$ if, and only if, there exist $v \in C^\infty(\nu_{S^7}^{1,0}(L))$ and $w \in C^\infty(\Lambda_h^{0,1} L \otimes \nu_{S^7}^{1,0}(L))$ satisfying

$$\begin{aligned}\bar{\partial}_h v &= (\lambda + 2 - i\nabla_\xi) w, \\ \bar{\partial}_h^* w &= \frac{1}{2}(\lambda + i\nabla_\xi) v,\end{aligned}$$

as claimed. □

6.2.4 An eigenproblem on the complex link

In Proposition 6.2.2 we characterised the set of exceptional weights $\mathcal{D}$ for which the operator (6.2.7) is not Fredholm in terms of an eigenproblem on the real link of a complex cone C. In this section we will introduce a trick used by Lotay [39, §6] to study an eigenvalue problem on the link of a coassociative cone which is a circle bundle over a complex curve in $\mathbb{C}P^2$. This will allow us to give an equivalent eigenvalue problem to (6.2.8)-(6.2.9) on the real link of C completely in terms of operators and vector bundles on the complex link of C.

Let C be a complex cone with real link L and complex link Σ. Suppose we have a problem of the following form: Find all of the functions f on L that satisfy

$$\mathcal{L}_\xi f = imf, \quad \bar{\partial}_h f = 0, \tag{6.2.12}$$

for some $m \in \mathbb{Z}$, where ξ is the Reeb vector field on C.

We would like to understand the relationship between the operator $\bar{\partial}_h$ on the real link of C and $\bar{\partial}_\Sigma$ on the complex link C.

Definition 6.2.4. Call a function, horizontal vector field or horizontal differential form f on L *basic* if

$$\mathcal{L}_\xi f = 0.$$

Basic functions, forms and vector fields are special because they are in one-one correspondence with functions, forms and vector fields on Σ. It follows from [49, Lem 1] that $\bar{\partial}_h$ acting on basic functions, forms or vector fields on L is equivalent to $\bar{\partial}_\Sigma$ acting on functions, forms or vector fields on Σ. In Problem (6.2.12), when $m \neq 0$, f is not basic. However, a simple trick allows us to pretend that f is basic.

By the definition of the complex link, we may identify the cone C with the vector bundle $\mathcal{O}_{\mathbb{C}P^3}(-1)|_\Sigma$, that is, the tautological line bundle over $\mathbb{C}P^3$ restricted to Σ. This is then a trivial (real) line bundle over L and therefore has a global section given by the map $x \mapsto s(x) = x$ for $x \in L$. It is easy to see that $\mathcal{L}_\xi s = is$, and therefore

$$f \otimes s^{-m},$$

is a section of the vector bundle $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$ satisfying

$$\mathcal{L}_\xi(f \otimes s^{-m}) = 0,$$

and therefore pushes down to a well-defined section of the vector bundle $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$. Since $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$ is a trivial line bundle over L, we can still consider $f \otimes s^{-m}$ as a function on L. Therefore we can rephrase Problem (6.2.12) as: Find all basic sections $\tilde{f}$ of $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma \to L$ satisfying

$$\bar{\partial}_h \tilde{f} = 0.$$

This is now equivalent to finding the sections $\tilde{f}$ of $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma \to \Sigma$ that satisfy

$$\bar{\partial}_\Sigma \tilde{f} = 0.$$

Therefore we have reduced Problem (6.2.12) to asking: How many holomorphic sections of the line bundle $\mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$ are there?

This problem is easily solved using the Hirzebruch–Riemann–Roch Theorem [17, Thm 5.1.1].

Theorem 6.2.3 (Hirzebruch–Riemann–Roch)**.** *Let Σ be a Riemann surface and let E be a vector bundle over Σ. Denote by $h^0(\Sigma, E)$ the dimension of the space of holomorphic sections of E. Let K_Σ denote the canonical bundle of Σ. Then*

$$h^0(\Sigma, E) = h^0(\Sigma, E^* \otimes K_\Sigma) + \deg(E) + \mathrm{rk}(E)(1-g),$$

where $\deg(E)$ *is the degree of the vector bundle E,* $\mathrm{rk}(E)$ *is the rank of the vector bundle and g is the genus of Σ.*

We will now apply the trick that we described above to rephrase the eigenvalue problem (6.2.8)-(6.2.9) on the real link of a cone as an eigenvalue problem on the complex link on a cone.

Proposition 6.2.4. *Let C be a complex cone in $\mathbb{C}^4$ with real link L and complex link Σ. Then given $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}$, pairs $v \in C^\infty(\nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma)$ and $w \in C^\infty(\Lambda^{0,1}\Sigma \otimes \nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma)$ so that*

$$\bar{\partial}_\Sigma v = (\lambda + 3 + m)w, \tag{6.2.13}$$

$$\bar{\partial}^*_\Sigma w = \frac{1}{2}(\lambda - 1 - m)v, \tag{6.2.14}$$

are in a one-one correspondence with pairs $\tilde{v} \in C^\infty(\nu^{1,0}_{S^7}(L))$ and $\tilde{w} \in C^\infty(\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L))$ satisfying

$$\mathcal{L}_\xi \tilde{v} = im\tilde{v}, \quad \mathcal{L}_\xi \tilde{w} = im\tilde{w},$$

where ξ is the Reeb vector field, and the eigenvalue problem (6.2.8)-(6.2.9).

Proof. We can pull back such v and w to basic sections of $\nu^{1,0}_{S^7}(L) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$ and $\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma$ over L respectively. As mentioned above, these sections are in one-one correspondence with sections $\tilde{v}$ and $\tilde{w}$ of $\nu^{1,0}_{S^7}(L)$ and $\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L)$ respectively satisfying

$$\mathcal{L}_\xi \tilde{v} = im\tilde{v}, \quad \mathcal{L}_\xi \tilde{w} = im\tilde{w}. \tag{6.2.15}$$

So we see that v and w are in one-one correspondence with $\tilde{v}$ and $\tilde{w}$ satisfying (6.2.15), and $\tilde{v}$ and $\tilde{w}$ satisfy

$$\bar{\partial}_h \tilde{v} = (\lambda + 3 + m)\tilde{w},$$

$$\bar{\partial}^*_h \tilde{w} = \frac{1}{2}(\lambda - 1 - m)\tilde{v}.$$

Finally, by [49, Lemma 3, §5], we see that any horizontal vector field X on S^7 viewed as a circle bundle over $\mathbb{C}P^3$ satisfies

$$\text{horizontal part}(\nabla_X \xi) = JX.$$

and so for any vector field of type $(1,0)$, we have that

$$\mathcal{L}_\xi v = \nabla_\xi v - \nabla_v \xi = \nabla_\xi v - iv.$$

Therefore (6.2.15) implies that

$$\nabla_\xi \tilde{v} = i(m+1)\tilde{v}, \quad \nabla_\xi \tilde{w} = i(m+1)\tilde{w},$$

and therefore

$$\begin{aligned}\bar{\partial}_h \tilde{v} &= (\lambda + 2 - i\nabla_\xi)\tilde{w},\\ \bar{\partial}_h^* \tilde{w} &= \frac{1}{2}(\lambda + i\nabla_\xi)\tilde{v},\end{aligned}$$

as required. □

6.3 Cone deformations: some calculations

Let C be a two-dimensional complex cone in $\mathbb{C}^4$. Let v be a normal vector field on C. If v is sufficiently small, we can apply the tubular neighbourhood theorem for cones 5.2.3 to identify v with a deformation of C. Write $v = v_1 \oplus v_2$, where $v_1 \in C^\infty(\nu^{1,0}_{\mathbb{C}^4}(C))$ and $v_2 \in C^\infty(\nu^{0,1}_{\mathbb{C}^4}(C))$. We know from Proposition 3.3.3 that v is an infinitesimal Cayley deformation of C if, and only if,

$$\bar{\partial} v_1 + \frac{1}{4}\bar{\partial}^*(v_2 \lrcorner \overline{\Omega}_0^\sharp) = 0,$$

where Ω_0 is the standard holomorphic volume form on $\mathbb{C}^4$ and $\sharp$ denotes the musical isomorphism $\nu^{*0,1}_{\mathbb{C}^4}(C) \to \nu^{1,0}_{\mathbb{C}^4}(C)$. Moreover by Proposition 3.4.2 v is an infinitesimal complex deformation of C if, and only if,

$$\bar{\partial} v_1 = 0 = \bar{\partial}^*(v_2 \lrcorner \overline{\Omega}_0^\sharp).$$

We would like to know what properties v must have in order for the deformation of C corresponding to v to be a cone itself. By Proposition 5.2.3, in which we constructed the tubular neighbourhood of a cone, we constructed a map

$$\Xi_C : V_C \to T_C,$$

where $V_C \subseteq \nu_{\mathbb{R}^8}(C)$ contains the zero section and $T_C \subseteq \mathbb{C}^4$ contains C. We constructed an action of $\mathbb{R}_+$ on $\nu_{\mathbb{C}^4}(C)$ satisfying $|t \cdot v| = t|v|$, and the map Ξ_C satisfies

$$\Xi_C(tr, l, tr \cdot v(r, l)) = t\Xi_C(r, l, v(r, l)).$$

Therefore, to guarantee that $\Xi_C \circ v$ is a cone in $\mathbb{C}^4$, we must have that $v(r, l) = r \cdot \hat{v}(l)$, for some $\hat{v} \in C^\infty(\nu_{S^7}(L))$. In this case,

$$\Xi_C(r, l, v(r, l)) = r\Xi_C(1, l, \hat{v}(l)),$$

for all $r \in \mathbb{R}_+$. Choosing a metric on $\nu_{\mathbb{C}^4}(C)$ that is independent of r, we see that $r \cdot \hat{v}(l) = r\hat{v}(l)$.

Therefore the dimension of the space of infinitesimal conical Cayley deformations of C is equal to the dimensions of the spaces of solutions to the eigenproblems (6.2.4) and (6.2.8)-(6.2.9) with $\lambda = 1$. As remarked after the statement of Proposition 6.2.1, this particular eigenspace can be identified with the space of infinitesimal associative deformations of the link of the cone in S^7 with its nearly parallel G_2-structure. This problem was studied by Kawai [30], who computed the dimension of these spaces for a range of examples. In terms of the work done here, this is equivalent to solving the eigenproblem (6.2.4) when $\lambda = 1$. In this section, we will study the eigenproblem (6.2.8)-(6.2.9) for the three examples of complex cones that were studied by Kawai in his paper. Our analysis will allow us to see directly the difference between the infinitesimal conical Cayley and complex deformations of a cone, and we hope that the complex geometry will make these calculations simpler.

Here we will describe the three complex cones in $\mathbb{C}^4$ whose infinitesimal Cayley and complex deformations we will study. Alongside each example, we will quote Kawai's calculation of the dimension of the space of infinitesimal associative deformations of the link in S^7, which in our notation will be equal to the dimension of the space of infinitesimal conical Cayley deformations of these cones in $\mathbb{C}^4$. In Section 6.3.4 we will apply the analysis in this thesis to compute the dimension of the space of infinitesimal conical complex deformations of these cones in $\mathbb{C}^4$. The links of these cones are all homogeneous associative submanifolds of S^7.

6.3.1 Example 1: $L_1 = S^3$

The first example is the simplest, being just a vector subspace (with the zero vector removed). We take

$$C_1 := \mathbb{C}^2 \backslash \{0\}, \quad L_1 := S^3, \quad \Sigma_1 := \mathbb{C}P^1,$$

where C_1 is the complex cone, L_1 is the real link of C_1 and Σ_1 is the complex link of C_1.

Proposition 6.3.1 ([30, §6.4.1]). *The space of infinitesimal associative deformations of L_1 in S^7 has dimension twelve.*

6.3.2 Example 2: $L_2 \cong SU(2)/\mathbb{Z}_2$

Our second example is a little less trivial. Take

$$C_2 := \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_4 = 0, z_1^2 + z_2^2 + z_3^2 = 0\}.$$

Then it can be shown [30, Ex 6.6] that the link of C_2, L_2, is isomorphic to the quotient group $SU(2)/\mathbb{Z}_2$.

The complex link of C_2 is

$$\Sigma_2 := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid z_0 = 0, z_1^2 + z_2^2 + z_3^2 = 0\}.$$

Proposition 6.3.2 ([39, Cor 5.12], [30, Prop 6.26]). *The space of infinitesimal associative deformations of L_2 in S^7 has dimension twenty-two.*

6.3.3 Example 3: $L_3 \cong SU(2)/\mathbb{Z}_3$

Our third example is the most complicated to state, but is certainly the most interesting.

Define the cone C_3 to be the cone over the submanifold L_3 of S^7 which is defined as follows: consider the following action of $SU(2)$ on $\mathbb{C}^4$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} a^3 z_1 + \sqrt{3}a^2 b z_2 + \sqrt{3}ab^2 z_3 + b^3 z_4 \\ -\sqrt{3}a^2\bar{b}z_1 + a(|a|^2 - 2|b|^2)z_2 + b(2|a|^2 - |b|^2)z_3 + \sqrt{3}\bar{a}b^2 z_4 \\ \sqrt{3}a\bar{b}^2 z_1 - \bar{b}(2|a|^2 - |b|^2)z_2 + \bar{a}(|a|^2 - 2|b|^2)z_3 + \sqrt{3}\bar{a}^2 b z_4 \\ -\bar{b}^3 z_1 + \sqrt{3}\bar{a}\bar{b}^2 z_2 - \sqrt{3}\bar{a}^2\bar{b}z_3 + \bar{a}^3 z_4 \end{pmatrix},$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. We define L_3 to be the orbit of the above action around the point $(1,0,0,0)^T$, that is,

$$L_3 := \begin{pmatrix} a^3 \\ -\sqrt{3}a^2\bar{b} \\ \sqrt{3}a\bar{b}^2 \\ -\bar{b}^3 \end{pmatrix},$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. We see that for

$$\mathbb{Z}_3 := \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} \in SU(2) \,|\, \zeta^3 = 1 \right\},$$

L_3 is invariant under the action of $\mathbb{Z}_3$, therefore $L_3 \cong SU(2)/\mathbb{Z}_3$. The complex link of the cone C_3 over L_3 is

$$\Sigma_3 := \{[x^3 : \sqrt{3}x^2 y : \sqrt{3}xy^2 : y^3] \in \mathbb{C}P^3 \,|\, [x : y] \in \mathbb{C}P^1\},$$

which is known as the *twisted cubic* in $\mathbb{C}P^3$.

This is a particularly interesting example for the following reason [38, Ex 5.8]. Define $L_3(\theta)$ to be the orbit of the above group action around the point $(\cos\theta, 0, 0, \sin\theta)^T$. Then $L_3(\theta)$ is associative for $\theta \in [0, \frac{\pi}{4}]$. As noted above, $L_3(0) = L_3$ is the real link of a complex cone, however, $L_3(\frac{\pi}{4})$ is the link of a special Lagrangian cone. Therefore there exists a family of Cayley cones in $\mathbb{C}^4$, including both a complex cone and a special Lagrangian cone, that are related by a group action.

Proposition 6.3.3 ([30, §6.3.2]). *The space of infinitesimal associative deformations of $L_3(\frac{\pi}{4})$ in S^7 has dimension thirty.*

6.3.4 Calculations

We will now study the eigenvalue problem (6.2.8)-(6.2.9) with $\lambda = 1$ for C_1, C_2 and C_3 defined above. Recall that by Proposition 6.2.4 we can study the eigenproblem (6.2.13)-(6.2.14) with $\lambda = 1$ on the complex link instead to make our calculations easier. We first explain how to count infinitesimal conical complex deformations and infinitesimal conical Cayley but non complex deformations of a complex cone.

Proposition 6.3.4. *Let C be a complex cone in $\mathbb{C}^4$ with real link L and complex link Σ. Infinitesimal complex conical deformations of C in $\mathbb{C}^4$ are given by holomorphic sections of $\nu^{1,0}_{\mathbb{C}P^3}(\Sigma)$. Infinitesimal Cayley conical deformations of C that are not complex are given by $v \in C^\infty(\nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma)$ satisfying*

$$\Delta_{\bar{\partial}_\Sigma} v = -\frac{1}{2} m(4+m) v, \tag{6.3.1}$$

where $-4 < m < 0$.

Proof. We know that infinitesimal complex deformations C will lie in the kernel of $\bar{\partial}_C$ or $\bar{\partial}^*_C$. Recall that Corollary 3.4.5 says that these spaces are isomorphic and so we expect them to have the same dimension. Examining the proof of Proposition 6.2.2 and comparing to Proposition 6.2.4, we see that infinitesimal complex deformations of C are given by holomorphic sections of $\nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(\lambda - 1)|_\Sigma$, and antiholomorphic sections of $\Lambda^{0,1}\Sigma \otimes \nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(-3-\lambda)$. Since infinitesimal conical deformations of C will correspond to $\lambda = 1$ here, we see that infinitesimal complex conical deformations of C correspond to holomorphic sections of

$$\nu^{1,0}_{\mathbb{C}P^3}(\Sigma),$$

and antiholomorphic sections of

$$\Lambda^{0,1}\Sigma \otimes \nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(-4)|_\Sigma \cong \nu^{*1,0}_{\mathbb{C}P^3}(\Sigma),$$

by the adjunction formula [17, Prop 2.2.17] since $K_{\mathbb{C}P^3}|_\Sigma = \mathcal{O}_{\mathbb{C}P^3}(-4)|_\Sigma$. So we see that infinitesimal conical complex deformations of C arise from holomorphic sections of the holomorphic normal bundle of the complex link in $\mathbb{C}P^3$. The dimension of the space of infinitesimal conical complex deformations of C is then equal to the real dimension (or twice the complex dimension) of the space of holomorphic sections of the holomorphic normal bundle of the complex link.

Finally, we see that any remaining infinitesimal conical Cayley deformations of C must satisfy the eigenproblem (6.2.13)-(6.2.14) with $\lambda = 1$ and $m \neq 0, -4$. Applying $\bar{\partial}^*_\Sigma$ to (6.2.13) and using (6.2.14), we see that the remaining infinitesimal conical Cayley deformations of C are given by $v \in C^\infty(\nu^{1,0}_{\mathbb{C}P^3}(\Sigma) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma)$ satisfying

$$\Delta_{\bar{\partial}_\Sigma} v = -\frac{1}{2} m(4+m)v.$$

$\square$

While we can apply the Hirzebruch–Riemann–Roch theorem 6.2.3 to count holomorphic sections of holomorphic vector bundles, solving eigenproblems for the Laplacian acting sections of vector bundles such as (6.3.1) is somewhat more difficult, especially since the degree of the line bundle we consider appears in the eigenvalue itself. Such problems have been studied, however, and we will make use of the following result of López Almorox and Tejero Prieto on eigenvalues of the $\bar{\partial}_\Sigma$-Laplacian acting on sections of holomorphic line bundles over $\mathbb{C}P^1$ equipped with a metric of constant scalar curvature.

Theorem 6.3.5 ([2, Thm 5.1])**.** *Let K be a Hermitian line bundle over Σ, where Σ is $\mathbb{C}P^1$ with metric of constant scalar curvature κ equipped with a unitary harmonic connection ∇_K of curvature $F^{\nabla_K} = -iB\omega_\Sigma$ for some $B \in \mathbb{R}$. Then the spectrum of the operator*

$$2\bar{\partial}^*_\Sigma \bar{\partial}_\Sigma : C^\infty(K) \to C^\infty(K),$$

is the set

$$\left\{ \lambda_q = \frac{\kappa}{2} \left[(q+a)^2 + (q+a)|\deg K + 1| \right] \mid q \in \mathbb{N} \cup \{0\} \right\},$$

where $a = 0$ if $\deg K \geq 0$, $a = 1$ otherwise.

The space of eigensections of $2\bar{\partial}_\Sigma^*\bar{\partial}_\Sigma$ *with eigenvalue* λ_q *is identified with the space of holomorphic sections of*

$$K_\Sigma^{-q} \otimes K,$$

when $\deg K \geq 0$*, or of holomorphic sections of*

$$K_\Sigma^{-q} \otimes K^{-1},$$

when $\deg K < 0$*. Therefore the multiplicity of* λ_q *is*

$$m(\lambda_q) = 1 + |\deg K| + 2q.$$

Example 1: $L_1 = S^3$

To calculate the dimension of the space of infinitesimal conical Cayley deformations of the cone $C_1 = \mathbb{C}^2$, which as real link $L_1 = S^3$ and complex link $\Sigma_1 = \mathbb{C}P^1$, we will apply Proposition 6.3.4. We first calculate the dimension of the space of holomorphic sections of

$$\nu^{1,0}_{\mathbb{C}P^3}(\Sigma_1) = \mathcal{O}_{\mathbb{C}P^3}(1)|_\Sigma \oplus \mathcal{O}_{\mathbb{C}P^3}(1)|_\Sigma,$$

which by the Hirzebruch–Riemann–Roch theorem 6.2.3 has dimension four. Therefore, the dimension of the space of infinitesimal conical complex deformations of C_1 is eight.

Now we study the eigenproblem

$$\Delta_{\bar{\partial}_\Sigma} v = -\frac{1}{2}m(4+m), \tag{6.3.2}$$

for $v \in C^\infty(\nu^{1,0}_{\mathbb{C}P^3}(\Sigma_1) \otimes \mathcal{O}_{\mathbb{C}P^3}(m)|_\Sigma) = C^\infty(\mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma \oplus \mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma)$ and $-4 < m < 0$. We can apply Theorem 6.3.5 to solve (6.3.2) as long as the connection on $\mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma \oplus \mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma$ takes the form

$$\begin{pmatrix} \nabla_1 & 0 \\ 0 & \nabla_2 \end{pmatrix},$$

where ∇_i are connections on $\mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma$. This is the case here, as can be seen from the relation between the connection on the normal bundle of Σ_1 in $\mathbb{C}P^3$ and the connection on the normal bundle of L_1 in S^7 (see [49, Lem 1]) and the fact that the normal bundle of L_1 in S^7 is trivial.

Therefore, by Theorem 6.3.5, solving (6.3.2) reduces to solving the algebraic equation

$$-m(4+m) = 4((q+a)^2 + (q+a)|m+2|),$$

for $m \in \mathbb{Z}$ and $q \in \mathbb{N} \cup \{0\}$ (since the scalar curvature of Σ_1 is eight) with $a = 0$ if $m \geq -1$ and $a = 1$ if $m \leq -2$. It can be checked that this has solution $(q, a, m) = (0, 1, -2)$, and so by Theorem 6.3.5 the dimension of eigensections of (6.3.2) has dimension $2 \times 2 = 4$. So we have a total of twelve infinitesimal conical Cayley deformations of C in $\mathbb{C}^4$.

We sum this up in a proposition.

Proposition 6.3.6. *The real dimension of the space of infinitesimal conical Cayley deformations of C_1 in $\mathbb{C}^4$ is twelve. The real dimension of the space of infinitesimal conical complex deformations of C_1 in $\mathbb{C}^4$ is eight.*

Remark. Recall that the stabiliser of a Cayley plane in $\mathbb{R}^8$ is isomorphic to $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ and that the dimension of $Spin(7)/((SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2)$ is twelve. The stabiliser of a two-dimensional complex plane in $\mathbb{C}^4$ is isomorphic to $U(2) \times U(2)$, and the dimension of $U(4)/(U(2) \times U(2))$ is equal to eight.

Example 2: $L_2 \cong SU(2)/\mathbb{Z}_2$

We now calculate the dimension of the space of infinitesimal conical Cayley deformations of the cone C_2 in $\mathbb{C}^4$ with link $L_2 \cong SU(2)/\mathbb{Z}_2$ and complex link Σ_2 as defined in Section 6.3.2. Again by Proposition 6.3.4 we compute the dimension of the space of holomorphic sections of

$$\nu^{1,0}_{\mathbb{C}P^3}(\Sigma_2) = \mathcal{O}_{\mathbb{C}P^3}(1)|_\Sigma \oplus \mathcal{O}_{\mathbb{C}P^3}(2)|_\Sigma,$$

which by the Hirzebruch–Riemann–Roch theorem 6.2.3 has dimension eight, and so we deduce that the space of infinitesimal conical complex deformations of C_2 has dimension sixteen.

Again since the normal bundle of L_2 in S^7 is trivial we may apply Theorem 6.3.5 to solve the eigenproblem

$$\Delta_{\bar{\partial}_\Sigma} v = -\frac{1}{2}m(m+4), \tag{6.3.3}$$

for $v \in C^\infty(\mathcal{O}_{\mathbb{C}P^3}(m+1)|_\Sigma \oplus \mathcal{O}_{\mathbb{C}P^3}(m+2)|_\Sigma)$ with $-4 < m < 0$. This reduces again to solving the equations for $m \in \mathbb{Z}$ and $q \in \mathbb{N} \cup \{0\}$

$$-m(m+4) = 2((q+a)^2 + (q+a)|2m+3|),$$

with $a = 0$ for $m \geq -1$ and $a = 1$ otherwise, which has solution $(q, a, m) = (0, 1, -2)$ and

$$-m(m+4) = 2((q+a)^2 + (q+a)|2m+5|),$$

with $a = 0$ for $m \geq -2$ and $a = 1$ otherwise, which has solution $(q, a, m) = (1, 0, -2)$. Therefore by Theorem 6.3.5 the dimension of the space of solutions to (6.3.3) has dimension $3 + 3 = 6$. Therefore, the dimension of the space of infinitesimal conical Cayley deformations of C_2 in $\mathbb{C}^4$ is twenty-two.

Proposition 6.3.7. *The real dimension of the space of infinitesimal conical Cayley deformations of C_2 in $\mathbb{C}^4$ is twenty-two. The real dimension of the space of infinitesimal conical complex deformations of C_2 in $\mathbb{C}^4$ is sixteen.*

Remark. The dimension of $Spin(7)/SU(4)$ is six, which implies that the six Cayley but not complex infinitesimal conical deformations of C_2 are just rigid motions induced by the action of $Spin(7)$ on $\mathbb{R}^8$.

Example 3: $L_3 \cong SU(2)/\mathbb{Z}_3$

Finally, we compute the dimension of the space of infinitesimal conical Cayley deformations of C_3 in $\mathbb{C}^4$, which has real link $L_3 \cong SU(2)/\mathbb{Z}_3$ and complex link Σ_3 as defined in Section 6.3.3.

The dimension of the space of holomorphic sections of

$$\nu^{1,0}_{\mathbb{C}P^3}(\Sigma_3) = \mathcal{O}_{\Sigma_3}(5) \oplus \mathcal{O}_{\Sigma_3}(5),$$

where $\mathcal{O}_{\Sigma_3}(n)$ denotes the line bundle of degree n over Σ_3. By the Hirzebruch–Riemann–Roch theorem 6.2.3, this space has dimension twelve, and so the dimension of the space of infinitesimal conical complex deformations of C_3 in $\mathbb{C}^4$ has dimension twenty-four.

So it remains to find $v \in C^\infty(\mathcal{O}_{\Sigma_3}(3m+5) \oplus \mathcal{O}_{\Sigma_3}(3m+5))$ satisfying

$$\Delta_{\bar{\partial}_\Sigma} v = -\frac{1}{2} m(4+m). \tag{6.3.4}$$

Unfortunately, for this example we cannot directly apply Theorem 6.3.5 to this problem, so we must find a different way to solve (6.3.4). We will do this by constructing a moving frame for L_3.

Proposition 6.3.8 ([30, §6.3.2]). *There exists an orthonormal frame of L_3, denoted $\{e_1, e_2, e_3\}$, where $Je_2 = e_3$ and e_1 is the Reeb vector field. We have that*

$$[e_1, e_2] = -\frac{2}{3} e_3, \quad [e_1, e_3] = \frac{2}{3} e_2, \quad [e_2, e_3] = -2e_1.$$

We extend this to a frame of S^7 as follows.

Lemma 6.3.9. *There exist orthonormal frames $\{e_1, e_2, e_3\}$ of L_3 and $\{f_4, f_5, f_6, f_7\}$ of $\nu_{S^7}(L_3)$ such that the structure equations of Proposition A.0.2 take the following form:*

$$\begin{aligned}
dx &= e_1\omega_1 + e_2\omega_2 + e_3\omega_3 + f_4\eta_4 + f_5\eta_5 + f_6\eta_6 + f_7\eta_7, \\
de_1 &= -\omega_1 x - \omega_3 e_2 + \omega_2 e_3 - \eta_5 f_4 + \eta_4 f_5 - \eta_7 f_6 + \eta_6 f_7, \\
de_2 &= -\omega_2 x + \omega_3 e_1 + \frac{\omega_1}{3} e_3 + \frac{2}{\sqrt{3}} \omega_2 f_4 + \frac{2}{\sqrt{3}} \omega_3 f_5, \\
de_3 &= -\omega_3 x - \omega_2 e_1 - \frac{\omega_1}{3} e_2 - \frac{2}{\sqrt{3}} \omega_3 f_4 + \frac{2}{\sqrt{3}} \omega_2 f_5, \\
df_4 &= -x\eta_4 + \eta_5 e_1 - \frac{2}{\sqrt{3}} \omega_2 e_2 + \frac{2}{\sqrt{3}} \omega_3 e_3 - \frac{\omega_1}{3} f_5 + \omega_2 f_6 + \omega_3 f_7, \\
df_5 &= -x\eta_5 - \eta_4 e_1 - \frac{2}{\sqrt{3}} \omega_3 e_2 - \frac{2}{\sqrt{3}} \omega_2 e_3 + \frac{\omega_1}{3} f_4 - \omega_3 f_6 + \omega_2 f_7, \\
df_6 &= -x\eta_6 + \eta_7 e_1 - \omega_2 f_4 + \omega_3 f_5 - \omega_1 f_7, \\
df_7 &= -x\eta_7 - \eta_6 e_1 - \omega_3 f_4 - \omega_2 f_5 + \omega_1 f_6,
\end{aligned}$$

where $Je_2 = e_3, Jf_4 = f_5, Jf_6 = f_7$, $\{\omega_1, \omega_2, \omega_3\}$ *is an orthonormal coframe of* L_3 $(\omega_i(e_j) = \delta_{ij})$ *and* $\{\eta_4, \eta_5, \eta_6, \eta_7\}$ *is an orthonormal coframe of the normal bundle of* L_2 *in* S^7 $(\eta_a(f_b) - \delta_{ab})$*. Further, the second structure equations of Proposition A.0.3 are also satisfied.*

Proof. Let ∇ denote the Levi-Civita connection of L_3. Again we take $\alpha_2 = \omega_2$ and $\alpha_3 = \omega_3$ as we may by Proposition A.0.4. We see that since, using the structure equations given in A.0.2,

$$-\alpha_1(e_1)e_3 - e_3 = \nabla_{e_1}e_2 - \nabla_{e_2}e_1 = [e_1, e_2] = -\frac{2}{3}e_3,$$

we must have that $\alpha_1 = -\frac{\omega_1}{3}$. We check that

$$-\frac{1}{3}e_2 + e_2 = \nabla_{e_1}e_3 - \nabla_{e_3}e_1 = [e_1, e_3] = \frac{2}{3}e_2,$$

and

$$-e_1 - e_1 = \nabla_{e_2}e_3 - \nabla_{e_3}e_2 = [e_2, e_3] = -2e_1.$$

Now Equation (A.0.3) tells us that we must have that

$$-2(\beta_2^4 \wedge \beta_2^5 + \beta_2^6 \wedge \beta_2^7) = -\frac{8}{3}\omega_2 \wedge \omega_3.$$

So we take $\beta_2^4 = \frac{2}{\sqrt{3}}\omega_2$ and $\beta_2^5 = \frac{2}{\sqrt{3}}\omega_3$, $\beta_2^6 = \beta_2^7 = 0$ and this is satisfied. To ensure that Equation (A.0.4) is satisfied, we seek γ so that

$$\begin{aligned}
d\beta_2^4 &= \frac{2}{\sqrt{3}}d\omega_2 = -\frac{4}{3\sqrt{3}}\omega_1 \wedge \omega_3 = -\frac{2}{\sqrt{3}}\omega_1 \wedge \omega_3 + \frac{1}{\sqrt{3}}\gamma_1 \wedge \omega_3, \\
d\beta_2^5 &= \frac{2}{\sqrt{3}}d\omega_3 = \frac{4}{3\sqrt{3}}\omega_1 \wedge \omega_2 = \frac{2}{\sqrt{3}}\omega_1 \wedge \omega_2 - \frac{1}{\sqrt{3}}\gamma_1 \wedge \omega_2, \\
d\beta_2^6 &= 0 = \frac{1}{\sqrt{3}}\gamma_3 \wedge \omega_3 - \frac{1}{\sqrt{3}}\gamma_2 \wedge \omega_2, \\
d\beta_2^7 &= 0 = -\frac{1}{\sqrt{3}}\gamma_3 \wedge \omega_2 - \frac{1}{\sqrt{3}}\gamma_2 \wedge \omega_2.
\end{aligned}$$

From this we see that we must have that $\gamma_1 = \frac{2}{3}\omega_1$, and $\gamma_2 = a\omega_2$ and $\gamma_3 = a\omega_3$. To determine a, we check Equation (A.0.5), which tells us that we must have

$$-\frac{1}{3}d\omega_1 = -\frac{2}{3}\omega_2 \wedge \omega_3 = \frac{a^2}{2}\omega_2 \wedge \omega_3 - \frac{8}{3}\omega_2 \wedge \omega_3,$$

and therefore we must have $a = 2$. It can be checked that the remaining parts of Equation (A.0.5) are satisfied with $\gamma = (\frac{2}{3}\omega_1, 2\omega_2, 2\omega_3)$. Therefore we choose $\{f_4, f_5, f_6, f_7\}$ so that the above choices of γ, β and α hold, and so the equations claimed hold. $\square$

We have that $\{f_4 - if_5, f_6 - if_7\}$ is a frame for the holomorphic tangent bundle of L_3 in S^7. We have that

$$\begin{aligned}\nabla_{e_1}(f_4 - if_5) &= -\frac{1}{3}f_5 - \frac{i}{3}f_4 = -\frac{i}{3}(f_4 - if_5),\\ \nabla_{e_1}(f_6 - if_7) &= -f_7 - if_6 = -i(f_6 - if_7).\end{aligned}$$

However,

$$\begin{aligned}(\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_3})(f_4 - if_5) &= 0,\\ (\nabla^{\perp}_{e_2} + i\nabla^{\perp}_{e_3})(f_6 - if_7) &= -2(f_4 - if_5),\\ (\nabla^{\perp}_{e_2} - i\nabla^{\perp}_{e_3})(f_4 - if_5) &= 2(f_6 - if_7),\\ (\nabla^{\perp}_{e_2} - i\nabla^{\perp}_{e_3})(f_6 - if_7) &= 0,\end{aligned}$$

and so we see explicitly that the connection on the normal bundle of L_3 in S^7 is not in a nice diagonal form as we had before. Since we have a moving frame of S^7, we will return to considering the eigenvalue problem (6.2.8)-(6.2.8). Writing a section of $\nu^{1,0}_{S^7}(L_3)$ as

$$g_1(f_4 - if_5) + g_2(f_6 - if_7),$$

where g_1, g_2 are functions on L_3 and sections of $\Lambda^{0,1}_h L \otimes \nu^{1,0}_{S^7}(L_3)$ as

$$\alpha_1 \otimes (f_4 - if_5) + \alpha_2 \otimes (f_6 - if_7),$$

where α_1, α_2 are sections of $\Lambda^{0,1}_h L$, we seek $g_1, g_2 \in C^\infty(L_3)$ and $\alpha_1, \alpha_2 \in C^\infty(\Lambda^{0,1}_h L)$ satisfying

$$\begin{aligned}\bar{\partial}_h g_1 - g_2(\omega_2 - i\omega_3) &= \left(\frac{8}{3} - i\nabla_{e_1}\right)\alpha_1,\\ \bar{\partial}^*_h \alpha_1 &= \frac{1}{2}\left(\frac{4}{3} + i\nabla_{e_1}\right)g_1,\end{aligned}$$

and

$$\bar{\partial}_h g_2 = (2 - i\nabla_{e_1})\alpha_2,$$
$$\bar{\partial}_h^* \alpha_2 + 2(e_2 \lrcorner \alpha_1) = \frac{1}{2}(2 + i\nabla_{e_1})g_2.$$

We must have that

$$g_2(\omega_2 - i\omega_3) = a\alpha_1,$$

for some $a \in \mathbb{C}$ (since if $\alpha_1 = 0$ then we find infinitesimal conical complex deformations of C_3), and so we may instead study the eigenvalue problems

$$\bar{\partial}_h g_1 = \left(\frac{8}{3} - i\nabla_{e_1} + a\right)\alpha_1, \tag{6.3.5}$$
$$\bar{\partial}_h^* \alpha_1 = \frac{1}{2}\left(\frac{4}{3} + i\nabla_{e_1}\right)g_1, \tag{6.3.6}$$

and

$$\bar{\partial}_h g_2 = (2 - i\nabla_{e_1})\alpha_2, \tag{6.3.7}$$
$$\bar{\partial}_h^* \alpha_2 = \frac{1}{2}\left(2 + \frac{4}{a} + i\nabla_{e_1}\right)g_2. \tag{6.3.8}$$

Using the structure equations given in Lemma 6.3.9, we see that the problem (6.3.7)-(6.3.8) is equivalent to the eigenproblem

$$\bar{\partial}_h(g_2(\omega_2 - i\omega_3)) = \left(\frac{8}{3} - i\nabla_{e_1}\right)\alpha_2 \otimes (\omega_2 - i\omega_3), \tag{6.3.9}$$
$$\bar{\partial}_h^*(\alpha_2 \otimes (\omega_2 - i\omega_3)) = \frac{1}{2}\left(\frac{4}{3} + \frac{4}{a} + i\nabla_{e_1}\right)g_2(\omega_2 - i\omega_3), \tag{6.3.10}$$

where we consider $g_2(\omega_2 - i\omega_3)$ as a $\Lambda_h^{0,1}L$-valued function, which becomes

$$a\bar{\partial}_h \alpha_1 = \left(\frac{8}{3} - i\nabla_{e_1}\right)\alpha_2, \tag{6.3.11}$$
$$\bar{\partial}_h^* \alpha_2 = \frac{a}{2}\left(\frac{4}{3} + \frac{4}{a} + i\nabla_{e_1}\right)\alpha_1, \tag{6.3.12}$$

where now α_2 is a section of $\Lambda_h^{0,1}L \otimes \Lambda_h^{0,1}L$. Supposing that

$$\mathcal{L}_{e_1} g_1 = img_1, \ \mathcal{L}_{e_1}\alpha_1 = im\alpha_1,$$

for $3m \in \mathbb{Z}$ we see that in order for the eigenproblem (6.3.11)-(6.3.12) to make sense we must have

$$\mathcal{L}_{e_1}\alpha_2 = im\alpha_2.$$

Write $\mathcal{O}_{\Sigma_3}(d)$ for the degree d line bundle over Σ_3. Then as explained in Section 6.2.4, we may replace the eigenvalue problems (6.3.5)-(6.3.6)-(6.3.11)-(6.3.12) with seeking $g_1 \in C^\infty(\mathcal{O}_{\Sigma_3}(3m))$, and $\alpha_1 \in C^\infty(\mathcal{O}_{\Sigma_3}(3m+2)), \alpha_2 \in C^\infty(\mathcal{O}_{\Sigma}(3m+4))$ satisfying

$$\bar{\partial}_{\Sigma_3} g_1 = \left(\frac{8}{3} + a + m\right)\alpha_1, \tag{6.3.13}$$

$$\bar{\partial}^*_{\Sigma_3}\alpha_1 = \frac{1}{2}\left(\frac{4}{3} - m\right) g_1, \tag{6.3.14}$$

and

$$a\bar{\partial}_{\Sigma_3}\alpha_1 = \left(\frac{8}{3} + m\right)\alpha_2, \tag{6.3.15}$$

$$\bar{\partial}^*_{\Sigma_3}\alpha_2 = \frac{a}{2}\left(\frac{4}{3} + \frac{4}{a} - m\right)\alpha_1. \tag{6.3.16}$$

We find that α_1 must simultaneously satisfy the following two eigenproblems: applying $\bar{\partial}_{\Sigma_3}$ to (6.3.14) and using (6.3.13) we find that

$$\bar{\partial}_{\Sigma_3}\bar{\partial}^*_{\Sigma_3}\alpha_1 = \frac{1}{2}\left(\frac{8}{3} + a + m\right)\left(\frac{4}{3} - m\right)\alpha_1, \tag{6.3.17}$$

and applying $\bar{\partial}^*_{\Sigma_3}$ to (6.3.15) and using (6.3.16) we have that

$$\bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}\alpha_1 = \frac{1}{2}\left(\frac{8}{3} + m\right)\left(\frac{4}{3} + \frac{4}{a} - m\right)\alpha_1. \tag{6.3.18}$$

Applying the formula [2, Lem 2.1, 2.2]

$$\bar{\partial}_{\Sigma_3}\bar{\partial}^*_{\Sigma_3}\alpha = \bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}\alpha + \frac{2}{3}(3m+2)\alpha,$$

where α is a section of $\mathcal{O}_{\Sigma_3}(3m+2)$, we see that

$$\begin{aligned}\bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}\alpha_1 &= \frac{1}{2}\left(\frac{8}{3} + m\right)\left(\frac{4}{3} + \frac{4}{a} - m\right)\alpha_1, \\ &= \frac{1}{2}\left[\left(\frac{8}{3} + a + m\right)\left(\frac{4}{3} - m\right) + \frac{4}{3}(3m+2)\right]\alpha_1,\end{aligned}$$

for $\alpha_1 \in C^\infty(\mathcal{O}_{\Sigma_3}(3m+2))$. Therefore $a \in \mathbb{C}$ must satisfy

$$\left(\frac{8}{3}+m\right)\left(\frac{4}{3}+\frac{4}{a}-m\right) = \left(\frac{8}{3}+a+m\right)\left(\frac{4}{3}-m\right) - \frac{4}{3}(3m+2).$$

Solving this equation for a, we find that for $m \neq 4/3$

$$a_\pm = \frac{4m+\frac{8}{3}\pm 8}{2(\frac{4}{3}-m)},$$

which simplifies to

$$a_+ = \frac{6m+16}{4-3m},\ a_- = -2.$$

First considering $a = a_+$ we apply Theorem 6.3.5 to see that

$$\frac{1}{2}\left(\frac{8}{3}+m\right)\left(\frac{4}{3}-m+\frac{4(4-3m)}{6m+16}\right),$$

is an eigenvalue of $\bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}$ acting on sections of $\mathcal{O}_{\Sigma_3}(3m+2)$ if, and only if, $m = -2/3$. In this case there are five $\alpha_1 \in C^\infty(\mathcal{O}_{\Sigma_3}(0))$ satisfying

$$\Delta_{\bar{\partial}_{\Sigma_3}}\alpha_1 = 4\alpha_1.$$

Taking $g_1 = \bar{\partial}^*_{\Sigma_3}\alpha_1$ and $\alpha_2 = \bar{\partial}_{\Sigma_2}\alpha_1$ completes this solution to the eigenproblem (6.3.13)-(6.3.14)-(6.3.15)-(6.3.16).

Secondly, when $a = a_- = -2$ Theorem 6.3.5 tells us that

$$\frac{1}{2}\left(\frac{8}{3}+m\right)\left(-\frac{2}{3}-m\right),$$

is an eigenvalue of $\bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}$ acting on sections of $\mathcal{O}_{\Sigma_3}(3m+2)$ if, and only if, $m = -2/3$, in which case we seek functions α_1 on Σ_3 satisfying

$$\Delta_{\bar{\partial}_{\Sigma_3}}\alpha_1 = 0.$$

Since Σ_3 is compact, α_1 must be holomorphic and further constant. Taking $g_1 = \alpha_2 = 0$ completes our analysis.

Finally, we check the case that $m = 4/3$. In this case, for the eigenvalues

$$\frac{1}{2}\left(\frac{8}{3}+\frac{4}{3}\right)\left(\frac{4}{a}\right) = -\frac{4}{6}(4+2),$$

we must have $a = -2$. However, in this case, the eigenvalue is equal to -4, which is negative and therefore not a possible eigenvalue of $\bar{\partial}^*_{\Sigma_3}\bar{\partial}_{\Sigma_3}$ on sections of $\mathcal{O}_{\Sigma_3}(6)$.

We have found a total of six infinitesimal conical Cayley deformations of C_3 that are not complex.

Proposition 6.3.10. *The real dimension of the space of infinitesimal conical Cayley deformations of C_3 in $\mathbb{C}^4$ is thirty. The real dimension of the space of infinitesimal conical complex deformations of C_3 in $\mathbb{C}^4$ is twenty-four.*

Remark. Similarly to Proposition 6.3.7 we have six infinitesimal conical Cayley deformations of C_3 which are not complex, which again implies that these deformations are just rigid motions.

6.4 Dimension of the moduli space of complex deformations of a CS complex surface

Now that we have discussed in more detail the set of exceptional weights $\mathcal{D}$ for the operator (6.2.6), we will apply the Atiyah–Patodi–Singer index theorem 4.4.1 to the operator $\bar{\partial} + \bar{\partial}^*$ to compare the dimension of the space of CS complex deformations of a CS complex surface in a Calabi–Yau four-fold to what we might expect to be the dimension of the space of all complex deformations of the complex surface from Kodaira's theorem 3.1.1.

Theorem 6.4.1. *Let N be a CS complex surface at $\hat{x}$ with cone C and rate μ inside a Calabi-Yau four-fold M. Write $\hat{N} := N\backslash\{\hat{x}\}$. Let, for $k > 4/p + 1$,*

$$\bar{\partial} + \bar{\partial}^* : L^p_{k+1,\mu}(\nu^{1,0}_M(\hat{N}) \oplus \Lambda^{0,2}\hat{N} \otimes \nu^{1,0}_M(\hat{N})) \to L^p_{k,\mu-1}(\Lambda^{0,1}\hat{N} \otimes \nu^{1,0}_M(\hat{N})), \qquad (6.4.1)$$

and denote the index of this operator by

$$\operatorname{ind}_\mu(\bar{\partial} + \bar{\partial}^*).$$

Then

$$\chi(N, \nu_M^{1,0}(N)) = \mathrm{ind}_\mu(\bar\partial + \bar\partial^*) + \sum_{\lambda \in (0,\mu)\cap\mathcal{D}} d(\lambda) + \frac{d(0)+\eta(0)}{2}, \tag{6.4.2}$$

where $\chi(N, \nu_M^{1,0}(N))$ *is the holomorphic Euler characteristic of* $\nu_M^{1,0}(N)$*,* $\mathcal{D}$ *is the set of* $\lambda \in \mathbb{R}$ *for which* (6.2.13)-(6.2.14) *has a nontrivial solution and then* $d(\lambda)$ *is the dimension of the solution space,* η *is the* η*-invariant which we can now define to be*

$$\eta(s) := \sum_{0\neq\lambda\in\mathcal{D}} d(\lambda)\frac{\mathrm{sign}(\lambda)}{|\lambda|^s}. \tag{6.4.3}$$

Remark. We interpret this as follows. The term $\chi(N, \nu_M^{1,0}(N))$ is interpreted as the dimension of the space of all complex deformations of N in M, since this is what we can expect if Kodaira's theorem 3.1.1 remains valid for complex varieties. Theorem 5.4.4 tells us that $\mathrm{ind}_\mu(\bar\partial + \bar\partial^*)$ is the expected dimension of the space of CS Cayley deformations of N in M (which by Proposition 5.5.3 we can interpret as the expected dimension of the space of CS complex deformations of N in M, although Theorem 5.5.2 tells us that in fact this should be equal to just the dimension of the kernel of (6.4.1), which is what we expect to happen generically anyway). The term $d(1)$ represents deformations of N that have a different tangent cone to N at $\hat{x}$.

Proof. This follows from Proposition 4.4.5, since in this case

$$\int_N \alpha_0(x)\ \mathrm{vol} = \chi(N, \nu_M^{1,0}(N)),$$

from [50, Thm 1.6]. □

6.4.1 Calculating the η-invariant for an example

The final calculation in this chapter is to compute the η-invariant for one of the examples we considered in Section 6.3. This will help us to calculate (what we expect to be) the codimension of the space of conically singular complex CS deformations of a CS complex surface N at C with rate μ in a Calabi–Yau manifold M inside the space of all complex deformations of N, for a certain cone C in $\mathbb{C}^4$, using Theorem 6.4.1.

We consider our simplest example of a two-dimensional complex cone in $\mathbb{C}^4$ which is $C_1 = \mathbb{C}^2$. Denote by Σ_1 the complex link of C_1, i.e., $\Sigma_1 = \mathbb{C}P^1$. Proposition 6.2.4 told us that the exceptional weights $\lambda \in \mathbb{R}$ satisfy an eigenproblem, and to calculate the η-invariant we must first find the dimension of the space of solutions to (6.2.13)-(6.2.14) for each $\lambda \in \mathbb{R}$. Setting $w = 0$ in (6.2.13)-(6.2.14), we seek holomorphic sections of $\nu^{1,0}_{\mathbb{C}P^3}(\Sigma_1) \otimes \mathcal{O}_{\mathbb{C}P^3}(\lambda - 1)|_{\Sigma_1} = \mathcal{O}_{\mathbb{C}P^3}(\lambda)|_{\Sigma_1} \oplus \mathcal{O}_{\mathbb{C}P^3}(\lambda)|_{\Sigma_1}$, for $\lambda \in \mathbb{N} \cup \{0\}$, which by the Hirzebruch–Riemann–Roch theorem 6.2.3 has dimension $2(\lambda+1)$. Similarly, setting $v = 0$ in (6.2.13)-(6.2.14), we seek antiholomorphic sections of $\mathcal{O}_{\mathbb{C}P^3}(-\lambda)|_{\Sigma_1} \oplus \mathcal{O}_{\mathbb{C}P^3}(-\lambda)|_{\Sigma_1}$, which again have dimension $2(\lambda + 1)$.

It remains to compute the multiplicity of λ as an eigenvalue of

$$2\bar{\partial}^*_{\Sigma_1}\bar{\partial}_{\Sigma_1} v = (\lambda - 1 - m)(\lambda + 3 + m)v, \tag{6.4.4}$$

where v is a section of $\mathcal{O}_{\mathbb{C}P^3}(m+1)|_{\Sigma_1} \oplus \mathcal{O}_{\mathbb{C}P^3}(m+1)|_{\Sigma_1}$ and $\lambda \neq 1 + m$ or $-3 - m$. Theorem 6.3.5 tells us that this is equivalent to solving the algebraic equation

$$(\lambda - 1 - m)(\lambda + 3 + m) = 4[q^2 + q|m + 2|],$$

where q is a positive integer.

It can be computed that the multiplicity of integer $\lambda > 0$ as an eigenvalue of (6.4.4) is $2\lambda(\lambda + 1)$ and the multiplicity of integer $\lambda < -2$ as an eigenvalue of (6.4.4) is $2(\lambda + 2)(\lambda + 1)$. So we have that

$$\begin{aligned} \eta(s) &= 4\sum_{\lambda=1}^{\infty} \frac{\lambda+1}{\lambda^s} + 2\sum_{\lambda=1}^{\infty} \frac{\lambda(\lambda+1)}{\lambda^s} - 2\sum_{\lambda=3}^{\infty} \frac{(-\lambda+2)(-\lambda+1)}{\lambda^s}, \\ &= 12\sum_{\lambda=1}^{\infty} \lambda^{1-s}, \end{aligned}$$

and so

$$\eta(0) = 12\zeta(-1) = -1,$$

where ζ is the Riemann zeta function.

We have that the multiplicity of the zero eigenvalue in this case in four. So we have found that

$$\frac{\eta(0)+h}{2}=\frac{3}{2}.$$

6.5 Concluding remarks

In Chapters 3–6 of this thesis we have considered the deformation theory of Cayley submanifolds, paying particular attention to the special case that the ambient manifold is a four-dimensional Calabi–Yau manifold and the Cayley submanifold is a two-dimensional complex submanifold. We saw in Chapter 3 that Cayley and complex deformations of a compact complex surface are the same, and in Chapter 5 we saw that conically singular Cayley and complex deformations of a conically singular complex surface were the same. We saw in Chapter 6 by considering complex cones in $\mathbb{C}^4$ that this is not the case in general. For example, we saw that we can deform a complex plane into a Cayley plane that is no longer complex.

There is still potential for further work on the deformation theory of conically singular Cayley submanifolds. In particular, it would be interesting to find an analytic justification for the heuristic explanation of the terms that appear in the index formula given by an application of the Atiyah–Patodi–Singer theorem in Theorem 6.4.1. This could be done by applying similar techniques to Joyce [22] and Lotay [37], who considered deformations of conically singular special Lagrangian and coassociative submanifolds respectively. It would, however, be interesting to see if new techniques could be developed to study more exotic deformations of conically singular Cayley submanifolds.

It would be interesting to try to exploit the relationship between Cayley and complex submanifolds explored in this thesis to find new techniques for problems such as desingularising Cayley submanifolds. Desingularising complex varieties is relatively simple using techniques from algebraic geometry and so it might be the case that the close relationship between Cayley and complex submanifolds could mean that such techniques could be applied to Cayley submanifolds.

Appendix A

Structure equations of $Spin(7)$

We will here give the structure equations of S^7 adapted to an associative submanifold of S^7. To do this, we will consider the sphere S^7 as the group quotient $Spin(7)/G_2$, that is, we can consider $Spin(7)$ as the G_2 frame bundle over S^7. Bryant [7, Prop 1.1] first wrote down the structure equations of $Spin(7)$, but we will quote them in the following useful form given by Lotay [38, §4].

Proposition A.0.1 ([38, Prop 4.2])**.** *We may write the Lie algebra* $\mathfrak{spin}(7)$ *of the Lie group* $Spin(7) \subseteq Gl(n, \mathbb{R})$ *as*

$$\mathfrak{spin}(7) = \left\{ \begin{pmatrix} 0 & -\omega^T & -\eta^T \\ \omega & [\alpha] & -\beta^T - \frac{1}{3}\{\eta\}^T \\ \eta & \beta + \frac{1}{3}\{\eta\} & \frac{1}{2}[\alpha - \omega]_+ + \frac{1}{2}[\gamma]_- \end{pmatrix} \middle| \begin{array}{c} \omega, \alpha, \gamma \in M_{3\times 1}(\mathbb{R}), \\ \eta \in M_{4\times 1}(\mathbb{R}), \\ \beta \in M_{4\times 3}(\mathbb{R}), \\ \beta_1^4 + \beta_2^7 + \beta_3^6 = 0,\ \beta_1^5 + \beta_2^6 - \beta_3^7 = 0, \\ \beta_1^6 - \beta_2^5 - \beta_3^4 = 0,\ \beta_1^7 - \beta_2^4 + \beta_3^5 = 0. \end{array} \right\},$$

where

$$[(x, y, z)^T] := \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix},$$

$$[(x,y,z)^T]_{\pm} := \begin{pmatrix} 0 & -x & -y & \pm z \\ x & 0 & z & \pm y \\ y & -z & 0 & \mp x \\ \mp z & \mp y & \pm x & 0 \end{pmatrix},$$

and

$$\{(p,q,r,s)^T\} := \begin{pmatrix} -q & -r & s \\ p & s & r \\ -s & p & -q \\ r & -q & -p \end{pmatrix}.$$

Now that we have the structure equations for $Spin(7)$, we may construct a moving frame for S^7 adapted to an associative three-fold. If we let $g : Spin(7) \to Gl(8,\mathbb{R})$ be the map taking $Spin(7)$ to the identity component of the Lie subgroup of $Gl(8,\mathbb{R})$ which has Lie algebra $\mathfrak{spin}(7)$, then we can write $g = (x\, e\, f)$, where for $p \in Spin(7)$ we have that $x(p) \in M_{8\times 1}(\mathbb{R}), e(p) = (e_1(p), e_2(p), e_3(p)) \in M_{8\times 3}(\mathbb{R})$ and $f(p) = (f_4(p), f_5(p), f_6(p), f_7(p)) \in M_{8\times 4}(\mathbb{R})$. We can choose our frame so that x represents a point of our associative three-fold L, e is an orthonormal frame for L and ω is an orthonormal coframe for L. Therefore f is an orthonormal frame for the normal bundle of L in S^7, η an orthonormal coframe. Then since the Maurer-Cartan form $\phi = g^{-1}dg$ takes values in $\mathfrak{spin}(7)$, we can write

$$\phi := \begin{pmatrix} 0 & -\omega^T & -\eta^T \\ \omega & [\alpha] & -\beta^T - \frac{1}{3}\{\eta\}^T \\ \eta & \beta + \frac{1}{3}\{\eta\} & \frac{1}{2}[\alpha - \omega]_+ + \frac{1}{2}[\gamma]_- \end{pmatrix}.$$

This yields the following results

Proposition A.0.2 ([38, Prop 4.3])**.** *Use the notation above. On the adapted frame bundle of an associative three-fold L in S^7, $x : L \to S^7$ and $\{e_1, e_2, e_3, f_4, f_5, f_6, f_7\}$ is* *a local oriented orthonormal basis for $TA \oplus NA$, so the first structure equations are*

$$\begin{aligned} dx &= e\omega; \\ de &= -x\omega^T + e[\alpha] + f\beta; \\ df &= -e\beta^T + \frac{1}{2} f([\alpha - \omega]_+ + [\gamma]_-). \end{aligned}$$

Proposition A.0.3 ([38, Prop 4.4])**.** *Use the notation above. On the adapted frame bundle of an associative three-fold in S^7, there exists a local tensor of functions $h = h^a_{jk} = h^a_{kj}$, for $1 \leq j, k \leq 3$ and $4 \leq a \leq 7$, such that the second structure equations are*

$$d\omega = -[\alpha] \wedge \omega; \tag{A.0.1}$$

$$\beta = h\omega; \tag{A.0.2}$$

$$d[\alpha] = -[\alpha] \wedge [\alpha] + \omega \wedge \omega^T + \beta^T \wedge \beta; \tag{A.0.3}$$

$$d\beta = -\beta \wedge [\alpha] - \frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-) \wedge \beta; \tag{A.0.4}$$

$$\frac{1}{2} d([\alpha - \omega]_+ + [\gamma]_-) = -\frac{1}{4}[\alpha - \omega]_+ \wedge [\alpha - \omega]_+ - \frac{1}{4}[\gamma]_- \wedge [\gamma]_- + \beta \wedge \beta^T. \tag{A.0.5}$$

Notice that $[\alpha]$ is the Levi-Civita connection of L and $\frac{1}{2}([\alpha - \omega]_+ + [\gamma]_-)$ defines the induced connection on the normal bundle of L in S^7. We have that h defines the second fundamental form $\mathbf{II}_L \in C^\infty(S^2T^*L; \nu(L))$ of L in S^7, writing

$$\mathbf{II}_L := h^a_{jk} f_a \otimes \omega_j \omega_k.$$

Since the associative submanifolds of S^7 that we are considering are S^1-bundles over complex curves, we may reduce the structure equations of L.

Proposition A.0.4 ([38, Ex 4.9])**.** *Let L be the link of complex cone C in $\mathbb{C}^4$. Then we can choose a frame of $TS^7|_L$ such that*

$$\alpha_2 = \omega_2, \ \alpha_3 = \omega_3 \text{ and } \beta^4_1 = \beta^5_3 = \beta^6_3 = \beta^7_3 = 0.$$

This implies that $\beta^4_3 = -\beta^5_2, \beta^5_3 = \beta^4_2, \beta^6_3 = -\beta^7_2$ and $\beta^7_3 = \beta^6_2$. Here e_1 defines the direction of the circle fibres of L over the complex link Σ of C.

Proof. This follows from supposing that the complex structure of $\mathbb{C}^4$ acts on C as follows:

$$Jx = e_1; \quad Je_2 = e_3; \quad Jf_4 = f_5; \quad Jf_6 = f_7.$$

□

Bibliography

[1] S. Akbulut and S. Salur. Deformations in G_2 manifolds. *Adv. Math.*, 217(5):2130–2140, 2008.

[2] A. López Almorox and C. Tejero Prieto. Holomorphic spectrum of twisted Dirac operators on compact Riemann surfaces. *J. Geom. Phys.*, 56(10):2069–2091, 2006.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[4] T. Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[5] T. Begley and K. Moore. On short time existence of Lagrangian mean curvature flow. *Mathematische Annalen*, pages 1–43, 2016.

[6] M. Berger. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[7] R. L. Bryant. Submanifolds and special structures on the octonians. *J. Differential Geom.*, 17(2):185–232, 1982.

[8] R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989.

[9] R.L. Bryant. Metrics with exceptional holonomy. *Ann. of Math. (2)*, 126(3):525–576, 1987.

[10] E. Calabi. The space of kähler metrics. In *Proceedings of the International Congress of Mathematicians*, volume 2, pages 206–207. Amsterdam, 1954.

[11] A. Chau, J. Chen, and W. He. Lagrangian mean curvature flow for entire Lipschitz graphs. *Calc. Var. Partial Differential Equations*, 44(1-2):199–220, 2012.

[12] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[13] T. Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[14] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[15] F. R. Harvey and H. B. Lawson. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.

[16] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[17] D. Huybrechts. *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005.

[18] T. Ilmanen, A. Neves, and F. Schulze. On short time existence for the planar network flow. *arXiv preprint arXiv:1407.4756*, 2014.

[19] Y. Imagi, D. D. Joyce, and J. Oliveira dos Santos. Uniqueness results for special Lagrangians and Lagrangian mean curvature flow expanders in $\mathbb{C}^m$. *Duke Math. J.*, 165(5):847–933, 2016.

[20] J. Jost. *Riemannian geometry and geometric analysis.* Universitext. Springer, Heidelberg, sixth edition, 2011.

[21] D. D. Joyce. *Compact manifolds with special holonomy.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[22] D. D. Joyce. Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications. *J. Differential Geom.*, 63(2):279–347, 2003.

[23] D. D. Joyce. Special Lagrangian submanifolds with isolated conical singularities. I. Regularity. *Ann. Global Anal. Geom.*, 25(3):201–251, 2004.

[24] D. D. Joyce. Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces. *Ann. Global Anal. Geom.*, 25(4):301–352, 2004.

[25] D. D. Joyce. *Riemannian holonomy groups and calibrated geometry*, volume 12 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2007.

[26] D. D. Joyce. Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow. *EMS Surv. Math. Sci.*, 2(1):1–62, 2015.

[27] D. D. Joyce, Y-I. Lee, and M-P. Tsui. Self-similar solutions and translating solitons for Lagrangian mean curvature flow. *J. Differential Geom.*, 84(1):127–161, 2010.

[28] D. D. Joyce and S. Salur. Deformations of asymptotically cylindrical coassociative submanifolds with fixed boundary. *Geom. Topol.*, 9:1115–1146, 2005.

[29] S. Karigiannis. Deformations of G_2 and $Spin(7)$ structures on manifolds. *arXiv preprint math/0301218*, 2003.

[30] K. Kawai. Deformations of associative submanifolds in nearly parallel G_2-manifolds. In *Proceedings of the 17th International Workshop on Differential Geometry and the 7th KNUGRG-OCAMI Differential Geometry Workshop [Vol. 17]*, pages 107–112. Natl. Inst. Math. Sci. (NIMS), Taejŏn, 2013.

[31] K. Kodaira. A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math. (2)*, 75:146–162, 1962.

[32] S. Lang. *Real analysis.* Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, second edition, 1983.

[33] S. Lang. *Introduction to differentiable manifolds.* Universitext. Springer-Verlag, New York, second edition, 2002.

[34] R. B. Lockhart. Fredholm, Hodge and Liouville theorems on noncompact manifolds. *Trans. Amer. Math. Soc.*, 301(1):1–35, 1987.

[35] R. B. Lockhart and R. C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[36] J. D. Lotay. 2-ruled calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$. *J. London Math. Soc. (2)*, 74(1):219–243, 2006.

[37] J. D. Lotay. Coassociative 4-folds with conical singularities. *Comm. Anal. Geom.*, 15(5):891–946, 2007.

[38] J. D. Lotay. Associative submanifolds of the 7-sphere. *Proc. Lond. Math. Soc. (3)*, 105(6):1183–1214, 2012.

[39] J. D. Lotay. Stability of coassociative conical singularities. *Comm. Anal. Geom.*, 20(4):803–867, 2012.

[40] J. D. Lotay and A. Neves. Uniqueness of Langrangian self-expanders. *Geom. Topol.*, 17(5):2689–2729, 2013.

[41] K. Mashimo. Homogeneous totally real submanifolds of S^6. *Tsukuba J. Math.*, 9(1):185–202, 1985.

[42] V. G. Maz′ja and B.A. Plamenevskiĭ. Estimates in L_p and in Hölder classes, and the Miranda-Agmon maximum principle for the solutions of elliptic boundary value problems in domains with singular points on the boundary. *Math. Nachr.*, 81:25–82, 1978.

[43] R. C. McLean. Deformations of calibrated submanifolds. *Comm. Anal. Geom.*, 6(4):705–747, 1998.

[44] C. B. Morrey. *Multiple integrals in the calculus of variations.* Die Grundlehren der mathematischen Wissenschaften, Band 130. Springer-Verlag New York, Inc., New York, 1966.

[45] A. Neves. Singularities of Lagrangian mean curvature flow: zero-Maslov class case. *Invent. Math.*, 168(3):449–484, 2007.

[46] A. Neves. Recent progress on singularities of Lagrangian mean curvature flow. In *Surveys in geometric analysis and relativity*, volume 20 of *Adv. Lect. Math. (ALM)*, pages 413–438. Int. Press, Somerville, MA, 2011.

[47] A. Neves. Finite time singularities for Lagrangian mean curvature flow. *Ann. of Math. (2)*, 177(3):1029–1076, 2013.

[48] M. Ohst. Deformations of compact Cayley submanifolds with boundary. *arXiv:1405.7886*, 2014.

[49] B. O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.

[50] V. K. Patodi. An analytic proof of Riemann-Roch-Hirzebruch theorem for Kaehler manifolds. *J. Differential Geometry*, 5:251–283, 1971.

[51] D. A. Salamon and T. Walpuski. Notes on the octonians. *arXiv preprint arXiv:1005.2820*, 2010.

[52] R. Schoen and J. Wolfson. Minimizing area among Lagrangian surfaces: the mapping problem. *J. Differential Geom.*, 58(1):1–86, 2001.

[53] K. Smoczyk. A canonical way to deform a Lagrangian submanifold. *dg-ga/9605005*, 1996.

[54] K. Smoczyk. Mean curvature flow in higher codimension: introduction and survey. In *Global differential geometry*, volume 17 of *Springer Proc. Math.*, pages 231–274. Springer, Heidelberg, 2012.

[55] R. P. Thomas and S-T. Yau. Special Lagrangians, stable bundles and mean curvature flow. *Comm. Anal. Geom.*, 10(5):1075–1113, 2002.

[56] B. White. A local regularity theorem for mean curvature flow. *Ann. of Math. (2)*, 161(3):1487–1519, 2005.

[57] J. Wolfson. Lagrangian homology classes without regular minimizers. *J. Differential Geom.*, 71(2):307–313, 2005.

[58] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.