On Gaussian Multiplicative Chaos

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To my beloved parents.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as specified in the text. Chapter 2 is based on the joint work [BWW18] with N. Berestycki and C. Webb, while Chapter 3 is based on [BW18] with G. Baverez. In all these collborations, the contribution of each collaborator was equal.

This dissertation is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

Gaussian multiplicative chaos was first constructed in Kahane's seminal paper in 1985 in an attempt to provide a mathematical foundation for Kolmogorov-Obukhov-Mandelbrot theory of energy dissipation in developed turbulence. It has attracted a lot of attentions from the mathematics community in the last decade, playing a pivotal role in the probabilistic formulation of Liouville conformal field theory, as well as showing up in different branches of mathematics such as analytic number theory where it describes the statistical behaviour of the Riemann zeta function on the critical line.

This thesis explores the theory of Gaussian multiplicative chaos in three different directions. We commence with a new connection with random matrix theory, showing that for large Hermitian matrices sampled from the one-cut-regular unitary ensemble, the absolute powers of the characteristic polynomial, when suitably normalised, converge in distribution to multiplicative chaos on the support of the limiting spectral distribution as the size of the matrix goes to infinity, and the limit is independent of the choice of the potential function. This is part of an ongoing programme of establishing Gaussian multiplicative chaos as a universal limit object in probability theory.

Next, we consider Gaussian multiplicative chaos in the context of Liouville conformal field theory and study the fusion estimate of the Liouville correlation function. More precisely, we derive the exact asymptotics for the Liouville four-point correlation when two points are merging and express the leading order coefficient in terms of DOZZ constants from the three-point correlation function. Our result is consistent with predictions from conformal bootstrap in theoretical physics, and has a geometric interpretation of surfaces being glued together, as hinted by the bootstrap equation.

Finally, we study the right tail of the mass of Gaussian multiplicative chaos and establish a formula for the leading order asymptotics under mild assumptions on the underlying log-correlated Gaussian field. The tail exponent satisfies a universal power-law profile, while the leading order coefficient can be described by the product of two constants, one capturing the dependence on the test set and any non-stationarity, and the other one encoding the universal properties of multiplicative chaos. This may be seen as a first step in understanding the full distributional properties of Gaussian multiplicative chaos.

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Contents

Chapter 1

Introduction

Gaussian multiplicative chaos, formally defined as the exponentiation of a log-correlated Gaussian field, is a one-parameter family of random measures with intriguing properties. First introduced by Mandelbrot [Man72] as a refinement of Kolmogorov-Obukhov's model of energy dissipation in turbulence, Gaussian multiplicative chaos is arguably the first example of multifractal measures (measures that are supported on sets of fractal dimension and that exhibit non-linear scaling relations) in the literature of intermittency modelling, but it was not until more than a decade later in Kahane's seminal paper [Kah85] that the first mathematical construction of multiplicative chaos was given.

Unlike other multifractal measures such as the cascade counterparts, Gaussian multiplicative chaos arises naturally in many different branches of mathematics, such as mathematical physics where it plays an indispensable role in the probabilistic formulation of Liouville conformal field theory, and probabilistic number theory where it is related to the description of the statistical behaviour of the Riemann zeta function on the critical line, to name but a few. Motivated by its significance, this thesis explores the theory of multiplicative chaos in three different directions with the goal of developing a better understanding of its universality and fundamental properties.

In this introductory chapter, we give an overview of Gaussian multiplicative chaos, starting with its construction and elementary properties. After that, we discuss some of the developments of the theory in the last two decades and highlight some recent applications in Liouville quantum gravity and intermediate sets of discrete log-correlated fields. We then explain the three directions that are explored in this thesis and an outline of the remaining chapters, and close the chapter with a discussion of future research.

1.1 Definition and elementary properties

1.1.1 Log-correlated Gaussian fields

In order to discuss the theory of Gaussian multiplicative chaos, we first need to explain the notion of log-correlated Gaussian fields.

Given a domain $D \subset \mathbb{R}^d$, we say that $X(\cdot)$ is a (centred) log-correlated Gaussian field on D if it is a Gaussian field with domain D and covariance of the form

$$K(x,y) = \mathbb{E}[X(x)X(y)] = -\log|x-y| + f(x,y) \qquad \forall x, y \in D \tag{1.1.1}$$

where $f(x,y)$ is some sufficiently regular function which remains bounded as $|x-y| \to 0$. The term "log-correlated" refers to the presence of logarithmic singularity along the diagonal of the covariance kernel (1.1.1), and as a consequence of this behaviour the field $X(\cdot)$ cannot be defined pointwise. There are, however, two ways of making sense of log-correlated Gaussian fields:

- **Stochastic process indexed by test functions.** We may view $X = (X(\phi))_{\phi \in \mathcal{F}}$ as a centred Gaussian process with index set given by some collection $\mathcal{F}$ of test functions, e.g. $\mathcal{F} = C_c^\infty(D)$, with the covariance structure given by

$$\mathbb{E}\left[X(\phi_1)X(\phi_2)\right] := \int_{D\times D} \phi_1(x)K(x,y)\phi_2(y)dxdy, \qquad \forall \phi_1, \phi_2 \in \mathcal{F}. \tag{1.1.2}$$

  If $K(x,y)$ is a positive-definite kernel, i.e. (1.1.2) is non-negative for any $\phi_1, \phi_2 \in C_c^\infty(D)$, then the existence of a Gaussian process with the aforementioned covariance is an immediate consequence of Kolmogorov's consistency criterion.

- **Gaussian generalised function.** Consider the operator $T : L^2(D) \to L^2(D)$ defined by the Fredholm integral

$$T(\phi)(x) := \int_D K(x,y)\phi(y)dy.$$

  Then T is a bounded symmetric operator and the spectral theory of self-adjoint compact operator implies that there exists eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ and an orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $L^2(D)$ such that $|\lambda_n| \to 0$ as $n \to \infty$ and $T(f_n) = \lambda_n f_n$. With a dominated convergence argument, the latter relation suggests that the eigenfunctions f_n are actually continuous. Since $K(x,y)$ is a positive-definite kernel, all the eigenvalues λ_n are non-negative and we can define a sequence of i.i.d. $N(0,1)$ random variables $(Z_k)_{k\in\mathbb{N}}$, and for each $n \in \mathbb{N}$ a continuous Gaussian field given by

$$X_n(x) := \sum_{k\le n} Z_k \sqrt{\lambda_k} f_k(x). \tag{1.1.3}$$

Given any two test functions ϕ_1, ϕ_2, it is straightforward to check that

$$(X_n(\phi_1), X_n(\phi_2)) := \left(\int_D X_n(x)\phi_1(x)dx, \int_D X_n(x)\phi_2(x)dx\right)$$

converges a.s. and in $L^2(\mathbb{P})$ to some Gaussian vector $(X(\phi_1), X(\phi_2))$ with the correct covariance

$$\begin{aligned}\mathbb{E}[X(\phi_1)X(\phi_2)] &= \int_{D\times D} \phi_1(x)\left(\sum_{k=1}^{\infty}\lambda_k f_k(x)f_k(y)\right)\phi_2(y)dxdy \\ &= \int_{D\times D}\phi_1(x)K(x,y)\phi_2(y)dxdy\end{aligned}$$

and the calculation extends to any finite collection of test functions. While the series

$$X(x) := \sum_{n=1}^{\infty} Z_n\sqrt{\lambda_n}f_n(x)$$

does not converge pointwise, we may still make sense of it by interpreting it as a random distribution in the sense of Schwartz.

1.1.2 Construction of Gaussian multiplicative chaos

Gaussian multiplicative chaos is formally defined as the random measure with density given by the exponentiation of a log-correlated Gaussian field, i.e.

$$M_{\gamma,\sigma}(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\sigma(dx), \qquad \gamma \in \mathbb{R} \tag{1.1.4}$$

where $\sigma(dx)$ is some reference Radon measure on D. For the purpose of our discussion, we shall restrict ourselves to the situation where $\sigma(dx) = g(x)dx$ for some non-negative continuous function g on D and abuse the notation to write $M_{\gamma,g}(dx) = M_{\gamma,\sigma}(dx)$. When $g(x) \equiv 1$ we simply write $M_\gamma(dx)$, or more generally $M_{\gamma,g}(dx) = g(x)M_\gamma(dx)$.

The first mathematical construction of Gaussian multiplicative chaos was due to Kahane [Kah85], based on a martingale approach. Assuming that the covariance kernel K can be decomposed into

$$K(x,y) = \sum_{n=1}^{\infty} K_n(x,y), \qquad \forall x,y \in D \tag{1.1.5}$$

where $(K_n)_{n\in\mathbb{N}}$ is a collection of covariance kernels of some independent continuous Gaussian fields $(Y_n)_{n\in\mathbb{N}}$ on D, then one may define $X_n(x) = \sum_{k\le n} Y_n(x)$ and hope that

$$M_{\gamma,g,n}(dx) = e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]} g(x)dx$$

converges to some random measure $M_{\gamma,g}(dx)$ as $n \to \infty$. Indeed, the sequence of random measures $(M_{\gamma,g,n})_{n\in\mathbb{N}}$ with respect to the filtration $\mathcal{G} = (\sigma(Y_k, k \le n))_n$ forms a measure-valued martingale, and the existence of the almost sure limit $M_{\gamma,g}$ is a consequence of the martingale convergence theorem.

Given the existence of the limit $M_{\gamma,g}$, two natural questions arise, namely

- whether $M_{\gamma,g}$ is a trivial measure; and
- whether $M_{\gamma,g}$ depends on the kernel decomposition (1.1.5).

The answer to the first question was already given by Kahane's paper: $M_{\gamma,g}$ is non-trivial if and only if $\gamma^2 < 2d$, which is now known as the subcritical regime of Gaussian multiplicative chaos. As for the second question, Kahane was only able to show that the limit is unique under σ-positivity, i.e. $M_{\gamma,g}$ is independent of the kernel decomposition if $K_n(x,y) \ge 0$ for all $x, y \in D$ and $n \in \mathbb{N}$. This is not very satisfactory because the condition is rather restrictive and cannot be verified easily, limiting the applicability of Kahane's theory. For instance, the kernel

$$K(x,y) = \log_+ \frac{L}{|x-y|} = \max\left(\log \frac{L}{|x-y|}, 0\right) \qquad (L > 0)$$

in dimension $d = 3$ was of interest to Kahane as it was proposed in the Kolmogorov-Obukhov model of turbulence to capture the intermittency phenomenon of energy dissipation, and the σ-positivity of which remains an open problem.

In recent years, a lot of activities have been centred around the study of log-correlated Gaussian fields. Motivated by new applications such as random planar geometry, people have been working towards a more robust theory of Gaussian multiplicative chaos and a new construction based on convolution has emerged. The idea is to pick a mollifier θ (i.e. non-negative function with compact support and $\int \theta(x)dx = 1$), and introduce a sequence of approximate Gaussian fields $X_\epsilon(x) = X * \theta_\epsilon(x)$ where $\theta_\epsilon(\cdot) = \epsilon^{-d}\theta(\cdot/\epsilon)$ and $\epsilon > 0$. Under minimal assumptions on θ, the field $X_\epsilon(\cdot)$ is a Borel measurable function for fixed ϵ, and one may try to define $M_{\gamma,g}$ as the limit

$$\lim_{\epsilon\to 0^+} M_{\gamma,g,\epsilon}(dx) = \lim_{\epsilon\to 0^+} g(x)M_{\gamma,\epsilon}(dx) = \lim_{\epsilon\to 0^+} g(x)e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}\left[X_\epsilon(x)^2\right]}dx.$$

The first convolution construction is due to Robert and Vargas [RV10b], under the condition that the covariance is translation invariant, and the method has been simplified by Berestycki and extended to deal with general kernels (1.1.1) with f being a continuous function in [Ber17]. The main result in the latter paper is that

Theorem 1.1.1. *Let $\gamma^2 < 2d$. Then the sequence $M_{\gamma,g,\epsilon}(dx) = e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}g(x)dx$ converges in probability to some random measure $M_{\gamma,g}$ in the weak* topology on D. Moreover, the limit is non-trivial and it does not depend on the choice of mollification.*[1]

As a by-product of the proof of the uniqueness part of the above theorem, Berestycki also showed that Kahane's martingale construction is equivalent to the convolution construction for general kernels, i.e. the limit arising from the martingale approach is the same as that from the regularisation approach.

To conclude our discussion here, let us mention the work of Shamov [Sha16], which adopts an equivalent but abstract approach of Gaussian Hilbert space to constructing Gaussian multiplicative chaos, and the work of Junilla and Saksman [JS17], which studies the uniqueness problem under the more general setting of smooth approximation.

1.1.3 Elementary properties of Gaussian multiplicative chaos

Let us focus on the subcritical regime $\gamma^2 < 2d$. Many properties of Gaussian multiplicative chaos are universal in the sense that they do not depend heavily on the function f that appears in the covariance kernel (1.1.1). An important example is the following criterion for the existence of moments: if $A \subset D$ is a non-empty bounded open set, then

$$\mathbb{E}\left[M_\gamma(A)^p\right] < \infty \qquad \Leftrightarrow \qquad p < \frac{2d}{\gamma^2}.$$

In particular the multiplicative chaos M_γ possesses some moment of order greater than 1. The result for the positive moments was already present in Kahane's work [Kah85], but that for the negative moments is more recent and due to Robert and Vargas [RV10b] who adapted the analysis of multiplicative cascades to the current setting.

Another interesting result is the multifractality of Gaussian multiplicative chaos: for any $p \in [0, \frac{2d}{\gamma^2})$ we have

$$\mathbb{E}\left[M_\gamma(B(x,r))^p\right] = \Theta(r^{\xi(p)}), \qquad r \to 0^+ \tag{1.1.6}$$

where ξ is the so-called structure exponent, given by $\xi(p) = (d + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2$. To see why this is true, consider the special case where $f \equiv L$ for some constant $L \in \mathbb{R}$, i.e. $\mathbb{E}[X(x)X(y)] = -\log|x-y| + L$, which is a positive definite kernel on sufficiently small ball $B(0,r)$. The corresponding field $X(\cdot)$ satisfies exact scale invariance, i.e. for any $c \in (0,1)$ we have the distributional equality

$$(X(cx))_{|x|\le r} \overset{d}{=} (X(x) + N_c)_{|x|\le r}$$

[1] Under minimal assumption on the mollifier θ, which is automatically satisfied if e.g. θ is in L^p for some $p > 1$, see [Ber17] for the details.

where N_c is an independent Gaussian random variable with zero mean and variance $\mathbb{E}[N_r^2] = -\log c$. This implies that

$$\begin{aligned} M_\gamma(B(0,cr)) &= \int_{|x|\le cr} e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}dx \\ &= c^d \int_{|u|\le r} e^{\gamma X(cu) - \frac{\gamma^2}{2}\mathbb{E}[X(cu)^2]}du \\ &\overset{d}{=} c^d e^{\gamma N_c - \frac{\gamma^2}{2}\mathbb{E}[N_c^2]} \int_{|u|\le r} e^{\gamma X(u) - \frac{\gamma^2}{2}\mathbb{E}[X(u)^2]}du = c^{d+\frac{\gamma^2}{2}} e^{\gamma N_c} M_\gamma(B(0,r)) \end{aligned}$$

and therefore

$$\mathbb{E}\left[M_\gamma(B(0,cr))^p\right] = \mathbb{E}[(c^{d+\frac{\gamma^2}{2}} e^{\gamma N_c})^p]\mathbb{E}\left[M_\gamma(B(0,r))^p\right] = c^{\xi(p)}\mathbb{E}\left[M_\gamma(B(0,r))^p\right].$$

For the general result, we invoke Kahane's convexity inequality, which, when specialised to log-correlated Gaussian fields, says that if $X_1(\cdot)$ and $X_2(\cdot)$ are two centred log-correlated Gaussian fields on D such that for any two different points $x, y \in D$

$$\mathbb{E}[X_1(x)X_1(y)] \le \mathbb{E}[X_2(x)X_2(y)],$$

then for any convex function $h : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}\left[h(M_\gamma^1(A))\right] \le \mathbb{E}\left[h(M_\gamma^2(A))\right]$$

where $M_\gamma^i(A)$ is the mass of a set $A \subset D$ with respect to the Gaussian multiplicative chaos associated with X_i, $i = 1, 2$. We may then take $h : x \mapsto x^p$, and compare the general kernel (1.1.1) with the exact kernel above (with two different choices of L to upper and lower bound the function $f(\cdot,\cdot)$ on $B(x,r)$) to obtain the same multifractal exponent $\xi(\cdot)$ for all multiplicative chaos.

Let us highlight yet another fundamental property, namely the support of M_γ. It is not difficult to see, under the usual topological definition, that M_γ is almost surely supported on the whole domain D. The following argument is due to [RV14]: for any open ball $B \subset D$, we have

$$\inf_{x\in B} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\widetilde{M}_{\gamma,n}(B) \le M_\gamma(B) \le \sup_{x\in B} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\widetilde{M}_{\gamma,n}(B)$$

where $\widetilde{M}_{\gamma,n}(dx) = \lim_{k\to\infty} e^{\gamma(X_k - X_n)(x) - \frac{\gamma^2}{2}\mathbb{E}\left[(X_k - X_n)^2(x)\right]}dx$ and $(X_n)_n$ are given by the truncated Karhunen-Loève expansion of $X(\cdot)$ (1.1.3)[2]. From the above inequality we

[2]Where $(Z_n)_n$ are i.i.d. $N(0,1)$ random variables coming from $Z_n = X(f_n)$

observe immediately that

$$\{M_\gamma(B) > 0\} = \bigcap_{n \ge 1} \{\widetilde{M}_{\gamma,n}(B) > 0\}$$

where the event on the RHS is in the tail σ-algebra generated by the i.i.d. random variables $(Z_n)_n$ and hence has probability 1 by Kolmogorov's 0-1 law.

On the other hand, one can also show that the mass of M_γ is concentrated on a random set of fractal dimension. Under the convolution construction of multiplicative chaos, we define the set of γ-thick points of $X(\cdot)$ as

$$T_\gamma := \left\{ x \in D : \lim_{\epsilon \to 0^+} \frac{X_\epsilon(x)}{-\log \epsilon} = \gamma \right\}.$$

By a simple calculation (see [Ber17, Lemma 3.5]) one can show that $\mathbb{E}[X_\epsilon(x)^2] = -\log \epsilon + O(1)$ as $\epsilon \to 0^+$, i.e. the set T_γ collects the exceptional points x at which the field X_ϵ blows up like $\gamma \mathrm{Var}(X_\epsilon(x))$. Furthermore, it is known (e.g. [RV14, Theorem 4.1-4.2]) that the set of γ-thick points gives full mass to M_γ in the sense that $M_\gamma(D \cap T_\gamma^c) = 0$ almost surely, and the Hausdorff dimension of T_γ is $d - \frac{\gamma^2}{2}$, which shows that Gaussian multiplicative chaos is not absolutely continuous with respect to the Lebesgue measure despite the formal expression (1.1.4). This also gives a partial explanation of why the measure M_γ becomes trivial when $\gamma^2 \ge 2d$. The idea of thick points was already hinted in Kahane's paper and the fact that the "support" of M_γ is of fractal dimension was reflected by the notion of measure with finite β-energy there, but these concepts have not been fully capitalised until recently in [Ber17] and lead to an elementary yet general construction of Gaussian multiplicative chaos in the subcritical phase.

1.2 Recent development in multiplicative chaos

The theory of Gaussian multiplicative chaos has attracted a lot of attention in the past decade thanks to new applications beyond intermittency modelling. In this section, we highlight some of the important applications of multiplicative chaos and survey various advancement made in the last few years.

1.2.1 Gaussian free field and Liouville quantum gravity

Many research activities in Gaussian multiplicative chaos in the last decade have been driven by the interest in random geometry – an area that studies the geometric characteristics of random curves/surfaces with special symmetry such as conformal invariance, as well as random discrete processes and their scaling limits.

We commence with the notion of Gaussian free field. The Gaussian free field on a bounded simply connected domain $D \subset \mathbb{R}^2$ with Dirichlet boundary condition is a centred

Gaussian field $X_D(\cdot)$ with covariance given by the Dirichlet Green's function $G_D(x,y)$ in the sense of (1.1.2), where $G_D(x,y)$ may be defined via

$$G_D(x,y) = \pi \int_0^\infty p_t^D(x,y)dt$$

with $p_t^D(x,y)$ being the transition density of a Brownian motion started from x and killed upon hitting $y \in \partial D$ at time t.

The Green's function $G_D(x,\cdot)$ satisfies the distributional equation $\Delta G_D(x,\cdot) = -2\pi\delta_x(\cdot)$ with Dirichlet boundary condition. It has the property of being a harmonic function in $D \setminus \{x\}$ and is an example of log-correlated Gaussian field:

$$G_D(x,y) = -\log|x-y| + \log R(x;D) + o(1), \qquad y \to x$$

where $R(x;D)$ is the conformal radius of x in D, defined by $R(x;D) = |m'(0)|$ for any conformal transformation $m : \mathbb{D} := \{x \in \mathbb{R}^2 : |x| < 1\} \to D$ with the property that $m(0) = x$. From the point of view of abstract Wiener space, the Gaussian free field may also be seen as a Gaussian Hilbert space indexed by the Sobolev space $H_0^1(D)$ of functions with compact support: for any $\phi \in H_0^1(D)$, we have

$$\mathrm{Var}\langle X_D, \phi\rangle_\nabla = ||\phi||_\nabla^2 = \frac{1}{2\pi}\int_D |\nabla\phi(x)|^2 dx. \tag{1.2.1}$$

Using this interpretation and the Karhunen-Loève expansion

$$X_D(\cdot) = \sum_n Z_n f_n(\cdot)$$

where $(f_n)_n$ is an orthonormal basis with respect to the Dirichlet inner product (1.2.1) and $(Z_n)_n$ is a collection of i.i.d. $N(0,1)$ random variables defined by $Z_n = \langle X_D, f_n\rangle_\nabla$, we see that the Gaussian free field may be realised as a random generalised function that lives in the negative Sobolev space $H_0^{-\epsilon}(D)$ for any $\epsilon > 0$. The two perspectives may be reconciled by an exercise of integration by parts: if we take

$$\rho(x) = -\frac{1}{2\pi}\Delta\phi(x), \qquad \phi \in C_c^\infty(D),$$

then we observe that

$$\begin{aligned}
\mathrm{Var}\langle X_D, \phi\rangle_\nabla = \frac{1}{2\pi}\int_D |\nabla\phi(x)|^2 dx &= -\frac{1}{2\pi}\int_D \phi(x)\Delta\phi(x)dx \\
&= \frac{1}{(2\pi)^2}\int_D \left(\int_D \Delta_y G_D(x,y)\phi(y)dy\right)\Delta_x\phi(x)dx \\
&= \frac{1}{(2\pi)^2}\int_D \left(\int_D G_D(x,y)\Delta_y\phi(y)dy\right)\Delta_x\phi(x)dx
\end{aligned}$$

$$= \int_{D\times D} \rho(x) G_D(x,y) \rho(y) dx dy = \mathbb{E}\left[X_D(\rho)^2\right]$$

and this can be immediately extended to an identity for the covariance structure (i.e. involving two test functions) by a standard polarisation argument.

The Gaussian free field has two remarkable properties that are essentially inherited from the Dirichlet inner product:

- Conformal invariance: if $T : D \to D'$ is a conformal map from D to D', then $G_{T(D)}(T(x), T(y)) = G_D(x,y)$ and hence

$$X_D(\cdot) \overset{d}{=} X_{T(D)}(T(\cdot)).$$

- Markov property: if $U \subset D$ is some fixed subdomain, then

$$X_D = X_0 + h$$

  where X_0 is a Gaussian free field in U with Dirichlet boundary condition and vanishes outside U, while h is independent of X_0 and harmonic in U.

These properties turn out to provide a characterisation of the Gaussian free field [BPR18].[3]

Motivated by Polyakov's work on two-dimensional quantum gravity [Pol81], the mathematics community has been trying to understand the geometry of random Riemann surfaces under natural probability measures. As a consequence of Riemann's uniformisation theorem, the study of a "random surface" may be reformulated as that of a random Riemannian metric

$$e^{\lambda(x)}(dx_1^2 + dx_2^2)$$

where $x = (x_1, x_2) \in D$ are the isothermal coordinates and $\lambda(\cdot)$ is randomly chosen. It happens that under the Liouville action and the non-interacting case where the cosmological constant μ is equal to zero (see also Section 1.2.2), the natural choice of $\lambda(\cdot)$ above is the Gaussian free field (up to some constant factor). This led Duplantier and Sheffield [DS11] to introduce the Liouville quantum gravity (LQG) measure

$$\begin{aligned} M_\gamma^{\mathrm{LQG}}(dx) &= R(x;D)^{\frac{\gamma^2}{2}} e^{\gamma X_D(x) - \frac{\gamma^2}{2}\mathbb{E}[X_D(x)^2]} dx \\ &= \lim_{\epsilon \to 0^+} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{D,\epsilon}(x)} dx \end{aligned} \tag{1.2.2}$$

and they were able to establish a relationship between Euclidean (i.e. with respect to the Lebesgue measure) and quantum (i.e. with respect to M_γ^{LQG}) scaling exponents, verifying

[3] Up to some moment conditions.

the Knizhnik-Polyakov-Zamolodchikov formula [KPZ88] from the physics literature. Many ongoing activities attempt to explore the connections between LQG measure and other objects such as Schramm-Loewner evolutions and scaling limits of random planar maps, see e.g. [DMS14, MS15, MS16a].

1.2.2 Liouville conformal field theory

More recently, David-Kupiainen-Rhodes-Vargas [DKRV16] provided the first rigorous construction of Liouville quantum field theory on the Riemann sphere $\widehat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. As mentioned in Section 1.2.1, this was proposed in the physics literature by Polyakov who was interested in developing a theory of path integral in dimension $d = 2$ with an exponential interaction term, and it is formally defined as the "Gibbs measure"[4]

$$\langle F \rangle = 2 \int F(X) e^{-S_L(X)} DX \tag{1.2.3}$$

where DX is the "Lebesgue measure" on the space of real functions $\widehat{\mathbb{C}} \to \mathbb{R}$, and S_L is the Liouville action[5]

$$S_L(X) = \frac{1}{4\pi} \int_{\widehat{\mathbb{C}}} \left(|\nabla_g X(x)|^2 + R_g(x) Q X(x) + 4\pi\mu e^{\gamma X(x)} \right) g(x) d^2 x. \tag{1.2.4}$$

Here $g(x) = |x|_+^{-4} = (|x| \vee 1)^{-4}$ is the background metric[6] with ∇_g and R_g being the associated gradient and curvature respectively; $\mu > 0$ is called the cosmological constant, $\gamma \in (0, 2)$ is a positive parameter and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. The first term of the Liouville action, or

$$\exp\left(-\frac{1}{4\pi} \int_{\widehat{\mathbb{C}}} |\nabla_g X(x)|^2 g(x) d^2 x \right) DX$$

hints that $X(\cdot)$ may be interpreted as some variant of Gaussian free field, and the third term $4\pi\mu e^{\gamma X(x)} g(x) d^2 x$ suggests an indispensable role of Gaussian multiplicative chaos in the mathematical definition of the functional (1.2.3).

A central object in Liouville quantum field theory is the correlation function. If $(z_k)_{k \le N}$ are N distinct points in $\mathbb{C}$ and $(\alpha_k)_{k \le N}$ are non-negative numbers such that the Seiberg bounds

$$s := \frac{\sum_{k=1}^N \alpha_k - 2Q}{\gamma} < 0 \qquad \text{and} \qquad \alpha_k \in (0, Q) \quad \forall k$$

[4]The factor 2 is for aesthetic purpose in order to match the DOZZ formula.

[5]The Liouville action has an extra curvature term which is omitted here for simplicity since it does not give any contribution when we restrict ourselves to our special choice of metric g.

[6]More precisely the background metric is $g(x)d^2x$. One can define LCFT with a different choice of background metric, and the Weyl anomaly formula provides a simple way to perform conformal changes of metrics, see [DKRV16, Section 3.5].

are satisfied, one may define the Liouville correlation function

$$\left\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \right\rangle = 2\int \left(\prod_{k=1}^{N} e^{\alpha_k \left(X(z_k) + \frac{Q}{2}\log g(z_k)\right)} \right) e^{-S_L(X)} DX,$$

under the probabilistic approach, by

$$\left\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \right\rangle = 2\mu^{-s}\gamma^{-1}\Gamma(s) \prod_{i<j} \frac{1}{|z_i - z_j|^{\alpha_i\alpha_j}} \mathbb{E}\left[\left(\int_{\mathbb{C}} F(x, \mathbf{z}) M_{\gamma,g}(d^2x) \right)^{-s} \right] \tag{1.2.5}$$

where

$$F(x, \mathbf{z}) = \prod_{k=1}^{N} \left(\frac{|x|_+}{|x - z_k|} \right)^{\gamma\alpha_k},$$

and $M_{\gamma,g}(d^2x) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} g(x) d^2x$ is the multiplicative chaos associated with the Gaussian free field with vanishing average over the unit circle:

$$\mathbb{E}[X(x)X(y)] = -\log|x-y| + \log|x|_+ + \log|y|_+.$$

The Liouville theory is a conformal field theory and the correlation function satisfies the property of conformal covariance (also known as the KPZ relation, again named after Knizhnik-Polyakov-Zamolodchikov): if ψ is any Möbius transform of the sphere, then

$$\left\langle \prod_{k=1}^{N} V_{\alpha_k}(\psi(z_k)) \right\rangle = \prod_{k=1}^{N} |\psi'(z_k)|^{-2\Delta_{\alpha_k}} \left\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \right\rangle$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ is called the conformal weight. Since any Möbius transform is uniquely determined by the image of three points, the above conformal symmetry allows us to express the three-point correlation function as

$$\left\langle \prod_{k=1}^{3} V_{\alpha_k}(z_k) \right\rangle = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} C_\gamma(\alpha_1, \alpha_2, \alpha_3)$$

where $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$, Δ_{23} and Δ_{13} are similarly defined, and the constant $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ is the three-point structure constant which may be seen as the three-point correlation evaluated at $(z_1, z_2, z_3) = (0, 1, \infty)$:

$$\begin{aligned} C_\gamma(\alpha_1, \alpha_2, \alpha_3) &= \lim_{z_3 \to \infty} |z_3|^{4\Delta_{\alpha_3}} \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(z_3) \rangle \\ &= 2\mu^{-s}\gamma^{-1}\Gamma(s)\mathbb{E}\left[\left(\int_{\mathbb{C}} \frac{|x|_+^{\gamma(\alpha_1+\alpha_2+\alpha_3)}}{|x|^{\gamma\alpha_1}|x-1|^{\gamma\alpha_2}} M_{\gamma,g}(d^2x) \right)^{-s} \right]. \end{aligned}$$

One research direction in Liouville conformal field theory is to compute the correlation function (1.2.5) and verify the formulae from the physics literature. In the case where $N = 3$, the celebrated DOZZ formula, proposed independently by Dorn-Otto and Zamolodchikov-Zamolodchikov, asserts that the three-point structure constant is given by

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \left(\pi\mu l(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{2-\frac{\gamma^2}{2}}\right)^{-\frac{\overline{\alpha}-2Q}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-Q)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_3)}$$

where $l(z) = \Gamma(z)/\Gamma(1-z)$, $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and $\Upsilon_{\frac{\gamma}{2}}(z)$ is Zamolodchikov's special holomorphic function defined on $\mathbb{C}$ which has the following integral representation when $\mathrm{Re}(z) \in (0, Q)$:

$$\log \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z\right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\right)\frac{dt}{t}.$$

The DOZZ formula has been, however, controversial within the physics community because of its invariance under the simultaneous change of parameters

$$\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}, \qquad \mu \leftrightarrow \widetilde{\mu} = \frac{(\mu\pi l(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi l(\frac{4}{\gamma^2})}$$

and such a symmetry is not apparent from the Liouvile action (1.2.4) a priori. This conjecture is finally resolved by Kupiainen, Rhodes and Vargas in their work [KRV15, KRV17], where the DOZZ formula was verified by deriving and solving the BPZ differential equations, which are satisfied by a degenerate four-point function with suitably chosen α's.

Let us mention that the interest in Liouville conformal field theory goes beyond the setting of Riemann sphere, and rigorous probabilistic constructions of the theory are also available for the complex tori [DRV16] and other compact Riemann surfaces of higher genus [GRV16], as well as non-compact surfaces such as the unit disc[7] [HRV18].

1.2.3 Discrete log-correlated Gaussian fields

Parallel to the research in the continuum, there has been a lot of work devoted to the study of discrete log-correlated Gaussian fields arising from physical models at criticality, an example of which is the membrane model in $d = 4$ [Kur09]. We shall focus on the discrete Gaussian free field in $d = 2$, which is closely related to random walks and is arguably the most extensively studied model in the literature.

For simplicity, consider a discrete Gaussian free field $X_N(\cdot)$ defined on $V_N = [0, N]^2 \cap \mathbb{Z}^2$ with Dirichlet boundary condition. This is a centred Gaussian function which is identically zero on the boundary ∂V_N (i.e. any points $x \in V_N$ with a nearest neighbour outside V_N),

[7] In which case there are additional boundary terms involved in the definition of Liouville action.

and is otherwise characterised by the covariance given by the discrete Green's function

$$\mathbb{E}_N[X_N(x)X_N(y)] = G_N(x,y) = E_x\left[\sum_{n=0}^{\tau_{\partial V_N}} 1_{\{S_n=y\}}\right], \qquad x,y \in \mathrm{Int}(V_N) = V_N \setminus \partial V_N.$$

Here $(S_n)_{n\ge 0}$ is a simple random walk starting from x under P_x[8], and $\tau_{\partial V_N} = \{n \ge 1 : S_n \in \partial V_n\}$ is the first exit time of $(S_n)_n$. Alternatively, the law of $X_N(\cdot)$ can be explicitly written as

$$\mathbb{P}_N(dX_N) \propto \exp\left(-\frac{1}{16}\sum_{x,y\in V: x\sim y}(X_N(x)-X_N(y))^2\right)\prod_{x\in\mathrm{Int}(V_N)} dX_N(x).$$

Viewing $\sum_{x,y}(X_N(x)-X_N(y))^2$ as the discrete analogue of the Dirichlet energy $\int_D |\nabla X(x)|^2 dx$, it should not be surprising that $X_N(\cdot)$, when suitably scaled (and extended), converges to a continuum Gaussian free field that was discussed in Section 1.2.1. Indeed, if we discretise the domain $D = [0,1]^2$ using a triangular mesh and project the continuum Gaussian free field to the σ-algebra generated by continuous functions that are affine on each triangle, then the resulting Gaussian field restricted to V_N is precisely the discrete Gaussian free field (see [She07, Section 4.2-4.3]).

It has been well-known since the work [BDG01] of Bolthausen, Deuschel and Giacomin that the maximum of the discrete Gaussian free field grows like

$$\max_{x\in V_N} X_N(x) \sim 2\sqrt{g}\log N, \qquad N\to\infty$$

where $g = 2/\pi$ describes the behaviour of the Green's function on the diagonal: $G_N(x,x) = g\log N + O(1)$ as $N\to\infty$ for any $x \in V_N$ sufficiently away from ∂V_N. Since then a lot of effort has been made in studying finer geometric properties of the field, such as the intermediate level sets

$$\{x \in V_N : X_N(x) \ge 2\sqrt{g}\lambda \log N\}, \qquad \lambda \in (0,1).$$

This is the subject of investigation in the paper [BL16b] by Biskup and Louidor, and the authors there proved that the scaling limit of the intermediate level sets is described by Gaussian multiplicative chaos:

Theorem 1.2.1 ([BL16b, Theorem 2.1 and Theorem 2.5]). *Let $(a_N)_{N\ge1}$ be any positive sequence such that $a_N \sim 2\sqrt{g}\lambda\log N$ for some $\lambda \in (0,1)$, $K_N := \frac{N^2}{\sqrt{\log N}}\exp\left(-\frac{a_N^2}{2g\log N}\right)$*

[8] $\mathbb{P}_N, \mathbb{E}_N$ are used for the law of the discrete Gaussian free field X_N, whereas P_x, E_x refer to the law of the symmetric random walk $(S_n)_n$.

and define

$$\eta_N := \frac{1}{K_N} \sum_{x \in V_N} \delta_{\frac{x}{N}} \otimes \delta_{X_N(x) - a_N}.$$

Then as $N \to \infty$,

$$\eta_N \xrightarrow{d} Z_{2\lambda}^D(dx) \otimes e^{-\frac{2}{\sqrt{g}}h} dh$$

with respect to the topology of vague convergence of measures on $D \times \mathbb{R}$. Here[9] *$Z_{2\lambda}^D$ is the Liouville quantum gravity measure defined in (1.2.2) up to a multiplicative factor, i.e.*

$$Z_{2\lambda}^D(dx) \overset{d}{=} cM_{\gamma=2\lambda}^{\mathrm{LQG}}(dx), \qquad x \in D$$

for some deterministic constant $c \in (0, \infty)$.

1.3 Outline of the thesis

In the following, we explain our contributions to three different aspects of the theory of Gaussian multiplicative chaos, and give an outline of the remaining chapters.

1.3.1 Gaussian multiplicative chaos as a universal limit

Log-correlated Gaussian fields and multiplicative chaos have, in recent years, appeared in different random models outside of their traditional applications (namely turbulence and Liouville theory). An example is the study of Riemann zeta function[10]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and Saksman and Webb [SW16] proved an intriguing result that connects the behaviour of the randomised Riemann zeta function on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ to that of a complex variant of Gaussian multiplicative chaos.

Here we are interested in establishing Gaussian multiplicative chaos as a limit object in different areas of mathematics. Our starting point is random matrix theory. In Chapter 2, we shall consider large random Hermitian matrices H_N sampled from the unitary ensemble

$$\mathbf{P}(dH_N) \propto e^{-N\mathrm{Tr}V(H_N)} dH_N$$

[9] Our parameters are different from those in [BL16b] where the authors use a different normalisation for Gaussian multiplicative chaos.

[10] The series representation is only valid when $\mathrm{Re}(s) > 1$; otherwise $\zeta(\cdot)$ is defined via meromorphic continuation.

where the potential function V is one-cut regular. This is an interesting class of random matrices that has been extensively studied in the physics literature for graph enumeration and discrete gravity [LZ13], and includes many important random matrix models such as the Gaussian unitary ensemble, which has been used since the work of Montgomery for different conjectures regarding the behaviour of the Riemann zeta function on the critical line.

Our main result concerns the characteristic polynomial of H_N: if $\mu_V(dx)$ is the equilibrium measure (which describes the asymptotic eigenvalue distribution) associated with the potential V, then the renormalised characteristic polynomial

$$\frac{|\det(H_N - xI_N)|^\beta}{\mathbb{E}|\det(H_N - xI_N)|^\beta}dx$$

converges in distribution, as the size N of the matrix goes to infinity, to a universal[11] Gaussian multiplicative chaos measure on $\mathrm{supp}(\mu_V)$ for sufficiently small[12] $\beta \geq 0$. To some extent, our result suggests that perhaps the centred logarithm of the characteristic polynomial, i.e.

$$x \mapsto \log|\det(H_N - xI_N)| - \mathbb{E}\log|\det(H_N - xI_N)|, \tag{1.3.1}$$

behaves asymptotically like a log-correlated Gaussian field. This may not be entirely surprising because of a well-known result of Johansson [Joh98] which states that the linear statistics of a one-cut regular ensemble satisfies a central limit theorem: if $\mathcal{T}: \mathbb{R} \to \mathbb{R}$ is some sufficiently regular function, $(\lambda_i)_{i \leq N}$ are the eigenvalues of H_N, then the quantity

$$\sum_{i=1}^N \mathcal{T}(\lambda_i) - N\int_{\mathbb{R}} \mathcal{T}(x)\mu_V(dx) \tag{1.3.2}$$

converges in distribution to some centred Gaussian random variable as N tends to infinity. What is special here is that (1.3.2) does not require a normalisation factor $1/\sqrt{N}$ that is present in the usual central limit theorem for i.i.d. random variables, and this would have relied on effective cancellation due to the regularity of eigenvalue distributions.

Unlike usual constructions of Gaussian multiplicative chaos, the field (1.3.1) has no martingale structure and Gaussianity only holds in the asymptotic sense which poses a huge challenge. A major part of our proof requires the derivation of the large-N asymptotics of mixed moments of the form

$$\mathbb{E}\left[e^{\sum_{j=1}^N \mathcal{T}(\lambda_j)}\prod_{i=1}^k |\det(H_N - x_iI_N)|^{\beta_i}\right]$$

[11] In the sense that if V_1 and V_2 are two one-cut regular potentials and the associated equilibrium measures are supported on the same interval, then the limit measure is the same for both potentials.

[12] In the L^2-regime of Gaussian multiplicative chaos.

which makes the notion of asymptotic Gaussianity more quantitative and recovers the structure of a log-correlated field.

1.3.2 Fusion estimates of Gaussian multiplicative chaos

Fusion estimates refer to the study of negative moments of a Gaussian multiplicative chaos integrated against merging singularities, or more precisely, expectations of the form

$$\mathbb{E}\left[\left(\int_D \frac{M_{\gamma,g}(dx)}{|x|^{\gamma\alpha_1}|x-z|^{\gamma\alpha_2}}\right)^{-s}\right], \qquad z \to 0 \tag{1.3.3}$$

where $s > 0$, $\gamma \in (0, \sqrt{2d})$, and $\alpha_1, \alpha_2 \in (0, Q)$ with $Q = \frac{\gamma}{2} + \frac{d}{\gamma}$. By a dominated convergence argument,

$$\lim_{z\to 0} \mathbb{E}\left[\left(\int_D \frac{M_{\gamma,g}(dx)}{|x|^{\gamma\alpha_1}|x-z|^{\gamma\alpha_2}}\right)^{-s}\right] = \mathbb{E}\left[\left(\int_D |x|^{-\gamma(\alpha_1+\alpha_2)} M_{\gamma,g}(dx)\right)^{-s}\right]$$

where the RHS is non-trivial (i.e. positive) if the merged singularity is not too strong. We are, however, more interested in the other case where the limit above is trivial and would like to understand the fusion asymptotics as z approaches the origin.

Variants of expectations of the form (1.3.3) appear naturally in many problems related to Gaussian multiplicative chaos. The first example is the study of annealed multifractal exponent of Gibbs measures associated with log-correlated Gaussian fields [Fyo09, Won17]. Suppose $(X_\epsilon)_{\epsilon>0}$ is a sequence of continuous Gaussian fields on a compact set $D \subset \mathbb{R}^d$ with covariance

$$\mathbb{E}\left[X_\epsilon(x)X_\epsilon(y)\right] = -\log\left(|x-y| \vee \epsilon\right) + f_\epsilon(x,y).$$

One may want to study the multifractality of the limiting Gibbs measure

$$\lim_{\epsilon\to 0^+} m_{\gamma,\epsilon}(u)du = \lim_{\epsilon\to 0^+} \frac{e^{\gamma X_\epsilon(u)}du}{\int_D e^{\gamma X_\epsilon(x)}dx}$$

via the annealed multifractal exponent $\tilde{\eta}_q$, which is defined, for each $q > 0$, by

$$\mathbb{E}\left[\int_D m_{\gamma,\epsilon}(u)^q du\right] = \mathbb{E}\left[\frac{Z_\epsilon(\gamma q)}{Z_\epsilon(\gamma)^q}\right] \overset{\epsilon\to 0^+}{\sim} \epsilon^{\tilde{\eta}_q}, \qquad Z_\epsilon(\beta) := \int_D e^{\beta X_\epsilon(u)}du.$$

To see the connection, observe that

$$\mathbb{E}\left[\frac{Z_\epsilon(\gamma q)}{Z_\epsilon(\gamma)^q}\right] = \epsilon^{-\frac{\gamma^2 q^2}{2}+\frac{\gamma^2 q}{2}} \int_D \mathbb{E}\left[\frac{e^{\gamma q X_\epsilon(u) - \frac{\gamma^2 q^2}{2}\mathbb{E}[X_\epsilon(u)^2]}}{\left(\int_D e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} e^{\frac{\gamma^2}{2}f_\epsilon(x,x)}dx\right)^q}\right] e^{\frac{\gamma^2 q^2}{2}f_\epsilon(u,u)}du$$

$$= \epsilon^{-\frac{\gamma^2 q^2}{2} + \frac{\gamma^2 q}{2}} \int_D \mathbb{E}\left[\left(\int_D \frac{e^{\frac{\gamma^2}{2} f_\epsilon(x,x) + \gamma q f_\epsilon(x,u)} M_{\gamma,\epsilon}(dx)}{(|x-u| \vee \epsilon)^{\gamma^2 q}}\right)^{-q}\right] e^{\frac{\gamma^2 q^2}{2} f_\epsilon(u,u)} du$$

where $M_{\gamma,\epsilon}(dx) = e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} dx$ and the second equality above follows from the Cameron-Martin theorem[13]. The integrand is reminiscent of (1.3.3) except that we now deal with the regularised chaos $M_{\gamma,\epsilon}$ and $(\epsilon, \gamma q)$ here plays the role of $(|z|, \alpha_1 + \alpha_2)$ there.

Another important application of fusion estimates comes from Liouville conformal field theory and is presented in Chapter 3. We consider the four-point correlation[14]

$$\begin{aligned}
&\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty)\rangle \\
&\quad = 2|z|^{-\alpha_1\alpha_2}|z-1|^{-\alpha_2\alpha_3}\mathbb{E}\left[\left(\int_{\mathbb{C}} \frac{|x|_+^{\gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}{|x|^{\gamma\alpha_1}|x-z|^{\gamma\alpha_2}|x-1|^{\gamma\alpha_3}} M_{\gamma,g}(d^2x)\right)^{-s}\right],
\end{aligned}$$

where $M_{\gamma,g}$ is the multiplicative chaos defined in Section 1.2.2 and $s = \frac{\sum_{k=1}^4 \alpha_k - 2Q}{\gamma}$, in the degenerate case when z is sent to the origin. We are able to derive the exact asymptotics for the fusion estimate, i.e. the rate in terms of $|z|$ at which the expectation above decays to 0 as well as the leading order coefficient. We show that the leading order coefficient may be expressed in terms of the DOZZ formula, and that our result is consistent with the predictions from conformal bootstrap in theoretical physics. In particular, in the "supercritical" regime where both $\alpha_1 + \alpha_2$ and $\alpha_3 + \alpha_4$ are greater than Q, the leading order coefficient can be factorised into the product of two DOZZ constants, demonstrating the philosophy of gluing surfaces together in the bootstrap approach to conformal field theory.

1.3.3 Distributional properties of Gaussian multiplicative chaos

The last topic we discuss in this thesis is the distributional properties of Gaussian multiplicative chaos. Despite being of fundamental importance, this topic is not actively explored in the literature. Answers to some very basic questions like whether the distribution of $M_\gamma(A)$ is non-atomic/has a density are still unknown except for the special case where the underlying Gaussian field is exact scale invariant [RV10b], and our knowledge about the distribution of $M_\gamma(A)$ has not gone much beyond the criterion for the existence of moments.

Driven by new applications in random geometry and random matrices, there has been some renewed effort in improving our understanding of the distribution of multiplicative

[13] The theorem says that if $(X(\cdot), Y)$ are centred and jointly Gaussian, then for any functional F we have

$$\mathbb{E}\left[e^{Y - \frac{\gamma^2}{2}\mathbb{E}[Y^2]} F(X(\cdot))\right] = \mathbb{E}\left[F(X(\cdot) + m(\cdot))\right], \qquad m(\cdot) = \mathbb{E}[X(\cdot)Y].$$

[14] Strictly speaking we consider the renormalised four-point correlation when z_4 is sent to infinity.

chaos. This includes the work of Ostrovsky [Ost16] which attempts to study $\mathbb{E}\left[F(M_{\gamma,g}(D))\right]$ by a formal expansion in the variable γ using his theory of intermittency differentiation. While the method employed in his work is not completely rigorous, Ostrovsky's computation has led to a lot of interesting conjectures, especially for multiplicative chaos associated with exactly scale invariant fields. Also, inspired by the work of Kupiainen-Rhodes-Vargas [KRV15, KRV17] on the proof of the DOZZ formula, Remy [Rem17] considers the total mass of the Gaussian multiplicative chaos measure associated with the Gaussian free field on the unit circle, i.e.

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = \frac{1}{2}\log\frac{1}{|e^{i\theta}-e^{i\theta'}|} \qquad \text{and} \qquad M_\gamma(\mathbb{T}) = \int_0^{2\pi} e^{\gamma X(e^{i\theta}) - \frac{\gamma^2}{2}\mathbb{E}[X(e^{i\theta})^2]}d\theta,$$

and he is able to verify the Fyodorov-Bouchaud formula [FB08a] from the physics literature, which says that

$$\mathbb{E}\left[M_\gamma(\mathbb{T})^p\right] = \frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p}, \qquad \forall p < \frac{4}{\gamma^2}.$$

In particular, the distribution of $M_\gamma(\mathbb{T})$ has an explicit density

$$\mathbb{P}(M_\gamma(\mathbb{T}) \in dm) = \frac{4\beta}{\gamma^2}(\beta m)^{-\frac{4}{\gamma^2}-1}e^{-(\beta m)^{-\frac{4}{\gamma^2}}}dm, \qquad m \ge 0$$

where $\beta = \Gamma(1-\frac{\gamma^2}{4})$, i.e. M_γ has the law of $\frac{1}{\beta}T^{-\frac{\gamma^2}{4}}$ where T is an Exp(1) random variable. With Zhu [RZ18] he extends the technique to the study of exactly scale invariant fields on the unit interval and verifies some of Ostrovsky's distributional conjectures. In general, however, there is no reason to believe that exact integrability results are possible for Gaussian multiplicative chaos even if we restrict ourselves to nice sets and kernels in $d=2$ where machinery from complex analysis can be employed. An example would be the total mass $M_\gamma(D)$ of the multiplicative chaos associated with the Gaussian free field on the unit disc, which seems to have all the possible symmetries one could hope for but little is known about it even at a heuristic level.

The final chapter of the thesis is devoted to another perspective on the distributional properties, namely the tail probability of general Gaussian multiplicative chaos. We consider the chaos measure $M_{\gamma,g}$ associated with general log-kernels (1.1.1), and derive the tail asymptotics

$$\mathbb{P}(M_{\gamma,g}(A) > t) \sim C_{\gamma,d,f,g}(A)t^{-\frac{2d}{\gamma^2}}, \qquad t \to \infty$$

for any bounded open sets A. The tail exponent should not be surprising as it is consistent with the criterion for the existence of moments. We are able to give a rather precise description of the tail coefficient $C_{\gamma,d,f,g}(A)$ including the dependence of the coefficient on

f, g and A. Using the exact integrability results [KRV15, KRV17, Rem17, RZ18], we are also able to provide a closed form expression for $C_{\gamma,d,f,g}(A)$ when the dimension is less than or equal to 2. This may be seen as a first step towards understanding finer universal distributional properties of Gaussian multiplicative chaos.

1.4 Future directions

To conclude this introductory chapter, we explain some of the general research themes that we intend to pursue, and present a selection of problems that have been inspired by the findings in this thesis and that we hope to address in the future.

Universality. There are two major goals we would like to achieve under the universality programme of Gaussian multiplicative chaos:

- We would like to establish Gaussian multiplicative chaos as a universal limit object in various different contexts. This could be extending our analysis to other models such as Wigner ensembles and Ginibre ensembles [WW18] in random matrix theory where evidence of logarithmic correlation and asymptotic Gaussianity is suggested by a central limit theorem for the linear statistics, or problems in integrable systems such as random partition of integers where Gaussian fluctuations have been observed. We hope that by establishing and exploiting new links between multiplicative chaos and large disordered systems, we are able to uncover new properties of both objects by translating the properties of one to those of the other.

- We hope to find an explanation for the universality of log-correlated Gaussian fields and multiplicative chaos, and study to what extent the convergence of a random function to a log-correlated Gaussian field is equivalent to that of the exponentiation of the random function to a multiplicative chaos.

Further distributional properties. Following the result in Chapter 4, we would like to investigate into the following problems.

- The leading order coefficient $C_{\gamma,d,f,g}(A)$ has a component $\overline{C}_{\gamma,d}$ called the reflection coefficient of Gaussian multiplicative chaos. This constant is unfortunately non-explicit in dimension $d \geq 3$ despite admitting various probabilistic representations, and we hope to find a way to adapt the Liouville conformal field theory techniques in [KRV15, KRV17, Rem17, RZ18] to obtain some exact integrability results in higher dimension.

- We would like to extend the investigation in Chapter 4 to lower order terms or full asymptotic series for the right tail probability if possible. It would also be interesting to consider the analogous problem for the left tail

$$\mathbb{P}(M_{\gamma,g}(A) \leq \epsilon), \qquad \epsilon \to 0^+$$

but this would require a very different approach, as the analysis of the right tail relies on a local argument which is completely irrelevant in the left tail problem.

- The interaction between Gaussian multiplicative chaos with different intermittency parameters γ is a topic that does not seem to have been explored very much before but it could be of fundamental importance if we would like to understand the scaling limit of several intermediate level sets of discrete Gaussian free field simultaneously. We would like to formulate a tail probability problem involving $M_{\gamma_1,g_1}(A_1)$ and $M_{\gamma_2,g_2}(A_2)$ where the chaos measures are defined with respect to the same Gaussian field but (γ_i, g_i, A_i) are different, and try to study how the knowledge of one of the variables affects the distribution of the other.

Beyond Gaussianity We would like to study non-Gaussian variants of multiplicative chaos which arise naturally in large disordered systems. This idea is inspired by the result in Chapter 2, where we show that the characteristic polynomial of a one-cut regular random matrix behaves asymptotically like a Gaussian multiplicative chaos. Without the one-cut condition, the (Gaussian) central limit theorem for the linear statistics of the random matrix is no longer true, and the convergence to Gaussian multiplicative chaos is conceptually impossible. We would therefore like to extend our study of random characteristic polynomials to more general unitary ensemble and hope to identify the multi-cut analogue of multiplicative chaos and study its properties there.

Chapter 2

Random Hermitian Matrices and Gaussian Multiplicative Chaos

Abstract. We prove that when suitably normalized, small enough powers of the absolute value of the characteristic polynomial of random Hermitian matrices, drawn from one-cut regular unitary invariant ensembles, converge in law to Gaussian multiplicative chaos measures. We prove this in the so-called L^2-phase of multiplicative chaos. Our main tools are asymptotics of Hankel determinants with Fisher-Hartwig singularities. Using Riemann-Hilbert methods, we prove a rather general Fisher-Hartwig formula for one-cut regular unitary invariant ensembles.

2.1 Introduction

2.1.1 Main result

Log-correlated Gaussian fields, namely Gaussian random generalized functions whose covariance kernels have a logarithmic singularity on the diagonal, are known to show up in various models of modern probability and mathematical physics – e.g. in combinatorial models describing random partitions of integers [IO02], random matrix theory [FKS16, HKO01, RV07], lattice models of statistical mechanics [Ken01], the construction of conformally invariant random planar curves such as stochastic Loewner evolution [AKSJ11, She16], and growth models [BF14] just to name a few examples. A recent and fundamental development in the theory of these log-correlated fields has been that while these fields are rough objects – distributions instead of functions – their geometric properties can be understood to some degree. For example, one can describe the behavior of the extremal values and level sets of the fields in a suitable sense – see e.g. [RV14, Section 4 and Section 6.4].

A fundamental tool in describing these geometric properties of the fields is a class of random measures, which can be formally written as an exponential of the field. As these fields are distributions instead of functions, exponentiation is not an operation one can naively perform, but through a suitable limiting and normalization procedure, these random measures can be rigorously constructed and they are known as Gaussian multiplicative chaos measures. These objects were introduced by Kahane in the 1980s [Kah85]. For a recent review, we refer the reader to [RV14] and for a concise proof of existence and uniqueness of these measures we refer to [Ber17].

A typical example of how log-correlated fields show up can be found in random matrix theory. For a large class of models of random matrix theory, the following is true: when the size of the matrix tends to infinity, the logarithm of the characteristic polynomial behaves like a log-correlated field. This is essentially equivalent to a suitable central limit theorem for the global linear statistics of the random matrix – see [FKS16, HKO01, RV07] for results concerning the GUE, Haar distributed random unitary matrices, and the complex Ginibre ensemble.

One would thus expect that the characteristic polynomial and powers of it should behave asymptotically like a multiplicative chaos measure. A related question was explored thoroughly though non-rigorously in [FK07, FS16]. The issue here is that the construction of the multiplicative chaos measure goes through a very specific approximation of the Gaussian field and typically uses things like independence and Gaussianity very strongly. In the random matrix theory situation these are present only asymptotically. Thus the precise extent of the connection between the theory of log-correlated processes and random matrix theory is far from fully understood. For rigorous results concerning multiplicative chaos and the study of extrema of approximately Gaussian log-correlated fields in random matrix theory we refer to [ABB17, CMN18, LOS18, LP18, PZ18, Web15].

In this article we establish a universality result showing that for a class of random Hermitian matrices, small enough powers of the absolute value of the characteristic polynomial can be described in terms of a Gaussian multiplicative chaos measure. More precisely, we prove the following result (for definitions of the relevant quantities, see Section 2.2).

Theorem 2.1.1. *Let H_N be a random $N \times N$ Hermitian matrix drawn from a one-cut regular, unitary invariant ensemble whose equilibrium measure is normalized to have support $[-1,1]$. Then for $\beta \in [0, \sqrt{2})$, the random measure*

$$\frac{|\det(H_N - x)|^\beta}{\mathbb{E}|\det(H_N - x)|^\beta} dx$$

on $(-1,1)$, converges in distribution with respect to the topology of weak convergence of measures on $(-1,1)$ to a Gaussian multiplicative chaos measure which can be formally written as $e^{\beta X(x) - \frac{\beta^2}{2}\mathbb{E}X(x)^2} dx$, where X is a centered Gaussian field with covariance kernel

$$\mathbb{E}X(x)X(y) = -\frac{1}{2}\log|2(x-y)|.$$

We note that in particular, this result holds for the Gaussian Unitary Ensemble (GUE) of random matrices, with a suitable normalization. The proof here is a generalization of that in [Web15] by the second author and relies on understanding the large N asymptotics of quantities which can be written in the form $\mathbb{E}[e^{\mathrm{Tr}\ \mathcal{T}(H_N)}\prod_{j=1}^k|\det(H_N-x_j)|^{\beta_j}]$ for a suitable function $\mathcal{T}:\mathbb{R}\to\mathbb{R}$, $x_j\in(-1,1)$ and $\beta_j\geq 0$.

It is easy to see, and we will recall the relevant derivations below, that such expectations can be written in terms of Hankel determinants with Fisher–Hartwig symbols, and while such quantities (and corresponding Toeplitz determinants) have been studied in great detail [CF16, DIK11, DIK14, Kra07], it seems that in the generality we require for Theorem 2.1.1, many of the results are lacking. Thus we give a proof of such results using Riemann–Hilbert techniques; see Proposition 2.2.10 for the precise result. This settles some conjectures due to Forrester and Frankel – see Remark 2.2.11 and [FF04, Conjecture 5 and Conjecture 8] for further information about their conjectures.

2.1.2 Motivations and related results

One of the main motivations for this work is establishing multiplicative chaos measures as something appearing universally when studying the global spectral behavior of random matrices. This is a new type of universality result in random matrix theory and also suggests that it should be possible to establish some of the geometric properties of log-correlated fields in the setting of random matrix theory as well. Perhaps on a more fundamental level, a further motivation for the work here is a general picture of when does the exponential of an approximation to a log-correlated field converge to a multiplicative chaos measure. Naturally we don't answer this question here, but the fact that our approach works so generally, suggests that part of this argument is something that transfers beyond random matrix theory to general models where one expects multiplicative chaos measures to play a role.

On a more speculative level, we also mention as motivation the connection to two-dimensional quantum gravity. It is well known that random matrix theory is related to a discretization of two-dimensional quantum gravity, namely the analysis of random planar maps – see e.g. [EM03] for a mathematically rigorous discussion of this connection. On the other hand, multiplicative chaos measures play a significant role in the study of Liouville quantum gravity [DKRV16, DS11] which is in some instances known to be the scaling limit of a suitable model of random planar maps [Gal13, Mie13, MS15, MS16a, MS16b]. The appearance of multiplicative chaos measures from random matrix theory seems like a curious coincidence from this point of view, and one that deserves further study.

One interpretation of Theorem 2.1.1 is that it gives a way of probing the (random fractal) set of points x where the recentered log characteristic polynomial $\log|\det(H_N-x)|-\mathbb{E}\log|\det(H_N-x)|$ is exceptionally large. In analogy with standard multiplicative chaos results (see e.g. [RV14, Theorem 4.1] or the approach of [Ber17]), one would expect that Theorem 2.1.1 implies that asymptotically, $\frac{|\det(H_N-x)|^\beta}{\mathbb{E}|\det(H_N-x)|^\beta}dx$ lives on the set of points x where

$$\lim_{N\to\infty} \frac{\log|\det(H_N-x)| - \mathbb{E}\log|\det(H_N-x)|}{\mathrm{Var}(\log|\det(H_N-x)|)} = \beta. \tag{2.1.1}$$

We emphasize that this really means that the (approximately Gaussian) random variable $\log|\det(H_N-x)| - \mathbb{E}\log|\det(H_N-x)|$ would be of the order of its variance instead of its standard deviation – as the variance is exploding, this is what motivates the claim of the log-characteristic polynomial taking exceptionally large values. Moreover, as it is known that the measure μ_β vanishes for $\beta \geq 2$, this connection suggests that for $\beta > 2$, there are no points where (2.1.1) is satisfied and that $\beta = 2$ corresponds to the scale of where the maximum of the field lives (note that it is rigorously known through other methods that the maximum is indeed on the scale of two times the variance of the field – see [LP18] and see also [ABB17, PZ18, CMN18] for analogous results in the case of ensembles of random unitary matrices). This suggests that suitable variants of Theorem 2.1.1 should provide a tool for studying extremal values of the characteristic polynomial, or even that more generally, existence of multiplicative chaos measures can be used to study the extremal behavior of log-correlated field. This is significant because maxima of logarithmically correlated fields (such as the log characteristic polynomial) are believed to display universality, and have as such been extensively studied in recent years (see e.g. [FHK12] and references below). In fact, the construction of Gaussian multiplicative chaos measures supported on points where the value of the field is a given fraction of the maximal value, may be viewed as part of the programme of establishing universality for such processes. While our results do not extend to the full range of values of β where one expects the result to be valid (roughly, we examine only the L^2 regime in Gaussian multiplicative chaos terminology), we believe that an appropriate modification of the methods of this paper eventually will yield the result in its full generality (for instance by combining it with a suitable modification of the approach in [Ber17]).

Regarding this programme, we mention the papers of Arguin, Belius and Bourgade [ABB17] which verify the leading order of the maximum of the CUE log characteristic polynomial, as well as Paquette and Zeitouni [PZ18] which refined this to obtain the second order, doubly logarithmic ("Bramson") correction. This is consistent with a prediction of Fyodorov, Hiary and Keating [FHK12]. In turn this was subsequently refined and generalized to the so-called circular β-ensemble by [CNN17] where tightness of the centered maximum was proved. For a large class of random Hermitian matrices, the leading order behavior was established recently by Lambert and Paquette [LP18], while in the case of the Riemann zeta function, the first order term was obtained (assuming the Riemann hypothesis) by Najnudel [Naj18] as well as (unconditionally) by Arguin et al. [ABB$^+$19]. In the case of the discrete Gaussian free field in two dimensions, the convergence in law of the recentered maximum was obtained recently in an important paper of Bramson, Ding and Zeitouni [BDZ16]. As for Gaussian multiplicative chaos measures (in the L^2-phase), the construction in the case of CUE random matrices was achieved by Webb [Web15]. Very recently, a related construction of a Gaussian multiplicative chaos measure was obtained by Lambert, Ostrovsky and Simm [LOS18] in the full L^1 regime of CUE random matrices, but for a slightly regularized version of the logarithm of the characteristic polynomial which is closer to a Gaussian field.

2.1.3 Organisation of the paper

The outline of the article is the following: in Section 2.2, we describe our model and objects of interest, our main results, and an outline of the proof. After this, in Section 2.3, we recall how the relevant moments can be expressed as Hankel determinants as well as how these determinants are related to orthogonal polynomials on the real line and Riemann-Hilbert problems. In this section we also recall from [DIK14] a differential identity for the relevant determinants. Then in Section 2.4 we go over the analysis of the relevant Riemann-Hilbert problem. This is very similar to the corresponding analysis in [Kra07, DIK14], but for completeness and due to slight differences in the proofs, we choose to present details of this in appendices. After this, in Section 2.5 we use the solution of the Riemann-Hilbert problem to integrate the differential identity to find the asymptotics of the relevant moments. Finally in Section 2.6, we put things together and prove our main results.

We have chosen to defer a number of technical proofs to the end of the paper in the form of multiple appendices. These contain proofs of results which might be considered in some sense routine calculations by experts in random matrix and integrable models, but which would require significant effort to readers not familiar with these techniques. Since we hope that the paper will be of interest to different communities, we have chosen to keep them in the paper at the cost of increasing its length.

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2.2 Preliminaries and outline of the proof

In this section, we describe the main objects we shall discuss in this article, state our main results, and give an outline of the proof of them.

2.2.1 One-cut regular ensembles of random Hermitian matrices

The basic objects we are interested in are $N \times N$ random Hermitian matrices H_N whose distribution can be written as

$$\mathbf{P}(dH_N) = \frac{1}{\widetilde{Z}_N(V)} e^{-N\mathrm{Tr}V(H_N)} dH_N, \tag{2.2.1}$$

where $dH_N = \prod_{j=1}^N dH_{jj} \prod_{1\leq i<j\leq N} d(\mathrm{Re}H_{ij})d(\mathrm{Im}H_{ij})$ denotes the Lebesgue measure on the space of $N \times N$ Hermitian matrices, $\mathrm{Tr}V(H_N)$ denotes $\sum_{j=1}^N V(\lambda_j)$, where (λ_j) are the eigenvalues of H_N (we drop the dependence on N from our notation), the potential $V : \mathbb{R} \to \mathbb{R}$ is a smooth function with nice enough growth at infinity so that this makes sense, and $\widetilde{Z}_N(V)$ is a normalizing constant. Perhaps the simplest model of such form is the Gaussian Unitary Ensemble for which $V(x) = 2x^2$. This corresponds to the diagonal entries of H_N being i.i.d. centered normal random variables with variance $1/(4N)$, and the entries above the diagonal being i.i.d. random variables whose real and imaginary parts are centered normal random variables with variance $1/(8N)$ and are independent of each other and of the diagonal entries. The entries below the diagonal are determined by the condition that the matrix is Hermitian.

The distribution (2.2.1) induces a probability distribution for the eigenvalues of H_N. In analogy with the GUE (see e.g. [AGZ09]) one finds that the distribution of the eigenvalues (on $\mathbb{R}^N$) is given by

$$\mathbb{P}(d\lambda_1, ..., d\lambda_N) = \frac{1}{Z_N(V)} \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{j=1}^N e^{-NV(\lambda_j)} d\lambda_j, \tag{2.2.2}$$

where $Z_N(V)$ is a normalizing constant called the partition function. Our main goal will be to describe the large N behavior of the characteristic polynomial of H_N, and more generally a power of this characteristic polynomial. To do this, we will have to impose further constraints on the function V. A general family of functions V for which our argument works is the class of one-cut regular potentials. We will review the relevant concepts here, but for more details, see [KM00].

First of all, we assume that V is real analytic on $\mathbb{R}$ and $\lim_{x\to\pm\infty} V(x)/\log|x| = \infty$. Further conditions on V are rather indirect as they are statements about the associated equilibrium measure μ_V which is defined as the unique minimizer of the functional

$$\mathcal{I}_V(\mu) = \int\int \log\frac{1}{|x-y|}\mu(dx)\mu(dy) + \int V(x)\mu(dx)$$

on the space of Borel probability measures on $\mathbb{R}$. For further information about μ_V, see e.g. [DKM98, ST97]. The measure μ_V can also be characterized in terms of Euler–Lagrange equations:

$$2\int \log|x-y|\mu_V(dy) = V(x) + \ell_V, \quad x \in \mathrm{supp}(\mu_V) \tag{2.2.3}$$

$$2\int \log|x-y|\mu_V(dy) \le V(x) + \ell_V, \quad x \notin \mathrm{supp}(\mu_V) \tag{2.2.4}$$

for some constant ℓ_V depending on V.

Our first constraint on V is that the support of μ_V is a single interval, and we normalize it to be $[-1,1]$. In this case, on $[-1,1]$, μ_V can be written as

$$\mu_V(dx) = d(x)\sqrt{1-x^2}dx, \tag{2.2.5}$$

where d is real analytic in some neighborhood of $[-1,1]$ – see [DKM98]. For one-cut regularity, we further assume that d is positive on $[-1,1]$ and that the inequality (2.2.4) is strict. We collect this all into a single definition.

Definition 2.2.1 (One-cut regular potentials)**.** *We say that the potential $V : \mathbb{R} \to \mathbb{R}$ is one-cut regular (with normalized support of the equilibrium measure) if it satisfies the following conditions:*

1. *V is real analytic.*
2. $\lim_{x\to\pm\infty} V(x)/\log|x| = \infty$.
3. *The support of the equilibrium measure μ_V is $[-1,1]$.*
4. *The inequality (2.2.4) is strict.*
5. *The real analytic function d from (2.2.5) is positive on $[-1,1]$.*

The condition that the support is $[-1,1]$ instead of say $[a,b]$ is not a real constraint since the general case can be mapped to this with a simple transformation. Moreover, note that the support of the equilibrium measure is where the eigenvalues accumulate asymptotically, as the size of the matrix tends to infinity. So in this limit, we expect that nearly all of the eigenvalues of H_N are in $[-1,1]$.

We also point out that this is a non-empty class of functions V, since for the GUE ($V(x)=2x^2$), it is known that all of the conditions of Definition 2.2.1 are satisfied – in particular $d(x)=2/\pi$ in this case.

2.2.2 The characteristic polynomial and powers of its absolute value

As mentioned, our main goal is to describe the large N behavior of the characteristic polynomial of H_N. There are several possibilities for what one might want to say. One could consider the characteristic polynomial at a single point, say inside the support of the equilibrium measure, in which case one might expect in analogy with random unitary matrices [KS00] that the logarithm of the characteristic polynomial should, as a linear statistic of eigenvalues, be asymptotically a Gaussian random variable with exploding variance. One could consider the behavior of the characteristic polynomial in a microscopic neighborhood of a fixed point, where one might expect it to be asymptotically a random analytic function as it is for the CUE – see [CNN17], or one could consider the logarithm of the absolute value of the characteristic polynomial on a macroscopic scale inside or outside the support of the equilibrium measure. For the GUE, on the macroscopic scale and in the support of the equilibrium measure, it is known [FKS16] that the recentered logarithm of the absolute value of the characteristic polynomial behaves like a random generalized function which is formally a Gaussian process with a logarithmic singularity in its covariance.

Our goal is to "exponentiate" this last statement. (Note that since the limiting process describing the logarithm of a the characteristic polynomial is only a generalized function, and not an actual function defined pointwise, taking its exponential is *a priori* highly nontrivial). More precisely, we make the following definitions.

Definition 2.2.2. *For $N\in\mathbb{Z}_+$, let H_N be distributed according to (2.2.1). For $x\in\mathbb{C}$, define*

$$P_N(x)=\det(H_N-x\mathbf{1}_{N\times N})=\prod_{j=1}^N(\lambda_j-x). \tag{2.2.6}$$

Moreover, let

$$X_N(x)=\log|P_N(x)|=\sum_{j=1}^N\log|\lambda_j-x|, \tag{2.2.7}$$

and for $\beta>0$, define the following measure on $(-1,1)$:

$$\mu_{N,\beta}(dx)=\frac{e^{\beta X_N(x)}}{\mathbb{E}e^{\beta X_N(x)}}dx=\frac{|P_N(x)|^\beta}{\mathbb{E}|P_N(x)|^\beta}dx. \tag{2.2.8}$$

While exponentiating a generalized function in general is impossible, it turns out that in our setting, the correct description of such a procedure is in terms of random measures known as Gaussian multiplicative chaos measures. We now describe some of the basics of the relevant theory.

2.2.3 Gaussian Multiplicative Chaos

Gaussian multiplicative chaos is a theory going back to Kahane [Kah85] with the aim of defining what the exponential of a Gaussian random (possibly generalized) function should mean when the covariance kernel of the Gaussian process has a suitable structure, as well as describing some geometric properties of these Gaussian processes.

Kahane proved, that if the covariance kernel has a logarithmic singularity, but otherwise has a particularly nice form, then with a suitable limiting and normalizing procedure, the exponential of the corresponding generalized function can be indeed understood as a random multifractal measure, known as a Gaussian multiplicative chaos measure. For a recent review of the theory, see [RV14] and for a concise proof for existence and uniqueness, see [Ber17].

Recently, these measures have found applications in constructing random SLE-like planar curves through conformal welding [AKSJ11, She16], quantum Loewner evolution [MS16b], the random geometry of two-dimensional quantum gravity [DKRV16, DS11] – see also the lecture notes [BerNotes], and even in models of mathematical finance [BKM13]. Complex variants of these objects are also connected to the statistical behavior of the Riemann zeta function on the critical line [SW16]. Perhaps their greatest importance is the role they are believed to play in describing the scaling limits of random planar maps embedded conformally – see [MS15, MS16a, MS16b] and [BerNotes]. In all of these cases, the covariance kernel of the Gaussian field has a logarithmic singularity on the diagonal.

In this section we will give a brief construction of the measures which are relevant to us. The random distribution we will be interested in is the whole-plane Gaussian free field restricted to the interval $(-1, 1)$ with a suitable choice of additive constant. Formally we will want to consider a Gaussian field X defined on $(-1, 1)$ such that it has a covariance kernel $\mathbb{E}X(x)X(y) = -\frac{1}{2}\log[2|x - y|]$. It can be shown that it is possible to construct such an object as a random variable taking values in a suitable Sobolev space of generalized functions, see [FKS16]. However, we will only need to work with approximations to this distribution which are well defined functions, so we will not need this fact. To motivate our definitions, we first recall a basic fact about expanding $\log|x - y|$ for $x, y \in (-1, 1)$ in terms of Chebyshev polynomials – see e.g. [Por90, Appendix C], [For10, Exercise 1.4.4], or [GP13, Lemma 3.1] for a proof.

Lemma 2.2.3. *Let $x, y \in (-1, 1)$ and $x \neq y$. Then*

$$\log|x - y| = -\log 2 - \sum_{n=1}^{\infty} \frac{2}{n} T_n(x)T_n(y), \tag{2.2.9}$$

where T_n is a Chebyshev polynomial of the first kind, i.e. it is the unique polynomial of degree n satisfying $T_n(\cos\theta) = \cos n\theta$ for all $\theta \in [0, 2\pi]$.

Thus formally, if $(A_k)_{k=1}^{\infty}$ were i.i.d. standard Gaussians and one defined

$$\mathcal{G}(x) = \sum_{j=1}^{\infty} \frac{A_j}{\sqrt{j}} T_j(x),$$

then one would have $\mathbb{E}\mathcal{G}(x)\mathcal{G}(y) = -\frac{1}{2}\log[2|x-y|]$. Motivated by this, we make the following definition.

Definition 2.2.4. *Let $(A_k)_{k=1}^{\infty}$ be i.i.d. standard Gaussian random variables. For $x \in (-1, 1)$ and $M \in \mathbb{Z}_+$, let*

$$\mathcal{G}_M(x) = \sum_{j=1}^{M} \frac{A_j}{\sqrt{j}} T_j(x). \tag{2.2.10}$$

We then want to understand $e^{\beta\mathcal{G}}$ (for suitable β) as a limit related to $e^{\beta\mathcal{G}_M}$ as $M \to \infty$. The precise statement is the following:

Lemma 2.2.5. *Consider the random measure*

$$\mu_\beta^{(M)}(dx) = e^{\beta\mathcal{G}_M(x) - \frac{\beta^2}{2}\mathbb{E}\mathcal{G}_M(x)^2} dx \tag{2.2.11}$$

on $(-1, 1)$. For $\beta \in (-\sqrt{2}, \sqrt{2})$, $\mu_\beta^{(M)}$ converges weakly almost surely (when the i.i.d. Gaussians are realized on the same probability space) to a non-trivial random measure μ_β on $(-1, 1)$, as $M \to \infty$.

This measure μ_β is the limiting object in Theorem 2.1.1. The basic idea is that the sequence $\mu_\beta^{(M)}$ is a measure-valued martingale, and it turns out that for $\beta \in (-\sqrt{2}, \sqrt{2})$, it is bounded in L^2 so by standard martingale theory it has a non-trivial limit. The L^2-boundedness is somewhat non-trivial and we will return to the details later.

Remark 2.2.6. *The measure μ_β exists actually for larger values of $|\beta|$ as well. It essentially follows from the standard theory of multiplicative chaos, or alternatively the approach of [Ber17], that a non-trivial limiting measure exists for $\beta \in (-2, 2)$. In fact, comparing with other log-correlated fields, it is natural to expect that with a suitable deterministic normalization, that differs from ours for some values of β, it is possible to construct a non-trivial limiting object for all $\beta \in \mathbb{C}$. However, for complex β, the limit might not be in general a measure (not even a signed measure), but only a distribution. We refer to [LRV15] for a study in complex multiplicative chaos and to [MRV16] for defining μ_β for large real β. Our approach for proving convergence relies critically on calculating second moments and it is known for example that the total mass of the measure μ_β has a finite second moment only for $\beta \in (-\sqrt{2}, \sqrt{2})$, so our approach is not directly possible* *for proving a corresponding result in the full range of values of β where we would expect the result to hold. However, combining our results, those of [CK15], and the approach of [LOS18] should yield the result for $\beta \in (0,2)$. This being said, we wish to point out that while the limiting object μ_β should exist for all complex β, one should not expect that $\mu_{N,\beta}$ converges to it if the real part of β is too negative – e.g. if $\beta \le -1$, then with overwhelming probability, $\int_{-1}^1 f(x)|P_N(x)|^\beta dx$ will be infinite and one can not hope for convergence. To avoid this type of complications, we focus on non-negative β.*

2.2.4 Outline of the proof

In this section we define the main objects we analyze in the proof of Theorem 2.1.1, and state the main results we need about them. Motivated by the approach in [Web15], we will consider an approximation to $\mu_{N,\beta}$, and we will denote this by $\widetilde{\mu}_{N,\beta}^{(M)}$, where M is an integer parametrizing the approximation. Using known results about the linear statistics of one-cut regular ensembles, it will be clear that as $N \to \infty$ for fixed M, $\widetilde{\mu}_{N,\beta}^{(M)} \to \mu_\beta^{(M)}$ in distribution. Thus our goal is to control the difference $\mu_{N,\beta} - \widetilde{\mu}_{N,\beta}^{(M)}$, when we first let $N \to \infty$ and then $M \to \infty$.

Let us begin by defining our approximation $\widetilde{\mu}_{N,\beta}^{(M)}$. It is essentially just truncating the Fourier-Chebyshev series of X_N, but we have to be slightly careful as the eigenvalues can be outside of $[-1,1]$ with non-zero probability.

Definition 2.2.7. *Fix $M \in \mathbb{Z}_+$ and $\epsilon > 0$ (small and possibly depending on M). Let $\widetilde{T}_j(x)$ be a $C^\infty(\mathbb{R})$-function with compact support such that $\widetilde{T}_j(x) = T_j(x)$ for each $x \in (-1-\epsilon, 1+\epsilon)$. Then define for $x \in (-1,1)$*

$$\widetilde{X}_{N,M}(x) = -\sum_{k=1}^{M} \frac{2}{k} \left[\sum_{j=1}^{N} \widetilde{T}_k(\lambda_j)\right] T_k(x), \tag{2.2.12}$$

and

$$\widetilde{\mu}_{N,\beta}^{(M)}(dx) = \frac{e^{\beta \widetilde{X}_{N,M}(x)}}{\mathbb{E} e^{\beta \widetilde{X}_{N,M}(x)}} dx. \tag{2.2.13}$$

Remark 2.2.8. *Our reasoning here is that if we pretended that all of the λ_j are in the interval $(-1,1)$, we could make use of Lemma 2.2.3. Then X_N would coincide with the above expansion for $M = \infty$ and $\widetilde{T}_j$ replaced by T_j. Outside of the interval, we have to use $\widetilde{T}_k$ instead of T_k, as otherwise $\mathbb{E} e^{\beta \widetilde{X}_{N,M}(x)}$ might not exist for all values of x and M.*

We will break our main statement down into parts now. The statement of our Theorem 2.1.1 is equivalent to saying that for each bounded continuous $\varphi : (-1,1) \to [0,\infty)$, $\mu_{N,\beta}(\varphi) := \int_{-1}^1 \varphi(x) \mu_{N,\beta}(dx)$ converges in distribution to $\mu_\beta(\varphi)$. It will actually be enough to assume that φ has compact support in $(-1,1)$, i.e. to prove vague convergence. We will be more detailed about these statements in the actual proof in Section 2.6. The way we will prove vague convergence is to write

$$\mu_{N,\beta}(\varphi) = [\mu_{N,\beta}(\varphi) - \widetilde{\mu}_{N,\beta}^{(M)}(\varphi)] + \widetilde{\mu}_{N,\beta}^{(M)}(\varphi).$$

By using standard central limit theorems for linear statistics of one-cut regular ensembles, and the definition of μ_β, we will see that the second term here tends to $\mu_\beta(\varphi)$ in the limit where first $N \to \infty$, and then $M \to \infty$. Our main result will then follow from showing that the second moment of the first term tends to zero in the same limit. We formulate this as a proposition.

Proposition 2.2.9. *If we first let $N \to \infty$ and then $M \to \infty$, then for $\beta \in (0, \sqrt{2})$ and each compactly supported continuous $\varphi : (-1,1) \to [0,\infty)$, $\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)$ converges in distribution to $\mu_\beta(\varphi)$, and*

$$\lim_{M\to\infty} \lim_{N\to\infty} \mathbb{E}|\mu_{N,\beta}(\varphi) - \widetilde{\mu}_{N,\beta}^{(M)}(\varphi)|^2 = 0. \tag{2.2.14}$$

Proving the second statement takes up most of this article. Expanding the square, we see that what is critical is having uniform asymptotics for $\mathbb{E}e^{\beta X_N(x)}$, $\mathbb{E}e^{\beta \widetilde{X}_{N,M}(x)}$, $\mathbb{E}e^{\beta(X_N(x)+X_N(y))}$, $\mathbb{E}e^{\beta(\widetilde{X}_{N,M}(x)+\widetilde{X}_{N,M}(y))}$, and $\mathbb{E}e^{\beta(X_N(x)+\widetilde{X}_{N,M}(y))}$. More precisely, we have:

$$\begin{aligned}\mathbb{E}|\mu_{N,\beta}(\varphi) - \widetilde{\mu}_{N,\beta}^{(M)}(\varphi)|^2 = &\iint \varphi(x)\varphi(y)\frac{\mathbb{E}(e^{\beta X_N(x)+\beta X_N(y)})}{\mathbb{E}(e^{\beta X_N(x)})\mathbb{E}(e^{\beta X_N(y)})}dxdy \\ &- 2\iint \varphi(x)\varphi(y)\frac{\mathbb{E}(e^{\beta X_N(x)+\beta \widetilde{X}_{N,M}(y)})}{\mathbb{E}(e^{\beta X_N(x)})\mathbb{E}(e^{\beta \widetilde{X}_{N,M}(y)})}dxdy \\ &+ \iint \varphi(x)\varphi(y)\frac{\mathbb{E}(e^{\beta \widetilde{X}_{N,M}(x)+\beta \widetilde{X}_{N,M}(y)})}{\mathbb{E}(e^{\beta \widetilde{X}_{N,M}(x)})\mathbb{E}(e^{\beta \widetilde{X}_{N,M}(y)})}dxdy.\end{aligned}$$

Each of these expectations here can be expressed as $\mathbb{E}\prod_{j=1}^N h(\lambda_j)$ for a suitable function $h : \mathbb{R} \to \mathbb{R}$. For instance,

$$e^{\beta X_N(x)+\beta \widetilde{X}_{N,M}(y)} = \prod_{j=1}^N |\lambda_j - x|^\beta e^{\mathcal{T}(\lambda_j)}; \text{ where } \mathcal{T}(\lambda) = \mathcal{T}(\lambda; y) = -\beta \sum_{k=1}^M \frac{2}{k}\widetilde{T}_k(\lambda)T_k(y).$$

As we will recall in Section 2.3, such quantities can be expressed in terms of Hankel determinants. Moreover, all of these Hankel determinants have a very specific type of symbol: one with so-called **Fisher–Hartwig** singularities. To explain what this means here, a Hankel matrix is a matrix in which the skew-diagonals are constant. They are closely related to Toeplitz matrices where the diagonals themselves are constant (these arise typically in the study of CUE and related random matrix ensembles rather than the GUE-type ensembles considered in this paper). In the case we will be interested in, the (i,j)th coefficient of the Hankel matrix will be of the form $\int_{\mathbb{R}} x^{i+j}h(x)e^{-NV(x)}dx$

where h is as above. When h is smooth enough and doesn't have any roots, then the asymptotic analysis of such determinants would follow from the classical strong Szegő theorem (actually this theorem applies in the Toeplitz case rather than the Hankel case, but here this isn't a crucial distinction). However in our situation h typically contains at least one root of the form $|x-x_i|^{\beta_i}$, which greatly complicates the task of analysing the corresponding determinant. This type of behavior is an example of a Fisher–Hartwig singularity. (In general a Fisher–Hartwig singularity might also include a jump at x_i corresponding to the symbol also having a term of the form $e^{\gamma \mathrm{Im} \log(x-x_i)}$.)

The asymptotics of Hankel determinants with Fisher–Hartwig singularities is still very much a subject of active research, and much information is already available using the steepest descent technique due to Deift and Zhou [DZ93]; see in particular the papers [DIK11, DIK14, Kra07, CF16] which play an important role in our proof. Yet results in the generality we need seem to still be lacking in the literature. What suffices for us is the following result (which we will only use with $k=1$ or $k=2$, but since there is no added difficulty in proving it for a general value of k we will do so).

Proposition 2.2.10. *Let $\mathcal{T} \in C^\infty(\mathbb{R})$ be real analytic in some neighborhood of $[-1,1]$ and have compact support. Let $k \in \mathbb{Z}_+$ be fixed, and let $\beta_1, ..., \beta_k \in [0,\infty)$ be fixed. Moreover, let $x_1, ..., x_k \in (-1,1)$ be distinct. Finally let H_N be a $N\times N$ random Hermitian matrix drawn from a one-cut regular unitary invariant ensemble with potential V. Then for $C(\beta) = 2^{\frac{\beta^2}{2}} \frac{G(1+\beta/2)^2}{G(1+\beta)}$, where G is the Barnes G function, we have as $N\to\infty$,*

$$\begin{aligned}
&\mathbb{E}\left[e^{\sum_{j=1}^N \mathcal{T}(\lambda_j)} \prod_{i=1}^k |\det(H_N - x_i)|^{\beta_i}\right] \qquad (2.2.15)\\
&= \prod_{j=1}^k C(\beta_j) \left(d(x_j)\frac{\pi}{2}\sqrt{1-x_j^2}\right)^{\frac{\beta_j^2}{4}} \left(\frac{N}{2}\right)^{\frac{\beta_j^2}{4}} e^{(V(x_j)+\ell_V)\frac{\beta_j}{2}N} \prod_{1\le i<j\le k} |2(x_i-x_j)|^{-\frac{\beta_i\beta_j}{2}}\\
&\quad\times e^{N\int_{-1}^1 \mathcal{T}(x) d(x)\sqrt{1-x^2}dx + \sum_{j=1}^k \frac{\beta_j}{2}\left[\int_{-1}^1 \frac{\mathcal{T}(x)}{\pi\sqrt{1-x^2}}dx - \mathcal{T}(x_j)\right]}\\
&\quad\times e^{\frac{1}{4\pi^2}\int_{-1}^1 dy \frac{\mathcal{T}(y)}{\sqrt{1-y^2}} P.V.\int_{-1}^1 \frac{\mathcal{T}'(x)\sqrt{1-x^2}}{y-x}dx}(1+o(1))
\end{aligned}$$

uniformly on compact subsets of $\{(x_1,...,x_k)\in(-1,1)^k : x_i \neq x_j \text{ for } i\neq j\}$. Here P.V. $\int$ denotes the Cauchy principal value integral. Moreover, if there exists a fixed $M\in\mathbb{Z}_+$, such that in some fixed neighborhood of $[-1,1]$, $\mathcal{T}(x) = \sum_{j=1}^M \alpha_j T_j(x)$, then the above asymptotics are uniform also in compact subsets of $\{(\alpha_1,...,\alpha_M)\in\mathbb{R}^M\}$.

Remark 2.2.11. *As mentioned in the introduction, this settles some conjectures due to Forrester and Frankel – see [FF04, Conjecture 5 and Conjecture 8] for more details. In terms of the potential V, we actually improve on the conjectures as these are only stated for polynomial V, but concerning the functions $\mathcal{T}$, our results are not as general as those* *appearing in the conjectures of Forrester and Frankel. This being said, one could easily relax some of our regularity assumptions on $\mathcal{T}$. In fact, the compact support or smoothness outside of a neighborhood of the interval $[-1,1]$ play essentially no role in our proof, but as this is a simple and clear way of stating the result, we do not attempt to state things in their greatest generality. Moreover, using techniques from [DIK14], one could attempt to generalize our estimates and prove a corresponding result when $\mathcal{T}$ is less smooth also on $[-1,1]$. Again, this is not necessary for our main goal, so we don't pursue this further.*

We also mention that after the first version of this article appeared, Charlier (in [Cha18]) proved an extension of this result to the case where the symbol can also have jump-type singularities.

We prove our results through Riemann–Hilbert methods. In particular, we first show that with a suitable differential identity, and some analysis of a Riemann–Hilbert problem, we can relate the $\mathcal{T}=0$ case to the $\mathcal{T}\neq 0$ case. Then with another differential identity (and further analysis of another Riemann–Hilbert problem) we relate the $\mathcal{T}=0$, general V-case to the GUE with $\mathcal{T}=0$. The asymptotics in the $\mathcal{T}=0$ case for the GUE have been obtained by Krasovsky [Kra07]. Using these, we are able to prove Proposition 2.2.10.

As we will need uniform asymptotics for $\mathbb{E}e^{\beta X_N(x)+\beta X_N(y)}$ and other terms, Proposition 2.2.10 is not quite enough for us. For uniform estimates, we will rely on a recent result of Claeys and Fahs [CF16], which combined with Proposition 2.2.10 will let us prove Proposition 2.2.9.

Next we review the connection between expectations of the form (2.2.15), Hankel determinants, and Riemann–Hilbert problems.

2.3 Hankel determinants and Riemann–Hilbert problems

In this section, we recall how the expectations we are interested in can be written as Hankel determinants, which are related to orthogonal polynomials, which in turn can be encoded into a Riemann–Hilbert problem. We also recall certain differential identities we will need for analyzing the expectations we are interested in. While our discussion is very similar to that in e.g. [DIK11, DIK14], there are some minor differences as we are dealing with Hankel determinants instead of Toeplitz ones. We choose to give some details for the convenience of a reader with limited experience with Riemann-Hilbert problems.

2.3.1 Hankel determinants and orthogonal polynomials

Terms of the form $\mathbb{E}\prod_{j=1}^N f(\lambda_j)$ can be written in determinantal form due to Andreief's identity – for a proof, one can use e.g. [AGZ09, Lemma 3.2.3] with the functions $f_i(x) = f(x)e^{-NV(x)}x^{i-1}$ and $g_i(x) = x^{i-1}$ as well as the product representation of the Vandermonde determinant.

Lemma 2.3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nice enough function (measurable and nice enough decay that all the relevant integrals converge absolutely). Then*

$$\mathbb{E}\prod_{j=1}^{N} f(\lambda_j) = \frac{N!}{Z_N(V)} \det\left(\int_{\mathbb{R}} x^{i+j} f(x) e^{-NV(x)} dx\right)_{i,j=0}^{N-1}. \tag{2.3.1}$$

where $Z_N(V)$ is as in (2.2.2).

Let us introduce some notation for the Hankel determinant here.

Definition 2.3.2. *For nice enough functions $f : \mathbb{R} \to \mathbb{R}$, (so that the integrals exist) let*

$$D_k(f) = D_k(f; V) = \det\left(\int_{\mathbb{R}} x^{i+j} f(x) e^{-NV(x)} dx\right)_{i,j=0}^{k}. \tag{2.3.2}$$

As the notation suggests, we will suppress the dependence on V when it's convenient. We suppress the dependence on N always.

It is a well known result in the theory of orthogonal polynomials, that such determinants can be written in terms of orthogonal polynomials. For the convenience of the reader, we offer a proof for the following result.

Lemma 2.3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be positive Lebesgue almost everywhere, have nice enough regularity and growth at infinity, and let $(p_j(x; f, V))_{j=0}^{\infty}$ be the sequence of real polynomials which have a positive leading order coefficient and which are orthonormal with respect to the measure $f(x)e^{-NV(x)}dx$ on $\mathbb{R}$ (we will write $p_j(x; f)$ when we wish to suppress the dependence on V and we will always suppress the dependence on N):*

$$\int_{\mathbb{R}} p_j(x; f) p_k(x; f) f(x) e^{-NV(x)} dx = \delta_{j,k}, \tag{2.3.3}$$

and $p_j(x; f) = \chi_j(f) x^j + \mathcal{O}(x^{j-1})$ as $x \to \infty$, where $\chi_j(f) > 0$. Then

$$D_k(f) = \prod_{j=0}^{k} \chi_j(f)^{-2}. \tag{2.3.4}$$

Note that due to our assumptions on f, the above polynomials do exist as we can construct them by applying the *determinantal representation* associated with the Gram–Schmidt procedure to the monomials.

Proof. Consider the space of real polynomials, equipped with an inner product given by the L^2 inner product on $\mathbb{R}$ with weight $f(x)e^{-NV(x)}$. A consequence of the Gram–Schmidt procedure applied to the sequence of monomials in this inner product space is the following:

for $j \geq 1$

$$p_j(x;f) = \frac{1}{\sqrt{D_{j-1}(f)D_j(f)}} \begin{vmatrix} \int f(y)e^{-NV(y)}dy & \cdots & \int y^j f(y)e^{-NV(y)}dy \\ \vdots & \ddots & \vdots \\ \int y^{j-1}f(y)e^{-NV(y)}dy & \cdots & \int y^{2j-1}f(y)e^{-NV(y)}dy \\ 1 & \cdots & x^j \end{vmatrix}. \quad (2.3.5)$$

where for $j = 0$ the determinant is replaced by 1, and $D_{-1}(f) = 1$.

Note that from our assumption on f and an easy generalization of Lemma 2.3.1, $D_j(f) > 0$ for all $j \geq 0$, so these polynomials exist. From (2.3.5) one sees that $\chi_j(f)$ – the coefficient of x^j in $p_j(x;f)$ – equals $\sqrt{D_{j-1}(f)/D_j(f)}$. The claim then follows as the product has a telescopic form, and we defined $D_{-1}(f) = 1$. □

2.3.2 Riemann-Hilbert problems and orthogonal polynomials

We now recall a result going back to Fokas, Its, and Kitaev [FIK92] about encoding orthogonal polynomials on the real line into a Riemann-Hilbert problem. In our setting, the relevant result is formulated in the following way.

Proposition 2.3.4 (Fokas, Its, and Kitaev)**.** *Let $\mathcal{T}$ be a real valued $C^\infty(\mathbb{R})$ function with compact support, let $(\beta_j)_{j=1}^k \in [0,\infty)^k$, $(x_j)_{j=1}^k \in (-1,1)^k$, and $x_i \neq x_j$ for $i \neq j$. Let V be some real analytic function on $\mathbb{R}$ satisfying $\lim_{x\to\pm\infty} V(x)/\log|x| = \infty$. For $\lambda \in \mathbb{R}$, define*

$$f(\lambda) = e^{\mathcal{T}(\lambda)} \prod_{j=1}^k |\lambda - x_j|^{\beta_j}, \quad (2.3.6)$$

and let $p_j(x;f)$ be as in Lemma 2.3.3, with the relevant measure being $f(\lambda)e^{-NV(\lambda)}d\lambda$ on $\mathbb{R}$. Consider the 2×2 matrix-valued function

$$Y(z) = Y_j(z;f,V) = \begin{pmatrix} \frac{1}{\chi_j(f)} p_j(z;f) & \frac{1}{\chi_j(f)} \int_{\mathbb{R}} \frac{p_j(\lambda;f)}{\lambda - z} \frac{f(\lambda)e^{-NV(\lambda)}d\lambda}{2\pi i} \\ -2\pi i \chi_{j-1}(f) p_{j-1}(z;f) & -\chi_{j-1}(f) \int_{\mathbb{R}} \frac{p_{j-1}(\lambda;f)}{\lambda - z} f(\lambda)e^{-NV(\lambda)}d\lambda \end{pmatrix}, \quad (2.3.7)$$

for $z \in \mathbb{C}\setminus\mathbb{R}$. Then Y is the unique solution to the following Riemann-Hilbert problem: find a function $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ such that

1. *Y is analytic.*
2. *On $\mathbb{R}$, Y has continuous boundary values $Y_\pm$, i.e. $Y_\pm(\lambda) = \lim_{\epsilon\to 0^+} Y(\lambda \pm i\epsilon)$ exists*

and is continuous for all $\lambda \in \mathbb{R}$. Moreover, $Y_\pm$ are related by the jump condition

$$Y_+(\lambda) = Y_-(\lambda) \begin{pmatrix} 1 & f(\lambda)e^{-NV(\lambda)} \\ 0 & 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{2.3.8}$$

3. As $z \to \infty$,

$$Y(z) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix}. \tag{2.3.9}$$

Remark 2.3.5. *Typically for Riemann-Hilbert problems related to Toeplitz and Hankel determinants with Fisher-Hartwig singularities (e.g. [DIK11, DIK14, CF16]) one says that the boundary values are continuous on the relevant contour minus the singularities x_j, and then imposes conditions on the behavior of Y near the singularities. This is relevant when one of the β_j is negative or non-real, but as we will shortly mention, in our case the boundary values are truly continuous on $\mathbb{R}$ and no further condition is needed.*

Sketch of proof. The proof for uniqueness is the standard one: one first looks at some solution to the RHP, say Y. From the jump condition, it follows that $\det Y$ is continuous across $\mathbb{R}$, so it is entire. From the behavior of Y at infinity, it follows that $\det Y$ is bounded, so by Liouville's theorem and the behavior at infinity, one sees that $\det Y = 1$. In particular, (as a matrix) Y is invertible and the inverse matrix Y^{-1} is analytic in $\mathbb{C} \setminus \mathbb{R}$. Now if $\widetilde{Y}$ is another solution, we see that $\widetilde{Y}Y^{-1}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and continuous across $\mathbb{R}$, so it is entire. From the behavior at infinity, $\widetilde{Y}(z)Y(z)^{-1} \to I$ (the 2×2 identity matrix) as $z \to \infty$, so again by Liouville, $\widetilde{Y} = Y$.

Consider then the statement that Y given in terms of the orthogonal polynomials is a solution. The analyticity condition is obvious. The continuity of the boundary values of the first column is obvious since we are dealing with polynomials. For the second column, the Sokhotski-Plemelj theorem implies that the boundary values of the second column can be expressed in terms of $p_j f e^{-NV}$ (or p_j replaced by p_{j-1}) and its Hilbert transform (see e.g. [Tit59, Chapter V] for an introduction to the Hilbert transform). The first term is obviously continuous. For the Hilbert transform, we note that $p_j f e^{-NV}$ is Hölder continuous, so as the Hilbert transform preserves Hölder regularity (see [Tit59, Chapter V.15]), we see that the boundary values of Y are continuous.

For the jump condition (2.3.8) and behavior at infinity (2.3.9), we refer to analogous problems in [Dei99, Section 3.2 and Section 7]. □

We next discuss how deforming V or $\mathcal{T}$ changes $D_{N-1}(f; V)$.

2.3.3 Differential identities

Let us fix our potential V (and drop dependence on it from our notation) and first consider how deforming $\mathcal{T}$ changes $D_{N-1}(f)$.

The proof of the following result is a minor modification of the proof of [DIK14, Proposition 3.3], but for completeness, we give a proof in Appendix 2.A. The role of this result is that if we know the asymptotics in the case $\mathcal{T} = 0$, instead of studying Y_j for all j, it's enough to study Y_N though with a one-parameter family of deformations of $\mathcal{T}$.

Lemma 2.3.6. *Let $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with compact support, let $(\beta_j)_{j=1}^k \in [0,\infty)^k$, $(x_j)_{j=1}^k \in (-1,1)^k$, and $x_i \neq x_j$ for $i \neq j$. For $t \in [0,1]$ and $\lambda \in \mathbb{R}$, define*

$$f_t(\lambda) = \left[1 - t + te^{\mathcal{T}(\lambda)}\right] \prod_{j=1}^k |\lambda - x_j|^{\beta_j}. \tag{2.3.10}$$

Let $Y(z,t)$ be as in (2.3.7) with $j = N$, $f = f_t$, and $p_l(x;f) = p_l(x;f_t)$ the orthonormal polynomials with respect to the measure $f_t(\lambda)e^{-NV(\lambda)}d\lambda$ on $\mathbb{R}$. Then

$$\partial_t \log D_{N-1}(f_t) = \frac{1}{2\pi i}\int_{\mathbb{R}} \left[Y_{11}(x,t)\partial_x Y_{21}(x,t) - Y_{21}(x,t)\partial_x Y_{11}(x,t)\right] \partial_t f_t(x) e^{-NV(x)} dx, \tag{2.3.11}$$

where the indices of Y refer to matrix entries.

The object we are interested in is $D_{N-1}(f_1)$ which we can analyze by writing

$$\log D_{N-1}(f_1) = \log D_{N-1}(f_0) + \int_0^1 \frac{\partial}{\partial t} \log D_{N-1}(f_t) dt.$$

For the GUE, the asymptotics of $D_{N-1}(f_0)$ – the case $\mathcal{T} = 0$ – were investigated in [Kra07], so a consequence of Lemma 2.3.6 is that if we understand the asymptotics of $Y(z,t)$ well enough, we are able to study the asymptotics of $D_{N-1}(f_1)$ in the GUE case.

The other deformation we will consider is what happens when we interpolate between the potentials $V_0(x) = 2x^2$ (the GUE) and $V_1(x) = V(x)$ in the $\mathcal{T} = 0$ case.

Lemma 2.3.7. *Let $(\beta_j)_{j=1}^k \in [0,\infty)^k$, $(x_j)_{j=1}^k \in (-1,1)^k$, and $x_i \neq x_j$ for $i \neq j$. Let f be defined by (2.3.6) with $\mathcal{T} = 0$ and let $V : \mathbb{R} \to \mathbb{R}$ be a real analytic function satisfying $\lim_{x\to\pm\infty} V(x)/\log|x| = \infty$. Define for $s \in [0,1]$*

$$V_s(x) = (1-s)2x^2 + sV(x). \tag{2.3.12}$$

Let us then write $Y(z;V_s)$ for Y defined as in (2.3.7) with $j = N$, $V = V_s$ and

$p_j(x;f) = p_j(x;f,V_s)$. *Then using the notation of* (2.3.2)

$$\begin{aligned}\partial_s \log D_{N-1}(f;V_s) \qquad\qquad (2.3.13)\\ = -N\frac{1}{2\pi i}\int_{\mathbb{R}} \left[Y_{11}(x;V_s)\partial_x Y_{21}(x;V_s) - Y_{21}(x;V_s)\partial_x Y_{11}(x;V_s)\right] f(x)[\partial_s V_s(x)]e^{-NV_s(x)}dx.\end{aligned}$$

Again, we give a proof in Appendix 2.A. The role of this differential identity is that if we understand the asymptotics of $Y(z;V_s)$ well enough, then by integrating (2.3.13), we can move from the GUE asymptotics to the general ones.

We mention that both of these identities are of course true for a much wider class of symbols than what we state in the results (in particular, in Lemma 2.3.7 the condition $\mathcal{T}=0$ is not necessary for anything). This is simply the generality we use them in. Next we move on to describing how to study the large N asymptotics of $Y(z,t)$ and $Y(z;V_s)$.

2.4 Solving the Riemann-Hilbert problem

In this section we will finally describe the asymptotic behavior of $Y(z,t)$ and $Y(z;V_s)$ as $N\to\infty$. The typical way this is done is through a series of transformations to the RHP, ultimately leading to a RHP where the jump matrix is asymptotically close to the identity matrix as $N\to\infty$, and the behavior at infinity is close to the identity matrix. Then using properties of the Cauchy-kernel, the final RHP can be solved in terms of a Neumann series solution of a suitable integral equation. Moreover, each term in the series expansion is of lower and lower order in N. We will go into further details about this part of the problem in Section 2.4.5, but we will start with transforming the problem.

While we never have both $s,t\in(0,1)$, we will find it notationally convenient to consider $Y(z)$ to be defined as in (2.3.7) with $f=f_t$ and $V=V_s$. We suppress all of this in our notation for Y. We will also focus on functions $\mathcal{T}$ with the regularity claimed in Proposition 2.2.10 which was stronger than what we stated in the differential identities.

2.4.1 Transforming the Riemann-Hilbert problem

Let us introduce some further notation to simplify things later on. Let $\mathcal{T}$ satisfy the conditions of Proposition 2.2.10, and let

$$\mathcal{T}_t(\lambda) = \log(1-t+te^{\mathcal{T}(\lambda)}) \qquad (2.4.1)$$

so that in the notation of Lemma 2.3.6

$$f_t(\lambda) = e^{\mathcal{T}_t(\lambda)}\prod_{j=1}^k |\lambda - x_j|^{\beta_j},$$

and let us assume that the singularities are ordered: $x_j < x_{j+1}$.

The series of transformations we will now start implementing is a minor modification of that in [Kra07, Section 4].

The first transformation

Our first transformation will change the asymptotic behavior of the solution to the RHP so that it is close to the identity as $z \to \infty$, as well as cause the distance between the jump matrix and the identity matrix to be exponentially small in N when we're off of the interval $[-1,1]$. The proofs of the statements of this section are either elementary or straightforward generalizations of standard ones in the RHP-literature, but for the convenience of readers unfamiliar with the literature, they are sketched in Appendix 2.B. Let us now make the relevant definitions.

Definition 2.4.1. *In the notation of* (2.2.5)*, for* $s \in [0,1]$ *as above, let*

$$d_s(\lambda) = (1-s)\frac{2}{\pi} + s d(\lambda), \tag{2.4.2}$$

and for $z \in \mathbb{C} \setminus (-\infty, 1]$*, let*

$$g_s(z) = \int_{-1}^{1} d_s(\lambda)\sqrt{1-\lambda^2}\log(z-\lambda)d\lambda, \tag{2.4.3}$$

where the branch of the logarithm is the principal one. We also define

$$\ell_s = (1-s)(-1-2\log 2) + s\ell_V, \tag{2.4.4}$$

where ℓ_V *is the constant from* (2.2.3) *and* (2.2.4)*. Finally, for* $z \in \mathbb{C} \setminus \mathbb{R}$*, let*

$$T(z) = e^{-N\ell_s\sigma_3/2}Y(z)e^{-N(g_s(z)-\ell_s/2)\sigma_3}, \tag{2.4.5}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad e^{q\sigma_3} = \begin{pmatrix} e^q & 0 \\ 0 & e^{-q} \end{pmatrix}.$$

Before describing the jump structure and normalization of T near infinity, we first point out some simple facts about the boundary values of g_s on $\mathbb{R}$ which follow from its definition and (2.2.3) (details may be found in Appendix 2.B).

Lemma 2.4.2. *For* $\lambda \in \mathbb{R}$*, let* $g_{s,\pm}(\lambda) = \lim_{\epsilon\to 0^+} g_s(\lambda \pm i\epsilon)$*. Then for* $\lambda \in (-1,1)$ *and* $s \in [0,1]$

$$g_{s,+}(\lambda) + g_{s,-}(\lambda) = V_s(\lambda) + \ell_s. \tag{2.4.6}$$

There exist $M, C > 0$ (independent of s) so that for $\lambda \in \mathbb{R} \setminus [-1,1]$,

$$g_{s,+}(\lambda) + g_{s,-}(\lambda) - V_s(\lambda) - \ell_s \leq \begin{cases} -C(|\lambda|-1)^{3/2}, & |\lambda| - 1 \in (0, M) \\ -\log|\lambda|, & |\lambda| - 1 > M \end{cases}. \tag{2.4.7}$$

For $\lambda \in \mathbb{R}$

$$g_{s,+}(\lambda) - g_{s,-}(\lambda) = \begin{cases} 2\pi i, & \lambda < -1 \\ 2\pi i \int_\lambda^1 d_s(x)\sqrt{1-x^2}dx, & |\lambda| < 1 \\ 0, & \lambda > 1 \end{cases}. \tag{2.4.8}$$

The function $g_{s,+} - g_{s,-}$ along with an analytic continuation of it will play a significant role in our analysis of the Riemann-Hilbert problem, so we give it a name.

Definition 2.4.3. *Let $U \subset \mathbb{C}$ be an open neighborhood of $\mathbb{R}$ into which d has an analytic continuation. For $z \in U \setminus ((-\infty, -1] \cup [1, \infty))$ and $s \in [0,1]$, let*

$$h_s(z) = -2\pi i \int_1^z d_s(w)\sqrt{1-w^2}dw, \tag{2.4.9}$$

where the square root is according to the principal branch (i.e. $\sqrt{1-w^2} = e^{\frac{1}{2}\log(1-w^2)}$ and the branch of the logarithm is the principal one), and the contour of integration is such that it stays in U and does not cross $(-\infty, -1] \cup [1, \infty)$.

The function h_s will often appear in the form $e^{\pm N h_s}$ and to estimate the size of such an exponential, we will need to know the sign of $\mathrm{Re}(h_s)$. For this, we use the following elementary fact.

Lemma 2.4.4. *In a small enough open neighborhood of $(-1,1)$ (independent of s) in the complex plane,*

$$\mathrm{Re}(h_s(z)) > 0 \qquad if \qquad \mathrm{Im}(z) > 0$$

and

$$\mathrm{Re}(h_s(z)) < 0 \qquad if \qquad \mathrm{Im}(z) < 0$$

for all $s \in [0,1]$, and if we restrict to a fixed set in the upper half plane such that the set is bounded away from the real axis, but inside this neighborhood of $(-1,1)$, we have e.g. $\mathrm{Re}(h_s(z)) \geq \epsilon > 0$ for some $\epsilon > 0$ independent of s. A similar result holds in the lower half plane.

Again, see Appendix 2.B for details on the proof of this and the next result, which describes the Riemann-Hilbert problem T solves.

Lemma 2.4.5. *The function $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ defined by (2.4.5) is the unique solution to the following Riemann-Hilbert problem.*

1. *$T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.*
2. *On $\mathbb{R}$, T has continuous boundary values $T_\pm$ and these are related by the jump conditions*

$$T_+(\lambda) = T_-(\lambda) \begin{pmatrix} e^{-Nh_s(\lambda)} & f_t(\lambda) \\ 0 & e^{Nh_s(\lambda)} \end{pmatrix}, \qquad \lambda \in (-1,1) \tag{2.4.10}$$

   and

$$T_+(\lambda) = T_-(\lambda) \begin{pmatrix} 1 & f_t(\lambda) e^{N(g_{s,+}(\lambda)+g_{s,-}(\lambda)-\ell_s-V_s(\lambda))} \\ 0 & 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}\backslash[-1,1]. \tag{2.4.11}$$

3. *As $z \to \infty$,*

$$T(z) = I + \mathcal{O}(|z|^{-1}). \tag{2.4.12}$$

The jump matrix given by (2.4.10) and (2.4.11) already looks good for $\lambda \notin [-1,1]$, in the sense that it is exponentially close to the identity, (compare (2.4.11) with (2.4.7)). However, the issue is that across $(-1,1)$, the jump matrix is not close to the identity in any way. We will next address this issue by performing a second transformation.

The second transformation

As customary in this type of problems, the next step is to "open lenses". That is, we will add further jumps to the problem off of the real line. Due to a nice factorization property of the jump matrix for T, the new jump matrix will be close to the identity on the new jump contours when we are not too close to the points ± 1 or x_j.

Before going into the details of this, we will define an analytic continuation of f_t into a subset of $\mathbb{C}$. Recall from our assumptions in Proposition 2.2.10 that on $(-1-\epsilon, 1+\epsilon)$, $\mathcal{T}(x)$ is real analytic. Thus $\mathcal{T}$ certainly has an analytic continuation to some neighborhood of $[-1,1]$. Moreover as it is real on $[-1,1]$, we see that in some small enough complex neighborhood of $[-1,1]$ (which is independent of t), $1 - t + te^{\mathcal{T}(z)}$ has no zeroes for any $t \in [0,1]$. Thus $\mathcal{T}_t$ (see (2.4.1)) has an analytic continuation to this neighborhood for all $t \in [0,1]$. We use this to define the analytic continuation of f_t.

Definition 2.4.6. *Let $U_{[-1,1]}$ be some neighborhood of $[-1,1]$ which is independent of t*

and in which $\mathcal{T}_t$ *is analytic for* $t \in [0,1]$*. In this domain, and for* $1 \le l \le k-1$*, let*

$$f_t(z) = e^{\mathcal{T}_t(z)} \times \begin{cases} \prod_{j=1}^k (x_j - z)^{\beta_j}, & \mathrm{Re}(z) < x_1 \\ \prod_{j=1}^l (x_j - z)^{\beta_j} \prod_{j=l+1}^k (z - x_j)^{\beta_j}, & \mathrm{Re}(z) \in (x_l, x_{l+1}), \\ \prod_{j=1}^k (z - x_j)^{\beta_j}, & \mathrm{Re}(z) > x_k \end{cases} \tag{2.4.13}$$

where the powers are according to the principal branch.

We will now impose some conditions on our new jump contours. Later on, we will be more precise about what we exactly want from them, but for now, we will ignore the details.

Definition 2.4.7. *For* $j = 1, ..., k+1$*, let* Σ_j^+ (Σ_j^-)*, be a smooth curve in the upper (lower) half plane from* x_{j-1} *to* x_j*, where we understand* x_0 *as* -1 *and* x_{k+1} *as* 1*. The curves are oriented from* x_{j-1} *to* x_j *and independent of* t*,* s*, and* N*. Moreover, they are contained in* $U_{[-1,1]}$*.*

The domain between Σ_j^+ *and* Σ_j^- *is called a lens. The domain between* Σ_j^+ *and* $\mathbb{R}$ *is called the top part of the lens, and that between* Σ_j^- *and* $\mathbb{R}$ *the bottom part of the lens. See Figure 2.1 for an illustration.*

Remark 2.4.8. *Our definition here and our coming construction implicitly assume that* $\beta_j \neq 0$ *for all* j*. If one (or more)* $\beta_j = 0$*, one simply ignores the corresponding* x_j *(so e.g. one connects* x_{j-1} *to* x_{j+1} *with a curve in the upper half plane etc).*

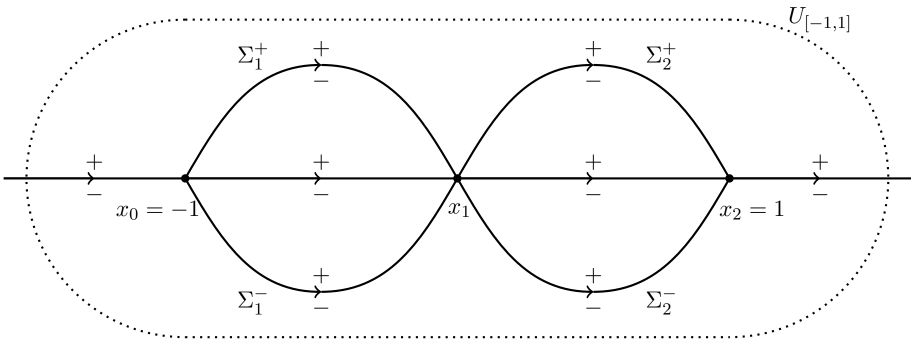

Figure 2.1: Opening of lenses, $k = 1$. The signs indicate the orientation of the curves: the $+$ side is the left side of the curve and $-$ the right.

We use these contours in our next transformation.

Definition 2.4.9. *For $z \notin \Sigma := \cup_{j=1}^{k+1}(\Sigma_j^+ \cup \Sigma_j^-) \cup \mathbb{R}$, let*

$$S(z) = \begin{cases} T(z), & \text{outside of the lenses} \\ T(z)\begin{pmatrix} 1 & 0 \\ -f_t(z)^{-1}e^{-Nh_s(z)} & 1 \end{pmatrix}, & \text{top part of the lenses} \\ T(z)\begin{pmatrix} 1 & 0 \\ f_t(z)^{-1}e^{Nh_s(z)} & 1 \end{pmatrix}, & \text{bottom part of the lenses} \end{cases}. \tag{2.4.14}$$

Remark 2.4.10. *Note that S depends on our choice of the contours Σ (as well as s, t, and N), but we suppress this in our notation. We also point out that as f_t has zeroes at the singularities, the entries in the first column of $S(z)$ blow up when z approaches a singularity from within the lens. Moreover, we see that we have discontinuities at the points ± 1. Thus the boundary values are no longer continuous on $\mathbb{R}$, but on $\mathbb{R} \setminus \{x_j : j = 0, ..., k+1\}$, where again $x_0 = -1$ and $x_{k+1} = 1$.*

Using the definition of S, the RHP for T, and the fact that

$$\begin{pmatrix} e^{-Nh_s(\lambda)} & f_t(\lambda) \\ 0 & e^{Nh_s(\lambda)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{Nh_s(\lambda)}f_t(\lambda)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{-Nh_s(\lambda)}f_t(\lambda)^{-1} & 1 \end{pmatrix}$$

it is simple to check what the Riemann–Hilbert problem for S should be; we omit the proof.

Lemma 2.4.11. *S is the unique solution to the following Riemann–Hilbert problem:*

1. *$S : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.*
2. *S has continuous boundary values on $\Sigma \setminus \{x_j\}_{j=0}^{k+1}$ and they are related by the jump conditions*

$$S_+(\lambda) = S_-(\lambda)\begin{pmatrix} 1 & 0 \\ f_t(\lambda)^{-1}e^{\mp Nh_s(\lambda)} & 1 \end{pmatrix}, \quad \lambda \in \cup_{j=1}^{k+1}\Sigma_j^{\pm} \setminus \{x_l\}_{l=0}^{k+1}, \tag{2.4.15}$$

$$S_+(\lambda) = S_-(\lambda)\begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}, \quad \lambda \in (-1,1) \setminus \{x_j\}_{j=1}^{k}, \tag{2.4.16}$$

and

$$S_+(\lambda) = S_-(\lambda) \begin{pmatrix} 1 & f_t(\lambda)e^{N(g_{s,+}(\lambda)+g_{s,-}(\lambda)-\ell_s-V_s(\lambda))} \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \setminus [-1,1]. \quad (2.4.17)$$

In (2.4.15) the $\mp$ and $\pm$ notation means that we have e^{-Nh_s} in the jump matrix when we cross Σ_j^+ and e^{Nh_s} when we cross Σ_j^-.

3. $S(z) = I + \mathcal{O}(|z|^{-1})$ *as* $z \to \infty$.
4. *For* $j = 1, ..., k$, $S(z)$ *is bounded as* $z \to x_j$ *from outside of the lenses, but when* $z \to x_j$ *from inside of the lenses,*

$$S(z) = \begin{pmatrix} \mathcal{O}(|z-x_j|^{-\beta_j}) & \mathcal{O}(1) \\ \mathcal{O}(|z-x_j|^{-\beta_j}) & \mathcal{O}(1) \end{pmatrix}. \quad (2.4.18)$$

Moreover, S is bounded at ± 1.

We are now in a situation where if we are on one of the $\Sigma_j^\pm$ or on $\mathbb{R} \setminus [-1,1]$ and not close to one of the points ± 1 or x_j, then the distance of the jump matrix from the identity matrix is exponentially small in N. We thus need to do something close to the points ± 1 and x_j as well as on the interval $(-1,1)$ to get a small norm problem, i.e. one that can be solved in terms of a Neumann series.

The way to proceed here is to construct functions which are solutions to approximations of the Riemann-Hilbert problem where we expect the approximations to be good if we are close to one of the points ± 1 or x_j, or then alternatively when we are far away from them and we expect the approximate problem related to the behavior on $(-1,1)$ to determine the behavior of S. We then construct an ansatz to the original problem in terms of these approximations. This will lead to a small norm problem.

These approximations are often called parametrices, and we will start with the solution far away from the points ± 1 and x_j. This case is often called the global parametrix.

2.4.2 The global parametrix

Our goal is to find a function $P^{(\infty)}(z)$ such that it has the same jumps as $S(z)$ across $(-1,1)$, is analytic elsewhere, and has the correct behavior at infinity. We won't go into great detail about how such problems are solved, but we will build on similar problems solved in [Kra07, Section 4.2] (see also for example [DKM+99, Section 5]). We will simply state the result here and sketch a proof in Appendix 2.C. Later on we will need some regularity properties of the solution considered here so we will state and prove the relevant facts here.

We now define our global parametrix.

Definition 2.4.12. *Let us write for $z \notin (-\infty, 1]$*

$$r(z) = (z-1)^{1/2}(z+1)^{1/2} \tag{2.4.19}$$

and

$$a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}, \tag{2.4.20}$$

where the powers are taken according to the principal branch. Then for $t \in [0,1]$ and $z \notin (-\infty, 1]$, let

$$\mathcal{D}_t(z) = (z + r(z))^{-\mathcal{A}} \exp\left[\frac{r(z)}{2\pi}\int_{-1}^{1} \frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}\frac{1}{z-\lambda}d\lambda\right]\prod_{j=1}^{k}(z-x_j)^{\beta_j/2} \tag{2.4.21}$$

where $\mathcal{A} = \sum_{j=1}^{k}\beta_j/2$ and the powers are according to the principal branch. Finally, for $z \notin (-\infty, 1]$ and $t \in [0,1]$, define the global parametrix

$$P^{(\infty)}(z) = P^{(\infty)}(z,t) = \frac{1}{2}\mathcal{D}_t(\infty)^{\sigma_3}\begin{pmatrix} a(z)+a(z)^{-1} & -i(a(z)-a(z)^{-1}) \\ i(a(z)-a(z)^{-1}) & a(z)+a(z)^{-1}\end{pmatrix}\mathcal{D}_t(z)^{-\sigma_3}, \tag{2.4.22}$$

where $\mathcal{D}_t(\infty) = \lim_{z\to\infty}\mathcal{D}_t(z) = 2^{-\mathcal{A}}e^{\frac{1}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}d\lambda}$.

Remark 2.4.13. *It's simple to check that r and a are continuous across $(-\infty,-1)$ so they can be analytically continued to $\mathbb{C}\setminus[-1,1]$. Using the fact that $r(\lambda)$ is negative for $\lambda < -1$, one can check that also $\mathcal{D}_t$ is continuous across $(-\infty,-1)$, so in fact $P^{(\infty)}$ is analytic in $\mathbb{C}\setminus[-1,1]$.*

We also point out that as $\mathcal{T}_0(\lambda) = 0$ (recall (2.4.1)) we can also write

$$P^{(\infty)}(z,t) = e^{\frac{\sigma_3}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}d\lambda}P^{(\infty)}(z,0)e^{-\sigma_3\frac{r(z)}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}\frac{d\lambda}{z-\lambda}}. \tag{2.4.23}$$

The relevance of this parametrix stems from the following lemma.

Lemma 2.4.14. *For each $t \in [0,1]$, $P^{(\infty)}(\cdot) = P^{(\infty)}(\cdot,t)$ satisfies the following Riemann–Hilbert problem.*

1. *$P^{(\infty)} : \mathbb{C}\setminus[-1,1] \to \mathbb{C}^{2\times 2}$ is analytic.*

2. *$P^{(\infty)}$ has continuous boundary values on $(-1,1)\setminus\{x_j\}_{j=1}^{k}$, and satisfies the jump*

condition

$$P_+^{(\infty)}(\lambda) = P_-^{(\infty)}(\lambda) \begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}, \qquad \lambda \in (-1,1) \setminus \{x_j\}_{j=1}^k. \tag{2.4.24}$$

3. As $z \to \infty$,

$$P^{(\infty)}(z) = I + \mathcal{O}(|z|^{-1}). \tag{2.4.25}$$

See Appendix 2.C for a proof. Later on, we will need some estimates on the regularity of the Cauchy transform appearing in (2.4.21) near the interval $[-1,1]$. The fact we need is the following one.

Lemma 2.4.15. *The function*

$$z \mapsto r(z) \int_{-1}^{1} \frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}} \frac{1}{z-\lambda} d\lambda$$

is bounded uniformly in $t \in [0,1]$ *and* z *in a small enough neighborhood of* $[-1,1]$*. Moreover, if in a neighborhood of* $[-1,1]$*,* $\mathcal{T}$ *is a real polynomial of fixed degree, and if we restrict its coefficients to be in some bounded set, then we have uniform boundedness of the above function in the coefficients of* $\mathcal{T}$ *as well.*

Proof. Let us fix a neighborhood of $[-1,1]$ such that for all $t \in [0,1]$, $\mathcal{T}_t$ is analytic in the closure of this neighborhood (this exists by similar reasoning as in the beginning of Section 2.4.1). Now write

$$\int_{-1}^{1} \frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}} \frac{1}{z-\lambda} d\lambda = \int_{-1}^{1} \frac{\mathcal{T}_t(\lambda)-\mathcal{T}_t(z)}{z-\lambda} \frac{1}{\sqrt{1-\lambda^2}} d\lambda + \mathcal{T}_t(z) \int_{-1}^{1} \frac{1}{\sqrt{1-\lambda^2}} \frac{1}{z-\lambda} d\lambda.$$

As $\mathcal{T}_t$ is analytic, the first term is of order $\mathcal{O}(\sup_{t\in[0,1]} ||\mathcal{T}_t'||_\infty)$ (the prime referring to the z-variable and the sup-norm is over z in the neighborhood we are considering) which is a finite constant depending on our neighborhood of $[-1,1]$ and the function $\mathcal{T}$. In the polynomial case, one can easily check that it is bounded uniformly in the coefficients when they are restricted to a compact set. The second integral can be calculated exactly:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-\lambda^2}} \frac{1}{z-\lambda} d\lambda = \frac{\pi}{r(z)}.$$

This can be seen for example by expanding the Cauchy kernel for large $|z|$ as a geometric series. The integrals resulting from this are simple to calculate and one can then also calculate the remaining sum exactly. The resulting quantity agrees with $\pi/r(z)$ on $(1,\infty)$ so by analyticity, the statement holds. The claim now follows from the uniform boundedness of $\mathcal{T}_t$ (for which the uniform boundedness in the polynomial case is again easy to check). □

2.4.3 Local parametrices near the singularities

We now wish to find functions approximating $S(z)$ well near the points x_j. We will thus look for functions that satisfy the same jump conditions as $S(z)$ in some fixed neighborhoods of the points x_j for $j = 1, ..., k$, but we will also want these approximations to be consistent with the global approximation, so we will replace a normalization at infinity with a matching condition, where we demand that the two approximations are close to each other on the boundary of the neighborhood we are looking at at. Our argument is built on [Kra07, Section 4.3], which in turn relies on [Van03, Section 4]. Again, we state the relevant facts here and give some further details in Appendix 2.D.

In this case, we will have to introduce a bit more notation before defining our actual object. We first introduce a change of coordinates that will blow up in a neighborhood of a singularity in a good way.

Definition 2.4.16. *Fix some $\delta > 0$ (independent of N, s, and t). Let us write U_{x_j} for the open δ-disk surrounding x_j. We assume that δ is small enough that the following conditions are satisfied:*

i) $|x_i - x_j| > 3\delta$ *for* $i \neq j$.

ii) $|x_j \pm 1| > 3\delta$ *for all* $j \in \{1, ..., k\}$.

iii) For all j, U'_{x_j} – the open $3\delta/2$-disk around x_j – is contained in U, which is some neighborhood of $\mathbb{R}$ into which d has an analytic continuation (see e.g. Definition 2.4.3).

For $z \in U'_{x_j}$, let

$$\zeta_s(z) = \pi N \int_{x_j}^{z} \left[\frac{2}{\pi}(1-s) + s d(w)\right] \sqrt{1-w^2} dw, \tag{2.4.26}$$

where the root is according to the principal branch, and the integration contour does not leave U'_{x_j}.

Remark 2.4.17. *The reason for introducing the two neighborhoods U_{x_j} and U'_{x_j}, is that we will want the local parametrices to be analytic functions approximately agreeing with $P^{(\infty)}$ on the boundary of U_{x_j}, but to ensure that they behave nicely near the boundary, we will construct them such that they are analytic in U'_{x_j}.*

We also point out that by taking δ smaller if needed, ζ_s can be seen to be injective as d is positive on $[-1, 1]$. More precisely, we see that $\zeta'_s(x_j) > cN$ for some constant c which is independent of s (but not necessarily of δ) and $|\zeta''_s(z)| \leq CN$ uniformly in $z \in U'_{x_j}$ for *some $C > 0$ independent of s (but not necessarily of δ). From this one sees that ζ_s is injective in a small enough (N- and s-independent) neighborhood of x_j.*

In addition to this change of coordinates, we will need to add further jumps to make our jump contour more symmetric, in order to obtain an approximate problem with a known solution.

Definition 2.4.18. *For $z \in U'_{x_j}$, let*

$$W_j(z) = W_j(z,t) \tag{2.4.27}$$
$$= e^{\mathcal{T}_t(z)/2} \prod_{l=1}^{j-1} (z - x_l)^{\beta_l/2} \prod_{l=j+1}^{k} (x_l - z)^{\beta_l/2} \times \begin{cases} (z - x_j)^{\beta_j/2}, & |\arg \zeta_s(z)| \in (\pi/2, \pi) \\ (x_j - z)^{\beta_j/2}, & |\arg \zeta_s(z)| \in (0, \pi/2) \end{cases},$$

where the roots are principal branch roots. Moreover, let

$$\phi_s(z) = \begin{cases} \frac{h_s(z)}{2}, & \mathrm{Im}(z) > 0 \\ -\frac{h_s(z)}{2}, & \mathrm{Im}(z) < 0 \end{cases}. \tag{2.4.28}$$

The precise form of ζ_s will be important for us to be able to see that the local parametrices indeed approximately agree with $P^{(\infty)}$ on the boundary of U_{x_j}. We also point out that for small enough δ, ζ_s is one-to-one, and it preserves the real axis (along with the orientation of the plane as it's conformal).

We also point out that W_j is almost identical to $f_t^{1/2}$, apart from the fact that it introduces some further branch cuts to it: along the imaginary axis in the ζ_s-plane, as well as on the real axis (recall that f_t has no branch cut along the real axis). These further branch cuts are useful in transforming the Riemann-Hilbert problem for the parametrix into one with certain constant jump matrices along a very special contour. This problem has been studied in [Van03].

We are now able to clarify our choice of the contours $\Sigma_j^{\pm}$ apart from the behavior near the end points ± 1.

Definition 2.4.19. *Let $(\Sigma_l^{\pm})_l$ be such that*

$$\zeta_s\left(\Sigma_{j-1}^{\pm} \cap U'_{x_j}\right) = \left[e^{\pm 3\pi i/4} \times [0,\infty)\right] \cap \zeta_s\left(U'_{x_j}\right) \tag{2.4.29}$$

and

$$\zeta_s\left(\Sigma_{j}^{\pm} \cap U'_{x_j}\right) = \left[e^{\pm \pi i/4} \times [0,\infty)\right] \cap \zeta_s\left(U'_{x_j}\right). \tag{2.4.30}$$

Outside of U'_{x_j} (apart from close to ± 1), we take $(\Sigma_l^{\pm})_l$ to be smooth, without self-intersections and the distance between them and the real axis to be bounded away from zero and of order δ, and such that the contours are contained in U – the neighborhood of $\mathbb{R}$ into which d has an analytic continuation. For an illustration, see Figure 2.2.

Using the injectivity of ζ_s we argued in Remark 2.4.17 and the Koebe quarter theorem, it is immediate that $\Sigma_j^{\pm}$ and $\Sigma_{j-1}^{\pm}$ are well defined for large enough N and small enough δ (large and small enough being independent of s).

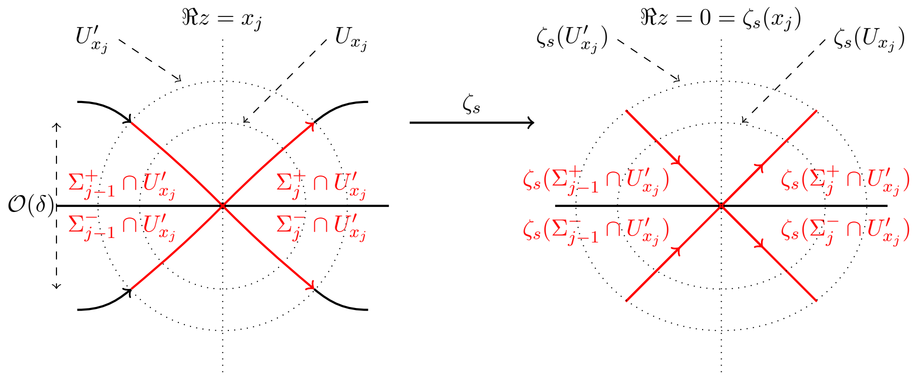

Figure 2.2: Choice of the jump contours near the singularities.

We still need one further ingredient before defining our local parametrix. This is a solution to a model Riemann-Hilbert problem – a problem where the jump contours and matrices are particularly simple and a solution can be given explicitly in terms of suitable special functions. We will give a rather compact definition here with a more detailed description in Appendix 2.D.

Definition 2.4.20. *Let us denote by Roman numerals the octants of the complex plane – so we write* $\mathrm{I} = \{re^{i\theta} : r > 0, \theta \in (0, \pi/4)\}$ *and so on. Denote by* Γ_l *the boundary rays of these octants: for* $1 \leq l \leq 8$, $\Gamma_l = \{re^{i\frac{\pi}{4}(l-1)}, r > 0\}$, *oriented as in Figure 2.3.*

For $\zeta \in \mathrm{I}$, *let*

$$\Psi(\zeta) = \frac{1}{2}\sqrt{\pi\zeta}\begin{pmatrix} H^{(2)}_{\frac{\beta_j+1}{2}}(\zeta) & -iH^{(1)}_{\frac{\beta_j+1}{2}}(\zeta) \\ H^{(2)}_{\frac{\beta_j-1}{2}}(\zeta) & -iH^{(1)}_{\frac{\beta_j-1}{2}}(\zeta) \end{pmatrix} e^{-\left(\frac{\beta_j}{2}+\frac{1}{4}\right)\pi i\sigma_3}, \tag{2.4.31}$$

where $H^{(i)}_\nu$ *are Hankel functions and the root is according to the principal branch. In other octants,* Ψ *satisfies the following Riemann-Hilbert problem:*

1. $\Psi : \mathbb{C} \setminus \cup_{l=1}^{8} \overline{\Gamma_l} \to \mathbb{C}^{2\times 2}$ *is analytic.*

2. Ψ *has continuous boundary values on each* Γ_l *and satisfies the following jump condition (again for the orientation, see Figure 2.3)* $\Psi_+(\zeta) = \Psi_-(\zeta)K(\zeta)$ *for* $\zeta \in$

$\cup_{l=1}^{8}\Gamma_l$, *where*

$$K(\zeta) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \Gamma_1 \cup \Gamma_5 \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \beta_j} & 1 \end{pmatrix}, & \zeta \in \Gamma_2 \cup \Gamma_6 \\ e^{\pi i \frac{\beta_j}{2}\sigma_3}, & \zeta \in \Gamma_3 \cup \Gamma_7 \\ \begin{pmatrix} 1 & 0 \\ e^{\pi i \beta_j} & 1 \end{pmatrix}, & \zeta \in \Gamma_4 \cup \Gamma_8 \end{cases} \tag{2.4.32}$$

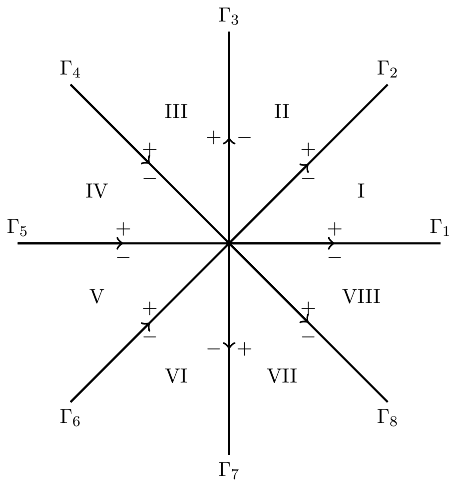

Figure 2.3: Jump contour of the model RHP

Uniqueness of such a Ψ can be argued in a similar manner as usual. First of all, one can check that for $\zeta \in \mathrm{I}$, $\det \Psi(\zeta) = 1$. As the jump matrices all have unit determinant, $\det \Psi$ is analytic in $\mathbb{C} \setminus \{0\}$, so $\det \Psi(\zeta) = 1$ for $\zeta \in \mathbb{C}$ (one can check that $\zeta = 0$ is a removable singularity). Consider then some other solution to the problem, say $\widetilde{\Psi}$. As $\det \Psi = \det \widetilde{\Psi} = 1$, $\Psi(\zeta)\widetilde{\Psi}(\zeta)^{-1}$ is analytic in $\mathbb{C} \setminus \cup_{l=1}\overline{\Gamma_l}$ and equals I for $\zeta \in \mathrm{I}$. Again it follows from the jump structure that $\Psi(\zeta)\widetilde{\Psi}(\zeta)^{-1}$ continues analytically to $\mathbb{C} \setminus \{0\}$ so it must equal I everywhere. For an explicit description of the solution, see Appendix 2.D.

The local parametrices will then be formulated in terms of this function Ψ, a coordinate change given by ζ_s, the function W_j, and an analytic ($\mathbb{C}^{2\times 2}$-valued) "compatibility matrix" E, which is needed for the matching condition to be satisfied. We now make the relevant definitions.

Definition 2.4.21. *For $z \in U'_{x_j} \cap \{\mathrm{Im}(z) > 0\}$, write*

$$E(z) = E(z,t,s) = P^{(\infty)}(z,t)W_j(z,t)^{\sigma_3} e^{N\phi_{s,+}(x_j)\sigma_3} e^{-(1\mp\beta_j)\pi i\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{2.4.33}$$

where the $-$ sign is in the domain $\{z \in \mathbb{C} : \arg(\zeta_s(z)) \in (0, \pi/2)\}$ and the $+$ sign is in the domain $\{z \in \mathbb{C} : \arg(\zeta_s(z)) \in (\pi/2, \pi)\}$. For $z \in U'_{x_j} \cap \{\mathrm{Im}(z) < 0\}$, write

$$E(z) = P^{(\infty)}(z)W_j(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{N\phi_{s,+}(x_j)\sigma_3} e^{-(1\mp\beta_j)\pi i\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{2.4.34}$$

where $-$ sign is in the domain $\{z \in \mathbb{C} : \arg(\zeta_s(z)) \in (-\pi/2, 0)\}$ and the $+$ sign is in the domain $\{z \in \mathbb{C} : \arg(\zeta_s(z)) \in (-\pi, -\pi/2)\}$.

Finally, for $z \in U'_{x_j} \setminus \Sigma$, let

$$P^{(x_j)}(z) = P^{(x_j)}(z,s,t) = E(z,s,t)\Psi(\zeta_s(z))W_j(z,t)^{-\sigma_3} e^{-N\phi_s(z)\sigma_3}. \tag{2.4.35}$$

Remark 2.4.22. *Using (2.4.27) – the definition of W_j – as well as (2.4.24) – the jump conditions of $P^{(\infty)}$, one can check that E has no jumps in U'_{x_j}. Moreover, using the behavior of both functions near x_j, one can check that E does not have an isolated singularity at x_j, so E is analytic in U'_{x_j}.*

We also point out that it follows directly from the definitions, i.e. (2.4.27), (2.4.33), (2.4.34), and (2.4.35), that for $z \in U'_{x_j} \setminus \Sigma$

$$P^{(x_j)}(z,t,s) = P^{(\infty)}(z,t)e^{\frac{1}{2}\mathcal{T}_t(z)\sigma_3} \left[P^{(\infty)}(z,0)\right]^{-1} P^{(x_j)}(z,0,s)e^{-\frac{1}{2}\mathcal{T}_t(z)\sigma_3}. \tag{2.4.36}$$

The main claim about $P^{(x_j)}$ is the following, whose proof we sketch in Appendix 2.D.

Lemma 2.4.23. *The function $P^{(x_j)}$ satisfies the following Riemann-Hilbert problem.*

1. *$P^{(x_j)} : U'_{x_j} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.*
2. *$P^{(x_j)}$ has continuous boundary values on $\Sigma \cap U'_{x_j} \setminus \{x_j\}$ and these satisfy the following jump conditions (with the same orientation as for S and same convention for the sign in $e^{\mp Nh_s(\lambda)}$): for $\lambda \in (U'_{x_j} \setminus \{x_j\}) \cap (\Sigma_{j-1}^+ \cup \Sigma_{j-1}^- \cup \Sigma_j^+ \cup \Sigma_j^{-1})$*

$$P_+^{(x_j)}(\lambda) = P_-^{(x_j)}(\lambda) \begin{pmatrix} 1 & 0 \\ f_t(\lambda)^{-1}e^{\mp Nh_s(\lambda)} & 1 \end{pmatrix}, \tag{2.4.37}$$

and for $\lambda \in \mathbb{R} \cap U'_{x_j} \setminus \{x_j\}$

$$P_+^{(x_j)}(\lambda) = P_-^{(x_j)}(\lambda) \begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}. \tag{2.4.38}$$

3. $P^{(x_j)}(z)$ *is bounded as* $z \to x_j$ *from outside of the lenses, but when* $z \to x_j$ *from inside of the lenses*

$$P^{(x_j)}(z) = \begin{pmatrix} \mathcal{O}(|z - x_j|^{-\beta_j}) & \mathcal{O}(1) \\ \mathcal{O}(|z - x_j|^{-\beta_j}) & \mathcal{O}(1) \end{pmatrix}. \tag{2.4.39}$$

4. *For* $z \in \partial U_{x_j}$

$$P^{(x_j)}(z) \left[P^{(\infty)}(z)\right]^{-1} = I + \mathcal{O}(N^{-1}), \tag{2.4.40}$$

where the $\mathcal{O}(N^{-1})$*-term is a* 2×2 *matrix whose entries are* $\mathcal{O}(N^{-1})$ *uniformly in* z, s, t, $\{|x_i - x_j| \geq 3\delta \text{ for } i \neq j\}$, *and* $\{|1 \pm x_j| \geq 3\delta \text{ for all } j \in \{1, ..., k\}\}$. *If in a neighborhood of* $[-1, 1]$, $\mathcal{T}$ *is a real polynomial of fixed degree, the error is also uniform in the coefficients once they are restricted to some bounded set.*

For our second differential identity, we will actually need more precise information about $P^{(x_j)}$ on ∂U_{x_j}. While we will only use it in the $\mathcal{T} = 0$ case, it is not more difficult to formulate the result in the general case.

Lemma 2.4.24. *For* $z \in \partial U_{x_j}$

$$P^{(x_j)}(z) \left[P^{(\infty)}(z)\right]^{-1} = I + \frac{\beta_j}{4\zeta_s(z)} E(z) \begin{pmatrix} 0 & 1 + \frac{\beta_j}{2} \\ 1 - \frac{\beta_j}{2} & 0 \end{pmatrix} E(z)^{-1} + \mathcal{O}\left(N^{-2}\right), \tag{2.4.41}$$

where the $\mathcal{O}(N^{-2})$*-term is a* 2×2 *matrix whose entries are* $\mathcal{O}(N^{-2})$ *uniformly in* z, s, *and* $\{|x_i - x_j| \geq 3\delta \text{ for } i \neq j\}$ *and* $\{|1 \pm x_j| \geq 3\delta \text{ for all } j \in \{1, ..., k\}\}$.

The $t = 0$, $s = 0$ case of these results has been proven in [Kra07, Section 4.3], though without focus on the uniformity relevant to us. Due to this, we will again sketch a proof in Appendix 2.D.

2.4.4 Local parametrices at the edge of the spectrum

The reasoning here is similar to the previous section – we wish to find a function approximating S near the points ± 1. We will do this by approximating the Riemann-Hilbert problem and imposing a matching condition. Our argument will follow [Kra07, Section 4.4], which in turn relies on [DKM+99]. We will focus on the approximation at 1, as the one at -1 is analogous. Again we will provide a sketch of the relevant proofs in Appendix 2.E. We will begin by introducing the relevant coordinate change in this case (analogous to ζ_s in the previous section).

Definition 2.4.25. *Let $\delta > 0$ satisfy the conditions of Definition 2.4.16. Denote by U_1 a δ-disk around 1 and U_1' denote a $3\delta/2$-disk around 1. We assume that δ is small enough that d has an analytic extension to U_1'. Moreover, we assume δ is small enough – though independent of s – so that with a suitable choice of the branch, the function*

$$\xi_s(z) = \left[-\frac{3}{2}N\phi_s(z)\right]^{2/3} \tag{2.4.42}$$

is analytic and injective in U_1', for all $s \in [0,1]$.

We will justify that this is indeed possible in Appendix 2.E. This conformal coordinate change allows us to define what $\Sigma_{k+1}^{\pm}$ looks like near 1. Let $\delta > 0$ be small enough to satisfy the conditions of Definition 2.4.25 and so that $\mathcal{T}_t$ is analytic in U_1' for all $t \in [0,1]$. We will define the local parametrix in U_1' and impose the matching condition on ∂U_1. Let us thus define $\Sigma_{k+1}^{\pm}$ in U_1'.

Definition 2.4.26. *Inside U_1', let $\Sigma_{k+1}^{\pm}$ be such that*

$$\xi_s(\Sigma_{k+1}^{\pm} \cap U_1') = \left[e^{\pm 2\pi i/3} \times [0,\infty)\right] \cap \xi_s(U_1'). \tag{2.4.43}$$

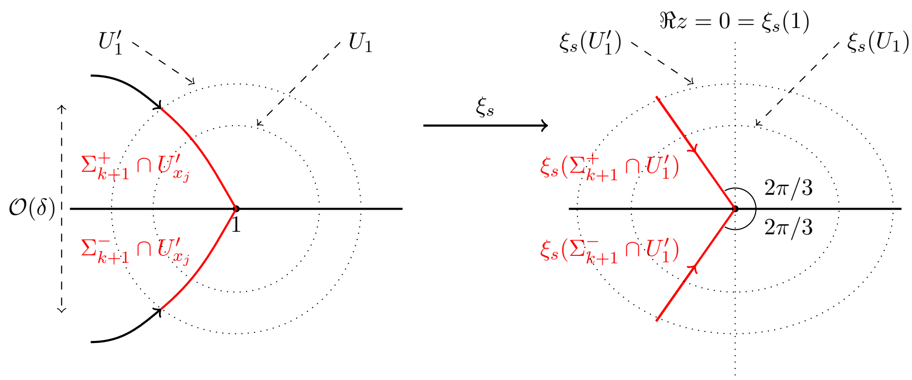

Figure 2.4: Choice of the jump contours near the edge of the spectrum.

Remark 2.4.27. *The angle $2\pi/3$ is slightly arbitrary here. In [DKM+99] the model Riemann-Hilbert problem relevant to us is constructed for a family of angle parameters $\sigma \in (\pi/3, \pi)$, and any angle here would work just as well for us, but we choose this for concreteness.*

Also we point out that the above definition is fine as we know that ξ_s is injective and we can apply the Koebe quarter theorem to ensure that the preimage of the rays is non-empty.

We are now also in a position to define our local parametrix. As in the previous section, we need for this a solution to a certain model RHP considered in [DKM+99] as well as a function which is analytic in U'_{x_j} which is required for the matching condition to hold.

Definition 2.4.28. *Let us write* $\mathrm{I} = \{re^{i\theta} : r > 0, \theta \in (0, 2\pi/3)\}$, $\mathrm{II} = \{re^{i\theta} : r > 0, \theta \in (2\pi/3, \pi)\}$, $\mathrm{III} = \{re^{i\theta} : r > 0, \theta \in (-\pi, -2\pi/3)\}$, *and* $\mathrm{IV} = \{re^{i\theta} : r > 0, \theta \in (-2\pi/3, 0)\}$. *Then define*

$$Q(\xi) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\xi) & \mathrm{Ai}(e^{4\pi i/3}\xi) \\ \mathrm{Ai}'(\xi) & e^{4\pi i/3}\mathrm{Ai}'(e^{4\pi i/3}\xi) \end{pmatrix} e^{-\pi i \sigma_3/6}, & \xi \in \mathrm{I} \\ \begin{pmatrix} \mathrm{Ai}(\xi) & \mathrm{Ai}(e^{4\pi i/3}\xi) \\ \mathrm{Ai}'(\xi) & e^{4\pi i/3}\mathrm{Ai}'(e^{4\pi i/3}\xi) \end{pmatrix} e^{-\pi i \sigma_3/6} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \xi \in \mathrm{II} \\ \begin{pmatrix} \mathrm{Ai}(\xi) & -e^{4\pi i/3}\mathrm{Ai}(e^{4\pi i/3}\xi) \\ \mathrm{Ai}'(\xi) & -\mathrm{Ai}'(e^{4\pi i/3}\xi) \end{pmatrix} e^{-\pi i \sigma_3/6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \xi \in \mathrm{III} \\ \begin{pmatrix} \mathrm{Ai}(\xi) & -e^{4\pi i/3}\mathrm{Ai}(e^{4\pi i/3}\xi) \\ \mathrm{Ai}'(\xi) & -\mathrm{Ai}'(e^{4\pi i/3}\xi) \end{pmatrix} e^{-\pi i \sigma_3/6}, & \xi \in \mathrm{IV} \end{cases}, \tag{2.4.44}$$

where Ai *is the Airy function.*

Morover, define another "compatibility matrix"

$$F(z) = F(z,t,s) = P^{(\infty)}(z,t) f_t(z)^{\sigma_3/2} e^{i\pi\sigma_3/4} \sqrt{\pi} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xi_s(z)^{\sigma_3/4} e^{-\pi i/12}, \tag{2.4.45}$$

where the roots are principal branch roots, and

$$P^{(1)}(z) = P^{(1)}(z,t,s) = F(z) Q(\xi_s(z)) e^{-N\phi_s(z)\sigma_3} f_t(z)^{-\sigma_3/2}. \tag{2.4.46}$$

Remark 2.4.29. *Note that we can write*

$$P^{(1)}(z,t,s) = P^{(\infty)}(z,t) e^{\mathcal{T}_t(z)\sigma_3/2} \left[P^{(\infty)}(z,0) \right]^{-1} P^{(1)}(z,0,s) e^{-\mathcal{T}_t(z)\sigma_3/2}. \tag{2.4.47}$$

Again the relevant fact about this function is that it satisfies a suitable Riemann-Hilbert problem. Part of this is the fact that F in (2.4.45) is an analytic function in U'_1. As before, we sketch the proof in Appendix 2.E.

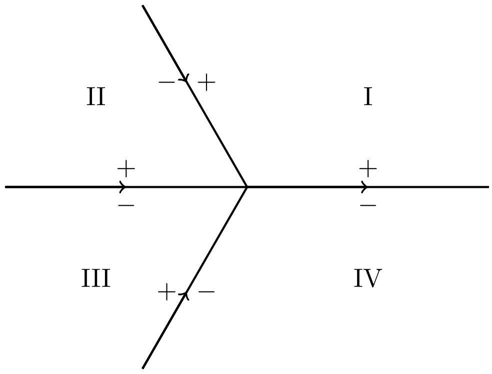

Figure 2.5: Jump contour of $Q(\xi)$

Lemma 2.4.30. *The function F from (2.4.45) is analytic in U_1' and the function $P^{(1)}(z)$ satisfies the following Riemann-Hilbert problem.*

1. *$P^{(1)}(z)$ is analytic in $U_1' \setminus (\Sigma_{k+1}^+ \cup \Sigma_{k+1}^- \cup \mathbb{R})$.*
2. *For $\lambda \in (-1,1) \cap U_1'$, $P^{(1)}$ satisfies*

$$P_+^{(1)}(\lambda) = P_-^{(1)}(\lambda) \begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}. \tag{2.4.48}$$

For $\lambda \in (1,\infty) \cap U_1'$, $P^{(1)}$ satisfies

$$P_+^{(1)}(\lambda) = P_-^{(1)}(\lambda) \begin{pmatrix} 1 & f_t(\lambda) e^{N(g_{+,s}(\lambda)+g_{s,-}(\lambda)-V_s(\lambda)-\ell_s)} \\ 0 & 1 \end{pmatrix}. \tag{2.4.49}$$

For $\lambda \in \Sigma_{k+1}^{\pm}$, $P^{(1)}$ satisfies

$$P_+^{(1)}(\lambda) = P_-^{(1)}(\lambda) \begin{pmatrix} 1 & 0 \\ f_t(\lambda)^{-1} e^{\mp N h_s(\lambda)} & 1 \end{pmatrix}. \tag{2.4.50}$$

3. *For $z \in \partial U_1$, $P^{(1)}$ satisfies the following matching condition,*

$$P^{(1)}(z) \left[P^{(\infty)}(z)\right]^{-1} = I + \mathcal{O}(N^{-1}), \tag{2.4.51}$$

where the entries of the $\mathcal{O}(N^{-1})$ matrix are $\mathcal{O}(N^{-1})$ uniformly in $z \in \partial U_1$, uniformly in $\{x_i\}$ for $|x_i - x_j| \geq 3\delta$ for $i \neq j$ and $|x_i \pm 1| \geq 3\delta$ for $j \in \{1,...,k\}$, uniformly *in $t \in [0,1]$, and uniformly in $s \in [0,1]$. If in a neighborhood of $[-1,1]$, $\mathcal{T}$ is a real polynomial with fixed degree, the error is also uniform in the coefficients once they are restricted to some bounded set.*

Again we will need finer asymptotics for our second differential identity and we will formulate them in the $\mathcal{T}=0$ case.

Lemma 2.4.31. *For $z \in \partial U_1$,*

$$P^{(1)}(z)\left[P^{(\infty)}(z)\right]^{-1} - I$$

$$= P^{(\infty)}(z) f(z)^{\sigma_3/2} e^{i\pi\sigma_3/4} \frac{1}{8} \begin{pmatrix} \frac{1}{6} & 1 \\ -1 & -\frac{1}{6} \end{pmatrix} e^{-i\pi\sigma_3/4} f(z)^{-\sigma_3/2} \left[P^{(\infty)}(z)\right]^{-1} \xi_s(z)^{-3/2} + \mathcal{O}(N^{-2})$$

where the $\mathcal{O}(N^{-2})$-term is a 2×2 matrix whose entries are $\mathcal{O}(N^{-2})$ uniformly in z, s, and $\{|x_i - x_j| \geq 3\delta \text{ for } i \neq j\}$ and $\{|1 \pm x_j| \geq 3\delta \text{ for all } j \in \{1,...,k\}\}$.

Remark 2.4.32. *Using the definition of F, one can check that this can be written also as*

$$P^{(1)}(z)\left[P^{(\infty)}(z)\right]^{-1} = I + F(z) \begin{pmatrix} 0 & \frac{5}{48}\xi_s(z)^{-2} \\ -\frac{7}{48}\xi_s(z)^{-1} & 0 \end{pmatrix} F(z)^{-1} + \mathcal{O}(N^{-2}).$$

From the previous representation of the matching condition matrix, one can easily see that the subleading term is indeed of order N. The benefit of this representation is that as F and F^{-1} are analytic in U_1, the subleading term is analytic in $U_1 \setminus \{1\}$ and has (at most) a second order pole at $z = 1$.

2.4.5 The final transformation and asymptotic analysis of the problem

We now perform the final transformation of the problem, and solve it asymptotically. The proofs of these statements are essentially standard in the RHP literature, but we don't know of a reference where the exact calculations we need exist and also issues such as uniformity in our relevant parameters are essential for us, but not usually stressed in the literature. Thus we provide proofs in Appendix 2.F.

Definition 2.4.33. *Let us fix some small $\delta > 0$ ("small" being independent of t and s and detailed in Section 2.4.3 and Section 2.4.4), and write $U_{\pm 1}$ for a δ-disk around ± 1 and U_{x_j} for a δ-disk around x_j. We also assume that for $i \neq j$, $|x_i - x_j| \geq 3\delta$ and for all*

$i \neq 0, k+1$, $|x_i \pm 1| \geq 3\delta$. We then define

$$R(z) = \begin{cases} S(z)\left[P^{(-1)}(z)\right]^{-1}, & z \in U_{-1} \setminus \Sigma \\ S(z)\left[P^{(x_j)}(z)\right]^{-1}, & z \in U_{x_j} \setminus \Sigma \text{ for some } j \\ S(z)\left[P^{(1)}(z)\right]^{-1}, & z \in U_1 \setminus \Sigma \\ S(z)\left[P^{(\infty)}(z)\right]^{-1}, & z \in \mathbb{C} \setminus \overline{U_{-1} \bigcup \cup_{j=1}^{k} U_{x_j} \bigcup U_1 \bigcup \Sigma} \end{cases}. \tag{2.4.52}$$

We now state what is the Riemann–Hilbert solved by R – for details, see Appendix 2.F.

Lemma 2.4.34. *For the δ in Definition 2.4.33, define*

$$\Gamma_\delta = (\mathbb{R} \setminus [-1-\delta, 1+\delta]) \bigcup \left(\cup_{j=1}^{k+1} (\Sigma_j^+ \cup \Sigma_j^-) \setminus \overline{U_{-1} \cup \cup_{j=1}^{k} U_{x_j} \cup U_1} \right) \bigcup \left(\partial U_{-1} \cup \cup_{j=1}^{k} \partial U_{x_j} \cup \partial U_1 \right), \tag{2.4.53}$$

where $\mathbb{R}$ and the lenses are oriented as before. ∂U_{x_j} and $\partial U_{\pm 1}$ are oriented in a clockwise manner – see Figure 2.6. Then R is the unique solution to the following Riemann-Hilbert problem:

1. *$R : \mathbb{C} \setminus \Gamma_\delta \to \mathbb{C}^{2\times 2}$ is analytic.*
2. *R satisfies the jump conditions $R_+(\lambda) = R_-(\lambda) J_R(\lambda)$ (with lenses and $\mathbb{R}$ oriented as before, and the circles are oriented clockwise), where the jump matrix J_R take the following form:*

   (i) For $\lambda \in \mathbb{R} \setminus [-1-\delta, 1+\delta]$,

$$J_R(\lambda) = P^{(\infty)}(\lambda) \begin{pmatrix} 1 & f_t(\lambda) e^{N(g_{s,+}(\lambda) + g_{s,-}(\lambda) - V_s(\lambda) - \ell_s)} \\ 0 & 1 \end{pmatrix} \left[P^{(\infty)}(\lambda)\right]^{-1}. \tag{2.4.54}$$

   (ii) For $\lambda \in \cup_{j=1}^{k+1} \Sigma_j^{\pm} \setminus \overline{U_{-1} \cup \cup_{j=1}^{k} U_{x_j} \cup U_1}$,

$$J_R(\lambda) = P^{(\infty)}(\lambda) \begin{pmatrix} 1 & 0 \\ f_t(\lambda)^{-1} e^{\mp N h_s(\lambda)} & 1 \end{pmatrix} \left[P^{(\infty)}(\lambda)\right]^{-1}. \tag{2.4.55}$$

   (iii) For $\lambda \in \partial U_{x_j} \setminus \cup_{j=1}^{k+1} (\Sigma_j^+ \cup \Sigma_j^-)$,

$$J_R(\lambda) = P^{(x_j)}(\lambda) \left[P^{(\infty)}(\lambda)\right]^{-1}. \tag{2.4.56}$$

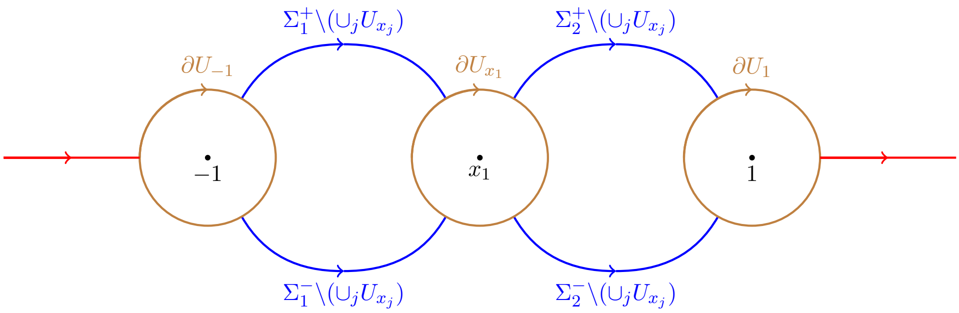

Figure 2.6: The jump contour of the Riemann–Hilbert problem for R, in the case $k = 1$.

(*iv*) *For* $\lambda \in \partial U_{\pm 1} \setminus (\mathbb{R} \cup \cup_{j=1}^{k+1}(\Sigma_j^+ \cup \Sigma_j^-)$,

$$J_R(\lambda) = P^{(\pm 1)}(\lambda) \left[P^{(\infty)}(\lambda)\right]^{-1}. \tag{2.4.57}$$

3. As $z \to \infty$,

$$R(z) = I + \mathcal{O}(|z|^{-1}). \tag{2.4.58}$$

The first ingredient to solving this Riemann–Hilbert problem is to show that the jump matrix of $R(z)$ is close to the identity matrix in a suitable sense.

Lemma 2.4.35. *For* $z \in \Gamma_\delta$*, write* $J_R(z) = I + \Delta_R(z) = I + \Delta$ *for the jump matrix of* R *as described in Lemma 2.4.34. Then for any* $p \geq 1$*, and large enough* N *("large enough" depending only on* V*)*

$$||\Delta||_{L^p(\Gamma_\delta)} = \mathcal{O}(N^{-1})$$

where the norm is any matrix norm, the L^p*-spaces are with respect to the Lebesgue measure on the jump contour, and the* $\mathcal{O}(N^{-1})$ *term is uniform in everything relevant (i.e.,* (x_i) *for* $|x_i - x_j| \geq 3\delta$*, for* $i \neq 0, k+1$*:* $|x_i \pm 1| \geq 3\delta$*, in* $s, t \in [0, 1]$*, and if* $\mathcal{T}$ *is a real polynomial in a neighborhood of* $[-1, 1]$*, then in its coefficients when restricted to a bounded set; but may depend on* δ*).*

See Appendix 2.F for a proof. We will want to show that R is close to the identity, and the tool which allows us to do this is the following representation of R as a solution to a suitable integral equation involving its jump matrix.

Proposition 2.4.36. *Let* $\delta > 0$ *be small enough ("small enough" being independent of* s *and* t*). For* N *sufficiently large (again independent of* s *and* t*), the unique solution of the Riemann–Hilbert problem for* R *(see Lemma 2.4.34) is given by*

$$R = I + C[\Delta + (I - C_\Delta)^{-1}(C_\Delta(I))\Delta] \tag{2.4.59}$$

where

$$C(f) := \frac{1}{2\pi i}\int_{\Gamma_\delta} f(s)\frac{ds}{s-z}$$

is the Cauchy operator on Γ_δ*, and* $C_\Delta(f) = C_-(f\Delta)$ *where* $C_-(f) = \lim_{z\to s} C(f)$ *as* z *approaches a point* $s \in \Gamma_\delta\backslash\{\text{intersection points}\}$ *from the* $-$*side of* Γ_δ *(for the orientation, see Lemma 2.4.34).*

Finally, what we want to show is that $R(z) = I + \mathcal{O}(N^{-1})$ uniformly in everything relevant and use this as well as the explicit form of our parametrices to analyze our differential identities. The precise statement we need is the following one.

Theorem 2.4.37. *For small enough* $\delta > 0$ *(again small enough being independent of relevant quantities) and large enough* N *(large enough being independent of everything relevant) with respect to any matrix norm* $|\cdot|$*, there exists a* $c > 0$ *such that*

$$|R(z) - I| \leq \frac{c}{N} \qquad \text{and} \qquad |R'(z)| \leq \frac{c}{N}$$

uniformly in (x_i) *for* $|x_i - x_j| \geq 3\delta$*,* $|x_i \pm 1| \geq 3\delta$ *for* $i \neq 0, k+1$*,* $t, s \in [0,1]$*,* $z \in \mathbb{C}\backslash\Gamma_\delta$*, and if* $\mathcal{T}$ *is a real polynomial in a neighborhood of* $[-1,1]$*, then the error is uniform in its coefficients when these are restricted to a bounded set.*

Moreover, for $\mathcal{T} = 0$*, we have*

$$R(z) = I + R_1(z) + o(1/N), \qquad R'(z) = R_1'(z) + o(1/N)$$

uniformly in (x_i) *for* $|x_i - x_j| \geq 3\delta$*,* $|x_i \pm 1| \geq 3\delta$ *for* $i \neq 0, k+1$*,* $s \in [0,1]$*, and* $z \in \mathbb{C}\backslash(\Gamma_\delta \cup \cup_{j=0}^{k+1} U_{x_j})$*. Here* $R_1(z) = \sum_{j=0}^{k+1} R_1^{(x_j)}(z)$ *with*

$$R_1^{(x_j)}(z) = \begin{cases} \frac{1}{x_j - z}\frac{\beta_j}{4\pi N d_s(x_j)\sqrt{1-x_j^2}} E^{(x_j)}(x_j)\begin{pmatrix} 0 & 1+\frac{\beta_j}{2} \\ 1-\frac{\beta_j}{2} & 0\end{pmatrix}\left[E^{(x_j)}(x_j)\right]^{-1}, & j \in \{1,...,k\} \\ -\operatorname*{Res}_{w=-1}\frac{1}{w-z}F^{(-1)}(w)\begin{pmatrix} 0 & -\frac{5}{48\xi_s^{(-1)}(w)^2} \\ -\frac{7}{48\xi_s^{(-1)}(w)} & 0\end{pmatrix}\left[F^{(-1)}(w)\right]^{-1}, & j = 0 \\ -\operatorname*{Res}_{w=1}\frac{1}{w-z}F^{(1)}(w)\begin{pmatrix} 0 & \frac{5}{48\xi_s^{(1)}(w)^2} \\ -\frac{7}{48\xi_s^{(1)}(w)} & 0\end{pmatrix}\left[F^{(1)}(w)\right]^{-1}, & j = k+1 \end{cases}.$$

where E *and* F *are the "compatibility matrices" from Definitions 2.4.21 and 2.4.28. In particular, we have*

$$\mathcal{J}^{(x_j)}(z) := \left([P^{(\infty)}(z)]^{-1}\left[R_1^{(x_j)}\right]'(z)P^{(\infty)}(z)\right)_{22}$$

$$
= \begin{cases}
\frac{1}{4}\frac{1}{(z-x_j)^2}\frac{i\beta_j}{4\pi N d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2}(c_{x_j,s}^2 + c_{x_j,s}^{-2} - \beta_j)\right. \\
\qquad \left. - \frac{a_+(x_j)^2}{a(z)^2}(c_{x_j,s}^2 + c_{x_j,s}^{-2} + \beta_j)\right], & j \in \{1,\dots,k\} \\
-\frac{1}{(z+1)^2}\frac{\sqrt{2}i}{8N}\left\{a(z)^{-2}\left[\frac{5+96\mathcal{A}^2}{48G_s^{(-1)}(-1)} - \frac{5\left[G_s^{(-1)}\right]'(-1)}{12G_s^{(-1)}(1)^2}\right] - a(z)^2\frac{7}{24G_s^{(-1)}(-1)}\right\} \\
\qquad + \frac{1}{(z+1)^3}\frac{5\sqrt{2}i}{48NG_s^{(-1)}(1)}a(z)^{-2}, & j = 0 \\
-\frac{1}{(z-1)^2}\frac{\sqrt{2}}{8N}\left\{a(z)^2\left[\frac{5+96\mathcal{A}^2}{48G_s^{(1)}(1)} - \frac{5\left[G_s^{(1)}\right]'(1)}{12G_s^{(1)}(1)^2}\right] - a(z)^{-2}\frac{7}{24G_s^{(1)}(1)}\right\} \\
\qquad - \frac{1}{(z-1)^3}\frac{5\sqrt{2}}{48NG_s^{(1)}(1)}a(z)^2, & j = k+1
\end{cases}
$$

where

$$
c_{x_j,s} = \left(x_j + i\sqrt{1-x_j^2}\right)^{\mathcal{A}} \exp\left(-i\sum_{k>j}\beta_k\pi/2 + N\phi_{s,+}(x_j) - (1+\beta_j)\pi i/4\right),
$$

$$
G_s^{(-1)}(-1) = -i\pi\sqrt{2}d_s(-1), \quad \left[G_s^{(-1)}\right]'(-1) = -\frac{3\pi i}{10\sqrt{2}}[4d_s'(-1) - d_s(-1)],
$$

$$
G_s^{(1)}(1) = \pi\sqrt{2}d_s(1), \quad \left[G_s^{(1)}\right]'(1) = \frac{3\pi}{10\sqrt{2}}[4d_s'(1) + d_s(1)].
$$

Remark 2.4.38. *As discussed in [Kra07], using the asymptotic expansions of the Airy function and Bessel functions, the matching conditions of the local parametrices can be extended into asymptotic expansions in inverse powers of N. These then can be used to prove a full asymptotic expansion for R and R'. We don't have use for this, so we won't discuss it further.*

2.5 Integrating the differential identities

In this section we will use our asymptotic solution and precise form of the parametrices to analyze the asymptotics of the differential identities (2.3.11) and (2.3.13), and finally integrate them. We will start with (2.3.11).

2.5.1 The differential identity (2.3.11)

Here we will give a (slightly simplified) variant of the argument in [DIK14, Section 5.3] to integrate the differential identity (2.3.11). As there are minor modifications due to the differences in the models and the argument being relevant for (2.3.13), we present a full proof here. The main goal we wish to prove is the following.

Proposition 2.5.1. *Let V be one-cut regular, $\mathcal{T}$ as in Proposition 2.2.10, and $\delta > 0$ small* *enough, but independent of N. Then as $N \to \infty$,*

$$\log \frac{D_{N-1}(f_1;V)}{D_{N-1}(f_0;V)} = N\int_{-1}^{1}\mathcal{T}(x)d(x)\sqrt{1-x^2}dx + \frac{\mathcal{A}}{\pi}\int_{-1}^{1}\frac{\mathcal{T}(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^{k}\frac{\beta_j}{2}\mathcal{T}(x_j)$$
$$+\frac{1}{4\pi^2}\int_{-1}^{1}dy\frac{\mathcal{T}(y)}{\sqrt{1-y^2}}P.V.\int_{-1}^{1}\frac{\mathcal{T}'(x)\sqrt{1-x^2}}{y-x}dx + o(1) \tag{2.5.1}$$

where $o(1)$ is uniform in $\{(x_j)_{j=1}^k : |x_i - x_j| \geq 3\delta, i \neq j$ and $|x_i \pm 1| \geq 3\delta\ \forall i\}$, and if in a neighborhood of $[-1,1]$, $\mathcal{T}$ is a real polynomial of fixed degree, then the error is also uniform in the coefficients of $\mathcal{T}$ when these are restricted to a bounded set.

The way we will do this is we'll express the integrand in (2.3.11) in a slightly different way which will allow deforming our integration contour in such a way that we can express Y in terms of R and the global parametrix $P^{(\infty)}$. The expression will be such that to leading order, we can treat R as the identity, and using the global parametrix, we can perform the relevant integrals explicitly.

Let us begin with expressing our integral in terms of the global parametrix. We first remind the reader that we denoted by $U_{[-1,1]}$ a fixed (independent of N and t) complex neighborhood of $[-1,1]$ into which $\mathcal{T}_t$ had analytic continuation for all $t \in [0,1]$. We also assumed that the lenses and neighborhoods $(U_{x_j})_{j=0}^{k+1}$ were inside $U_{[-1,1]}$.

Lemma 2.5.2. *Let $\tau_+ : [0,1] \to \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\} \cap U_{[-1,1]}$ be a smooth simple curve independent of N. We also assume that $\tau_+(0) < -1$, $\tau_+(1) > 1$, and that $\tau(s)$ is outside of the lenses and neighborhoods $(U_{x_j})_{j=0}^{k+1}$ for all s. We also define τ_- in a similar way but in the lower half plane and with the assumption that $\tau_-(0) = \tau_+(0)$ as well as $\tau_-(1) = \tau_+(1)$. See Figure 2.7 for an illustration.*

Then for $t \in [0,1]$

$$\frac{1}{2\pi i}\int_{\mathbb{R}}\left[Y_{11}(x,t)\partial_x Y_{21}(x,t) - Y_{21}(x,t)\partial_x Y_{11}(x,t)\right]\partial_t f_t(x) e^{-NV(x)}dx$$
$$= N\int_{-1}^{1} d(x)\sqrt{1-x^2}\frac{\partial_t f_t(x)}{f_t(x)}dx + \frac{1}{2\pi i}\left[\int_{\tau_+} - \int_{\tau_-}\right]\frac{\mathcal{D}_t'(z)}{\mathcal{D}_t(z)}\frac{\partial_t f_t(z)}{f_t(z)}dz + o(1),$$

where $o(1)$ is uniform in $t \in [0,1]$, $\{(x_j)_{j=1}^k : |x_i - x_j| \geq 3\delta, i \neq j$ and $|x_i \pm 1| \geq 3\delta\ \forall i\}$, and if in a neighborhood of $[-1,1]$, $\mathcal{T}$ is a real polynomial of fixed degree, then the error is also uniform in the coefficients of $\mathcal{T}$ when these are restricted to a bounded set.

Proof. Let us write $Y' = \partial_x Y$. We first note that an elementary calculation using (2.3.8) and the fact that the first column of Y consists of polynomials which have no jump across

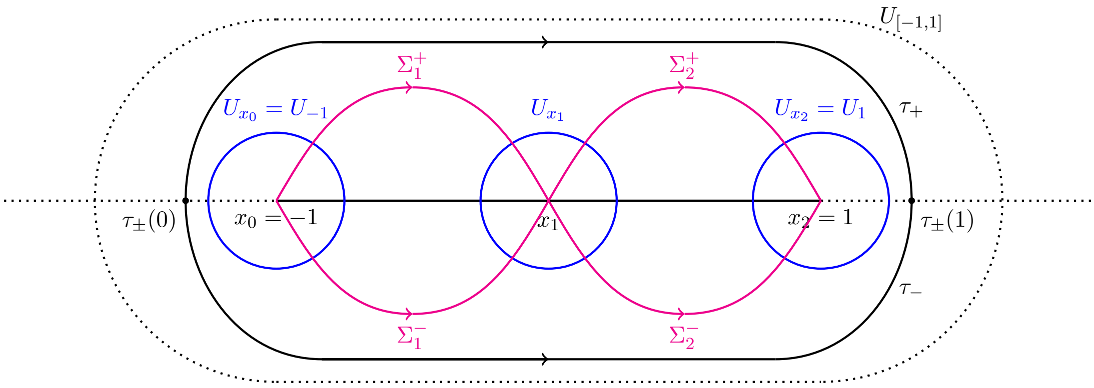

Figure 2.7: Deforming the integration contour, $k = 1$.

$\mathbb{R}$, show that for $\lambda \in \mathbb{R}$,

$$f_t e^{-NV}(Y_{11}Y_{21}' - Y_{21}Y_{11}') = \left(Y_{22,-}Y_{11}' - Y_{12,-}Y_{21}'\right) - \left(Y_{22,+}Y_{11}' - Y_{12,+}Y_{21}'\right). \tag{2.5.2}$$

Now recall that $Y_{12,\pm}$ and $Y_{22,\pm}$ have continuous boundary values on $\mathbb{R}$ so we see that the terms $Y_{22}Y_{11}' - Y_{12}Y_{21}'$ are analytic in $\mathbb{C} \setminus \mathbb{R}$ and are continuous up to the boundary. Moreover, by our construction, $f_t(z)^{-1}\partial_t f_t(z)$ is analytic in $U_{[-1,1]}$. We can thus argue by Cauchy's integral theorem to deform the integration contour. In particular, plugging (2.5.2) into (2.3.11), we find

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathbb{R}} \left[Y_{11}(x,t)\partial_x Y_{21}(x,t) - Y_{21}(x,t)\partial_x Y_{11}(x,t)\right]\partial_t f_t(x) e^{-NV(x)} dx \\
&\quad = \frac{1}{2\pi i}\int_{(-\infty,\tau_+(0)]\cup[\tau_+(1),\infty)} \left[Y_{11}(x,t)Y_{21}'(x,t) - Y_{21}(x,t)Y_{11}'(x,t)\right]\partial_t f_t(x) e^{-NV(x)} dx \\
&\qquad - \frac{1}{2\pi i}\left[\int_{\tau_+} - \int_{\tau_-}\right]\left(Y_{22}(z,t)Y_{11}'(z,t) - Y_{12}(z,t)Y_{21}'(z,t)\right)\frac{\partial_t f_t(z)}{f_t(z)} dz.
\end{aligned}$$

Notice that

$$Y_{11}Y_{21}' - Y_{21}Y_{11}' = [Y^{-1}Y']_{21}, \qquad Y_{22}Y_{11}' - Y_{12}Y_{21}' = [Y^{-1}Y']_{11}.$$

Unravelling our transformations, we note as we are not inside the lenses or the neighborhoods, we have on $\mathbb{R} \setminus [\tau_+(0), \tau_+(1)]$ and on $\tau_\pm$

$$\begin{aligned}
Y^{-1}Y' &= \left[e^{N\ell_1\sigma_3/2} S e^{N(g_1-\ell_1/2)\sigma_3}\right]^{-1}\left[e^{N\ell_1\sigma_3/2} S e^{N(g_1-\ell_1/2)\sigma_3}\right]' \\
&= Ng_1'\sigma_3 + e^{-N(g_1-\ell_1/2)\sigma_3} S^{-1} S' e^{N(g_1-\ell_1/2)\sigma_3} \\
&= Ng_1'\sigma_3 + e^{-N(g_1-\ell_1/2)\sigma_3}\left[\left(P^{(\infty)}\right)^{-1} R^{-1}\left(RP^{(\infty)}\right)'\right] e^{N(g_1-\ell_1/2)\sigma_3}
\end{aligned} \tag{2.5.3}$$

where we have used the global parametrix in the last equality. Since the $P^{(\infty)}$-RHP implies that $P^{(\infty)}(z)$ is complex analytic when $z \notin [-1,1]$, $I + \mathcal{O}(|z|^{-1})$ as $z \to \infty$, and $\det P^{(\infty)} \equiv 1$, we see that both $\left(P^{(\infty)}\right)^{-1}$ and $\left(P^{(\infty)}\right)'$ are bounded when we are away from a (complex) neighbourhood of $[-1,1]$. One can easily check that they are in fact uniformly bounded in all our relevant parameters. Combined with the estimates

$$R(z,t) = I + \mathcal{O}(N^{-1}), \qquad R'(z,t) = \mathcal{O}(N^{-1})$$

in Theorem 2.4.37, we have $S^{-1}S' = \left(P^{(\infty)}\right)^{-1}\left(P^{(\infty)}\right)' + \mathcal{O}(N^{-1})$.

Consider first the integral along $\mathbb{R} \setminus [\tau_+(0), \tau_+(1)]$. Using the specific form (2.4.22) of $P^{(\infty)}$, (2.5.3), and the fact that terms containing R give something $o(1)$, a direct calculation shows that

$$\begin{aligned}
[Y(z,t)^{-1}Y'(z,t)]_{21} &= e^{N(2g_1(z)-\ell_1)}\left[P_{11}^{(\infty)}(z,t)\partial_z P_{21}^{(\infty)}(z,t) - P_{21}^{(\infty)}(z,t)\partial_z P_{11}^{(\infty)}(z,t) + o(1)\right] \\
&= \frac{ie^{N(2g_1(z)-\ell_1)}}{4\mathcal{D}_t^2(z)}\left[((a(z)^2 + a(z)^{-2})(a(z)^2 - a(z)^{-2})' - (a(z)^2 - a(z)^{-2})(a(z)^2 + a(z)^{-2})' + o(1)\right] \\
&= \frac{ie^{N(2g_1(z)-\ell_1)}}{\mathcal{D}_t(z)^2}\left[\frac{1}{z^2-1} + o(1)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&[Y_{11}(x,t)\partial_x Y_{21}(x,t) - Y_{21}(x,t)\partial_x Y_{11}(x,t)]\,\partial_t f_t(x) e^{-NV(x)} \\
&\quad = \left[\frac{e^{\mathcal{T}(x)}-1}{\mathcal{D}_t(x)^2(x^2-1)}\prod_{j=1}^k |x-x_j|^{\beta_j} + o(1)\right] e^{N(g_{1,+}(x)+g_{1,-}(x)-\ell_1-V(x))}
\end{aligned}$$

and one finds from (2.4.7) that as $N \to \infty$, the integral along $\mathbb{R} \setminus [\tau_+(0), \tau_+(1)]$ is $o(1)$ uniformly in everything relevant.

Consider then the integrals along $\tau_\pm$. A similar direct calculation shows that

$$\begin{aligned}
[Y(z,t)^{-1}Y'(z,t)]_{11} &= Ng_1'(z) + P_{22}^{(\infty)}(z,t)\partial_z P_{11}^{(\infty)}(z,t) - P_{12}^{(\infty)}(z,t)\partial_z P_{21}^{(\infty)}(z,t) + o(1) \\
&= Ng_1'(z) + \frac{1}{4}\left[\frac{\partial_z \mathcal{D}_t(z)^{-1}}{\mathcal{D}_t(z)^{-1}}\left((a(z)^2 + a(z)^{-2})^2 - (a(z)^2 - a(z)^{-2})^2\right)\right] + o(1) \\
&= Ng_1'(z) - \frac{\mathcal{D}_t'(z)}{\mathcal{D}_t(z)} + o(1)
\end{aligned}$$

and hence

$$\left(Y_{22}(z,t)Y_{11}'(z,t) - Y_{12}(z,t)Y_{21}'(z,t)\right)\frac{\partial_t f_t(z)}{f_t(z)} = Ng_1'(z)\frac{\partial_t f_t(z)}{f_t(z)} - \frac{\mathcal{D}_t'(z)}{\mathcal{D}_t(z)}\frac{\partial_t f_t(z)}{f_t(z)} + o(1),$$

where again $o(1)$ is uniform in everything relevant. This yields the claim once we notice that by contour deformation and (2.4.8)

$$-\frac{1}{2\pi i}\left[\int_{\tau_+}-\int_{\tau_-}\right] g_1'(z)\frac{\partial_t f(z)}{f_t(z)}dz = \int_{-1}^{1} d(x)\sqrt{1-x^2}\frac{\partial_t f_t(x)}{f_t(x)}dx.$$

□

Our next task is to calculate the $\tau_\pm$ integrals. To do this, we introduce some notation.

Definition 2.5.3. *For $z \in \mathbb{C}\setminus(-\infty,1]$, let*

$$q_{FH}(z) = \log\left[(z+r(z))^{-\mathcal{A}}\prod_{j=1}^{k}(z-x_j)^{\beta_j/2}\right], \tag{2.5.4}$$

where the logarithm is with the principal branch, $\mathcal{A} = \sum_{j=1}^{k}\beta_j/2$, and FH refers to Fisher-Hartwig. We also define for $z \in \mathbb{C}\setminus[-1,1]$

$$q_{Sz}(z) = q_{Sz}(z,t) = \frac{r(z)}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}\frac{1}{z-\lambda}d\lambda, \tag{2.5.5}$$

where $r(z)$ is as in (2.4.19) and Sz refers to Szegő.

Note that we have $\mathcal{D}_t'/\mathcal{D}_t = q_{FH}' + q_{Sz}'$. We will need the following fact before proving Proposition 2.5.1. The following is an analogue of a result in [Dei99b] in the case of the circle.

Lemma 2.5.4. *Write $\tau_\pm$ be as in Lemma 2.5.2. We have*

$$\int_0^1\frac{1}{2\pi i}\left[\int_{\tau_+}-\int_{\tau_-}\right] q_{Sz}'(z,t)\frac{\partial_t f_t(z)}{f_t(z)}dzdt = -\frac{1}{4\pi^2}\int_{-1}^{1}dy\frac{\mathcal{T}(y)}{\sqrt{1-y^2}}P.V.\int_{-1}^{1}\frac{\mathcal{T}'(x)\sqrt{1-x^2}}{x-y}dx. \tag{2.5.6}$$

Proof. Let us recall that we saw in the proof of Lemma 2.4.15 that off of $[-1,1]$ we can write

$$q_{Sz}(z,t) = \frac{r(z)}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)-\mathcal{T}_t(z)}{z-\lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}}+\frac{\mathcal{T}_t(z)}{2}$$

which implies that q_{Sz} is bounded in a neighborhood of $[-1,1]$ and $q_{Sz}(\pm1,t) = \frac{1}{2}\mathcal{T}_t(\pm1)$. Moreover, we see from this that

$$\begin{aligned} q_{Sz}'(z,t) = {} & \frac{r'(z)}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(\lambda)-\mathcal{T}_t(z)}{z-\lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}} \\ & + \frac{r(z)}{2\pi}\int_{-1}^{1}\frac{\mathcal{T}_t(z)-\mathcal{T}_t(\lambda)-\mathcal{T}_t'(z)(z-\lambda)}{(z-\lambda)^2}\frac{d\lambda}{\sqrt{1-\lambda^2}}+\frac{\mathcal{T}_t'(z)}{2}. \end{aligned}$$

This in turn implies that q_{Sz}' is bounded except at $z=\pm1$ where it has singularities of order $|z\mp1|^{-1/2}$; in particular these are integrable ones. Due to the singularities being integrable, we can perform contour deformation and integrate by parts in the z-integral in the left hand side of (2.5.6). Noting that $f_t^{-1}\partial_t f_t = \partial_t \mathcal{T}_t =: \dot{\mathcal{T}}_t$ (we will use a dot here and below to indicate time derivatives below when there is no risk of confusion), we see that

$$I := \int_0^1 dt \left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{dz}{2\pi i}\dot{\mathcal{T}}_t(z) q'_{Sz}(z,t) = -\int_0^1 dt \int_{-1}^1 \frac{dx}{2\pi i}\dot{\mathcal{T}}_t'(x)\left[q_{Sz,+}(x,t) - q_{Sz,-}(x,t)\right]. \tag{2.5.7}$$

Let us write for $x \in (-1,1)$, $s(x) = \sqrt{1-x^2}$. As for $x \in (-1,1)$, $r_\pm(x) = \pm i s(x)$, we see by Sokhotski–Plemelj that

$$q_{Sz,+}(x,t) - q_{Sz,-}(x,t) = is(x)\frac{1}{\pi}P.V.\int_{-1}^1 \frac{\mathcal{T}_t(y)}{x-y}\frac{dy}{s(y)} =: is(x)[\mathcal{H}(\mathbf{1}_{(-1,1)}\mathcal{T}_t/s)](x),$$

where $\mathbf{1}_{(-1,1)}$ is the indicator function of the interval $(-1,1)$, and $\mathcal{H}$ denotes the Hilbert transform (note that the Hilbert transform is well defined as $\mathbf{1}_{(-1,1)}\mathcal{T}_t/s \in L^p(\mathbb{R})$ for $p \in [1,2)$).

To simplify notation slightly, let us write $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$. Integrating by parts in the t integral in (2.5.7) we see that

$$\begin{aligned} I &= -\int_0^1 \frac{1}{2\pi}\left\langle \dot{\mathcal{T}}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}\left(\mathbf{1}_{(-1,1)}\mathcal{T}_t/s\right)\right\rangle dt \qquad (2.5.8)\\ &= -\frac{1}{2\pi}\left\langle \mathcal{T}', \mathbf{1}_{(-1,1)} s\mathcal{H}\left(\mathbf{1}_{(-1,1)}\mathcal{T}/s\right)\right\rangle + \int_0^1 \frac{1}{2\pi}\left\langle \mathcal{T}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}\left(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s\right)\right\rangle dt. \end{aligned}$$

Our aim is now to show that actually $\frac{1}{2\pi}\int_0^1 \langle \mathcal{T}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s)\rangle dt = -I$ so we would have $I = -\langle \mathcal{T}', \mathbf{1}_{(-1,1)} s\mathcal{H}(\mathbf{1}_{(-1,1)}\mathcal{T}/s)\rangle/4\pi$, which we will see to be equivalent to our claim. To see that indeed $\frac{1}{2\pi}\int_0^1 \langle \mathcal{T}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s)\rangle dt = -I$, we note first that

$$\frac{s(x)}{s(y)}\frac{1}{x-y} = \frac{s(y)}{s(x)}\frac{1}{x-y} - \frac{x+y}{s(x)s(y)}$$

implying that for say a continuous $f : [-1,1] \to \mathbb{R}$ and $x \in (-1,1)$

$$s(x)\left[\mathcal{H}\left(\mathbf{1}_{(-1,1)}f/s\right)\right](x) = \frac{1}{s(x)}\left[\mathcal{H}\left(\mathbf{1}_{(-1,1)}fs\right)\right](x) - \frac{1}{\pi}\int_{-1}^1 \frac{x+y}{s(x)s(y)}f(y)dy. \tag{2.5.9}$$

Using the definition of the Cauchy principal value integral, one can also check easily that for a smooth $f : [-1,1] \to \mathbb{R}$ and $x \in (-1,1)$

$$\left[\mathcal{H}(\mathbf{1}_{(-1,1)}fs)\right]'(x) = \left[\mathcal{H}(\mathbf{1}_{(-1,1)}(fs)')\right](x). \tag{2.5.10}$$

Thus integrating by parts in the x integral, using the fact that $q_+(\pm 1,t) = q_-(\pm 1,t)$, and (2.5.10), we see that

$$\begin{aligned}\Big\langle \mathcal{T}_t', \mathbf{1}_{(-1,1)} s \mathcal{H}\left(\mathbf{1}_{(-1,1)} \dot{\mathcal{T}}_t/s\right)\Big\rangle \\ = \int_{-1}^{1} dx \mathcal{T}_t(x)\frac{s'(x)}{s(x)^2}\left(\left[\mathcal{H}\left(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t s\right)\right](x) - \int_{-1}^{1}\frac{x+y}{\pi s(y)}\dot{\mathcal{T}}_t(y)dy\right) \\ - \int_{-1}^{1} dx \mathcal{T}_t(x)\frac{1}{s(x)}\left(\left[\mathcal{H}\left(\mathbf{1}_{(-1,1)}(\dot{\mathcal{T}}_t s)'\right)\right](x) - \int_{-1}^{1}\frac{\dot{\mathcal{T}}_t(y)}{\pi s(y)}dy\right).\end{aligned} \tag{2.5.11}$$

We then note that

$$\begin{aligned}[\mathcal{H}(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t s')](x) - \frac{1}{\pi}\int_{-1}^{1}\frac{\dot{\mathcal{T}}_t(y)}{s(y)}dy &= \frac{1}{\pi}P.V.\int_{-1}^{1}\frac{\dot{\mathcal{T}}_t(y)}{s(y)}\left(\frac{-y}{x-y}-1\right)dy \\ &= -x[\mathcal{H}(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s)](x)\end{aligned}$$

and

$$\begin{aligned}\left[\mathcal{H}\left(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t s\right)\right](x) - \frac{1}{\pi}\int_{-1}^{1}\frac{x+y}{s(y)}\dot{\mathcal{T}}_t(y)dy &= \frac{1}{\pi}P.V.\int_{-1}^{1}\frac{\dot{\mathcal{T}}_t(y)}{s(y)}\frac{\left[s(y)^2-(x^2-y^2)\right]}{x-y}dy \\ &= s(x)^2[\mathcal{H}(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s)](x).\end{aligned}$$

Plugging these into (2.5.11), using the fact that $s'(x) = -x/s(x)$ along with the anti-self adjointness of $\mathcal{H}$ we see that

$$\begin{aligned}\frac{1}{2\pi}\int_0^1\Big\langle \mathcal{T}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}\left(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t/s\right)\Big\rangle dt &= -\frac{1}{2\pi}\int_0^1\Big\langle \mathcal{T}_t, \mathbf{1}_{(-1,1)} s^{-1}\mathcal{H}\left(\mathbf{1}_{(-1,1)}\dot{\mathcal{T}}_t' s\right)\Big\rangle dt \\ &= \frac{1}{2\pi}\int_0^1\Big\langle \dot{\mathcal{T}}_t', \mathbf{1}_{(-1,1)} s\mathcal{H}(\mathbf{1}_{(-1,1)}\mathcal{T}_t/s)\Big\rangle dt \\ &= -I.\end{aligned} \tag{2.5.12}$$

Note that $1/s \notin L^2(-1,1)$ so we can't use the anti-self adjointness of the Hilbert transform on the space L^2, but we use the fact that if $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, where p' is the Hölder conjugate of p, then $\int g\mathcal{H}f = -\int f\mathcal{H}g$ – see e.g. [Tit59, Theorem 102].

Plugging (2.5.12) into (2.5.8), we find our previous claim that

$$I = -\frac{1}{4\pi}\langle \mathcal{T}', \mathbf{1}_{(-1,1)} s\mathcal{H}(\mathbf{1}_{(-1,1)}\mathcal{T}/s)\rangle$$

Making use of the anti-self adjointness of $\mathcal{H}$ again, this translates into

$$I = \frac{1}{4\pi^2}\int_{-1}^{1} dy \frac{\mathcal{T}(y)}{\sqrt{1-y^2}}P.V.\int_{-1}^{1}\frac{\mathcal{T}'(x)\sqrt{1-x^2}}{y-x}dx$$

which is our claim. $\square$

We are now in a position to finish the proof.

Proof of Proposition 2.5.1. We start with the result of Lemma 2.5.2. Consider first the integral along $[-1,1]$. Here we note that by the definition of f_t, $\int_0^1 f_t(x)^{-1}\partial_t f_t(x)dt = \log f_1(x) - \log f_0(x) = \mathcal{T}(x)$. This yields the $\mathcal{O}(N)$-term in (2.5.1).

Let us now consider the $\mathcal{D}_t'/\mathcal{D}_t$-terms. The contribution from q_{Sz} is calculated in Lemma 2.5.4, so we need to understand the contribution of q_{FH}. As q_{FH} is independent of t, we find that

$$\int_0^1 dt \left[\int_{\mathcal{T}_+} - \int_{\mathcal{T}_-}\right] \frac{dz}{2\pi i} q_{FH}'(z)\frac{\dot{f}_t(z)}{f_t(z)} = \left[\int_{\mathcal{T}_+} - \int_{\mathcal{T}_-}\right] \frac{dz}{2\pi i}\mathcal{T}(z)q_{FH}'(z). \tag{2.5.13}$$

Now as

$$q_{FH}'(z) = -\frac{\mathcal{A}}{r(z)} + \sum_{j=1}^k \frac{\beta_j}{2}\frac{1}{z-x_j}$$

we see by Cauchy's integral theorem, the fact that $r_\pm(x) = \pm i\sqrt{1-x^2}$ for $x\in(-1,1)$, and Sokhotski-Plemelj that

$$\int_0^1 dt \left[\int_{\mathcal{T}_+} - \int_{\mathcal{T}_-}\right] \frac{dz}{2\pi i} q_{FH}'(z)\frac{\dot{f}_t(z)}{f_t(z)} = \frac{\mathcal{A}}{\pi}\int_{-1}^1 \frac{\mathcal{T}(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^k \frac{\beta_j}{2}\mathcal{T}(x_j). \tag{2.5.14}$$

Thus combining (2.5.14), (2.5.6), our reasoning about the $\mathcal{O}(N)$ term, and Lemma 2.5.2, yields

$$\begin{aligned}\log D_{N-1}(f_1) - \log D_{N-1}(f_0) = N\int_{-1}^1 \mathcal{T}(x)d(x)\sqrt{1-x^2}dx + \frac{\mathcal{A}}{\pi}\int_{-1}^1 \frac{\mathcal{T}(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^k \frac{\beta_j}{2}\mathcal{T}(x_j)\\ + \frac{1}{4\pi^2}\int_{-1}^1 dy \frac{\mathcal{T}(y)}{\sqrt{1-y^2}} P.V.\int_{-1}^1 \frac{\mathcal{T}'(x)\sqrt{1-x^2}}{y-x}dx + o(1),\end{aligned}$$

where $o(1)$ is uniform in everything relevant. This is precisely the claim. $\square$

2.5.2 The differential identity (2.3.13)

The main goal of this section is to prove the following identity.

Proposition 2.5.5. *Let V be one-cut regular, $\mathcal{T}$ as in Proposition 2.2.10, $\delta > 0$ small* *enough but independent of N. Then as $N \to \infty$,*

$$\begin{aligned}
&\log D_{N-1}(f_0;V_1) - \log D_{N-1}(f_0;V_0)\\
&\quad = -\frac{N^2}{2}\int_{-1}^{1}\left(\frac{2}{\pi}+d(x)\right)(V(x)-2x^2)\sqrt{1-x^2}dx\\
&\qquad -\mathcal{A}\frac{N}{\pi}\int_{-1}^{1}\frac{V(x)-2x^2}{\sqrt{1-x^2}}dx + N\sum_{j=1}^{k}\frac{\beta_j}{2}(V(x_j)-2x_j^2)\\
&\qquad + \sum_{j=1}^{k}\frac{\beta_j^2}{4}\log\left(\frac{\pi}{2}d(x_j)\right) - \frac{1}{24}\log\left(\frac{\pi^2}{4}d(1)d(-1)\right) + o(1),
\end{aligned}$$

where $o(1)$ is uniform in $\{(x_j)_{j=1}^k : |x_i - x_j| \geq 3\delta, i \neq j$ and $|x_i \pm 1| \geq 3\delta\ \forall i\}$.

The arguments are largely similar to those related to the differential identity (2.3.11) so we will be less detailed here. The arguments in the proof of Lemma 2.5.2 can be repeated in this case with the only difference being that we replace $\partial_t f_t$ by $-Nf\partial_s V_s$ and d with d_s etc, apart from approximating R by the identity – we'll need the $\mathcal{O}(N^{-1})$ contribution from R here as well. We will also need to assume that our lenses and neighborhoods of the singularities are chosen so that V is analytic in some neighborhood of them, but as we assumed V to be real analytic, we can of course do this. We will also assume that $\tau_\pm$ are inside this domain where V can be analytically continued to. Repeating the arguments from the previous section in such a setting leads to the following lemma.

Lemma 2.5.6. *Let $\tau_\pm$ be as in Lemma 2.5.2 with the difference that we assume that the contours are within the domain where V is analytic in. Then for $s \in [0,1]$*

$$\begin{aligned}
&-\frac{N}{2\pi i}\int_{\mathbb{R}}\left[Y_{11}(x;V_s)\partial_x Y_{21}(x;V_s) - Y_{21}(x;V_s)\partial_x Y_{11}(x;V_s)\right] f(x)e^{-NV_s(x)}\partial_s V_s(x)dx\\
&\quad = -N^2\int_{-1}^{1} d_s(x)\sqrt{1-x^2}\partial_s V_s(x)dx - \frac{N}{2\pi i}\left[\int_{\tau_+} - \int_{\tau_-}\right]\mathcal{J}_s(z)\partial_s V_s(z)dz + o(1),
\end{aligned}$$

where $o(1)$ is uniform in $s \in [0,1]$, $\{(x_j)_{j=1}^k : |x_i - x_j| \geq 3\delta, i \neq j$ and $|x_i \pm 1| \geq 3\delta\ \forall i\}$ and

$$\mathcal{J}_s(z) = -Y_{22}(z;V_s)Y_{11}'(z;V_s) + Y_{12}(z;V_s)Y_{21}'(z;V_s).$$

The proof is essentially identical to that of Lemma 2.5.2 and we omit it. We now consider the asymptotics of the integral of this from $s = 0$ to $s = 1$. Let us first consider the order N^2 term.

Lemma 2.5.7. *We have*

$$\int_0^1 ds(-N^2)\int_{-1}^{1} d_s(x)\partial_s V_s(x)\sqrt{1-x^2}dx = -\frac{N^2}{2}\int_{-1}^{1}\left(\frac{2}{\pi}+d(x)\right)(V(x)-2x^2)\sqrt{1-x^2}dx.$$

Proof. This follows immediately from the definitions: $\partial_s V_s(x) = V(x) - 2x^2$ and $d_s(x) = (1-s)\frac{2}{\pi} + sd(x)$. $\square$

For $\mathcal{J}$-terms, we note that we now need to take into account $\mathcal{O}(N^{-1})$ terms in the expansion of R – these will result in $\mathcal{O}(1)$ terms in the differential identity. We first focus on the $\mathcal{O}(N)$ terms which come from the $\mathcal{O}(1)$ terms in the expansion of R. For this, repeating our argument from the previous section results in the $\mathcal{O}(N)$ term being

$$\frac{N}{2\pi i}\int_0^1 ds \oint_\gamma \frac{\mathcal{D}'(x)}{\mathcal{D}(x)}\partial_s V_s(x)dx = \frac{N}{2\pi i}\oint_\gamma \frac{\mathcal{D}'(x)}{\mathcal{D}(x)}(V(x)-2x^2)dx,$$

where γ is a nice curve enclosing $[-1,1]$ inside which everything relevant is analytic. We again have $\mathcal{D}'(z)/\mathcal{D}(z) = q_{Sz}'(z,0) + q_{FH}'(z,0) = q_{FH}'(z,0)$ (as $q_{Sz}(z,0)=0$). Recalling that

$$q_{FH}'(z) = -\frac{\mathcal{A}}{r(z)} + \sum_{j=1}^k \frac{\beta_j}{2}\frac{1}{z-x_j},$$

an application of Sokhotski-Plemelj shows that the order N terms combine into the following quantity

$$\frac{N}{2\pi i}\oint_\gamma \frac{\mathcal{D}'(x)}{\mathcal{D}(x)}(V(x)-2x^2)dx = -\frac{N}{2\pi i}\int_{-1}^1 (q_{FH,+}'(x) - q_{FH,-}'(x))(V(x)-2x^2)dx \tag{2.5.15}$$

$$= -\mathcal{A}\frac{N}{\pi}\int_{-1}^1 \frac{V(x)-2x^2}{\sqrt{1-x^2}}dx + N\sum_{j=1}^k \frac{\beta_j}{2}(V(x_j)-2x_j^2).$$

Finally, let us consider the $\mathcal{O}(1)$ terms. We will make use of the following lemma (whose variants are surely well known in the literature, but as we don't know of a reference exactly in our setting we will sketch a proof of it).

Lemma 2.5.8. *For $x \in (-1,1)$ and one-cut regular potential V,*

$$P.V.\int_{-1}^1 V'(\lambda)\frac{\sqrt{1-\lambda^2}}{\lambda-x}d\lambda = -2\pi + 2\pi^2 d(x)(1-x^2) \tag{2.5.16}$$

and

$$\int_x^1 d(\lambda)\sqrt{1-\lambda^2}d\lambda = \frac{\sqrt{1-x^2}}{2\pi^2}P.V.\int_{-1}^1 \frac{V(\lambda)}{x-\lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}} + \frac{1}{\pi}\arccos(x). \tag{2.5.17}$$

Proof. For (2.5.16), define the function $H : \mathbb{C}\setminus[-1,1] \to \mathbb{C}$

$$H(z) = 2\pi(z-1)^{1/2}(z+1)^{1/2}\int_{-1}^1 \frac{d(\lambda)\sqrt{1-\lambda^2}}{\lambda-z}d\lambda + \int_{-1}^1 \frac{V'(\lambda)\sqrt{1-\lambda^2}}{\lambda-z}d\lambda.$$

Using Sokhotksi-Plemelj and (2.2.3), one can check that this function is continuous across $(-1,1)$. One also sees easily that H is bounded at ± 1 so we conclude that it is entire. Finally as $H(\infty) = -2\pi$, Liouville implies that $H(z) = -2\pi$. An application of Sokhotski-Plemelj then implies (2.5.16).

We note that as a consequence of (2.5.16), one can check that what's required for (2.5.17) is to prove the identity

$$\underbrace{\int_x^1 \frac{1}{\sqrt{1-y^2}} P.V. \int_{-1}^1 \frac{V'(\lambda)}{\lambda - y}\sqrt{1-\lambda^2}d\lambda dy}_{=:p(x)} = \underbrace{\sqrt{1-x^2} P.V. \int_{-1}^1 \frac{V(\lambda)}{x-\lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}}}_{=:q(x)}. \tag{2.5.18}$$

One can easily check that these are both smooth functions of x and satisfy $p(1) = q(1) = 0$, so it's enough for us to check that $p'(x) = q'(x)$. For this, let us first write

$$q(x) = \frac{1}{\sqrt{1-x^2}} P.V. \int_{-1}^1 \frac{V(\lambda)}{x-\lambda}\sqrt{1-\lambda^2}d\lambda - \frac{1}{\sqrt{1-x^2}}\int_{-1}^1 \frac{(x+\lambda)V(\lambda)}{\sqrt{1-\lambda^2}}d\lambda.$$

We again make use of the fact that differentiation commutes with the Hilbert transform so one can check that

$$\begin{aligned}
q'(x) = p'(x) &- \frac{1}{\sqrt{1-x^2}} P.V. \int_{-1}^1 \frac{\lambda V(\lambda)}{x-\lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}} + \frac{x}{(1-x^2)^{3/2}} P.V. \int_{-1}^1 \frac{V(\lambda)}{x-\lambda}\sqrt{1-\lambda^2}d\lambda \\
&- \frac{x}{(1-x^2)^{3/2}}\int_{-1}^1 \frac{(x+\lambda)V(\lambda)}{\sqrt{1-\lambda^2}}d\lambda - \frac{1}{\sqrt{1-x^2}}\int_{-1}^1 \frac{V(\lambda)}{\sqrt{1-\lambda^2}}d\lambda \\
&= p'(x) + \frac{x}{\sqrt{1-x^2}} P.V. \int_{-1}^1 \frac{V(\lambda)}{x-\lambda}\left[-\frac{1}{\sqrt{1-\lambda^2}} + \frac{\sqrt{1-\lambda^2}}{1-x^2} - \frac{x^2-\lambda^2}{(1-x^2)\sqrt{1-\lambda^2}}\right]d\lambda \\
&= p'(x).
\end{aligned}$$

We conclude that $p = q$ and (2.5.17) is true. $\square$

Now to get a hold of the $\mathcal{O}(1)$-terms we are interested in, we need the $\mathcal{O}(N^{-1})$ term in the expansion of $\mathcal{J}_s$ for the $\tau_\pm$-integrals. Again by Theorem 2.4.37, we know that

$$R(z) = I + \underbrace{R_1(z)}_{\mathcal{O}(N^{-1})} + o(N^{-1}), \quad \Rightarrow \quad R(z)^{-1} = I - R_1(z) + o(N^{-1})$$

where the claim about R^{-1} follows by Neumann series expansion. Inspecting (2.5.3), one realizes that the extra $\mathcal{O}(N^{-1})$ correction is indeed given by

$$-\left(\left[P^{(\infty)}\right]^{-1} R_1' P^{(\infty)}\right)_{11}.$$

Let us consider first the contributions from the $R_1^{(x_j)}$ terms with $j \in \{1, ..., k\}$ (recall

Theorem 2.4.37 for the definition of this and $\mathcal{J}^{(x_j)}$ below).

Lemma 2.5.9. *Let $\tau_\pm$ be as in Lemma 2.5.4 and $j \in \{1, ..., k\}$. Then*

$$-\int_0^1 ds \frac{N}{2\pi i}\left[\int_{\tau_+} - \int_{\tau_-}\right] \mathcal{J}^{(x_j)}(z)\partial_s V_s(z)dz = \frac{\beta_j^2}{4}\log\left[\frac{\pi}{2}d(x_j)\right] + \mathcal{O}(N^{-1}) \tag{2.5.19}$$

uniformly in $x_j \in (-1+\epsilon, 1-\epsilon)$.

Proof. Recall first of all from Theorem 2.4.37 that for $j \in \{1, ...k\}$

$$\begin{aligned} N\mathcal{J}^{(x_j)}(z) = &-\frac{1}{4}\frac{1}{(z-x_j)^2}\frac{i\beta_j^2}{4\pi d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2} + \frac{a_+(x_j)^2}{a(z)^2}\right] \\ &+\frac{1}{4}\frac{1}{(z-x_j)^2}\frac{i\beta_j(c_{x_j,s}^2 + c_{x_j,s}^{-2})}{4\pi d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2} - \frac{a_+(x_j)^2}{a(z)^2}\right] \end{aligned}$$

where

$$c_{x_j,s} = \left(x_j + i\sqrt{1-x_j^2}\right)^{\mathcal{A}} \exp\left(-i\sum_{k>j}\beta_k\pi/2 + N\phi_{s,+}(x_j) - (1+\beta_j)\pi i/4\right).$$

Let us first focus on the z-integral in the statement of the lemma. Note first that

$$\frac{a(z)^2}{a_+(x_j)^2} + \frac{a_+(x_j)^2}{a(z)^2} = \frac{2i(1-x_jz)}{(z-1)^{1/2}(z+1)^{1/2}\sqrt{1-x_j^2}} \tag{2.5.20}$$

and

$$\frac{a(z)^2}{a_+(x_j)^2} - \frac{a_+(x_j)^2}{a(z)^2} = \frac{2i(x_j-z)}{(z-1)^{1/2}(z+1)^{1/2}\sqrt{1-x_j^2}}. \tag{2.5.21}$$

Using (2.5.20) and (2.5.21) one can check with direct calculations that

$$\frac{1}{(x_j-z)^2}\left[\frac{a(z)^2}{a_+(x_j)^2} + \frac{a_+(x_j)^2}{a(z)^2}\right] = \frac{2i}{\sqrt{1-x_j^2}}\frac{d}{dz}\frac{(z-1)^{1/2}(z+1)^{1/2}}{z-x_j}$$

and

$$\frac{1}{(x_j-z)^2}\left[\frac{a(z)^2}{a_+(x_j)^2} - \frac{a_+(x_j)^2}{a(z)^2}\right] = \frac{2i}{\sqrt{1-x_j^2}}\frac{1}{x_j-z}\frac{1}{(z-1)^{1/2}(z+1)^{1/2}}.$$

Recalling that $\partial_s V_s(z) = V(z) - 2z^2$, we thus see by integration by parts, contour deformation, and Sokhotski-Plemelj that

$$\begin{aligned}\left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{1}{(x_j - z)^2} \left[\frac{a(z)^2}{a_+(x_j)^2} + \frac{a_+(x_j)^2}{a(z)^2}\right] \partial_s V_s(z) \frac{dz}{2\pi i} \\ = -\frac{1}{\pi}\frac{1}{\sqrt{1-x_j^2}} \left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{(z-1)^{1/2}(z+1)^{1/2}}{z - x_j}(V'(z) - 4z)dz \\ = -\frac{2i}{\pi\sqrt{1-x_j^2}} P.V. \int_{-1}^{1} (V'(\lambda) - 4\lambda)\frac{\sqrt{1-\lambda^2}}{\lambda - x_j} d\lambda\end{aligned} \tag{2.5.22}$$

and simply by Sokhotski-Plemelj that

$$\begin{aligned}\left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{1}{(x_j - z)^2} \left[\frac{a(z)^2}{a_+(x_j)^2} - \frac{a_+(x_j)^2}{a(z)^2}\right] \partial_s V_s(z) \frac{dz}{2\pi i} \\ = \frac{2}{\pi i}\frac{1}{\sqrt{1-x_j^2}} P.V. \int_{-1}^{1} \frac{V(\lambda) - 2\lambda^2}{x_j - \lambda}\frac{d\lambda}{\sqrt{1-\lambda^2}}.\end{aligned} \tag{2.5.23}$$

Let us first focus on the integral of the first term. We have from (2.5.22) and (2.5.16)

$$\begin{aligned}-\int_0^1 ds \left[\int_{\tau_+} - \int_{\tau_-}\right] \left(-\frac{1}{4}\frac{1}{(z-x_j)^2}\frac{i\beta_j^2}{4\pi d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2} + \frac{a_+(x_j)^2}{a(z)^2}\right]\right) \partial_s V_s(z) \frac{dz}{2\pi i} \\ = \frac{\beta_j^2}{4}\left(d(x_j) - \frac{2}{\pi}\right)\int_0^1 \frac{ds}{d_s(x_j)} = \frac{\beta_j^2}{4}\log\left[\frac{\pi}{2}d(x_j)\right].\end{aligned} \tag{2.5.24}$$

Let us now turn to the second term. We have from (2.5.23) and (2.5.17) that

$$\begin{aligned}-\int_0^1 ds \left[\int_{\tau_+} - \int_{\tau_-}\right] \left(\frac{1}{4}\frac{1}{(z-x_j)^2}\frac{i\beta_j(c_{x_j,s}^2 + c_{x_j,s}^{-2})}{4\pi d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2} - \frac{a_+(x_j)^2}{a(z)^2}\right]\right) \partial_s V_s(z) \frac{dz}{2\pi i} \\ = -(1-x_j)^{-3/2}\frac{\beta_j}{4}\int_{x_j}^1 \left(d(\lambda) - \frac{2}{\pi}\right)\sqrt{1-\lambda^2}d\lambda \int_0^1 ds \frac{c_{x_j,s}^2 + c_{x_j,s}^{-2}}{d_s(x_j)}.\end{aligned}$$

Let us note that we can write $c_{x_j,s}^2 = e^{i\theta_N(x_j)}e^{2\pi i N s\int_{x_j}^1\left(d(\lambda)-\frac{2}{\pi}\right)\sqrt{1-\lambda^2}}d\lambda$, where $e^{i\theta_N(x_j)}$ is a complex number of unit length and independent of s. Thus

$$\begin{aligned}\int_{x_j}^1 \left(d(\lambda) - \frac{2}{\pi}\right)\sqrt{1-\lambda^2}d\lambda \int_0^1 ds \frac{c_{x_j,s}^{\pm 2}}{d_s(x_j)} \\ = \pm e^{\pm i\theta_N(x_j)}\frac{1}{2\pi i N}\int_0^1 \frac{1}{d_s(x_j)}\frac{d}{ds}e^{\pm 2\pi i N s\int_x^1\left(d(\lambda)-\frac{2}{\pi}\right)\sqrt{1-\lambda^2}d\lambda}ds.\end{aligned}$$

Integrating this by parts, noting that $\frac{d}{ds}d_s(x) = d(x) - \frac{2}{\pi}$ is bounded and $1/d_s(x)^2$ is bounded in x and s, we see that

$$-\int_0^1 ds\left[\int_{\tau_+}-\int_{\tau_-}\right]\left(\frac{1}{(z-x_j)^2}\frac{i\beta_j(c_{x_j,s}^2+c_{x_j,s}^{-2})}{d_s(x_j)\sqrt{1-x_j^2}}\left[\frac{a(z)^2}{a_+(x_j)^2}-\frac{a_+(x_j)^2}{a(z)^2}\right]\right)\partial_s V_s(z)dz \tag{2.5.25}$$

is $\mathcal{O}(N^{-1})$ uniformly in $x_j\in(-1+\epsilon,1-\epsilon)$. Combining (2.5.24) and (2.5.25), yields the claim (2.5.19). □

Let us now treat the integrals associated to $\mathcal{J}^{(\pm 1)}$.

Lemma 2.5.10. *We have*

$$-\int_0^1 ds\frac{N}{2\pi i}\left[\int_{\tau_+}-\int_{\tau_-}\right]\mathcal{J}^{(1)}(z)\partial_s V_s(z)dz=-\frac{1}{24}\log\left(\frac{\pi}{2}d(1)\right), \tag{2.5.26}$$
$$-\int_0^1 ds\frac{N}{2\pi i}\left[\int_{\tau_+}-\int_{\tau_-}\right]\mathcal{J}^{(-1)}(z)\partial_s V_s(z)dz=-\frac{1}{24}\log\left(\frac{\pi}{2}d(-1)\right).$$

Proof. We only prove the first equality. From Theorem 2.4.37 we have

$$\begin{aligned}\mathcal{J}^{(1)}(z)=-\frac{1}{(z-1)^2}\frac{2^{1/2}}{8N}\Bigg\{a(z)^2\left[\frac{1}{48}\left(G_s^{(1)}(1)\right)^{-1}(5+96\mathcal{A}^2)-\frac{5}{12}\left(G_s^{(1)}(1)\right)^{-2}\left(\left[G_s^{(1)}\right]'(1)\right)\right]\\ -a(z)^{-2}\frac{7}{24}\left(G_s^{(1)}(1)\right)^{-1}\Bigg\}-\frac{1}{(z-1)^3}\frac{5\sqrt{2}}{48NG_s^{(1)}(1)}a(z)^2\end{aligned}$$

where $G_s^{(1)}$ is defined in (2.E.2) and we have $G_s^{(1)}(1)=\pi\sqrt{2}d_s(1)$. Note that

$$\frac{a(z)^2}{(z-1)^2}=-\frac{d}{dz}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\qquad\text{and}\qquad\frac{a(z)^2}{(z-1)^3}=\frac{1}{3}\frac{d}{dz}\frac{(z-2)(z+1)^{1/2}(z-1)^{1/2}}{(z-1)^2}.$$

Thus integrating by parts, contour deformation, and a simple application of Lemma 2.5.8 imply that

$$\begin{aligned}\left[\int_{\tau_+}-\int_{\tau_-}\right]\frac{a(z)^2}{(z-1)^2}V(z)dz&=-\left[\int_{\tau_+}-\int_{\tau_-}\right]V(z)\frac{d}{dz}\left(\frac{z+1}{z-1}\right)^{1/2}dz\\&=\left[\int_{\tau_+}-\int_{\tau_-}\right]V'(z)\left(\frac{z+1}{z-1}\right)^{1/2}dz\\&=2i\int_{-1}^1\frac{\sqrt{1-x^2}}{x-1}V'(x)dx=-4\pi i\end{aligned}$$

and

$$\left[\int_{\tau_+}-\int_{\tau_-}\right]\frac{a(z)^2}{(z-1)^2}\partial_s V_s(z)dz=\left[\int_{\tau_+}-\int_{\tau_-}\right]\frac{a(z)^2}{(z-1)^2}(V(z)-2z^2)dz=0. \tag{2.5.27}$$

In a similar manner and with an application of Lemma 2.5.8,

$$\begin{aligned}
\left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{a(z)^2}{(z-1)^3} V(z)dz &= -\left[\int_{\tau_+} - \int_{\tau_-}\right] V'(z)\frac{1}{3}\frac{(z-2)(z+1)^{1/2}(z-1)^{1/2}}{(z-1)^2}dz \\
&= -\frac{1}{3}\left[\int_{\tau_+} - \int_{\tau_-}\right] V'(z)\frac{(z+1)^{1/2}(z-1)^{1/2}}{z-1}dz \\
&\quad + \frac{1}{3}\left[\int_{\tau_+} - \int_{\tau_-}\right] V'(z)\frac{(z+1)^{1/2}(z-1)^{1/2}}{(z-1)^2}dz \\
&= -\frac{1}{3}\left[\int_{\tau_+} - \int_{\tau_-}\right] V'(z)\frac{(z+1)^{1/2}(z-1)^{1/2}}{z-1}dz \\
&\quad + \frac{1}{3}\left.\frac{d}{dx}\right|_{x=1}\left[\int_{\tau_+} - \int_{\tau_-}\right] V'(z)\frac{(z+1)^{1/2}(z-1)^{1/2}}{z-x}dz \\
&= \frac{2i}{3}\int_{-1}^{1} V'(\lambda)\sqrt{\frac{1+\lambda}{1-\lambda}}d\lambda + \frac{2i}{3}\left.\frac{d}{dx}\right|_{x=1} P.V. \int_{-1}^{1} V'(\lambda)\frac{\sqrt{1-\lambda^2}}{\lambda - x}d\lambda \\
&= \frac{4\pi i}{3} - \frac{8\pi^2 i}{3}d(1),
\end{aligned}$$

which implies

$$\left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{a(z)^2}{(z-1)^3}\partial_s V_s(z)dz = -\frac{8\pi^2 i}{3}\left(d(1) - \frac{2}{\pi}\right). \tag{2.5.28}$$

Consider finally the $a(z)^{-2}$ term. One can easily check that

$$\frac{a(z)^{-2}}{(z-1)^2} = -\frac{2}{3}\frac{\partial}{\partial z}\left[\frac{(z-1)^{1/2}(z+1)^{1/2}}{(z-1)^2}\right] + \frac{1}{3}\frac{a(z)^2}{(z-1)^2}.$$

We can safely ignore the second term on the RHS, as we saw that it will integrate to zero. Moreover, we essentially calculated the integral related to the first term already:

$$-\frac{2}{3}\left[\int_{\tau_+} - \int_{\tau_-}\right] V(z)\frac{\partial}{\partial z}\left[\frac{(z-1)^{1/2}(z+1)^{1/2}}{(z-1)^2}\right]dz = -\frac{16}{3}\pi^2 i d(1)$$

and we find

$$\left[\int_{\tau_+} - \int_{\tau_-}\right] \frac{a(z)^{-2}}{(z-1)^2}\partial_s V_s(z)dz = -\frac{16\pi^2 i}{3}\left(d(1) - \frac{2}{\pi}\right). \tag{2.5.29}$$

Putting together (2.5.27), (2.5.28), and (2.5.29) a direct calculation leads to

$$-\int_0^1 ds \frac{N}{2\pi i}\left[\int_{\tau_+} - \int_{\tau_-}\right] \mathcal{J}^{(1)}(z)\partial_s V_s(z)dz = -\frac{1}{24}\log\left(\frac{\pi}{2}d(1)\right).$$

□

Proof of Proposition 2.5.5. This is simply a combination of Lemma 2.5.6, Lemma 2.5.7, (2.5.15), Lemma 2.5.9, and Lemma 2.5.10. □

We are now in a position to apply these results.

2.6 Proof of Theorem 2.1.1

As discussed earlier, we do this through Proposition 2.2.9. Before proving this, we will need to recall Krasovsky's result for the GUE from [Kra07] and a result of Claeys and Fahs [CF16] which we need to control the situation when the singularities are close to each other. Let us begin with Krasovsky's result [Kra07, Theorem 1].

Theorem 2.6.1 (Krasovsky). *Let $(x_j)_{j=1}^k$ be distinct points in $(-1,1)$, let $\beta_j > -1$, and let H_N be a GUE matrix (i.e. $V(x) = 2x^2$). Then as $N \to \infty$*

$$\mathbb{E}\prod_{j=1}^k |\det(H_N - x_j)|^{\beta_j}$$
$$= \prod_{j=1}^k C(\beta_j)(1-x_j^2)^{\frac{\beta_j^2}{8}}\left(\frac{N}{2}\right)^{\frac{\beta_j^2}{4}} e^{(2x_j^2-1-2\log 2)\frac{\beta_j}{2}N}\prod_{i<j}|2(x_i-x_j)|^{-\frac{\beta_i\beta_j}{2}}(1+\mathcal{O}(\log N/N))$$

uniformly in compact subsets of $\{(x_1,...,x_k) \in (-1,1)^k : x_i \neq x_j \text{ for } i \neq j\}$. Here $C(\beta) = 2^{\frac{\beta^2}{2}}\frac{G(1+\beta/2)^2}{G(1+\beta)}$, and G is the Barnes G function.

We mention that Krasovsky's result is actually valid for complex β_j with real part greater than -1, and he used a slightly different normalization, but obtaining this formulation follows after trivial scaling. Also his formulation of the result does not stress the uniformity, but it can easily be checked through uniform bounds on the jump matrices which are similar to the ones we have considered.

Combining this with Proposition 2.5.5 yields the following result.

Proposition 2.6.2. *Let H_N be drawn from a one-cut regular ensemble with potential V and support of the equilibrium measure normalized to $[-1,1]$. If $(x_j)_{j=1}^k$ are distinct points in $(-1,1)$ and $\beta_j \geq 0$ for all j, then*

$$\mathbb{E}\prod_{j=1}^k |\det(H_N - x_j)|^{\beta_j} = \prod_{j=1}^k C(\beta_j)\left(d(x_j)\frac{\pi}{2}\sqrt{1-x_j^2}\right)^{\frac{\beta_j^2}{4}}\left(\frac{N}{2}\right)^{\frac{\beta_j^2}{4}} e^{(V(x_j)+\ell_V)\frac{\beta_j}{2}N}$$
$$\times\prod_{i<j}|2(x_i-x_j)|^{-\frac{\beta_i\beta_j}{2}}(1+o(1))$$

uniformly in compact subsets of $\{(x_1,...,x_k) \in (-1,1)^k : x_i \neq x_j \text{ for } i \neq j\}$.

Proof. Let us write $\mathbb{E}_V$ for the expectation with respect to an ensemble with potential V. Note that from (2.3.1) setting $f = 1$, we have

$$\frac{Z_N(V)}{N!} = D_{N-1}(1; V)$$

so we see from Proposition 2.5.5 that for $f(\lambda) = \prod_{j=1}^k |\lambda - x_j|^{\beta_j}$ and $V_0(x) = 2x^2$

$$\begin{aligned}
&\log \mathbb{E}_V \prod_{j=1}^k |\det(H_N - x_j)|^{\beta_j} - \log \mathbb{E}_{V_0} \prod_{j=1}^k |\det(H_N - x_j)|^{\beta_j} \qquad (2.6.1)\\
&= \log D_{N-1}(f; V) - \log D_{N-1}(f; V_0) - \log D_{N-1}(1; V) + \log D_{N-1}(1; V_0)\\
&= -N \sum_{j=1}^k \frac{\beta_j}{2} \left[\frac{1}{\pi}\int_{-1}^1 \frac{V(x) - 2x^2}{\sqrt{1-x^2}}dx - (V(x_j) - 2x_j^2)\right] + \sum_{j=1}^k \frac{\beta_j^2}{4} \log\left(\frac{\pi}{2} d(x_j)\right) + o(1),
\end{aligned}$$

where we have the desired uniformity.

Let us now recall the logarithmic potential of the arcsine law (see e.g. [ST97, Section 1.3: Example 3.5]): $\frac{1}{\pi}\int_{-1}^1 \log|x-y|/\sqrt{1-x^2}dx = -\log 2$ for all $y \in (-1, 1)$. This along with (2.2.3) imply that

$$\frac{1}{\pi}\int_{-1}^1 \frac{V(x)}{\sqrt{1-x^2}}dx + \ell_V = -2\log 2.$$

This in turn implies that

$$(2x_j^2 - 1 - 2\log 2) - \frac{1}{\pi}\int_{-1}^1 \frac{V(x) - 2x^2}{\sqrt{1-x^2}}dx + (V(x_j) - 2x_j^2) = V(x_j) + \ell_V.$$

Combining this with Theorem 2.6.1 and (2.6.1) yields the claim. ☐

We now recall the result of Claeys and Fahs that we will need, namely [CF16, Theorem 1.1].

Theorem 2.6.3 (Claeys and Fahs)**.** *Let V be one-cut regular and let the support of the associated equilibrium measure be $[a, b]$ with $a < 0 < b$. Let $\beta > 0$, $u > 0$, and $f_u(x) = |x^2 - u|^\beta$. Then*

$$\begin{aligned}
\log D_{N-1}(f_u; V) = \log D_{N-1}(f_0; V) + \int_0^{s_{N,u}} \frac{\sigma_\beta(s) - \beta^2}{s}ds + \frac{\beta}{2}s_{N,u}\\
+ N\frac{\beta}{2}(V(\sqrt{u}) + V(-\sqrt{u}) - 2V(0)) + \mathcal{O}(\sqrt{u}) + \mathcal{O}(N^{-1})
\end{aligned}$$

uniformly as $u \to 0$ and $N \to \infty$. Here

$$s_{N,u} = -2\pi i N \int_{-\sqrt{u}}^{\sqrt{u}} d(s)\sqrt{(s-a)(b-s)}ds$$

and $\sigma_\beta(s)$ is analytic on $-i\mathbb{R}_+$, independent of V, N, and u and satisfies:

$$\sigma_\beta(s) = \begin{cases} \beta^2 + o(1), & s \to -i0^+ \\ \frac{\beta^2}{2} - \frac{\beta}{2}s + \mathcal{O}(|s|^{-1}), & s \to -i\infty \end{cases} \tag{2.6.2}$$

Moreover, the integral involving σ_β is taken along $-i\mathbb{R}_+$.

Much more is in fact known about σ_β. For example, it is known to satisfy a Painlevé V equation. A generalization of it was studied extensively in [CK15]. Theorem 2.6.3 and Proposition 2.6.2 let us prove the convergence of $\mathbb{E}[\mu_N(f)^2]$ – the argument is similar to analogous ones in [CF16, CK15].

Proposition 2.6.4. *Let $\varphi : (-1,1) \to [0,\infty)$ be continuous and have compact support. Moreover, let $\beta \in (0,\sqrt{2})$. Then*

$$\lim_{N\to\infty} \mathbb{E}[\mu_{N,\beta}(\varphi)^2] = \int_{-1}^{1}\int_{-1}^{1} \varphi(x)\varphi(y)(2|x-y|)^{-\frac{\beta^2}{2}}dxdy$$

Proof. This is very similar to the proof of [CF16, Corollary 1.11] where a more general statement was proven for the GUE. Let us fix some small $\epsilon > 0$, $\alpha \in (\beta^2/2, 1)$, and write the relevant moment in the following way:

$$\begin{aligned}\mathbb{E}[\mu_N(\varphi)^2] &= \left[\int_{|x-y|\geq\epsilon} + \int_{2N^{-\alpha}\leq|x-y|<\epsilon} + \int_{|x-y|\leq 2N^{-\alpha}}\right]\varphi(x)\varphi(y) \\ &\quad\times \frac{\mathbb{E}\left[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta\right]}{\mathbb{E}|\det(H_N-x)|^\beta\mathbb{E}|\det(H_N-y)|^\beta}dxdy \\ &=: A_{N,1}(\epsilon) + A_{N,2}(\epsilon) + A_{N,3}.\end{aligned}$$

It follows immediately from Proposition 2.6.2 that if there is some $\epsilon > 0$ such that $|x-y| \geq \epsilon$ and $x, y \in (-1+\epsilon, 1-\epsilon)$ then uniformly in such x, y

$$\frac{\mathbb{E}\left[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta\right]}{\mathbb{E}|\det(H_N-x)|^\beta\mathbb{E}|\det(H_N-y)|^\beta} = \frac{1}{(2|x-y|)^{\frac{\beta^2}{2}}}(1+o(1)).$$

As φ has compact support in $(-1,1)$, this is precisely the situation for the integral in $A_{N,1}(\epsilon)$. We conclude that

$$\lim_{N\to\infty} A_{N,1}(\epsilon) = \int_{|x-y|\geq\epsilon}\varphi(x)\varphi(y)\frac{1}{(2|x-y|)^{\frac{\beta^2}{2}}}dxdy \xrightarrow{\epsilon\to 0^+} \int_{-1}^{1}\int_{-1}^{1}\varphi(x)\varphi(y)\frac{1}{(2|x-y|)^{\frac{\beta^2}{2}}}dxdy.$$

Let us now consider $A_{N,3}$. Here we find by Cauchy–Schwarz and Proposition 2.6.2 that there exists some finite $B(\beta)$ (uniform in the relevant x, y) such that

$$\frac{\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta]}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]} \leq \frac{\sqrt{\mathbb{E}_V[|\det(H_N-x)|^{2\beta}]\mathbb{E}_V[|\det(H_N-y)|^{2\beta}]}}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]} \leq B(\beta)N^{\beta^2/2}$$

so we see that as $N \to \infty$

$$A_{N,3} = \int_{|x-y|\leq 2N^{-\alpha}} \varphi(x)\varphi(y)\frac{\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta]}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]}dxdy \lesssim N^{-\alpha+\frac{\beta^2}{2}} \to 0$$

since we chose $\alpha > \beta^2/2$.

Thus to conclude the proof, it's enough to show that

$$\lim_{\epsilon\to 0^+}\limsup_{N\to\infty} A_{N,2}(\epsilon) = 0.$$

Let us begin doing this by noting that if we write $u = \frac{(x-y)^2}{4}$ and $V_{x,y}(\lambda) = V(\lambda + (x+y)/2)$, then in the notation of Theorem 2.6.3

$$\mathbb{E}_V\left[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta\right] = \frac{D_{N-1}(f_u;V_{x,y})}{D_{N-1}(1;V)}.$$

This follows from (2.2.2) through the change of variables $\lambda_i = \mu_i + \frac{x+y}{2}$. The goal is to make use of Theorem 2.6.3 to estimate $D_{N-1}(f_u;V_{x,y})$. There are several issues we need to check to justify this. First of all, we need $V_{x,y}$ to be one-cut regular and the interior of the support of its equilibrium measure to contain the point 0. This is simple to justify as one can check from the Euler-Lagrange equations that the equilibrium measure associated to $V_{x,y}$ is simply $d(u+\frac{x+y}{2})\sqrt{1-(u+\frac{x+y}{2})^2}du$ and its support is $[-1-\frac{x+y}{2}, 1-\frac{x+y}{2}]$. The remaining conditions for one-cut regularity are easy to check with this representation.

It is less obvious that we can use Theorem 2.6.3 to study the asymptotics of $D_{N-1}(f_u;V_{x,y})$ as now $V_{x,y}$ depends on x and y and we would need the errors in the theorem to be uniform in V as well. As mentioned in [CF16] for the GUE, for $x, y \in (-1+\epsilon, 1-\epsilon)$, this can be checked by going through the relevant estimates in the proof. This is true also for general one-cut regular ensembles. As checking this may be non-trivial for a reader with little background in Riemann-Hilbert problems, we outline how to do this in Appendix 2.G.

We may therefore use Theorem 2.6.3, and so we have

$$\begin{aligned}
&\log \mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta] \\
&\quad= \log D_{N-1}(f_0;V_{x,y}) - \log D_{N-1}(1;V) + \int_0^{s_{N,u}} \frac{\sigma_\beta(s)-\beta^2}{s}ds + \frac{\beta}{2}s_{N,u} \\
&\qquad+ N\frac{\beta}{2}(V_{x,y}(\sqrt{u}) + V_{x,y}(-\sqrt{u}) - 2V_{x,y}(0)) + \mathcal{O}(\sqrt{u}) + \mathcal{O}(N^{-1}),
\end{aligned}$$

where the error estimates are uniform in $|x-y|<\epsilon$ and $x,y\in(-1+\epsilon,1-\epsilon)$. Note that now

$$\begin{aligned} s_{N,u} &= -2\pi i N\int_{-\sqrt{u}}^{\sqrt{u}} d_{x,y}(s)\sqrt{1-\left(s+\frac{x+y}{2}\right)^2}ds \\ &= -4\pi i N\sqrt{u}d\left(\frac{x+y}{2}\right)\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\mathcal{O}(Nu) \end{aligned}$$

again uniformly in the relevant values of x and y.

Recall that we're considering u such that $\sqrt{u}<2\epsilon$ but $\sqrt{u}>N^{-\alpha}$ with $\frac{\beta^2}{2}<\alpha<1$. We then have $s_{N,u}\to -i\infty$ uniformly in the relevant x,y and using [CK15, equation (1.26)] one has

$$\lim_{N\to\infty}\left[\int_0^{s_{N,u}}\frac{\sigma_\beta(s)-\beta^2}{s}ds+\frac{\beta}{2}s_{N,u}+\frac{\beta^2}{2}\log|s_{N,u}|\right]=\log\frac{G(1+\frac{\beta}{2})^4G(1+2\beta)}{G(1+\beta)^4}$$

uniformly for $x,y\in(-1+\epsilon,1-\epsilon)$ and $2N^{-\alpha}<|x-y|<\epsilon$.

On the other hand, reversing our mapping from V to $V_{x,y}$, we see that

$$\log D_{N-1}(f_0;V_{x,y})-\log D_{N-1}(1;V)=\log\mathbb{E}_V\left|\det\left(H_N-\frac{x+y}{2}\right)\right|^{2\beta}.$$

Thus we see that uniformly for $x,y\in(-1+\epsilon,1-\epsilon)$ and $2N^{-\alpha}<|x-y|<\epsilon$

$$\begin{aligned} &\log\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta] \\ &\quad=\log\mathbb{E}_V\left|\det\left(H_N-\frac{x+y}{2}\right)\right|^{2\beta}+\log\frac{G(1+\frac{\beta}{2})^4G(1+2\beta)}{G(1+\beta)^4} \\ &\quad\quad-\frac{\beta^2}{2}\log\left[4\pi N\sqrt{u}d\left(\frac{x+y}{2}\right)\sqrt{1-\left(\frac{x+y}{2}\right)^2}\right]+N\frac{\beta}{2}(V_{x,y}(\sqrt{u})+V_{x,y}(-\sqrt{u})-2V_{x,y}(0)) \\ &\quad\quad+\mathcal{O}(\sqrt{u})+o(1), \end{aligned}$$

where $o(1)$ means something that tends to zero as $N\to\infty$. Using these estimates, we can write for such x,y

$$\begin{aligned} &\frac{\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta]}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]} \\ &\quad=\frac{G(1+\frac{\beta}{2})^4G(1+2\beta)}{G(1+\beta)^4}\frac{\mathbb{E}_V\left|\det\left(H_N-\frac{x+y}{2}\right)\right|^{2\beta}}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]} \\ &\quad\quad\times N^{-\frac{\beta^2}{2}}(2|x-y|)^{-\frac{\beta^2}{2}}\left[\pi d\left(\frac{x+y}{2}\right)\sqrt{1-\left(\frac{x+y}{2}\right)^2}\right]^{-\frac{\beta^2}{2}} \end{aligned}$$

$$\times\, e^{\frac{N\beta}{2}\left(V_{x,y}(\sqrt{u})+V_{x,y}(-\sqrt{u})-2V_{x,y}(0)\right)}e^{\mathcal{O}(\sqrt{u})}(1+o(1))$$

uniformly in $x, y \in (-1+\epsilon, 1-\epsilon)$ and $2N^{-\alpha} < |x-y| < \epsilon$. Plugging in Proposition 2.6.2, we see that this becomes

$$\begin{aligned}
&\frac{\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta]}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]} \\
&= \frac{\left(d\left(\frac{x+y}{2}\right)\sqrt{1-\left(\frac{x+y}{2}\right)^2}\right)^{\beta^2/2}}{\left(d(x)\sqrt{1-x^2}d(y)\sqrt{1-y^2}d(y)\right)^{\frac{\beta^2}{4}}}(2|x-y|)^{-\frac{\beta^2}{2}}(1+o(1))(1+\mathcal{O}(\sqrt{u})) \\
&= (2|x-y|)^{-\frac{\beta^2}{2}}(1+o(1))(1+\mathcal{O}(\sqrt{u})).
\end{aligned}$$

We conclude that

$$\lim_{\epsilon\to 0^+}\limsup_{N\to\infty}\int_{2N^{-\alpha}<|x-y|<\epsilon}\varphi(x)\varphi(y)\frac{\mathbb{E}_V[|\det(H_N-x)|^\beta|\det(H_N-y)|^\beta]}{\mathbb{E}_V[|\det(H_N-x)|^\beta]\mathbb{E}_V[|\det(H_N-y)|^\beta]}dxdy = 0,$$

which was the missing part of the proof. □

Next we need to study the cross term $\mathbb{E}\mu_{N,\beta}(\varphi)\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)$ along with the fully truncated term $\mathbb{E}[\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)^2]$. For this, we need Proposition 2.2.10, so let us finish the proof of it.

Proof of Proposition 2.2.10. We have now

$$\mathbb{E}e^{\sum_{j=1}^N \mathcal{T}(\lambda_j)}\prod_{j=1}^k|\det(H_N-x_j)|^{\beta_j} = \frac{D_{N-1}(f;V)}{D_{N-1}(1;V)},$$

where $f(\lambda) = f_1(\lambda) = e^{\mathcal{T}(\lambda)}\prod_{j=1}^k|\lambda-x_j|^{\beta_j}$. Since we know the asymptotics of this for $\mathcal{T}=0$, we can apply Proposition 2.5.1 to get the relevant asymptotics for $\mathcal{T}\neq 0$:

$$\begin{aligned}
\frac{D_{N-1}(f_1;V)}{D_{N-1}(1;V)} &= \frac{D_{N-1}(f_0;V)}{D_{N-1}(1;V)}e^{N\int_{-1}^1\mathcal{T}(x)d(x)\sqrt{1-x^2}dx+\sum_{j=1}^k\frac{\beta_j}{2}\left[\int_{-1}^1\frac{\mathcal{T}(x)}{\pi\sqrt{1-x^2}}dx-\mathcal{T}(x_j)\right]} \\
&\quad\times e^{\frac{1}{4\pi^2}\int_{-1}^1 dy\frac{\mathcal{T}(y)}{\sqrt{1-y^2}}P.V.\int_{-1}^1\frac{\mathcal{T}'(x)\sqrt{1-x^2}}{y-x}dx}(1+o(1))
\end{aligned}$$

uniformly in everything relevant. Applying Proposition 2.6.2 to this yields the claim. □

We now apply this to understanding the remaining terms.

Proposition 2.6.5. *Let $\beta\in(0,\sqrt{2})$ and $\varphi:(-1,1)\to[0,\infty)$ be continuous with compact support. Then for fixed $M\in\mathbb{Z}_+$*

$$\lim_{N\to\infty}\mathbb{E}[\mu_{N,\beta}(\varphi)\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)] = \lim_{N\to\infty}\mathbb{E}[\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)^2] = \int_{-1}^1\int_{-1}^1\varphi(x)\varphi(y)e^{\beta^2\sum_{k=1}^M\frac{1}{k}T_k(x)T_k(y)}dxdy.$$

Proof. Let us first consider the cross term. We write this as

$$\mathbb{E}[\mu_{N,\beta}(\varphi)\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)] = \int_{-1}^{1}\int_{-1}^{1}\varphi(x)\varphi(y)\frac{\mathbb{E}|\det(H_N - x)|^\beta e^{\beta \widetilde{X}_{N,M}(y)}}{\mathbb{E}|\det(H_N - x)|^\beta \mathbb{E}e^{\beta\widetilde{X}_{N,M}(y)}}dxdy.$$

Let us begin by calculating the numerator. Note that as we have only one singularity, Proposition 2.2.10 gives us asymptotics which are uniform in x throughout the whole integration region. To apply Proposition 2.2.10, we point out that we now have $\mathcal{T}(\lambda) = \mathcal{T}(\lambda; y) = -\beta\sum_{k=1}^M \frac{2}{k}\widetilde{T}_k(\lambda)T_k(y)$. We need uniformity in y, but this is ensured by the fact that in a neighborhood of $[-1,1]$, $\mathcal{T}$ is a polynomial of fixed degree and its coefficients are uniformly bounded for fixed M. Using the facts that $\int_{-1}^1 T_k(y)/\sqrt{1-y^2}dy = 0$ for $k \geq 1$, $P.V.\frac{1}{\pi}\int_{-1}^1 T_k'(y)\sqrt{1-y^2}/(x-y)dy = kT_k(x)$, and the orthogonality of the Chebyshev polynomials: $2\int_{-1}^1 T_k(\lambda)T_l(\lambda)/(\pi\sqrt{1-\lambda^2})d\lambda = \delta_{k,l}$ for $k, l \geq 1$, we see that

$$\begin{aligned}\mathbb{E}[|\det(H_N - x)|^\beta e^{\beta\widetilde{X}_{N,M}(y)}] &= \mathbb{E}[|\det(H_N-x)|^\beta] e^{-\beta N\sum_{k=1}^M \frac{2}{k}T_k(y)\int_{-1}^1 T_k(\lambda)d(\lambda)\sqrt{1-\lambda^2}d\lambda} \\ &\quad \times e^{\frac{\beta^2}{2}\sum_{k=1}^M \frac{1}{k}T_k(y)^2 + \beta^2\sum_{k=1}^M \frac{1}{k}T_k(x)T_k(y)}(1+o(1))\end{aligned}$$

uniformly in $x, y \in (-1+\epsilon, 1-\epsilon)$. We see that the $\mathbb{E}[|\det(H_N-x)|^\beta]$-term in the denominator will cancel, but we still need to understand the $\mathbb{E}e^{\beta\widetilde{X}_{N,M}(y)}$-term. This now has no singularity, so we get the asymptotics from Proposition 2.2.10 by setting $\beta_j = 0$ for all j. Thus we find with a similar argument that

$$\mathbb{E}e^{\beta\widetilde{X}_{N,M}(y)} = e^{-\beta N\sum_{k=1}^M \frac{2}{k}T_k(y)\int_{-1}^1 T_k(\lambda)d(\lambda)\sqrt{1-\lambda^2}d\lambda + \frac{\beta^2}{2}\sum_{k=1}^M \frac{1}{k}T_k(y)^2}(1+o(1)),$$

uniformly in y, and we conclude that

$$\lim_{N\to\infty}\mathbb{E}[\mu_{N,\beta}(\varphi)\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)] = \int_{-1}^1\int_{-1}^1 f(x)f(y)e^{\beta^2\sum_{k=1}^M \frac{1}{k}T_k(x)T_k(y)}dxdy.$$

For the fully truncated term one argues in a similar way: in this case

$$\mathcal{T}(\lambda) = \mathcal{T}(\lambda; x, y) = -\beta\sum_{j=1}^M \frac{2}{j}\widetilde{T}_j(\lambda)(T_j(x) + T_j(y))$$

and only the part quadratic in $\mathcal{T}$ affects the leading order asymptotics. Going through the calculations one finds

$$\lim_{N\to\infty}\mathbb{E}[\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)^2] = \int_{-1}^1\int_{-1}^1 \varphi(x)\varphi(y)e^{\beta^2\sum_{k=1}^M \frac{1}{k}T_k(x)T_k(y)}dxdy.$$

□

Before proving Proposition 2.2.9, we need to know that μ_β exists, namely we need to prove Lemma 2.2.5.

Proof of Lemma 2.2.5. As discussed earlier, this boils down to showing that $(\mu_\beta^{(M)}(\varphi))_{M=1}^\infty$ is bounded in L^2 for continuous $\varphi : [-1,1] \to [0,\infty)$. From the definition of $\mu_\beta^{(M)}$ (see (2.2.11)), we see that

$$\mathbb{E}[\mu_\beta^{(M)}(\varphi)^2] = \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) e^{\beta^2 \sum_{j=1}^M \frac{1}{j} T_j(x) T_j(y)} dx dy.$$

Now from Proposition 2.6.4 and Proposition 2.6.5, we see that if φ had compact support in $(-1,1)$, then

$$\begin{aligned} 0 \leq \lim_{N\to\infty} \mathbb{E}[(\mu_{N,\beta}(\varphi) - \widetilde{\mu}_{N,\beta}^{(M)}(\varphi))^2] &= \int_{-1}^1 \int_{-1}^1 \frac{\varphi(x)\varphi(y)}{|2(x-y)|^{\beta^2/2}} dx dy \\ &\quad - \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) e^{\beta^2 \sum_{k=1}^M \frac{1}{k} T_k(x) T_k(y)} dx dy, \end{aligned}$$

so for fixed $M \in \mathbb{Z}_+$ and continuous, compactly supported in $(-1,1)$, non-negative φ

$$\int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) e^{\beta^2 \sum_{k=1}^M \frac{1}{k} T_k(x) T_k(y)} dx dy \leq \int_{-1}^1 \int_{-1}^1 \frac{\varphi(x)\varphi(y)}{|2(x-y)|^{\beta^2/2}} dx dy < \infty$$

as $\beta^2/2 < 1$. For continuous $\varphi : [-1,1] \to [0,\infty)$, we get the same inequality simply by approximating φ by a compactly supported one. We conclude that $\mu_\beta^{(M)}(\varphi)$ is indeed bounded in L^2 and thus (as it is a martingale as a function of M), a limit $\mu_\beta(\varphi)$ exists in $L^2(\mathbb{P})$. ☐

We are now in a position to prove Proposition 2.2.9.

Proof of Proposition 2.2.9. As noted, Proposition 2.6.4 and Proposition 2.6.5 imply that

$$\lim_{N\to\infty} \mathbb{E}[(\mu_{N,\beta}(\varphi) - \widetilde{\mu}_{N,\beta}^{(M)}(\varphi))^2] = \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) \left[\frac{1}{|2(x-y)|^{\beta^2/2}} - e^{\beta^2 \sum_{k=1}^M \frac{1}{k} T_k(x) T_k(y)}\right] dx dy.$$

As this is a limit of a second moment, it is non-negative and we see that

$$\limsup_{M\to\infty} \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) e^{\beta^2 \sum_{k=1}^M \frac{1}{k} T_k(x) T_k(y)} dx dy \leq \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) (2|x-y|)^{-\frac{\beta^2}{2}} dx dy.$$

On the other hand, Lemma 2.2.3 and Fatou's lemma imply that

$$\int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) (2|x-y|)^{-\frac{\beta^2}{2}} dx dy \leq \liminf_{M\to\infty} \int_{-1}^1 \int_{-1}^1 \varphi(x)\varphi(y) e^{\beta^2 \sum_{k=1}^M \frac{1}{k} T_k(x) T_k(y)} dx dy,$$

so we see actually that

$$\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{E}[(\mu_{N,\beta}(\varphi)-\widetilde{\mu}_{N,\beta}^{(M)}(\varphi))^2]=0.$$

We still need to prove that when we first let $N\to\infty$ and then $M\to\infty$, $\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)$ converges in law to $\mu_\beta(\varphi)$. As $\mu_\beta(\varphi)$ is constructed as a limit of $\mu_\beta^{(M)}(\varphi)$, this will follow from showing that $\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)$ converges to $\mu_\beta^{(M)}(\varphi)$ in law if we let $N\to\infty$ for fixed M. For this, consider the function $F:\mathbb{R}^M\to[0,\infty)$

$$F(u_1,...,u_M)=\int_{-1}^1\varphi(\lambda)e^{\beta\sum_{k=1}^M\frac{1}{\sqrt{k}}u_kT_k(\lambda)-\frac{\beta^2}{2}\sum_{k=1}^M\frac{1}{k}T_k(\lambda)^2}d\lambda.$$

We now have

$$F\left(\left(-\frac{2}{\sqrt{k}}\mathrm{Tr}\widetilde{T}_k(H_N)+\frac{2}{\sqrt{k}}N\int_{-1}^1T_k(\lambda)\mu_V(d\lambda)\right)_{k=1}^M\right)=\widetilde{\mu}_{N,\beta}^{(M)}(\varphi)(1+o(1)),$$

where $o(1)$ is deterministic. Moreover, if $(A_k)_{k=1}^M$ are the i.i.d. standard Gaussians used in the definition of $\mu_\beta^{(M)}$, then $F(A_1,...,A_M)=\mu_\beta^{(M)}(\varphi)$. It follows easily from the dominated convergence theorem that F is a continuous function, so if we knew that

$$\left(-\frac{2}{\sqrt{k}}\mathrm{Tr}\widetilde{T}_k(H_N)+\frac{2}{\sqrt{k}}N\int_{-1}^1T_k(\lambda)\mu_V(d\lambda)\right)_{k=1}^M\xrightarrow{d}(A_1,...,A_M)$$

as $N\to\infty$, we would be done. This is of course well known and follows from more general results such as [Joh98] for polynomial potentials or [BG13] for more general ones. Nevertheless, we point out that it also follows from our analysis. If one looks at the function $\mathcal{T}(\lambda)=\sum_{j=1}^M\alpha_j\frac{2}{\sqrt{j}}(\widetilde{T}_j(\lambda)-\int T_j(u)\mu_V(du))$, one can then check that it follows from Proposition 2.2.10 (setting $\beta_j=0$ for all j) that

$$\mathbb{E}e^{\sum_{j=1}^N\mathcal{T}(\lambda_j)}=e^{\frac{1}{2}\sum_{k=1}^M\alpha_j^2},$$

which implies the claim. □

Theorem 2.1.1 is essentially a direct corollary of Proposition 2.2.9.

Proof of Theorem 2.1.1. It is a standard probabilistic argument that Proposition 2.2.9 implies that also $\mu_{N,\beta}(\varphi)$ converges in law to $\mu_\beta(\varphi)$ as $N\to\infty$ (for compactly supported continuous $\varphi:(-1,1)\to[0,\infty)$) – see e.g. [Kal02, Theorem 4.28]. Upgrading to weak convergence is actually also very standard. One can simply approximate general continuous $\varphi:[-1,1]\to[0,\infty)$ by ones with compact support in $(-1,1)$ and argue by Markov's inequality. For further details, we refer to e.g. [Kal83, Section 4]. □

Appendix 2.A Proof of differential identities

In this appendix we prove Lemma 2.3.6 and Lemma 2.3.7.

Proof of Lemma 2.3.6. First of all, note that all of the appearing objects are differentiable functions of t as can be seen from the determinantal representation of the polynomials (2.3.5).

Recall from (2.3.4) that $\log D_j(f_t) = -2\sum_{k=0}^{j} \log \chi_k(f_t)$. Also from (2.3.3), we see that all polynomials of degree less than j are orthogonal to p_j, so

$$\int_{\mathbb{R}} \chi_j(f_t) x^j p_j(x; f_t) f_t(x) e^{-NV(x)} dx = 1$$

and

$$\begin{aligned}\int [\partial_t p_j(x; f_t)]\, p_j(x; f_t) f_t(x) e^{-NV(x)} dx &= \int [\partial_t \chi_j(f_t)]\, x^j p_j(x; f_t) f_t(x) e^{-NV(x)} dx \\ &= \frac{\partial_t \chi_j(f_t)}{\chi_j(f_t)}.\end{aligned}$$

Thus we see that

$$\partial_t \log D_j(f_t) = -\int \partial_t \left[\sum_{l=0}^{j} p_l(x; f_t)^2\right] f_t(x) e^{-NV(x)} dx. \tag{2.A.1}$$

The Christoffel-Darboux identity (see e.g. [Dei99, page 55]) states that

$$\sum_{l=0}^{j} p_l(x; f_t)^2 = \frac{\chi_j(f_t)}{\chi_{j+1}(f_t)} [p_{j+1}'(x; f_t) p_j(x; f_t) - p_j'(x; f_t) p_{j+1}(x; f_t)]. \tag{2.A.2}$$

Here $'$ denotes differentiation with respect to x. Plugging this into (2.A.1), we see that

$$\begin{aligned}\partial_t \log D_j(f_t) &= -\int \partial_t \left[\frac{\chi_j(f_t)}{\chi_{j+1}(f_t)} [p_{j+1}'(x; f_t) p_j(x; f_t) - p_j'(x; f_t) p_{j+1}(x; f_t)]\right] f_t(x) e^{-NV(x)} dx \\ &= -\partial_t \int \frac{\chi_j(f_t)}{\chi_{j+1}(f_t)} [p_{j+1}'(x; f_t) p_j(x; f_t) - p_j'(x; f_t) p_{j+1}(x; f_t)] f_t(x) e^{-NV(x)} dx \\ &\quad + \int \frac{\chi_j(f_t)}{\chi_{j+1}(f_t)} [p_{j+1}'(x; f_t) p_j(x; f_t) - p_j'(x; f_t) p_{j+1}(x; f_t)] \partial_t f_t(x) e^{-NV(x)} dx.\end{aligned}$$

Using (2.3.3), one finds that the first integral equals $j+1$ (note that the term corresponding to $p_j' p_{j+1}$ integrates to zero by orthogonality) so its derivative equals zero.

Recalling that for $Y(z,t) = Y_{j+1}(z,t)$, we have

$$Y(z,t) = \begin{pmatrix} \frac{1}{\chi_{j+1}(f_t)} p_{j+1}(z, f_t) & * \\ -2\pi i \chi_j(f_t) p_j(z, f_t) & * \end{pmatrix},$$

where we ignore the second column of the matrix as it's not relevant right now. Thus we see the claim by replacing p_j and p_{j+1} by the entries of Y and setting $j = N-1$. □

We now prove our second differential identity.

Proof of Lemma 2.3.7. The beginning of the proof is identical to the proof of Lemma 2.3.6. Indeed, we can repeat everything up to (2.A.1) to get

$$\partial_s \log D_j(f, V_s) = -\int_{\mathbb{R}} \partial_s \left[\sum_{l=0}^{j} p_l(x; f, V_s)^2 \right] f(x) e^{-NV_s(x)} dx.$$

Again making use of Christoffel-Darboux and orthogonality, we find

$$\begin{aligned} &\partial_s \log D_j(f; V_s) \\ &= \int \frac{\chi_j(f; V_s)}{\chi_{j+1}(f; V_s)} [p'_{j+1}(x; f, V_s) p_j(x; f, V_s) - p'_j(x; f, V_s) p_{j+1}(x; f, V_s)] f(x) \partial_s e^{-NV_s(x)} dx, \end{aligned}$$

which yields the claim when we set $j = N-1$. □

Appendix 2.B Proofs for the first transformation

In this appendix we prove Lemma 2.4.2, Lemma 2.4.4, and Lemma 2.4.5. Variants of Lemma 2.4.2 are certainly well known in Riemann-Hilbert literature (see e.g. [DKM+99, Proposition 5.4]), but to have it in precisely the form we need it, we sketch a proof.

Proof of Lemma 2.4.2. The first statement – (2.4.6) – is simply linearity and making use of the fact that for the GUE, one has $\ell_{GUE} = -1 - 2\log 2$ in our normalization. This amounts to simply calculating the logarithmic potential (or noncommutative entropy) of the semi-circle law. This is a standard calculation and we omit the proof, see e.g. Theorem 4.1 in [GP13] or alternatively one can integrate (2.2.3) against the arcsine law and use the logarithmic potential of the arcsine law [ST97, Section 1.3: Example 3.5].

For (2.4.7) consider first the case where $|\lambda| - 1 > M$. Here we note that $g_{s,+}(\lambda) + g_{s,-}(\lambda) = 2\log|\lambda| + \mathcal{O}(1)$ as $|\lambda| \to \infty$ (uniformly in s), but we know that $V(\lambda)/\log|\lambda| \to \infty$ as $|\lambda| \to \infty$, so we see that by choosing M large enough (independent of s), $g_{s,+}(\lambda) + g_{s,-}(\lambda) - V_s(\lambda) - \ell_s \leq -\log|\lambda|$.

For the $|\lambda|-1<M$-case, note that the left side of (2.4.7) is a continuous function, and if we take $M'<M$, then our function is a continuous function which is (uniformly in s) negative on $[M',M]$. Thus it's enough to consider the situation where M is small. In particular, we can assume it's so small, that d is positive in $|\lambda|-1\in(0,M)$. Let us focus on the $\lambda>1$ case. The $\lambda<-1$ case is similar.

Let us suppress the dependence on s and write $F(\lambda)=g_+(\lambda)+g_-(\lambda)-V(\lambda)-\ell$. As $F(1)=0$, we have by using the Euler-Lagrange equation (2.2.3) at the point $x=1$

$$
\begin{aligned}
F(\lambda)=F(\lambda)-F(1)&=2\int_{-1}^{1}(\log(\lambda-x)-\log(1-x))\mu_V(dx)-V'(1)(\lambda-1)+\mathcal{O}((\lambda-1)^2)\\
&=2\int_{-1}^{1}\int_1^\lambda\frac{du}{u-x}\mu_V(dx)-2\int_{-1}^1\frac{\lambda-1}{1-x}\mu_V(dx)+\mathcal{O}((\lambda-1)^2)\\
&=2\int_{-1}^{1}\int_1^\lambda\left[\frac{1}{u-x}-\frac{1}{1-x}\right]du\mu_V(dx)+\mathcal{O}((\lambda-1)^2)\\
&=-2\int_1^\lambda(u-1)\int_{-1}^1\frac{d(x)\sqrt{1-x^2}}{(u-x)(1-x)}dxdu+\mathcal{O}((\lambda-1)^2).
\end{aligned}
$$

In the x-integral, let us make the change of variables, $1-x=(u-1)y$. We find

$$
\begin{aligned}
\int_{-1}^1\frac{d(x)\sqrt{1-x^2}}{(u-x)(1-x)}dx&=(u-1)\int_0^{\frac{2}{u-1}}\frac{d(1-(u-1)y)\sqrt{(u-1)y}\sqrt{2-(u-1)y}}{(u-1)^2y(1+y)}dy\\
&=\sqrt{2}d(1)(u-1)^{-1/2}\int_0^{\frac{2}{u-1}}\frac{dy}{\sqrt{y}(1+y)}+\mathcal{O}\left(\int_0^{\frac{2}{u-1}}\frac{\sqrt{(u-1)y}}{(1+y)}dy\right)\\
&=\mathcal{O}((u-1)^{-1/2}).
\end{aligned}
$$

We conclude that $F(\lambda)=-\int_1^\lambda\mathcal{O}(\sqrt{u-1})du+\mathcal{O}((\lambda-1)^2)$ which implies the claim in (2.4.7).

For (2.4.8), we note that for $\lambda\in\mathbb{R}$ and $x\in(-1,1)$

$$
\lim_{\epsilon\to0^+}[\log(\lambda+i\epsilon-x)-\log(\lambda-i\epsilon-x)]=\begin{cases}2\pi i, & \lambda<x\\ 0, & \lambda>x\end{cases}.
$$

Thus for $\lambda\in\mathbb{R}$

$$
g_{s,+}(\lambda)-g_{s,-}(\lambda)=\begin{cases}2\pi i, & \lambda<-1\\ 2\pi i\int_\lambda^1\left[(1-s)\frac{2}{\pi}+sd(x)\right]\sqrt{1-x^2}dx, & |\lambda|<1\\ 0, & \lambda>1\end{cases}
$$

which is (2.4.8).

□

We now move on to prove Lemma 2.4.4.

Proof of Lemma 2.4.4. Let $\lambda \in (-1,1)$ and $\epsilon > 0$ be small. We have

$$\begin{aligned} h_s(\lambda + i\epsilon) = &-2\pi i \int_1^\lambda \left[(1-s)\frac{2}{\pi} + sd(x)\right]\sqrt{1-x^2}dx \\ &- 2\pi i \int_0^\epsilon \left[(1-s)\frac{2}{\pi} + sd(\lambda+iu)\right]\sqrt{1-(\lambda+iu)^2} i du. \end{aligned}$$

The first term is purely imaginary. The second term is an analytic function of ϵ (in a small enough λ-dependent neighborhood of the origin), it vanishes at $\epsilon = 0$, its derivative at $\epsilon = 0$ is positive, and second derivative in a neighborhood of zero is bounded. From this one can conclude that for small enough $\epsilon > 0$, the real part of $h_s(\lambda + i\epsilon) > 0$. A similar argument works for the claim about the real part of $h_s(\lambda - i\epsilon)$. Such an argument is easily extended into a uniform one in this case. □

Finally we prove Lemma 2.4.5.

Proof of Lemma 2.4.5. Uniqueness can be argued as for Y. The analyticity condition comes from analyticity of Y and g_s, so let us look at the jump conditions. Consider first $\lambda \in (-1,1)$. Then from (2.4.5), (2.3.8), (2.4.8), (2.4.6), and some elementary matrix calculations one finds

$$\begin{aligned} T_+(\lambda) &= e^{-N\ell_s\sigma_3/2}Y_-(z)\begin{pmatrix} 1 & f_t(\lambda)e^{-NV_s(\lambda)} \\ 0 & 1 \end{pmatrix} e^{-N\left(g_{s,-}(\lambda)+2\pi i\int_\lambda^1\left[(1-s)\frac{2}{\pi}+sd(x)\right]\sqrt{1-x^2}dx-\ell_s/2\right)\sigma_3} \\ &= T_-(\lambda)\begin{pmatrix} 1 & e^{2Ng_{s,-}(\lambda)-N\ell_s}f_t(\lambda)e^{-NV_s(\lambda)} \\ 0 & 1 \end{pmatrix} e^{-Nh_s(\lambda)\sigma_3} \\ &= T_-(\lambda)\begin{pmatrix} e^{-Nh_s(\lambda)} & f_t(\lambda) \\ 0 & e^{Nh_s(\lambda)}. \end{pmatrix} \end{aligned}$$

For $|\lambda| > 1$, we note that by (2.4.8), $g_{s,+}(\lambda) - g_{s,-}(\lambda) \in \{0, 2\pi i\}$, and a similar argument results in

$$T_+(\lambda) = T_-(\lambda)\begin{pmatrix} 1 & e^{N(g_{+,s}(\lambda)+g_{s,-}(\lambda)-\ell_s-V_s(\lambda))}f_t(\lambda) \\ 0 & 1 \end{pmatrix}$$

which is precisely (2.4.11).

For the behavior at infinity, we note that as $z \to \infty$ (uniformly for z not on the negative real axis) $g_s(z) = \log z + \mathcal{O}(|z|^{-1})$. Thus we see from (2.3.9) and (2.4.5) that indeed

(2.4.12) is satisfied (with behavior on the negative real axis coming from continuity up to the boundary). $\square$

Appendix 2.C The RHP for the global parametrix

In this appendix we will sketch a proof of Lemma 2.4.14. We will make use of the fact that the result is proven for $t=0$, i.e. the case when $\mathcal{T}=0$, in [Kra07, Section 4.2] (which relies on a similar result in [KMVAV04, Section 5], which again makes use of results in e.g. [Dei99]).

Sketch of a proof of Lemma 2.4.14. The analyticity condition was already argued in Remark 2.4.13. The normalization at infinity is easy to see from the fact that the a-matrix (in right hand side of (2.4.22)) is $2I+\mathcal{O}(|z|^{-1})$ and $\mathcal{D}_t(z)=\mathcal{D}_t(\infty)+\mathcal{O}(|z|^{-1})$ as $z\to\infty$. Thus the jump condition is the main one to check.

This would be a fairly short calculation to check directly, but we make use of it being known for $t=0$ and the representation (2.4.23). We start by noting that by the Sokhotski-Plemelj formula and (2.4.23), for $\lambda\in(-1,1)\setminus\{x_j\}_{j=1}^k$

$$P_\pm^{(\infty)}(\lambda,t)=e^{\frac{\sigma_3}{2\pi}\int_{-1}^1\frac{\mathcal{T}_t(x)}{\sqrt{1-x^2}}dx}P_\pm^{(\infty)}(\lambda,0)e^{-\sigma_3\frac{r_\pm(\lambda)}{2\pi}\left[\pm\pi i\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}+P.V.\int_{-1}^1\frac{\mathcal{T}_t(x)}{\sqrt{1-x^2}}\frac{dx}{\lambda-x}\right]},$$

where $P.V.$ denotes the Cauchy principal value integral. Thus from the jump condition of $P^{(\infty)}(z,0)$ (note that $\det P^{(\infty)}(z,t)=1$ so everything makes sense)

$$\left[P_-^{(\infty)}(\lambda,t)\right]^{-1}P_+^{(\infty)}(\lambda,t)=e^{\sigma_3\frac{r_-(\lambda)}{2\pi}\left[-\pi i\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}+P.V.\int_{-1}^1\frac{\mathcal{T}_t(x)}{\sqrt{1-x^2}}\frac{dx}{\lambda-x}\right]}\begin{pmatrix}0 & f_0(\lambda)\\ -f_0(\lambda)^{-1} & 0\end{pmatrix}$$
$$\times\, e^{-\sigma_3\frac{r_+(\lambda)}{2\pi}\left[\pi i\frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}+P.V.\int_{-1}^1\frac{\mathcal{T}_t(x)}{\sqrt{1-x^2}}\frac{dx}{\lambda-x}\right]}$$

Noting that (from the definition of r; see (2.4.19)) $r_+(\lambda)=i\sqrt{1-\lambda^2}$ and $r_-(\lambda)=-i\sqrt{1-\lambda^2}$ so with a simple calculation

$$\left[P_-^{(\infty)}(\lambda,t)\right]^{-1}P_+^{(\infty)}(\lambda,t)=\begin{pmatrix}0 & e^{\mathcal{T}_t(\lambda)}f_0(\lambda)\\ -e^{-\mathcal{T}_t(\lambda)}f_0(\lambda)^{-1} & 0\end{pmatrix},$$

which is precisely the claim as $f_0e^{\mathcal{T}_t}=f_t$. $\square$

Appendix 2.D The RHP for the local parametrix near a singularity

Here we give further details about the local parametrix near a singularity. First of all, we give a full description of the solution to the model RHP - the function Ψ.

Definition 2.D.1. *Recall that we use Roman numerals for the octants of the plane:* $\mathrm{I} = \{re^{i\theta} : r > 0, \theta \in (0, \pi/4)\}$ *and so on. We also write* I_ν *and* K_ν *for the modified Bessel functions of the first and second kind, as well as* $H_\nu^{(1)}$ *and* $H_\nu^{(2)}$ *for the Hankel functions of the first and second kind. We then define (again roots are principal branch roots)*

$$\Psi(\zeta) = \frac{1}{2}\sqrt{\pi\zeta}\begin{pmatrix} H^{(2)}_{\frac{\beta_j+1}{2}}(\zeta) & -iH^{(1)}_{\frac{\beta_j+1}{2}}(\zeta) \\ H^{(2)}_{\frac{\beta_j-1}{2}}(\zeta) & -iH^{(1)}_{\frac{\beta_j-1}{2}}(\zeta) \end{pmatrix} e^{-\left(\frac{\beta_j}{2}+\frac{1}{4}\right)\pi i\sigma_3} \quad \zeta \in \mathrm{I}, \tag{2.D.1}$$

$$\Psi(\zeta) = \sqrt{\zeta}\begin{pmatrix} \sqrt{\pi}I_{\frac{\beta_j+1}{2}}(-i\zeta) & -\frac{1}{\sqrt{\pi}}K_{\frac{\beta_j+1}{2}}(-i\zeta) \\ -i\sqrt{\pi}I_{\frac{\beta_j-1}{2}}(-i\zeta) & -\frac{i}{\sqrt{\pi}}K_{\frac{\beta_j-1}{2}}(-i\zeta) \end{pmatrix} e^{-\frac{\beta_j}{4}\pi i\sigma_3} \quad \zeta \in \mathrm{II}, \tag{2.D.2}$$

$$\Psi(\zeta) = \sqrt{\zeta}\begin{pmatrix} \sqrt{\pi}I_{\frac{\beta_j+1}{2}}(-i\zeta) & -\frac{1}{\sqrt{\pi}}K_{\frac{\beta_j+1}{2}}(-i\zeta) \\ -i\sqrt{\pi}I_{\frac{\beta_j-1}{2}}(-i\zeta) & -\frac{i}{\sqrt{\pi}}K_{\frac{\beta_j-1}{2}}(-i\zeta) \end{pmatrix} e^{\frac{\beta_j}{4}\pi i\sigma_3} \quad \zeta \in \mathrm{III}, \tag{2.D.3}$$

$$\Psi(\zeta) = \frac{1}{2}\sqrt{-\pi\zeta}\begin{pmatrix} iH^{(1)}_{\frac{\beta_j+1}{2}}(-\zeta) & -H^{(2)}_{\frac{\beta_j+1}{2}}(-\zeta) \\ -iH^{(1)}_{\frac{\beta_j-1}{1}}(-\zeta) & H^{(2)}_{\frac{\beta_j-1}{2}}(-\zeta) \end{pmatrix} e^{\left(\frac{\beta_j}{2}+\frac{1}{4}\right)\pi i\sigma_3} \quad \zeta \in \mathrm{IV}, \tag{2.D.4}$$

$$\Psi(\zeta) = \frac{1}{2}\sqrt{-\pi\zeta}\begin{pmatrix} -H^{(2)}_{\frac{\beta_j+1}{2}}(-\zeta) & -iH^{(1)}_{\frac{\beta_j+1}{2}}(-\zeta) \\ H^{(2)}_{\frac{\beta_j-1}{2}}(-\zeta) & iH^{(1)}_{\frac{\beta_j-1}{2}}(-\zeta) \end{pmatrix} e^{-\left(\frac{\beta_j}{2}+\frac{1}{4}\right)\pi i\sigma_3} \quad \zeta \in \mathrm{V}, \tag{2.D.5}$$

$$\Psi(\zeta) = \sqrt{\zeta}\begin{pmatrix} -i\sqrt{\pi}I_{\frac{\beta_j+1}{2}}(i\zeta) & -\frac{i}{\sqrt{\pi}}K_{\frac{\beta_j+1}{2}}(i\zeta) \\ \sqrt{\pi}I_{\frac{\beta_j-1}{2}}(i\zeta) & -\frac{1}{\sqrt{\pi}}K_{\frac{\beta_j-1}{2}}(i\zeta) \end{pmatrix} e^{-\frac{\beta_j}{4}\pi i\sigma_3} \quad \zeta \in \mathrm{VI}, \tag{2.D.6}$$

$$\Psi(\zeta)=\sqrt{\zeta}\begin{pmatrix} -i\sqrt{\pi}I_{\frac{\beta_j+1}{2}}(i\zeta) & -\frac{i}{\sqrt{\pi}}K_{\frac{\beta_j+1}{2}}(i\zeta) \\ \sqrt{\pi}I_{\frac{\beta_j-1}{2}}(i\zeta) & -\frac{1}{\sqrt{\pi}}K_{\frac{\beta_j-1}{2}}(i\zeta) \end{pmatrix} e^{\frac{\beta_j}{4}\pi i\sigma_3} \quad \zeta\in \mathrm{VII}, \tag{2.D.7}$$

$$\Psi(\zeta)=\frac{1}{2}\sqrt{\pi\zeta}\begin{pmatrix} -iH^{(1)}_{\frac{\beta_j+1}{2}}(\zeta) & -H^{(2)}_{\frac{\beta_j+1}{2}}(\zeta) \\ -iH^{(1)}_{\frac{\beta_j-1}{1}}(\zeta) & -H^{(2)}_{\frac{\beta_j-1}{2}}(\zeta) \end{pmatrix} e^{\left(\frac{\beta_j}{2}+\frac{1}{4}\right)\pi i\sigma_3} \quad \zeta\in \mathrm{VIII}. \tag{2.D.8}$$

In [Van03, Theorem 4.2] it is shown that this function indeed satisfies the problem we used in Definition 2.4.20. An important fact about the function Ψ is its behavior near the origin. The following was also part of [Van03, Theorem 4.2]: as $\zeta\to 0$

$$\Psi(\zeta)=\begin{cases} \begin{pmatrix} \mathcal{O}(|\zeta|^{\beta_j/2}) & \mathcal{O}(|\zeta|^{-\beta_j/2}) \\ \mathcal{O}(|\zeta|^{\beta_j/2}) & \mathcal{O}(|\zeta|^{-\beta_j/2}) \end{pmatrix}, & \zeta\in \mathrm{II},\mathrm{III},\mathrm{VI},\mathrm{VII} \\ \begin{pmatrix} \mathcal{O}(|\zeta|^{-\beta_j/2}) & \mathcal{O}(|\zeta|^{-\beta_j/2}) \\ \mathcal{O}(|\zeta|^{-\beta_j/2}) & \mathcal{O}(|\zeta|^{-\beta_j/2}) \end{pmatrix}, & \zeta\in \mathrm{I},\mathrm{IV},\mathrm{V},\mathrm{VIII} \end{cases}. \tag{2.D.9}$$

We also mention that the function Ψ could be expressed in terms of the confluent hypergeometric function of the second kind as in [DIK11, DIK14]. Let us now sketch the proof of Lemma 2.4.23.

Sketch of a proof of Lemma 2.4.23. Consider first the analyticity condition. As we mentioned in Remark 2.4.22, one can check that E is analytic in U'_{x_j}, so the jumps of $P^{(x_j)}$ come from those of $\Psi(\zeta_s(z))$, $W_j(z)^{-\sigma_3}$ and $e^{-N\phi_s(z)\sigma_3}$.

As ζ_s preserves the real axis, and Σ was chosen so that under ζ_s, $\Sigma\cap U'_{x_j}$ is mapped to the real axis and lines intersecting origin at angles $\pm\pi/4$. Thus from Definition 2.4.20, $\Psi(\zeta_s(z))$ has jumps on Σ and $\{z:\mathrm{Re}(\zeta_s(z))=0\}$.

From (2.4.27) – the definition of W_j – we see that W_j has jumps only across $\mathbb{R}$ and $\{z:\mathrm{Re}(\zeta_s(z))=0\}$. Also from (2.4.28) and (2.4.9) we see that ϕ only has a jump across $\mathbb{R}$.

Thus to see that $P^{(x_j)}(z,t,s)$ is analytic in $U'_{x_j}\setminus\Sigma$, we need to check that the jump of $W_j(z)^{-\sigma_3}$ cancels that of $\Psi(\zeta_s(z))$ along $\{z:\mathrm{Re}(\zeta_s(z))=0\}$. Let us look at for example the jump across $\{z:\mathrm{Re}(\zeta_s(z))=0,\mathrm{Im}(\zeta_s(z))>0\}=\zeta_s^{-1}(\Gamma_3)$. From (2.4.27) we find that for $\lambda\in\zeta_s^{-1}(\Gamma_3)$ (where the orientation is as for Γ_3)

$$W_{j,+}(\lambda)W_{j,-}(\lambda)^{-1}=\frac{(\lambda-x_j)^{\beta_j/2}}{(x_j-\lambda)^{\beta_j/2}}=e^{i\pi\frac{\beta_j}{2}}.$$

Combining this with (2.4.32)

$$\Psi_+(\zeta_s(\lambda))W_{j,+}(\lambda)^{-\sigma_3} = \Psi_-(\zeta_s(\lambda))e^{i\pi\frac{\beta_j}{2}\sigma_3}e^{-i\pi\frac{\beta_j}{2}\sigma_3}W_{j,-}(\lambda) = \Psi_-(\zeta_s(\lambda))W_{j,-}(\lambda),$$

so we see that $P^{(x_j)}(z)$ is continuous across $\zeta_s^{-1}(\Gamma_3)$. The argument is similar for the jump across $\zeta_s^{-1}(\Gamma_7)$. We conclude that $P^{(x_j)}$ is analytic in $U'_{x_j} \setminus \Sigma$.

Consider now the jump structure. The existence of continuous boundary values is inherited from the corresponding properties of Ψ, W_j and ϕ_s. As W_j and ϕ_s have no jumps across $\Sigma_{j-1}^{\pm}$ or $\Sigma_j^{\pm}$, the jumps here come from the jumps of Ψ. Let us consider for example $\lambda \in \zeta_s^{-1}(\Gamma_2)$. Here using the jump condition of Ψ, an elementary matrix calculation shows that

$$\begin{aligned}
P_+^{(x_j)}(\lambda) &= P_-^{(x_j)}(\lambda)W_j(\lambda)^{\sigma_3}e^{N\phi_s(\lambda)}\begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta_j} & 1\end{pmatrix}W_j(\lambda)^{-\sigma_3}e^{-N\phi_s(\lambda)} \\
&= P_-^{(x_j)}(\lambda)\begin{pmatrix} 1 & 0 \\ f_t(\lambda)^{-1}e^{-Nh_s(\lambda)} & 1\end{pmatrix}.
\end{aligned}$$

Calculating the jump matrix across $\Sigma_{j-1}^{\pm}$ and Σ_j^{-} is similar. For the jump across $\mathbb{R}$, we have for example for $\lambda \in U'_{x_j} \cap (x_j, \infty)$, from (2.4.27), (2.4.28), the analyticity of h_s across $U'_{x_j} \cap \mathbb{R}$, along with Definition 2.4.20:

$$\begin{aligned}
P_+^{(x_j)}(\lambda) &= P_-^{(x_j)}(\lambda)\begin{pmatrix} 0 & e^{Nh_s(\lambda)-i\pi\frac{\beta_j}{2}}W_{j,-}(\lambda)^2e^{2N\phi_{s,-}(\lambda)} \\ -e^{-Nh_s(\lambda)+i\pi\beta_j}W_{j,-}(\lambda)^{-2}e^{-2N\phi_{s,-}(\lambda)} & 0\end{pmatrix} \\
&= P_-^{(x_j)}(\lambda)\begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0\end{pmatrix}.
\end{aligned}$$

The calculation for the jump across $U'_{x_j} \cap (-\infty, x_j)$ is similar.

To see (2.4.39), note first that as $z \to x_j$, $\zeta_s(x) = \mathcal{O}(|z-x_j|)$ (the implicit constant depending on x_j, N, and s, but this doesn't matter now) and $W_j(z) = \mathcal{O}(|z-x_j|^{\beta_j/2})$. So we have from (2.D.9) that for $z \in \zeta_s^{-1}(\mathrm{I})$ and $z \to x_j$

$$\Psi(\zeta_s(z))W_j(z)^{-\sigma_3} = \begin{pmatrix} \mathcal{O}(|z-x_j|^{-\beta_j}) & \mathcal{O}(1) \\ \mathcal{O}(|z-x_j|^{-\beta_j}) & \mathcal{O}(1)\end{pmatrix}.$$

As E is analytic in U'_{x_j}, it is in particular bounded at x_j, so as multiplying from the left doesn't mix the columns, we have the same behavior for $E(z)\Psi(\zeta_s(z))W_j(z)^{-\sigma_3}$. Now also ϕ_s is bounded at x_j and again multiplying by a diagonal matrix doesn't mix the columns so we have the claimed asymptotics for $P^{(x_j)}(z)$ as $z \to x_j$ from $\zeta_s^{-1}(\mathrm{I})$. The other regions are similar.

Let us now focus on the matching condition (2.4.40). We note that as d is positive on $[-1,1]$, we see that for $z \in \partial U_{x_j}$ (and for δ small enough), $|\zeta_s(z)| \asymp N$ where the implied constants are uniform in $x_j \in (-1+3\delta, 1-3\delta)$, $s \in [0,1]$, and $z \in \partial U_{x_j}$. Thus to study $\Psi(\zeta_s(z))$, we can make use of the large argument expansion of Bessel functions. We won't go into great detail here, but simply refer the reader to [Van03, Section 4.3] and references therein.

For simplicity, we focus on the domain $\{z : \arg \zeta_s(z) \in (0, \pi/2)\}$. In the other domains, one has different asymptotics for Ψ, but the argument is similar. The relevant asymptotics here are

$$\Psi(\zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I + \mathcal{O}(|\zeta|^{-1})\right] e^{\frac{\pi i}{4}\sigma_3} e^{-i\zeta\sigma_3} e^{-\pi i \frac{\beta_j}{4}\sigma_3}, \tag{2.D.10}$$

where the implied constant in $\mathcal{O}(|\zeta|^{-1})$ is uniform in the first quadrant. Here and below, the $\mathcal{O}$-notation will refer to a 2×2 matrix whose entries satisfy the relevant bound. Noting from (2.4.26), (2.4.9), and (2.4.28), that for $z \in U'_{x_j} \cap \{\mathrm{Im}(z) > 0\}$

$$\zeta_s(z) = -Ni(\phi_{s,+}(x_j) - \phi_s(z)).$$

It then follows from this and (2.D.10) that for $z \in \zeta_s^{-1}(\mathrm{I} \cup \mathrm{II}) \cap \partial U_{x_j}$

$$\begin{aligned} \Psi(\zeta_s(z))W_j(z)^{-\sigma_3} e^{-N\phi_s(z)\sigma_3} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I + \mathcal{O}(N^{-1})\right] e^{\frac{\pi i}{4}\sigma_3} e^{-N(\phi_{s,+}(x_j) - \phi_s(z))\sigma_3} e^{-\pi i \frac{\beta_j}{4}\sigma_3} \\ &\quad \times W_j(z)^{-\sigma_3} e^{-N\phi_s(z)\sigma_3} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \left[I + \mathcal{O}(N^{-1})\right] e^{i\frac{\pi}{4}(1-\beta_j)\sigma_3} e^{-N\phi_{s,+}(x_j)\sigma_3} W_j(z)^{-\sigma_3}, \end{aligned}$$

where the $\mathcal{O}(N^{-1})$ term is uniform in everything relevant. Using (2.4.33) and (2.4.35), we see that for $z \in \zeta_s^{-1}(\mathrm{I} \cup \mathrm{II}) \cap \partial U_{x_j}$

$$P^{(x_j)}(z) \left[P^{(\infty)}(z)\right]^{-1} = A(z)(I + \mathcal{O}(N^{-1}))A(z)^{-1} = I + A(z)\mathcal{O}(N^{-1})A(z)^{-1},$$

where the $\mathcal{O}(N^{-1})$ term is uniform in everything relevant and

$$A(z) = P^{(\infty)}(z)W_j(z)^{\sigma_3}e^{N\phi_{s,+}(x_j)\sigma_3}e^{-i\frac{\pi}{4}(1-\beta_j)\sigma_3} = E(z)\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right]^{-1}$$

The claim (in this sector of the boundary) will then follow if we show that A is uniformly bounded in everything relevant. As $\phi_{s,+}(x_j)$ is purely imaginary (see (2.4.9)), we see that the relevant question is the boundedness of $P^{(\infty)}(z)W_j(z)^{\sigma_3}$ and its inverse. Looking at (2.4.22), we see that this is equivalent to $\mathcal{D}_t(z)^{-1}W_j(z)$ being uniformly bounded and uniformly bounded away from zero. Let us write this quantity out. From (2.4.21) and (2.4.27) we have

$$\left|\mathcal{D}_t(z)^{-1}W_j(z)\right| = \left|(z+r(z))^{\mathcal{A}}e^{-\frac{r(z)}{2\pi}\int_{-1}^1 \frac{\mathcal{T}_t(\lambda)}{\sqrt{1-\lambda^2}}\frac{d\lambda}{z-\lambda}}e^{\frac{1}{2}\mathcal{T}_t(z)}\right|.$$

Since $z+r(z)$ is obviously bounded for z in a compact set, the integral term is uniformly bounded in everything relevant by Lemma 2.4.15, and the last term is bounded as $\mathcal{T}_t$ is uniformly bounded in everything relevant. Similarly we see uniform boundedness away from zero. This concludes the proof for $z \in \zeta_s^{-1}(\mathrm{I}\cup\mathrm{II})\cap\partial U_{x_j}$. The proof in the remaining parts of the boundary are similar. $\square$

We now move on to considering the proof of Lemma 2.4.24.

Proof of Lemma 2.4.24. Here we simply need to take into account the next term in the asymptotic expansion of Ψ. The argument is otherwise as in the proof of Lemma 2.4.23. For simplicity, we will focus on the case where ζ is in the first quadrant. Other quadrants are handled in a similar manner. We refer to the discussion around [Van03, equation (5.9)] for the following asymptotics:

$$\Psi(\zeta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I - i\frac{\beta_j}{4\zeta}\begin{pmatrix} \frac{\beta_j}{2} & i \\ i & -\frac{\beta_j}{2} \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-2}\right)\right]e^{i\left(\frac{\pi}{4}-\frac{\beta_j\pi}{4}-\zeta\right)\sigma_3}, \qquad (2.D.11)$$

where the error $\mathcal{O}(|\zeta|^{-2})$ is uniform for ζ in the first quadrant. Then arguing as in the previous proof, we see that

$$P^{(x_j)}(z)\left[P^{(\infty)}(z)\right]^{-1} = I - i\frac{\beta_j}{4\zeta_s(z)}A(z)\begin{pmatrix} \frac{\beta_j}{2} & i \\ i & -\frac{\beta_j}{2} \end{pmatrix}A(z)^{-1} + \mathcal{O}\left(|\zeta_s(z)|^{-2}\right),$$

where we used the uniform boundedness of A and A^{-1}. Noting that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \frac{\beta_j}{2} & i \\ i & -\frac{\beta_j}{2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 0 & \left(1+\frac{\beta_j}{2}\right)i \\ \left(1-\frac{\beta_j}{2}\right)i & 0 \end{pmatrix},$$

making use of $\zeta_s(z) \asymp N$ uniformly in everything relevant for $z \in \partial U_{x_j}$ and the fact that the asymptotic expansion of Ψ is uniform, we see the claim. Again, the argument in the other regions is similar. □

Appendix 2.E The RHP for the local parametrix near the edge of the spectrum

In this section we will give some further details about the parametrices near the edge of the spectrum. First we will justify the definition of the function ξ_s from (2.4.42).

Justification of the definition of ξ_s. The argument is essentially as in [DKM+99, Section 7]. Let us first recall some properties of ϕ_s. From (2.4.28) and (2.4.9), we note that ϕ_s has a jump across $U_1' \cap (-1,1)$ but is continuous across $U_1' \cap (1,\infty)$, so it is analytic in $U_1' \setminus [-1,1]$. Moreover, in $U_1' \setminus [-1,1]$ we can write

$$\frac{3\pi}{2} d_s(z)(z+1)^{1/2}(z-1)^{1/2} = \widetilde{G}_s^{(1)}(z)(z-1)^{1/2}, \tag{2.E.1}$$

where $\widetilde{G}_s^{(1)}$ is analytic in U_1'. Expanding $\widetilde{G}_s^{(1)}$ as a series, integrating, and taking into account the branch structure of ϕ_s, we can write

$$-\frac{3}{2}\phi_s(z) = G_s^{(1)}(z)(z-1)^{3/2}, \tag{2.E.2}$$

where the power is according to the principal branch and $G_s^{(1)}$ is analytic in U_1'. If we expand $G_s^{(1)}(z) = \sum_{k=0}^{\infty} G_{s,k}^{(1)}(z-1)^k$ and $\widetilde{G}_s^{(1)}(w) = \sum_{k=0}^{\infty} \widetilde{G}_{s,k}^{(1)}(w-1)^k$, then

$$G_{s,k}^{(1)} = \frac{2}{3+2k}\widetilde{G}_{s,k}^{(1)}.$$

Now as $\widetilde{G}_{s,0}^{(1)} = \frac{3\pi}{\sqrt{2}} d_s(1)$ is uniformly bounded away from zero, we see from the above display that the same holds for $G_{s,0}^{(1)}$. By Cauchy's integral formula (for derivatives),

$$\left|\widetilde{G}_{s,k}^{(1)}\right| \le (3\delta/2)^{-k} \sup_{|z-1|=\delta}\left|\frac{3}{2}\sqrt{z+1}\left[sd(z) + (1-s)\frac{2}{\pi}\right]\right| \le C_\delta (3\delta/2)^{-k},$$

for some constant C_δ independent of s, so we again get a similar bound for $G_{s,k}^{(1)}$. From this type of estimate, one can easily argue that by possibly decreasing δ by some s-independent factor, $G_s^{(1)}$ is zero free in U_1'. Thus with a suitable convention for the branch of the power, the function

$$\xi_s(z) = N^{2/3}(z-1)G_s^{(1)}(z)^{2/3}$$

is analytic in U_1'.

For injectivity, note that the derivative of the function $z \mapsto (z-1)G_s^{(1)}(z)^{2/3}$ at $z=1$ is uniformly (in s) bounded away from zero and its second derivative is uniformly bounded in s and in a small enough (s independent) neighborhood of 1. Thus by decreasing δ if needed (in an s independent manner), we have univalence of ξ_s.

□

We now sketch the proof of Lemma 2.4.30.

Sketch of a proof of Lemma 2.4.30. Let us first of all consider the analyticity of F. $P^{(\infty)}$ is analytic in $U_1' \setminus [-1,1]$, $f^{1/2}$ is analytic in U_1', and as $\zeta_s(1) = 0$, $\zeta_s^{1/4}$ has a branch cut in U_1. We note from (2.E.2) that as one can check (from (2.4.9)) that $-\phi_s(\lambda) > 0$ for $\lambda > 1$, $G_s(\lambda) > 0$ for $\lambda > 1$. Thus G_s is real on $\mathbb{R} \cap U_1'$. As we argued above that it's zero free, it must be positive on $\mathbb{R} \cap U_1'$, so we see that $\xi_s(\lambda) < 0$ for $\lambda < 1$. As we are dealing with the principal branch, the cut of $\xi_s^{1/4}$ is along $U_1' \cap (-1,1)$. It's thus enough to check that F is continuous across $(-1,1) \cap U_1'$ and does not have an isolated singularity at $z=1$.

For the continuity across $(-1,1)$, let $\lambda \in (-1,1) \cap U_1'$. We have from (2.4.24) and the jump for $\xi_s^{1/4}$: for $\lambda \in (-1,1) \cap U_1'$

$$[\xi_s]_+^{1/4}(\lambda) = i[\xi_s]_-^{1/4}(\lambda),$$

so that

$$\begin{aligned}
F_-(\lambda)^{-1}F_+(\lambda) &= \left([\xi_s]_-^{1/4}(\lambda)\right)^{-\sigma_3} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3} f_t(\lambda)^{-\sigma_3/2}\left[P_-^{(\infty)}(\lambda)\right]^{-1} P_+^{(\infty)}(\lambda) \\
&\quad \times f_t(\lambda)^{\sigma_3/2} e^{i\frac{\pi}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\left([\xi_s]_+^{1/4}(\lambda)\right)^{\sigma_3} \\
&= \left([\xi_s]_-^{1/4}(\lambda)\right)^{-\sigma_3} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\left([\xi_s]_+^{1/4}(\lambda)\right)^{\sigma_3} \\
&= I.
\end{aligned}$$

Thus F is continuous across $(-1,1) \cap U_1'$.

For the absence of an isolated singularity, we note that the entries of $\xi_s(z)^{\sigma_3/4}$ behave at worst like $|z-1|^{-1/4}$ as $z \to 1$. From (2.4.22) and Lemma 2.4.15, we see that the entries of $P^{(\infty)}(z)$ behave at worst like $|z-1|^{-1/4}$ as well. As $f^{1/2}(z)$ is bounded at $z=1$, the entries of $F(z)$ behave at worst like $|z-1|^{-1/2}$. This is not strong enough to be a pole, so there can be no isolated singularity at $z=1$ and F is analytic.

Towards checking the analyticity of $P^{(1)}$ on $U_1' \setminus \Sigma$, we refer to [DKM+99, Section 7] on the following matter (in their notation $Q = \Psi^\sigma$): $Q(\xi_s(z))$ is analytic on $U_1' \setminus \Sigma$ and it satisfies the following jump conditions:

$$Q_+(\xi_s(\lambda)) = Q_-(\xi_s(\lambda)) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \lambda \in \Sigma_{k+1}^{\pm} \cap U_1', \tag{2.E.3}$$

$$Q_+(\xi_s(\lambda)) = Q_-(\xi_s(\lambda)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \in (-1,1)^{\pm} \cap U_1', \tag{2.E.4}$$

and

$$Q_+(\xi_s(\lambda)) = Q_-(\xi_s(\lambda)) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in (1,\infty)^{\pm} \cap U_1'. \tag{2.E.5}$$

As $f_t^{\pm 1/2}$ is analytic in U_1' as if F, and ϕ_s has a jump along $(-1,1) \cap U_1'$, we see that $P^{(1)}$ indeed is analytic in U_1'.

The jump conditions come from those of Q. Let us check for example the one across $(-1,1) \cap U_1'$ – (2.4.48). For $\lambda \in (-1,1) \cap U_1'$, we have

$$\begin{aligned}
\left[P_-^{(1)}(\lambda)\right]^{-1} P_+^{(1)}(\lambda) &= f_t(\lambda)^{\sigma_3/2} e^{N\phi_{s,-}(\lambda)} Q_-(\xi_s(\lambda)) Q_+(\xi_s(\lambda) e^{-N\phi_{s,+}(\lambda)} f_t(\lambda)^{-\sigma_3/2} \\
&= f_t(\lambda)^{\sigma_3/2} e^{-\frac{1}{2}Nh_s(\lambda)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2}Nh_s(\lambda)\sigma_3} f_t(\lambda)^{-\sigma_3/2} \\
&= \begin{pmatrix} 0 & f_t(\lambda) \\ -f_t(\lambda)^{-1} & 0 \end{pmatrix}.
\end{aligned}$$

The other jump conditions are similar.

Let us then check the matching condition. Let $z \in \partial U_1$. For small enough δ (independent of s), it is clear from (2.4.9) and (2.4.28) that $|\phi_s(z)|$ is bounded away from zero uniformly in s and uniformly in $z \in \partial U_1$. Thus $|\xi_s(z)| \asymp N^{2/3}$ where the implied constants are uniform in z and s. We can thus make use of the large $|\xi|$ asymptotics of $\mathrm{Ai}(\xi)$ and

$\mathrm{Ai}'(\xi)$ to obtain asymptotics for $Q(\xi_s(z))$. For this, we will again refer to [DKM+99] – in particular [DKM+99, equation (7.30)]: for $z \in \partial U_1$

$$Q(\xi_s(z))e^{\frac{2}{3}\xi_s(z)^{3/2}\sigma_3} = \frac{e^{\pi i/12}}{2\sqrt{\pi}}\xi_s(z)^{-\sigma_3/4}\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3} + \mathcal{O}(N^{-1})\right],$$

where the error is uniform in z and s. Recalling that the construction of ξ_s was precisely so that $\frac{2}{3}\xi_s(z)^{3/2} = -N\phi_s(z)$, we see that

$$Q(\xi_s(z))e^{-N\phi_s(z)\sigma_3}f_t(z)^{-\sigma_3/2} = \frac{e^{\pi i/12}}{2\sqrt{\pi}}\xi_s(z)^{-\sigma_3/4}\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3} + \mathcal{O}(N^{-1})\right] f_t(z)^{-\sigma_3/2},$$

with the $\mathcal{O}(N^{-1})$-term being uniform in everything relevant. Thus

$$P^{(1)}(z)\left[P^{(\infty)}(z)\right]^{-1} = I + P^{(\infty)}(z)f_t(z)^{\sigma_3/2}\mathcal{O}(N^{-1})f_t(z)^{-\sigma_3/2}\left[P^{(\infty)}(z)\right]^{-1}.$$

As $f_t(z)^{\pm 1}$ as well as the entries of $[P^{(\infty)}]^{\pm 1}$ are uniformly (in everything relevant) bounded on ∂U_1, the claim follows. □

We will also give a proof of Lemma 2.4.31.

Proof of Lemma 2.4.31. This is again proven as the matching condition, but using finer asymptotics of the Airy function. In particular, one has (see [DKM+99, equation (7.30)])

$$\begin{aligned} &Q(\xi_s(z))e^{-N\phi_s(z)\sigma_3} \\ &= \frac{e^{\pi i/12}}{2\sqrt{\pi}}\xi_s(z)^{-\sigma_3/4}\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{5}{48} & \frac{5}{48} \\ -\frac{7}{48} & -\frac{7}{48} \end{pmatrix}\xi_s(z)^{-3/2} + \mathcal{O}(|\xi_s(z)|^{-3})\right] e^{-i\frac{\pi}{4}\sigma_3}, \end{aligned}$$

where the constant implied by the $\mathcal{O}$ notation is uniform in everything relevant. Thus arguing as in the previous proof, we see that for $z \in \partial U_1$

$$\begin{aligned} &P^{(1)}(z)\left[P^{(\infty)}(z)\right]^{-1} \\ &= I + P^{(\infty)}(z)f(z)^{\sigma_3/2}e^{i\pi\sigma_3/4}\frac{1}{8}\begin{pmatrix} \frac{1}{6} & 1 \\ -1 & -\frac{1}{6} \end{pmatrix} e^{-i\pi\sigma_3/4}f(z)^{-\sigma_3/2}\left[P^{(\infty)}(z)\right]^{-1}\xi_s(z)^{-3/2} + \mathcal{O}(N^{-2}) \end{aligned}$$

uniformly in everything relevant. □

Appendix 2.F Proofs concerning the final transformation and solving the R-RHP

In this section we sketch proofs concerning the final transformation and the solution of the R-RHP. We start with checking that R indeed solves the RHP of Lemma 2.4.34.

Proof of Lemma 2.4.34. Uniqueness follows from S being the unique solution to its problem. The last condition is immediate to check as for large $|z|$, $R(z) = S(z)[P^{(\infty)}(z)]^{-1}$ and both of these terms are asymptotically $I + \mathcal{O}(|z|^{-1})$. The jump conditions simply make use of the definition of R and the jump conditions of S – these are direct to check and we skip this.

For the analyticity condition we begin with the domain $U_{\pm 1}$. Here the construction of $P^{(\pm 1)}$ was such that it would have the same jumps as S so R has no branch cuts inside of $U_{\pm 1}$. We are left with the possibility that R would have an isolated singularity at $z = \pm 1$. Recall that $S(z)$ is bounded as $z \to \pm 1$, while Lemma 2.4.15 implies that the entries of $[P^{(\infty)}(z)]^{-1}$ can blow up at most like $|z \mp 1|^{-1/4}$ as $z \to \pm 1$. Thus the possible isolated singularity of R is not strong enough to be a pole (or essential), so it is removable, and R is analytic in $U_{\pm 1}$.

Consider now a neighborhood U_{x_j}. Again, by the construction of the parametrix, there are no jumps here, and the only possible singularity is an isolated singularity at x_j. Recall now that as $z \to x_j$ from outside of the lenses, $S(z) = \mathcal{O}(1)$, and as $z \to x_j$ from inside of the lenses,

$$S(z) = \begin{pmatrix} \mathcal{O}(|z - x_j|^{-\beta_j}) & \mathcal{O}(1) \\ \mathcal{O}(|z - x_j|^{-\beta_j}) & \mathcal{O}(1) \end{pmatrix}.$$

$P^{(x_j)}(z)$ has similar behavior near x_j. To estimate it's inverse, we note that $\det P^{(x_j)}(z) = 1$ for all $z \in U_{x_j}$ - which follows directly from the definitions once one knows that $\det \Psi = 1$ (which we argued following Definition 2.4.20, or one could check directly using the explicit representation of Ψ from Appendix 2.D).

We thus see that as $z \to x_j$ from outside of the lenses, $[P^{(x_j)}(z)]^{-1}$ remains bounded, and as $z \to x_j$ from inside the lenses, we have

$$\left[P^{(x_j)}(z)\right]^{-1} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(|z - x_j|^{-\beta_j}) & \mathcal{O}(|z - x_j|^{-\beta_j}) \end{pmatrix}$$

so we conclude that from the inside of the lens, the entries of the matrix $S(z)[P^{(x_j)}(z)]^{-1}$ have singularities of order $\mathcal{O}(|z - x_j|^{-\beta_j})$ at worst. Now we see that as $S(z)[P^{(x_j)}(z)]^{-1}$ remains bounded as $z \to x_j$ from outside of the lenses, it can't have a pole at x_j. But as the degree of the singularity is bounded (we can find an integer k such that

$(z-x_j)^k S(z)[P^{(x_j)}(z)]^{-1}$ tends to zero as $z \to x_j$), the singularity can't be essential either. Thus the only possibility is that the singularity is removable, and $R(z)$ is analytic in U_{x_j}. Thus we see that R indeed solves the Riemann-Hilbert problem. $\square$

We next prove the relevant estimate for the jump matrix.

Proof of Lemma 2.4.35. Let us first consider the jump matrix on $\mathbb{R} \setminus [-1-\delta, 1+\delta]$. Here we have

$$\Delta(\lambda) = P^{(\infty)}(\lambda) \begin{pmatrix} 0 & f_t(\lambda) e^{N(g_{s,+}(\lambda)+g_{s,-}(\lambda)-V_s(\lambda)-\ell_s)} \\ 0 & 0 \end{pmatrix} \left[P^{(\infty)}(\lambda)\right]^{-1}.$$

First of all, we note that the entries of $P^{(\infty)}(\lambda)$ and $[P^{(\infty)}(\lambda)]^{-1}$ are bounded (uniformly in everything relevant) in this area, and $f_t(\lambda)$ grows like $|\lambda|^{\sum_{j=1}^k \beta_j}$ as $|\lambda| \to \infty$. From (2.4.7), we see that there exist constants $C, M > 0$ depending only on V such that for $|\lambda| > 1+M$, $e^{N(g_{s,+}(\lambda)+g_{s,-}(\lambda)-V_s(\lambda)-\ell_s)} \leq |\lambda|^{-N}$ and for $|\lambda| - 1 \in (0, M)$, $e^{N(g_{s,+}(\lambda)+g_{s,-}(\lambda)-V_s(\lambda)-\ell_s)} \leq e^{-NC(|\lambda|-1)^{3/2}}$. From these estimates, it's easy to see that any L^p norm on $\mathbb{R}\setminus[-1-\delta, 1+\delta]$ is exponentially small in N.

Consider next the part of the contour lying on the boundaries of the lenses. More precisely, we have for $\lambda \in \cup_{j=1}^{k+1}\Sigma_j^\pm \setminus \overline{U_{-1} \cup \cup_{j=1}^k U_{x_j} \cup U_1}$,

$$\Delta(\lambda) = P^{(\infty)}(\lambda) \begin{pmatrix} 0 & 0 \\ f_t(\lambda)^{-1} e^{\mp N h_s(\lambda)} & 0 \end{pmatrix} \left[P^{(\infty)}(\lambda)\right]^{-1}.$$

We now refer to Lemma 2.4.4, which states that for example for $\lambda \in \Sigma_j^+ \setminus \overline{U_{-1} \cup \cup_{l=1}^k U_{x_l} \cup U_1}$, there exists an $\epsilon > 0$ independent of s and λ such that $\mathrm{Re}(h_s(\lambda)) > \epsilon$ (we assume that the distance between this part of the contour and the real axis is bounded away from zero uniformly in everything relevant). Moreover, $f_t(\lambda)^{-1}$ is uniformly bounded here so we again get exponential smallness for any L^p norm uniformly in everything relevant for this part of the contour (as the contour has finite length). The Σ_j^--case is identical.

For ∂U_{x_j} and $\partial U_{\pm 1}$ the bounds come from the matching conditions in Lemma 2.4.23 and Lemma 2.4.30. Combining the estimates from the different parts of the contour is elementary and we find the claim. $\square$

The next proof we consider is the representation of R in terms of a certain Neumann-series. The proof follows [DKM+99, Theorem 7.8], and while it is a standard fact, we record it here for completeness.

Proof of Proposition 2.4.36. By the Sokhotski-Plemelj theorem, we see that the function $\widehat{R} = I + C(R_+ - R_-)$ satisfies $\widehat{R}_+ - \widehat{R}_- = R_+ - R_-$ across $\Gamma_\delta \setminus \{\text{intersection points}\}$ (note that from our proof of Lemma 2.4.35, we see that $R_+ - R_- = R_-\Delta$ has nice enough decay at infinity for $\widehat{R}$ to be well defined). Thus the function $\widehat{R} - R$ has no jump across $\Gamma_\delta \setminus \{\text{intersection points}\}$. By construction, both functions are bounded at the intersection points of the different parts of the contour, and behave like $I + \mathcal{O}(|z|^{-1})$ as $z \to \infty$, so by Liouville's theorem

$$R = I + C(R_+ - R_-) = I + C(R_-\Delta).$$

In particular, taking the limit from the $-$ side, we obtain

$$R_- - I = C_-(R_-\Delta) = C_\Delta(R_-) \quad \Leftrightarrow \quad (I - C_\Delta)(R_- - I) = C_\Delta(I).$$

It is well known that C_- is a bounded operator from $L^2(\Gamma_\delta)$ to $L^2(\Gamma_\delta)$ – see e.g. the discussion and references in [DKM+99, Appendix A]. Given the estimate in Lemma 2.4.35 the operator norm of C_Δ is of order $\mathcal{O}(1/N)$, $I - C_\Delta$ is invertible (and the inverse can be expanded as a Neumann series) for N sufficiently large and the result follows. □

Finally we prove the main result concerning R. Our proof is a minor modification of that in [Kra07].

Proof of Theorem 2.4.37. Note that since $(I - C_\Delta)(R_- - I) = C_\Delta(I)$ and since the L^2-boundedness of C_- implies that $||C_\Delta(I)||_{L^2(\Gamma_\delta)} = \mathcal{O}(N^{-1})$ (uniformly in everything relevant), we have

$$||R_- - I||_{L^2(\Gamma_\delta)} \leq ||(I - C_\Delta)^{-1}||_{L^2(\Gamma_\delta)\to L^2(\Gamma_\delta)}||C_\Delta(I)||_{L^2(\Gamma_\delta)} \leq \frac{c_1}{N}$$

for some $c_1 > 0$ (independent of the relevant quantities).

Now fix some small $\epsilon > 0$, and suppose z is at least ϵ away from the jump contour Γ_δ. Recall that in the proof of (2.4.59), we saw that $(I - C_\Delta)^{-1}C_\Delta(I) = R_- - I$, so we have (for c_2, c_3, c_4 depending on ϵ but not on $t, s, \ldots$)

$$\begin{aligned} |R - I| &\leq |C(\Delta)| + |C((R_- - I)\Delta)| \\ &\leq \frac{c_2}{N} + c_3||R_- - I||_{L^2(\Gamma_\delta)}||\Delta||_{L^2(\Gamma_\delta)} \leq \frac{c_4}{N}, \end{aligned}$$

where we used Cauchy-Schwarz in the second step and the facts that R_- is bounded on Γ_δ and behaves like $I + \mathcal{O}(|\lambda|^{-1})$, as $\lambda \to \infty$.

For $z \in \mathbb{C}\setminus\Gamma_\delta$ that is within a distance of ϵ from Γ_δ but not close to any intersection points, we use the usual trick of contour deformation. First note that we can analytically continue the jump matrix J_R to, without loss of generality, a (2ϵ)-neighbourhood of Γ_δ, with the estimates in Lemma 2.4.35 remaining true (up to a change of constants).

We may assume that z lies on the $+$ side of Γ_δ. Let $\widetilde{\Gamma}_\delta$ be the contour in Figure 2.8, obtained from Γ_δ with the dotted part replaced by a half circle of radius ϵ, and $\widetilde{R}$ be

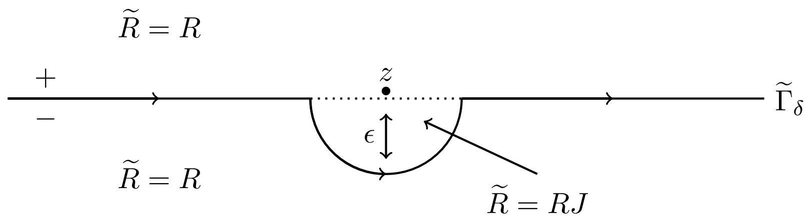

Figure 2.8: Deforming the R-RHP.

defined as shown, where J is the analytic continuation of J_R. Then $\widetilde{R}(z)$ satisfies the same Riemann-Hilbert problem as $R(z)$ except on the new contour $\widetilde{\Gamma}_\delta$. Repeating our argument for the case where z is at distance at least ϵ from the contour, we see that

$$|R(z) - I| = |\widetilde{R}(z) - I| \leq \frac{c_5}{N},$$

for a c_5 which is uniform in the relevant quantities. Now note that all estimates established so far are also uniform in $\delta \in K \subset (0, \delta_0]$ for some compact set K and $\delta_0 > 0$, see [DKM+99, Section 7.2]. If z is close to any intersection points we may then deform our contour by varying δ.

For the derivative, let us consider the case where the distance between z and the jump contour is greater than ϵ. Then by Cauchy's integral formula we have

$$R'(z) = \frac{1}{2\pi i} \int_{|w-z|=\epsilon} \frac{R(w)}{(w-z)^2} dw = \frac{1}{2\pi i} \int_{|w-z|=\epsilon} \frac{R(w) - I}{(w-z)^2} dw = \mathcal{O}(N^{-1})$$

where the last equality follows from the uniform estimates for $R(w) - I$. For z close to the contour we argue by contour deformation again.

We now want to extract the second order asymptotics when $\mathcal{T} = 0$. Since

$$R = I + C(\Delta) + C((R_- - I)\Delta),$$

repeating our argument with minor modifications we see that

$$R - I - C(\Delta) = \mathcal{O}(N^{-2}) \quad \text{and} \quad R' - C(\Delta)' = \mathcal{O}(N^{-2})$$

uniformly off of Γ_δ and uniformly in everything relevant. Now by definition, we have

$$[C(\Delta)](z) = \int_{\Gamma_\delta} \frac{\Delta(w)}{w - z} \frac{dw}{2\pi i}.$$

With similar arguments as in the proof of Lemma 2.4.35, one can easily see (e.g. using Cauchy-Schwarz and a L^2-norm bound on the jump matrix on the unbounded part of the contour and a L^∞-norm bound on the part of the contour on the boundary of the lenses) that the contribution from the part of the contour on $\mathbb{R}$ and on the boundary of the lenses has uniformly (in everything relevant) exponentially small contribution to $C(\Delta)$. Thus we have for z not on the jump contour

$$[C(\Delta)](z) = \sum_{j=0}^{k+1} \oint_{\partial U_{x_j}} \frac{\Delta(w)}{w-z} \frac{dw}{2\pi i} + \mathcal{O}(N^{-2}) =: \sum_{j=0}^{k+1} R_1^{(x_j)}(z) + \mathcal{O}(N^{-2}),$$

where the orientation of the contours is in the clockwise direction and the $\mathcal{O}(N^{-2})$ is uniform in everything relevant. From Lemma 2.4.24, Lemma 2.4.31, and Remark 2.4.32, we can then write (again for z off of the jump contour)

$$R_1^{(x_j)}(z) = \frac{1}{2\pi i} \oint_{\partial U_{x_j}} \frac{dw}{w-z} \frac{\beta_j}{4\zeta_s^{(x_j)}(w)} E^{(x_j)}(w) \begin{pmatrix} 0 & 1+\frac{\beta_j}{2} \\ 1-\frac{\beta_j}{2} & 0 \end{pmatrix} \left[E^{(x_j)}(w)\right]^{-1}, \quad 1 \le j \le k$$

$$R_1^{(\pm1)}(z) = \frac{1}{2\pi i} \oint_{\partial U_{\pm 1}} \frac{dw}{w-z} F^{(\pm1)}(w) \begin{pmatrix} 0 & \pm\frac{5}{48}\left[\xi_s^{(\pm1)}(w)\right]^{-2} \\ -\frac{7}{48}\left[\xi_s^{(\pm1)}(w)\right]^{-1} & 0 \end{pmatrix} \left[F^{(\pm1)}(w)\right]^{-1}$$

where the superscripts have been added to underline that the functions depend on the singularity we are considering.

Consider now $z \notin U_{x_j}$ with $j \in \{1, ..., k\}$. Then as E, E^{-1} are analytic in U_{x_j} and $1/\zeta_s^{(x_j)}(w)$ has a simple pole at $w = x_j$ (and no other singularities in U_{x_j}), we see that

$$R_1^{(x_j)}(z) = \frac{1}{z-x_j} \frac{\beta_j}{4\pi N\left(\frac{2}{\pi}(1-s)+sd(x_j)\right)\sqrt{1-x_j^2}} E^{(x_j)}(x_j) \begin{pmatrix} 0 & 1+\frac{\beta_j}{2} \\ 1-\frac{\beta_j}{2} & 0 \end{pmatrix} \left[E^{(x_j)}(x_j)\right]^{-1},$$

where, by writing $b_{x_j} = a_+(x_j)^2 + a_+(x_j)^{-2}$ and $\bar{b}_{x_j} = a_+(x_j)^2 - a_+(x_j)^{-2}$, one finds (after an elementary calculation)

$$\begin{aligned} &E^{(x_j)}(x_j) \begin{pmatrix} 0 & 1+\frac{\beta_j}{2} \\ 1-\frac{\beta_j}{2} & 0 \end{pmatrix} \left[E^{(x_j)}(x_j)\right]^{-1} \\ &= \frac{1}{8} \begin{pmatrix} -i[2(c_{x_j}^2 + c_{x_j}^{-2})b_{x_j}\bar{b}_{x_j} + \beta_j(b_{x_j}^2 + \bar{b}_{x_j}^2)] & 2\mathcal{D}(\infty)^2[(c_{x_j}^2 b_{x_j}^2 + c_{x_j}^{-2}\bar{b}_{x_j}^2) + \beta_j b_{x_j}\bar{b}_{x_j}] \\ 2\mathcal{D}(\infty)^{-2}[(c_{x_j}^{-2} b_{x_j}^2 + c_{x_j}^2 \bar{b}_{x_j}^2) + \beta_j b_{x_j}\bar{b}_{x_j}] & i[2(c_{x_j}^2 + c_{x_j}^{-2})b_{x_j}\bar{b}_{x_j} + \beta_j(b_{x_j}^2 + \bar{b}_{x_j}^2)] \end{pmatrix}. \end{aligned}$$

Here we made use of the fact that $E^{(x_j)}$ is analytic at x_j so we can evaluate $E^{(x_j)}(x_j)$ using the formula (2.4.33).

For $R_1^{(\pm 1)}(z)$ with $z \notin U_{\pm 1}$ the residue calculations are more involved (but still straightforward) because of the presence of a second order pole. We just summarize here that

$$R_1^{(-1)}(z) = -\frac{2^{1/2}}{2N}\frac{1}{(-1-z)^2}\frac{5}{48G_s^{(-1)}(-1)}\begin{pmatrix} -i & \mathcal{D}(\infty)^2 \\ \mathcal{D}(\infty)^{-2} & i \end{pmatrix}$$

$$+\frac{2^{1/2}}{8N}\frac{1}{z+1}\begin{pmatrix} i\left[\frac{9-96\mathcal{A}^2}{48G_s^{(-1)}(-1)} - \frac{5\left[G_s^{(-1)}\right]'(-1)}{12G_s^{(-1)}(-1)^2}\right] & \mathcal{D}(\infty)^2\left[\frac{19+96\mathcal{A}(1+\mathcal{A})}{48G_s^{(-1)}(-1)} + \frac{5\left[G_s^{(-1)}\right]'(-1)}{12G_s^{(-1)}(-1)^2}\right] \\ \frac{i}{\mathcal{D}(\infty)^2}\left[\frac{19-96\mathcal{A}(1-\mathcal{A})}{48G_s^{(-1)}(-1)} + \frac{5\left[G_s^{(-1)}\right]'(-1)}{12G_s^{(-1)}(-1)^2}\right] & -i\left[\frac{9-96\mathcal{A}^2}{48G_s^{(1)}(1)} - \frac{5\left[G_s^{(-1)}\right]'(-1)}{12G_s^{(-1)}(-1)^2}\right] \end{pmatrix},$$

$$R_1^{(1)}(z) = -\frac{2^{1/2}}{2N}\frac{1}{(1-z)^2}\frac{5}{48G_s^{(1)}(1)}\begin{pmatrix} 1 & -i\mathcal{D}(\infty)^2 \\ -i\mathcal{D}(\infty)^{-2} & -1 \end{pmatrix}$$

$$-\frac{2^{1/2}}{8N}\frac{1}{1-z}\begin{pmatrix} \frac{9-96\mathcal{A}^2}{48G_s^{(1)}(1)} + \frac{5\left[G_s^{(1)}\right]'(1)}{12G_s^{(1)}(1)^2} & i\mathcal{D}(\infty)^2\left[\frac{19+96\mathcal{A}+96\mathcal{A}^2}{48G_s^{(1)}(1)} - \frac{5\left[G_s^{(1)}\right]'(1)}{12G_s^{(1)}(1)^2}\right] \\ i\mathcal{D}(\infty)^{-2}\left[\frac{19-96\mathcal{A}+96\mathcal{A}^2}{48G_s^{(1)}(1)} - \frac{5\left[G_s^{(1)}\right]'(1)}{12G_s^{(1)}(1)^2}\right] & -\frac{9-96\mathcal{A}^2}{48G_s^{(1)}(1)} - \frac{5\left[G_s^{(1)}\right]'(1)}{12G_s^{(1)}(1)^2} \end{pmatrix},$$

where the functions $G_s^{(\pm 1)}(z)$ come from

$$\xi_s^{(-1)}(z) = e^{-i\pi}N^{2/3}G_s^{(-1)}(z)^{2/3}(z+1), \qquad \xi_s^{(1)}(z) = N^{2/3}G_s^{(1)}(z)^{2/3}(z-1),$$

(see Appendix 2.E). $\mathcal{J}^{(x_j)}(z)$ may now be obtained by direct calculation.

□

Appendix 2.G Uniformity of the asymptotics in Theorem 2.6.3

In this appendix we will give a brief outline of how to check that the asymptotics in Theorem 2.6.3 are still uniform when we replace V by $V_{x,y}$ when $x, y \in (-1+\epsilon, 1-\epsilon)$ (in the notation of Section 2.6). We will not try to be self contained here and we will use notations both from [CF16] and ones we've adopted earlier in this article. We won't provide all of the relevant definitions from [CF16]. We will simply try to provide a map of how to go over the argument.

Let us write $u = (x-y)^2/4 \geq 0$ (which in the notation of [CF16] is t) and $v = (x+y)/2 \in (-1+\epsilon, 1-\epsilon)$, where ϵ is determined by the support of our non-negative test function. We also write $V_v(z) = V(z+v)$. In the notation of Section 2.6, we are interested in the asymptotics of $D_{N-1}(f_u; V_v)$, which in the notation of [CF16] would be $\widehat{Z}_N(u, \beta, V_v)/N!$. Note that in the notation of [CF16], β is replaced by α.

Let us write Y for the solution of the RHP related to $D_{N-1}(f_u; V_v)$. Y depends on u and v, but as usual, we suppress this dependence in our notation. Then as the "center of mass" and "relative motion" coordinates decouple, or $\partial_u V_v = 0$ for all u and v, the proof of [CF16, Proposition 4.1] carries through word to word and one finds

$$\partial_u \log D_{N-1}(f_u; V_v) = -\frac{\beta}{2\sqrt{u}} \left[(Y(\sqrt{u})^{-1} Y'(\sqrt{u}))_{22} - (Y(-\sqrt{u})^{-1} Y'(-\sqrt{u}))_{22} \right]. \quad (2.G.1)$$

The goal will be to integrate this from zero to some positive u. Even though $\pm\sqrt{u}$ lie on the jump contour of Y, this quantity in fact does not have a jump so the notation is justified. Moreover one can calculate the relevant quantities at a point z and then let $z \to \pm\sqrt{u}$ – in particular the point z can be taken to be outside of the relevant lenses and for simplicity in the lower half plane (see [CF16, Figure 8]). In [CF16, Section 6], using results of [CK15], it is argued that near the points $\pm\sqrt{u}$, but outside of the lenses, one can write

$$Y(z) = e^{-N\frac{\ell_v}{2}\sigma_3} \left(R_v(z) E_v(z) \Psi^{(2)}(\lambda_v(z); s_{N,u}) W_v(z) \right) e^{N g_v(z)\sigma_3} e^{\frac{N\ell_v}{2}\sigma_3}, \quad (2.G.2)$$

where ℓ_v and g_v refer to the ℓ- and g-quantities constructed from the potential V_v. If we restrict to points z outside of the lenses and in the lower half plane, then one has

$$W_v(z) = \left[(z^2 - u)^{-\beta/2} e^{\frac{-\pi i \beta}{2}} e^{N\phi_v(z)} \right]^{\sigma_3}, \quad (2.G.3)$$

where (see the discussion around [CF16, equation (4.13)] for details about the branch and integration contour – note that in the notation of [CF16], d is h and the support of the equilibrium measure is $[a, b]$ instead of our $[-1 - v, 1 - v]$)

$$\phi_v(z) = \pi \int_{1-v}^{z} d_v(\xi)((\xi + v) + 1)((\xi + v) - 1))^{1/2} d\xi. \quad (2.G.4)$$

λ_v is a coordinate change which for z in the lower half plane is defined by (see [CF16, equation (6.2)])

$$\lambda_v(z) = -iN \left(-\phi_v(z) - \frac{\phi_{v,+}(\sqrt{u}) + \phi_{v,+}(-\sqrt{u})}{2} \right). \quad (2.G.5)$$

The main reason the uniformity of the asymptotics holds is that varying $v \in (-1+\epsilon, 1-\epsilon)$ does not change the qualitative behavior of the asymptotics of $\lambda_v(z)$. If one were to allow $v = \pm 1$, then the situation would be different.

For the definition of $\Psi^{(2)}(\lambda, s)$, we refer to [CF16, Section 3], but point out here that while it depends on β, it does not depend on x, y, or V. The function E_v is analytic in a neighborhood of zero (containing the points $\pm\sqrt{u}$) and for the values of z we are interested in, it can be written as (see [CF16, Section 6.4])

$$E_v(z) = \mathcal{N}_v(z)W_v(z)^{-1}e^{-i\lambda_v(z)\sigma_3} = \mathcal{N}_v(z)\left[(z^2-u)^{\beta/2}e^{\pi i\beta/2}\right]^{\sigma_3} e^{\frac{N}{2}\left(\phi_{v,+}(\sqrt{u})+\phi_{v,+}(-\sqrt{u})\right)\sigma_3}, \tag{2.G.6}$$

where $\mathcal{N}_v(z)$ is the global parametrix which is of similar form as the one we consider in Section 2.4.2 apart from the support of the equilibrium measure now being $[-1-v, 1-v]$ which changes the formulas slightly. See also around [CF16, equations (5.5) and (6.1)] for details. In particular, as $z \to \pm\sqrt{u}$ for a fixed N, $\mathcal{N}_v(z) \sim (z \mp \sqrt{u})^{-\frac{\beta}{2}\sigma_3}$ uniformly in v. This combined with the fact that $\phi_{v,+}(\pm\sqrt{u})$ is purely imaginary implies that in a neighborhood of the origin, E_v, E_v^{-1}, and E_v' are bounded uniformly in $v \in (-1+\epsilon, 1-\epsilon)$.

Finally R_v is a solution to a small norm RHP. As pointed out in [CF16], the analysis of R_v and its RHP is essentially carried out in [CK15]. While verifying in full detail the asymptotic behavior of R_v is not something we will do, we will briefly sketch part of the argument, namely uniform asymptotics for the jump matrix across part of the boundary of a neighborhood of the origin. Analyzing the jump matrix of R in the remaining part of the contour is similar and with a standard argument one finds that R is uniformly close to the identity and its derivative is uniformly small.

From the definition of R_v in [CF16, Section 6.5] we see for z on the boundary of some neighborhood of the origin containing the points $\pm\sqrt{u}$

$$R_{v,+}(z) = R_{v,-}(z)E_v(z)\Psi^{(2)}(\lambda_v(z); s_{N,u})W_v(z)\mathcal{N}_v(z)^{-1} \tag{2.G.7}$$

Following the notation in [CF16, Section 3], we note that we can write

$$\Psi^{(2)}(\lambda, s) = \Psi_{CK}\left(-\frac{4\lambda}{|s|}i; s\right)\chi(\lambda), \tag{2.G.8}$$

where Ψ_{CK} is the solution to the RHP in [CK15, Section 3] and $\chi(\lambda)$ is defined in [CF16, equation (3.12)]. We note that as u is always small for us, $|\lambda_v(z)/|s|| \sim u^{-1/2}$ is large if z is at a fixed distance from $\pm\sqrt{u}$. We thus want to know the $\lambda \to \infty$ asymptotics of $\Psi_{CK}(\lambda, s)$ for all values of s. This was studied in [CK15]. For the relevant asymptotics for $\Psi_{CK}(\zeta; s)$, we refer to the discussion relevant to [CK15, equations (3.6), (5.25), and (6.32)]. For $\Psi^{(2)}(\lambda; s)$ these asymptotics translate into the following: for large $|\lambda|$

$$\Psi^{(2)}(\lambda; s) = \begin{cases} (I + \mathcal{O}(|s||\lambda|^{-1}))e^{i\lambda\sigma_3}, & s \to -i0^+ \\ (I + \mathcal{O}(|\lambda|^{-1}))e^{i\lambda\sigma_3}, & s = \mathcal{O}(1) \\ (I + \mathcal{O}(|s\lambda|^{-1}))e^{i\lambda\sigma_3}, & s \to -i\infty \end{cases}.$$

Using (2.G.6) and fact that E_v and E_v^{-1} are uniformly bounded, we thus see that for all u and uniformly in v, the jump matrix along this part of the jump contour is

$$I + E_v(z)\mathcal{O}(\min(|s|, |s|^{-1})|\lambda_v(z)|^{-1})E_v(z)^{-1} = I + \mathcal{O}(\min(|s|, |s|^{-1})|\lambda_v(z)|^{-1}).$$

Going over such an argument in full detail would then imply that R_v can be solved through the general small-norm approach and one has uniform asymptotics for R_v, e.g. $R_v(z) = I + \mathcal{O}(N^{-1})$ and $R_v'(z) = \mathcal{O}(N^{-1})$ uniformly in z and $v \in (-1+\epsilon, 1-\epsilon)$.

Let us now return to the differential identity (2.G.1). With a basic matrix algebra argument, one finds from (2.G.2) as in [CF16, Section 5]

$$\begin{aligned}\left(Y^{-1}(z)Y'(z)\right)_{22} &= (B(z))_{22} - \frac{N}{2}V_v'(z) + \frac{\beta}{2}\left[\frac{1}{z-\sqrt{u}} + \frac{1}{z+\sqrt{u}}\right] \\ &\quad + \left[\Psi^{(2)}(\lambda_v(z); s_{N,u})\frac{d}{dz}\Psi^{(2)}(\lambda_v(z); s_{N,u})\right]_{22},\end{aligned}$$

where

$$B(z) = \Psi^{(2)}(\lambda_v(z); s_{N,u})^{-1}(R_v(z)E_v(z))^{-1}(R_v(z)E_v(z))'\Psi^{(2)}(\lambda_v(z); s_{N,u}).$$

For the asymptotics of the $\frac{d}{dz}\Psi^{(2)}$-term, one can argue exactly like in [CF16, Section 6.4] (see also [CF16, equations (5.27) and (5.28); Lemma 5.3]) to find that as $z \to \pm\sqrt{u}$,

$$\begin{aligned}\left(\Psi^{(2)}(\lambda_v(z); s)^{-1}\frac{d}{dz}\Psi^{(2)}(\lambda_v(z); s)\right)_{22} &= \pm 2i\frac{\lambda_v'(\pm\sqrt{u})}{s_{N,u}}\left(\frac{\sigma_\beta(s_{N,u}) - \frac{\beta^2}{2}}{\beta} + \frac{s_{N,u}}{2}\right) - \frac{\beta}{2}\frac{1}{z\mp\sqrt{u}} \\ &\quad + \mathcal{O}(1),\end{aligned}$$

where $\mathcal{O}(1)$ is uniform in v.

Thus what remains is the B-term. For this, by what we've argued about R and E, we see that $(RE)^{-1}(RE)' = \mathcal{O}(1)$ uniformly in v in a neighborhood of zero. Thus it is enough to show that as $z \to \pm\sqrt{u}$, $((\Psi^{(2)})^{-1}\mathcal{O}(1)\Psi^{(2)})_{22} = \mathcal{O}(1)$ uniformly in v. Here again the asymptotics of $\Psi^{(2)}$ come from [CK15], and in fact the uniformity in v follows from the argument for a fixed v as in [CF16, Section 5.6 and Section 6.6] and the uniform behavior of λ_v.

Chapter 3

Fusion Asymptotics for Liouville Correlation Functions

Abstract Under the probabilistic framework for the path integral approach to Liouville Conformal Field Theory (LCFT) introduced by David-Kupiainen-Rhodes-Vargas, we compute fusion estimates for the four-point correlation function on the Riemann sphere and find that it is consistent with predictions from the framework of conformal bootstrap in theoretical physics. This result fits naturally into the famous KPZ conjecture which relates the four-point function to the expected density of points around the root of a large random planar map weighted by some statistical mechanics model. From a purely probabilistic point of view, we establish non-trivial results on negative moments of Gaussian Multiplicative Chaos, giving exact formulae based on the DOZZ formula in the Liouville case and a probabilistic representation of the limit in other cases. Finally, we also show how to extend our results to boundary LCFT, treating the cases of the fusion of two boundary or bulk insertions as well as the absorption of a bulk insertion on the boundary.

3.1 Introduction

3.1.1 Path integral

The Liouville action on the Riemann sphere $\mathbb{S}^2 \cong \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the action functional $S_L : \Sigma \to \mathbb{R}$ (where Σ is some function space to be determined) defined by[1]

$$S_{\mathrm{L}}(X) = \frac{1}{4\pi} \int_{\mathbb{S}^2} (|\nabla_g X|^2 + R_g(z) Q X(z) + 4\pi\mu e^{\gamma X}) g(z) d^2 z \tag{3.1.1}$$

where $g(z) = |z|_+^{-4} := (|z| \vee 1)^{-4}$ is the background metric, $\gamma \in (0, 2)$ is the parameter

[1] $\nabla_g = \frac{1}{g}\nabla$ is the gradient associated to g, and $R_g(z) = -\frac{1}{g(z)}\Delta \log g(z)$ is the associated curvature. Since we will consider metrics whose curvature concentrates on the unit circle, the curvature term will not play an important role here.

of the theory, and $\mu > 0$ is the cosmological constant (whose value is irrelevant in this paper). Another important parameter is the so-called *background charge* which is defined by $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$. From here, Liouville Conformal Field Theory (LCFT) is the "Gibbs measure" associated to S_L, which is formally defined in the physics literature by

$$\langle F \rangle := \int F(X) e^{-S_{\mathrm{L}}(X)} DX \tag{3.1.2}$$

for all continuous functional F on Σ. Here DX stands for "Lebesgue measure" on $C^\infty(\mathbb{S}^2)$, which of course does not make sense mathematically. Nonetheless, it is possible to define (3.1.2) in a rigorous framework using the Gaussian Free Field (GFF) and Kahane's theory of Gaussian Multiplicative Chaos (GMC) – see [DKRV16] and Sections 3.2.1 and 3.2.2 of this paper. Roughly speaking, the GFF X on $\mathbb{S}^2$ is the Gaussian field corresponding to the "Gaussian measure" $e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2}|\nabla X|^2} DX$. We will write $\mathbb{P}$ for its probability measure and $\mathbb{E}$ for the associated expectation. The GFF lives $\mathbb{P}$-a.s. in the topological dual of the Sobolev space $H^1(\mathbb{S}^2, g)$ and is therefore defined as a distribution (in the sense of Schwartz). In this context, GMC is the random measure M^γ on $\mathbb{S}^2$ defined for all $\gamma \in (0,2)$ and making sense of the exponential of the GFF (which is *a priori* ill-defined). This can be constructed through a regularisation of the field and we will loosely write $dM^\gamma(z) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} g(z) d^2 z$ to refer to the limiting measure, even though X is only defined as a distribution.

The main observables in LCFT are the *vertex operators* $V_\alpha(z) := e^{\alpha X(z)}$, giving rise to the *correlation functions*, which can be thought of as the Laplace transform of the field defined by the measure (3.1.2):

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \right\rangle = \int \prod_{i=1}^N e^{\alpha_i X(z_i)} e^{-S_{\mathrm{L}}(X)} DX. \tag{3.1.3}$$

On the sphere, these are defined for all pairwise disjoint *insertions* $(z_1, ..., z_n) \in \widehat{\mathbb{C}}^N$ and *Liouville momenta* $(\alpha_1, ..., \alpha_n) \in \mathbb{R}_+^N$ satisfying the *Seiberg bounds*

$$\sigma := \sum_{i=1}^N \frac{\alpha_i}{Q} - 2 > 0, \qquad \forall i,\ \alpha_i < Q. \tag{3.1.4}$$

In particular, this implies that the correlation function exists only if $N \geq 3$.

For fixed $z_0 \in \widehat{\mathbb{C}}$, the vertex operator $V_\alpha(z_0)$ has a geometric interpretation, as it inserts a conical singularity of order α/Q at z_0 in the physical metric ([Sei90, HMW11], Appendix 3.B). Thus the second Seiberg bound is there to make the singularity integrable around z_0. On the other hand by Gauss-Bonnet theorem, the first bound is equivalent to asking that the surface $\mathbb{S}^2 \setminus \{z_1, ..., z_N\}$ with conical singularities of order α_i/Q at z_i has negative total curvature.

The correlation functions satisfy some conformal covariance under Möbius transformation, namely if ψ is such a map, then [DKRV16]

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(\psi(z_i)) \right\rangle = \prod_{i=1}^{N} |\psi'(z_i)|^{-2\Delta_i} \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle$$

where $\Delta_i = \Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is called the *conformal dimension* of $V_{\alpha_i}(\cdot)$. This property implies that the three-point correlation function $\langle \prod_{i=1}^{3} V_{\alpha_i}(z_i)\rangle$ is determined by $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$ since there is a unique Möbius transformation sending (z_1, z_2, z_3) to $(0, 1, \infty)$. The three-point correlation functions play a central role in the conformal bootstrap approach to CFTs (see Section 3.1.2). For LCFT, they are given by the celebrated DOZZ formula, a proof of which was given for the first time in [KRV17], where the authors rigorously implemented the method known as Teschner's trick [Tes95] (see [DO94, ZZ96] for the original derivation of the formula which uses a different approach).

We now turn to the four-point function. By conformal covariance, we can take the insertions to be at $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ with $z \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ being the free parameter. In this paper, we will take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the Seiberg bounds and will be concerned about the behaviour of the four-point function as $z \to 0$ (the other fusions being easily deduced from conformal invariance). In the framework of [DKRV16] using the GFF and GMC, the four-point function has the following expression for $|z| \leq 1$:

$$\left\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle = 2\gamma^{-1}\mu^{-\frac{Q\sigma}{\gamma}}\Gamma\left(\frac{Q\sigma}{\gamma}\right)|z|^{-\alpha_1\alpha_2}|1-z|^{-\alpha_2\alpha_3}\mathbb{E}\left[\left(\int_{\mathbb{C}} e^{\gamma\sum_{i=1}^{4}\alpha_i G(z_i,\cdot)}dM^{\gamma}\right)^{-\frac{Q\sigma}{\gamma}}\right] \tag{3.1.5}$$

where $G = G(\cdot, \cdot)$ is Green's function on $(\mathbb{S}^2, g)$. The main feature of (3.1.5) is that, up to explicit factors, it is expressed using negative moments of GMC. One of our main results (Theorem 3.1.1) gives the exact asymptotic behaviour of (3.1.5) as $z \to 0$ using the integrability result of the DOZZ formula. Now the reader will notice that the negative exponent in the definition of (3.1.5) depends on the α_i's, so the DOZZ formula does not give integrability results for *all* moments of GMC but only for the one corresponding to the Liouville correlation function. However, in our framework, we lose nothing in promoting σ to a free parameter, so we were able to find the asymptotic behaviour of all negative moments (Theorems 3.1.2 and 3.1.3) but only in the Liouville case did we get an exact expression for the limit. In this special case, we were able to confirm a prediction coming from the bootstrap approach to LCFT, which we review now.

3.1.2 Conformal bootstrap

The foundations of the conformal bootstrap were laid in [BPZ84] and since then it has been acknowledged in the physics community as a powerful tool to analyse two dimensional

CFTs. However it is still a challenge to make sense of the theory in a rigorous mathematical framework. One of the goals of this paper is to recover some aspects of the bootstrap predictions in the probabilistic formulation of LCFT.

The conformal bootstrap is an algebraic approach based on the axiom that the vertex operator V_α can be associated to a highest-weight representation of the Virasoro algebra [Rib14]. It turns out that this assumption constrains the correlation functions drastically through some identities like the the Ward or BPZ equations (a null-vector equation at level 2). The constraints of local conformal invariance imply that all correlation functions can be constructed from more fundamental objects:

1. The *spectrum* $\mathcal{S} \subset \mathbb{C}$. For $\alpha \in \mathcal{S}$, the vertex operator $V_\alpha(\cdot)$ is called a *primary field*. In Liouville CFT, the spectrum is the line $Q + i\mathbb{R}$. It is important to notice that the conformal bootstrap assumes that vertex operators are defined for all $\alpha \in \mathbb{C}$ and not necessarily for α in the "physical region" defined by the Seiberg bounds.

2. The 3-point correlation functions, a.k.a. the *structure constants*. In Liouville CFT, these are given by the DOZZ formula $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$, which is meromorphic in each one of the α_i's.

Another key idea of the conformal bootstrap is that local fields should satisfy a so-called *Operator Product Expansion* (OPE), which can be understood analytically as a Taylor expansion of vertex operators in the z variable. In other words, the OPE of the local operators $V_{\alpha_1}(0)V_{\alpha_2}(z)$ describes the fusion of the two insertions as $z \to 0$. The fusion rule is particularly simple in the case where the Verma module associated to $V_{\alpha_2}(z)$ is reducible (i.e. $\alpha_2 \in -\frac{\gamma}{2}\mathbb{N}^* - \frac{2}{\gamma}\mathbb{N}^*$), but in the case of α_1, α_2 in the spectrum, it has the following form ([BZ06], equation (1.18))

$$V_{\alpha_1}(0)V_{\alpha_2}(z) = \frac{1}{8\pi}\int_{\mathbb{R}} |z|^{2(\Delta_P - \Delta_1 - \Delta_2)} C_\gamma(\alpha_1, \alpha_2, Q - iP) V_{Q+iP}(0) |f_{\gamma,P}^{\alpha_{12}}(z)|^2 dP \tag{3.1.6}$$

where $\Delta_P = \frac{Q^2}{4} + \frac{P^2}{4}$ is the conformal dimension of V_{Q-iP} and $f_{\gamma,P}^{\alpha_{12}}(z) = 1 + o_{z\to 0}(1)$ is a so-called *conformal block*, a holomorphic function of z depending only on $P, \gamma, \alpha_1, \alpha_2$. Plugging this into the four-point correlation function yields[2]

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} &= \frac{1}{8\pi}|z|^{2(\frac{Q^2}{4} - \Delta_1 - \Delta_2)} \\ &\quad \times \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1, \alpha_2, Q - iP) C_\gamma(Q + iP, \alpha_3, \alpha_4) |\mathcal{F}_{\gamma,P}^{\alpha_{1234}}(z)|^2 dP\end{aligned} \tag{3.1.7}$$

where $\mathcal{F}_{\gamma,P}^{\alpha_{1234}}(\cdot)$ is the four-point conformal block coming from the contribution of the OPE conformal block. It is also holomorphic in z and universal in the sense that it depends

[2] We add the superscript $^{\mathrm{cb}}$ for "conformal bootstrap", in order to differentiate it with the correlation function given by the path integral.

only on $\gamma, P, \alpha_1, \alpha_2, \alpha_3$ and α_4. We call this formula the *conformal bootstrap equation*. The term "bootstrap" refers to the fact that one can recursively compute all the correlation functions on any Riemann surface of any genus by "bootstrapping" the structure constants using the spectrum and the conformal blocks.

Let us stress again that formula (3.1.7) is far from having a mathematical justification. In general, one way to make sense of the bootstrap predictions is to recover them from the rigorous probabilistic framework of DKRV. This is usually a hard matter, but first steps have been made in this direction, notably in [KRV15, KRV17] where the authors showed the validity of Ward identities and BPZ differential equations and gave a proof of the DOZZ formula. At this stage, we are still far from having a probabilistic interpretation of formula (3.1.7) because the spectrum and the conformal blocks are not properly understood in the path integral approach. However, we will see that in the limit where $z \to 0$, these two objects disappear from the equation and we are left with DOZZ formula which is well understood.

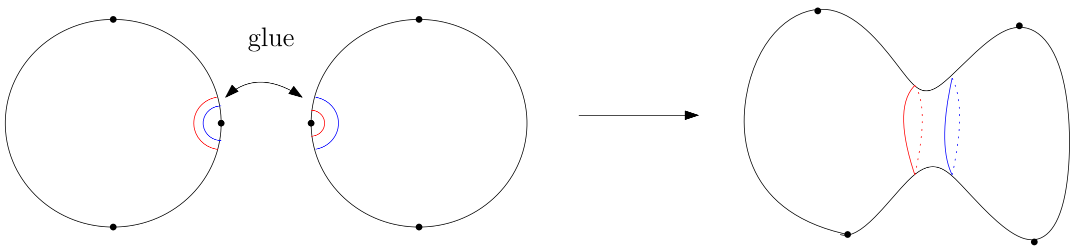

Figure 3.1: The gluing of two instances of the thrice-punctured sphere, producing a four-punctured sphere.

There is a geometric interpretation of equation (3.1.7). Indeed, one can produce a four-punctured sphere by gluing together two instances of the thrice-punctured sphere along annuli neighbourhoods of one puncture (see Figure 3.1 and [TV15] for details of this procedure). The bootstrap equation is the CFT counterpart of this gluing procedure since the integrand is a product of DOZZ formulae. We will see in Section 3.1.3 that the factorisation becomes exact in the $z \to 0$ limit. The problem of factorisation of surfaces is an old one and was stressed by Seiberg ([Sei90, p.336]) as the most important open problem in Liouville CFT, at a time where the DOZZ formula was not yet known (nor even guessed). This paper gives a partial answer to the problem since we will show rigorously that the state factorises into two independent states as $z \to 0$.

Finally, let us briefly comment on the place of this work within the existing literature. The recent proof of the DOZZ formula [KRV17] made an extensive use of the BPZ equation, a second order ODE satisfied by the correlation function $z \mapsto \langle V_{-\frac{\gamma}{2}}(z) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle$, which was established in the earlier paper [KRV15] and solved explicitly using hypergeometric functions. The reason why such an equation was expected to hold in the first place is that the representation of the Virasoro algebra associated to the field $V_{-\frac{\gamma}{2}}(\cdot)$ is expected to be degenerate, with a null vector at level two in the Verma module. This drastically simplifies the fusion rule for the fields $V_{-\frac{\gamma}{2}}(z)V_{\alpha_1}(0)$, and using the interpretation of Virasoro generators as differential operators, this leads to the second order BPZ equation. In this paper on the contrary, we study the general form of the fusion rule, for which the associated representation should not be degenerate in general, thus not leading to a differential equation. To our knowledge, there is no rigorous construction of representations of the Virasoro algebra in Liouville CFT yet, but there are works addressing the question and exploiting null vectors in the context of boundary CFT. For instance, it was shown in [Dub15] that SLE partitions functions can be constructed from highest-weight representations of the Virasoro algebra. In general, some BPZ and Ward-type identities appear in SLE related martingales as the condition making the drift term in Itô's formula vanish [Fri04].

3.1.3 Main results

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be satisfying the Seiberg bounds (3.1.4). In particular, this implies that either $\alpha_1+\alpha_2 > Q$ or $\alpha_3+\alpha_4 > Q$ (or both), and we assume without loss of generality that $\alpha_3 + \alpha_4 > Q$. Notice that these conditions are equivalent to having the Seiberg bounds being satisfied by (α_1, α_2, Q) (with the exception of the $\alpha_3 = Q$ saturation).

Suppose for now that $\alpha_1 + \alpha_2 \geq Q$. Then equation (3.1.7) is expected to hold, i.e. we should have

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} &= \frac{1}{8\pi}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)} \\ &\times \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1, \alpha_2, Q-iP)C_\gamma(Q+iP, \alpha_3, \alpha_4)|\mathcal{F}_{\gamma,P}^{\alpha_{1234}}(z)|^2 dP.\end{aligned} \tag{3.1.8}$$

At the geometrical level, we can produce a four-punctured sphere by gluing together two copies of the thrice-punctured sphere (see Figure 3.1) by picking one puncture on each sphere and identifying together annuli neighbourhoods of these punctures. The form of equation (3.1.7) reveals this gluing construction: the four-point function is a factorisation of three-point functions.

Assume $\alpha_1 + \alpha_2 > Q$. Taking $\mathcal{F}_{\gamma,P}^{\alpha_{1234}}(z) \equiv 1$ uniformly as $P \to 0$, making the change of variable $P \mapsto P\sqrt{\log\frac{1}{|z|}}$, equation (3.1.7) gives

$$\begin{aligned}
8\pi|z|^{2(\Delta_1+\Delta_2-\frac{Q}{4}^2)}&\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}}\\
&= \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4)|\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(z)|^2 dP\\
&= \frac{1}{\sqrt{\log\frac{1}{|z|}}}\int_{\mathbb{R}} e^{-\frac{P^2}{2}} C_\gamma\left(\alpha_1,\alpha_2,Q-i\frac{P}{\sqrt{\log\frac{1}{|z|}}}\right)\\
&\qquad\times C_\gamma\left(Q+i\frac{P}{\sqrt{\log\frac{1}{|z|}}},\alpha_3,\alpha_4\right)\left|\mathcal{F}^{\alpha_{1234}}_{\gamma,\frac{P}{\sqrt{\log\frac{1}{|z|}}}}(z)\right|^2 dP\\
&\underset{|z|\to 0}{\sim} \left(\log\frac{1}{|z|}\right)^{-3/2}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\int_{\mathbb{R}} P^2 e^{-\frac{P^2}{2}}dP\\
&= \sqrt{2\pi}\left(\log\frac{1}{|z|}\right)^{-3/2}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4).
\end{aligned} \tag{3.1.9}$$

Hence

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} \underset{z\to 0}{\sim} \frac{|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}}{4\sqrt{2\pi}\log^{3/2}\frac{1}{|z|}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4). \tag{3.1.10}$$

There are two important features in this asymptotic behaviour

- There is a $\left(\log\frac{1}{|z|}\right)^{-3/2}$ term correcting the polynomial rate $|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}$
- The limit is expressed as a product of two derivative DOZZ formulae. Geometrically speaking, this means that we are sewing two instances of the thrice-punctured spheres, each one presenting a cusp at the $\alpha = Q$ singularity. The fact that we have a product means that we have two "independent" surfaces.

In the case $\alpha_1+\alpha_2 = Q$, the computation of Appendix 3.A shows that:

$$\lim_{P\to 0} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4) = -4\partial_1 C_\gamma(Q,\alpha_3,\alpha_4). \tag{3.1.11}$$

Going back to the bootstrap equation and noticing that $2(\frac{Q^2}{4}-\Delta_1-\Delta_2) = -\alpha_1\alpha_2$, we can apply the same change of variables as in (3.1.9), and get in this case

$$\begin{aligned}
\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} &\underset{z\to 0}{\sim} -\frac{|z|^{-\alpha_1\alpha_2}}{2\pi\sqrt{\log\frac{1}{|z|}}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\int_{\mathbb{R}} e^{-\frac{P^2}{2}}dP\\
&= -\frac{1}{\sqrt{2\pi}}\frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)
\end{aligned} \tag{3.1.12}$$

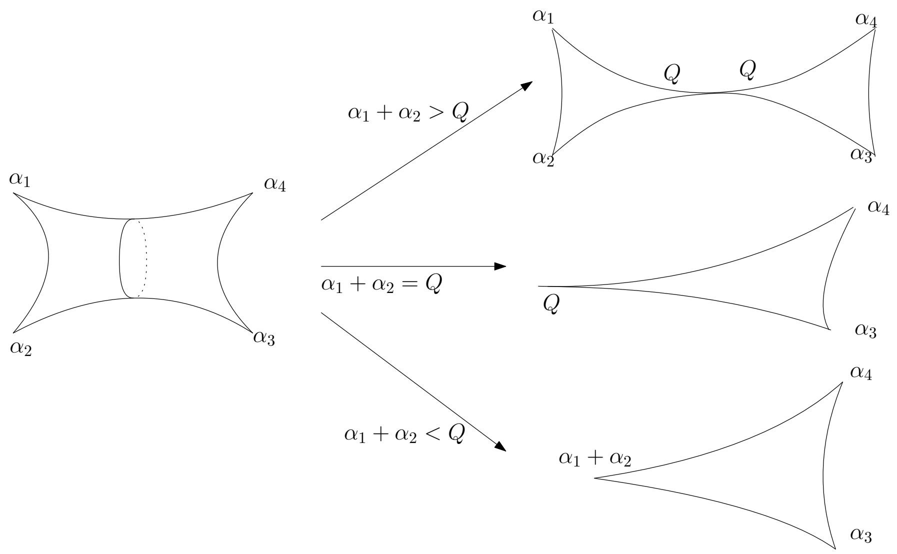

Figure 3.2: The three different regimes depending on the sign of $\alpha_1+\alpha_2-Q$. **Up:** Case $\alpha_1+\alpha_2>Q$. The surface on the left is a four-punctured sphere with conical singularities of order $(\frac{\alpha_1}{Q},\frac{\alpha_2}{Q},\frac{\alpha_3}{Q},\frac{\alpha_4}{Q})$ at $(0,z,1,\infty)$. The limiting surface is a pair of thrice-punctured sphere: one with singularities $(\frac{\alpha_1}{Q},\frac{\alpha_2}{Q},1)$ at $(0,1,\infty)$ (the singularity at ∞ is a cusp), the other with singularities $(1,\frac{\alpha_3}{Q},\frac{\alpha_4}{Q})$ at $(0,1,\infty)$. **Middle:** Case $\alpha_1+\alpha_2=Q$. The limiting surface is a thrice-punctured sphere with singularities of order $(1,\frac{\alpha_3}{Q},\frac{\alpha_4}{Q})$ at $(0,1,\infty)$. **Bottom:** Case $\alpha_1+\alpha_2<Q$. The limiting surface is a thrice-punctured sphere with singularities $(\frac{\alpha_1+\alpha_2}{Q},\frac{\alpha_3}{Q},\frac{\alpha_4}{Q})$ at $(0,1,\infty)$.

Again, let us notice two important features of this asymptotic behaviour

- There is a $\left(\log\frac{1}{|z|}\right)^{-1/2}$ correction term to be compared with the power $-3/2$ found in the supercritical case $\alpha_1+\alpha_2>Q$ in (3.1.9). This is explained by the fact that there is only one cusp and one limiting surface (so no extra zero mode).
- The limit is expressed with only one derivative DOZZ block, to be compared with the product found in (3.1.9). Intuitively, this means that in this critical case $\alpha_1+\alpha_2=Q$, we see only one surface with two conical singularities and one cusp.

Finally we turn to the case $\alpha_1+\alpha_2<Q$. In this case, equation (3.1.7) does not hold in this form and there is a need for "discrete corrections" (see [BZ06, Section 8] for a thorough discussion of the phenomenon). This is linked with the fact that the contour of integration in (3.1.7) includes poles of the DOZZ formula, and the discrete corrections are merely residues. In particular, the leading order as $z\to 0$ is simply

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} \underset{z\to 0}{\sim} |z|^{-\alpha_1\alpha_2}C_\gamma(\alpha_1+\alpha_2,\alpha_3,\alpha_4)$$

so that the geometric interpretation is that the two singularities add up together. This makes sense since $(\alpha_1+\alpha_2,\alpha_3,\alpha_4)$ satisfies the Seiberg bounds. In this last case, the spectrum is "hidden" behind the discrete leading-order terms. In order to see the spectrum in our probabilistic framework, one would need to push the asymptotic expansion further. It should be possible to do so using similar techniques as in [KRV17, Section 6] but we restrict ourselves to the leading order for now.

Theorem 3.1.1. *Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the Seiberg bounds and such that $\alpha_3 + \alpha_4 > Q$. The asymptotic behaviour as $z \to 0$ of the correlation function $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle$ depends on the sign of $\alpha_1 + \alpha_2 - Q$ and is described by the following three cases.*

1. Supercritical case:

   If $\alpha_1 + \alpha_2 > Q$, then

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle& \\ \underset{z\to 0}{\sim} \frac{1}{4\sqrt{2\pi}} \frac{|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}}{\log^{3/2}\frac{1}{|z|}} &\partial_3 C_\gamma(\alpha_1, \alpha_2, Q)\partial_1 C_\gamma(Q, \alpha_3, \alpha_4).\end{aligned} \tag{3.1.13}$$

2. Critical case:

   If $\alpha_1 + \alpha_2 = Q$, then

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \underset{z\to 0}{\sim} -\frac{1}{\sqrt{2\pi}} \frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}} \partial_1 C_\gamma(Q, \alpha_3, \alpha_4) \tag{3.1.14}$$

3. Subcritical case[3]:

   If $\alpha_1 + \alpha_2 < Q$, then

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \underset{z\to 0}{\sim} |z|^{-\alpha_1\alpha_2} C_\gamma(\alpha_1 + \alpha_2, \alpha_3, \alpha_4) \tag{3.1.15}$$

The different regimes appearing in the statement of Theorem 3.1.1 have a natural geometric explanation (see Figure 3.2 for an illustration of the phenomenon). First, notice that the condition $\alpha_3 + \alpha_4 - Q > 0$ corresponds to having the Seiberg bounds satisfied for (Q, α_3, α_4), except that the first coefficient saturates the second bound. When $\alpha_1 + \alpha_2 < Q$, the two singularities add up and the limit is non-trivial. When $\alpha_1 + \alpha_2 = Q$, the second Seiberg bound is saturated and it is natural [DKRV17, Bav18] to expect the factor $(\log \frac{1}{|z|})^{-1/2}\partial_1 C_\gamma(Q, \alpha_3, \alpha_4)$ since the 0-th order is trivial in this case. When $\alpha_1 + \alpha_2 - Q > 0$, this also explains the factor $(\log \frac{1}{|z|})^{-1}\partial_3 C_\gamma(\alpha_1, \alpha_2, Q)\partial_1 C_\gamma(Q, \alpha_3, \alpha_4)$. The extra $(\log \frac{1}{|z|})^{-1/2}$ term has a more subtle origin. Since both (α_1, α_2, Q) and (Q, α_3, α_4) satisfy the Seiberg bounds, we expect to see the two spheres split and form a disconnected pair of surfaces in the limit. In this limit, the GFF should have two zero modes (given e.g. by the mean on each independent surface). Roughly speaking, upon splitting, the mean

[3] This was already proved in [KRV17, Section 6.1] and essentially follows from dominated convergence.

on the right surface conditioned on the mean on the total surface is a Gaussian random variable with large variance which – when properly rescaled – produces the extra zero mode. This rescaling explains the extra $(\log\frac{1}{|z|})^{-1/2}$ term appearing in (3.1.13).

Theorem 3.1.1 can be equivalently reformulated in terms of GMC. Since our proof does not depend on the particular choice of $(-\frac{Q\sigma}{\gamma})$-moment in the four-point correlation, we may promote σ to a free parameter and study fusion estimates for arbitrary negative moments of GMC that could be of independent interest. We first record the decay rate in the theorem below.

Theorem 3.1.2. *Let $\kappa > 0$, $\gamma \in (0,2)$ and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in \mathbb{R}^4_+$ be such that the Seiberg bound is satisfied. Also let $(z_1,z_2,z_3,z_4) = (0,z,1,\infty)$ with $z \in \mathbb{C}\setminus\{0\}$. Then there exists some constant $E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4) > 0$ such that*

$$\lim_{z\to 0}\frac{1}{I^{\gamma,\kappa}_{\alpha_1+\alpha_2}(z)}\mathbb{E}\left[M^\gamma\left(e^{\gamma\sum_{j=1}^4\alpha_j G(z_j,\cdot)}\right)^{-\kappa}\right] = E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \tag{3.1.16}$$

where the rate function $I^{\gamma,\kappa}_\alpha$ is given by

$$I^{\gamma,\kappa}_\alpha(z) = \begin{cases} 1 & \alpha - Q < 0,\\ \sqrt{\log\frac{1}{|z|}} & \alpha - Q = 0,\\ |z|^{\frac{(\alpha-Q)^2}{2}}\left(\log\frac{1}{|z|}\right)^{3/2} & \alpha - Q \in (0,\kappa\gamma),\\ |z|^{\frac{(\alpha-Q)^2}{2}}\sqrt{\log\frac{1}{|z|}} & \alpha - Q = \kappa\gamma,\\ |z|^{\frac{(\alpha-Q)^2}{2}-\frac{(\kappa\gamma-(\alpha-Q))^2}{2}} & \alpha - Q > \kappa\gamma. \end{cases}$$

As mentioned in Section 3.1.1, LCFT gives an exact expression for $E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ in terms of the DOZZ formula when $\kappa = \frac{\sum_{i=1}^4\alpha_i - 2Q}{\gamma}$. While this is not the case in general, we can still provide a probabilistic representation of the constant based on the radial/angular decomposition of the GFF on the infinite cylinder (see Section 3.2.1). For this it is useful to introduce the random functional

$$\begin{aligned} F_{a_1,a_2}(u,f(\cdot)) &= e^{-\gamma u}\int_{|x|\geq 1}\frac{dM^\gamma(x)}{|x|^{4-\gamma(a_1+a_2)}|x-1|^{\gamma a_1}} + \int_{\mathbb{R}_{s\geq 0}\times\mathbb{S}^1_\theta} e^{-\gamma(f(s)-a_1G(1,e^{-s-i\theta}))}d\widehat{M}^\gamma(s,\theta)\\ &= \int_{\mathcal{C}_\infty} e^{\gamma\left((-u+B_s+(Q-a_2)s)1_{\{s\leq 0\}}-f(s)1_{\{s\geq 0\}}+a_1G(1,e^{-s-i\theta})\right)}d\widehat{M}^\gamma(s,\theta) \end{aligned} \tag{3.1.17}$$

where $(B_s)_{s\geq 0}$ is a Brownian motion independent of the GMC $d\widehat{M}^\gamma(s,\theta)$ associated with the lateral noise of GFF (see Lemma 3.2.1). We will also write $(\widetilde{\beta}^u_s)_{s\geq 0}$ to denote a $\widetilde{\mathrm{BES}}_u(3)$-process (see Definition 3.2.7).

Theorem 3.1.3. *Let $\alpha_1+\alpha_2-Q \geq 0$. The constant $E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ in (3.1.16) has the following probabilistic representations.*

- *If $\alpha_1 + \alpha_2 - Q = 0$, then*

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{\kappa\gamma}\sqrt{\frac{2}{\pi}}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau, \widetilde{\beta}^\tau_\cdot)\right)^{-\kappa}\right] \tag{3.1.18}$$

where $\tau \sim \mathrm{Exp}(\kappa\gamma)$.

- *If $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$, then*

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{\gamma}\frac{B\left(\frac{\alpha_1+\alpha_2-Q}{\gamma}, \kappa - \frac{\alpha_1+\alpha_2-Q}{\gamma}\right)}{(\alpha_1 + \alpha_2 - Q)(\kappa\gamma - (\alpha_1 + \alpha_2 - Q))}\sqrt{\frac{2}{\pi}}$$
$$\times \mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau, \widetilde{\beta}^\tau_\cdot)\right)^{-(\kappa - \frac{\alpha_1+\alpha_2-Q}{\gamma})}\right]\mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(\mathcal{T}, \widetilde{\beta}^{\mathcal{T}}_\cdot)\right)^{-\frac{\alpha_1+\alpha_2-Q}{\gamma}}\right] \tag{3.1.19}$$

where $\tau \sim \mathrm{Exp}(\kappa\gamma - (\alpha_1 + \alpha_2 - Q))$, $\mathcal{T} \sim \mathrm{Exp}(\alpha_1 + \alpha_2 - Q)$ and $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

- *If $\alpha_1 + \alpha_2 - Q = \kappa\gamma$, then*

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{\kappa\gamma}\sqrt{\frac{2}{\pi}}\mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(\mathcal{T}, \widetilde{\beta}^{\mathcal{T}}_\cdot)\right)^{-\kappa}\right]. \tag{3.1.20}$$

where $\mathcal{T} \sim \mathrm{Exp}(\kappa\gamma)$.

- *If $\alpha_1 + \alpha_2 - Q > \kappa\gamma$, then*

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(0, -B_\cdot^{-(\alpha_1+\alpha_2-Q-\kappa\gamma)})\right)^{-\kappa}\right] \tag{3.1.21}$$

where $(B_s^{-(\alpha_1+\alpha_2-Q-\kappa\gamma)})_{s\geq 0}$ is a Brownian motion with negative drift $-(\alpha_1+\alpha_2-Q-\kappa\gamma)$.

Remark 3.1.4. *When $\alpha_1 + \alpha_2 - Q > \kappa\gamma$, we can easily rewrite (3.1.21) as*

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{E}\left[\left(\int_{\mathbb{C}} \frac{|x^{-1}|_+^{(\kappa+1)\gamma^2} dM^\gamma(x)}{|x|^{4-\gamma(\alpha_1+\alpha_2)}|x-1|^{\gamma\alpha_2}}\right)^{-\kappa}\right]$$

which is very similar to the subcritical regime $\alpha_1 + \alpha_2 - Q < 0$ where

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{E}\left[\left(\int_{\mathbb{C}} \frac{dM^\gamma(x)}{|x|^{\gamma(\alpha_1+\alpha_2)}|x-1|^{\gamma\alpha_3}|x|_+^{4-\gamma\sum_{j=1}^4 \alpha_j}}\right)^{-\kappa}\right]$$

can be obtained immediately by dominated convergence.

3.1.4 Conjectured link with random planar maps

The result of Theorem 3.1.1 has an interesting counterpart in the world of 2d discretised quantum gravity via the famous KPZ conjecture which was originally formulated in the physics literature by Knizhnik, Polyakov and Zamolodchikov [KPZ88]. Roughly speaking, the authors conjectured that, in some sense, LCFT should be the scaling limit of large random planar maps weighted by some statistical mechanics model.

We start by recalling some facts about planar maps, using the setting of [Kup16, Section 1] (see also [DKRV16, Section 5.3]). A *planar map* is a graph together with an embedding into the sphere such that no two edges cross and viewed up to orientation preserving homeomorphisms.

For concreteness, we will work with triangulations, meaning that all the faces in the map are triangles. Let $\mathcal{T}_{N,3}$ be the set of planar triangulations with N faces and 3 extra marked points (called *roots*). The combinatorics of $\mathcal{T}_{N,3}$ is well known since the work of Tutte [Tut63] and we have

$$\#\mathcal{T}_{N,3} \underset{N\to\infty}{\asymp} N^{-1/2}e^{-\mu_c N}$$

for some $\mu_c > 0$. We mention that a wide class of planar maps fall into the same universality class (e.g. $2p$-angulations), meaning that they scale like $N^{-1/2}e^{-\mu_c N}$ where μ_c depends on the model.

There is a way to conformally embed any triangulation $(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ into the sphere by first turning it into a topological manifold and second specifying complex coordinate charts. This endows the triangulation with a structure of Riemann surface with conical singularities at vertices with $n \neq 6$ neighbours, and this embedding is unique if we add the extra requirement that the marked points $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ are sent to $(0, 1, \infty)$ (see e.g. [Kup16]). Concretely, if $\triangle \subset \mathbb{C}$ is an equilateral triangle with unit (Lebesgue) volume, the embedding provides a conformal map $\psi_t : \triangle \to \widehat{\mathbb{C}}$ for each triangle t in the map. For all $a > 0$, we consider the pushforward measure $d\nu_{t,a}(z) = a^2|(\psi_t^{-1})'(z)|^2dz$ on $\psi_t(\triangle)$, which assigns a mass a^2 to each triangle of $\mathbf{t}$. The collection of $(\nu_{t,a})_{t\in\mathbf{t}}$ defines a measure $\nu_a^{\mathbf{t}}$ on $\widehat{\mathbb{C}}$, and in particular $\nu_a^{\mathbf{t}}(\widehat{\mathbb{C}}) = Na^2$ for all $\mathbf{t} \in \mathcal{T}_{N,3}$.

The model becomes interesting when we choose the triangulation randomly. The simplest example is the case of *pure gravity*, which amounts in sampling the triangulation with respect to the probability measure defined by

$$\mathbb{P}_a(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \frac{1}{Z_a}e^{-\mu|\mathbf{t}|}$$

where $\mu := (1 + a^2)\mu_c$, $|\mathbf{t}|$ is the number of faces of $\mathbf{t}$ and Z_a is a normalising constant. Notice that $Z_a \to \infty$ as we send $a \to 0$, which means that the measure selects larger and larger maps. When $(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is sampled under $\mathbb{P}_a$, the KPZ conjecture states that the random measure $\nu_a = \nu_a^{\mathbf{t}}$ converges in distribution (with respect to the topology of weak convergence of measures) as $a \to 0$ to a random Radon measure ν on $\mathbf{S}^2$. This limiting measure is expected to be given by the Liouville measure (see [DKRV16, Section 3.3] for a definition) and in particular, it should satisfy the property that for all measurable $A \subset \widehat{\mathbb{C}}$,

$$\mathbb{E}\left[\frac{\nu(A)}{\nu(\widehat{\mathbb{C}})}\right] = \int_A f_{\sqrt{8/3},\mu_c}$$

where we have defined the probability density function

$$f_{\gamma,\mu}(z) := \frac{\mu\gamma}{3\gamma - 2Q} \frac{\langle V_\gamma(0)V_\gamma(z)V_\gamma(1)V_\gamma(\infty)\rangle}{C_\gamma(\gamma,\gamma,\gamma)} \tag{3.1.22}$$

for all $\gamma \in (0,2)$ and $\mu > 0$ (see Appendix 3.C for the derivation of the normalising constant).

The critical case of Theorem 3.1.1 is given by $\gamma = \frac{2}{\sqrt{3}}$[4], so that $\gamma = \sqrt{\frac{8}{3}}$ falls into the supercritical case. Thus we have the asymptotic behaviour (note that $\Delta_\gamma = \frac{\gamma}{2} \times \frac{2}{\gamma} = 1$)

$$f_{\gamma,\mu}(z) \underset{z\to 0}{\sim} \frac{\mu\gamma}{2\sqrt{2\pi}(3\gamma - 2Q)} |z|^{\frac{Q^2}{2}-4} \left(\log\frac{1}{|z|}\right)^{-3/2} \frac{(\partial_3 C_\gamma(\gamma,\gamma,Q))^2}{C_\gamma(\gamma,\gamma,\gamma)}. \tag{3.1.23}$$

If we integrate this formula on a small disc of radius ε, we find

$$\int_0^\varepsilon r^{\frac{Q^2}{2}-4}\left(\log\frac{1}{r}\right)^{-3/2} rdr = (Q^2/2-2)^{1/2}\int_{(Q^2/2-2)\log\frac{1}{\varepsilon}}^\infty e^{-u}u^{-3/2}du \underset{\varepsilon\to 0}{\sim} 2\frac{\varepsilon^{\frac{Q^2}{2}-2}}{\sqrt{\log\frac{1}{\varepsilon}}}$$

so that

$$\int_{|z|\le\varepsilon} f_{\gamma,\mu}(z)dz \underset{\varepsilon\to 0}{\sim} \frac{\sqrt{2\pi}\mu\gamma}{3\gamma - 2Q}\frac{(\partial_3 C_\gamma(\gamma,\gamma,Q))^2}{C_\gamma(\gamma,\gamma,\gamma)}\frac{\varepsilon^{\frac{Q^2}{2}-2}}{\sqrt{\log\frac{1}{\varepsilon}}}. \tag{3.1.24}$$

If the conjecture holds true, the asymptotic behaviour (3.1.24) gives the expected fraction of vertices which are close to 0 in a large planar map. In particular, the exponent of ε is $\frac{Q^2}{2} - 2 = 1/12$ for pure gravity.

Similar conjectures hold for random maps coupled with some statistical mechanics model (such as the Potts model, see e.g. [DKRV16]). The conjectures are essentially the same in each case except that the value of γ and μ may vary (e.g. Ising model corresponds to $\gamma = \sqrt{3}$). However one can still plug the good value of γ in formula (3.1.24) to conjecture the expected density of points around 0.

[4] We notice that this is a special value of γ from the random maps perspective since it corresponds to the scaling limit of bipolar-oriented maps, see [KMSW15]

3.1.5 Outline

The remainder of this article is organised as follows. In the next section, we provide a summary of GFF and GMC for the construction of Liouville correlation functions, and then explain the main idea of our proofs. Section 4.3 is devoted to the proof of Theorem 3.1.1 (on the four-point correlation) and Theorem 3.1.2 (on the decay of arbitrary negative moments of GMC), while that of Theorem 3.1.3 (on the probabilistic representations of the limiting constants) is treated in Section 3.4. In the appendices we collect the DOZZ formula, discuss our work from the perspective of surfaces with conical singularities and explain how to normalise the four-point correlation to a probability distribution.

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3.2 Background

In this section, we recall the mathematical foundation for the Liouville measure (3.1.2) and the derivation for the 4-point function, and explain the main idea of our approach. To commence with, we quickly review GFF and GMC and mention several facts about them.

3.2.1 Gaussian Free Field

Let $H_0^1(\mathbb{S}^2, g)$ (or simply H_0^1) be the Sobolev space of functions with distributional derivatives in $L^2(\mathbb{S}^2, g)$ and vanishing g-mean. This space is equipped with the norm

$$||X||_\nabla^2 := \frac{1}{2\pi}\int_{\mathbb{S}^2} |\nabla X|^2 = -\frac{1}{2\pi}\int_{\mathbb{S}^2} \Delta X \cdot X$$

that we call the *Dirichlet energy*. Hence we can interpret the formal measure $\frac{1}{Z_{\mathrm{GFF}}}\int e^{-\frac{1}{2}||X||_\nabla^2} DX$ as a Gaussian probability measure on the space H_0^1 (where Z_{GFF} is a "normalising constant" which we will explain at the end of this section). Thus if $(e_n)_{n\geq 1}$ is an orthonormal basis of H_0^1, we define the formal series

$$X = \sum_{n\geq 1} \alpha_n e_n$$

where $(\alpha_n)_{n\geq 1}$ is a sequence of i.i.d. normal random variables. It can be shown that this series converges in H_0^{-1}, the topological dual of H_0^1. In particular, it is not defined as a function but rather as a distribution in the sense of Schwartz. We call this field the

Gaussian Free Field (GFF). We write $\mathbb{P}$ for the probability measure of the GFF and $\mathbb{E}$ the associated expectation. The covariance kernel of the GFF is given by Green's function $G := (-\frac{1}{2\pi}\Delta)^{-1}$, i.e. we formally write

$$\mathbb{E}[X(x)X(y)] = G(x,y)$$

where the kernel of Green's function is explicitly given by

$$G(x,y) = \log\frac{1}{|x-y|} + \log|x|_+ + \log|y|_+.$$

Thus the "normalising constant" Z_{GFF} that we are looking for should be given by $Z_{\mathrm{GFF}} := (\det(-\frac{1}{2\pi}\Delta))^{1/2}$, which is obtained via zeta-regularisation [OPS88].

There is a convenient choice of basis for H^1_0, which is the family $(\sqrt{\frac{2\pi}{\lambda_n}}\varphi_n)_{n\geq 1}$ where $(\varphi_n)_{n\geq 0}$ is an orthonormal basis of L^2 of eigenfunctions of $-\Delta$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq ... \leq \lambda_n....$ This gives an L^2 decomposition of the GFF, except that we are missing the zero mode (the coefficient in front of the constant function $\varphi_0 \equiv \mathrm{Vol}_g(\mathbb{S}^2)^{-1/2}$). This should be a Gaussian with infinite variance and we interpret this as Lebesgue measure, since $\sqrt{\frac{2\pi}{\lambda}}$ times the law of a Gaussian random variable with variance λ^{-1} converges vaguely to Lebesgue measure as $\lambda \to 0$. So our final interpretation of the measure $e^{-\frac{1}{2}||X||^2_\nabla}DX$ is that we set for all continuous functional $F : H^{-1} \to \mathbb{R}$

$$\int F(X)e^{-\frac{1}{2}||X||^2_\nabla}DX = \left(\frac{\det(-\frac{1}{2\pi}\Delta)}{\mathrm{Vol}_g(\mathbb{S}^2)}\right)^{-1/2}\int_{\mathbb{R}}\mathbb{E}[F(X+c)]dc.$$

Throughout the paper, we will make an extensive use of the so-called *radial/angular* decomposition of the GFF, which is better understood in cylinder coordinates. Let $\mathcal{C}_\infty := \mathbb{R}_s \times \mathbb{S}^1_\theta$ be the complete cylinder. Under the conformal change of coordinates $\psi : z \mapsto -\log z$, the Riemann sphere $(\widehat{\mathbb{C}} \setminus \{0,\infty\}, g)$ endowed with the crêpe metric is mapped to $(\mathcal{C}, g_\psi)$ with $g_\psi(s,\theta) = e^{-2|s|}$. From now on, we write G for Green's function on $(\mathcal{C}_\infty, g_\psi)$ with vanishing mean on $\{0\} \times \mathbb{S}^1$.

Lemma 3.2.1. *Let $X(s,\theta)$ be a GFF on $\mathcal{C}_\infty$. Then we can write $X(s,\theta) = B_s + Y(s,\theta)$ where*

1. *$(B_s)_{s\in\mathbb{R}}$ is a two-sided Brownian motion. We will call this process the* radial part *of the field.*

2. *Y is a* log*-correlated field with covariance kernel*

$$H(s,\theta,s',\theta') := \mathbb{E}[Y(s,\theta)Y(s',\theta')] = \log\frac{e^{-s}\vee e^{-s'}}{|e^{-s-i\theta} - e^{-s'-i\theta'}|}. \tag{3.2.1}$$

We will call this field the lateral noise *or* angular part *of the field. Notice that the* *law of Y is translation invariant.*

3. *B is independent of Y.*

Otherwise stated, Lemma 3.2.1 enables to rewrite Green's function (on the cylinder) as

$$\begin{aligned} G(s,\theta,s',\theta') &= (|s|\wedge|s'|)1_{ss'\geq 0} + H(s,\theta,s',\theta') \\ &= (|s|\wedge|s'|)1_{ss'\geq 0} + H(0,0,s'-s,\theta'-\theta) \\ &= (|s|\wedge|s'|)1_{ss'\geq 0} + G(0,0,s'-s,\theta'-\theta). \end{aligned} \tag{3.2.2}$$

Remark 3.2.2. *We will sometimes abuse notations and write the more compact form $G(s+i\theta,s'+i\theta')$ (resp. $H(s+i\theta,s'+i\theta')$) for $G(s,\theta,s',\theta')$ (resp. $H(s+i\theta,s'+i\theta')$).*

3.2.2 Gaussian Multiplicative Chaos

Recall that a GFF is only defined as a distribution, so the exponential term $e^{\gamma X}$ is ill-defined *a priori*. However it is possible to make sense of the measure $e^{\gamma X(x)}g(x)d^2x$ using a regularising procedure based on Kahane's theory of Gaussian Multiplicative Chaos (GMC) (see [RV14, Ber17] for more detailed reviews).

We use the regularisation called the circle average. For $\varepsilon>0$, let $X_{g,\varepsilon}$ be the average of X on the geodesic circle of radius ε in the metric g. The field X_ε is continuous, so the measure

$$dM^\gamma_{g,\varepsilon}(x) := e^{\gamma X_{g,\varepsilon}(x)-\frac{1}{2}\gamma^2\mathbb{E}[X_{g,\varepsilon}(x)^2]}d^2x$$

is well defined for all $\gamma\in(0,2)$, and it is known that the sequence of measures $M^\gamma_{g,\varepsilon}$ converges weakly in probability to a (random) Radon measure M^γ_g with no atoms.

An important property of GMC measure is its conformal covariance [DKRV16, DRV16, GRV16] under conformal multiplication

Proposition 3.2.3. *Let $\omega\in C^\infty(\mathbb{S}^2,g)$. Let X be a GFF on $(\mathbb{S}^2,g)$ and $M^\gamma_{\tilde{g}}$ be the GMC measure obtained when regularising the field with circle averages in the metric $\tilde{g}:=e^\omega g$. Then $M^\gamma_{\tilde{g}} = e^{\frac{\gamma Q}{2}}M^\gamma_g$.*

Remark 3.2.4. *For notational convenience, when the regularising metric is the background metric $g(x)=|x|_+^{-4}$ on $\widehat{\mathbb{C}}$, we will drop the subscript and write $M^\gamma = M^\gamma_g$.*

Another useful tool of GMC is Kahane's convexity inequality [Kah85].

Theorem 3.2.5 (Kahane 1985)**.** *Let X and Y be two continuous Gaussian fields on $D\subset\mathbb{S}^2$ such that for all $x,y\in D$*

$$\mathbb{E}[X(x)X(y)]\leq\mathbb{E}[Y(x)Y(y)].$$

Then for all convex function $F : \mathbb{R}_+ \to \mathbb{R}$ with at most polynomial growth at infinity,

$$\mathbb{E}\left[F\left(\int_D e^{\gamma X(x)-\frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}d^2x\right)\right] \leq \mathbb{E}\left[F\left(\int_D e^{\gamma Y(x)-\frac{\gamma^2}{2}\mathbb{E}[Y(x)^2]}d^2x\right)\right].$$

In practice, one can apply this theorem to the GMC measure associated with log-correlated fields like the GFF after using the regularising procedure.

Now suppose X, Y are log-correlated fields with $|\mathbb{E}[X(x)X(y) - \mathbb{E}[Y(x)Y(y)]| \leq \varepsilon$ and write M^γ, N^γ for their respective chaos measure. In particular we have

$$\mathbb{E}[X(x)X(y)] \leq \mathbb{E}[Y(x)Y(y)] + \varepsilon.$$

Notice that the field $Z(x) = Y(x) + \sqrt{\varepsilon}\delta$ – with $\delta \sim \mathcal{N}(0,1)$ independent of everything – has covariance kernel $\mathbb{E}[Y(x)Y(y)] + \varepsilon$. Hence by Kahane's convexity inequality, we have for all $\kappa > 0$

$$\mathbb{E}[M^\gamma(D)^{-\kappa}] \leq \mathbb{E}[e^{-r\gamma\sqrt{\varepsilon}\delta}N^\gamma(D)^{-\kappa}] = e^{\frac{1}{2}\gamma^2 r^2\varepsilon}\mathbb{E}[N^\gamma(D)^{-\kappa}].$$

By the symmetry of the roles played by X and Y, the converse inequality is also true, so

$$\mathbb{E}[M^\gamma(D)^{-\kappa}] = \mathbb{E}[N^\gamma(D)^{-\kappa}](1 + O_{\varepsilon\to 0}(\varepsilon)).$$

Similarly, we have for all $c \in \mathbb{R}$,

$$\mathbb{E}\left[\exp(-\mu e^{\gamma c}M^\gamma(D))\right] = \mathbb{E}\left[\exp(-\mu e^{\gamma c}N^\gamma(D))\right](1 + O_{\varepsilon\to 0}(\varepsilon)).$$

3.2.3 Derivation of the correlation function

Using the GFF and GMC we are ready to state the definition of the correlation functions on the sphere. For $\varepsilon > 0$, we can regularise the vertex operator $V_{\alpha_i}(z_i)$ by defining $V_{\alpha_i,\varepsilon}(z_i) = e^{\alpha_i X_\varepsilon(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_\varepsilon(z_i)^2]}$. By Cameron-Martin theorem, we have (recall $\sigma = \sum_{i=1}^N \frac{\alpha_i}{Q} - 2 > 0$)

$$\left\langle \prod_{i=1}^N V_{\alpha,\varepsilon}(z_i)\right\rangle = 2e^{C_\varepsilon(\mathbf{z})}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\widehat{\mathbb{C}}} e^{\gamma\sum_{i=1}^N \alpha_i G_\varepsilon(z_i,\cdot)}dM^\gamma\right)\right]dc \qquad (3.2.3)$$

where $C_\varepsilon(\mathbf{z}) = \sum_{i<j}\alpha_i\alpha_j G_\varepsilon(z_i,z_j)$. This regularised correlation function (3.2.3) converges to a positive finite limit as $\varepsilon \to 0$ as long as the Seiberg bounds are satisfied as the GMC measure integrates the singularities around each insertion. We take this limit as our

definition of the correlation function

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \right\rangle = 2e^{C(\mathbf{z})} \int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^N \alpha_i G(z_i,\cdot)} dM^\gamma\right)\right] dc$$
$$= 2e^{C(\mathbf{z})}\gamma^{-1}\mu^{-\frac{Q\sigma}{\gamma}}\Gamma\left(\frac{Q\sigma}{\gamma}\right)\mathbb{E}\left[\left(\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^N \alpha_i G(z_i,\cdot)} dM^\gamma\right)^{-\frac{Q\sigma}{\gamma}}\right] \tag{3.2.4}$$

after making the change of variable $u = e^{\gamma c}$. As can be seen from expression (3.2.4), the finiteness of the correlation function in our probabilistic formulation is equivalent to the finiteness of the moments of the GMC measure. This holds provided the *extended Seiberg bounds* are satisfied [KRV17]

$$-\frac{Q\sigma}{\gamma} < \frac{4}{\gamma^2} \wedge \min_{1\le i\le N}(Q-\alpha_i) \qquad \text{and} \qquad \alpha_i < Q \quad \forall i.$$

In particular, if $N = 3$ with insertions at $(0, 1, \infty)$ and Liouville momenta $(\alpha_1, \alpha_2, \alpha_3)$ satisfying the Seiberg bounds, the expression is simply

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = 2\gamma^{-1}\mu^{-\frac{Q\sigma}{\gamma}}\Gamma\left(\frac{Q\sigma}{\gamma}\right)\mathbb{E}\left[\left(\int_{\widehat{\mathbb{C}}} e^{\gamma(\alpha_1 G(0,\cdot)+\alpha_2 G(1,\cdot)+\alpha_3 G(\infty,\cdot))} dM^\gamma\right)^{-\frac{Q\sigma}{\gamma}}\right] \tag{3.2.5}$$

and this expression equals the DOZZ formula $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ [KRV17].

As for the four-point correlation function with insertions at $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ with $|z| < 1$, we find

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle$$
$$= \frac{2}{|z|^{\alpha_1\alpha_2}|1-z|^{\alpha_2\alpha_3}} \int_{\mathbb{R}} e^{-Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^4 \alpha_i G(z_i,\cdot)} dM^\gamma\right)\right] dc. \tag{3.2.6}$$

3.2.4 Main idea

We now explain our approach which is inspired by [DKRV17]. By applying the radial/angular decomposition of the GFF as we will see in Section 3.3.1, we can effectively transform our problem to the study of exponential functionals of Brownian motion.

To be more precise consider the following toy model. Let $(B_s^\lambda)_{s\ge 0}$ be a Brownian motion with drift λ, and suppose $C_1, C_2 > 0$ are two fixed constants. Our goal is to understand the asymptotics of

$$\mathbb{E}\left[\left(C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda}\right)^{-\kappa}\right] \tag{3.2.7}$$

as $t \to \infty$. In order to extract the leading order in (3.2.7), we have to play the game of balancing energy (i.e. asking our drifted Brownian motion $(B_s^\lambda)_s$ to remain small) and entropy (i.e. paying a multiplicative cost given by the probability of such event).

- When $\lambda < 0$, we don't have to do anything because $B_s^\lambda \xrightarrow{s\to\infty} -\infty$ anyway, and

$$\mathbb{E}\left[\left(C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda}\right)^{-\kappa}\right] \xrightarrow{t\to\infty} \mathbb{E}\left[\left(C_1 + \int_0^\infty e^{\gamma B_s^\lambda} ds\right)^{-\kappa}\right]$$

  by dominated convergence easily.

- When $\lambda = 0$, we should demand our Brownian motion to never exceed an $O(1)$ threshold. On the event that $\{\sup_{s\le t} B_s \le N\}$, $(N - B_s)_{s\le t}$ behaves like a $\mathrm{BES}_N(3)$-process and drifts to $-\infty$, and therefore for suitably chosen $t' \ll t$ we see that

$$C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda} \approx C_1 + \int_0^{t'} e^{\gamma B_s^\lambda} ds$$

  is expected to be $O(1)$ while the entropy cost is given by

$$\mathbb{P}\left(\sup_{s\le t} B_s \le N\right) \sim \sqrt{\frac{2}{\pi}}\frac{N}{\sqrt{t}} = O\left(t^{-\frac{1}{2}}\right).$$

- When $\lambda \in (0, \kappa\gamma)$, we still demand our drifted Brownian motion B_t^λ to remain below an $O(1)$ threshold, which requires an entropy cost of

$$\mathbb{P}\left(\sup_{s\le t} B_s^\lambda \le N\right) \sim \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{\lambda^2}{2}t}}{\lambda^2 t^{\frac{3}{2}}} N e^{\lambda N} = O\left(e^{-\frac{\lambda^2}{2}t} t^{-\frac{3}{2}}\right).$$

  The structural difference here is that even though B_s^λ is rather negative in the intermediate time interval $s \in [t', t - t']$, the terminal value B_t^λ is typically $O(1)$:

$$\mathbb{P}\left(B_t^\lambda \le x \,\middle|\, \sup_{s\le t} B_s^\lambda \le N\right) \xrightarrow{t\to\infty} e^{-\lambda(N-x)}(1 + \lambda(N-x)), \qquad x \le N.$$

  Therefore for the purpose of deriving the renormalised constant, we will have to keep

$$C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda} \approx C_1 + \int_0^{t'} e^{\gamma B_s^\lambda} ds + \int_{t-t'}^t e^{\gamma B_s^\lambda} ds + e^{\gamma B_t^\lambda} C_2.$$

  which is $O(1)$ as $(B_s^\lambda)_{s\le t'}$ and $(B_{t-s}^\lambda - B_t^\lambda)_{s\le t'}$ behave like the negation of two independent BES(3)-processes.

- Moving beyond, we can only ask the B_s^λ not to drift faster than $\lambda - \kappa\gamma$ or else the entropy cost would be too expensive. To proceed we first apply Cameron-Martin theorem to rewrite (3.2.7) as

$$\begin{aligned}&\mathbb{E}\left[e^{\kappa\gamma B_t-\frac{\kappa^2\gamma^2}{2}t}\left(C_1+\int_0^t e^{\gamma B_s^{\lambda-\kappa\gamma}}ds+C_2e^{\gamma B_t^{\lambda-\kappa\gamma}}\right)^{-\kappa}\right]\\&\qquad=e^{-\kappa\gamma\lambda t+\frac{\kappa^2\gamma^2}{2}t}\mathbb{E}\left[\left(C_1e^{-\gamma B_t^{\lambda-\kappa\gamma}}+\int_0^t e^{\gamma(B_s^{\lambda-\kappa\gamma}-B_t^{\lambda-\kappa\gamma})}ds+C_2\right)^{-\kappa}\right].\end{aligned}\tag{3.2.8}$$

If $\lambda=\kappa\gamma$, there isn't any drift in the expectation. The observation from the case $\lambda=0$ suggests that we may want to demand $B_{t-s}-B_t$ to not exceed an $O(1)$ threshold for $s\le t$. This would imply again an entropy cost of $O(t^{-\frac{1}{2}})$, and we expect that

$$C_1e^{-\gamma B_t^{\lambda-\kappa\gamma}}+\int_0^t e^{\gamma(B_s-B_t)}ds+C_2\approx\int_0^{t'} e^{\gamma(B_s-B_t)}ds+C_2$$

is $O(1)$ because $(B_{t-s}-B_t)_{s\le t'}$ behaves like the negation of a BES(3)-process as before. If $\lambda>\kappa\gamma$, the story is simpler because $B_{t-s}^{\lambda-\kappa\gamma}-B_t^{\lambda-\kappa\gamma}$ may be seen as a Brownian motion with negative drift. Similar to the earlier case where $\lambda<0$,

$$C_1e^{-\gamma B_t^{\lambda-\kappa\gamma}}+\int_0^t e^{\gamma(B_s^{\lambda-\kappa\gamma}-B_t^{\lambda-\kappa\gamma})}ds+C_2\approx\int_0^{t'} e^{\gamma(B_s^{\lambda-\kappa\gamma}-B_t^{\lambda-\kappa\gamma})}ds+C_2.$$

is already $O(1)$ without incurring any further entropy cost.

3.2.5 Path decomposition of BES(3)-processes

Before we proceed to the proofs, we collect Williams' path decomposition theorem [Wil74] for 3-dimensional Bessel processes (abbreviated as BES(3)-processes) which will be helpful when we study the probabilistic representations of the renormalised constant (3.1.16).

Theorem 3.2.6 (Williams 1974)**.** *Fix $x>0$, and consider the following independent objects:*

- *$(B_s)_{s\ge0}$ is a standard Brownian motion (starting from 0).*
- *U is a* Uniform$[0,1]$ *random variable.*
- *$(\beta_s^0)_{s\ge0}$ is a 3-dimensional Bessel process starting from 0.*

Then the process $(\widehat{\beta}_s^x)_{s\ge0}$ defined by

$$\widehat{\beta}_s^x=\begin{cases}x+B_s & s\le T_{-x(1-U)},\\ xU+\beta^0_{s-T_{-x(1-U)}} & s\ge T_{-x(1-U)}\end{cases}\tag{3.2.9}$$

with

$$T_{-x(1-U)} = \inf\{s > 0 : B_s = -x(1-U)\} = \inf\{s > 0 : x + B_s = xU\}$$

is a 3-dimensional Bessel process starting from x (written as $\mathrm{BES}_x(3)$-process).

In view of Theorem 3.2.6, we introduce the following definition.

Definition 3.2.7. *Let $(B_s)_{s\geq 0}$ and $(\beta_s^0)_{s\geq 0}$ be as in Theorem 3.2.6, and $x \geq 0$ an independent random variable. Then the process $(\widetilde{\beta}_s^x)_{s\geq 0}$ defined by*

$$\widetilde{\beta}_s^x = \begin{cases} x + B_s & s \leq T_{-x}, \\ \beta^0_{s-T_{-x}} & s \geq T_{-x} \end{cases} \tag{3.2.10}$$

with

$$T_{-x} = \inf\{s > 0 : x + B_s = 0\}$$

is called a 3-dimensional Bessel process starting from x conditioned on hitting 0, written as $\widetilde{\mathrm{BES}}_x(3)$-process.

3.3 Proof of Theorem 3.1.1

3.3.1 Supercritical case

We set the insertions at $(z_1, z_2, z_3, z_4) := (0, z, 1, \infty)$ with Liouville momenta $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the Seiberg bounds, and we write $-\log z = t + i\phi$ with $t > 0$ and $\phi \in [0, 2\pi)$. We assume that both $\alpha_3 + \alpha_4 - Q > 0$ and $\alpha_1 + \alpha_2 - Q > 0$ which corresponds to the supercritical case of Theorem 3.1.1. Notice that this corresponds precisely to having (α_1, α_2, Q) and (Q, α_3, α_4) satisfying the Seiberg bounds (with respectively the 3-rd and the 1-st momenta saturating the second Seiberg bound).

Proof of (3.1.13). Let $X(s,\theta) = B_s + Y(s,\theta)$ be a GFF on $\mathcal{C}_\infty = \mathbb{R}_s \times \mathbb{S}^1_\theta$. By the conformal covariance of GMC, it is equivalent to study the chaos measure of X with respect to g_ψ or to consider the field $X(s,\theta) + \frac{Q}{2}\log g_\psi(s,\theta) = X(s,\theta) - Q|s|$ and do the regularisation with respect to Lebesgue measure.

From now on, we write $d\widehat{M}^\gamma(s,\theta)$ for GMC measure of the lateral noise with respect to Lebesgue measure on $\mathcal{C}_\infty$ (while $dM^\gamma(x)$ will be used for GMC measure of the entire GFF in spherical coordinates).

We are interested in the total GMC mass

$$
\begin{aligned}
W_t &:= \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1-Q)s1_{s>0}-(\alpha_4-Q)s1_{s<0}+\alpha_3 G(0,s+i\theta)+\alpha_2 G(t+i\phi,s+i\theta))} d\widehat{M}^\gamma(s,\theta) \\
&= \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1+\alpha_2 1_{s<t}-Q)s1_{s>0}-(\alpha_4-Q)s1_{s<0}+\alpha_3 G(0,s+i\theta)+\alpha_2 G(0,s-t+i(\theta-\phi)))} d\widehat{M}^\gamma(s,\theta).
\end{aligned}
\tag{3.3.1}
$$

The behaviour of this integral is essentially governed by the radial process. From the expression above, we can see that on the negative real line the process is $(B_{-s}+(\alpha_4-Q)s)_{s\geq 0}$ which is a Brownian motion with negative drift so the integrand is integrable at $s=-\infty$. On the positive real line, the radial process has a positive drift $\alpha_1+\alpha_2-Q$ up to time t, then a negative drift α_4-Q from t to ∞.

The first step is to apply Cameron-Martin theorem to get rid of the $(\alpha_1+\alpha_2-Q)$ drift term in $[0,t]$, so that for all continuous and bounded function $F:\mathbb{R}\to\mathbb{R}$

$$
\mathbb{E}\left[F(W_t)\right] = \mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t-\frac{1}{2}(\alpha_1+\alpha_2-Q)^2 t}F(Z_t)\right] \tag{3.3.2}
$$

where Z_t is the random variable defined by

$$
Z_t := \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1-Q)(t-s)1_{s>t}-(\alpha_4-Q)s1_{s<0}+\alpha_2 G(0,t-s+i(\phi-\theta))+\alpha_3 G(0,s+i\theta))} d\widehat{M}^\gamma(s,\theta). \tag{3.3.3}
$$

Hence the correlation function takes the form (recall $t=\log\frac{1}{|z|}$)

$$
\begin{aligned}
&\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \\
&\qquad = 2|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}|1-z|^{-\alpha_2\alpha_3}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c}Z_t)\right]dc
\end{aligned}
\tag{3.3.4}
$$

where the exponent for $|z|$ was found by noticing that $\frac{1}{2}(\alpha_1+\alpha_2-Q)^2-\alpha_1\alpha_2 = 2(\frac{Q^2}{4}-\Delta_1-\Delta_2)$.

Remark 3.3.1. *The change of measure (3.3.2) becomes trivial if $\alpha_1+\alpha_2=Q$. This is the reason why there is a phase transition at this value and why the case is easier to treat.*

Remark 3.3.2. *From a geometric point of view, the change of measure (3.3.2) has the effect of changing the background metric from a cone to a cylinder as illustrated in Figure 3.3 (see also Appendix 3.B for links between changes of metrics and changes of probability measures).*

We can sample the radial part $(B_s)_{0\leq s\leq t}$ by the independent sum $B_s = Br_s + \frac{\delta}{\sqrt{t}}s$ where $(Br_s)_{0\leq s\leq t}$ is a standard Brownian bridge and $\delta\sim\mathcal{N}(0,1)$ (see Figure 3.4). We write $(\tilde{B}_s)_{0\leq s\leq t}$ the process on $\mathbb{R}$ where

1. $(\tilde{B}_{-s})_{s\geq 0}$ and $(\tilde{B}_s)_{s\geq t}$ are independent Brownian motions.

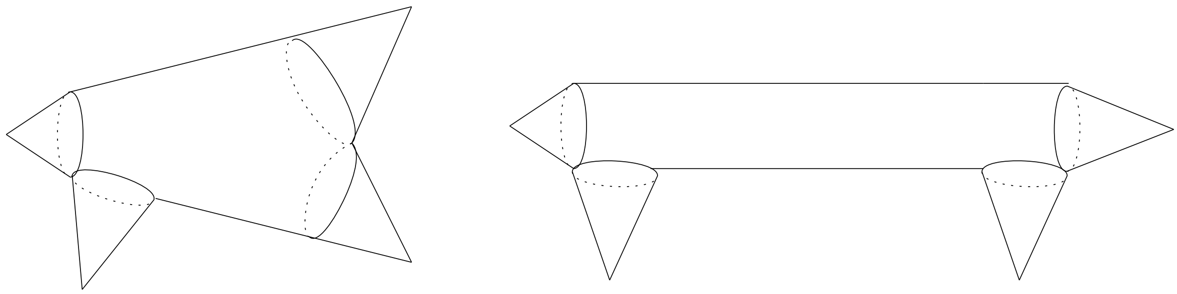

Figure 3.3: Change of measure from the cone to the cylinder

2. $(\tilde{B}_s)_{0\le s\le t}$ is a Brownian bridge in $[0,t]$ independent of the two other processes.

Similarly, we write $\tilde{Z}_t$ for the GMC mass defined similarly as Z_t but with $\tilde{B}$ instead of B. The result will follow from an analysis of the behaviour of $\tilde{Z}_t$.

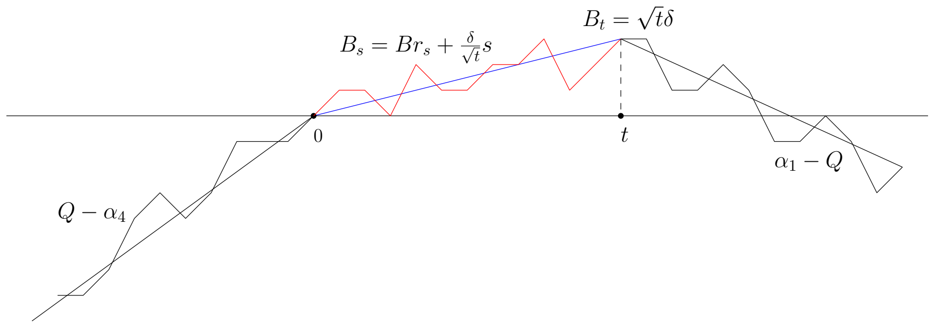

Figure 3.4: The radial process in $(0,t)$ is the independent sum of a Brownian bridge (red) and a random drift (blue).

Let $\eta \in (0,1/2)$. We split $\tilde{Z}_t$ into three parts and write $\tilde{Z}_t = \tilde{L}_t + \tilde{C}_t + \tilde{R}_t$ where $\tilde{L}_t$, $\tilde{C}_t$ and $\tilde{R}_t$ are obtained by restricting the domain of integration to $(-\infty, t^{1/2-\eta}) \times \mathbb{S}^1$, $(t^{1/2-\eta}, t - t^{1/2-\eta}) \times \mathbb{S}^1$ and $(t - t^{1/2-\eta}, \infty) \times \mathbb{S}^1$ respectively. We define $Z_t = L_t + C_t + R_t$ similarly. These random variables are the "left", "central", and "right" parts of the $\tilde{Z}_t$ and Z_t.

For $b > 0$, we introduce the event $\tilde{A}_{b,t} := \left\{ \sup_{0\le s\le t} \tilde{B}_s \le b \right\}$. This event has probability

$$\mathbb{P}(\tilde{A}_{b,t}) = 1 - e^{-2b^2/t} =: f(b/\sqrt{t}).$$

Notice that $\lim_{x\to\infty} f(x) = 1$ and $f(x) \underset{x\to 0}{\sim} 2x^2$.

Conditioning on $\tilde{A}_{b,t}$, the processes $(b - \tilde{B}_s)_{0\le s\le t/2}$ and $(b - \tilde{B}_{t-s})_{0\le s\le t/2}$ are absolutely continuous with respect to a $\mathrm{BES}_b(3)$-process. Hence there exists $\eta' > 0$ such that with high probability as $t \to \infty$, we have $\sup_{t^{1/2-\eta}\le s\le t-t^{1/2-\eta}} \tilde{B}_s \le -t^{1/2-\eta'}$. It follows that $\tilde{C}_t \to 0$ in probability as $t \to \infty$ when conditioned on $\tilde{A}_{b,t}$.

Let $\mathbb{P}_b$ the law of a field $X(s,\theta) = B_s + Y(s,\theta)$ where

1. Y is a standard lateral noise.
2. $(B_{-s})_{s\geq 0}$ is a standard Brownian motion.
3. $(b-B_s)_{s\geq 0}$ is a $\mathrm{BES}_b(3)$-process independent of $(B_{-s})_{s\geq 0}$.

We now describe the behaviour of $\tilde{L}_t$ and $\tilde{R}_t$. On $\tilde{A}_{b,t}$, the law of the process $(b-\tilde{B}_s)_{0\leq s\leq t^{1/2-\eta}}$ is absolutely continuous with respect to that of a $\mathrm{BES}_b(3)$-process, and the Radon-Nikodym derivative tends to 1 a.s. and in L^1 as $t\to\infty$ (see e.g. [MY16, Exercise 9.4]). Hence the pair of processes $((b-\tilde{B}_s)_{0\leq s\leq t^{1/2-\eta}},(b-\tilde{B}_{t-s})_{0\leq s\leq t^{1/2-\eta}})$ converges in distribution to a pair of $\mathrm{BES}_b(3)$-processes, and it is clear that these limit processes are independent of each other.

As for the angular part, notice that for all $s<t^{1/2-\eta}$ and $s'>t-t^{1/2-\eta}$, we have for all $\theta,\theta'\in\mathbb{S}^1$,

$$H(s+i\theta,s'+i\theta')=\log\frac{1}{|1-e^{-(s'-s)-i(\theta'-\theta)}|}\leq\log\frac{1}{1-e^{-(t-2t^{1/2-\eta})}}=O(e^{-t/2})\quad(3.3.5)$$

Now let Y^+,Y^- be independent lateral noises on $\mathcal{C}_\infty$ and define $Y'(s,\theta):=Y^+(s,\theta)1_{s<t/2}+Y^-(s,\theta)1_{s\geq t/2}$. Let $\tilde{L}_t^-$ (resp $\tilde{R}_t^+$) be the random variable defined like $\tilde{L}_t$ (resp. $\tilde{R}_t^-$) except we use Y' rather than Y for the lateral noise. Then under $\tilde{A}_{b,t}$, the pair $(\tilde{L}_t^-,\tilde{R}_t^+)$ converges in distribution to a pair of independent random variables (L_∞,R_∞) with

$$\begin{aligned}L_\infty&\overset{\text{law}}{=}\int_{\mathcal{C}_\infty}e^{\gamma(B_s-(\alpha_4-Q)s1_{s\leq 0}+\alpha_3G(0,s+i\theta))}dM^\gamma(s,\theta),\\R_\infty&\overset{\text{law}}{=}\int_{\mathcal{C}_\infty}e^{\gamma(B_s-(\alpha_1-Q)s1_{s\leq 0}+\alpha_2G(0,s+i\theta))}dM^\gamma(s,\theta)\end{aligned}$$

where the field is sampled from $\mathbb{P}_b$ in both cases.

Using the estimate (3.3.5) and Kahane's convexity inequality, we have for all $c\in\mathbb{R}$

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}(\tilde{L}_t+\tilde{R}_t)\right)|\tilde{A}_{b,t}\right]&=\mathbb{E}_b\left[\mathcal{E}_t\exp\left(-\mu e^{\gamma c}(\tilde{L}_t^-+\tilde{R}_t^+)\right)\right](1+O(e^{-t/2}))\\&\underset{t\to\infty}{\to}\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}(L_\infty+R_\infty))\right]\\&=\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}R_\infty)\right].\end{aligned}$$

Putting pieces together, we find for all $c\in\mathbb{R}$

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\left[\exp(-\mu e^{\gamma c}\tilde{Z}_t)|\tilde{A}_{b,t}\right]&=\lim_{t\to\infty}\mathbb{E}\left[\exp(-\mu e^{\gamma c}\tilde{L}_t)\exp(-\mu e^{\gamma c}\tilde{C}_t)\exp(-\mu e^{\gamma c}\tilde{R}_t)|\tilde{A}_{b,t}\right]\\&=\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}R_\infty)\right].\end{aligned}$$

To conclude we need to relate the behaviour of $\tilde{Z}_t$ with that of Z_t as $t\to\infty$. To this end we will condition on the value of the drift $\delta\sim\mathcal{N}(0,1)$. For fixed $\delta\in\mathbb{R}$, we have $\frac{\delta}{\sqrt{t}}t^{1/2-\eta}=\delta t^{-\eta}$, and this will be sufficient to show that up to time $t^{1/2-\eta}$, the radial part of the GFF $(B_{t-s}-\frac{\delta}{\sqrt{t}}s)_{0\leq s\leq t^{1/2-\eta}}$ does not "feel" the drift and therefore looks like a Brownian motion started from $\sqrt{t}\delta$. More precisely, we have

$$e^{-\gamma|\delta|t^{-\eta}}\tilde{R}_t\leq e^{-\gamma\sqrt{t}\delta}R_t\leq e^{\gamma|\delta|t^{-\eta}}\tilde{R}_t.$$

Taking expectations and rescaling δ by $t^{-1/2}$, we get for all $c\in\mathbb{R}$

$$\begin{aligned}
&\sqrt{t}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c}R_t)|\tilde{A}_{b,t}\right]\\
&\quad=\int_{\mathbb{R}}e^{(\alpha_1+\alpha_2-Q)\delta}\mathbb{E}\left[\exp(-\mu e^{\gamma(c+\delta+\delta O(t^{-1/2-\eta}))}\tilde{R}_t)|\tilde{A}_{b,t}\right]\frac{e^{-\frac{t\delta^2}{2}}}{\sqrt{2\pi}}d\delta\\
&\quad\underset{t\to\infty}{\to}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{(\alpha_1+\alpha_2-Q)\delta}\mathbb{E}_b[\exp(-\mu e^{\gamma(c+\delta)}R_\infty]d\delta
\end{aligned}$$

where we applied the dominated convergence theorem in the last line.

Remark 3.3.3. *The take-home message of this computation is that as t gets large the value of the radial part at t is distributed like $\sqrt{t}\delta$, so when properly rescaled, its law converges vaguely to Lebesgue measure. Hence the field in the right part looks like a usual GFF plus a constant which is "distributed" with Lebesgue measure, so δ plays the role of an extra zero mode in the limit. This translates the fact that we see two independent surfaces in the limit.*

Recalling the expression of the correlation function (3.3.4), we make the change of variable $(c,\delta)=(u,v-u)$ (with Jacobian equal to 1) and find

$$\begin{aligned}
&\sqrt{t}\int_{\mathbb{R}}e^{Q\sigma c}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c}Z_t)|\tilde{A}_{b,t}\right]dc\\
&\quad=\sqrt{t}\int_{\mathbb{R}}e^{Q\sigma c}\int_{\mathbb{R}}e^{(\alpha_1+\alpha_2-Q)\sqrt{t}\delta}\mathbb{E}\left[\exp(-\mu e^{\gamma c}Z_t)|\tilde{A}_{b,t}\right]\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}d\delta dc\\
&\quad\underset{t\to\infty}{\to}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2}e^{(\alpha_1+\alpha_2-Q)(c+\delta)}e^{(\alpha_3+\alpha_4-Q)c}\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma(c+\delta)}R_\infty)\right]d\delta dc\\
&\quad=\frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}e^{(\alpha_3+\alpha_4-Q)u}\mathbb{E}_b\left[\exp(-\mu e^{\gamma u}L_\infty)\right]du\right)\left(\int_{\mathbb{R}}e^{(\alpha_1+\alpha_2-Q)v}\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}R_\infty)\right]dv\right).
\end{aligned}\tag{3.3.6}$$

Thus we have for each $b>0$

$$\begin{aligned}
&\lim_{t\to\infty}t^{3/2}\int_{\mathbb{R}}e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)|\tilde{A}_{b,t}]\mathbb{P}(\tilde{A}_{b,t})dc\\
&\quad=\sqrt{\frac{2}{\pi}}b^2\left(\int_{\mathbb{R}}e^{(\alpha_1+\alpha_2-Q)u}\mathbb{E}_b[\exp(-\mu e^{\gamma u}R_\infty)]du\right)\left(\int_{\mathbb{R}}e^{(\alpha_3+\alpha_4-Q)v}\mathbb{E}_b[\exp(-\mu e^{\gamma v}L_\infty)]dv\right).
\end{aligned}$$

It is shown in [DKRV17] that $b\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right]$ has a non-trivial limit as $b\to\infty$

and, exchanging limits, the authors conclude that

$$\lim_{b\to\infty} b\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right] = \lim_{t\to\infty}\sqrt{\frac{\pi t}{2}}\mathbb{E}\left[\exp(-\mu e^{\gamma v}L_t)\right]. \tag{3.3.7}$$

On the other hand, one can recover the $\mathrm{BES}_b(3)$-process by conditioning a Brownian motion with negative drift to stay below b forever and letting the drift tend to 0. More precisely, if $\tau_{\alpha,b} = \inf\{s \geq 0,\ B_s + (\alpha - Q)s \geq b\}$, then we have $\mathbb{P}(\tau_{\alpha,b} = \infty) \underset{\alpha\to Q^-}{\sim} 2(Q-\alpha)b$. Now adding the drift $\alpha - Q$ in the definition of L_∞ gives the correlation function $\frac{1}{2}C_\gamma(\alpha, \alpha_3, \alpha_4)$. In the end (see [Bav18] for details), we have the alternative characterisation of the limit (3.3.7)

$$\lim_{b\to\infty} b\int_{\mathbb{R}} e^{(\alpha_3+\alpha_4-Q)v}\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right]dv = -\frac{1}{4}\lim_{\alpha\to Q}\frac{C_\gamma(\alpha,\alpha_3,\alpha_4)}{\alpha - Q} = -\frac{1}{4}\partial_1 C_\gamma(\alpha,\alpha_3,\alpha_4). \tag{3.3.8}$$

A similar statement holds for the L_∞ term, so we have

$$\lim_{b\to\infty}\lim_{t\to\infty} t^{3/2}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)1_{\tilde{A}_{b,t}}]dc = \frac{1}{8\sqrt{2\pi}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4).$$

From [DKRV17], the family of functions $\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)1_{\tilde{A}_{b,t}}]$ converges uniformly with respect to t as $b\to\infty$, enabling us to exchange limits in b an in t. Hence

$$\begin{aligned}\lim_{t\to\infty} t^{3/2}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)]dc &= \lim_{b\to\infty}\lim_{t\to\infty} t^{3/2}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)1_{\tilde{A}_{b,t}}]dc \\ &= \frac{1}{8\sqrt{2\pi}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4).\end{aligned}$$

Recall equation (3.3.4) to find

$$\begin{aligned}&\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \\ &\quad\underset{z\to 0}{\sim}\frac{1}{4\sqrt{2\pi}}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}|1-z|^{-\alpha_2\alpha_3}(\log\frac{1}{|z|})^{-3/2}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4).\end{aligned}$$

□

3.3.2 Critical case

We conclude the proof of Theorem 3.1.1 by proving the asymptotic formula (3.1.14), i.e. we assume $\alpha_1 + \alpha_2 = Q$.

Proof of (3.1.14). The analysis of Section 3.3.1 fails only because the limit identified in (3.3.6) becomes trivial in this case because the triplet (α_1, α_2, Q) violates the first Seiberg bound. Geometrically, the random variable R_t does not have enough mass as $t\to\infty$ in order to produce another surface.

However, the analysis is still valid up to equation (3.3.2) and the expression of Z_t is the same with this new set of parameters. Consider the same decomposition $Z_t = L_t + C_t + R_t$ and write $\xi_t := C_t + R_t$ with the same $\eta > 0$.

As before, we condition the radial part no to exceed a given value. For $b > 0$, we define the event

$$A_{b,t} := \left\{ \sup_{0 \le s \le t} B_s \le b \right\}.$$

It is well-known that

$$\mathbb{P}(A_{b,t}) = \sqrt{\frac{2}{\pi}} \int_0^{b/\sqrt{t}} e^{-\frac{x^2}{2}} dx =: g(b/\sqrt{t}).$$

Notice that $g(x) \underset{x\to\infty}{\to} 1$ and $g(x) \underset{x\to 0}{\sim} \sqrt{\frac{2}{\pi}}x$. The process $(B_s)_{s\ge 0}$ conditioned on $A_{b,t}$ has the law of a $\mathrm{BES}_b(3)$-process. Repeating the argument of the previous subsection, we find that $\xi_t \to 0$ in probability as $t \to \infty$ when conditioned on $A_{b,t}$.

As for the radial part, we have the following estimate for $s < t^{1/2-\eta}$ and $\theta \in \mathbb{S}^1$

$$|H(s+i\theta, t+i\phi)| = \log \frac{1}{|1 - e^{-(t-s)-i(\phi-\theta)}|} = O(e^{-t/2}).$$

Let $\mathbb{P}_b$ be the law of the field when the radial part $(B_s)_{s\ge 0}$ is conditioned not to exceed b. Applying exactly the same framework as before, we have for all $\kappa > 0$

$$\begin{aligned} \lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[Z_t^{-\kappa}\right] &= \lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[L_t^{-\kappa}\right] \\ &= \sqrt{\frac{2}{\pi}} \lim_{b\to\infty} b\mathbb{E}_b\left[L_\infty^{-\kappa}\right]. \end{aligned} \tag{3.3.9}$$

So it follows from the result of [Bav18] that

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^{-Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c} \int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^4 \alpha_i G(z_i,\cdot)} dM^\gamma\right)\right] dc = -\frac{1}{2\sqrt{2\pi}} \partial_1 C_\gamma(Q, \alpha_3, \alpha_4) \tag{3.3.10}$$

which concludes the proof. □

3.3.3 Proof of Theorem 3.1.2

As mentioned in Section 3.1.3, Theorem 3.1.2 follows easily from Theorem 3.1.1 by taking σ to be arbitrary. We will use the notations in Section 3.3.1 and 3.3.2, outlining the differences with the Liouville case and leaving the details to the reader.

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be such that the Seiberg bound is satisfied. If $\alpha_1 + \alpha_2 - Q < \kappa\gamma$, the previous analysis applies immediately modulo the obvious substitution $\frac{Q\sigma}{\gamma} \leftrightarrow \kappa$ in the relevant places. If $\alpha_1 + \alpha_2 - Q \ge \kappa\gamma$, however, we only apply Cameron-Martin to partially offset the positive drift in $[0,t]$ by $\kappa\gamma$, as motivated in Section 3.2.4. This leads to

$$\mathbb{E}\left[W_t^{-\kappa}\right] = e^{-\kappa\gamma(\alpha_1+\alpha_2-Q)t+\frac{\kappa^2\gamma^2}{2}t}\mathbb{E}\left[\left(e^{-\gamma(B_t+(\alpha_1+\alpha_2-Q-\kappa\gamma)t)}\widehat{Z}_t\right)^{-\kappa}\right] \tag{3.3.11}$$

where W_t is defined in (3.3.1) and $\widehat{Z}_t$ is defined suitably. Notice that (3.3.11) is identical to (3.3.2) when $\alpha_1+\alpha_2-Q=\kappa\gamma$, the analysis of which is similar to that of Section 3.3.2 except that here we consider the event

$$A'_{b,t} := \left\{\sup_{0\le s\le t}(B_{t-s}-B_t)\le b\right\}$$

so that L_t becomes irrelevant in the limit while R_t survives as $t\to\infty$ instead. The case $\alpha_1+\alpha_2-Q>\kappa\gamma$ is straightforward because $e^{-\gamma(B_t+(\alpha_1+\alpha_2-Q-\kappa\gamma)t)}\widehat{Z}_t$ is an integral involving the exponentiation of a two-sided Brownian motion with negative drifts in both directions, and we can even obtain (3.1.21) by dominated convergence directly.

□

3.4 Proof of Theorem 3.1.3

This section is devoted to the proof of Theorem 3.1.3 which gives probabilistic representations for the limits (3.3.8) and (3.3.9) for which we do not have exact formulae outside of the Liouville case. We will not discuss (3.1.21) which is basically explained in the last section.

3.4.1 Infinite series representation of $E_\kappa^\gamma(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$

In order to obtain Theorem 3.1.3 we need the following intermediate result.

Lemma 3.4.1. *Fix $h>0$. When $\alpha_1+\alpha_2-Q\in[0,\kappa\gamma]$, the constant $E_\kappa^\gamma(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ in (3.1.16) has the following representations.*

- *If $\alpha_1+\alpha_2-Q=0$, we have*

$$E_\kappa^\gamma(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^\infty nhe^{-\kappa\gamma nh}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(nh,\beta_\cdot^{nh})\right)^{-\kappa}1_{\{\min_{s>0}\beta_s^{nh}\le h\}}\right] \tag{3.4.1}$$

where $(\beta_s^u)_{s\ge0}$ is a $\mathrm{BES}_u(3)$-process.

- *If* $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$,

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^\infty \frac{nhe^{-(\kappa\gamma - (\alpha_1+\alpha_2-Q))nh}}{(\alpha_1+\alpha_2-Q)^2} \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta_{L,s}^{nh} \le h\} \cup \{\min_{s>0} \beta_{R,s}^{\mathcal{T}} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^\kappa} \right] \tag{3.4.2}$$

where $(\beta_{L,s}^u)_{s\ge 0}$ *and* $(\beta_{R,s}^{\mathcal{T}})_{s\ge0}$ *are independent* $\mathrm{BES}_u(3)$*- and* $\mathrm{BES}_{\mathcal{T}}(3)$*-processes respectively with* $\mathcal{T} \sim \mathrm{Gamma}(2, \alpha_1+\alpha_2-Q)$*, and* F' *is an independent copy of* F*.*

- *If* $\alpha_1 + \alpha_2 - Q = \kappa\gamma$,

$$E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^\infty nhe^{-\kappa\gamma nh} \mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(nh, \beta_\cdot^{nh})\right)^{-\kappa} 1_{\{\min_{s>0} \beta_s^{nh} \le h\}} \right] \tag{3.4.3}$$

where $(\beta_s^u)_{s\ge0}$ *is a* $\mathrm{BES}_u(3)$*-process.*

Proof. For the sake of brevity we only sketch the proof for the case $h=1$ here and leave the details to the reader. The key idea is the partitioning of

$$A_{n,t} = \left\{ \sup_{0\le s\le t} B_s \le n \right\} = \bigcup_{k\le n} \left\{ \sup_{0\le s\le t} B_s \in [(k-1), k] \right\} = \bigcup_{k\le n} \left\{ \min_{0\le s\le t} k - B_s \in [0,1] \right\}.$$

When $\alpha_1 + \alpha_2 - Q = 0$, our claim essentially follows from Proposition 3.1 and Lemma 3.2 in [DKRV17], where a dominated convergence argument (see the paragraph after Lemma 3.2 and Section 5.0.3 in that article) implies that the renormalised constant is given by

$$\sum_{n=1}^\infty \lim_{t\to\infty} \left(\sqrt{t} \mathbb{E}\left[L_t^{-\kappa} 1_{\{\min_{0\le s\le t} n - B_s \le 1\}} \middle| A_{n,t} \right] \mathbb{P}(A_{n,t}) \right) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^\infty n \mathbb{E}_n \left[L_\infty^{-\kappa} 1_{\{\min_{s\ge0} n - B_s \le 1\}} \right]$$

which is equivalent to (3.4.1). The proof of (3.4.3) is similar.

To apply the same dominated convergence approach to (3.4.2), we need a control analogous to [DKRV17, equation (3.18)] when $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$. Indeed the same argument there suggests that

$$t^{3/2} \mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t} (L_t + R_t)^{-\kappa} 1_{\{\sup_{0\le s\le t} B_s \in [(n-1), n]\}} \right] \le Ce^{-(\kappa\gamma - (\alpha_1+\alpha_2-Q))n}$$

for some constant $C > 0$ independent of t and n, and therefore $E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ again has an infinite series representation of the form

$$\sum_{n=1}^\infty \lim_{t\to\infty} \left(t^{3/2} \mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t} (L_t + R_t)^{-\kappa} 1_{\{\sup_{0\le s\le t} B_s \in [n-1, n]\}} \right] \right). \tag{3.4.4}$$

Let us highlight several observations.

- For every $n \in \mathbb{N}$, the event $\{\sup_{0\leq s\leq t} B_s \in [n-1, n]\}$ may be replaced by

$$\underbrace{\left\{\sup_{0\leq s\leq t} B_s \leq n\right\}}_{=A_{n,t}} \cap \underbrace{\left(\left\{\min_{0\leq s\leq t^{1/2-\eta}} n - B_s \leq 1\right\} \cup \left\{\min_{0\leq s\leq t^{1/2-\eta}} n - B_t - (B_{t-s} - B_t) \leq 1\right\}\right)}_{=:\overline{A}_{n,t}}$$

up to a cost of $o(1)$ for neglecting the unlikely event $\left\{\sup_{s\in[t^{1/2-\eta}, t-t^{1/2-\eta}]} B_s \geq n-1\right\}$.

- Similar to the proof of Theorem 3.1.1, if we condition on the event $A_{n,t}$ and $B_t = x$, then

$$(n - B_s)_{0\leq s\leq t^{1/2-\eta}}, \qquad (n - B_t - (B_{t-s} - B_t))_{0\leq s\leq t^{1/2-\eta}}$$

converge in distribution to independent $\mathrm{BES}_n(3)$- and $\mathrm{BES}_{n-x}(3)$-processes $(\beta^n_{L,s})_{s\geq 0}$ and $(\beta^{n-x}_{R,s})_{s\geq 0}$ respectively. Consequently L_t and R_t converge in distribution to $e^{\gamma n} F_{\alpha_3,\alpha_4}(n, \beta^n_{L,\cdot})$ and $e^{\gamma n} F'_{\alpha_2,\alpha_1}(n-x, \beta^{n-x}_{R,\cdot})$ respectively.

We now compute

$$\begin{aligned}
&\mathbb{E}\left[1_{A_{n,t}\cap\overline{A}_{n,t}} \middle| (B_s)_{s\in(-\infty, t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right] \\
&\quad = 1_{\left\{\min_{0\leq s\leq t^{1/2-\eta}} n - B_s \leq 1\right\}\cup\left\{\min_{0\leq s\leq t^{1/2-\eta}} n-B_t-(B_{t-s}-B_t)\leq 1\right\}} \\
&\qquad \times \mathbb{P}\left(A_{n,t} \middle| (B_s)_{s\in(-\infty,t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{P}\left(A_{n,t} \middle| (B_s)_{s\in(-\infty,t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right) &= 1_{\{\sup_{0\leq s\leq t^{1/2-\eta}} B_s \leq n\}} 1_{\{\sup_{0\leq s\leq t^{1/2-\eta}} B_{t-s} - B_t \leq n - B_t\}} \\
&\quad \times \mathbb{P}\left(\sup_{t^{1/2-\eta}\leq s\leq t-t^{1/2-\eta}} B_s \leq n \middle| B_{t^{1/2-\eta}}, B_{t-t^{1/2-\eta}}\right)
\end{aligned}$$

and

$$\mathbb{P}\left(\sup_{t^{1/2-\eta}\leq s\leq t-t^{1/2-\eta}} B_s \leq n \middle| B_{t^{1/2-\eta}}, B_{t-t^{1/2-\eta}}\right) = 1 - e^{-\frac{2}{t-2t^{1/2-\eta}}(n-B_{t^{1/2-\eta}})(n-B_t-(B_{t-t^{1/2-\eta}}-B_t))}$$

is asymptotically $\frac{2}{t}(n - B_{t^{1/2-\eta}})(n - B_t - (B_{t-t^{1/2-\eta}} - B_t))$ when t is large. In particular

$$\mathbb{P}\left(A_{n,t} \middle| B_t = x\right) \sim \frac{2}{t} n(n-x) + o(t^{-1}), \qquad t \to \infty.$$

Substituting this into the summand in (3.4.4), we obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{3/2}\int_{-\infty}^{n}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)x}(L_t+R_t)^{-\kappa}1_{\overline{A}_{n,t}}\middle|A_{n,t},B_t=x\right]\mathbb{P}(A_{n,t}|B_t=x)\mathbb{P}(B_t\in dx)\\
&=\frac{e^{(\alpha_1+\alpha_2-Q)n}}{\sqrt{2\pi}}\lim_{t\to\infty} t\int_{-\infty}^{n}\mathbb{E}\left[e^{-(\alpha_1+\alpha_2-Q)(n-x)}(L_t+R_t)^{-\kappa}1_{\overline{A}_{n,t}}\middle|A_{n,t},B_t=x\right]\\
&\qquad\times\mathbb{P}(A_{n,t}|B_t=x)e^{-\frac{x^2}{2t}}dx\\
&=\frac{2e^{(\alpha_1+\alpha_2-Q)n}}{\sqrt{2\pi}}\int_{-\infty}^{n}\mathbb{E}\left[\frac{e^{-(\alpha_1+\alpha_2-Q)(n-x)}1_{\{\min_{s\geq 0}\beta^n_{L,s}\leq 1\}\cup\{\min_{s\geq0}\beta^{n-x}_{R,s}\leq 1\}}}{(e^{\gamma n}F_{\alpha_3,\alpha_4}(n,\beta^n_{L,\cdot})+e^{\gamma n}F'_{\alpha_3,\alpha_4}(n-x,\beta^{n-x}_{R,\cdot}))^{\kappa}}\right]n(n-x)dx
\end{aligned}$$

where the last line follows by dominated convergence, and is equal to

$$\sqrt{\frac{2}{\pi}}ne^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))n}\int_0^{\infty}\mathbb{E}\left[\frac{1_{\{\min_{s\geq 0}\beta^n_{L,s}\leq 1\}\cup\{\min_{s\geq 0}\beta^x_{R,s}\leq 1\}}}{(F_{\alpha_3,\alpha_4}(n,\beta^n_{L,\cdot})+F'_{\alpha_3,\alpha_4}(x,\beta^x_{R,\cdot}))^{\kappa}}\right]xe^{-(\alpha_1+\alpha_2-Q)x}dx$$

so we are done.

□

Remark 3.4.2. *The careful reader may notice that the proof above when $\alpha_1+\alpha_2-Q\in(0,\kappa\gamma)$ differs slightly from that in Section 3.3.1 where one considers the event $\widetilde{A}_{n,t}=\left\{\sup_{0\leq s\leq t}\widetilde{B}_s\leq n\right\}$ instead of $A_{n,t}=\left\{\sup_{0\leq s\leq t}B_s\leq n\right\}$. The current approach, which addresses the partitioning of probability space instead of factorisation in the first place, may have the drawback that (3.4.2) does not give a product of two negative moments immediately but it allows for an easier side-by-side comparison with the analysis in [DKRV17].*

3.4.2 Proof of Theorem 3.1.3

The infinite series representation in Lemma 3.4.1 is reminiscent of Riemann sums. We now explain how to obtain the simplified expressions in Theorem 3.1.3.

Proof of (3.1.18) and (3.1.20). We begin with $\alpha_1+\alpha_2-Q=0$. Fix some $N>0$, and without loss of generality choose a sequence of $h\to 0^+$ such that h always divides both N^{-1} and N. Then by Lemma 3.4.1 we have

$$E^{\gamma}_{\kappa}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\sqrt{\frac{2}{\pi}}\sum_{n=1/Nh+1}^{N/h}nhe^{-\kappa\gamma nh}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{\cdot})\right)^{-\kappa}1_{\{\min_{s>0}\beta^{nh}_s\leq h\}}\right]+C_N \tag{3.4.5}$$

for some constant $C_N>0$ which depends on N and the other parameters but not on h, with the property that $\lim_{N\to\infty}C_N=0$.

Recall (3.1.17) for the definition of the random functional F. By Theorem 3.2.6, we can rewrite the sum in (3.4.5) as

$$\sum_{n=1/Nh+1}^{N/h} nhe^{-\kappa\gamma nh}\int_0^{\frac{1}{n}}\mathbb{E}\Bigg[\Bigg(e^{-\gamma nh}\int_{|x|\geq 1}\frac{dM^\gamma(x)}{|x|^{4-\gamma(\alpha_3+\alpha_4)}|x-1|^{\gamma\alpha_3}}$$
$$+\int_{\mathbb{R}_{s\geq 0}\times\mathbb{S}^1_\theta} e^{-\gamma((nh+B_s)1_{\{s\leq T_{-nh(1-u)}\}}+(nhu+\beta^0_{s-T_{-nh(1-u)}})1_{\{s\geq T_{-nh(1-u)}\}}-\alpha_3 G(1,e^{-s-i\theta}))}d\widehat{M}^\gamma(s,\theta)\Bigg)^{-\kappa}\Bigg]du$$
$$\overset{x=nh(1-u)}{=}\sum_{n=1/Nh+1}^{N/h} e^{-\kappa\gamma nh}\int_{(n-1)h}^{nh}\mathbb{E}\Bigg[\Bigg(e^{-\gamma nh}\int_{|x|\geq 1}\frac{dM^\gamma(x)}{|x|^{4-\gamma(\alpha_3+\alpha_4)}|x-1|^{\gamma\alpha_3}}$$
$$+\int_{\mathbb{R}_{s\geq 0}\times\mathbb{S}^1_\theta} e^{-\gamma((nh+B_s)1_{\{s\leq T_{-x}\}}+(nh-x+\beta^0_{s-T_{-x}})1_{\{s\geq T_{-x}\}}-\alpha_3 G(1,e^{-s-i\theta}))}d\widehat{M}^\gamma(s,\theta)\Bigg)^{-\kappa}\Bigg]dx$$
$$=(1+o(1))\int_{1/N}^{N} e^{-\kappa\gamma x}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(x,\widetilde{\beta}^x_\cdot)\right)^{-\kappa}\right]dx$$

where the $o(1)$ error is with respect to $h\to 0^+$ and comes from the fact that

$$e^{-\gamma nh}=(1+o(1))e^{-\gamma x},\qquad e^{-\gamma(nh-x)}=(1+o(1))$$

uniformly in $h>0$ and $n\in\mathbb{N}$ for all $x\in[(n-1)h,nh]$. The desired formula (3.1.18) is recovered by sending $h\to 0^+$ and $N\to\infty$. The proof of (3.1.20) is similar. □

The case where $\alpha_1+\alpha_2-Q\in(0,\kappa\gamma)$ is slightly more involved and the following elementary formula will be useful.

Lemma 3.4.3. *Fix $\kappa,\gamma,\lambda>0$ such that $\lambda<\kappa\gamma$. Let X,Y be independent non-negative random variables and T an independent* $\mathrm{Exp}(\lambda)$ *random variable. Provided that all the moments below exist, we have*

$$\mathbb{E}\left[\left(X+e^{-\gamma T}Y\right)^{-\kappa}\right]=\frac{\lambda}{\gamma}B\left(\frac{\lambda}{\gamma},\kappa-\frac{\lambda}{\gamma}\right)\mathbb{E}\left[X^{-(\kappa-\frac{\lambda}{\gamma})}\right]\mathbb{E}\left[Y^{-\frac{\lambda}{\gamma}}\right].\tag{3.4.6}$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the beta function.

The proof of Lemma 3.4.3 follows from the same change-of-variable argument in (3.3.6) and is skipped here. For a sanity check one may quickly verify that both the LHS and RHS of (3.4.6) converge to $\mathbb{E}[X^{-\kappa}]$ as $\lambda/\gamma\to 0$.

Proof of (3.1.19). Our starting point is (3.4.2) from Lemma 3.4.1. It is clear that

$$\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{nh}_{L,s}\leq h\}\cup\{\min_{s>0}\beta^{\mathcal{T}}_{R,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]=\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{nh}_{L,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]$$
$$+\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{\mathcal{T}}_{R,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]-\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{nh}_{L,s}\leq h\}\cap\{\min_{s>0}\beta^{\mathcal{T}}_{R,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]$$

where the last term is $O(h^2)$ and may be safely ignored. Arguing as before, we see that

$$\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{nhe^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))nh}}{(\alpha_1+\alpha_2-Q)^2}\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{nh}_{L,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]$$
$$=\frac{\sqrt{2/\pi}}{(\alpha_1+\alpha_2-Q)^2(\kappa\gamma-(\alpha_1+\alpha_2-Q))}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau,\widetilde{\beta}^{\tau}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{-\kappa}\right]+o(1) \tag{3.4.7}$$

where $\tau\sim\mathrm{Exp}(\kappa\gamma-(\alpha_1+\alpha_2-Q))$ and $\mathcal{T}\sim\mathrm{Gamma}(2,\alpha_1+\alpha_2-Q)$. Recall that if U is an independent Uniform$[0,1]$ random variable, then $(\mathcal{T}_1,\mathcal{T}_2):=(\mathcal{T}U,\mathcal{T}(1-U))$ is a pair of independent $\mathrm{Exp}(\alpha_1+\alpha_2-Q)$ random variables. Combining this fact with Theorem 3.2.6, we obtain

$$F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\stackrel{d}{=}e^{-\gamma\mathcal{T}_1}F'_{\alpha_2,\alpha_1}(\mathcal{T}_2,\widetilde{\beta}^{\mathcal{T}_2}_{R,\cdot})$$

and we can rewrite the expectation in (3.4.7) as

$$\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau,\widetilde{\beta}^{\tau}_{L,\cdot})+e^{-\gamma\mathcal{T}_1}F'_{\alpha_2,\alpha_1}(\mathcal{T}_2,\widetilde{\beta}^{\mathcal{T}_2}_{R,\cdot})\right)^{-\kappa}\right].$$

Similarly, if we let τ_1,τ_2 be independent $\mathrm{Exp}(\kappa\gamma-(\alpha_1+\alpha_2-Q))$, then

$$\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{nhe^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))nh}}{(\alpha_1+\alpha_2-Q)^2}\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{\mathcal{T}}_{R,s}\leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]$$
$$=\frac{\sqrt{2/\pi}}{(\alpha_1+\alpha_2-Q)(\kappa\gamma-(\alpha_1+\alpha_2-Q))^2}\mathbb{E}\left[\left(e^{-\gamma\tau_1}F_{\alpha_3,\alpha_4}(\tau_2,\widetilde{\beta}^{\tau_2}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T}_2,\widetilde{\beta}^{\mathcal{T}_2}_{R,\cdot})\right)^{-\kappa}\right]+o(1). \tag{3.4.8}$$

The claim then follows by sending $h\to 0^+$ and applying Lemma 3.4.3 to (3.4.7) and (3.4.8). $\square$

3.5 Fusion in boundary Liouville Conformal Field Theory

3.5.1 Boundary Liouville Conformal Field Theory

Boundary LCFT is LCFT on proper simply connected domains $D \subset \mathbb{C}$. We start by a brief review of the theory and refer to [HRV18] for details. Like LCFT on the sphere, the theory is conformally invariant, so by the Riemann uniformisation theorem, it is enough to study it on the upper-half plane $\mathbb{H} := \{\mathrm{Im} z > 0\}$ (the unit disc $\mathbb{D}$ is also a common choice) equipped with some background metric g. In this context, the Liouville action with boundary term is given by[5]

$$S_{\mathrm{L}}(X,g) := \frac{1}{4\pi}\int_{\mathbb{H}} \left(|\nabla X|^2 + 4\pi\mu e^{\gamma X} g(z)\right) d^2z + \mu_\partial \int_{\mathbb{R}} e^{\frac{\gamma}{2}X} g(x)^{1/2} dx \tag{3.5.1}$$

where $\mu_\partial > 0$ is the boundary cosmological constant. One recognises the Dirichlet energy in the first term of the action, giving rise to a GFF which we take to have Neumann boundary conditions. The GFF is weighted by its bulk GMC mass $M^\gamma(\mathbb{H})$ and its boundary GMC mass $M_\partial^\gamma(\mathbb{R})$, where the boundary GMC is formally

$$dM_\partial^\gamma(x) = e^{\frac{\gamma}{2}X(x) - \frac{\gamma^2}{8}\mathbb{E}[X(x)^2]} g(x)^{1/2} dx$$

and is obtained via a regularisation of the field using semi-circle averages.

As in the sphere case, the observables are the vertex operators $V_\alpha(z)$ for insertions $z \in \mathbb{H}$. The main difference is that one can consider insertions on the boundary, which we formally write

$$B_\beta(x) := e^{\frac{\beta}{2}X(x)}$$

for $x \in \mathbb{R}$ and β in a range to be determined. The correlation functions $\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}(x_j)\rangle$ exist if and only if the Seiberg bounds are satisfied, which in this context are given by

$$\begin{gathered} \sigma := \sum_{i=1}^N \frac{\alpha_i}{Q} + \sum_{j=1}^M \frac{\beta_j}{2Q} - 1 > 0, \\ \forall i,\ \alpha_i < Q \quad \text{and} \quad \beta_j < Q. \end{gathered} \tag{3.5.2}$$

If these are satisfied, the correlation function has the following form[6] [HRV18]:

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}(x_j) \right\rangle = 2e^{C(\mathbf{z},\mathbf{x})} \int_{\mathbb{R}} e^{Q\sigma c} \mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\mathbb{H}} e^{\gamma H} dM^\gamma - \mu_\partial e^{\frac{\gamma}{2}c}\int_{\mathbb{R}} e^{\frac{\gamma}{2}H} dM_\partial^\gamma\right)\right] dc \tag{3.5.3}$$

[5] We omit the Ricci and geodesic curvature terms.

[6] We chose the prefactor 2 so that the asymptotic behaviour of the bulk 1-point function with $\mu = 0$ coincides with that of [FZZ00, equation (2.24)].

where H and $C(\mathbf{z},\mathbf{x})$ are the functions defined by

$$\begin{aligned} H &= \sum_{i=1}^{N} \alpha_i G(z_i,\cdot) + \sum_{j=1}^{M} \frac{\beta_j}{2} G(x_j,\cdot) \\ C(\mathbf{z},\mathbf{x}) &= \sum_{i<i'} \alpha_i \alpha_{i'} G(z_i, z_i') + \sum_{i,j} \frac{\alpha_i \beta_j}{2} G(z_i, x_j) + \sum_{j<j'} \frac{\beta_j \beta_{j'}}{4} G(x_j, x_j') \end{aligned} \tag{3.5.4}$$

with G being Green's function with Neumann boundary conditions on $(\mathbb{H}, g)$. Notice that the usual change of variable $u = e^{\gamma c}$ does not give a nicer expression in this case since the exponential term in the expectation is quadratic in $e^{\frac{\gamma}{2}c}$.

Correlation functions are conformally covariant, and if $\psi : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation, then (recall that $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$)

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(\psi(z_i)) \prod_{j=1}^{M} B_{\beta_j}(\psi(x_j)) \right\rangle = \prod_{i=1}^{N} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \prod_{j=1}^{M} |\psi'(x_j)|^{-\Delta_{\beta_j}} \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B_{\beta_j}(x_j) \right\rangle.$$

Möbius transforms of $\mathbb{H}$ have three real parameters, so when the location of the insertions have less than (or exactly) three real parameters, the correlation functions are determined by conformal invariance, and we have the following structure constants

1. Bulk-boundary two-point function

$$\langle V_\alpha(z) B_\beta(x) \rangle = \frac{R(\alpha,\beta)}{|z-\bar{z}|^{2\Delta_\alpha - \Delta_\beta} |z-x|^{2\Delta_\beta}}.$$

   As a special case of this equation for $\beta = 0$, we have the bulk one-point function

$$\langle V_\alpha(z) \rangle = \frac{U(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}}. \tag{3.5.5}$$

2. Boundary three-point function

$$\langle B_{\beta_1}(x_1) B_{\beta_2}(x_2) B_{\beta_3}(x_3) \rangle = \frac{c(\beta_1,\beta_2,\beta_3)}{|x_1-x_2|^{\Delta_{\beta_1}+\Delta_{\beta_2}-\Delta_{\beta_3}} |x_2-x_3|^{\Delta_{\beta_2}+\Delta_{\beta_3}-\Delta_{\beta_1}} |x_3-x_1|^{\Delta_{\beta_3}+\Delta_{\beta_1}-\Delta_{\beta_2}}}. \tag{3.5.6}$$

Remark 3.5.1. *There is also a definition for a boundary two-point function, which we omit here since we will not be needing it for the purpose of this paper. Let us just mention that this object is to the reflection coefficient of [KRV17] what the boundary three-point function is to the DOZZ formula.*

The above structure constants are to be understood as meromorphic functions of the parameters and they arise naturally in the bootstrap formalism. Physicists have conjectured exact formulae for the values of these structure constants [FZZ00, PT02], and there are works in progress by Gwynne and Remy establishing the validity of (3.5.5) and Remy and Zhu addressing (3.5.6).

3.5.2 Main results

The cases we treat are the fusion on two boundary-insertions, the absorption of a bulk-insertion on the boundary and the fusion of two bulk-insertions.

Theorem 3.5.2 (Boundary four-point). *Let $(\beta_1, \beta_2, \beta_3, \beta_4)$ satisfying the Seiberg bounds and suppose that $\beta_3 + \beta_4 > Q$. Then the following asymptotic holds:*

1. Supercritical case

   If $\beta_1 + \beta_2 > Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} \frac{1}{4\sqrt{\pi}} \frac{|x|^{\frac{Q^2}{4}-\Delta_{\beta_1}-\Delta_{\beta_2}}}{\log^{3/2}\frac{1}{|x|}} \partial_3 c(\beta_1, \beta_2, Q)\partial_1 c(Q, \beta_3, \beta_4).$$

2. Critical case

   If $\beta_1 + \beta_2 = Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} -\frac{1}{\sqrt{\pi}} \frac{|x|^{-\frac{1}{2}\beta_1\beta_2}}{\log^{1/2}\frac{1}{|x|}} \partial_1 c(Q, \beta_3, \beta_4). \tag{3.5.7}$$

3. Subcritical case

   If $\beta_1 + \beta_2 < Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} |x|^{-\frac{1}{2}\beta_1\beta_2} c(\beta_1 + \beta_2, \beta_3, \beta_4).$$

The next theorem is about the fusion in the bulk two-point function.

Theorem 3.5.3 (Bulk two-point: Fusion). *Let $(\alpha_1, \alpha_2, \beta)$ satisfying the Seiberg bounds. Then the following asymptotics hold:*

1. *If $\beta = 0$, then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)\rangle \underset{z\to 0}{\sim} -\frac{2^{-\alpha_1\alpha_2}}{\sqrt{2\pi}} \frac{|z|^{2(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}}{\log^{1/2}\frac{1}{|z|}} \partial_3 C_\gamma(\alpha_1, \alpha_2, Q). \tag{3.5.8}$$

2. *If $\beta > 0$, then*

   (a) Supercritical case

If $\alpha_1 + \alpha_2 > Q$, *then*

$$\begin{aligned}&\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \\ &\underset{z\to 0}{\sim} \frac{2^{\Delta_\beta - \frac{Q^2}{2} - \alpha_1\alpha_2}}{4\sqrt{2\pi}} \frac{|z|^{2(\frac{Q^2}{4} - \Delta_{\alpha_1} - \Delta_{\alpha_2})}}{\log^{3/2}\frac{1}{|z|}} \partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 R(Q,\beta).\end{aligned}$$

(b) Critical case

If $\alpha_1 + \alpha_2 = Q$, *then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \underset{z\to 0}{\sim} -\frac{2^{\Delta_\beta - \frac{Q^2}{2} - \alpha_1\alpha_2}}{\sqrt{2\pi}} \frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}} \partial_1 R(Q,\beta).$$

(c) Subcritical case

If $\alpha_1 + \alpha_2 < Q$, *then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \underset{z\to 0}{\sim} 2^{\Delta_\beta - \frac{Q^2}{2} - \alpha_1\alpha_2}|z|^{-\alpha_1\alpha_2} R(\alpha_1+\alpha_2,\beta). \tag{3.5.9}$$

Another interesting limit of the bulk two-point function is sending one insertion to the boundary.

Theorem 3.5.4 (Bulk two-point: Absorption)**.** *Let* (α_1,α_2) *satisfying the Seiberg bounds, and suppose* $\alpha_1 > \frac{Q}{2}$. *Then the following asymptotic holds:*

1. Supercritical case

If $\alpha_2 > \frac{Q}{2}$, *then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} \frac{2^{2(\frac{Q^2}{4} - \Delta_{\alpha_1} - \Delta_{\alpha_2})}}{4\sqrt{\pi}} \frac{|z|^{(\alpha_2 - \frac{Q}{2})^2}}{\log^{3/2}\frac{1}{|z|}} \partial_2 R(\alpha_1,Q)\partial_2 R(\alpha_2,Q).$$

2. Critical case

If $\alpha_2 = \frac{Q}{2}$, *then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} -\frac{2^{\frac{Q^2}{2} - 2\Delta_{\alpha_1}}}{\sqrt{\pi}\log^{1/2}\frac{1}{|z|}} \partial_2 R(\alpha_1,Q).$$

3. Subcritical case

If $\alpha_2 < \frac{Q}{2}$, *then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} \frac{R(\alpha_1,2\alpha_2)}{2^{2\Delta_{\alpha_1} - \Delta_{2\alpha_2}}}. \tag{3.5.10}$$

We now turn to the bulk-boundary three-point function $\langle V_\alpha(z)B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$. There is not much to say about the merging of the bulk insertion with a boundary insertion since for all $r>0$ and $\theta\in(0,\pi)$, the correlation function $\langle V_\alpha(re^{i\theta})B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$ is deduced from $\langle V_\alpha(e^{i\theta})B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$ by scaling. The non-trivial parameter we can vary is θ, and the limit $\theta\to 0$ corresponds to the absorption of an bulk insertion on a boundary point which is not an insertion. Thus we will study the correlation function $\langle V_\alpha(z)B_{\beta_1}(1)B_{\beta_2}(\infty)\rangle$ in the limit $z\to 0$. Notice that by Möbius invariance, this is the same as studying the function $\langle V_\alpha(i)B_{\beta_1}(0)B_{\beta_2}(x)\rangle$ in the limit $x\to 0$, i.e. merging the two boundary insertions.

Theorem 3.5.5 (Bulk-boundary three-point)**.** *Let (α,β_1,β_2) satisfying the Seiberg bounds, and assume that $\beta_1+\beta_2>\frac{Q}{2}$. Then the following asymptotic holds*

1. Supercritical case

   If $\alpha>\frac{Q}{2}$, then

$$\langle V_\alpha(z)B_{\beta_1}(1)B_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} \frac{2^{\frac{Q^2}{4}-2\Delta_\alpha}}{4\sqrt{\pi}}\frac{|z|^{(\alpha-\frac{Q}{2})^2}}{\log^{3/2}\frac{1}{|z|}}\partial_2 R(\alpha,Q)\partial_1 c(Q,\beta_1,\beta_2).$$

2. Critical case

   If $\alpha=\frac{Q}{2}$, then

$$\langle V_\alpha(z)B_{\beta_1}(1)B_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} -\frac{1}{\sqrt{\pi}\log^{1/2}\frac{1}{|z|}}\partial_1 c(Q,\beta_1,\beta_2).$$

3. Subcritical case

   If $\alpha<\frac{Q}{2}$, then

$$\langle V_\alpha(z)V_{\beta_1}(1)V_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} c(2\alpha,\beta_1,\beta_2).$$

Remark 3.5.6. *More generally, the fusion rules in the supercritical case are the following:*

1. *Fusion of boundary-boundary (β_1,β_2)-insertions produces a boundary three-point function $\partial_3 c(\beta_1,\beta_2,Q)$.*

2. *Absorption of a bulk α-insertion produces a bulk-boundary function $\partial_2 R(\alpha,Q)$.*

3. *Fusion of bulk-bulk (α_1,α_2)-insertions produces a DOZZ formula $\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)$.*

This rule, as well as the rate functions of the above theorems, can be used to compute the asymptotic behaviour of all correlation functions upon fusion of insertions, and express the limit with a lower order correlation function.

We haven't said anything about the fusion of bulk-boundary insertions. This is because it can be seen as a two-step procedure of first absorbing the bulk insertion into the boundary and then merging the boundary insertions. Hence the operation does not produce a structure constant. As an example, consider the correlation function $\langle V_\alpha(z)B_{\beta_1}(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle$ in the limit $z\to 0$, for $(\alpha,\beta_1,\beta_2,\beta_3)$ satisfying the Seiberg bounds, and suppose that both $\beta_3+\beta_4>Q$ and $2\alpha+\beta_1>Q$, so that we are in the supercritical case. Then the asymptotic is given by

$$\begin{aligned}&\langle V_\alpha(z)B_{\beta_1}(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle\\&\underset{z\to0}{\sim}\frac{1}{4\sqrt{\pi}}\frac{|z|^{(\alpha-\frac{Q}{2})^2-\alpha\beta_1}}{\log^{3/2}\frac{1}{|z|}}\frac{\partial}{\partial\beta}\langle V_\alpha(i)B_{\beta_1}(0)B_\beta(\infty)\rangle_{|\beta=Q}\partial_1 c(Q,\beta_2,\beta_3).\end{aligned}$$

Remark 3.5.7. *Even though the correlation functions can no longer be expressed in terms of negative moments of GMC (unless $\mu\mu_\partial=0$), it is still possible to give probabilistic representations of the renormalised constants in the aforementioned theorems by performing the same partitioning-of-probability-space procedure on*

$$\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\mathbb{H}}e^{\gamma H}dM^\gamma-\mu_\partial e^{\frac{\gamma}{2}c}\int_{\mathbb{R}}e^{\frac{\gamma}{2}H}dM_\partial^\gamma\right)\right].$$

as we did in Section 3.4. We omit the details here.

We now turn to proving Theorems 3.5.2, 3.5.3, 3.5.4 and 3.5.5. We only deal with Theorems 3.5.2 and 3.5.3 since the other cases are similar.

Subcritical cases follow from dominated convergence so we won't treat them. The rest of the proofs are very similar to that of Theorem 3.1.1 so we will be brief.

Proof of Theorem 3.5.2. The setting is the upper-half plane $\mathbb{H}$ equipped with the metric $g(z)=4|z|_+^{-4}$. We use the same procedure as for the sphere and apply the conformal change of coordinate $\psi: z\mapsto e^{-z/2}$ from the infinite strip $\mathcal{S}:=\mathbb{R}\times(0,2\pi)$ to $\mathbb{H}$. Then Green's function on the strip is given by the even part of Green's function on the cylinder, i.e. if X is a GFF on $\mathbb{R}_s\times(0,2\pi)_\theta$, we have (recall (3.2.2))

$$\begin{aligned}\mathbb{E}[X(s,\theta)X(s',\theta')]&=G(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},\frac{\theta'}{2})+G(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},-\frac{\theta'}{2})\\&=(|s|\wedge|s'|)1_{ss'\geq0}+H(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},\frac{\theta'}{2})+H(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},-\frac{\theta'}{2})\\&=(|s|\wedge|s'|)1_{ss'\geq0}+G(0,0,\frac{s'-s}{2},\frac{\theta'-\theta}{2})+G(0,0,\frac{s'-s}{2},\frac{\theta'+\theta}{2}).\end{aligned}\tag{3.5.11}$$

Hence the field can be decomposed into the independent sum $X=B+Y$ where $(B_s)_{s\in\mathbb{R}}$ is standard two-sided Brownian motion and Y is a log-correlated field whose covariance kernel is given by the sum of G functions on the right-hand side of the previous equation. It is also clear from the definition that the law of Y is translation invariant. The pullback measure of g on the strip is $g_\psi(s,\theta)=e^{-|s|}$ so we can take the GMC measure of Y with respect to Lebesgue measure on $\mathcal{S}$ and take the drifted process $B_s - \frac{Q}{2}|s|$ for the radial part of the GFF.

First we have to explain how to make sense of boundary (derivative) Q-insertions. A boundary insertion with momentum β at ∞ (on the strip) amounts in adding a positive drift $\frac{\beta}{2}$ to the radial process (on the positive real line), so the total drift vanishes when $\beta = Q$. For $t > 0$, define $\mathbb{H}_t := \mathbb{H}\backslash(e^{-t/2}\mathbb{D})$ (resp. $\mathbb{R}_t := \mathbb{R}\backslash(-e^{-t/2}, e^{-t/2})$) and $\langle B_Q(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle_t$ the correlation function where we integrate the bulk (resp. boundary) GMC measure of (3.5.3) on $\mathbb{H}_t$ (resp. $\mathbb{R}_t$) instead of $\mathbb{H}$ (resp. $\mathbb{R}$). Viewed in the strip, this is the same as taking $\mathcal{S}_t := (-\infty, t) \times (0, 2\pi)$ and $(-\infty, t) \times \{0, 2\pi\}$ as domains of integration for the bulk and boundary measures.

For fixed $b > 0$, we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0\leq s\leq t} B_s \leq b\right) &\underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}}b, \\ \mathbb{P}\left(\sup_{t\geq 0} B_s + \frac{1}{2}(\beta - Q)s \leq b\right) &\underset{\beta\to Q^-}{\sim} (Q-\beta)b \end{aligned} \tag{3.5.12}$$

and so by previous arguments we have

$$\lim_{t\to\infty}\sqrt{\frac{\pi t}{2}}\langle B_Q(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle_t = \lim_{\beta\to Q^-}\frac{1}{Q-\beta}\langle B_\beta(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle = -\partial_1 c(Q, \beta_2, \beta_3).$$

The critical case (3.5.7) follows easily from this equality.

Now we turn to the supercritical case. We write $t := 2\log\frac{1}{|x|}$. The radial process has a positive drift $\frac{1}{2}(\beta_1 + \beta_2 - Q)$ in $(0, t)$, which we kill by Cameron-Martin's theorem (recall (3.3.2), yielding the Radon-Nikodym derivative $e^{\frac{1}{2}(\beta_1+\beta_2-Q)B_t - \frac{1}{8}(\beta_1+\beta_2-Q)^2 t}$. This accounts for the polynomial rate in $|x|$.

Similarly as in Figure 3.4, we condition on value of the process at time t and introduce $B_t = \sqrt{t}\delta$ with $\delta \sim \mathcal{N}(0, 1)$ independent of everything. Thus the process in $[0, t]$ is the sum of a random drift $\frac{\delta}{\sqrt{t}}$ and an independent Brownian bridge in $[0, t]$ (see Figure 3.5). Conditioning the Brownian bridge in $(0, t)$ to stay below b, we get a contribution of $\sqrt{\frac{2}{\pi}}t^{-3/2} = \frac{1}{2\sqrt{2\pi}\log^{3/2}\frac{1}{|x|}}$. Taking $t \to \infty$ then $b \to \infty$, the limiting integral on the left is a strip with a β_4-insertions at $-\infty$, a β_3-insertion at 0 and a (derivative) Q-insertion at $+\infty$ (see Figure 3.5), hence the limit is $-\frac{1}{2}\partial_1 c(Q, \beta_3, \beta_4)$ (recall the prefactor 2 in the definition of (3.5.3)). Similarly the limiting integral on the left is $-\frac{1}{2}\partial_1 c(\beta_1, \beta_2, Q)$, which yields the result.

□

Proof of Theorem 3.5.3. In this proof, we use the flat disc $(\mathbb{D}, dz)$ as set-up, which is mapped to the semi-infinite cylinder $\mathcal{C}_+ = \mathbb{R}_+ \times \mathbb{S}^1$ equipped with the metric $g(s, \theta) = e^{-2s}$ under the conformal transformation $z \mapsto e^{-z}$. So the GFF decomposes as the sum of a

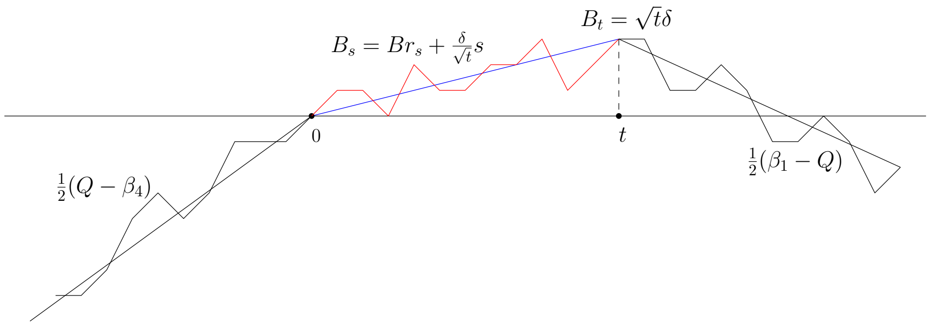

Figure 3.5: The radial process on the strip in $[0,t]$ is the sum of a Brownian bridge (red) and a random independent drift (blue).

drifted Brownian motion $(B_s - Qs)_{s\geq 0}$ and an independent lateral noise Y from which we take the GMC measure with respect to Lebesgue measure.

We treat the case $\beta > 0$ and $\alpha_1 + \alpha_2 > Q$, the others being similar. Let $t := \log \frac{1}{|z|}$. With the presence of the insertions, the radial part has a positive drift $\alpha_1 + \alpha_2 - Q$ in $(0,t)$ and negative drift $\alpha_1 - Q$ in (t,∞). Killing the drift in $(0,t)$ with Cameron-Martin's theorem gives the exponent in $|z|$. Conditioning on the value of $B_t = \sqrt{t}\delta$ and conditioning the Brownian bridge not to exceed some $b > 0$ gives a prefactor of $\sqrt{\frac{2}{\pi}}t^{-3/2}$. Taking $t \to \infty$ then $b \to \infty$, we find that the integral on the right is an infinite cylinder with insertions (α_1, α_2, Q) at $(+\infty, 0, -\infty)$, so its value is $-\frac{1}{4}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)$ (the lateral noise is close to the one used before in this region and can be dealt with using Kahane's convexity inequality). On the other hand, the integral on the left is a semi-infinite cylinder with a Q-insertion at ∞ and a β-insertion on the boundary, so its value is $-\frac{1}{4}\frac{\partial}{\partial\alpha}\langle V_\alpha(i)B_\beta(0)\rangle_{|\alpha=Q}$. $\square$

3.5.3 Links with random planar maps

The above results can be interpreted with respect to the KPZ conjecture on random planar maps with the topology of the disc. For concreteness, let $\mathcal{T}_{n,m}$ be the set of triangulations of the disc with n internal vertices and $m+2$ boundary vertices, with two marked vertices (one internal and one on the boundary). Then it is known [AS03] that there exists $\mu^c, \mu^c_\partial > 0$ such that

$$\#\mathcal{T}_{n,m} \asymp e^{\mu^c n} e^{\mu^c_\partial m} m^{1/2} n^{-5/2}.$$

We suppose that for a triangulation $(\mathbf{t},\mathbf{z},\mathbf{x})$, we have conformal mapped $\mathbf{t}$ to $\mathbf{H}$ (in the manner of section 3.1.4) and that $\mathbf{z}$ is mapped to i and $\mathbf{x}$ is mapped to 0. For each such triangulation and $a > 0$, we can construct measures $\nu^{\mathbf{t},a}$ (resp. $\nu^{\mathbf{t},a}_\partial$) giving mass a^2 (resp. a) to each triangle (resp. each boundary edge). Now we let $\mu := (1+a^2)\mu^c$ and

$\mu_\partial := (1+a)\mu_\partial^c$, and sample the triangulations at random with the probability measure

$$\mathbb{P}_a(\mathbf{t},\mathbf{z},\mathbf{x}) = \frac{1}{Z_a} e^{-\mu|\mathbf{t}|} e^{-\mu_\partial \ell(\mathbf{t})}$$

where Z_a is the normalising constant and $\ell(\mathbf{t})$ is the boundary length of $\mathbf{t}$. Additionally we choose the internal marked vertex uniformly in the internal vertices of $\mathbf{t}$ and similarly for the boundary marked vertex.

It is conjectured [HRV18] that the pair of random measures $(\nu^{\mathbf{t},a}, \nu_\partial^{\mathbf{t},a})$ converges in distribution to a pair of random measures on $(\mathbb{D}, \partial\mathbb{D})$, and the limit (ν, ν_∂) should be given by (some form of) LQFT on the disc. In particular, it should be the case that for all measurable sets $A \subset \mathbb{H}, B \subset \mathbb{R}$,

$$\mathbb{E}\left[\frac{\nu(A)}{\nu(\mathbb{H})}\right] = \int_A f_{\sqrt{\frac{8}{3}},\mu^c,\mu_\partial^c}(z) d^2 z$$
$$\mathbb{E}\left[\frac{\nu_\partial(B)}{\nu_\partial(\mathbb{R})}\right] = \int_B \lambda_{\sqrt{\frac{8}{3}},\mu^c,\mu_\partial^c}(x) dx$$

where we define for all $\gamma \in (0,2)$ and $\mu, \mu_\partial > 0$,

$$\begin{aligned} f_{\gamma,\mu,\mu_\partial}(z) &:= \frac{1}{Z}\langle V_\gamma(z) V_\gamma(i) B_\gamma(0)\rangle \\ \lambda_{\gamma,\mu,\mu_\partial}(x) &:= \frac{1}{Z_\partial}\langle B_\gamma(x) V_\gamma(i) B_\gamma(0)\rangle \end{aligned} \tag{3.5.13}$$

where Z, Z_∂ are normalising constants whose values are discussed in Appendix 3.C.

Similarly to the discussion of section 3.1.4, the result of Theorems 3.5.5 and 3.5.3 gives precise estimates on the expected density of vertices in different settings: internal or boundary vertices around the marked point on the boundary, internal vertices around the internal marked point, and internal vertices around the boundary.

Finally, we mention that one can formulate other conjectures involving different values of γ (e.g. by weighting the measure $\mathbb{P}_a$ by some statistical mechanics model), μ and μ_∂ (e.g. by considering other types of maps).

Appendix 3.A The DOZZ formula

The DOZZ formula is the expression of the 3-point correlation function on the sphere $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2}$. The formula reads

$$\begin{aligned} C_\gamma(\alpha_1,\alpha_2,\alpha_3) = & \left(\pi\mu\left(\frac{\gamma}{2}\right)^{2-\frac{\gamma^2}{2}}\frac{\Gamma(\gamma^2/4)}{\Gamma(1-\gamma^2/4)}\right)^{-\frac{\overline{\alpha}-2Q}{\gamma}} \\ & \times\frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}-2Q}{2}\right)\Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2}-\alpha_1\right)\Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2}-\alpha_2\right)\Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2}-\alpha_3\right)} \end{aligned} \tag{3.A.1}$$

where $\overline{\alpha}=\alpha_1+\alpha_2+\alpha_3$ and $\Upsilon_{\frac{\gamma}{2}}$ is Zamolodchikov's special function. It has the following integral representation for $\Re z\in(0,Q)$

$$\log\Upsilon(z)=\int_0^\infty\left(\left(\frac{Q}{2}-z\right)^2e^{-t}-\frac{\sinh^2\left(\left(\frac{Q}{2}-z\right)\frac{t}{2}\right)}{\sinh\left(\frac{\gamma t}{4}\right)\sinh\left(\frac{t}{\gamma}\right)}\right)\frac{dt}{t}$$

and it is extended holomorphically to $\mathbb{C}$.

It satisfies the functional relation $\Upsilon(Q-z)=\Upsilon(z)$ and it has a simple zero at 0 if $\gamma^2\in\mathbb{R}\setminus\mathbb{Q}$[7], so it has a simple zero at Q too and $\Upsilon'(Q)=-\Upsilon'(0)\neq 0$.

Let us introduce the notation

$$\overline{C}_\gamma(\alpha_1,\alpha_2,\alpha_3)=\frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\overline{\alpha}}{2}-Q)\Upsilon(\frac{\overline{\alpha}}{2}-\alpha_1)\Upsilon(\frac{\overline{\alpha}}{2}-\alpha_2)\Upsilon(\frac{\overline{\alpha}}{2}-\alpha_3)}.$$

Now we assume $\alpha_1+\alpha_2=Q$ and $\alpha_3=Q-iP$ and show the limit (3.1.11). Then $\overline{\alpha}=2Q-iP$ and

$$\overline{C}_\gamma(\alpha_1,Q-\alpha_1,Q-iP)=\frac{\Upsilon'(0)\Upsilon(\alpha_1)^2\Upsilon(iP)}{\Upsilon(-\frac{iP}{2})\Upsilon(\alpha_1+\frac{iP}{2})\Upsilon(\alpha_1-\frac{iP}{2})\Upsilon(\frac{iP}{2})}\underset{P\to0}{\sim}\frac{4i}{P}.$$

The product of DOZZs appearing in the bootstrap equation (3.1.7) becomes

$$\begin{aligned} &\overline{C}_\gamma(\alpha_1,\alpha_2,Q-iP)\overline{C}_\gamma(Q+iP,\alpha_3,\alpha_4) \\ &\underset{P\to0}{\sim}\frac{4\Upsilon'(0)^2\Upsilon(\alpha_3)\Upsilon(\alpha_4)}{\Upsilon(\frac{\alpha_3+\alpha_4-Q+iP}{2})\Upsilon(\frac{\alpha_3+\alpha_4-Q-iP}{2})\Upsilon(\frac{\alpha_4+Q+iP-\alpha_3}{2})\Upsilon(\frac{\alpha_3+Q+iP-\alpha_4}{2})} \\ &\underset{P\to0}{\sim}-4\partial_1\overline{C}_\gamma(Q,\alpha_3,\alpha_4). \end{aligned} \tag{3.A.2}$$

Notice that $2Q-(\alpha_1+\alpha_2+Q-iP)\underset{P\to0}{\to}0$, so the prefactor in the DOZZ formula with

[7] This is not really a restriction since the theory is continuous in γ

Liouville momenta $(\alpha_1, Q-\alpha_1, Q-iP)$ is simply 1 in this limit. Hence

$$\lim_{P\to 0} C_\gamma(\alpha_1, Q-\alpha_1, Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4) = 4\partial_1 C_\gamma(Q,\alpha_3,\alpha_4).$$

Appendix 3.B Conical singularities

Here we reproduce [Bav18, Appendix B] for the commodity of the reader.

We study the effect of a change of measure with respect to the Liouville field. Let X be a GFF on $\mathbb{S}^2$ with some background metric g and dM_g^γ be the associated chaos measure (regularised in g). Let $\omega \in H_0^1$ be a function such that $e^{\frac{Q}{2}\omega} \in L^1(dM_g^\gamma)$. Let $\hat{g} := e^{\omega} g$ and $dM_{\hat{g}}^\gamma$ be the chaos of X regularised in $\hat{g}$. Then for all $\kappa > 0$, applying successively Girsanov's theorem and conformal covariance, we find

$$\mathbb{E}\left[e^{\langle X, \frac{Q}{2}\omega\rangle_\nabla - \frac{Q^2}{8}||\omega||_\nabla^2} M_g^\gamma(\mathbb{S}^2)^{-\kappa}\right] = \mathbb{E}\left[\left(\int_{\mathbb{S}^2} e^{\frac{\gamma Q}{2}\omega} dM_g^\gamma\right)^{-\kappa}\right] = \mathbb{E}\left[M_{\hat{g}}^\gamma(\mathbb{S}^2)^{-\kappa}\right] \quad (3.B.1)$$

In particular, the vertex operator which is formally written $V_\alpha(z) = e^{\alpha X(z) - \frac{\alpha^2}{2}\mathbb{E}[X(z)^2]}$ is a special case of the previous setting with $\omega = \frac{2\alpha}{Q} G(z,\cdot)$. Hence, after regularising, we find that adding a vertex operator is the same as conformally multiplying the metric by Green's function, i.e. we have $\hat{g} = e^{\frac{2\alpha}{Q}G(z,\cdot)} g$. Hence the metric behaves like $|x-z|^{-\frac{2\alpha}{Q}}$ near 0 so it has a conical singularity of order α/Q.

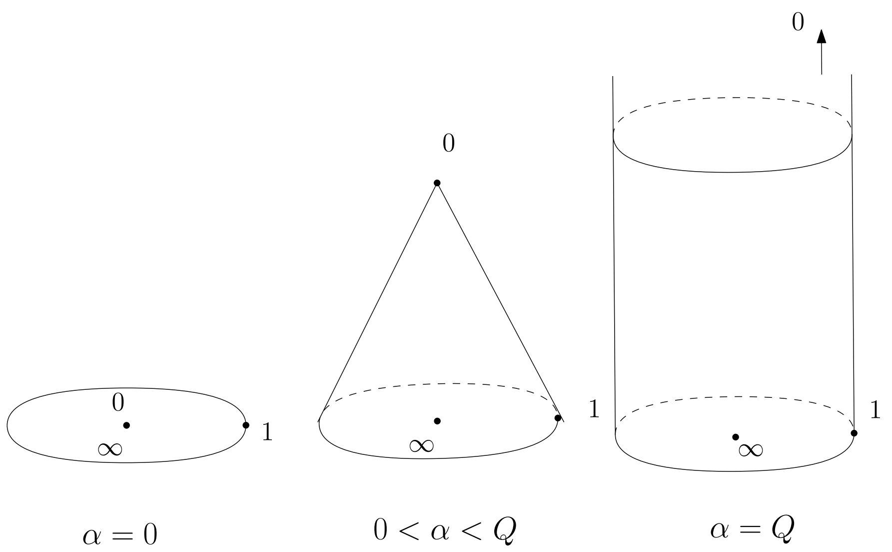

Figure 3.6: The effect of the the vertex operator $V_\alpha(0)$ in the crêpe metric.

If $\alpha = Q$, the singularity is no longer integrable, so the volume is infinite and the surface has a semi-infinite cylinder. Loosely, we will refer to this situation as a cusp, even though the hyperbolic cusp has finite volume because of the extra log-correction in the metric:

$$\log \hat{g}(z+h) = -2\log|h| - 2\log\log\frac{1}{|h|} + O(1).$$

The reason for this abuse of terminology is that we are interested in GMC measure. Indeed, suppose $z = 0$ in the sphere coordinates. By conformal covariance, if we use the cylinder coordinates, the log-correction term is the same as shifting the radial part of the GFF from the Brownian motion $(B_s)_{s\geq 0}$ to $(B_s - Q\log(1+s))_{s\geq 0}$. Up to time t, this corresponds to a change of measure given by the exponential martingale $e^{-Q\int_0^t \frac{dB_s}{1+s} - \frac{Q^2}{2}\int_0^t \frac{1}{(1+s)^2}ds}$, which is uniformly integrable since $\int_0^\infty \frac{1}{(1+t)^2}dt < \infty$. So the new field is absolutely continuous with respect to the old one, meaning that GMC does not make a difference between a Euclidean cylinder and a hyperbolic cusp.

Another way to see this is to look at the curvature, which reads in the distributional sense

$$K_{\hat{g}} = e^{-\frac{2\alpha}{Q}G(z,\cdot)}\left(K_g + \frac{4\pi\alpha}{Q}\left(\delta_z - \frac{1}{\mathrm{Vol}_g(\mathbb{S}^2)}\right)\right)$$

where $\mathrm{Vol}_g(\mathbb{S}^2)$ is the volume of the sphere in the metric g. Thus the metric has an atom of curvature at z, meaning it has a conical singularity.

Of course, when $\alpha = Q$, the singularity is no longer integrable and the metric looks like a semi-infinite (flat) cylinder near 0.

Appendix 3.C The normalising constant in (3.1.22) and (3.5.13)

We present the computation of the normalising constant for $f_{\gamma,\mu}$ (in a more general setting). The idea is that integrating over the location of a γ-insertion is the same as differentiating with respect to the cosmological constant. We present the main steps and leave the details to the reader.

Let $N \geq 3$ and $z_1, ..., z_N \in \widehat{\mathbb{C}}$ pairwise disjoint and $(\alpha_1, ..., \alpha_N)$ satisfying the Seiberg bounds. For notational convenience, we write $\mathcal{G}(x) := \sum_{i=1}^N \alpha_i G(z_i, x)$ and as usual $\sigma = \sum_{i=1}^N \frac{\alpha_i}{Q} - 2$.

Using Cameron-Martin's theorem to go from the second to third line we find

$$\begin{aligned}
&\frac{1}{2}e^{-\sum\limits_{1\leq i<j}\alpha_i\alpha_j G(z_i,z_j)}\int_{\widehat{\mathbb{C}}}\left\langle V_\gamma(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle dz\\
&=\int_{\widehat{\mathbb{C}}}e^{\gamma\mathcal{G}(z)}\int_{\mathbb{R}}e^{(Q(\sigma+\frac{\gamma}{Q})c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}M^\gamma\left(e^{\gamma(\mathcal{G}+\gamma G(z,\cdot))}\right)\right)\right]dcd^2z\\
&=\mathbb{E}\left[\int_{\mathbb{R}}e^{Q\sigma c}e^{\gamma c}M^\gamma\left(e^{\gamma\mathcal{G}}\right)\exp\left(-\mu e^{\gamma c}M^\gamma\left(e^{\gamma\mathcal{G}}\right)\right)dc\right]\\
&=-\frac{1}{2}e^{-\sum\limits_{1\leq i<j}\alpha_i\alpha_j G(z_i,z_j)}\frac{\partial}{\partial\mu}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle
\end{aligned}\tag{3.C.1}$$

so that in the end

$$\int_{\widehat{\mathbb{C}}}\left\langle V_\gamma(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle d^2z=-\frac{\partial}{\partial\mu}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle=\frac{Q\sigma}{\gamma\mu}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle\tag{3.C.2}$$

where we simply used that $\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\rangle$ is equal to $\mu^{-\frac{Q\sigma}{\gamma}}$ times some quantity independent of μ. In particular this yields (3.1.22) for $N=3$ and $(\alpha_1,\alpha_2,\alpha_3)=(\gamma,\gamma,\gamma)$.

Similarly, in the disc case, we find that for $(\alpha_1,...,\alpha_N,\beta_1,...,\beta_M)$ satisfying the Seiberg bounds, we have

$$\int_{\mathbb{H}}\left\langle V_\gamma(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}(x_j)\right\rangle d^2z=-\frac{\partial}{\partial\mu}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)B_{\beta_j}(x_j)\right\rangle$$

and

$$\int_{\mathbb{R}}\left\langle B_\gamma(x)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}(x_j)\right\rangle dx=-\frac{\partial}{\partial\mu_\partial}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)B_{\beta_j}(x_j)\right\rangle.$$

In general, this does not simplify as nicely as (3.C.2) but if e.g. $\mu=0$, then we have for instance

$$\int_{\mathbb{R}}\langle B_\gamma(x)V_\gamma(i)B_\gamma(0)\rangle dx=\frac{3\gamma-2Q}{2\gamma\mu}R(\gamma,\gamma).$$

Chapter 4

Tail Asymptotics of General Gaussian Multiplicative Chaos

Abstract. In this chapter we study the tail probability of the mass of Gaussian multiplicative chaos. With the novel use of a Tauberian argument and Goldie's implicit renewal theorem, we provide a unified approach to general log-correlated Gaussian fields in arbitrary dimension and derive precise first order asymptotics of the tail probability, resolving a conjecture of Rhodes and Vargas. The leading order is described by a universal constant that captures the generic property of Gaussian multiplicative chaos, and may be seen as the analogue of the Liouville unit volume reflection coefficients in higher dimensions.

4.1 Introduction

Gaussian multiplicative chaos (GMC) was first constructed by Kahane [Kah85] in an attempt to provide a mathematical framework for the Kolmogorov-Obukhov-Mandelbrot model of energy dissipation in turbulence. The theory of (subcritical) GMC consists of defining and studying, for each $\gamma \in (0, \sqrt{2d})$, the random measure

$$M_\gamma(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}dx, \tag{4.1.1}$$

where $X(\cdot)$ is a (centred) log-correlated Gaussian field on some domain $D \subset \mathbb{R}^d$. The expression (4.1.1) is formal because $X(\cdot)$ is not defined pointwise; instead it is only a random generalised function. It is now, however, well understood that M_γ may be defined via a limiting procedure of the form

$$M_\gamma(dx) = \lim_{\epsilon \to 0} M_{\gamma,\epsilon}(dx) = \lim_{\epsilon \to 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}dx$$

where $X_\epsilon(\cdot)$ is some suitable sequence of smooth Gaussian fields that converges to $X(\cdot)$ as $\epsilon \to 0$. We refer the readers to e.g. [Ber17] for more details about the construction.

In recent years the theory of GMC has attracted a lot of attention in the mathematics and physics communities due to its wide array of applications – it plays a central role in random planar geometry [DMS14, DS11] and the mathematical formulation of Liouville conformal field theory (LCFT) [DKRV16], appears as a universal limit in other areas such as random matrix theory [Web15, BWW18, LOS18, NSW18], and is even used as a model for Riemann zeta function in probabilistic number theory [SW16] or stochastic volatility in quantitative finance [DRV12].

In spite of the importance of the theory, not much is known about the distributional properties of GMC. For instance, given a bounded open set $A \subset D$, one may ask what the exact distribution of $M_\gamma(A)$ is, but nothing is known except in very specific cases where specialised LCFT tools are applicable [KRV17, Rem17, RZ18]. Indeed even the regularity of the distribution (e.g. whether it has a density or not) is not known except for kernels with exact scale invariance [RV10b].

4.1.1 Main results

Define $M_{\gamma,g}(dx) = g(x)M_\gamma(dx)$ where $g(x) \ge 0$ is continuous on $\overline{D}$. The goal of this paper is to derive the leading order asymptotics for

$$\mathbb{P}\left(M_{\gamma,g}(A) > t\right) \tag{4.1.2}$$

for non-trivial[1] bounded open sets $A \subset D$ as $t \to \infty$. This may be seen as a first step towards the goal of understanding the full distribution of $M_{\gamma,g}(A)$, and will also highlight a new universality phenomenon of GMC. It is a standard fact in the literature that

$$\mathbb{E}\left[M_{\gamma,g}(A)^p\right] < \infty \quad \Leftrightarrow \quad p < \frac{2d}{\gamma^2}$$

and this suggests the possibility that the right tail (4.1.2) may satisfy a power law with exponent $2d/\gamma^2$. Our main result confirms this behaviour.

Theorem 4.1.1. *Let $\gamma \in (0, \sqrt{2d})$, $Q = \frac{\gamma}{2} + \frac{d}{\gamma}$ and $M_{\gamma,g}$ be the subcritical GMC associated with the Gaussian field $X(\cdot)$ with covariance*

$$\mathbb{E}[X(x)X(y)] = -\log|x-y| + f(x,y), \qquad \forall x, y \in D \tag{4.1.3}$$

where f is a continuous function on $\overline{D} \times \overline{D}$. Suppose f can be decomposed into

$$f(x,y) = f_+(x,y) - f_-(x,y) \tag{4.1.4}$$

where f_+, f_- are covariance kernels for some continuous Gaussian fields on $\overline{D}$. Then there exists some constant $\overline{C}_{\gamma,d} > 0$ independent of f and g such that for any bounded open set

[1] In the sense that $\int_A g(x)dx > 0$. In particular A has non-trivial Lebesgue measure.

$A \subset D$,

$$\mathbb{P}\left(M_{\gamma,g}(A) > t\right) \overset{t\to\infty}{=} \left(\int_A e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)} g(v)^{\frac{2d}{\gamma^2}} dv\right) \frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1} \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}}} + o(t^{-\frac{2d}{\gamma^2}}) \tag{4.1.5}$$

While the decomposition condition (4.1.4) may look intractable at first glance, it is implied by a more convenient criterion regarding higher regularity of f (see Lemma 4.2.3 or [JSW18] for more details about local Sobolev spaces H^s_{loc}). This is satisfied by the important example of the Liouville quantum gravity measure in dimension 2, i.e.

$$\mu_\gamma^{\mathrm{LQG}}(dx) = R(x; D)^{\frac{\gamma^2}{2}} M_\gamma(dx)$$

where $M_\gamma(dx)$ is the GMC measure associated with the Gaussian free field with Dirichlet boundary conditions on ∂D, in which case $f(x,x) = R(x;D)$ is the conformal radius of x in D. This is not covered in any previously known results.

Corollary 4.1.2. *If $f \in H^s_{\mathrm{loc}}(\mathcal{O} \times \mathcal{O})$ for some $s > d$ and open set $\mathcal{O} \supset \overline{D}$, then the decomposition condition* (4.1.4) *is satisfied. In particular the tail asymptotics* (4.1.5) *holds.*

The constant $\overline{C}_{\gamma,d}$ that appears in the tail asymptotics (4.1.5) has various probabilistic representations which are summarised in Corollary 4.3.3, and we shall call it the reflection coefficient of Gaussian multiplicative chaos[2] as it may be seen as the d-dimensional analogue of the reflection coefficient in Liouville conformal field theory (LCFT), see Section 4.A. Based on existing exact integrability results, we can even provide an explicit expression for $\overline{C}_{\gamma,d}$ when $d = 1$ and $d = 2$.

Corollary 4.1.3 (cf. [RV17, Section 4])**.** *The constant $\overline{C}_{\gamma,d}$ in* (4.1.5) *is given by*

$$\overline{C}_{\gamma,d} = \begin{cases} \dfrac{(2\pi)^{\frac{2}{\gamma}(Q-\gamma)}}{\frac{\gamma}{2}(Q-\gamma)\Gamma\left(\frac{\gamma}{2}(Q-\gamma)\right)^{\frac{2}{\gamma^2}}}, & d=1, \\[2ex] -\dfrac{\left(\pi\Gamma(\frac{\gamma^2}{4})/\Gamma(1-\frac{\gamma^2}{4})\right)^{\frac{2}{\gamma}(Q-\gamma)}}{\frac{2}{\gamma}(Q-\gamma)} \dfrac{\Gamma(-\frac{\gamma}{2}(Q-\gamma))}{\Gamma(\frac{\gamma}{2}(Q-\gamma))\Gamma(\frac{2}{\gamma}(Q-\gamma))}, & d=2. \end{cases} \tag{4.1.6}$$

Proof. The $d = 2$ case follows from [RV17] which proves (4.1.5) when $f \equiv 0$ and $g \equiv 1$. By Theorem 4.1.1, our constant $\overline{C}_{\gamma,d}$ is independent of f and therefore coincides with the Liouville unit volume reflection coefficient evaluated at γ, the value of which is given by the formula in (4.1.6).

For $d = 1$, this follows from [Rem17] which verifies the Fyodorov-Bouchaud formula [FB08a] that gives the exact distribution of the total mass of the GMC (associated with Gaussian free field with vanishing average over the unit circle) on the circle. □

[2]evaluated at γ; see the general definition of $\overline{C}_{\gamma,d}(\alpha)$ in Section 4.A.

4.1.2 Previous work and our approach

Despite being a very fundamental question, the tail probability of GMC has not been investigated very much in the literature. To our knowledge, the first result in this direction is established by Barral and Jin [BJ14] for the GMC associated with the exactly scale invariant kernel $\mathbb{E}[X(x)X(y)] = -\log|x-y|$ on the unit interval $[0,1]$:

$$\mathbb{P}(M_\gamma([0,1]) > t) = \frac{C_*}{t^{\frac{2}{\gamma^2}}} + o(t^{-\frac{2}{\gamma^2}})$$

where the constant $C_* > 0$ is given by

$$C_* = \frac{2\gamma^2}{2-\gamma^2} \frac{\mathbb{E}\left[M_\gamma([0,1])^{\frac{2}{\gamma^2}-1} M_\gamma([0,\frac{1}{2}]) - M_\gamma([0,\frac{1}{2}])^{\frac{2}{\gamma^2}}\right]}{\log 2}.$$

The issue about their approach is that they rely heavily on the exact scale invariance of the kernel and the symmetry of the unit interval in order to derive a stochastic fixed point equation. Such derivation of leading tail coefficient results in the inexplicit constant C_*. It is not clear how their method can be adapted to general kernels in higher dimension, let alone arbitrary open test sets A.

A recent paper [RV17] by Rhodes and Vargas, who consider the whole-plane Gaussian free field (GFF) restricted to the unit disc (i.e. $\mathbb{E}[X(x)X(y)] = -\log|x-y|$ on $D = \{x \in \mathbb{R}^2 : |x| < 1\}$), offers a new perspective for the tail problem. Their starting point is the localisation trick

$$\mathbb{P}\left(M_{\gamma,g}(A) > t\right) = \int_A \mathbb{E}\left[\frac{1_{\{M_{\gamma,g}(v,A)>t\}}}{M_{\gamma,g}(v,A)}\right] g(v)dv, \quad M_{\gamma,g}(v,A) := \int_A \frac{e^{\gamma^2 f(x,v)} M_{\gamma,g}(dx)}{|x-v|^{\gamma^2}}$$

which effectively pins down the γ-thick points of $X(\cdot)$, allowing one to express the dependence of the leading tail coefficient on the test set A in a very explicit way. Their proof then makes use the polar decomposition of the GFF, which can be adapted to the case when the function f is positive definite and sufficiently regular[3] when $d \leq 2$.

Our strategy is inspired by the ideas from both approaches and we make several additional input. Instead of working directly with $\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1} 1_{\{M_{\gamma,g}(v,A)>t\}}\right]$ in the localisation trick like [RV17], we shall apply Tauberian arguments and consider the equivalent problem of the asymptotics of

$$\mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)} e^{-\lambda/M_{\gamma,g}(v,A)}\right] \tag{4.1.7}$$

as $\lambda \to \infty$. The advantage of working with this expression is that it is more amenable to further analysis with Kahane's interpolation formula and ultimately allows us to reduce our

[3] This extension is covered in [RV17, v3] with f being locally Hölder.

problem to the case where the underlying kernel is exact (i.e. $\mathbb{E}[X(x)X(y)] = -\log|x-y|$). Then all we need is the precise asymptotics of the tail probability

$$\mathbb{P}\left(\int_{|x|\le r} |x|^{-\gamma^2} M_{\gamma,g}(dx) > t\right) \tag{4.1.8}$$

which can be obtained using a coupling argument and a result by Goldie from the literature of random recursive equations. Unlike many other estimates such as moment bounds in GMC, the expectation (4.1.7) we are studying here concerns a function $F: x \mapsto x^{-1}e^{-\lambda x}$ which is not convex or concave. The lack of a convenient convex/concave modification of F without affecting the behaviour of the expectation as $\lambda \to \infty$ means that the popular convexity inequality (4.2.9) is not applicable, and Kahane's full interpolation formula (4.2.8) plays an indispensable role in our analysis.

The novel use of Tauberian arguments and Goldie's result helps us bypass many tedious estimates in existing approaches, and our proof requires no special decomposition of the log-kernel (such as the cone construction in [BJ14] or the polar decomposition of GFF in [RV17]), providing a unified framework for general kernels in all dimensions[4]. Our philosophy is that once we obtain the tail probability of a particular GMC, we can extrapolate the result to all other GMCs in the same dimension, as far as the leading order term is concerned. The end result suggests that the power law of $M_{\gamma,g}(A)$ is a consequence of approximate scale invariance of log-correlated fields.

We note that our result generalises that of [RV17] not only to general kernels in arbitrary dimension, but also to sets that do not necessarily have C^1 boundary. Theorem 4.1.1 shares the same spirit of the result in [RV17] in the sense that we have successfully separated the dependence on the test set A and the functions f, g from the rest of the tail coefficient, and the constant $\overline{C}_{\gamma,d}$ captures any remaining dependence on d and γ and generic feature of GMC. The fact that we are unable to provide an explicit formula for $\overline{C}_{\gamma,d}$ for $d \ge 3$ should not be seen as a drawback of our approach – explicit expressions are known for $d = 1$ and $d = 2$ only because the constant has an LCFT interpretation, and their formulae are found (independently of the study of tail probability) by LCFT tools which do not seem to have natural generalisation to higher dimension at the moment.

[4]The paper [BJ14] used Goldie's ideas in a very different way. Indeed the authors were not aware of the localisation trick and therefore had to revisit Goldie's proof to relax the independence assumption. Such an approach was not robust enough to treat higher dimensions d, general test sets A or arbitrary densities g, and also required a proof of $C_* < \infty$ which was involved.

4.1.3 On the relevance of the kernel decomposition

Based on the continuity assumption of f, it is always possible to decompose f into the difference of two positive definite functions: indeed

$$T_f : h(\cdot) \mapsto \int_D f(\cdot, y) h(y) dy$$

is a symmetric Hilbert-Schmidt operator that maps $L^2(D)$ to $L^2(D)$ and by the standard spectral theory of compact self-adjoint operators there exist $\lambda_n \in \mathbb{R}$ and $\phi_n \in L^2(D)$ such that $(T_f \phi_n)(x) = \lambda_n \phi_n(x)$, $|\lambda_n| \xrightarrow{n\to\infty} 0$ and

$$\begin{aligned} f(x,y) &= \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\phi_n(y) \\ &= \underbrace{\left(\sum_{n=1}^{\infty} |\lambda_n| \phi_n(x)\phi_n(y) 1_{\{\lambda_n > 0\}}\right)}_{=:f_+(x,y)} - \underbrace{\left(\sum_{n=1}^{\infty} |\lambda_n| \phi_n(x)\phi_n(y) 1_{\{\lambda_n < 0\}}\right)}_{=:f_-(x,y)} \end{aligned}$$

in $L^2(D)$.Therefore, the relevant question is to determine the least regularity on $f_\pm$ for the power-law profile (4.1.5) to hold. Our decomposition condition (4.1.4) requires $f_\pm$ to be kernels of some continuous Gaussian fields. As it turns out, we only use this technical assumption to obtain the following estimate (see for instance Corollary 4.3.5(ii)):

- There exists some $r > 0$ and $C > 0$ such that for all $v \in D$ and $s \in [0,1]$

$$\mathbb{P}\left(\int_{B(v,r)\cap D} \frac{M_\gamma^s(dx)}{|x-v|^{\gamma^2}} > t\right) \le \frac{C}{t^{\frac{2d}{\gamma^2}-1}} \qquad \forall t > 0 \tag{4.1.9}$$

  where $M_\gamma^s(dx) = e^{\gamma Z_s(x) - \frac{\gamma^2}{2}\mathbb{E}[Z_s(x)^2]}dx$ is the Gaussian multiplicative chaos associated with the log-correlated field Z_s with covariance $\mathbb{E}[Z_s(x)Z_s(y)] = -\log|x-y| + sf(x,y)$.

Inspecting the proof in Section 4.3, this is the only assumption (other than the continuity of f) we need in order to apply dominated convergence in several places (such as (4.3.19)) which ultimately yields the desired power law. In other words our decomposition condition (4.1.4) may be relaxed so long as (4.1.9) is satisfied, e.g. we may assume instead that

- The Gaussian fields $G_\pm$ associated with the kernels $f_\pm$ satisfy

$$\mathbb{P}\left(\sup_{x\in D} |G_\pm(x)| < \infty\right) > 0 \tag{4.1.10}$$

  (see Section 4.2.1 for various implications).

All the proofs in Section 4.3 will go through without any modification to cover this slightly more general setting (which obviously includes the case where $G_\pm$ are continuous on $\overline{D}$). We choose not to phrase Theorem 4.1.1 this way because (4.1.10) is less tractable and not necessarily much more general. Indeed when $f_\pm(x,y) = f_\pm(x-y)$ are continuous shift-invariant kernels, a classical result by Belayev [Bel61] states that $G_\pm$ are either continuous or unbounded on any non-empty open sets[5], and so (4.1.10) is equivalent to the original condition (4.1.4) in the stationary setting. We also think that the decomposition condition (4.1.4) is a very natural assumption because for any $s \ge 0$, $\epsilon > 0$ and symmetric function $f(\cdot,\cdot) \in H^s(\mathbb{R}^{2d})$, one can always find some symmetric function $\widetilde{f}(\cdot,\cdot) \in C_c^\infty(\mathbb{R}^{2d})$, say by truncating suitable basis expansion (see also [JSW18, Lemma 2.2]), such that $||f - \widetilde{f}||_{H^s(\mathbb{R}^{2d})} < \epsilon$ and that the operator $T_{\widetilde{f}}$ is of finite rank, i.e. the decomposition condition (4.1.4) is satisfied by a "dense collection" of covariance kernels of the form (4.1.3).

To understand the importance of continuity at the level of the fields $G_\pm$, let us consider the simpler situation where $f = f_+$. We have

$$\mathbb{E}\left[X(x)X(y)\right] = -\log|x-y| + f(x,y) \approx -\log|x-y| + f(v,v)$$

on a ball of small radius $r > 0$ centred around $v \in A$. This says that $X(\cdot)$ is the sum of an exactly scale invariant field Y (with covariance $\mathbb{E}[Y(x)Y(y)] = K(x,y) = -\log|x-y|$) and an independent field G_+ which locally behaves like an independent random variable $N_v \sim \mathcal{N}(0, f(v,v))$, and this leads to

$$\begin{aligned} &\mathbb{P}\Bigg(\underbrace{\int_A \frac{e^{\gamma^2 f(x,v)} M_{\gamma,g}(dx)}{|x-v|^{\gamma^2}}}_{=:M_{\gamma,g}(v,A)} > t\Bigg) \\ &\quad \approx \mathbb{P}\Bigg(\underbrace{\int_{|x-v|\le r} \frac{e^{\gamma^2 f(x,v)} M_{\gamma,g}(dx)}{|x-v|^{\gamma^2}}}_{=:M_{\gamma,g}(v,r)} > t\Bigg) \sim e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)} g(v)^{\frac{2d}{\gamma^2}-1} \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}-1}} \end{aligned} \tag{4.1.11}$$

(see Corollary 4.3.5 and Remark 4.3.6). This allows us to interpret

$$\mathbb{P}\left(M_{\gamma,g}(A) > t\right) \sim \left(\int_A e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)} g(v)^{\frac{2d}{\gamma^2}} dv\right) \frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1} \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}}}$$

in the following way: if $M_{\gamma,g}(A)$ is extremely large, then most of its mass comes from a small neighbourhood $B(v,r) \subset A$ of some γ-thick point $v \in A$ of $X(\cdot)$, and this point v is more likely to come from regions of higher density with respect to g and/or of higher values of f, i.e. where G_+ has higher variance near v.

[5] The theorem of Belayev actually concerns stationary kernels in $d = 1$, but this implies the statement in higher dimension because we may view $G_\pm$, with $d-1$ coordinates fixed, as Gaussian fields in 1 dimension.

When G_+ is not continuous, the localisation intuition is not valid anymore and our method breaks down because (4.1.10) is possibly false by Belayev's dichotomy mentioned earlier. It may happen that (4.1.9) is still valid, in which case the power-law profile will still hold, but it is unclear how to proceed with a Gaussian field G_+ that is only guaranteed to have a separable and measurable version but nothing else. We conjecture that the power law (4.1.5) remains true without the generalised decomposition condition (4.1.10) based on two heuristics:

- Despite the possibility that $G_\pm$ are unbounded in every non-empty open set, $G_\pm$ are still measurable and Lusin's theorem suggests some "approximate" continuity of the fields which is much weaker than the usual notion of continuity but is perhaps sufficient for studying integrals.
- The construction of the GMC measure involves the mollification of the underlying log-correlated field. When $G_\pm$ are convolved with a smooth mollifier $\theta \in C_c^\infty(\mathbb{R}^d)$, the new covariance kernels are differentiable which implies that the resulting fields are actually continuous.

4.1.4 Critical GMCs and extremal processes: heuristics

Let us abuse the notation and denote by $M_{\sqrt{2d}}$ the critical GMC (via Seneta–Heyde renormalisation[6])

$$M_{\sqrt{2d}}(dx) = \lim_{\epsilon \to 0^+} \sqrt{\frac{\pi}{2}} \left(\mathbb{E}[X_\epsilon(x)^2]\right)^{\frac{1}{2}} e^{\sqrt{2d}X_\epsilon(x) - d\mathbb{E}\left[X_\epsilon(x)^2\right]}dx$$

and similarly $M_{\sqrt{2d},g}(dx) = g(x)M_{\sqrt{2d}}(dx)$. While a similar criterion for the existence of moments [DRSV14b]

$$\mathbb{E}\left[M_{\sqrt{2d},g}(A)^p\right] < \infty \qquad \Leftrightarrow \qquad p < 1$$

has been known for critical GMC associated with general fields, previous attempts to understand the tail probability $\mathbb{P}(M_{\sqrt{2d},g}(A) > t)$ are again restricted to exact kernels so that the derivation via stochastic fixed point equation may be applied [BKN+15]. By combining the techniques in this paper with additional ingredients including fusion estimates of GMC that have appeared in [DKRV17, BW18], it is possible to prove that

$$\mathbb{P}\left(M_{\sqrt{2d},g}(A) > t\right) \overset{t \to \infty}{=} \frac{\int_A g(v)dv}{t\sqrt{2d}} + o(t^{-1}). \tag{4.1.12}$$

[6] Our definition differs from the usual one by the factor $\sqrt{\pi/2}$ for aesthetic purpose.

The precise statement and the proof of the result will be discussed separately in a forthcoming article in order not to overload the present paper. Nevertheless, let us provide a heuristic proof of (4.1.12) in the case $d = 2$ based on Theorem 4.1.1. Recall that for $\gamma \in (0, 2)$ we have

$$\overline{C}_{\gamma,2} = -\frac{\pi^{\frac{4}{\gamma^2}-1}\left(\Gamma(\frac{\gamma^2}{4})/\Gamma(1-\frac{\gamma^2}{4})\right)^{\frac{4}{\gamma^2}-1}}{\frac{4}{\gamma^2}-1}\frac{\Gamma(\frac{\gamma^2}{4}-1)}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(\frac{4}{\gamma^2}-1)}.$$

Using the property[7] that

$$\frac{M_\gamma(dx)}{2-\gamma} \xrightarrow{\gamma\to 2^-} 2M_2(dx)$$

and that $\Gamma(x) = x^{-1}\Gamma(1+x) \overset{x\to 0}{\sim} x^{-1}$, we should expect

$$\begin{aligned}\mathbb{P}\left(M_{2,g}(A) > t\right) &\overset{\gamma\to 2^-}{\approx} \mathbb{P}\left(M_{\gamma,g}(A) > (2-\gamma)2t\right)\\ &\overset{\gamma\to 2^-}{\sim} \left(\frac{4}{\gamma^2}-1\right)\left(\frac{1-\frac{\gamma^2}{4}}{\frac{\gamma^2}{4}}\right)^{\frac{4}{\gamma^2}-1}\frac{\int_A g(v)dv}{((2-\gamma)\cdot 2t)^{\frac{4}{\gamma^2}}} \overset{\gamma\to 2^-}{\sim} \frac{\int_A g(v)dv}{2t}.\end{aligned}$$

Unfortunately it seems impossible to justify the interchanging of the limits $\gamma \to 2^-$ and $t \to \infty$ to turn the above argument into a rigorous proof, and this is actually not the approach adopted in the separate paper. On the other hand, the constant $\overline{C}_{\gamma,d}$ is not explicitly known in higher dimension $d \geq 3$ but the heuristic here suggests the existence of a non-trivial limit:

$$\lim_{\gamma\to\sqrt{2d}^-}(\sqrt{2d}-\gamma)^{\frac{2d}{\gamma^2}}\overline{C}_{\gamma,d} = \lim_{\gamma\to\sqrt{2d}^-}(\sqrt{2d}-\gamma)\overline{C}_{\gamma,d} \in (0,\infty).$$

Connection to discrete Gaussian free field The tail probability of critical chaos is not only interesting in its own right but is also closely related to the study of extrema of log-correlated Gaussian fields, which has been an active area of research in the last two decades. For instance, it is known that the extremal process of a discrete Gaussian free field (DGFF) in $d = 2$ converges to a Poisson point process with intensity $e^{-2x} \otimes Z(dx)$ for some random measure $Z(dx)$ [BL14, BL16a, BL18] which has long been conjectured to be some constant multiple of the critical LQG measure μ_2^{LQG}, i.e.

$$Z(dx) \propto \mu_2^{\mathrm{LQG}}(dx) = R(x; D)^2 M_2(dx), \qquad x \in D \tag{4.1.13}$$

where $M_2(dx)$ is the critical GMC associated with Gaussian free field with Dirichlet

[7] This was first proved in $d = 2$, for GFF with Dirichlet boundary conditions in [APS18], and subsequently extended in [JSW18] to log-correlated fields (4.1.3) with $f \in H^{d+\epsilon}_{\mathrm{loc}}$ in dimension $d = 2$.

boundary condition. The random measure $Z(dx)$ is characterised (up to a deterministic multiplicative factor) by a set of properties, among which the Laplace-type estimate

$$\lim_{\lambda \to 0^+} \frac{\mathbb{E}\left[Z(A)e^{-\lambda Z(A)}\right]}{-\log \lambda} = c \int_A R(x; D)^2 dx, \tag{4.1.14}$$

(where $c > 0$ is independent of A) has been left unverified by μ_2^{LQG} for several years until very recently in the revision of [BL14]. Here we suggest an approach slightly different from that in [BL14]: it is sufficient to first establish the statement that

$$\mathbb{P}\left(\mu_2^{\mathrm{LQG}}(A) > t\right) \overset{t\to\infty}{=} \frac{c\int_A R(x; D)^2 dx}{t} + o(t^{-1}) \tag{4.1.15}$$

from which we conclude that the Laplace-type estimate holds by straightforward computation. We would like to point out that (4.1.15) is a strictly stronger statement and cannot be deduced from the estimate (4.1.14) without additional assumption.

4.1.5 On the critical case in Karamata's Tauberian theory

The second version of [BL14] claims to have obtained the tail probability (4.1.15) as an easy consequence of (4.1.14) through the use of Tauberian theorem (cf. [BL14, Corollary 2.10]). This would have relied on a result of the following form[8]: for a non-negative random variable U and $q > 0$

$$\mathbb{P}(U > t) = \frac{C}{t^q} + o(t^{-1}) \qquad \Leftrightarrow \qquad \lim_{\lambda \to 0^+} \frac{\mathbb{E}\left[U^q e^{-\lambda U}\right]}{-\log \lambda} = Cq. \tag{4.1.16}$$

While the forward implication of (4.1.16) can be verified by straightforward computation (Lemma 4.2.13), the backward implication (which is the direction of interest in [BL14]) is, unfortunately, false in general, as seen by the simple counter-example $q = 1$ and $\mathbb{P}(U > t) = (1 + 0.1\sin(\log t))/t$ for $t \ge 1$. To understand what the backward implication is really suggesting, first recall that

$$\mathbb{E}\left[Ue^{-\lambda U}\right] \overset{\lambda\to 0^+}{\sim} -C\log\lambda \qquad \Leftrightarrow \qquad \mathbb{E}\left[U1_{\{U\le t\}}\right] \overset{t\to\infty}{\sim} C\log t$$

by standard Tauberian theorem, and in the notation of Theorem 4.2.10 we are in the critical case of Karamata's Tauberian theory where $\rho = 0$ and $L(x) = \log x$. Since $\mathbb{E}\left[U1_{\{U\le t\}}\right] = -t\mathbb{P}(U > t) + \int_0^t \mathbb{P}(U > s)ds$ by Fubini, if we can ignore the negative term (which would be subleading anyway if $\mathbb{P}(U > t)$ were supposed to be $o(t^{-1}\log t)$) then we

[8] [BL14] only requires $q = 1$, but if such claim were true for $q = 1$ it would be true for any $q > 0$ by a simple reduction argument.

have

$$\int_0^t \mathbb{P}(U > s)ds \overset{t\to\infty}{\sim} C \log t \tag{4.1.17}$$

The backward implication of (4.1.16) is thus, to some extent, equivalent to the question of whether we can "differentiate" the above asymptotics and obtain $\mathbb{P}(U > t) \sim C/t$, and the same counter-example we mentioned just now provides a negative answer to this. Indeed, even under (4.1.17), it is still not possible to prove the existence of some $C' > 0$ such that for all $t > 0$ sufficiently large

$$\mathbb{P}(U > t) \leq \frac{C}{t}$$

or an analogous lower bound – whether U has a density function or not, one can always construct counter-examples such that these bounds are not satisfied.

The necessary and sufficient conditions for the backward implication are related to the notion of de Haan class from the higher-order theory of regular variation (see [dH76] or e.g. [BGT89, Chapter 3]), which requires better control over subleading order terms in $\mathbb{E}\left[U^q e^{-\lambda U}\right]$ as $\lambda \to 0^+$. Such control is unavailable with the method in [BL14], and this explains why the asymptotics of the tail probability of subcritical/critical GMC is more subtle than that of the corresponding Laplace-type estimate.

Note, however, that once we prove an asymptotic power law for a random variable U, we can rely on the forward implication of (4.1.16) to study the leading order coefficient C in the asymptotics. For our purpose, this provides an alternative probabilistic representation of $\overline{C}_{\gamma,d}$ (see Corollary 4.3.3) which may be more useful in $d \geq 3$ for the derivation of an explicit formula in the future.

4.1.6 Outline of the paper

The remainder of the article is organised as follows.

In Section 4.2 we compile a list of results that will be used in the proof of Theorem 4.1.1. This includes a collection of facts regarding separable Gaussian processes, log-correlated Gaussian fields and GMCs, Karamata's Tauberian theorem and auxiliary asymptotics, and random recursive equations.

In Section 4.3 we present the proof of Theorem 4.1.1 which is divided into two parts. After sketching the idea of the localisation trick, we first establish the tail asymptotics for GMCs associated with exact kernels. We then apply Kahane's interpolation and generalise the result to general kernels (4.1.3).

We conclude the article with Section 4.A where we define the reflection coefficient $\overline{C}_{\gamma,d}(\alpha)$ of Gaussian multiplicative chaos and prove that it is equivalent to the Liouville reflection coefficients in $d = 1$ and $d = 2$.

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4.2 Preliminaries

4.2.1 Basic facts of Gaussian processes

We collect a few standard results (see e.g. [GN16, Chapter 2]) regarding Gaussian processes in the following theorem.

Theorem 4.2.1. *Let $(G_t)_{t\in\mathcal{T}}$ be a separable centred Gaussian process such that*

$$\mathbb{P}\left(\sup_{t\in\mathcal{T}}|G_t| < \infty\right) > 0.$$

Then the following statements are true.

- *Zero-one law:* $\mathbb{P}\left(\sup_{t\in\mathcal{T}}|G_t| < \infty\right) = 1$.
- *Finite moments:* $\mathbb{E}\left[\sup_{t\in\mathcal{T}}|G_t|\right] < \infty$ *and* $\sigma^2 = \sigma^2(G) = \sup_{t\in\mathcal{T}}\mathbb{E}\left[G_t^2\right] < \infty$.
- *Concentration: there exists some* $c > 0$ *such that for any* $t \geq 0$,

$$\mathbb{P}\left(\left|\sup_{t\in\mathcal{T}}|G_t| - \mathbb{E}\left[\sup_{t\in\mathcal{T}}|G_t|\right]\right| > t\right) \leq 2e^{-c\frac{u^2}{\sigma^2}}. \tag{4.2.1}$$

The lemma below is an easy consequence of Theorem 4.2.1.

Lemma 4.2.2. *Let $G(\cdot)$ be a continuous Gaussian field on some compact domain $K \subset \mathbb{R}^d$, then the following are true.*

(i) There exists some $c > 0$ such that

$$\mathbb{P}\left(\sup_{x\in K}|G(x)| > t\right) \leq \frac{1}{c}e^{-ct^2}, \qquad \forall t \geq 0. \tag{4.2.2}$$

(ii) Let $x \in \mathrm{int}(K)$. For any monotone functions $\Psi : \mathbb{R} \to \mathbb{R}$ with at most exponential growth at infinity,

$$\lim_{r\to 0^+}\mathbb{E}\left[\Psi\left(\sup_{y\in B(x,r)} G(y)\right)\right] = \lim_{r\to 0^+}\mathbb{E}\left[\Psi\left(\inf_{y\in B(x,r)} G(y)\right)\right] = \mathbb{E}\left[\Psi\left(G(x)\right)\right] \tag{4.2.3}$$

Proof. Since $G(\cdot)$ is continuous on K, it is separable and satisfies $\sup_{x\in K}|G(x)| < \infty$ almost surely. By Theorem 4.2.1 we have $\mathbb{E}\left[\sup_{x\in K}|G(x)|\right] < \infty$ and $\sigma^2(G) < \infty$. The tail in (i) can thus be obtained from the concentration inequality (4.2.1).

For (ii), note that by monotonicity we can split Ψ into positive and negative parts $\Psi = \Psi_+ - \Psi_-$, such that $\Psi_\pm$ are monotone functions with at most exponential growth at infinity. Since we can deal with Ψ_+ and Ψ_- separately, we may as well assume without loss of generality that Ψ is non-negative. Now take $r_0 > 0$ such that $B(x, r_0) \Subset K$, and consider the case where Ψ is non-decreasing. By (4.2.2) and the assumption on the growth of Ψ at infinity, we have

$$\mathbb{E}\left[\Psi\left(\sup_{y\in B(x,r_0)} G(y)\right)\right] < \infty.$$

But then for any $r \in (0, r_0)$,

$$0 \le \inf_{y\in B(x,r)} \Psi(G(y)) \le \sup_{y\in B(x,r)} \Psi(G(y)) \le \sup_{y\in B(x,r_0)} \Psi(G(y))$$

and (4.2.3) follows from the continuity of G and dominated convergence. The case where Ψ is non-increasing is similar. □

4.2.2 Decomposition of Gaussian fields

We mention a result concerning the decomposition of symmetric functions from the very recent paper [JSW18]. Let $f(x, y)$ be a symmetric function on $D \times D$ for some domain $D \subset \mathbb{R}^d$. We say f is in the local Sobolev space $H^s_{\mathrm{loc}}(D \times D)$ of index $s > 0$ if κf is in $H^s(D \times D)$ for any $\kappa \in C_c^\infty(D \times D)$, i.e.

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{(\kappa f)}(\xi)|^2 d\xi < \infty$$

where $\widehat{(\kappa f)}$ is the Fourier transform of κf (see more details in [JSW18, Section 2]). Then

Lemma 4.2.3 (cf. [JSW18, Lemma 3.2])**.** *If $f \in H^s_{\mathrm{loc}}(D \times D)$ for some $s > d$, then there exist two centred, Hölder-continuous Gaussian processes $G_\pm$ on $\mathbb{R}^d$ such that*

$$\mathbb{E}[G_+(x)G_+(y)] - \mathbb{E}[G_-(x)G_-(y)] = f(x, y), \qquad \forall x, y \in \mathcal{O}' \tag{4.2.4}$$

for any bounded open set D' such that $\overline{D'} \subset D$.

This decomposition result has various important implications, one of which is the positive-definiteness of the logarithmic kernel. The following result may be seen as a trivial special case of [JSW18, Theorem B] and has been known since [RV10a].

Lemma 4.2.4. *For each $L \in \mathbb{R}$, there exists $r_d(L) > 0$ such that the kernel*

$$K_L(x, y) = -\log|x - y| + L \tag{4.2.5}$$

is positive definite on $B(0, r_d(L)) \subset \mathbb{R}^d$. In particular, for any $R > 0$ there exists some $L > 0$ such that K_L is positive definite on $B(0, R)$.

For the sake of convenience, we shall from now on call (4.2.5) the L-exact kernel, and when $L = 0$ we simply call $K_0(\cdot, \cdot)$ the exact kernel and write $r_d = r_d(0)$. The exact kernel will play a pivotal role as the reference point from which we extrapolate our tail result to general kernels in the subcritical regime.

4.2.3 Gaussian multiplicative chaos

Given a log-correlated Gaussian field 4.1.3, there are various equivalent constructions of the GMC measure M_γ. In the subcritical case $\gamma \in (0, \sqrt{2d})$, one approach is the regularisation procedure, which is first suggested in [RV10b] and then generalised/simplified in [Ber17]. The idea is to pick any suitable mollifier $\theta(\cdot)$ and define

$$M_{\gamma,\epsilon}(dx) = e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}dx \tag{4.2.6}$$

where $X_\epsilon(\cdot) = X * \theta_\epsilon(\cdot)$ is a continuous Gaussian field on D. Then

Theorem 4.2.5. *For $\gamma \in (0, \sqrt{2d})$, the sequence of measures $M_{\gamma,\epsilon}$ converges in probability to some measure M_γ in the weak* topology as $\epsilon \to 0^+$. The limit M_γ is independent of the choice of the mollification θ.*

We collect a few standard results in the literature of GMC. The first is the celebrated interpolation principle by Kahane.

Lemma 4.2.6 ([Kah85])**.** *Let ρ be a Radon measure on D, $X(\cdot)$ and $Y(\cdot)$ be two continuous centred Gaussian fields, and $F : \mathbb{R}_+ \to \mathbb{R}$ be some smooth function with at most polynomial growth at infinity. For $t \in [0, 1]$, define $Z_t(x) = \sqrt{t}X(x) + \sqrt{1-t}Y_t(x)$ and*

$$\varphi(t) := \mathbb{E}\left[F(W_t)\right], \qquad W_t := \int_D e^{Z_t(x) - \frac{1}{2}\mathbb{E}[Z_t(x)^2]}\rho(dx). \tag{4.2.7}$$

Then the derivative of φ is given by

$$\begin{aligned}\varphi'(t) = \frac{1}{2}\int_D\int_D &\left(\mathbb{E}[X(x)X(y)] - \mathbb{E}[Y(x)Y(y)]\right) \\ &\times \mathbb{E}\left[e^{Z_t(x) + Z_t(y) - \frac{1}{2}\mathbb{E}[Z_t(x)^2] - \frac{1}{2}\mathbb{E}[Z_t(y)^2]}F''(W_t)\right]\rho(dx)\rho(dy).\end{aligned} \tag{4.2.8}$$

In particular, if

$$\mathbb{E}[X(x)X(y)] \le \mathbb{E}[Y(x)Y(y)] \qquad \forall x, y \in D,$$

then for any convex $F : \mathbb{R}_+ \to \mathbb{R}$

$$\mathbb{E}\left[F\left(\int_D e^{X(x)-\frac{1}{2}\mathbb{E}[X(x)^2]}\rho(dx)\right)\right] \le \mathbb{E}\left[F\left(\int_D e^{Y(x)-\frac{1}{2}\mathbb{E}[Y(x)^2]}\rho(dx)\right)\right]. \tag{4.2.9}$$

and the inequality is reversed if F *is concave instead.*

While Lemma 4.2.6 is stated for continuous fields, it may be extended to log-correlated fields if we first apply it to mollified fields X_ϵ and Y_ϵ and take the limit $\epsilon \to 0^+$. Such argument will work immediately for comparison principles (4.2.9) and we shall make no further remarks on that. For the interpolation principle (4.2.8) we only need the following weaker statement which may be extended to log-correlated fields in the same way.

Corollary 4.2.7. *Under the same assumptions and notations in Lemma 4.2.6, if there exists some* $C > 0$ *such that*

$$|\mathbb{E}[X(x)X(y)] - \mathbb{E}[Y(x)Y(y)]| \le C \qquad \forall x, y \in D,$$

then

$$|\varphi'(t)| \le \frac{C}{2}\mathbb{E}\left[(W_t)^2|F''(W_t)|\right]$$

and consequently

$$|\varphi(1) - \varphi(0)| \le \frac{C}{2}\int_0^1 \mathbb{E}\left[(W_t)^2|F''(W_t)|\right] dt.$$

The next result is a generalised criterion for the existence of moments of GMC.

Lemma 4.2.8. *Let* $\gamma \in (0, \sqrt{2d})$, $Q = \frac{\gamma}{2} + \frac{d}{\gamma}$, $\alpha \in [0, Q)$ *and* $B(0,r) \subset D$. *Then*

$$\mathbb{E}\left[\left(\int_{|x|\le r} |x|^{-\gamma\alpha} M_\gamma(dx)\right)^s\right] < \infty \tag{4.2.10}$$

if $s < \frac{2d}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha)$. *In particular*

$$\mathbb{E}\left[\left(\int_{|x|\le r} M_\gamma(dx)\right)^s\right] < \infty, \qquad \forall s < \frac{2d}{\gamma^2},$$

$$\textit{and} \qquad \mathbb{E}\left[\left(\int_{|x|\le r} |x|^{-\gamma^2} M_\gamma(dx)\right)^s\right] < \infty, \qquad \forall s < \frac{2d}{\gamma^2} - 1.$$

Remark 4.2.9. *The bound on (4.2.10) is uniform among the class of fields (4.1.3) with* $\sup_{x,y\in D}|f(x,y)| \le C$ *for some* $C > 0$ *by Gaussian comparison (Lemma 4.2.6).*

4.2.4 Tauberian theorem and related auxiliary results

Let us record the classical Tauberian theorem of Karamata.

Theorem 4.2.10 ([Fel71, Theorem XIII.5.3])**.** *Let $f(d\cdot)$ be a non-negative measure on $\mathbb{R}_+$, $F(t) := \int_0^t f(ds)$ and suppose*

$$\widetilde{F}(\lambda) := \int_0^\infty e^{-\lambda t} f(dt)$$

exists for $\lambda > 0$. If L is slowly varying at zero and $\rho \in [0, \infty)$, then

$$\widetilde{F}(\lambda) \overset{\lambda \to \infty}{\sim} \lambda^{-\rho} L(\lambda^{-1}) \qquad \Leftrightarrow \qquad F(\epsilon) \overset{\epsilon \to 0^+}{\sim} \frac{1}{\Gamma(1+\rho)} \epsilon^\rho L(\epsilon). \tag{4.2.11}$$

The above is also true when we consider the asymptotics $\lambda \to 0^+$ and $\epsilon \to \infty$, and L being slowly varying at infinity.

Our use of Theorem 4.2.10 is summarised in the following corollary.

Corollary 4.2.11. *Let U be a non-negative random variable, $C > 0$ and $q > 0$. Then*

$$\mathbb{E}\left[U^{-1} 1_{\{U > t\}}\right] \overset{t \to \infty}{\sim} \frac{C}{t^q} \qquad \Leftrightarrow \qquad \mathbb{E}\left[U^{-1} e^{-\lambda/U}\right] \overset{\lambda \to \infty}{\sim} \frac{C\Gamma(1+q)}{\lambda^q}. \tag{4.2.12}$$

Proof. Let $V = U^{-1}$. In the notation of Theorem 4.2.10, we choose $f(ds) = s\mathbb{P}(V \in ds)$, $L \equiv C\Gamma(1+q)$ and $\epsilon = t^{-1}$ such that $\widetilde{F}(\lambda) = \mathbb{E}\left[U^{-1} e^{-\lambda/U}\right]$ and $\widetilde{F}(\epsilon) = \mathbb{E}\left[U^{-1} 1_{\{U>t\}}\right]$, and our claim is now immediate. □

To save ourselves from repeated calculations, we shall collect a few basic estimates below. The first one concerns the Laplace transform estimate of a random variable with power-law tail.

Lemma 4.2.12. *If U is a non-negative random variable such that*

$$\mathbb{P}(U > t) \overset{t \to \infty}{\sim} \frac{C}{t^q}$$

for some $C > 0$ and $q > 0$, then for any $p > 0$

$$\mathbb{E}\left[U^{-p} e^{-\lambda/U}\right] \overset{\lambda \to \infty}{\sim} \frac{q}{p+q} \frac{C\Gamma(p+q+1)}{\lambda^{p+q}}. \tag{4.2.13}$$

If $\mathbb{P}(U > t) \le Ct^{-q}$ for all $t > 0$ instead, then there exists some $C' > 0$ such that

$$\mathbb{E}\left[U^{-p} e^{-\lambda/U}\right] \le \frac{C'}{\lambda^{p+q}}, \qquad \forall \lambda > 0. \tag{4.2.14}$$

Proof. For any $t_0 > 0$, it is not difficult to see that there exists $c_0 > 0$ such that

$$\mathbb{E}\left[U^{-p}e^{-\lambda/U}1_{\{U\le t_0\}}\right] = O(e^{-c_0\lambda}).$$

For any $\epsilon > 0$, choose $t_0 > 0$ such that for all $t > t_0$ we have

$$\frac{C(1-\epsilon)}{t^q} \le \mathbb{P}(U>t) \le \frac{C(1+\epsilon)}{t^q}.$$

Using Fubini, we have

$$\begin{aligned}\mathbb{E}\left[U^{-p}e^{-\lambda/U}1_{\{U\ge t_0\}}\right] &= \frac{1}{t_0^p}e^{-\lambda/t_0}\mathbb{P}(U>t_0) + \int_{t_0}^\infty e^{-\lambda/t}\left(-\frac{p}{t^{p+1}}+\frac{\lambda}{t^{p+2}}\right)\mathbb{P}(U>t)dt\\ &\le O(e^{-\lambda/t_0}) + C\int_{t_0}^\infty e^{-\lambda/t}\left(-\frac{p(1-\epsilon)}{t^{p+q+1}}+\frac{\lambda(1+\epsilon)}{t^{p+q+2}}\right)dt.\end{aligned}$$

Note that for any $m > 0$ we have

$$\int_{t_0}^\infty \frac{e^{-\lambda/t}}{t^{m+2}}dt = \lambda^{-(1+m)}\int_0^{\lambda/t_0} s^m e^{-s}ds \overset{\lambda\to\infty}{=} (1+o(1))\Gamma(1+m)\lambda^{-(m+1)}$$

and therefore

$$\begin{aligned}\mathbb{E}\left[U^{-p}e^{-\lambda/U}\right] &\le \frac{C}{\lambda^{p+q}}\left[-p(1-\epsilon)\Gamma(p+q)+(1+\epsilon)\Gamma(p+q+1)\right] + o(\lambda^{-(p+q)})\\ &\le \left(\frac{Cq}{p+q}+(p+1)\epsilon\right)\frac{\Gamma(p+q+1)}{\lambda^{p+q}} + o(\lambda^{-(p+q)}).\end{aligned}$$

Similarly we have

$$\mathbb{E}\left[U^{-p}e^{-\lambda/U}\right] \ge \left(\frac{Cq}{p+q}-(p+1)\epsilon\right)\frac{\Gamma(p+q+1)}{\lambda^{p+q}} + o(\lambda^{-(p+q)}).$$

This means that

$$\begin{aligned}\left(\frac{Cq}{p+q}-(p+1)\epsilon\right)\Gamma(p+q+1) &\le \liminf_{\lambda\to\infty}\lambda^{q+1}\mathbb{E}\left[U^{-p}e^{-\lambda/U}\right]\\ &\le \limsup_{\lambda\to\infty}\lambda^{q+1}\mathbb{E}\left[U^{-p}e^{-\lambda/U}\right] \le \left(\frac{Cq}{p+q}+(p+1)\epsilon\right)\Gamma(p+q+1).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \to 0^+$ and obtain (4.2.13). The claim (4.2.14) is similar. □

We collect another Laplace transform estimate discussed in Section 4.1.5. The proof of the result is similar to that of Lemma 4.2.12 and is omitted.

Lemma 4.2.13. *If U is a non-negative random variable such that*

$$\mathbb{P}(U>t) \overset{t\to\infty}{\sim} \frac{C}{t^q}$$

for some $C > 0$ and $q > 0$, then

$$\lim_{\lambda \to 0^+} \frac{\mathbb{E}\left[U^q e^{-\lambda U}\right]}{-\log \lambda} = Cq. \tag{4.2.15}$$

If $\mathbb{P}(U > t) \le Ct^{-q}$ for all t sufficiently large instead, then (4.2.15) may be replaced by the statement that the limit superior is upper bounded by Cq.

We also need the following elementary result, the proof of which is again skipped.

Lemma 4.2.14. *Let U, V be two non-negative random variables. Suppose there exists some $C > 0$ and $q > 0$ such that*

$$\begin{aligned} &(i) \qquad \mathbb{P}(U > t) \overset{t \to \infty}{\sim} Ct^{-q}, \\ &(ii) \qquad \mathbb{P}(V > t) \overset{t \to \infty}{\sim} o(t^{-p}) \qquad \forall p > 0. \end{aligned}$$

Then the tail behaviour of UV is given by

$$(iii) \qquad \mathbb{P}(UV > t) \overset{t \to \infty}{\sim} C\mathbb{E}[V^q]t^{-q}.$$

Remark 4.2.15. *The converse of Lemma 4.2.14 is false: in general if we are given only conditions (ii) and (iii), we can only show that there exists some $C' > 0$ such that*

$$\mathbb{P}(U > t) \le C' t^{-q}$$

which follows immediately from $\mathbb{P}(UV > t) \ge \mathbb{P}(U > t/a)\mathbb{P}(V > a)$ for any $a > 0$ such that $\mathbb{P}(V > a) \neq 0$.

4.2.5 Random recursive equation

Here we collect Goldie's implicit renewal theorem [Gol91] from the literature of random recursive equation.

Theorem 4.2.16. *Let M and R be two independent non-negative random variables. Suppose there exists some $q > 0$ such that*

- *(i)* $\mathbb{E}[M^q] = 1$.
- *(ii)* $\mathbb{E}[M^q \log M] < \infty$.
- *(iii) The conditional law of $\log M$ given $M \neq 0$ is non-arithmetic.*
- *(iv)* $\int_0^\infty |\mathbb{P}(R > t) - \mathbb{P}(MR > t)| t^{q-1} dt < \infty$.

Then $\mathbb{E}[M^q \log M] \in (0, \infty)$ *and as* $t \to \infty$,

$$\mathbb{P}(R > t) = \frac{C}{t^q} + o(t^{-q})$$

where the constant $C > 0$ *is given by*

$$C = \frac{1}{\mathbb{E}[M^q \log M]} \int_0^\infty \left(\mathbb{P}(R > t) - \mathbb{P}(MR > t)\right) t^{q-1} dt. \tag{4.2.16}$$

Theorem 4.2.16 will be used alongside the following lemma.

Lemma 4.2.17. *Let* U, V *be two non-negative random variables and* $q > 0$. *Then*

$$\int_0^\infty \left|\mathbb{P}(U > t) - \mathbb{P}(V > t)\right| t^{q-1} dt \le \frac{1}{q} \mathbb{E}\left|U^q - V^q\right|. \tag{4.2.17}$$

Moreover, for any coupling of (U, V) *such that* $\mathbb{E}|U^q - V^q| < \infty$,

$$\int_0^\infty \left[\mathbb{P}(U > t) - \mathbb{P}(V > t)\right] t^{q-1} dt = \frac{1}{q} \mathbb{E}\left[U^q - V^q\right]. \tag{4.2.18}$$

Proof. Suppose U, V are bounded by some constant $M > 0$. The inequality (4.2.17) is then a simple consequence of

$$\begin{aligned}
&|\mathbb{P}(U > t) - \mathbb{P}(V > t)| \\
&\quad = |\mathbb{P}(U > t, V > t) + \mathbb{P}(U > t, V \le t) - \mathbb{P}(U > t, V > t) - \mathbb{P}(U \le t, V > t)| \\
&\quad = |\mathbb{P}(U > t, V \le t) - \mathbb{P}(U \le t, V > t)| \\
&\quad \le \mathbb{P}(U > t, V \le t) + \mathbb{P}(U \le t, V > t) \\
&\quad = \mathbb{P}(\max(U, V) > t) - \mathbb{P}(\min(U, V) > t)
\end{aligned}$$

combined with the fact that

$$\begin{aligned}
\mathbb{E}\left|U^q - V^q\right| &= \mathbb{E}\left[\max(U, V)^q - \min(U, V)^q\right] \\
&= q \int_0^\infty t^{q-1} \left[\mathbb{P}(\max(U, V) > t) - \mathbb{P}(\min(U, V) > t)\right] dt.
\end{aligned}$$

The equality (4.2.18) is trivial because $\mathbb{E}[U^q], \mathbb{E}[V^q]$ are all finite.

For U, V that are not necessarily bounded but $\mathbb{E}|U^q - V^q| < \infty$ (otherwise (4.2.17) is trivial), we introduce a cutoff $M > 0$ and write $U_M = \min(U, M), V_M = \min(V, M)$. Then the previous discussion implies that

$$\begin{aligned}
&\int_0^M \left|\mathbb{P}(U > t) - \mathbb{P}(V > t)\right| t^{q-1} dt = \int_0^\infty \left|\mathbb{P}(U_M > t) - \mathbb{P}(V_M > t)\right| t^{q-1} dt \\
&\quad \le \frac{1}{q} \mathbb{E}\left|\max(U_M, V_M)^q - \min(U_M, V_M)^q\right|
\end{aligned}$$

$$\leq \frac{1}{q}\mathbb{E}\left|(U^q - V^q)1_{\{\max(U,V)\leq M\}}\right| + \frac{1}{q}\mathbb{E}\left[\left(M^q - \min(U,V)^q\right)1_{\{\max(U,V)\geq M\}}\right]$$
$$\xrightarrow{M\to\infty} \frac{1}{q}\mathbb{E}\left|U^q - V^q\right|$$

by dominated convergence since both

$$\left|(U^q - V^q)1_{\{\max(U,V)\leq M\}}\right|, \qquad \left(M^q - \min(U,V)^q\right)1_{\{\max(U,V)\geq M\}}$$

are bounded by $|U^q - V^q|$. We send $M \to \infty$ on the LHS of the above inequality and obtain (4.2.17) by monotone convergence. The equality (4.2.18) may be proved by a similar cutoff argument. □

4.3 Proof of Theorem 4.1.1

This section is devoted to the proof of the tail asymptotics of subcritical GMC measures. As advertised earlier, our proof of Theorem 4.1.1 consists of two steps.

(i) Tail asymptotics of reference measure (Section 4.3.1): we consider the chaos measure $\overline{M}_{\gamma,g}$ associated with the exact kernel as the reference measure and derive the leading order term of $\mathbb{P}\left(\int_{|x|\leq r}|x|^{-\gamma^2}\overline{M}_{\gamma,g}(dx) > t\right)$ as $t\to\infty$.

(ii) Tail extrapolation principle (Section 4.3.2): the leading order tail behaviour of $M_{\gamma,g}$ can be expressed in terms of that of $\overline{M}_{\gamma,g}$.

Before we start, let us highlight the localisation trick from [RV17]

Lemma 4.3.1. *Let $A \subset D$ be a non-empty open subset. Then*[9]

$$\mathbb{P}\left(M_{\gamma,g}(A) > t\right) = \int_A \mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)>t\}}\right]g(v)dv \tag{4.3.1}$$

where

$$M_{\gamma,g}(v,A) := \int_A \frac{e^{\gamma^2 f(x,v)}M_{\gamma,g}(dx)}{|x-v|^{\gamma^2}}.$$

Sketch of proof. For each $\epsilon > 0$, let X_ϵ be the mollified field with covariance $\mathbb{E}[X_\epsilon(x)X_\epsilon(y)] = -\log(|x-y|\vee\epsilon) + f_\epsilon(x,y)$ where $f_\epsilon(x,y) \xrightarrow{\epsilon\to 0^+} f(x,y)$ pointwise (cf. [Ber17, Lemma

[9] Actually it is not known whether the distribution of $M_{\gamma,g}(v,A)$ is continuous everywhere and hence the correct statement should be

$$\int_A \mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)>t\}}\right]g(v)dv \leq \mathbb{P}\left(M_{\gamma,g}(v,A) > t\right) \leq \int_A \mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)\geq t\}}\right]g(v)dv.$$

We are cheating here so that we do not have to keep the lower and upper bounds everywhere but for the purpose of evaluating the tail asymptotics as $t\to\infty$ it does not make any difference.

3.4]). If $M_{\gamma,\epsilon}(dx)$ is the GMC associated to X_ϵ and $M_{\gamma,g,\epsilon}(dx) = g(x)M_{\gamma,\epsilon}(dx)$, then

$$\begin{aligned}
\mathbb{P}(M_{\gamma,g}(A) > t) &= \lim_{\epsilon\to 0^+} \mathbb{P}(M_{\gamma,g,\epsilon}(A) > t) \\
&= \lim_{\epsilon\to 0^+} \int_A \mathbb{E}\left[\frac{M_{\gamma,g,\epsilon}(A)}{M_{\gamma,g,\epsilon}(A)} 1_{\{M_{\gamma,g,\epsilon}(A)>t\}}\right] dv \\
&= \lim_{\epsilon\to 0^+} \int_A \mathbb{E}\left[\frac{e^{\gamma X_\epsilon(v) - \frac{\gamma^2}{2}\mathbb{E}\left[X_\epsilon(v)^2\right]}}{M_{\gamma,g,\epsilon}(A)} 1_{\{M_{\gamma,g,\epsilon}(A)>t\}}\right] g(v)dv. \qquad (4.3.2)
\end{aligned}$$

One may interpret $e^{\gamma X_\epsilon(v) - \frac{\gamma^2}{2}\mathbb{E}\left[X_\epsilon(v)^2\right]}$ as a Radon-Nikodym derivative, and by applying the Cameron-Martin theorem, we can remove this exponential by shifting the mean of $X(\cdot)$ by $\mathbb{E}\left[X_\epsilon(\cdot)\gamma X_\epsilon(v)\right] = \gamma\left(-\log\left(|\cdot - v| \vee \epsilon\right) + f(\cdot, v)\right)$, i.e.

$$\mathbb{E}\left[\frac{e^{\gamma X_\epsilon(v) - \frac{\gamma^2}{2}\mathbb{E}\left[X_\epsilon(v)^2\right]}}{M_{\gamma,g,\epsilon}(A)} 1_{\{M_{\gamma,g,\epsilon}(A)\}>t\}}\right] = \mathbb{E}\left[\frac{1}{M_{\gamma,g,\epsilon}(v, A)} 1_{\{M_{\gamma,g,\epsilon}(v,A)>t\}}\right] \qquad (4.3.3)$$

where

$$M_{\gamma,g,\epsilon}(v, A) = \int_A e^{\gamma X_\epsilon(x) + \gamma\mathbb{E}[X_\epsilon(x)X_\epsilon(v)] - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} g(x)dx = \int_A \frac{e^{\gamma^2 f_\epsilon(x,v)} M_{\gamma,g,\epsilon}(dx)}{\left(|x - v| \vee \epsilon\right)^{\gamma^2}}.$$

Then (4.3.3) converges to the integrand in (4.3.1) as $\epsilon \to 0^+$, and we can interchange the limit and integral in (4.3.2) by dominated convergence since the expectation is always upper-bounded by $1/t$.

$\square$

4.3.1 The reference measure $\overline{M}_\gamma$

Let $\overline{M}_\gamma^L$ be the GMC associated with the log-correlated field Y_L with covariance $\mathbb{E}[Y_L(x)Y_L(y)] = K_L(x, y) = -\log|x - y| + L$, which by Lemma 4.2.4 is positive definite on $B(0, r_d(L))$. We shall suppress the dependence on L when we are referring to the exact kernel, i.e. $L = 0$. The main result in this subsection concerns the tail probability of $\overline{M}_\gamma(0, r) := \int_{|x|\le r} |x|^{-\gamma^2}\overline{M}_\gamma(dx)$.

Lemma 4.3.2. *There exists some constant $\overline{C}_{\gamma,d} > 0$ such that for any $r \in (0, r_d]$ and as $t \to \infty$,*

$$\mathbb{P}\left(\overline{M}_\gamma(0, r) > t\right) = \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}-1}} + o(t^{-\frac{2d}{\gamma^2}+1}). \qquad (4.3.4)$$

Proof. Pick $c \in (0, 1)$. Using the fact that

$$(Y(cx))_{|x|\le r} \overset{d}{=} (Y(x) + N_c)_{|x|\le r}$$

where $N_c \sim N(0, -\log c)$ is an independent random variable, we obtain

$$\begin{aligned}
\overline{M}_\gamma(0, cr) &= \int_{|x|<|cr|} e^{\gamma Y(x) - \frac{\gamma^2}{2}\mathbb{E}[Y(x)^2]} \frac{dx}{|x|^{\gamma^2}} \\
&= c^d \int_{|u|<|r|} e^{\gamma Y(cu) - \frac{\gamma^2}{2}\mathbb{E}[Y(cu)^2]} \frac{du}{|cu|^{\gamma^2}} \\
&\overset{d}{=} c^{d-\gamma^2} e^{\gamma N_c - \frac{\gamma^2}{2}\mathbb{E}[N_c^2]} \int_{|u|<|r|} e^{\gamma Y(u) - \frac{\gamma^2}{2}\mathbb{E}[Y(u)^2]} \frac{du}{|u|^{\gamma^2}} \\
&= c^{d-\frac{\gamma^2}{2}} e^{\gamma N_c} \overline{M}_\gamma(0, r). && (4.3.5)
\end{aligned}$$

For convenience, set $q = \frac{2d}{\gamma^2} - 1$ and write $M = c^{d-\frac{\gamma^2}{2}} e^{\gamma N_c} = c^{\frac{\gamma^2}{2}q} e^{\gamma N_c}$ and $R = \overline{M}_\gamma(0, r)$. We only need to show that conditions (i) – (iv) in Theorem 4.2.16 are satisfied to obtain our desired tail behaviour. Conditions (ii) and (iii) are trivial, while

$$\mathbb{E}\left[M^q\right] = c^{\frac{\gamma^2}{2}q^2} c^{-\frac{\gamma^2}{2}q^2} = 1$$

and so condition (i) is also satisfied. If we take $U = \overline{M}_\gamma(0, r)$, $V = \overline{M}_\gamma(0, cr)$, and

$$W = U - V = \int_{|x|\in[cr,r)} e^{\gamma Y(x) - \frac{\gamma^2}{2}\mathbb{E}[Y(x)^2]} \frac{dx}{|x|^{\gamma^2}} \le |cr|^{-\gamma^2} \overline{M}_\gamma(B(0, r)).$$

then

$$\begin{aligned}
\int_0^\infty |\mathbb{P}(R > t) - \mathbb{P}(MR > t)| t^{q-1} dt &= \int_0^\infty |\mathbb{P}(U > t) - \mathbb{P}(V > t)| t^{q-1} dt \\
&\le \frac{1}{q} \mathbb{E}\left|(V + W)^q - V^q\right| \\
&\le 2^{q-1} \mathbb{E}\left[V^{q-1} W + W^q\right] && (4.3.6)
\end{aligned}$$

where the first inequality follows from Lemma 4.2.17 and the second inequality from the elementary estimate

$$(V + W)^q - V^q \le q(V + W)^{q-1} W \le q 2^{q-1}\left(V^{q-1} W + W^q\right).$$

Since $\mathbb{E}[W^{q+1-\epsilon}] < \infty$ for any $\epsilon > 0$ (in particular that $\mathbb{E}[W^q] < \infty$), we see that

$$\mathbb{E}[V^{q-1} W] \le \mathbb{E}\left[V^{\frac{(q-1)(q+1-\epsilon)}{q-\epsilon}}\right]^{1-\frac{1}{q+1-\epsilon}} \mathbb{E}\left[W^{q+1-\epsilon}\right]^{\frac{1}{q+1-\epsilon}} < \infty$$

for ϵ sufficiently small so that $(q-1)(q+1-\epsilon)/(q-\epsilon) < q$. Then (4.3.6) is finite and condition (iv) is also satisfied, and by Theorem 4.2.16 (and again Lemma 4.2.17)

$$\mathbb{P}(\overline{M}_\gamma(0, r) > t) = \frac{\overline{C}_{\gamma,d}}{t^q} + o(t^{-q}).$$

□

We summarise various probabilistic representations of $\overline{C}_{\gamma,d}$ in the following corollary.

Corollary 4.3.3. *The constant $\overline{C}_{\gamma,d}$ has the following equivalent representations.*

$$\begin{aligned}\overline{C}_{\gamma,d} &= \lim_{t\to\infty} t^{\frac{2d}{\gamma^2}-1}\mathbb{P}\left(\overline{M}_\gamma(0,r) > t\right) \\ &= \lim_{\lambda\to 0^+} \frac{1}{\frac{2d}{\gamma^2}-1} \frac{\mathbb{E}\left[\overline{M}_\gamma(0,r)^{\frac{2d}{\gamma^2}-1} e^{-\lambda \overline{M}_\gamma(0,r)}\right]}{-\log\lambda} \qquad (4.3.7)\\ &= \frac{1}{-\frac{2}{\gamma^2}\left(d-\frac{\gamma^2}{2}\right)^2\log c}\mathbb{E}\left[\overline{M}_\gamma(0,r)^{\frac{2d}{\gamma^2}-1} - \overline{M}_\gamma(0,cr)^{\frac{2d}{\gamma^2}-1}\right], \qquad \forall c\in(0,1). \qquad (4.3.8)\end{aligned}$$

Proof. The first representation is an immediate consequence of Lemma 4.3.2, and the second representation follows from Lemma 4.2.13. For the third representation, the proof of Lemma 4.3.2 and Theorem 4.2.16 suggests that

$$\begin{aligned}&\lim_{t\to\infty} t^q\mathbb{P}\left(\overline{M}_\gamma(0,r) > t\right) \\ &\quad= \frac{1}{\mathbb{E}\left[c^{\frac{\gamma^2}{2}q^2}e^{\gamma q N_c}\left(\frac{\gamma^2}{2}q\log c + \gamma N_c\right)\right]}\frac{1}{q}\mathbb{E}\left[\overline{M}_\gamma(0,r)^q - \overline{M}_\gamma(0,cr)^q\right]\end{aligned}$$

for $q = \frac{2d}{\gamma^2} - 1$ and any $c \in (0,1)$. Then it is straightforward to check that

$$\mathbb{E}\left[c^{\frac{\gamma^2}{2}q^2}e^{\gamma q N_c}\left(\frac{\gamma^2}{2}q\log c + \gamma N_c\right)\right] = \frac{\gamma^2}{2}q\log c + \gamma\mathbb{E}\left[\gamma q N_c^2\right] = -\frac{\gamma^2}{2}q\log c$$

which implies (4.3.8). □

Remark 4.3.4. *The fact that (4.3.8) holds regardless of $c \in (0,1)$ is not surprising. Indeed when $c = 2^{-N}$, we have*

$$\mathbb{E}\left[\overline{M}_\gamma(0,r)^{\frac{2d}{\gamma^2}-1} - \overline{M}_\gamma(0,cr)^{\frac{2d}{\gamma^2}-1}\right] = \sum_{n=1}^N \mathbb{E}\left[\overline{M}_\gamma(0,2^{-(n-1)}r)^{\frac{2d}{\gamma^2}-1} - \overline{M}_\gamma(0,2^{-n}r)^{\frac{2d}{\gamma^2}-1}\right]$$

and the summand on the RHS does not change with n because of the scaling property (4.3.5). The scaling property also explains why (4.3.8) is independent of $r \in (0, r_d)$ (as long as the exact kernel remains positive definite on $B(0,r)$).

Lemma 4.3.2 has several useful implications.

Corollary 4.3.5. *The following are true.*

(i) *For any $L \in \mathbb{R}$ and $r \in (0, r_d(L)]$, let $\overline{M}_\gamma^L(0,r) = \int_{|x|\le r} |x|^{-\gamma^2} e^{\gamma^2 L} \overline{M}_\gamma^L(dx)$. We have, as $t \to \infty$,*

$$\mathbb{P}\left(\overline{M}_\gamma^L(0,r) > t\right) = e^{\frac{2d}{\gamma}(Q-\gamma)L} \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}-1}} + o(t^{-\frac{2d}{\gamma^2}+1}). \tag{4.3.9}$$

(ii) *Let X be the log-correlated field in Theorem 4.1.1, and $A \subset D$ be a fixed, non-trivial open set. Then there exists some $C > 0$ independent of $v \in A$ such that*

$$\mathbb{P}\left(M_{\gamma,g}(v, A) > t\right) \le \frac{C}{t^{\frac{2d}{\gamma^2}-1}} \qquad \forall t > 0. \tag{4.3.10}$$

Remark 4.3.6. *The importance of Corollary 4.3.5 is as follows.*

- *The tail (4.3.9) in (i) suggests how $\mathbb{P}\left(M_{\gamma,g}(v, A) > t\right)$ should behave asymptotically as $t \to \infty$. As we shall see in the proof, we can pick any $r > 0$ such that $B(v,r) \subset A$ and consider instead $\mathbb{P}\left(M_{\gamma,g}(v, r) > t\right)$ without changing the asymptotic behaviour. When r is small, the covariance structure of X looks like $-\log|x-y| + f(v,v) = K_{f(v,v)}(x,y)$ locally in $B(v,r)$ and we should expect*

$$\mathbb{P}\left(M_{\gamma,g}(v, r) > t\right) \sim e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)} g(v)^{\frac{2d}{\gamma^2}-1} \frac{\overline{C}_{\gamma,d}}{t^{\frac{2d}{\gamma^2}-1}}. \tag{4.3.11}$$

  It is not hard to verify this claim when f is the covariance of some continuous Gaussian field. The situation becomes slightly more tricky under the setting of Theorem 4.1.1 when we only assume that $f = f_+ - f_-$ is the difference of two such covariance kernels and we shall not attempt to prove (4.3.11) here.

- *The uniform bound (4.3.10) in (ii) provides an estimate sufficient for an application of dominated convergence: since*

$$t^{\frac{2d}{\gamma^2}} \mathbb{E}\left[\frac{1_{\{M_{\gamma,g}(v,A) > t\}}}{M_{\gamma,g}(v, A)}\right] \le t^{\frac{2d}{\gamma^2}} \left[\frac{1}{t}\mathbb{P}\left(M_{\gamma,g}(v, A) > t\right)\right] \le C \qquad \forall v \in A,$$

  we have, by the localisation trick (4.3.1)

$$\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}} \mathbb{P}\left(M_{\gamma,g}(v, A) > t\right) = \int_A \left(\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}} \mathbb{E}\left[\frac{1_{\{M_{\gamma,g}(v,A) > t\}}}{M_{\gamma,g}(v, A)}\right]\right) g(v) dv \tag{4.3.12}$$

  provided that the limit on the RHS exists for g-almost every $v \in A$. Note that the existence of this limit is not known a priori. If we were allowed to assume (4.3.11) *though, the existence of such limit would not be an issue because*

$$\begin{aligned}
&\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}1_{\{M_{\gamma,g}(v,A)>t\}}\right]\\
&= \frac{1}{\Gamma(1+\frac{2d}{\gamma^2})}\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\lambda/M_{\gamma,g}(v,A)}\right] \\
&= e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}-1}\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}
\end{aligned} \tag{4.3.13}$$

where the first equality follows from Corollary 4.2.11 and the second from Lemma 4.2.12 (with the fact that $\frac{2d}{\gamma^2}-1=\frac{2}{\gamma}(Q-\gamma)$), and this would yield Theorem 4.1.1. Our proof, however, will adopt a more direct approach of evaluating the Laplace estimate (4.3.13) without assuming the general tail behaviour (4.3.11).

Proof of Corollary 4.3.5. For convenience, let $q=\frac{2d}{\gamma^2}-1=\frac{2}{\gamma}(Q-\gamma)$.

(i) For any $c,\theta\in(0,1)$, we have

$$\begin{aligned}
&\mathbb{P}\left(\overline{M}_\gamma^L(0,cr)>t\right)\le\mathbb{P}\left(\overline{M}_\gamma^L(0,r)>t\right)\\
&\qquad\le\mathbb{P}\left(\overline{M}_\gamma^L(0,cr)>(1-\theta)t\right)+\mathbb{P}\left(\overline{M}_\gamma^L(0,B(0,r)\setminus B(0,cr))>\theta t\right)
\end{aligned}$$

where $\overline{M}_\gamma^L(0,A):=\int_A|x|^{-\gamma^2}e^{\gamma^2L}\overline{M}_\gamma^L(dx)$. Since

$$\mathbb{E}\left[\overline{M}_\gamma^L(0,B(0,r)\setminus B(0,cr))^p\right]\le(cr)^{-p\gamma^2}\mathbb{E}\left[\overline{M}_\gamma^L(B(0,r))^p\right]<\infty\qquad\forall p<\frac{2d}{\gamma^2},$$

the tail probability of the random variable $\overline{M}_\gamma^L(0,B(0,r)\setminus B(0,cr))$ decays faster than t^{-q} as $t\to\infty$ by Markov's inequality, and therefore

$$\begin{aligned}
&\liminf_{t\to\infty}t^q\mathbb{P}\left(\overline{M}_\gamma^L(0,cr)>t\right)\le\liminf_{t\to\infty}t^q\mathbb{P}\left(\overline{M}_\gamma^L(0,r)>t\right)\\
&\qquad\le\limsup_{t\to\infty}t^q\mathbb{P}\left(\overline{M}_\gamma^L(0,r)>t\right)\le\limsup_{t\to\infty}t^q\mathbb{P}\left(\overline{M}_\gamma^L(0,cr)>(1-\theta)t\right).
\end{aligned}$$

As $\theta\in(0,1)$ is arbitrary, if $\mathbb{P}\left(\overline{M}_\gamma(0,r)>t\right)\sim Ct^{-q}$ for some $C>0$, then C must be independent of $r\in(0,r_d(L)]$. We may thus assume $r>0$ to be as small as we like (but independent of t) without loss of generality.

If $L\ge0$, we may interpret $K_L(x,y)=K_0(x,y)+L$ as the sum of the exact kernel and the variance of an independent random variable $\mathcal{N}_L\sim\mathcal{N}(0,L)$, and hence

$$\mathbb{P}\left(\overline{M}_\gamma^L(0,r)>t\right)=\mathbb{P}\left(e^{\gamma\mathcal{N}_L-\frac{\gamma^2}{2}L}\overline{M}_\gamma(0,r)>t\right)\sim\frac{\overline{C}_{\gamma,d}\mathbb{E}\left[\left(e^{\gamma\mathcal{N}_L-\frac{\gamma^2}{2}L}\right)^q\right]}{t^q}$$

by Lemma 4.2.14, and $\mathbb{E}\left[\left(e^{\gamma\mathcal{N}_L-\frac{\gamma^2}{2}L}\right)^q\right]=e^{\frac{2d}{\gamma}(Q-\gamma)L}$.

If $L<0$, we instead interpret $K_L(x,y)=-\log\left|e^{-L}(x-y)\right|$ as the exact kernel with coordinates scaled by e^{-L}. If we restrict ourselves to $x,y\in B(0,e^{-L}r_d)$ or equivalently $r\in(0,e^{-L}r_d]$, then

$$\begin{aligned}\mathbb{P}\left(\overline{M}_\gamma^L(0,r)>t\right)&=\mathbb{P}\left(\int_{|x|\le r}|e^{-L}x|^{-\gamma^2}e^{\gamma Y(e^{-L}x)-\frac{\gamma^2}{2}\mathbb{E}\left[Y(e^{-L}x)^2\right]}dx>t\right)\\&=\mathbb{P}\left(e^{dL}\overline{M}_\gamma(0,e^Lr)>t\right)\sim\frac{\overline{C}_{\gamma,d}e^{dqL}}{t^q}\end{aligned}$$

where $e^{dqL}=e^{\frac{2d}{\gamma}(Q-\gamma)L}$ as expected.

(ii) Let $r=r_d$. Then

$$\mathbb{P}\left(M_{\gamma,g}(v,A)>t\right)\le\mathbb{P}\left(M_{\gamma,g}(v,B(v,r)\cap D)>\frac{t}{2}\right)+\mathbb{P}\left(|r|^{-\gamma^2}e^{\gamma^2L}M_{\gamma,g}(D)>\frac{t}{2}\right).$$

Since $\mathbb{E}\left[M_{\gamma,g}(D)^q\right]<\infty$ by Lemma 4.2.8, Markov's inequality implies that we only need to verify $\mathbb{P}\left(M_{\gamma,g}(v,B(v,r)\cap D)>t\right)\le Ct^{-q}$ uniformly in v.

By (i), let $C>0$ be such that

$$\mathbb{P}\left(\overline{M}_\gamma(0,r)>t\right)\le\frac{C}{t^q}\qquad\forall t>0.$$

To go beyond exact kernels, we utilise the decomposition condition of f. Let $G_\pm(\cdot)$ be independent continuous Gaussian fields on $\overline{D}$ with covariance $f_\pm$, and introduce the random variables

$$R_+=e^{\gamma\sup_{x\in D}G_+(x)+\gamma^2\sup_{y,z\in D}|f(y,z)|},\qquad R_-=e^{\gamma\inf_{x\in D}G_-(x)-\frac{\gamma^2}{2}\sup_{y\in D}|f_-(y,y)|}$$

which possess moments of all orders by Lemma 4.2.2. Let $a>0$ be such that

$$P_{R_-}:=\mathbb{P}(R_->a)>0.$$

Since $\mathbb{E}[X(x)X(y)]+f_-(x,y)=K_0(x-v,y-v)+f_+(x,y)$, we have

$$\begin{aligned}\mathbb{P}(M_{\gamma,g}(v,B(v,r)\cap D)>t)&\le P_{R_-}^{-1}\mathbb{P}(R_-M_{\gamma,g}(v,B(v,r)\cap D)>at)\\&\le P_{R_-}^{-1}\mathbb{P}\left(\int_{B(v,r)\cap D}\frac{e^{\gamma^2f(x,v)}e^{\gamma G_-(x)-\frac{\gamma^2}{2}\mathbb{E}[G_-(x)^2]}}{|x-v|^{\gamma^2}}M_\gamma(dx)>\frac{at}{||g||_\infty}\right)\end{aligned}$$

$$= P_{R_-}^{-1}\mathbb{P}\left(\int_{B(0,r)\cap(D-v)} e^{\gamma^2 f(x+v,v)}e^{\gamma G_+(x+v)-\frac{\gamma^2}{2}\mathbb{E}[G_+(x+v)^2]}\frac{\overline{M}_\gamma(dx)}{|x|^{\gamma^2}} > \frac{at}{||g||_\infty}\right)$$
$$\leq P_{R_-}^{-1}\mathbb{P}\left(R_+\overline{M}_\gamma(0,r) > \frac{at}{||g||_\infty}\right)$$
$$\leq P_{R_-}^{-1}\mathbb{E}\left[\mathbb{P}\left(\overline{M}_\gamma(0,r) > \frac{at}{||g||_\infty R_+}\middle| R_+\right)\right] \leq P_{R_-}^{-1}\frac{C(||g||_\infty/a)^q\mathbb{E}\left[R_+^q\right]}{t^q}.$$

The coefficient $P_{R_-}^{-1}C(||g||_\infty/a)^q\mathbb{E}\left[R_+^q\right] < \infty$ is independent of v so we are done.

□

4.3.2 The tail extrapolation principle

Based on the discussion in Remark 4.3.6, we have actually proved Theorem 4.1.1 when $\mathbb{E}[X(x)X(y)] = K_L(x,y)$ is the L-exact kernel, and in this subsection we shall show the existence of the limit

$$\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\lambda/M_{\gamma,g}(v,A)}\right]$$

and evaluate the value of it.

Step 1: removal of non-singularity. We show that

Lemma 4.3.7. *For any $r > 0$ such that $B(v,r) \subset A$,*

$$\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\lambda/M_{\gamma,g}(v,A)}\right] \overset{\lambda\to\infty}{=} \mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\lambda/M_{\gamma,g}(v,r)}\right] + o(\lambda^{-\frac{2d}{\gamma^2}}) \qquad (4.3.14)$$

Proof. Starting with the localisation trick (4.3.1), we know by the uniform bound (4.3.10) from Corollary 4.3.5 that

$$\mathbb{P}(M_{\gamma,g}(A) > t) \leq \int_A \frac{1}{t}\mathbb{P}\left(M_{\gamma,g}(v,A) > t\right)g(v)dv \leq \frac{C\int_A g(v)dv}{t^{\frac{2d}{\gamma^2}}}$$

for all $t > 0$. In particular

$$\mathbb{P}\left(M_{\gamma,g}(v, A\setminus B(v,r)) > t\right) \leq \mathbb{P}\left(|r|^{-\gamma^2}M_{\gamma,g}(A) > t\right) \leq \frac{C_{r,g}}{t^{\frac{2d}{\gamma^2}}} \qquad \forall t > 0$$

for some $C_{r,g} > 0$.

To finish our proof we only need to show matching upper/lower bounds for (4.3.14).

For a lower bound, pick $\delta \in (0,1)$ and

$$\begin{aligned}
\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\lambda/M_{\gamma,g}(v,A)}\right] &\geq \mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)+M_{\gamma,g}(v,A\setminus B(v,r))}\right] \\
&\geq \mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}\left(1+\frac{\lambda^{1-\delta}}{M_{\gamma,g}(v,r)}\right)^{-1}1_{\{M_{\gamma,g}(v,r)\}\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(v,r))\leq\lambda^{1-\delta}\}}\right] \\
&\geq \left(1-\lambda^{-\frac{3\delta}{4}}\right)\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}1_{\{M_{\gamma,g}(v,r)\}\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(v,r))\leq\lambda^{1-\delta}\}}\right] \\
&= \left(1-\lambda^{-\frac{3\delta}{4}}\right)\left\{\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}\right]-\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}1_{\{M_{\gamma,g}(v,r)\leq\lambda^{1-\frac{\delta}{4}}\}}\right]\right. \\
&\qquad\left.-\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}1_{\{M_{\gamma,g}(v,r)\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(v,r))\geq\lambda^{1-\delta}\}}\right]\right\}
\end{aligned}$$

where

$$\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}1_{\{M_{\gamma,g}(v,r)\leq\lambda^{1-\frac{\delta}{4}}\}}\right]\leq\lambda^{-(1-\delta/4)}e^{-\lambda^{3\delta/4}}=o(\lambda^{-\frac{2d}{\gamma^2}})$$

and

$$\begin{aligned}
&\mathbb{E}\left[\frac{e^{-\lambda/M_{\gamma,g}(v,r)}}{M_{\gamma,g}(v,r)}1_{\{M_{\gamma,g}(v,r)\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(v,r))\geq\lambda^{1-\delta}\}}\right]\Bigg\} \\
&\quad\leq\lambda^{-(1-\delta/4)}\mathbb{P}\left(M_{\gamma,g}(v,A\setminus B(v,r))\geq\lambda^{1-\delta}\right)\leq C_r\lambda^{-(1-\delta)\left(\frac{2d}{\gamma^2}+1\right)}
\end{aligned}$$

and so we just pick $\delta>0$ small enough satisfying $(1-\delta)\left(\frac{2d}{\gamma^2}+1\right)>\frac{2d}{\gamma^2}$ for our desired lower bound.

As for the upper bound,

$$\begin{aligned}
\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\lambda/M_{\gamma,g}(v,A)}\right] &= \mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}e^{-\frac{\lambda}{M_{\gamma,g}(v,r)}\left(1+\frac{M_{\gamma,g}(v,A\setminus B(v,r))}{M_{\gamma,g}(v,r)}\right)^{-1}}\right] \\
&\leq \mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\frac{\lambda}{M_{\gamma,g}(v,r)}\left(1+\lambda^{-\frac{3\delta}{4}}\right)^{-1}}1_{\{M_{\gamma,g}(v,r)\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(0,r))\leq\lambda^{1-\delta}\}}\right] \\
&\quad+\underbrace{e^{-\frac{\lambda^{\delta/4}}{2}}\mathbb{E}\left[M_{\gamma,g}(v,A)^{-1}\right]+\lambda^{-(1-\delta)}\mathbb{P}\left(M_{\gamma,g}(v,A\setminus B(0,r))>\lambda^{1-\delta}\right)}_{=o(\lambda^{-2d/\gamma^2})}
\end{aligned}$$

where

$$
\begin{aligned}
&\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\frac{\lambda}{M_{\gamma,g}(v,r)}\left(1+\lambda^{-\frac{3\delta}{4}}\right)^{-1}}1_{\{M_{\gamma,g}(v,r)\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(0,r))\leq\lambda^{1-\delta}\}}\right]\\
&\quad\leq\underbrace{e^{\lambda^{-\delta/2}}}_{=1+o(1)}\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\frac{\lambda}{M_{\gamma,g}(v,r)}}1_{\{M_{\gamma,g}(v,r)\geq\lambda^{1-\frac{\delta}{4}},M_{\gamma,g}(v,A\setminus B(0,r))\leq\lambda^{1-\delta}\}}\right]\\
&\quad\leq(1+o(1))\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\frac{\lambda}{M_{\gamma,g}(v,r)}}\right]+o(\lambda^{-\frac{2d}{\gamma^2}})
\end{aligned}
$$

where the last inequality follows from similar calculations in the proof of the lower bound. This concludes the proof of (4.3.14). □

Step 2: tail extrapolation. For $s\in[0,1]$, define $Z_s(x)=\sqrt{s}X(x)+\sqrt{1-s}Y_{f(v,v)}(x-v)$, $M_\gamma^s(dx)=e^{\gamma Z_s(x)-\frac{\gamma^2}{2}\mathbb{E}[Z_s(x)^2]}dx$ and

$$
M_{\gamma,g}^s(v,r):=\int_{B(v,r)}\frac{e^{\gamma^2 f(v,v)}g(v)M_\gamma^s(dx)}{|x-v|^{\gamma^2}},\qquad \varphi(s):=\mathbb{E}\left[\frac{1}{M_{\gamma,g}^s(v,r)}e^{-\lambda/M_{\gamma,g}^s(v,r)}\right]\tag{4.3.15}
$$

where $r\in(0,r_d(f(v,v))]$. Our goal is to prove the following extrapolation result.

Lemma 4.3.8. *Suppose $v\in D$ satisfies $g(v)>0$. We have*

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(1)&=\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(0)\\
&=\Gamma\left(1+\frac{2d}{\gamma^2}\right)e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}-1}\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}.
\end{aligned}\tag{4.3.16}
$$

In particular,

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\lambda/M_{\gamma,g}(v,r)}\right]\\
&\quad=\Gamma\left(1+\frac{2d}{\gamma^2}\right)e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}-1}\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}.
\end{aligned}\tag{4.3.17}
$$

Proof. We first recall that the definition of $\varphi(t)$ depends on r but the limits (4.3.16), if exist, do not because of Lemma 4.3.7. Also

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(0)&=\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[\left(g(v)\overline{M}_\gamma^{f(v,v)}(v,r)\right)^{-1}e^{-\lambda/\left(g(v)\overline{M}_\gamma^{f(v,v)}(v,r)\right)}\right]\\
&=\Gamma\left(1+\frac{2d}{\gamma^2}\right)e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}-1}\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}
\end{aligned}
$$

by combining Corollary 4.3.5 ($L = f(v,v)$) with Lemma 4.2.12. From now on we shall focus on the equality of the two limits (4.3.16).

For any $\epsilon > 0$ there exists some $r = r(\epsilon) \in (0, r_d(f(v,v))]$ be such that

$$|f(x,y) - f(v,v)| \le \epsilon \tag{4.3.18}$$

for all $x, y \in B(v,r)$ by continuity. If we write $F(x) = x^{-1}e^{-\lambda/x}$, then $F''(x) = e^{-\lambda/x}\left(\frac{2}{x^3} - \frac{4\lambda}{x^4} + \frac{\lambda^2}{x^5}\right)$, and Corollary 4.2.7 yields

$$|\varphi(1) - \varphi(0)| \le \frac{\epsilon}{2}\int_0^1 \mathbb{E}\left[e^{-\lambda/M_{\gamma,g}^s(v,r)}\left(\frac{2}{M_{\gamma,g}^s(v,r)} + \frac{4\lambda}{M_{\gamma,g}^s(v,r)^2} + \frac{\lambda^2}{M_{\gamma,g}^s(v,r)^3}\right)\right] ds. \tag{4.3.19}$$

Going through the argument in the proof of Corollary 4.3.5(ii), we can check that there exists some $C > 0$ independent of $s \in [0,1]$ and $v \in D$ such that

$$\mathbb{P}\left(M_{\gamma,g}^s(v,r) > t\right) \le \frac{C}{t^{\frac{2d}{\gamma^2}-1}} \qquad \forall t > 0.$$

By Lemma 4.2.12, the integrand in (4.3.19) is uniformly bounded by $C'\lambda^{-\frac{2d}{\gamma^2}}$ for some $C' > 0$ which means that

$$\limsup_{\lambda\to\infty} \lambda^{\frac{2d}{\gamma^2}}|\varphi(1) - \varphi(0)| \le \frac{C'\epsilon}{2}.$$

Since $\epsilon > 0$ is arbitrary, we have $\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(1) = \lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(0)$.

Finally, let $\epsilon, r > 0$ be chosen according to (4.3.18) and the additional constraint that

$$\left|\frac{g(x)}{g(v)} - 1\right| \le \epsilon \qquad \forall x \in B(v,r)$$

which is possible because $g(v) > 0$ and g is continuous. Then

$$\begin{aligned}
&\liminf_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\lambda/M_{\gamma,g}(v,r)}\right]\\
&\qquad \ge \lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}(1+\epsilon)^{-1}e^{-\gamma^2\epsilon}\mathbb{E}\left[M_{\gamma,g}^1(v,r)^{-1}e^{-\lambda(1+\epsilon)e^{\gamma^2\epsilon}/M_{\gamma,g}^1(v,r)}\right]\\
&\qquad = \left((1+\epsilon)e^{\gamma^2\epsilon}\right)^{-\left(1+\frac{2d}{\gamma^2}\right)}\lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\varphi(1),\\
&\limsup_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}\mathbb{E}\left[M_{\gamma,g}(v,r)^{-1}e^{-\lambda/M_{\gamma,g}(v,r)}\right]\\
&\qquad \le \lim_{\lambda\to\infty}\lambda^{\frac{2d}{\gamma^2}}(1+\epsilon)e^{\gamma^2\epsilon}\mathbb{E}\left[M_{\gamma,g}^1(v,r)^{-1}e^{-\lambda(1+\epsilon)^{-1}e^{-\gamma^2\epsilon}/M_{\gamma,g}^1(v,r)}\right]
\end{aligned}$$

$$= \left((1+\epsilon)e^{\gamma^2\epsilon}\right)^{\left(1+\frac{2d}{\gamma^2}\right)} \lim_{\lambda\to\infty} \lambda^{\frac{2d}{\gamma^2}}\varphi(1).$$

Given that the lim inf/lim sup do not depend on r by Lemma 4.3.7 and ϵ is arbitrary, the claim (4.3.17) follows and this concludes the proof. □

Proof of Theorem 4.1.1. Since

$$t^{\frac{2d}{\gamma^2}}\mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)>t\}}\right] \le t^{\frac{2d}{\gamma^2}-1}\mathbb{P}\left(M_{\gamma,g}(v,A)>t\right)$$

is uniformly bounded in $v\in A$ by Corollary 4.3.5, and

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}}\mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)>t\}}\right] &= \lim_{\lambda\to\infty}\frac{\lambda^{\frac{2d}{\gamma^2}}}{\Gamma(1+\frac{2d}{\gamma^2})}\mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}e^{-\lambda/M_{\gamma,g}(v,A)}\right]\\ &= e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}-1}\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}\end{aligned}$$

g-almost everywhere by Corollary 4.2.11, Lemma 4.3.7 and Lemma 4.3.8, we conclude that

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}}\mathbb{P}\left(M_{\gamma,g}(A)>t\right) &= \int_A\left(\lim_{t\to\infty} t^{\frac{2d}{\gamma^2}}\mathbb{E}\left[\frac{1}{M_{\gamma,g}(v,A)}1_{\{M_{\gamma,g}(v,A)>t\}}\right]\right)g(v)dv\\ &= \left(\int_A e^{\frac{2d}{\gamma}(Q-\gamma)f(v,v)}g(v)^{\frac{2d}{\gamma^2}}dv\right)\frac{\frac{2}{\gamma}(Q-\gamma)}{\frac{2}{\gamma}(Q-\gamma)+1}\overline{C}_{\gamma,d}\end{aligned}$$

by dominated convergence. □

Appendix 4.A Reflection coefficient of GMC

In this appendix we explain why $\overline{C}_{\gamma,d}$ should be seen as a natural d-dimensional analogue of the Liouville reflection coefficients evaluated at γ. To commence with, we define $\overline{C}_{\gamma,d}(\alpha)$, which we call the reflection coefficient of GMC, for each $\alpha\in(\frac{\gamma}{2},Q)$ as follows.

Proposition 4.A.1. *Let $\overline{M}_{\gamma,\alpha}(0,r)=\int_{|x|\le r}|x|^{-\gamma\alpha}\overline{M}_\gamma(dx)$ for $\alpha\in(\frac{\gamma}{2},Q)$. Then there exists some constant $\overline{C}_{\gamma,d}(\alpha)>0$ independent of $r\in(0,r_d)$ such that*

$$\begin{aligned}\overline{C}_{\gamma,d}(\alpha) &= \lim_{t\to\infty} t^{\frac{2}{\gamma}(Q-\alpha)}\mathbb{P}\left(\overline{M}_{\gamma,\alpha}(0,r)>t\right)\\ &= \lim_{\lambda\to 0^+}\frac{1}{\frac{2}{\gamma}(Q-\alpha)}\frac{\mathbb{E}\left[\overline{M}_{\gamma,\alpha}(0,r)^{\frac{2}{\gamma}(Q-\alpha)}e^{-\lambda\overline{M}_{\gamma,\alpha}(0,r)}\right]}{-\log\lambda}.\end{aligned} \tag{4.A.1}$$

Proof. The first equality can be obtained by repeating the proof of Lemma 4.3.2, and the second equality follows from Lemma 4.2.13. □

We now show that $\overline{C}_{\gamma,d}(\alpha)$ coincides with the Liouville reflection coefficients[10].

Proposition 4.A.2. *When $d = 2$, the reflection coefficient $\overline{C}_{\gamma,2}(\alpha)$ of GMC is equivalent to the unit volume Liouville reflection coefficient $\overline{R}(\alpha)$ defined in [RV17].*

Proof. Using the notations in [RV17], we can write

$$\overline{M}_{\gamma,\alpha}(0,1) \stackrel{d}{=} e^{\gamma M} \int_{-L_{-M}}^{\infty} e^{\gamma \mathcal{B}_s^\alpha} Z_s ds =: e^{\gamma M} \mathcal{I}(L_{-M})$$

where

- $Z_s ds$ is the GMC associated with the lateral noise of GFF;
- $(\mathcal{B}_s^\alpha)_{s\in\mathbb{R}}$ an independent two-sided Brownian motion with negative drift $\alpha - Q$ conditioned to stay non-positive;
- M is an independent $\mathrm{Exp}(2(Q-\alpha))$ random variable; and
- L_{-M} is the last time $(\mathcal{B}_s^\alpha)_{s\geq 0}$ hits $-M$.

Applying (4.A.1) and the decomposition above, we have

$$\overline{C}_{\gamma,2}(\alpha) = \lim_{\lambda\to 0^+} \frac{1}{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\mathcal{I}(L_{-M})^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{(e^{\gamma M})^{\frac{2}{\gamma}(Q-\alpha)} e^{-\lambda e^{\gamma M}\mathcal{I}(L_{-M})}}{-\log\lambda}\right)\right].$$

When $\lambda \to 0^+$, the above expectation is dominated by the event that the exponential variable M is large, in which case L_{-M} is very large and $\mathcal{I}(L_{-M})$ behaves like $\mathcal{I}(\infty)$ which does not depend on M. To make this rigorous we aim to prove matching upper/lower bounds. Since $\mathbb{P}(e^{\gamma M} > t) = t^{-\frac{2}{\gamma}(Q-\alpha)}$ for $t \geq 1$, a straightforward computation shows that

$$\mathbb{E}\left[\left(e^{\gamma M}\right)^{\frac{2}{\gamma}(Q-\alpha)} e^{-\lambda e^{\gamma M}}\right] = -\frac{2}{\gamma}(Q-\alpha)e^{-\lambda}\log\lambda + O(1)$$

where the error $O(1)$ is bounded independently of $\lambda > 0$. Using the fact that $\mathcal{I}(\infty)$ has moments of all orders smaller than $\frac{4}{\gamma^2}$ ([KRV17, Lemma 2.8]), we deduce that

$$\begin{aligned}\overline{C}_{\gamma,2}(\alpha) &\leq \lim_{\lambda\to 0^+} \frac{1}{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\mathcal{I}(\infty)^{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\left(\frac{(e^{\gamma M})^{\frac{2}{\gamma}(Q-\alpha)} e^{-\lambda e^{\gamma M}\mathcal{I}(0)}}{-\log\lambda}\right)\middle|\mathcal{I}(0)\right]\right] \\ &= \mathbb{E}\left[\mathcal{I}(\infty)^{\frac{2}{\gamma}(Q-\alpha)}\right]\end{aligned}$$

[10] We only focus on $d = 2$; for $d = 1$ the same proof shows that $\overline{C}_{\gamma,1}$ coincides with the boundary unit volume reflection coefficient, see [RV17, Section 4.3].

which is the desired upper bound. Now fix any $T > 0$, we have

$$\begin{aligned}\overline{C}_{\gamma,2}(\alpha) &\geq \lim_{\lambda\to 0^+} \frac{1}{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\mathcal{I}(L_{-T})^{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\left(\frac{(e^{\gamma M})^{\frac{2}{\gamma}(Q-\alpha)} e^{-\lambda e^{\gamma M}\mathcal{I}(\infty)}}{-\log\lambda}\right) \Big| \mathcal{I}(\infty)\right]\right] \\ &\quad - \lim_{\lambda\to 0^+} \frac{1}{\frac{2}{\gamma}(Q-\alpha)} \mathbb{E}\left[\mathcal{I}(\infty)^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{(e^{\gamma M})^{\frac{2}{\gamma}(Q-\alpha)} e^{-\lambda e^{\gamma M}\mathcal{I}(\infty)}}{-\log\lambda}\right) 1_{\{M\leq T\}}\right] \\ &= \mathbb{E}\left[\mathcal{I}(L_{-T})^{\frac{2}{\gamma}(Q-\alpha)}\right].\end{aligned}$$

Since T is arbitrary, we may send $T \to \infty$ so that $L_{-T} \to \infty$ and obtain $\overline{C}_{\gamma,2}(\alpha) \geq \mathbb{E}\left[\mathcal{I}(\infty)^{\frac{2}{\gamma}(Q-\alpha)}\right]$. This matches our upper bound and is precisely the probabilistic definition of the Liouville reflection coefficient $\overline{R}(\alpha)$ in [RV17, equation (1.10)].

□

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