# Boundary Value Problems for the Laplace Equation on 

Bounded Convex Domains with

## Analytic Boundary



A dissertation submitted to the University of Cambridge for the degree of Doctor of Philosophy

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To my parents and family for their tireless dedication and support.

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## Preface

## Declaration

I declare that the work contained in this thesis is either entirely my own, or a result of discussion with my supervisor, Dr. Anthony Ashton. No part of this dissertation has been submitted for any other qualification or degree.

# Boundary Value Problems for the Laplace Equation on Bounded Convex Domains with Analytic Boundary 

By Parousia Rockstroh

In this thesis we study boundary value problems for the Laplace equation on bounded convex domains with analytic boundary. It is well-established that the solution of a BVP for the Laplace equation is determined by the Dirichlet and Neumann boundary data. Indeed, given the Dirichlet and Neumann data for a specified BVP, the standard representation formula for the Laplace equation can be used to specify the solution at any point in the interior of the domain. Our approach in this thesis will be to develop an invertible relation between the Dirichlet and Neumann data for a given BVP. This will allow us to solve for the unknown boundary data, given the known boundary data, thereby finding a solution to the Laplace equation.

We begin the thesis by discussing a number of classical techniques for solving the Laplace equation, along with their respective strengths and weaknesses. Following this, we introduce a method by T. Fokas that was originally developed in the context of integrable systems. In recent years this method has been used to analyze and solve a range of elliptic PDEs in $\mathbb{C} \cong \mathbb{R}^{2}$. A key component of this method is a relation, known as the global relation, that couples the boundary data for a given BVP. To date, this relation has primarily been applied to PDEs on polygonal domains. In this thesis we extend the use of the global relation to more general domains with analytic boundary.

In Chapter 3 we introduce a new transform, denoted by $\mathcal{F}_{p}$, that is an analogue of the Fourier transform on analytic concave curves. This transform naturally arises in the context of the global relation on domains with analytic boundary. In the remainder of the chapter we show that the $\mathcal{F}_{p}$-transform is a bounded operator.

In Chapter 4 we perform a spectral analysis of a differential equation which is an eigenvalue problem associated to the $\mathcal{F}_{p}$-transform. In performing this analysis, we solve a Riemann-Hilbert problem that enables us to reconstruct a solution to the differential equation that is sectionally analytic in the complex plane. Using this
sectionally analytic function, we derive an inverse to the $\mathcal{F}_{p}$-transform.
In Chapter 5 we construct an operator $T$ that allows us to write the global relation as an operator equation. Using results from the previous two chapters, we show that the operator $T$ is continuous and bounded below. From this, we conclude that the global relation defines a map between the Dirichlet and Neumann data that is continuously invertible. Hence, this allows us to find a solution to the Laplace equation for a given BVP on a domain with analytic boundary.

Finally, in Chapter 6 we construct a numerical method to find the Neumann data, given the Dirichlet data, for a specified BVP on a domain with analytic boundary. This is done by extending a previous method by Fornberg and Flyer, [16], that was developed for polygonal domains. We run the method on three test cases and discuss the corresponding convergence properties. We observe that the method is particularly well-suited for domains where the boundary has low curvature.

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## List of Symbols

$\lesssim A \quad$ Asymptotically less than or equal to with constant depending on $A$. See page 34 for formal definition.
$\Omega \quad$ A bounded convex domain in $\mathbb{R}^{2}$ with analytic boundary.
$\Gamma \quad$ The boundary $\partial \Omega$.
$\Gamma_{j} \quad$ The $j$ th component of the boundary $\Gamma=\partial \Omega$.
$\Sigma$ The interval $[-\sigma, \sigma]$, where $0<\sigma<\infty$.
$\Sigma_{j} \quad$ The interval $\left[-\sigma_{j}, \sigma_{j}\right]$, corresponding to the boundary segment $\Gamma_{j}$.
$\varphi, \varphi_{j} \quad$ A function in $L^{2}(\Sigma)$ or $L^{2}\left(\Sigma_{j}\right)$, respectively.
$\Lambda \quad$ The interval $(0, \infty)$.
$\boldsymbol{\Sigma} \quad$ The space $L^{2}\left(\Sigma_{1}\right) \times \ldots \times L^{2}\left(\Sigma_{n}\right)$.
$\boldsymbol{\Lambda}$ The space $\underbrace{L^{2}(\Lambda) \times \ldots \times L^{2}(\Lambda)}_{n-\text { copies }}$.
$\tilde{\varphi}(\lambda)$ The transform $\tilde{\varphi}(\lambda)=\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)(1+$ $\left.p^{\prime}(x)\right) d x$. Referred to as the $\mathcal{F}_{p}$-transform.
$\mathcal{D} \quad$ The $\mathcal{F}_{p}$-transform of Dirichlet data on $\Gamma$.
$\mathcal{N} \quad$ The $\mathcal{F}_{p}$-transform of Neumann data on $\Gamma$.
$\alpha_{j}(\tau)$ The angle that the tangent vector to the boundary segment $\Gamma_{j}$ makes with the real axis at the point $\tau$.
$\Psi \quad$ The vector $\left(\tilde{\varphi}_{1}\left(e^{-i \alpha_{1}(0)} \lambda\right), \ldots, \tilde{\varphi}_{n}\left(e^{-i \alpha_{n}(0)} \lambda\right)\right)$.
$\Psi^{t} \quad$ The tangential component of the vector $\Psi$.
$\Psi^{n} \quad$ The normal component of the vector $\Psi$.

## List of Terminology

- Dirichlet-Neumann Map: Given a harmonic function $u$ in a bounded domain $\Omega \subset \mathbb{R}^{n}$, the Dirichlet-Neumann operator maps the function values $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial \Omega$ (i.e., the Dirichlet data) to the normal derivative on the boundary of the domain, i.e. $\partial u / \partial \boldsymbol{n}(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial \Omega$.
- Poincare-Steklov Operator: The Dirichlet-Neumann map can be realized via the Poincare-Steklov operator, which is defined by:

$$
S:\left.\left.u\right|_{\partial \Omega} \mapsto \frac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial \Omega}
$$

- Representation Formula: If a function $u \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ satisfies the PDE $\Delta u(\boldsymbol{x})=0$ for all $\boldsymbol{x} \in \Omega$, then an explicit formula for the solution for the Laplace equation in the domain $\Omega$ is given by the representation formula, which states:

$$
u(\boldsymbol{x})=\int_{\Gamma} \Phi(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{y})-u(\boldsymbol{y}) \frac{\partial \Phi}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y}) d \sigma(\boldsymbol{y})
$$

where $\Phi(\boldsymbol{x}, \boldsymbol{y})$ is the fundamental solution of the Laplace equation, and $d \sigma$ is the surface measure on $\Gamma=\partial \Omega$.

- Global Relation: The global relation is a relation that couples the boundary data for a given BVP for the Laplace equation. The global relation depends on
a spectral parameter $\lambda \in \mathbb{C}$ and may take one of the following forms:
(1) $\sum_{j=1}^{n} \rho_{j}(\lambda)=0, \quad$ where $\rho_{j}(\lambda)=\int_{\Gamma_{j}} e^{-i z \lambda} \frac{\partial u}{\partial z} d z$,
(2) $0=\int_{\partial \Omega} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left(i(\boldsymbol{\lambda} \cdot \boldsymbol{x}) u(\boldsymbol{x})+\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x})\right) d \sigma(\boldsymbol{x}), \quad \lambda \in Z_{p}=\left\{\boldsymbol{\lambda} \in \mathbb{C}^{n}: \Delta(\boldsymbol{\lambda})=0\right\}$,
(3) $T(\mathcal{N}-i \mathcal{D})=0$,
where the operator $T$ takes the form $T=I+K$, with

$$
K \Phi_{j}=\sum_{k \neq j} e^{i e^{-i \alpha_{j}}\left(m_{j}-m_{k}\right) \lambda} \Phi_{k}\left(e^{-i\left(\alpha_{j}-\alpha_{k}\right)} \lambda\right), \quad 1 \leq j \leq n,
$$

where $\Phi$ is an $n$-dimensional vector.

- Hilbert-Schmidt Operator: A Hilbert-Schmidt operator is an integral operator $K: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ given by:

$$
(K u)(x)=\int_{\Omega} k(x, y) u(y) d y,
$$

such that the kernel $k$ is in $L^{2}(\Omega) \times L^{2}(\Omega)$.

- Complete Contour: A curve $\Gamma$ is said to be a complete contour if $\Gamma$ can be oriented so that $\mathbb{C} \backslash \Gamma$ can be decomposed into left and right components. That is, $\mathbb{C} \backslash \Gamma$ can be written in the form $\mathbb{C} \backslash \Gamma=\Omega^{+} \cup \Omega^{-}$, where $\Omega^{+} \cap \Omega^{-}=\emptyset$.
- Sectionally Analytic: Let $\Gamma$ be a complete contour that divides the complex plane into regions $\Omega^{+}$and $\Omega^{-}$which lie to the left and right of the contour $\Gamma$, respectively. A function $\Phi$ defined in the complex plane, except possibly along a complete contour $\Gamma$, is said to be sectionally analytic if the following two properties hold:
- The function $\Phi(z)$ is analytic in each of the regions $\Omega^{+}$and $\Omega^{-}$, except possibly at $z=\infty$.
- The following limits exist:

$$
\begin{align*}
& \Phi^{+}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{+}}} \Phi(z), \quad t \in \Gamma  \tag{0.0.1}\\
& \Phi^{-}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{-}}} \Phi(z), \quad t \in \Gamma, \tag{0.0.2}
\end{align*}
$$

where we assume the limit is taken along a path that is contained entirely in $\Omega^{+}$or $\Omega^{-}$, respectively, and is not tangent to $\Gamma$ in each case.

- Riemann-Hilbert Problem: Given a complete contour $\Gamma$ in the complex plane, and jump functions $G: \Gamma \rightarrow \mathbb{C}$ and $F: \Gamma \rightarrow \mathbb{C}$, find a function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow$ $\mathbb{C}$ such that:
$-\Phi(z)$ is sectionally analytic, i.e. $\Phi(z)$ is analytic for all $z \in \mathbb{C} \backslash \Gamma$ and the limits $\Phi^{+}(t)$ and $\Phi^{-}(t)$ defined by (0.0.1)-(0.0.2) exist for all $t \in \Gamma$.
- $\Phi(z)$ is bounded at $z=\infty$, and in particular, the following bound holds:

$$
\limsup _{|z| \rightarrow \infty}|z|^{n}|\Phi(z)|<\infty,
$$

for some finite integer $n$.

- $\Phi(z)$ satisfies the following jump condition:

$$
\Phi^{+}(t)=\Phi^{-}(t) G(t)+F(t) \quad \text { for } t \in \Gamma,
$$

- Generalized Analytic Function: Let $\Phi(z, \bar{z})$ be a function defined in the region $\Omega \subset \mathbb{C}$. The function $\Phi(z, \bar{z})$ is said to be a generalized analytic function if:

$$
\frac{\partial \Phi}{\partial \bar{z}}=f(z, \bar{z}), \quad z \in \Omega .
$$

In the case where $\partial \Phi / \partial \bar{z}=0$, the function $\Phi$ is analytic as a consequence of the Cauchy-Riemann equations.

- Generalized Sectionally Analytic: Let $\Gamma$ be a complete contour that divides the complex plane into regions $\Omega^{+}$and $\Omega^{-}$which lie to the left and right of the
contour $\Gamma$, respectively. A function $\Phi(z, \bar{z})$ defined in the complex plane, except possibly along a complete contour $\Gamma$, is referred to as a generalized sectionally analytic if the following two properties hold:
$-\Phi(z, \bar{z})$ is a generalized analytic function in each of the regions $\Omega^{+}$and $\Omega^{-}$, except possibly at $z=\infty$.
- The following limits exist:

$$
\begin{aligned}
& \Phi^{+}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{+}}} \Phi(z), \quad t \in \Gamma \\
& \Phi^{-}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{-}}} \Phi(z), \quad t \in \Gamma
\end{aligned}
$$

where we assume the limit is taken along a non-tangential path that is contained entirely in $\Omega^{+}$or $\Omega^{-}$, respectively.

- $\bar{\partial}$-Problem: Given a complete contour $\Gamma$ in the complex plane, and jump function $F: \Gamma \rightarrow \mathbb{C}$, find a function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow \mathbb{C}$ such that:
$-\Phi(z)$ is a generlized sectionally analytic function,
$-\Phi(z)$ has finite degree at infinity,
$-\Phi(z)$ satisfies the following jump condition:

$$
\Phi^{+}(t)-\Phi^{-}(t)=F(t) \quad \text { for } t \in \Gamma
$$

- Positive Operator: An operator $T: X \rightarrow Y$ between two ordered vector spaces is called positive if $T x \geq 0$ for all $x \geq 0$.


## Introduction: The Laplace Equation

### 1.1 Problem Statement

When dealing with elliptic PDEs on bounded domains in $\mathbb{R}^{n}$, such as the Laplace equation:

$$
\begin{equation*}
\Delta u(\mathbf{x})=0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^{n}, \tag{1.1.1}
\end{equation*}
$$

there are a number of different boundary conditions that may be assigned to form a boundary value problem corresponding to the given PDE. The two most common types of boundary conditions are the Dirichlet and Neumann:

- Dirichlet: The function $u(\mathbf{x})$ is known for $\mathbf{x} \in \partial \Omega$.
- Neumann: The normal derivative $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})$ is known for $\mathbf{x} \in \partial \Omega$, where $\frac{\partial u}{\partial \mathbf{n}}=\nabla u \cdot \boldsymbol{n}$, and $\boldsymbol{n}$ is the outward unit normal on $\partial \Omega$.

Other boundary conditions may arise, such as the Robin boundary condition which is a combination of the Dirichlet and Neumann conditions, i.e. $\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})+u(\mathbf{x})$ is given for $\mathbf{x} \in \partial \Omega$. Mixed boundary conditions may also arise in which a Dirichlet condition is prescribed on part of the boundary while a Neumann condition is prescribed on the remaining part. In this thesis we will be concerned primarily with Dirichlet and

Neumann boundary conditions.
One often wants to find the Neumann data corresponding to the Dirichlet problem for an elliptic PDE on a given domain, or similarly the Dirichlet data given the Neumann data. We give two examples below

- Suppose there is a steady-state distribution of temperature in a domain $\Omega \subset \mathbb{R}^{2}$ and the temperature distribution on the boundary of the domain $\partial \Omega$ is known, then we may also want to know the heat flux through the boundary. The correspondence between the temperature on the boundary and the heat flux required to maintain the temperature distribution is given by the correspondence known as the Dirichlet-Neumann map, which maps the Dirichlet boundary values to the Neumann boundary values for a given elliptic PDE on the bounded domain $\Omega \subset \mathbb{R}^{2}$.
- Suppose that a medium is in a steady state of electrical conduction in a bounded domain $\Omega \subset \mathbb{R}^{2}$, and suppose the voltage potential at the boundary and induced current flux through the boundary of the domain are known. Then we may want to find the electrical conductivity of the medium in the domain $\Omega$. This is the central problem of Electrical Impedance Tomography (EIT). In this example, the Dirichlet condition is given by the voltage potential at the boundary, and the Neumann condition is the current flux through the boundary. Moreover, the Dirichlet-Neumann map gives the correspondence between the voltage potential at the boundary and the induced current flux through the boundary. The Dirichlet-Neumann map is essential in finding the coefficient of the corresponding elliptic PDE in divergence form (and hence finding the electrical conductivity of the medium).

In this thesis we are concerned with analyzing the Laplace equation and corresponding Dirichlet-Neumann map associated to the Dirichlet problem given by

$$
\begin{cases}\Delta u(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega  \tag{1.1.2}\\ u(\mathbf{x})=f(\mathbf{x}) & \text { for } \mathbf{x} \in \Gamma,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{2}$ is a bounded convex region with an analytic boundary $\Gamma=\partial \Omega, u \in$ $C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$, and $f \in C^{1}(\Gamma)$ is given (Dirichlet) boundary data on $\Gamma$.

In the remainder of this chapter we will review classical methods for solving the Laplace equation on bounded domains in $\mathbb{R}^{2}$. In Chapter 2 we will introduce a more recent method that has been developed for solving the Laplace equation on polygonal domains in $\mathbb{C} \cong \mathbb{R}^{2}$. This method will provide the theoretical framework that will be used for the remainder of the thesis.

### 1.2 Classical Methods

In this section we give a brief summary of several classical methods for solving the Laplace equation in a bounded domain and comment on the incorporation of the boundary data into the solution for each case.

### 1.2.1 Separation of Variables

Perhaps the most common method of solution for the Laplace equation is separation of variables, which naturally leads to a Fourier series representation of the solution. The method of separation of variables proceeds as follows:

1. Choose a coordinate system such that the PDE is separable - this will depend on both the PDE operator and the domain on which the PDE is being solved.
2. Write the (proposed) solution as the product of two functions (each of which is a function of only one of the variables from the original PDE). Substitute this back into the original PDE, and separate the variables. This produces two ODEs - one for each of the variables.
3. Substitute the (proposed) solution into the boundary conditions.
4. One of the ODEs will be a BVP - solve this to determine the eigenvalues and eigenfunctions for the given problem.
5. Solve the second ODE, and use this to simplify the solution if possible.
6. Use the principle of superposition to write a solution that solves the PDE and satisfies the given boundary conditions. The solution generally comes in the form
of a Fourier series such as

$$
\begin{equation*}
u(x, y)=\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{s}_{j, k} e^{i j x} e^{i k y} \tag{1.2.1}
\end{equation*}
$$

The method of separation of variables results in a solution that is given in terms of a Fourier series which naturally incorporates the boundary conditions. This is a result of the fact that the boundary conditions were used in constructing the series (i.e., steps 3-4 above). This approach gives a compact way of expressing the solution, however, since an entirely new series must be constructed every time the boundary conditions change, it is not an efficient way to study boundary value problems. In addition, the domain must be 'nice' enough to find a coordinate system such that the PDE is separable and such that the boundary conditions can be used in the solution (i.e., step 1 above). Hence, for more complex domains it is difficult to use this method to solve BVPs.

One benefit of separation of variables is that when the solution is sufficiently smooth and a convergent Fourier series can be constructed for the given PDE and domain, this leads to an accurate and stable numerical method. These are known as spectral methods, and they exhibit exponential convergence, [20], [35].

### 1.2.2 Conformal Mapping

Let $\Phi(z)=u(x, y)+i v(x, y)$ be an analytic function in the domain $\Omega$. As a consequence of the fact that $\Phi(z)$ satisfies the Cauchy-Riemann equations, the real and imaginary parts $u(x, y)$ and $v(x, y)$ are both harmonic, i.e. satisfy (1.1.1), in $\Omega$. Therefore, the problem of solving the Laplace equation in the domain $\Omega$ reduces to the problem of finding an analytic function that satisfies the desired boundary conditions on $\partial \Omega$.

When the domain is the upper half plane or the unit disk, the process of finding an analytic function $\Phi(z)$ that satisfies the prescribed boundary conditions is simplified. Thus, we desire a change of variables of the form $w=f(z)$ such that the region $\Omega$ in the $z$-plane is mapped to the upper half plane or the unit circle in the $w$-plane. Further, we require that the function $f(z)$ be analytic and that its derivative $d f / d z$ is never zero. The mapping $w=f(z)$ is known as a conformal map, and has the
property that angles between intersecting curves are preserved by the transformation.
As a consequence of the fact that $f(z)$ is analytic and its derivative is never zero, the mapping $w=f(z)$ is univalent, i.e. one-to-one and holomorphic. Hence, the mapping can be inverted so that for every $w \in \Omega^{*}$ there exists a unique $z \in \Omega$ such that $z=f^{-1}(w)$. Moreover, if $\phi(z)$ is a function defined on $\Omega$, then it can be mapped to the domain $\Omega^{*}$ via the function $\psi(w)$, defined by

$$
\begin{equation*}
\psi(w)=\phi\left(f^{-1}(w)\right), \tag{1.2.2}
\end{equation*}
$$

for each $w \in \Omega^{*}$. Furthermore, it can be verified via the Cauchy-Riemann equations that if $\phi(z)$ is harmonic in $\Omega$, then $\psi(w)$ will be harmonic in $\Omega^{*}$. In addition, the boundary values of $\phi(z)$ on $\partial \Omega$ can be mapped to the boundary values of $\psi(w)$ on $\partial \Omega^{*}$.

The Riemann mapping theorem states that if $\Omega$ is a non-empty simply connected open subset of the complex plane, and is not the entire complex plane, then there is a bijective holomorphic mapping $f: \Omega \rightarrow D$ from the domain $\Omega$ to the unit disk $D$, and furthermore, the inverse $f^{-1}: D \rightarrow \Omega$ is also holomorphic. This guarantees the existence of a conformal map for any such domain $\Omega$, however, the proof of the Riemann mapping theorem is non-constructive and in practice it is often difficult to find such a mapping.

In some special cases a conformal transformation of the form mentioned above can be constructed. One such case is that of a polygonal domain in which the upper half plane is mapped to the interior of a polygon while the real axis is mapped to the boundary. This mapping is called a Schwarz-Christoffel mapping, and is realized by the function $f$ given by:

$$
\begin{equation*}
f(z)=C_{1} \int_{0}^{z}\left(w-x_{1}\right)^{\alpha_{1} / \pi}\left(w-x_{2}\right)^{\alpha_{2} / \pi} \cdots\left(w-x_{n}\right)^{\alpha_{n} / \pi} \mathrm{d} w+C_{2} . \tag{1.2.3}
\end{equation*}
$$

Here $C_{1}$ and $C_{2}$ are constants and $\alpha_{1}, \ldots, \alpha_{n}$ represent the interior angles of the polygon. The function $f(z)$ maps the points $x_{1}, \ldots, x_{n}$ on the real axis to the corners
$w_{1}, \ldots, w_{n}$ of the polygon, with the correspondences

$$
\begin{equation*}
f\left(x_{1}\right)=w_{1}, \quad f\left(x_{2}\right)=w_{2}, \quad \ldots, \quad f\left(x_{n-1}\right)=w_{n-1}, \quad f(\infty)=w_{n} . \tag{1.2.4}
\end{equation*}
$$

For an exposition on the Schwarz-Christoffel mapping along with some numerical methods for solving the corresponding integrals, see [8]. The Schwarz-Christoffel mapping technique is an effective way of solving the Laplace equation in a general polygonal domain, however it often results in integrals that are difficult to evaluate analytically or numerically. More generally, the method of conformal mapping is effective in the cases where a mapping to the upper half plane or unit circle is known, however such a mapping is only known for a limited number of domains. For further reading on conformal mapping see [1].

### 1.2.3 Integral Representation via Fundamental Solution

Consider the BVP for the Laplace equation given by

$$
\begin{cases}\Delta u(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega \subset \mathbb{R}^{n}  \tag{1.2.5}\\ u(\mathbf{x})=f(\mathbf{x}) & \text { for } \mathbf{x} \in \Gamma\end{cases}
$$

where the boundary of the domain $\Omega$ is $C^{2}$. A fundamental solution corresponding to the operator $\Delta$ is a function $\Phi(\boldsymbol{x}, \boldsymbol{y})$ defined on $\mathbb{R}^{n}$ such that

$$
\begin{equation*}
\Delta \Phi(\boldsymbol{x}, \boldsymbol{y})=\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{1.2.6}
\end{equation*}
$$

where $\delta$ is the Dirac delta distribution. The fundamental solution for the Laplace equation is given by

$$
\Phi(\boldsymbol{x}, \boldsymbol{y})= \begin{cases}\frac{1}{2 \pi} \ln (|\boldsymbol{x}-\boldsymbol{y}|) & \text { for } n=2  \tag{1.2.7}\\ \frac{1}{\omega_{n}(n-2)} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{n-2}}, & \text { for } n \geq 3\end{cases}
$$

where $\omega_{n}=2 \pi^{n / 2} / \Gamma(n / 2)$, which is the surface area of the unit sphere in $\mathbb{R}^{n}$, and $\Gamma$ is the gamma function.

If the function $u \in C^{2}(\bar{\Omega})$ satisfies the PDE $\Delta u(\boldsymbol{x})=0$ for all $\boldsymbol{x} \in \Omega$, then Green's third identity tells us that $u(\boldsymbol{x})$ may be expressed as:

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma} \Phi(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{y})-u(\boldsymbol{y}) \frac{\partial \Phi}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y}) d \sigma(\boldsymbol{y}), \tag{1.2.8}
\end{equation*}
$$

where $\Phi(\boldsymbol{x}, \boldsymbol{y})$ is a fundamental solution as defined above, $d \sigma(\boldsymbol{y})$ is the surface measure on $\Gamma$, and $d V(\boldsymbol{y})$ is the volume measure in $\Omega$. As before, $\partial u / \partial \boldsymbol{n}$ is the normal derivative, i.e. $\partial u / \partial \boldsymbol{n}=\nabla u \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the outward pointing unit normal on $\Gamma$. Equation (1.2.8) is referred to as a representation formula. The representation formula depends only on the Dirichlet and Neumann data, and therefore the solution of the Laplace equation is completely determined by specifying the Dirichlet and Neumann conditions for a given BVP. One way of doing this is by finding a DirichletNeumann map, which can be realized via the Poincare-Steklov operator given by:

$$
\begin{equation*}
S:\left.\left.u\right|_{\partial \Omega} \mapsto \frac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial \Omega} . \tag{1.2.9}
\end{equation*}
$$

Using the operator $S$ and the representation formula given in (1.2.8), the solution to the Laplace equation can be written as:

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma} \Phi(\boldsymbol{x}-\boldsymbol{y}) S f(\boldsymbol{y})-f(\boldsymbol{y}) \frac{\partial \Phi}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y}) d \sigma(\boldsymbol{y}) \tag{1.2.10}
\end{equation*}
$$

given sufficient regularity of the Dirichlet data $f$ (in our case, the assumption $f \in$ $C^{1}(\Gamma)$ is sufficient). Therefore, given sufficient regularity for $f$, finding a well-defined Dirichlet-Neumann map is equivalent to finding a solution to the Laplace equation for a given well-posed BVP, [3].

We will now discuss two ways in which the representation formula in (1.2.8) can be used to find a solution to the Laplace equation for a given BVP. The first relies on modifying the fundamental solution $\Phi(\boldsymbol{x}-\boldsymbol{y})$ so that the specified boundary conditions are satisfied. The second relies on finding the unknown boundary data - this approach is related to the idea of finding a Dirichlet-Neumann map.

As noted above, the integral in (1.2.8) depends on both $u(\boldsymbol{y})$ and $\partial u / \partial \boldsymbol{n}(\boldsymbol{y})$ for $\boldsymbol{y} \in$ $\Gamma$, and hence both the Dirichlet and the Neumann data must be known in order for $u$ to be a solution of the BVP given in (1.2.5). In some cases this may be circumvented by constructing a function $G(\boldsymbol{x}, \boldsymbol{y})$, known as a Green's function, which is a modification of the fundamental solution that satisfies the boundary conditions. In particular, let $\psi \in C^{2}(\bar{\Omega})$ be a harmonic function that satisfies the BVP

$$
\left\{\begin{align*}
\Delta \psi(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega  \tag{1.2.11}\\
\psi(\mathbf{x})=\Phi(\boldsymbol{x}-\boldsymbol{y}) & \text { for } \mathbf{x} \in \Gamma
\end{align*}\right.
$$

then the function $G(\boldsymbol{x}, \boldsymbol{y}):=\Phi(\boldsymbol{x}-\boldsymbol{y})-\psi(\boldsymbol{x})$ satisfies the BVP given in (1.2.5). Using (1.2.8) and $G(\boldsymbol{x}, \boldsymbol{y})$, the solution $u \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ to (1.2.5) can be written as

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma} \frac{\partial G}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) d \sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega \tag{1.2.12}
\end{equation*}
$$

A similar Green's function can also be constructed for the corresponding Neumann problem, and takes the form

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma} G(\boldsymbol{x}-\boldsymbol{y}) g(\boldsymbol{y}) d \sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega \tag{1.2.13}
\end{equation*}
$$

where $g(\boldsymbol{x})$ is the specified Neumann data. When a suitable Green's function can be found for a given BVP, this is an effective method for constructing a solution. However, the Green's function depends on the shape of the domain $\Omega$ and unfortunately is only known analytically for a limited number of simple domains. For an overview of some domains for which the Green's function can be computed analytically, see [9].

Now we will mention another class of methods which allows us to circumvent the construction of a Green's function for a given BVP. We introduce the following integral operator $T$ given by:

$$
\begin{equation*}
T \mu(\boldsymbol{x})=\int_{\Gamma} \frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{y}} \mu(\boldsymbol{y}) d \sigma(\boldsymbol{y}), \quad x \in \Omega \cup \Omega^{c}, \tag{1.2.14}
\end{equation*}
$$

where $\Phi(\boldsymbol{x}-\boldsymbol{y})$ is the (known) fundamental solution given by (1.2.7), $\mu$ is a density
function associated to $T$, and $\Omega^{c}=\mathbb{R}^{2} \backslash \Omega$. Equation (1.2.14) is known as a doublelayer potential. Similarly, we introduce the integral operator $S$ given by:

$$
\begin{equation*}
S \rho(\boldsymbol{x})=\int_{\Gamma} \Phi(\boldsymbol{x}-\boldsymbol{y}) \rho(\boldsymbol{y}) d \sigma(\boldsymbol{y}), \quad x \in \Omega \cup \Omega^{c}, \tag{1.2.15}
\end{equation*}
$$

where $\rho$ is a density function associated to the operator $S$. Equation (1.2.15) is known as a single-layer potential. We note that the representation formula (1.2.8) is comprised of a double- and single-layer potential, for $x \in \Omega$, where the densities $\mu$ and $\rho$ correspond to the Dirichlet and Neumann data, respectively. For a given BVP only either the Dirichlet or Neumann data is specified, and, as previously mentioned, one way to solve the BVP is to find the unknown boundary data. The Boundary Integral Equation (BIE) method is a class of techniques that uses the known boundary data, along with a set of integral relations, to find a density function, $\mu$ or $\rho$, corresponding to the unknown boundary data. Comparing this approach to the Green's function method, we note that the integral operators $T$ and $S$ have a similar structure to equations (1.2.12) and (1.2.13), respectively. However, equations (1.2.12) and (1.2.13) require the computation of a Green's function which is suitable for the given domain, whereas the double- and single-layer potentials use the known fundamental solution $\Phi(\boldsymbol{x}-\boldsymbol{y})$ as a kernel function. In the Green's function approach, the Green's function must be found for the given domain, whereas in the BIE method, the unknown density function, $\mu$ or $\rho$, must be found for the specified BVP.

The BIE method proceeds by taking the limit as $x$ approaches $\Gamma$ in the representation formula and its normal derivative. To do this, we first take the limit as $x$ approaches $\Gamma$ for the single- and double-layer potentials, which leads to the following operators:

$$
\begin{array}{rlrl}
V \mu(\boldsymbol{x}) & =-\lim _{\boldsymbol{\xi} \rightarrow \boldsymbol{x} \in \Gamma} \nabla_{\boldsymbol{\xi}} T \mu(\boldsymbol{\xi}) \cdot \boldsymbol{n}_{\boldsymbol{x}}, & & \boldsymbol{\xi} \in \Omega \\
K \mu(\boldsymbol{x}) & =\lim _{\boldsymbol{\xi} \rightarrow \boldsymbol{x} \in \Gamma} T \mu(\boldsymbol{\xi})+\frac{1}{2} \mu(\boldsymbol{x}), & & \boldsymbol{\xi} \in \Omega \\
S \rho(\boldsymbol{x}) & =\lim _{\boldsymbol{\xi} \rightarrow \boldsymbol{x} \in \Gamma} S \rho(\boldsymbol{\xi}), & \boldsymbol{\xi} \in \Omega \\
K^{*} \rho(\boldsymbol{x}) & =\lim _{\boldsymbol{\xi} \rightarrow \boldsymbol{x} \in \Gamma} \nabla_{\boldsymbol{\xi}} S \rho(\boldsymbol{\xi}) \cdot \boldsymbol{n}_{\boldsymbol{x}}-\frac{1}{2} \rho(\boldsymbol{x}), & & \boldsymbol{\xi} \in \Omega . \tag{1.2.19}
\end{array}
$$

It is well-known from classical analysis that the limits in (1.2.16)-(1.2.19) exist for the Laplace equation, [24], [28]. In particular, if the boundary $\Gamma$ is $C^{2}$ and $\mu$ and $\rho$ are continuous, then the limits in (1.2.17)-(1.2.19) exist uniformly for all $x \in \Gamma$ and can be expressed explicitly as the following integral operators on the boundary:

$$
\begin{align*}
K \mu(\boldsymbol{x}) & =\int_{\Gamma \backslash\{\boldsymbol{x}\}} \frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}} \mu(\boldsymbol{y}) d \sigma(\boldsymbol{y}),  \tag{1.2.20}\\
S \rho(\boldsymbol{x}) & =\int_{\Gamma \backslash\{\boldsymbol{x}\}} \Phi(\boldsymbol{x}-\boldsymbol{y} \in \Gamma, \rho(\boldsymbol{y}) d \sigma(\boldsymbol{y}),  \tag{1.2.21}\\
K^{*} \rho(\boldsymbol{x}) & \text { for } \boldsymbol{x} \in \Gamma,  \tag{1.2.22}\\
\int_{\Gamma \backslash\{\boldsymbol{x}\}} \frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}} \rho(\boldsymbol{y}) d \sigma(\boldsymbol{y}), & \text { for } \boldsymbol{x} \in \Gamma .
\end{align*}
$$

For the operator $V$, we quote the following lemma (given as Lemma 1.2.2 in [24]) which gives an explicit representation for the limit in (1.2.16):

Lemma. Let $\Gamma \in C^{2}$ and let $\mu$ be a Holder continuously differentiable function. Then the limit in (1.2.16) exists uniformly with respect to all $x \in \Gamma$ and all $\mu$ with $\|\mu\|_{C^{1+\alpha}} \leq$ 1. ${ }^{1}$ Moreover, for dimension $n=2$, the operator $V$ can be expressed as a composition of tangential derivatives and the single-layer potential operator $S$ :

$$
\begin{equation*}
V \mu(\boldsymbol{x})=-\frac{d}{d \sigma(\boldsymbol{x})} S\left(\frac{d \mu}{d \sigma}\right)(\boldsymbol{x}) . \tag{1.2.24}
\end{equation*}
$$

The kernel functions in (1.2.20) and (1.2.22) are given by:

$$
\begin{align*}
& \frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}=\frac{1}{2 \pi(n-1)} \frac{(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{x}-\boldsymbol{y}|^{n}},  \tag{1.2.25}\\
& \frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}=-\frac{1}{2 \pi(n-1)} \frac{(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{x}}}{|\boldsymbol{x}-\boldsymbol{y}|^{n}}, \tag{1.2.26}
\end{align*}
$$

where $n$ is the dimension of $\Omega$. The integral operators $K$ and $K^{*}$ as defined in (1.2.20) and (1.2.22) are weakly singular in the sense that the kernels (1.2.25) and (1.2.26) obey

[^0]the following bounds for all $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$ :
\[

$$
\begin{align*}
& \left|\frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{y}}}\right| \leq C_{1}|\boldsymbol{x}-\boldsymbol{y}|^{-\lambda_{1}}  \tag{1.2.27}\\
& \left|\frac{\partial \Phi(\boldsymbol{x}-\boldsymbol{y})}{\partial \boldsymbol{n}_{\boldsymbol{x}}}\right| \leq C_{2}|\boldsymbol{x}-\boldsymbol{y}|^{-\lambda_{2}} \tag{1.2.28}
\end{align*}
$$
\]

where $C_{1}, C_{2}, \lambda_{1}<n-1$, and $\lambda_{2}<n-1$ are constants. In the case $n=2$, both kernels in (1.2.27) and (1.2.28) can be extended to a $C^{0}$-function for $\boldsymbol{y} \rightarrow \boldsymbol{x}$, [28]. Furthermore, it is well-established that $K$ and $K^{*}$, as defined in (1.2.20) and (1.2.22), are compact operators, [28].

Taking the limit as $x$ approaches $\Gamma$ in the representation formula (1.2.8) and its normal derivative, we get:

$$
\begin{align*}
u(\boldsymbol{x}) & =\left(\frac{1}{2} I-K\right) u(\boldsymbol{x})+S \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x})  \tag{1.2.29}\\
\frac{\partial u}{\partial \boldsymbol{n}} & =V u(\boldsymbol{x})+\left(\frac{1}{2} I+K^{*}\right) \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) \tag{1.2.30}
\end{align*}
$$

These integral equations give a relation between the Dirichlet and Neumann data. Using these two relations, we can formulate the boundary integral problem for finding the unknown boundary data in at least two different ways. This is useful as one of the formulations often has benefits over the other either in terms of analysis or numerical implementation.

If we consider the Dirichlet problem (1.2.5), i.e., given $\mu=\left.u\right|_{\Gamma}$ find $\rho=\partial u /\left.\partial \boldsymbol{n}\right|_{\Gamma}$, then we may use either (1.2.29) or (1.2.30) to find the (unknown) Neumann data. Using the relation (1.2.29), the unknown density $\rho$ (and hence the Neumann boundary data) may be found by solving the integral equation

$$
\begin{equation*}
S \rho(\boldsymbol{x})=F_{1}(\boldsymbol{x}), \quad \text { for } \boldsymbol{x} \in \Gamma \tag{1.2.31}
\end{equation*}
$$

where $F_{1}(\boldsymbol{x})=\frac{1}{2} \mu(\boldsymbol{x})+K \mu(\boldsymbol{x})$ is determined by the Dirichlet data, and therefore known. Equation (1.2.31) takes the form of a Fredholm integral equation of the first kind.

Instead of using the relation in (1.2.29), we may use the relation (1.2.30), in which
case the unknown density $\rho$ (and hence the Neumann boundary data) may be found by solving the integral equation

$$
\begin{equation*}
\frac{1}{2} \rho(\boldsymbol{x})-K^{*} \rho(\boldsymbol{x})=F_{2}(\boldsymbol{x}), \quad \text { for } \boldsymbol{x} \in \Gamma, \tag{1.2.32}
\end{equation*}
$$

where $F_{2}(\boldsymbol{x})=V \mu(\boldsymbol{x})$ is determined by the Dirichlet data, and therefore known. Equation (1.2.32) takes the form of a Fredholm integral equation of the second kind, which is a well-studied class of integral equations, [21], [38]. This shows that for a given BVP, there is often at least two ways of formulating the boundary integral equations. In this case, the formulation resulting from (1.2.32) is more amenable to analysis and numerical implementation.

If we consider the corresponding Neumann problem, i.e. given the Neumann data $\mu$ find the Dirichlet data $\rho$, then using the relation (1.2.29) results in an integral equation of the form:

$$
\begin{equation*}
\frac{1}{2} \mu(\boldsymbol{x})+K \mu(\boldsymbol{x})=G_{1}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{1.2.33}
\end{equation*}
$$

where $G_{1}(\boldsymbol{x})=S \rho(\boldsymbol{x})$. Equation (1.2.33) is a Fredholm integral equation of the second kind, which is the same class as (1.2.32). If we use the relation (1.2.30) for the Neumann problem, then we get an integral equation of the form:

$$
\begin{equation*}
V \mu(\boldsymbol{x})=G_{2}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{1.2.34}
\end{equation*}
$$

where $G_{2}(\boldsymbol{x})=\frac{1}{2} \rho\left(\boldsymbol{x}-K^{*} \rho(\boldsymbol{x})\right)$. Equation (1.2.34) is a hypersingular boundary integral equation of the first kind. Again we see for the Neumann problem that there are two ways of formulating the boundary integral equations. In this case, it is more advantageous to use the boundary integral formulation given in (1.2.33). It is often the case that for a given BVP one of the boundary integral formulations is more advantageous to use.

For more general equations, the BIE method often involves integral operators with strongly singular or hypersingular kernels. In these cases, further analysis is generally required. For a synopsis of the singular integrals that are encountered when using the BIE method, see [22].

### 1.3 Summary

In this chapter we discussed several ways of solving the Laplace equation on domains with analytic boundary, each with advantages and disadvantages. The method of separation of variables gives an algorithmic way of finding the solution to a given BVP while at the same time leading to an efficient numerical method. However, separation of variables can only be performed on a limited number of domains and requires a carefully chosen coordinate system. The method of conformal mapping is also an effective way of finding a solution to a BVP for the Laplace equation, however, it is often difficult to find an appropriate conformal map for many domains. Of the methods presented here, the BIE method is the most versatile as it can be applied to a wide variety of domains and it naturally leads to a variety of effective numerical methods. However, the BIE method requires the use of singular integrals, which often requires further analysis.

In this thesis we will use (and extend) a recent method, known as the 'Fokas' or 'Unified Transform' method, that can be thought of as a spectral analogue of the BIE approach. This method uses Green's second identity to derive a relationship that couples the Dirichlet and Neumann boundary data. For a given BVP, this relationship can then be used to find the unknown boundary data. Unlike the conformal mapping and separation of variables approaches, this method can be used on a diverse set of domains. To date, the method has successfully been applied to the class of convex polygons in $\mathbb{C} \cong \mathbb{R}^{2}$, and in this thesis it will be extended to the more general class of convex domains with analytic boundary in $\mathbb{C} \cong \mathbb{R}^{2}$. This method also has the benefit of avoiding the use of singular integrals that are needed for the BIE approach. In the chapter that follows we will introduce and further explore this method.

As a note, we intentionally restrict our attention to the Laplace equation in this study. It is in keeping with the literature in this field to first perform a thorough analysis of the Laplace equation before moving to more general elliptic PDEs (e.g., the Helmholtz and modified Helmholtz equations). There are well-established methods for extending from the Laplace equation to more general elliptic PDEs for the case of polygonal domains that will likely carry over for domains with analytic boundary. These extensions are briefly mentioned in the conclusion as a matter for future study.

## CHAPTER 2

## The Fokas Method and Global Relation

In the previous chapter we identified three classes of methods for studying the Laplace equation, and the corresponding boundary conditions, in a bounded domain $\Omega \subset \mathbb{R}^{2}$. While all of the methods produce valid solutions in certain cases, as identified, each method also has limitations. In this chapter we introduce a method that allows us to deal with more general domains than with separation of variables or conformal mapping, while at the same time reducing the amount of work needed to find a map between the Dirichlet and Neumann data. In this chapter we will use this method to derive some initial results for polygonal domains. This will allow us to introduce the method in a similar context to what will be covered in this thesis while at the same time building some ideas that will be used later.

The aforementioned method was originally developed in the context of studying integrable PDEs. In the 1960s the Inverse Scattering Transform (IST) was developed for solving the initial value problem for the KdV and non-linear Schrodinger equations. In subsequent years, the IST was further applied to find soliton solutions to other nonlinear PDEs which admit a "Lax pair formulation", that is they can be written as the compatibility condition of two eigenvalue equations. Such equations are called "integrable". In [14], Fokas developed the "unified method", or "Fokas method", which extended the Inverse Scattering Transform method from initial value problems
to boundary value problems for non-linear and linear integrable PDEs. This is done by performing a simultaneous spectral analysis of both parts of the Lax pair for the given integrable PDE, which involves solving a Riemann-Hilbert problem, and by analyzing a relation that encodes information about the boundary values of the given BVP, known as the 'global relation'. In this thesis we will focus on the latter of these two, i.e. the global relation.

### 2.1 The Laplace Equation on a Polygon

In order to illustrate the relevant features of the unified method of Fokas, we will show how the method can be used to study the Laplace equation on polygonal domains in $\mathbb{R}^{2}$. In particular, we will briefly explain how the unified method leads to an integral representation for the solution of the Laplace equation in a polygon. We will then study a key relationship that holds for the boundary data and discuss how this can be used to understand the relationship between the Dirichlet and Neumann data. This relation will be the focus of the thesis.

To be precise, let $\Omega \subset \mathbb{C} \cong \mathbb{R}^{2}$ be the closed $n$-sided polygon with corners given by $z_{1}, \ldots, z_{n}$, where $z_{j} \in \mathbb{C}$ for each $j \in\{1, \ldots, n\}$. Also, we denote the side from $z_{j}$ to $z_{j+1}$ by $l_{j}$, where $z_{n+1}=z_{1}$. The problem that we are concerned with analyzing is that of the Laplace equation in the polygon with Dirichlet boundary data given by:

$$
\begin{cases}\Delta u(z)=0 & \text { for } z \in \Omega \subset \mathbb{C}  \tag{2.1.1}\\ u(z)=f(z) & \text { for } z \in \Gamma=\partial \Omega,\end{cases}
$$

where the boundary data, $f$, for each side is given by $u(z)=f_{j}(z)$ for $z \in l_{j}$. We seek a solution $u \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$, and assume $f_{j} \in C^{1}\left(l_{j}\right)$ for each $j \in\{1, \ldots, n\}$.

Now we introduce some notation. Let $z=x+i y$ so that the conjugate of $z$ is given by $\bar{z}=x-i y$, and define the partial derivatives $\partial_{z}$ and $\partial_{\bar{z}}$ by

$$
\begin{equation*}
\partial_{z}=\frac{1}{2}\left(\partial_{x}-i \partial_{y}\right) \quad \text { and } \quad \partial_{\bar{z}}=\frac{1}{2}\left(\partial_{x}+i \partial_{y}\right) \tag{2.1.2}
\end{equation*}
$$

We use the notation $f_{z}=\partial_{z}(f)$ and $f_{\bar{z}}=\partial_{\bar{z}}(f)$ to denote the respective derivatives of
the function $f$.
In the unified method of Fokas, we proceed by viewing the Laplace equation as an integrable PDE. The corresponding Lax pair for the Laplace equation is given by:

$$
\begin{equation*}
\mu_{z}-i \lambda \mu=U(z) \quad \text { and } \quad \mu_{\bar{z}}=0, \tag{2.1.3}
\end{equation*}
$$

where $U(z)$ is defined by $U(z)=u_{z}=u_{x}-i u_{y}$, and $\mu=\mu(z, \lambda)$ with $z \in \Omega$ and $\lambda \in \mathbb{C}$. Moreover, the compatibility condition for this system is $U_{\bar{z}}=u_{z \bar{z}}=\frac{1}{4} \Delta u=0$, which is the analyticity condition for $U(z)$. The particular solution to (2.1.3) can be written for each side $l_{j}$ as follows:

$$
\begin{equation*}
\mu_{j}(z, \lambda)=\int_{z_{j}}^{z} e^{i \lambda\left(z-z^{\prime}\right)} U\left(z^{\prime}\right) d z^{\prime}, \quad j=1, \ldots, n, \quad \lambda \in S_{j}, \tag{2.1.4}
\end{equation*}
$$

where $z_{j}$ are the corners of the polygon, $z$ is a point along the segment connecting $z_{j}$ and $z_{j+1}$, and $S_{j}:=\left\{\lambda \in \mathbb{C}: \arg (\lambda) \in\left[-\arg \left(z_{j-1}-z_{j}\right), \pi-\arg \left(z_{j+1}-z_{j}\right)\right]\right\}$ is the set of values of $\lambda$ in the complex plane for which (2.1.4) is bounded. The regions $\left\{S_{j}\right\}_{j=1}^{n}$ form a partition of the complex plane and a solution which is analytic for all $\lambda \in \mathbb{C}$ can be defined as follows:

$$
\begin{equation*}
\mu(z, \lambda)=\mu_{j}(z, \lambda), \quad \lambda \in S_{j}, \quad j=1, \ldots, n \tag{2.1.5}
\end{equation*}
$$

By performing integration by parts on each of the functions $\mu_{j}(z, \lambda)$, we get the following decay estimate for large $\lambda$ :

$$
\begin{equation*}
\mu(z, \lambda)=O\left(\frac{1}{\lambda}\right), \quad \text { as } \lambda \rightarrow \infty . \tag{2.1.6}
\end{equation*}
$$

Furthermore, by subtracting the adjacent $\mu_{j}$ 's, we get the following relations:

$$
\begin{equation*}
\mu_{j+1}(z, \lambda)-\mu_{j}(z, \lambda)=e^{i \lambda z} \int_{z_{j+1}}^{z_{j}} e^{-i \lambda z} U(z) d z, \quad \lambda \in L_{j} \tag{2.1.7}
\end{equation*}
$$

where $L_{j}$ is the ray $L_{j}:=\left\{\lambda \in \mathbb{C}: \arg (\lambda)=-\arg \left(z_{j}-z_{j+1}\right)\right\}$, oriented toward infinity.

We now wish to express the function $\mu(z, \lambda)$ in terms of an integral relation which is analytic for all $\lambda \in \mathbb{C}$ and that satisfies the conditions (2.1.6)-(2.1.7). This takes the form of a Riemann-Hilbert problem, This is referred to as a Riemann-Hilbert problem ${ }^{1}$, and it was shown in [11] and [12] that this problem has a unique solution which is given by the following Cauchy-type integral:

$$
\begin{equation*}
\partial_{z} u=\frac{1}{2 \pi} \sum_{j=1}^{n} \int_{L_{j}} e^{i \lambda z} \rho_{j}(\lambda) d \lambda, \quad \text { where } \quad \rho_{j}(\lambda)=\int_{l_{j}} e^{-i \lambda z^{\prime}} \frac{\partial q}{\partial z^{\prime}} d z^{\prime} \tag{2.1.8}
\end{equation*}
$$

The functions $\left\{\rho_{j}(\lambda)\right\}_{j=1}^{n}$ are known as spectral functions. As we will see in the discussion that follows, the spectral functions incorporate information about both the Dirichlet and Neumann data.

### 2.1.1 The Global Relation for the Laplace Equation on a Polygon

So far we have shown that the solution to the Laplace equation on a polygon can be expressed in terms of an integral representation. This integral representation is written in terms of spectral functions $\left\{\rho_{j}(\lambda)\right\}_{j=1}^{n}$, which we will now explore further.

It was shown in [10] that the following differential 1 -form is associated with the Laplace equation in $\mathbb{C}$ :

$$
\begin{equation*}
\eta(z, \bar{z}, \lambda)=e^{-i \lambda z} u_{z} d z . \tag{2.1.9}
\end{equation*}
$$

The exterior derivative of the differential form $\eta$ is given by

$$
\begin{equation*}
d \eta=e^{-i \lambda z} u_{z \bar{z}} d \bar{z} \wedge d z \tag{2.1.10}
\end{equation*}
$$

Since $u$ satisfies the Laplace equation, i.e. $u_{z \bar{z}}=0$ in $\Omega$, it follows that $d \eta=0$, and hence $\eta$ is a closed 1 -form. By Stoke's theorem, we have

$$
\begin{equation*}
\oint_{\partial \Omega} \eta=0, \quad \lambda \in \mathbb{C} . \tag{2.1.11}
\end{equation*}
$$

[^1]Equation (2.1.11) is referred to as the global relation, and it encodes information about both the Dirichlet and Neumann boundary conditions and depends meromorphically on the spectral parameter $\lambda$.

For the case of a polygon, (2.1.11) is simply a line integral of $\eta(z, \bar{z}, \lambda)$ along each segment $l_{j}$, and hence can be expressed as a sum of the corresponding spectral functions $\left\{\rho_{j}(\lambda)\right\}_{j=1}^{n}$, i.e.

$$
\begin{equation*}
\sum_{j=1}^{n} \rho_{j}(\lambda)=0, \quad \lambda \in \mathbb{C} \tag{2.1.12}
\end{equation*}
$$

where the spectral functions are defined in (2.1.8).
The relation given in (2.1.12) reduces the study of the global relation to understanding the spectral functions $\left\{\rho_{j}(\lambda)\right\}_{j=1}^{n}$. As we will show, each of the spectral functions can be written in terms of tangential and normal derivatives, and hence encodes information about both the Dirichlet and Neumann data. This in turn allows us to establish a relationship between the two types of boundary data for a given BVP. Indeed, it has been shown that the global relation completely determines the Dirichlet-Neumann map for the Laplace equation in the case of polygonal domains, [5],[4].

We will now give an example to demonstrate how the global relation can be used to establish a relationship between the Dirichlet and Neumann data on a square. For the sake of presentation we will leave the relationship between the Dirichlet and Neumann data in implicit form. Following the example, we will show how the global relation can be used to explicitly obatin the Neumann data given the Dirichlet data (or vice-versa).

### 2.1.2 An Example

Let the domain $\Omega$ be the square of side-length two centered at the origin. We will analyze the BVP for the Laplace equation where the Dirichlet data is given for each edge. The corners of the square are given by:

$$
\begin{equation*}
z_{1}=(1+i), \quad z_{2}=-1+i, \quad z_{3}=\bar{z}_{2}, \quad z_{4}=\bar{z}_{1} . \tag{2.1.13}
\end{equation*}
$$

The solution to Laplace's equation in this domain is given by (2.1.8) with $n=4$. For the sake of symmetry, we assume that the Dirichlet data is the same for each side so
that $u(z)=f_{j}(z)$ for $z \in l_{j}$ and $f_{j}=f$ for each $j \in\{1, \ldots, 4\}$. Similarly, the Neumann data will be denoted by $\partial_{\boldsymbol{n}} u(z)=g_{j}(z)$ for $z \in l_{j}$, where $\partial_{\boldsymbol{n}}$ is the normal derivative to the surface at the given point. We also assume that the Neumann data is the same on all sides so that $g_{j}=g$ for each $j \in\{1, \ldots, 4\}$.

First we note that for the side $l_{3}$ the presence of the tangential and normal derivatives in the vector field $\partial_{z}$ is particularly apparent. Since $l_{2}$ is parallel to the real-axis, the vector field $\partial_{z}$ on this part of the boundary becomes:

$$
\begin{equation*}
\partial_{z}=\frac{1}{2}\left(\partial_{x}-i \partial_{y}\right)=\frac{1}{2}\left(\partial_{\boldsymbol{t}}+i \partial_{\boldsymbol{n}}\right), \tag{2.1.14}
\end{equation*}
$$

where $\partial_{t}$ denotes the tangential derivative at the boundary (corresponding to the Dirichlet data), and $\partial_{\boldsymbol{n}}$ denotes the normal derivative at the boundary (corresponding to the Neumann data).

We now proceed by writing the spectral functions in terms of the tangential and normal components. First, we parametrize the sides of the square. The side $l_{3}$ may be parametrized by:

$$
\begin{equation*}
z(t)=(t-i), \quad \text { for } \quad t \in[-1,1] . \tag{2.1.15}
\end{equation*}
$$

Each of the respective sides may be similarly parametrized by rotating by $e^{-2 \pi i / 4}=-i$. Hence, the parametrizations of the four sides are given by:

$$
\begin{array}{lll}
l_{1}: & z_{1}(t)=-(t-i), & t \in[-1,1], \\
l_{2}: & z_{2}(t)=-i(t-i), & t \in[-1,1], \\
l_{3}: & z_{3}(t)=(t-i), & t \in[-1,1], \\
l_{4}: & z_{4}(t)=i(t-i), & t \in[-1,1] . \tag{2.1.19}
\end{array}
$$

Similarly, the differentials $d z_{j}$ are given by:

$$
\begin{equation*}
d z_{1}=-d t, \quad d z_{2}=-i d t, \quad d z_{3}=d t, \quad d z_{4}=i d t . \tag{2.1.20}
\end{equation*}
$$

We obtain the corresponding vector fields $\partial_{z}$ by rotating by $e^{2 \pi i / 4}=i$ to get:

$$
\begin{array}{ll}
l_{1}: & \partial_{z}=-\frac{1}{2}\left(\partial_{\boldsymbol{t}}+i \partial_{\boldsymbol{n}}\right) \\
l_{2}: & \partial_{z}=-\frac{1}{2} i\left(\partial_{\boldsymbol{t}}+i \partial_{\boldsymbol{n}}\right) \\
l_{3}: & \partial_{z}=\frac{1}{2}\left(\partial_{\boldsymbol{t}}+i \partial_{\boldsymbol{n}}\right) \\
l_{4}: & \partial_{z}=\frac{1}{2} i\left(\partial_{\boldsymbol{t}}+i \partial_{\boldsymbol{n}}\right) \tag{2.1.24}
\end{array}
$$

The spectral functions can now be written in terms of the Dirichlet and Neumann data as:

$$
\begin{align*}
& \rho_{1}(\lambda)=\frac{e^{\lambda}}{2} \int_{-1}^{1} e^{i \lambda t}\left(\frac{d f}{d t}+i g\right) d t  \tag{2.1.25}\\
& \rho_{2}(\lambda)=\frac{e^{i \lambda}}{2} \int_{-1}^{1} e^{-\lambda t}\left(\frac{d f}{d t}+i g\right) d t  \tag{2.1.26}\\
& \rho_{3}(\lambda)=\frac{e^{-\lambda}}{2} \int_{-1}^{1} e^{-i \lambda t}\left(\frac{d f}{d t}+i g\right) d t  \tag{2.1.27}\\
& \rho_{4}(\lambda)=\frac{e^{-i \lambda}}{2} \int_{-1}^{1} e^{\lambda t}\left(\frac{d f}{d t}+i g\right) d t . \tag{2.1.28}
\end{align*}
$$

Using the notation

$$
\begin{equation*}
\mathcal{G}(\lambda):=\int_{-1}^{1} \exp (-i \lambda t) g d t \quad \mathcal{F}(\lambda):=\int_{-1}^{1} \exp (-i \lambda t) \frac{d f}{d t} d t \tag{2.1.29}
\end{equation*}
$$

along with the representations of the spectral functions given in (2.1.25)-(2.1.28), the global relation may be expressed as:

$$
\begin{align*}
e^{\lambda} \mathcal{G}(-\lambda) & +e^{i \lambda} \mathcal{G}(-i \lambda)+e^{-\lambda} \mathcal{G}(\lambda)+e^{-i \lambda} \mathcal{G}(i \lambda) \\
& =-i\left[e^{\lambda} \mathcal{F}(-\lambda)+e^{i \lambda} \mathcal{F}(-i \lambda)+e^{-\lambda} \mathcal{F}(\lambda)+e^{-i \lambda} \mathcal{F}(i \lambda)\right] \quad \lambda \in \mathbb{C} \tag{2.1.30}
\end{align*}
$$

This representation of the global relation establishes an explicit relationship between the Dirichlet and Neumann data. Indeed, the right-hand-side of (2.1.30) is expressed in terms of the Dirichlet data, which is known, while the left-hand-side is expressed in terms of the Neumann data, which is unknown. This suggests that the global relation
is useful for finding the Neumann data. We explore this idea in the discussion that follows as we return to the setting of the general polygon.

The same procedure outlined in this example can be used to establish a similar relation to (2.1.30) for a general convex polygon. While this relation is useful, it is insufficient on its own for determining the unknown Neumann data. For an $n$-sided polygon we must find the Neumann data for each of the $n$ sides, which suggests that $n$ equations similar to (2.1.30) are needed.

For a general $n$-sided convex polygon let $j \in\{1, \ldots, n\}$ be fixed and let $m_{j}$ denote the midpoint of the $j$ th side and let $\alpha_{j}:=\arg \left(z_{j}-z_{j+1}\right)$. If we multiply the global relation by $e^{i \lambda m_{j}}$ and make the replacement $\lambda \mapsto e^{-i \alpha_{j}} \lambda$, then this generates another equation that is independent of the original global relation. Performing this process for each $j \in\{1, \ldots, n\}$ generates $n$ independent equations. Using this procedure and a local parametrization that is symmetric with respect to the midpoint of each edge of the polygon, Ashton showed, in [5], that the global relation can be written as ${ }^{2}$

$$
\begin{equation*}
T(\mathcal{N}-i \mathcal{D})=0, \tag{2.1.31}
\end{equation*}
$$

where $\mathcal{D}$ is a vector that contains the Fourier transform of the Dirichlet data on each edge of the polygon, similarly $\boldsymbol{\mathcal { N }}$ is a vector that contains the Fourier transform of the Neumann data, and $T$ is a linear operator of the form $T=I+K$, where $I$ is the identity and $K$ is a compact operator given by:

$$
\begin{equation*}
K \Phi_{j}=\sum_{k \neq j} e^{i e^{-i \alpha_{j}}\left(m_{j}-m_{k}\right) \lambda} \Phi_{k}\left(e^{-i\left(\alpha_{j}-\alpha_{k}\right)} \lambda\right), \quad 1 \leq j \leq n \tag{2.1.32}
\end{equation*}
$$

where $\Phi$ is an $n$-dimensional vector (i.e., $\mathcal{D}$ or $\boldsymbol{\mathcal { N }}$ ). Further, it was shown that (2.1.31) defines a continuous linear map between the spectral Dirichlet data $\mathcal{D}$ and the spectral Neumann data $\boldsymbol{\mathcal { N }}$, thus establishing a spectral Dirichlet-Neumann map. Indeed, the operator $T$ can be viewed as a spectral Poincare-Steklov operator for the Laplace equation on polygonal domains, [5].

Through analysis of the operator $T$, it was shown that the global relation leads

[^2]to a family of well-posed weak problems that can be approximated using Galerkin techniques. This, in turn, is used to find the entries of the vector $\boldsymbol{\mathcal { N }}$, from which the Neumann data can be recovered by performing an inverse Fourier transform. In addition, a number of other numerical methods have been developed that use the global relation to recover the Neumann data, given the Dirichlet data, on a polygonal domain, [16], [17], [32], [33]. Motivated by the success of these methods, in this thesis we will extend the ideas presented here to more general domains that have an analytic boundary.

### 2.2 The Global Relation

We now turn our attention to exploring the global relation in a more general framework. In particular, we will derive a global relation that is valid for the domain $\Omega$ (with analytic boundary) given in the boundary value problem (1.1.2). We will also demonstrate how this can be done for a more general PDEs than the Laplace equation.

Let $P(D)$ be a general linear second-order differential operator with constant coefficients of the form

$$
\begin{equation*}
P(D)=\sum_{i=1}^{2} \sum_{j=1}^{2} a_{i j} D_{i} D_{j}+\sum_{j=1}^{2} b_{j} D_{j}+C, \tag{2.2.1}
\end{equation*}
$$

where $C \in \mathbb{C}$ is a constant. In this case, $P(D)$ is self-adjoint. The symbol of the operator $P(D)$ is denoted by $P(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \in \mathbb{C}^{2}$. We will denote the zero set of this polynomial by $Z_{P}=\left\{\boldsymbol{\lambda} \in \mathbb{C}^{2}: P(\boldsymbol{\lambda})=0\right\}$. We will further assume that the boundary of the domain $\partial \Omega$ is $C^{2}$.

The divergence theorem states that for the differential operator $P(D)$ and for functions $\varphi, \psi \in C^{2}(\bar{\Omega})$ we have:

$$
\begin{equation*}
\int_{\Omega} \psi P(D) \varphi-\varphi P(D) \psi d V=\int_{\partial \Omega} \boldsymbol{F}(\varphi, \psi) \cdot \boldsymbol{n} d \sigma \tag{2.2.2}
\end{equation*}
$$

where the term $\boldsymbol{F}(\varphi, \psi)$ depends on $\varphi$ and $\psi$ as well as their derivatives up to order 1,
and can be computed explicitly by:

$$
\begin{equation*}
\boldsymbol{F}_{i}=\sum_{j=1}^{n}\left[a_{i j} \psi D_{j} \varphi-\varphi D_{j}\left(a_{i j} \psi\right)\right]+b_{i} \varphi \psi \tag{2.2.3}
\end{equation*}
$$

Now we let $\varphi=u$ be a solution corresponding to the operator $P(D)$ and let $\psi=e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}$ with $\boldsymbol{\lambda} \in Z_{P}$. Substituting these values into (2.2.2) gives us:

$$
\begin{equation*}
\int_{\Omega} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}} P(D) u-u P(D)\left(e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right) d V=\int_{\partial \Omega} \boldsymbol{F}\left(u, e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right) \cdot \boldsymbol{n} d \sigma, \quad \boldsymbol{\lambda} \in Z_{P} \tag{2.2.4}
\end{equation*}
$$

Clearly $P(D) u=0$ since $u$ is a solution corresponding to the operator $P(D)$. Now considering the second term on the left hand side of (2.2.4), we note that by the chain rule $P(D)\left(e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right)=e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}} P(\boldsymbol{\lambda})$. Since we assumed $\boldsymbol{\lambda} \in Z_{P}$, it follows that $P(\boldsymbol{\lambda})=0$ and hence $P(D)\left(e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right)=0$. Therefore, the global relation may be written as

$$
\begin{equation*}
0=\int_{\partial \Omega} \boldsymbol{F}\left(u, e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right) \cdot \boldsymbol{n} d \sigma, \quad \boldsymbol{\lambda} \in Z_{P} . \tag{2.2.5}
\end{equation*}
$$

The integral above is over the boundary $\partial \Omega$, and the term $\boldsymbol{F}\left(u, e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right)$ is a relation that couples the boundary data, as can be seen from equation (2.2.3).

In order to make this more concrete, we demonstrate the above result with the example of the Laplace equation.

Example 1. Consider the Laplace operator which takes the form $P(D)=D^{2}$, where $D=-i \partial$. The corresponding symbol associated with this differential operator is $P(\boldsymbol{\lambda})=\sum_{k=1}^{n} \lambda_{k}^{2}$, where $\boldsymbol{\lambda} \in \mathbb{R}^{n}$. As above, we denote the zero set of this polynomial by $Z_{p}=\left\{\boldsymbol{\lambda} \in \mathbb{R}^{n}: P(\boldsymbol{\lambda})=0\right\}$.

Now let $\varphi$ and $\psi$ be functions on $\Omega$ with $\varphi, \psi \in C^{2}(\Omega) \cap C(\bar{\Omega})$, the divergence theorem in $\mathbb{R}^{2}$ takes the form of Green's second identity which states

$$
\int_{\Omega}(\varphi \Delta \psi-\psi \Delta \varphi) d V=\int_{\partial \Omega}\left(\varphi \frac{\partial \psi}{\partial \boldsymbol{n}}-\psi \frac{\partial \varphi}{\partial \boldsymbol{n}}\right) d \sigma,
$$

where $d \sigma$ is the surface measure on the boundary $\partial \Omega$. Now we set $\varphi=u$, where $u$ is a solution of the Laplace equation, and set $\psi=e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}$, with $\boldsymbol{\lambda} \in Z_{P}$. This yields the following expression

$$
\begin{equation*}
0=\int_{\partial \Omega} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left(i(\boldsymbol{\lambda} \cdot \boldsymbol{n}) u(\boldsymbol{x})+\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x})\right) d \sigma(\boldsymbol{x}), \quad \text { for } \boldsymbol{\lambda} \in Z_{p} . \tag{2.2.6}
\end{equation*}
$$

The left hand side of the above expression is zero because $\Delta u=0$ by assumption, and $\Delta\left(e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\right)=0$ by the chain rule and the fact that $\boldsymbol{\lambda} \in Z_{P}$.

Continuing the discussion from Example 1, note that if we take $\boldsymbol{\lambda}=(\lambda, i \lambda)$, and let $\boldsymbol{x}=(x, p(x))$, where $p(x)$ is an analytic concave function, then the following transform naturally arises:

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi_{j} \rightarrow \tilde{\varphi}_{j}(\lambda)=\int_{\Gamma_{j}} e^{-i \lambda x+\lambda p(x)} \varphi_{j}(x)\left[1+i p^{\prime}(x)\right] d x \tag{2.2.7}
\end{equation*}
$$

where $\Gamma_{j}$ is a segment of the boundary $\partial \Omega$, and $\varphi_{j} \in L^{2}\left(\Gamma_{j}\right)$. This will be referred to as the $\mathcal{F}_{p^{\prime}}$-transform, and can be thought of as a perturbation of the standard Fourier transform. To see this, set $p(x)=0$ in (2.2.7), and the integral reduces to the Fourier transform over a straight line segment. The $\mathcal{F}_{p}$-transform will be instrumental in our analysis of the global relation for domains with analytic boundary, and its study will occupy much of the first part of the thesis.

We finish this section by noting that the global relation has an intimate connection to the solution of boundary value problems for the Laplace equation on domains with analytic boundary. In this thesis we seek to study the BVP for the Laplace equation given by:

$$
\begin{cases}\Delta u(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega  \tag{2.2.8}\\ u(\mathbf{x})=f(\mathbf{x}) & \text { for } \mathbf{x} \in \Gamma,\end{cases}
$$

where $\Omega \subset \mathbb{C} \cong \mathbb{R}^{2}$ is a bounded convex region with an analytic boundary $\Gamma=\partial \Omega$, $u \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$, and $f \in C^{1}(\Gamma)$ is given (Dirichlet) boundary data on $\Gamma$. The following theorem from [4] makes the connection between the BVP (2.2.8) and the global relation clear.

Theorem. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded, convex domain with smooth boundary $\Gamma$. Suppose there exists a function $g \in C(\Gamma)$ such that

$$
\begin{equation*}
\int_{\Gamma} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left[g(\boldsymbol{x})+i\left(\boldsymbol{\lambda} \cdot \boldsymbol{n}_{\boldsymbol{x}}\right) f(\boldsymbol{x})\right] d \Gamma_{\boldsymbol{x}}=0, \quad \boldsymbol{\lambda} \in Z_{p} \tag{2.2.9}
\end{equation*}
$$

for a given function $f \in C(\Gamma)$. Then, there exists a solution to the corresponding BVP for the Laplace equation, i.e. equation (2.2.8), and $g$ corresponds to the unknown Neumann boundary value.

Therefore, by finding a solution to the global relation, we can solve the corresponding BVP for the Laplace equation. This is the strategy that we employ in this thesis.

### 2.3 Summary of Thesis

In this section we will begin by giving a brief summary of the work that is done in this thesis. Following this, we will give a detailed outline of how the work covered in Chapters 3-5 can collectively be used to show that the global relation defines a continuous map between the Dirichlet and Neumann data (i.e., a Dirichlet-Neumann map), and, moreover, that if the Dirichlet data is given, then the global relation can be solved for the Neumann data. This will serve as a guide to unify the material in the first part of the thesis.

### 2.3.1 Summary of Work

This thesis is structured as follows. In Chapter 3 we formally define and analyze the $F_{p}$-transform. We show that the map $\mathcal{F}_{p}: \varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}([-\sigma, \sigma])$ to $L^{2}([0, \infty])$, in the case that $p(x)$ is an analytic concave function with $p(0)=p^{\prime}(0)=0$ and $x \in[-\sigma, \sigma]$. In Chapter 4, we find an inverse of the transform $\mathcal{F}_{p}: \varphi \rightarrow \tilde{\varphi}(\lambda)$ by solving a Riemann-Hilbert problem. Chapters 3 and 4 can be read independently of the rest of the thesis as they focus solely on the $\mathcal{F}_{p}$-transform and its inverse.

In Chapter 5, we derive the global relation for the Laplace equation on a domain, $\Omega \subset \mathbb{C} \cong \mathbb{R}^{2}$, with analytic boundary. Similar to (2.1.31), we will show that the global relation on $\Omega$ can be written in the form

$$
\begin{equation*}
T(\mathcal{N}-i \mathcal{D})=0, \tag{2.3.1}
\end{equation*}
$$

where $\mathcal{D}$ is the $\mathcal{F}_{p}$-transform of the Dirichlet data and $\mathcal{N}$ is the $\mathcal{F}_{p}$-transform of the Neumann data. The operator $T$ again can be written as $T=I+K$, where $K$ takes a similar form to (2.1.32). Since we deal with domains with a curved boundary, the $\mathcal{F}_{p}$-transform naturally arises in this context, and we make use of the results from Chapters 3 and 4 which serve as a basis for analyzing the global relation. We use properties of the operator $T$ to show that the global relation defines a continuous map between the Dirichlet and Neumann data and that this relation can be solved to find the Neumann data, given the Dirichlet data. This argument is outlined in detail in Section 2.3.2.

Finally, in Chapter 6 we construct a spectrally accurate collocation method for recovering the Neumann data, given the Dirichlet data, for a specified BVP for the Laplace equation on a convex domain with analytic boundary. This is done by extending an existing numerical method developed by Fornberg and Flyer in [16].

### 2.3.2 Outline of Results in Chapters 3-5

We will now give an outline of how the results in Chapters 3-5 will be used to show that the global relation defines a continuous map between the Dirichlet and Neumann data, and that this relation can be solved to find the Neumann data, given the Dirichlet data. This will serve as a guide for the first part of the thesis.

Let $\Omega$ be a convex domain in $\mathbb{C}$ with an analytic boundary that is partitioned into $n$ segments $\left\{\Gamma_{j}\right\}_{j=1}^{n}$. Let $\varphi=\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ be a complex-valued vector where the $j$ th component contains the tangential derivative of the Dirichlet data on $\Gamma_{j}$ as the real part and the Neumann data on $\Gamma_{j}$ as the imaginary part. Let $\boldsymbol{\Psi}=\left(\Psi_{1}, \ldots, \Psi_{n}\right)$ be the vector ${ }^{3}$ that contains the $\mathcal{F}_{p}$-transform of each of the components of $\varphi$.

Our study of the operator $T$, given in (2.3.1), will be based on the following sequence

[^3]of maps:
\[

$$
\begin{equation*}
\varphi \xrightarrow{A} \Psi \xrightarrow{B} T \Psi . \tag{2.3.2}
\end{equation*}
$$

\]

We will now discuss the maps $A$ and $B$.

1. Our analysis of the map $A$ relies on the following properties which are listed below.
(a) In Chapter 3, we will show that for any $\varphi \in L^{2}([-\sigma, \sigma])$ the map $\mathcal{F}_{p}$ : $\varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}([-\sigma, \sigma])$ to $L^{2}([0, \infty])$, where $p(x)$ is an analytic concave function with $p(0)=p^{\prime}(0)=0$, and $x \in[-\sigma, \sigma]$. This result establishes that the map $\varphi_{j} \mapsto \Psi_{j}$ is bounded from $L^{2}\left(\left[-\sigma_{j}, \sigma_{j}\right]\right)$ to $L^{2}([0, \infty])$ for each $j \in\{1, \ldots, n\}$. Therefore, the map $A$ is bounded from $L^{2}\left(\left[-\sigma_{1}, \sigma_{1}\right]\right) \times \cdots \times L^{2}\left(\left[-\sigma_{n}, \sigma_{n}\right]\right)$ to $\underbrace{L^{2}([0, \infty]) \times \ldots \times L^{2}([0, \infty])}_{n-\text { copies }}$.
(b) In Chapter 4 we will construct an inverse to the $\mathcal{F}_{p}$-operator which will be denoted by $\mathcal{F}_{p}^{-1}$. This shows that the map $A$ is injective and surjective on its range, and therefore is an isomorphism.
(c) By the Banach bounded inverse theorem, since the $F_{p}$-transform is bounded, the inverse $\mathcal{F}_{p}^{-1}$ is also bounded.
2. Our study of the map $B$ relies on the following properties which are listed below.
(a) In Chapter 5 we will show that the operator $T$ takes the form $T=I+$ $K$, where $K$ is an operator that has a similar form to (2.1.32). Further, we will show that each component of the operator $K$ can be written as a composition of $\mathcal{F}_{p}^{-1}$ and a compact operator. Since $\mathcal{F}_{p}^{-1}$ is bounded, and the composition of a bounded operator with a compact operator gives a compact operator, it follows that each component of the operator $K$ (and therefore the operator $K$ itself) is compact.
(b) Since the operator $K$ is compact, it follows that $T=I+K$ is a Fredholm operator of index zero, [26].
(c) In [4], Ashton proved that the solution to the global relation is unique. This implies that the operator $T$ is injective.

It follows from properties $2(\mathrm{~b})$ and $2(\mathrm{c})$ above that $T$ is a bounded injective linear operator with closed range. Furthermore, the operator $T$ is bounded below, since every bounded injective linear operator with closed range is bounded below, [2]. Since $T$ is bounded below, it is continuously invertible on its range $\operatorname{Ran}(T),[2]$. This means that if the Dirichlet data is given, then the global relation can be solved for the Neumann data. We conclude with the following theorem, which is given as Theorem 5 in Chapter 5.

Theorem. The solution of the global relation (2.3.1) corresponding to the BVP (2.2.8) exists, is unique, and depends continuously on the Dirichlet data. Moreover, the global relation can be solved to find the (unknown) Neumann data.

Given that we know the global relation can be solved to find the unknown Neumann data, we construct a numerical method to do so in Chapter 6.

## CHAPTER 3

## The $\mathcal{F}_{p}$-transform

In this chapter we formally introduce the $\mathcal{F}_{p}$-transform which was mentioned in the introduction and prove that it is a bounded operator.

Definition $1\left(\mathcal{F}_{p}\right.$-transform). Let $\Gamma=\{z \in \mathbb{C}: z=x+i p(x), x \in[-\sigma, \sigma]\}$, where $0<\sigma<\infty$, and assume $p(x)$ satisfies the following conditions:

- The function $p(x)$ is analytic on the interval $[-\sigma, \sigma]$, i.e. $p \in C^{\infty}([-\sigma, \sigma])$.
- The function $p(x)$ is concave and $p(0)=p^{\prime}(0)=0$.

Then, for $\varphi \in L^{2}([-\sigma, \sigma])$, the $\mathcal{F}_{p}$-transform is defined by:

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi \rightarrow \tilde{\varphi}(\lambda)=\int_{\Gamma} e^{-i \lambda z} \varphi(z) d z \tag{3.0.1}
\end{equation*}
$$

where $\lambda \in(0, \infty)$ is real.
We may also write the $\mathcal{F}_{p}$-transform as

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi \rightarrow \tilde{\varphi}(\lambda)=\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)\left(1+i p^{\prime}(x)\right) d x . \tag{3.0.2}
\end{equation*}
$$

Remark 1. As a note, we will specify that $\operatorname{Re}\left(e^{-i \lambda x+\lambda p(x)}\right)<0$ for the sake of exponential decay. Hence, the function $p(x)$ must be concave. Alternatively, we may choose $p(x)$ to be convex, in which case the exponential term takes the form $e^{-i \lambda x-\lambda p(x)}$.

Remark 2. For the sake of proving boundedness, it suffices to assume that $p(x)<0$. However, we additionally assume concavity because this will be needed when proving results for the global relation in Chapter 5. In particular, in order for a domain $\Omega$ to be convex, each of the segments that forms the boundary, i.e. each $\Gamma_{j}$ for $j=$ $1, \ldots, n$, must be concave. We will therefore prove the results in this chapter under the assumption that $p(x)$ is concave.

An example curve on which the $\mathcal{F}_{p}$-transform may be defined is illustrated in Figure 3.1 below:


Figure 3.1: An analytic concave curve in the complex plane with $p(0)=p^{\prime}(0)=0$.

As mentioned in the introduction, the $\mathcal{F}_{p}$-transform can be seen as a perturbation of the Fourier transform. To see this, set $p(x)=0$ and observe that $\Gamma$ becomes the segment $[-\sigma, \sigma]$, and the $\mathcal{F}_{p}$-transform becomes the Fourier transform over this interval.

We will now make a further connection between the $\mathcal{F}_{p}$-transform and the Fourier transform. Let $\Phi(z)$ be an entire function, and suppose the function $\varphi(x)$ in (3.0.2) is the restriction of $\Phi(z)$ to the curve $\Gamma$. Since $\Phi(z)$ is entire, Cauchy's theorem states
(see Figure 3.2 on the following page):

$$
\begin{equation*}
\int_{\Gamma} e^{-i \lambda z} \Phi(z) d z-\int_{-\sigma+i p(-\sigma)}^{\sigma+i p(\sigma)} e^{-i \lambda z} \Phi(z) d z=0 \tag{3.0.3}
\end{equation*}
$$

Since $\varphi(x)$ is the restriction of $\Phi(z)$ to the curve $\Gamma$, we may write the first integral in the form of (3.0.2) and rearrange to get:

$$
\begin{equation*}
\int_{-\sigma}^{\sigma} e^{-i \lambda x+i, p(x)} \varphi(x)\left(1+i p^{\prime}(x)\right) d x=\int_{-\sigma+i p(-\sigma)}^{\sigma+i p(\sigma)} e^{-i \lambda z} \Phi(z) d z \tag{3.0.4}
\end{equation*}
$$

This implies that the $\mathcal{F}_{p}$-transform of $\Phi(z)$ along the curve $\Gamma$ is equivalent to the Fourier transform of $\Phi(z)$ along the segment $[-\sigma+i p(-\sigma), \sigma+i p(\sigma)]$.


Figure 3.2: The curve $\Gamma$ (oriented counter-clockwise) and the line segment along $[-\sigma+$ $i p(-\sigma), \sigma+i p(\sigma)$ ] (oriented clockwise) form a closed curve in $\mathbb{C}$ in which the function $\Phi(z)$ has no singularity.

We observe the relation in (3.0.4) explicitly in the following example:

Example 2. Let $p(x)=-x^{2}$, and let

$$
\begin{equation*}
\Phi(z)=e^{z}, \quad \text { where } \quad z=x+i y \tag{3.0.5}
\end{equation*}
$$

and let $\varphi(x)$ be the restriction of $\Phi(z)$ to the curve $\Gamma=\left\{z \in \mathbb{C}: z=x-i x^{2}, x \in\right.$ $[-\sigma, \sigma]\}$. Then the $\mathcal{F}_{p}$-transform is given by:

$$
\begin{equation*}
\tilde{\varphi}(\lambda)=\int_{-\sigma-i \sigma^{2}}^{\sigma-i \sigma^{2}} e^{-i \lambda z} \Phi(z) d z \tag{3.0.6}
\end{equation*}
$$

which is the Fourier transform of $\Phi(z)$ over the segment $\left[-\sigma-i \sigma^{2}, \sigma-i \sigma^{2}\right]$.

### 3.1 Idea of Boundedness Proof for $\mathcal{F}_{p}$-transform

Now we wish to show that the map $\varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$, where

$$
\begin{equation*}
\Sigma:=[-\sigma, \sigma] \quad \text { and } \quad \Lambda:=[0, \infty] . \tag{3.1.1}
\end{equation*}
$$

The idea for the proof is as follows. First we use a bump function to partition the domain of integration into a neighborhood around the origin and a region away from the origin. Since the $\mathcal{F}_{p}$-transform decays exponentially in $\lambda$ away from the origin, the estimate for the region away from the origin is straightforward (see (3.2.7)). The integral over the origin is re-written using the Fourier inversion theorem and estimated using integration by parts and Young's inequality. The result that will be proved is stated in the theorem below:

Theorem 1. The following bound holds for all $\varphi \in L^{2}(\Sigma)$ :

$$
\begin{equation*}
\|\tilde{\varphi}(\lambda)\|_{L^{2}(\Lambda)} \lesssim\|\varphi\|_{L^{2}(\Sigma)^{1}}{ }^{1} \tag{3.1.2}
\end{equation*}
$$

and therefore the operator $\mathcal{F}_{p}: \varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$.
The work for proving Theorem 1 occupies the remainder of this chapter and is organized as follows. In Section 3.2 we partition the domain of integration into a region containing the origin and a region away from the origin, as mentioned above. We then show that the integral over the region away from the origin is bounded, and we establish some additional results that hold for a general analytic and concave $p(x)$. In Section 3.3, we show that the map $\varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$ for the case where $p(x)=-x^{2 n}$, with $n \in\{1,2, \ldots\}$. This allows us to develop some results needed for the more general case. Finally, in Section 3.4, we show that $\varphi \mapsto \tilde{\varphi}(\lambda)$ is

[^4]bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$ for a general analytic and concave $p(x)$.

### 3.2 Initial Results for General Analytic and Concave $p(x)$

We will begin by establishing some initial results for the general case of an analytic and concave $p(x)$. In subsequent sections, we will start from these results and prove the estimates for the respective cases.

Assume that $\varphi \in L^{2}(\Sigma)$ and that $p(x)$ in equation (3.0.1) is analytic and concave. Fix a bump function $\rho(x) \in C_{c}^{\infty}(\Sigma)$ with $\rho=1$ when $|x|<\delta / 2$, and $\rho=0$ when $|x|>\delta$. We may now split the integral operator into a component that contains the origin and another component away from the origin using the bump function $\rho(x):^{2}$

$$
\begin{equation*}
\tilde{\varphi}(\lambda)=\underbrace{\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x) \rho(x) d x}_{K_{1} \varphi(\lambda)}+\underbrace{\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)[1-\rho(x)] d x}_{K_{2} \varphi(\lambda)} . \tag{3.2.1}
\end{equation*}
$$

### 3.2.1 Estimate for the Operator $K_{2}: L^{2}(\Sigma) \rightarrow L^{2}(\Lambda)$

We will now show that the operator $K_{2} \varphi(\lambda)$ is bounded. First, we estimate the integrand of $\left\|K_{2} \varphi(\lambda)\right\|_{L^{2}(\Lambda)}^{2}$ below:

$$
\begin{align*}
\mid \int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)[1 & -\rho(x)]\left.d x\right|^{2} \\
& \leq \int_{-\sigma}^{\sigma}|\varphi(x)|^{2} d x \int_{-\sigma}^{\sigma}\left|e^{-i \lambda x+\lambda p(x)}[1-\rho(x)]\right|^{2} d x  \tag{3.2.2}\\
& =\|\varphi\|_{L^{2}(\Sigma)}^{2} \int_{-\sigma}^{\sigma}\left|e^{-i \lambda x}\right|^{2}\left|e^{\lambda p(x)}\right|^{2}[1-\rho(x)]^{2} d x  \tag{3.2.3}\\
& =\|\varphi\|_{L^{2}(\Sigma)}^{2} \int_{-\sigma}^{\sigma} e^{2 \lambda p(x)}[1-\rho(x)]^{2} d x  \tag{3.2.4}\\
& \leq\|\varphi\|_{L^{2}(\Sigma)}^{2} \int_{\delta / 2<|x|<\sigma} e^{2 \lambda p(x)} d x  \tag{3.2.5}\\
& \leq\|\varphi\|_{L^{2}(\Sigma)}^{2}(2 \sigma-\delta) e^{2 \lambda p(\delta / 2)}  \tag{3.2.6}\\
& \lesssim\|\varphi\|_{L^{2}(\Sigma)}^{2} e^{2 \lambda p(\delta / 2)} . \tag{3.2.7}
\end{align*}
$$

[^5]In the final inequality above, $e^{2 \lambda p(\delta / 2)}$ is bounded since $p(x)$ is concave. Hence, using the estimates above, we bound $\left\|K_{2} \varphi(\lambda)\right\|_{L^{2}(\Lambda)}$ as follows ${ }^{3}$

$$
\begin{equation*}
\int_{0}^{\infty}\left|K_{2} \varphi(\lambda)\right|^{2} d \lambda \lesssim\|\varphi\|_{L^{2}(\Sigma)}^{2} \int_{0}^{\infty} e^{2 \lambda p(\delta / 2)} d \lambda \lesssim \delta\|\varphi\|_{L^{2}(\Sigma)}^{2} \tag{3.2.8}
\end{equation*}
$$

### 3.2.2 The Operator $K_{1}: L^{2}(\Sigma) \rightarrow L^{2}(\Lambda)$

Now we turn our attention to the term $K_{1} \varphi(\lambda)$. We are interested in the large $\lambda$ behavior so for $R>0$ fixed, we have

$$
\begin{equation*}
\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(0, \infty)}^{2}=\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(0, R)}^{2}+\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(R, \infty)}^{2} . \tag{3.2.9}
\end{equation*}
$$

The first term in the expression above can be bounded by applying the Cauchy-Schwarz inequality to get:

$$
\begin{equation*}
\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(0, R)}^{2} \lesssim R\|\varphi\|_{L^{2}(\Sigma)}^{2} . \tag{3.2.10}
\end{equation*}
$$

Hence, we find

$$
\begin{equation*}
\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(0, \infty)}^{2} \lesssim R\|\varphi\|_{L^{2}(\Sigma)}^{2}+\left\|K_{1} \varphi(\lambda)\right\|_{L^{2}(R, \infty)}^{2} \tag{3.2.11}
\end{equation*}
$$

and therefore we may focus our attention on the second term in the expression above. Without loss of generality, we will assume $R>1$ throughout.

In the discussion that follows, we will assume $\varphi$ is analytic and has compact support on $\Sigma$, i.e. $\varphi \in C_{c}^{\infty}(\Sigma)$, so that $\hat{\varphi}$ has rapid decay. We will prove the desired estimate under this assumption, and the result can be extended to a larger class of functions (i.e., $L^{2}(\Sigma)$ ) using a density argument.

Using the Fourier inversion theorem, $\varphi$ may be written as

$$
\begin{equation*}
\varphi(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i y x} \hat{\varphi}(y) d y, \tag{3.2.12}
\end{equation*}
$$

[^6]and hence $K_{1} \varphi(\lambda)$ becomes
\[

$$
\begin{equation*}
K_{1} \varphi(\lambda)=\int_{-\infty}^{\infty} e^{-i \lambda x+\lambda p(x)} \rho(x)\left[\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i y x} \hat{\varphi}(y) d y\right] d x \tag{3.2.13}
\end{equation*}
$$

\]

Owing to the rapid decay of $\hat{\varphi}$, we can use Fubini's theorem to interchange the order of integration to get:

$$
\begin{align*}
K_{1} \varphi(\lambda) & =\int_{-\infty}^{\infty}\left[\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-i \lambda x+\lambda p(x)} \rho(x) e^{i y x} \hat{\varphi}(y) d x\right] d y  \tag{3.2.14}\\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-i(\lambda-y) x+\lambda p(x)} \rho(x) d x\right] \hat{\varphi}(y) d y . \tag{3.2.15}
\end{align*}
$$

We define the function $H(\lambda, y)$ as the inner integral above

$$
\begin{equation*}
H(\lambda, y)=\int_{-\infty}^{\infty} e^{-i(\lambda-y) x+\lambda p(x)} \rho(x) d x \tag{3.2.16}
\end{equation*}
$$

If $|\lambda-y| \leq 1$, then since $p(x) \leq 0$ and $\rho(x)$ has compact support on $[-\sigma, \sigma]$, we have the bound

$$
\begin{equation*}
|H(\lambda, y)| \leq \int_{-\infty}^{\infty}|\rho(x)| d x \lesssim \frac{1}{1+1} \lesssim \frac{1}{1+|\lambda-y|^{N}}, \quad|\lambda-y| \leq 1, \tag{3.2.17}
\end{equation*}
$$

for any $N \geq 0$.
Recall that an analytic concave function can be expressed in the form $p(x)=$ $-c_{2 n} x^{2 n}+c_{2 n+1} x^{2 n+1} \psi(x)$, where $\psi(x)$ is an analytic function, and $c_{2 n}$ and $c_{2 n+1}$ are constants. Therefore, it is sufficient to show that $K_{1} \varphi(\lambda)$ is bounded for such a representation of $p(x)$. To this end, we first consider the case $p(x)=-x^{2 n}$, where $n \in\{1,2,3, \ldots\}$. This will allow us to establish some initial results that will be used in the more general setting. We will then consider the general case $p(x)=$ $-c_{2 n} x^{2 n}+c_{2 n+1} x^{2 n+1} \psi(x)$, and prove an estimate of the form (3.2.17) for the case $|\lambda-y|>1$. This estimate will allow us to show that $K_{1} \varphi$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$. Since we have already shown that $K_{2} \varphi$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$, this will imply that the map $\varphi \mapsto \tilde{\varphi}$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$.

### 3.3 Estimate for the Case $p(x)=-x^{2 n}$

In this section we will establish the necessary estimates to show that the map $\varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$ in the case where $p(x)=-x^{2 n}$ for $n \in\{1,2, \ldots\}$. Since in the general case, $p(x)$ is assumed to be analytic ${ }^{4}$ and concave, this implies that the first term in the Taylor series expansion will be of the form $p(x) \sim-x^{2 n}$ for some fixed positive integer $n$. Hence, the case $p(x)=-x^{2 n}$ for $n \in\{1,2, \ldots\}$ is a first-order approximation of the more general $p(x)$.

### 3.3.1 Estimate for $H(\lambda, y)$ with $p(x)=-x^{2 n}$

We will now show that $H(\lambda, y)$ is bounded in the case where $p(x)=-x^{2 n}$. We use the change of variable $x=\varepsilon \tau$, where $\varepsilon=\lambda^{-1 / 2 n}$. Note that since $\lambda>1$, this implies that $\varepsilon<1$. Using this change of variable $H(\lambda, y)$ may be written as:

$$
H(\lambda, y)=\int_{-\infty}^{\infty} e^{-i(\lambda-y) \varepsilon \tau} e^{-\tau^{2 n}} \rho(\varepsilon \tau) \varepsilon d \tau .
$$

Integrating $H(\lambda, y)$ by parts $N=2 n$ times results in the following:

$$
\begin{equation*}
H(\lambda, y)=\frac{\varepsilon}{[-i \varepsilon(\lambda-y)]^{N}} \int_{-\infty}^{\infty} e^{-i(\lambda-y) \varepsilon \tau} \frac{d^{N}}{d \tau^{N}}\left[e^{-\tau^{2 n}} \rho(\varepsilon \tau)\right] d \tau . \tag{3.3.2}
\end{equation*}
$$

Applying the product rule for derivatives, we find

$$
\begin{equation*}
\frac{d^{N}}{d \tau^{N}}\left[e^{-\tau^{2 n}} \rho(\varepsilon \tau)\right]=\sum_{k=0}^{N}\binom{N}{k} \frac{d^{N-k}}{d \tau^{N-k}}\left[e^{-\tau^{2 n}}\right] \frac{d^{k}}{d \tau^{k}}[\rho(\varepsilon \tau)] . \tag{3.3.3}
\end{equation*}
$$

In order to further understand the right-hand-side of equation (3.3.3), we will compute and re-write $\frac{d^{N-k}}{d \tau^{N-k}}\left[e^{-\tau^{2 n}}\right]$ in the lemma below.

[^7]Lemma 1. Let $P_{m}(x)$ denote a polynomial of order $m$ in the variable $x$, then

$$
\begin{equation*}
\frac{d^{N-k}}{d \tau^{N-k}}\left[e^{-\tau^{2 n}}\right]=P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}} \tag{3.3.4}
\end{equation*}
$$

Proof. We will establish the claim by induction, and to simplify notation, let $m=$ $N-k$. As a base case, when $m=1$ we find:

$$
\begin{align*}
\frac{d}{d \tau}\left[e^{-\tau^{2 n}}\right] & =\left[1-2 n \tau^{2 n-1}\right] e^{-\tau^{2 n}}  \tag{3.3.5}\\
& =P_{1(2 n-1)}(\tau) e^{-\tau^{2 n}} \tag{3.3.6}
\end{align*}
$$

Now suppose the claim holds for a fixed $m$, then differentiating in $\tau$ yields

$$
\begin{align*}
\frac{d}{d \tau}\left[P_{m(2 n-1)}\right. & \left.(\tau) e^{-\tau^{2 n}}\right] \\
& =\left[\left(\sum_{k=0}^{m(2 n-1)} a_{k} \tau^{k}\right)\left(-2 n \tau^{2 n-1}\right)+\sum_{k=1}^{m(2 n-1)} k a_{k} \tau^{k-1}\right] e^{-\tau^{2 n}}  \tag{3.3.7}\\
& =\left[\left(\sum_{k=0}^{m(2 n-1)}\left(-2 n a_{k}\right) \tau^{k+(2 n-1)}\right)+\sum_{k=1}^{m(2 n-1)} k a_{k} \tau^{k-1}\right] e^{-\tau^{2 n}}  \tag{3.3.8}\\
& \left.=\left[\left(\sum_{k=0}^{(m+1)(2 n-1)} \tilde{a}_{k} \tau^{k}\right)+\sum_{k=1}^{m(2 n-1)} k a_{k} \tau^{k-1}\right] e^{-\tau^{2 n}}\right]  \tag{3.3.9}\\
& =\left(\sum_{k=0}^{(m+1)(2 n-1)} c_{k} \tau^{k}\right) e^{-\tau^{2 n}} \tag{3.3.10}
\end{align*}
$$

for new $\left\{c_{k}\right\}_{k=0}^{(m+1)(2 n-1)}$. The final expression above is a polynomial of order $(m+$ 1) $(2 n-1)$ in $\tau$, i.e.

$$
\begin{equation*}
\frac{d^{m+1}}{d \tau^{m+1}}\left[e^{-\tau^{2 n}}\right]=P_{(m+1) \cdot(2 n-1)}(\tau) e^{-\tau^{2 n}} \tag{3.3.11}
\end{equation*}
$$

and hence the claim is established.

Using Lemma 1, equation (3.3.3) may be written as

$$
\begin{equation*}
\frac{d^{N}}{d \tau^{N}}\left[e^{-\tau^{2 n}} \rho(\varepsilon \tau)\right]=\sum_{k=0}^{N}\binom{N}{k}\left[P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}}\right] \varepsilon^{k} \rho^{(k)}(\varepsilon \tau) \tag{3.3.12}
\end{equation*}
$$

Substituting this back into equation (3.3.2), we find

$$
\begin{equation*}
|H(\lambda, y)| \lesssim \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{k=0}^{N} \varepsilon^{k} \int_{-\infty}^{\infty}\left[P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}}\right] \rho^{(k)}(\varepsilon \tau) d \tau \tag{3.3.13}
\end{equation*}
$$

Note that in the term $P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}}$, the exponential will dominate the polynomial, and will rapidly decay as $|\tau| \rightarrow \infty$. In addition, $\rho(\varepsilon \tau)$ and all of its derivatives $\rho^{(k)}(\varepsilon \tau)$ for $k \geq 0$ are bounded. Hence, we have

$$
\begin{equation*}
\int_{-\infty}^{\infty} P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}} \rho^{(k)}(\varepsilon \tau) d \tau \leq c_{k}, \quad \text { for } \quad k \in\{0,1, \ldots, N\} \tag{3.3.14}
\end{equation*}
$$

Thus, equation (3.3.13) becomes:

$$
\begin{align*}
|H(\lambda, y)| & \lesssim \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{k=0}^{N} \varepsilon^{k} \int_{-\infty}^{\infty} P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}} \rho^{(k)}(\varepsilon \tau) d \tau  \tag{3.3.15}\\
& \leq \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{k=0}^{N} \varepsilon^{k} c_{k}  \tag{3.3.16}\\
& \lesssim \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}}\left(\frac{1-\varepsilon^{N+1}}{1-\varepsilon}\right)  \tag{3.3.17}\\
& \lesssim N \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}}  \tag{3.3.18}\\
& \lesssim \frac{\varepsilon}{1+|\varepsilon(\lambda-y)|^{N}} \tag{3.3.19}
\end{align*}
$$

This gives a bound for $H(\lambda, y)$ which we will use for showing that $K_{1} \varphi(\lambda)$ is bounded.

### 3.3.2 Estimate for the Operator $K_{1}: L^{2}(\Sigma) \rightarrow L^{2}(\Lambda)$

Now we wish to show that the map $K_{1} \varphi$ is bounded from $L^{2}(1, \infty)$ to $L^{2}(\Sigma)$; in particular we will show

$$
\begin{equation*}
\left\|K_{1} \varphi\right\|_{L^{2}(1, \infty)}=\left\|\int_{-\infty}^{\infty} H(\cdot, y) \hat{\varphi}(y) d y\right\|_{L^{2}(1, \infty)} \lesssim\|\hat{\varphi}\|_{L^{2}(\mathbb{R})}=\|\varphi\|_{L^{2}(\Sigma)} . \tag{3.3.20}
\end{equation*}
$$

We will obtain the estimate through a careful application of Young's inequality. By establishing the following two estimates:

$$
\begin{array}{ll}
\text { [A] } & \sup _{\lambda>1} \int_{-\infty}^{\infty}|H(\lambda, y)| d y \lesssim 1  \tag{3.3.21}\\
\text { [B] } & \sup _{y \in \mathbb{R}} \int_{1}^{\infty}|H(\lambda, y)| d \lambda \lesssim 1,
\end{array}
$$

Young's inequality will immediately imply

$$
\begin{equation*}
\left\|\int_{-\infty}^{\infty} H(\cdot, y) \hat{\varphi}(y) d y\right\|_{L^{2}((1, \infty))} \lesssim\|\hat{\varphi}\|_{L^{2}(\mathbb{R})} . \tag{3.3.22}
\end{equation*}
$$

For reference, Young's inequality can be found in most analysis textbooks such as [31].

## Estimate A

From (3.3.19), we have

$$
\begin{equation*}
\sup _{\lambda>1} \int_{-\infty}^{\infty}|H(\lambda, y)| d y \lesssim \sup _{\lambda>1} \int_{-\infty}^{\infty} \frac{\varepsilon}{1+|\varepsilon(\lambda-y)|^{N}} d y . \tag{3.3.23}
\end{equation*}
$$

We make the following substitution

$$
\begin{equation*}
z=\varepsilon(\lambda-y) \quad \Rightarrow \quad d z=\varepsilon d y \tag{3.3.24}
\end{equation*}
$$

which gives us the integral below

$$
\begin{equation*}
\sup _{\lambda>1} \int_{-\infty}^{\infty} \frac{\varepsilon}{1+|\varepsilon(\lambda-y)|^{N}} d y=\sup _{\lambda>1} \int_{-\infty}^{\infty} \frac{1}{1+|z|^{N}} d z \tag{3.3.25}
\end{equation*}
$$

Since we are assuming $|\lambda-y|>1$, it follows that

$$
\begin{equation*}
\sup _{\lambda>1} \int_{-\infty}^{\infty} \frac{1}{1+|z|^{N}} d z \leq \sup _{\lambda>1} \int_{-\infty}^{\infty} \frac{1}{1+|z|^{2}} d z \lesssim 1 \tag{3.3.26}
\end{equation*}
$$

This establishes Estimate A.

## Estimate B

From (3.3.19), we have

$$
\begin{equation*}
\sup _{y \in \mathbb{R}} \int_{1}^{\infty}|H(\lambda, y)| d \lambda \lesssim \sup _{y \in \mathbb{R}} \int_{1}^{\infty} \frac{\lambda^{-1 / 2 n}}{1+\left|\left(\lambda^{1-1 / 2 n}-\frac{y}{\lambda^{1 / 2 n}}\right)\right|^{N}} d \lambda, \quad \text { for } N \geq 2 \tag{3.3.27}
\end{equation*}
$$

For $y \leq 0$ the following inequality holds:

$$
\begin{equation*}
\frac{1}{1+\left|\lambda^{1-1 / 2 n}-\frac{y}{\lambda^{1 / 2 n}}\right|^{N}} \leq \frac{1}{1+\left|\lambda^{1-1 / 2 n}\right|^{N}} \tag{3.3.28}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\sup _{y \leq 0} \int_{1}^{\infty} \frac{\lambda^{-1 / 2 n}}{1+\left|\left(\lambda^{1-1 / 2 n}-\frac{y}{\lambda^{1 / 2 n}}\right)\right|^{N}} d \lambda \leq \int_{1}^{\infty} \frac{\lambda^{-1 / 2 n}}{1+\left|\lambda^{1-1 / 2 n}\right|^{N}} d \lambda \lesssim 1 \tag{3.3.29}
\end{equation*}
$$

which follows from the simple substitution $z=\lambda^{1-1 / 2 n}$.
For $y>0$ we can use the substitution

$$
\begin{equation*}
z=\lambda^{1-1 / 2 n}-\frac{y}{\lambda^{1 / 2 n}} \tag{3.3.30}
\end{equation*}
$$

We note that since $y>0$ and $d z / d \lambda>0, \lambda=\lambda(z)$ is well-defined. Further, we have

$$
\begin{equation*}
\lambda^{-1 / 2 n} d \lambda=\frac{d z}{\left[\left(1-\frac{1}{2 n}\right)+\frac{y}{2 n \lambda}\right]} \tag{3.3.31}
\end{equation*}
$$

This results in the following integral

$$
\begin{equation*}
\sup _{y>0} \int_{1}^{\infty} \frac{\lambda^{-1 / 2 n}}{1+\left|\left(\lambda^{1-1 / 2 n}-\frac{y}{\lambda^{1 / 2 n}}\right)\right|^{N}} d \lambda=\sup _{y>0} \int_{1-y}^{\infty} \frac{1}{1+|z|^{N}} \frac{d z}{\left[\left(1-\frac{1}{2 n}\right)+\frac{y}{2 n \lambda}\right]} \tag{3.3.32}
\end{equation*}
$$

Since $y>0$ and $\lambda=\lambda(z)>1$, we have

$$
\begin{equation*}
\frac{1}{\left[\left(1-\frac{1}{2 n}\right)+\frac{y}{2 n \lambda}\right]} \leq \frac{1}{1-\frac{1}{2 n}} \tag{3.3.33}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\sup _{y>0} \int_{1-y}^{\infty} \frac{1}{1+|z|^{N}} \frac{d z}{\left[\left(1-\frac{1}{2 n}\right)+\frac{y}{2 n \lambda}\right]} \leq \sup _{y>0} \int_{1-y}^{\infty} \frac{1}{1+|z|^{N}} \frac{d z}{1-\frac{1}{2 n}} \lesssim 1 \tag{3.3.34}
\end{equation*}
$$

From equations (3.3.26) and (3.3.34), we obtain

$$
\begin{equation*}
\sup _{y \in \mathbb{R}} \int_{1}^{\infty}|H(\lambda, y)| d \lambda \lesssim 1 \quad \text { for } N \geq 2 \tag{3.3.35}
\end{equation*}
$$

Summarizing our results, for $p(x)=-x^{2 n}$ we have shown the following bounds

$$
\begin{equation*}
\sup _{\lambda>1} \int_{-\infty}^{\infty}|H(\lambda, y)| d y \lesssim 1 \quad \text { and } \quad \sup _{y \in \mathbb{R}} \int_{1}^{\infty}|H(\lambda, y)| d \lambda \lesssim 1 \tag{3.3.36}
\end{equation*}
$$

and therefore by Young's inequality we have

$$
\begin{equation*}
\left\|K_{1} \varphi\right\|_{L^{2}(1, \infty)}=\left\|\int_{-\infty}^{\infty} H(\cdot, y) \hat{\varphi}(y) d y\right\|_{L^{2}(1, \infty)} \lesssim\|\hat{\varphi}\|_{L^{2}(\mathbb{R})}=\|\varphi\|_{L^{2}(\Sigma)} \tag{3.3.37}
\end{equation*}
$$

where the final equality is by Parseval's theorem. This shows that $K_{1} \varphi$ is bounded for the case $p(x)=-x^{2 n}$. Since we have already established that $K_{2} \varphi$ is bounded, this
implies that the $\mathcal{F}_{p}$-transform is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$, that is

$$
\begin{equation*}
\|\tilde{\varphi}\|_{L^{2}(\Lambda)} \lesssim\|\varphi\|_{L^{2}(\Sigma)} . \tag{3.3.38}
\end{equation*}
$$

This result was shown for a concave function of the form $p(x)=-x^{2 n}$, in the section that follows we will use similar methods to show the same bound for a general analytic concave $p(x)$.

### 3.4 Estimate for Analytic and Concave $p(x)$

Now we will assume that $p(x)$ takes the form $p(x)=-x^{2 n}+x^{2 n+1} \psi(x)$, where $\psi(x)$ is analytic. By Taylor's theorem this is the general form of an analytic concave function up to a constant multiple. We will show that the corresponding integral in equation (3.2.16), i.e.

$$
\begin{equation*}
H(\lambda, y)=\int_{-\infty}^{\infty} e^{-i(\lambda-y) x-\lambda\left(x^{2 n}+x^{2 n+1} \psi(x)\right)} \rho(x) d x \tag{3.4.1}
\end{equation*}
$$

is bounded. Using the change of variable $x=\lambda^{-1 / 2 n} \tau$, we rewrite $H(\lambda, y)$ as

$$
\begin{equation*}
H(\lambda, y)=\int_{-\infty}^{\infty} e^{-i(\lambda-y) \varepsilon \tau} e^{-\tau^{2 n}} e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \rho(\varepsilon \tau) \varepsilon d \tau, \tag{3.4.2}
\end{equation*}
$$

where $\varepsilon=\lambda^{-1 / 2 n}$. Integrating $H(\lambda, y)$ by parts N times yields

$$
\begin{equation*}
H(\lambda, y)=\frac{\varepsilon}{[-i \varepsilon(\lambda-y)]^{N}} \int_{-\infty}^{\infty} e^{-i(\lambda-y) \varepsilon \tau} \frac{d^{N}}{d \tau^{N}}\left[e^{-\tau^{2 n}} e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \rho(\varepsilon \tau)\right] d \tau \tag{3.4.3}
\end{equation*}
$$

Using the product rule for derivatives, the term $A(\tau)=e^{-\tau^{2 n}} e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \rho(\varepsilon \tau)$ may be expressed as follows

$$
\begin{align*}
& \frac{d^{N}}{d \tau^{N}}[A(\tau)] \\
& =\sum_{k=0}^{N}\binom{N}{k}\left[\frac{d^{N-k}}{d \tau^{N-k}} e^{-\tau^{2 n}}\right] \frac{d^{k}}{d \tau^{k}}\left[e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \rho(\varepsilon \tau)\right]  \tag{3.4.4}\\
& =\sum_{k=0}^{N}\binom{N}{k} \frac{d^{N-k}}{d \tau^{N-k}}\left[e^{-\tau^{2 n}}\right] \sum_{m=0}^{k}\binom{k}{m} \frac{d^{m}}{d \tau^{m}}\left[e^{-\varepsilon \tau^{2 n+1}} \psi(\varepsilon \tau)\right] \frac{d^{k-m}}{d \tau^{k-m}}[\rho(\varepsilon \tau)]  \tag{3.4.5}\\
& =\sum_{k=0}^{N} \sum_{m=0}^{k}\binom{N}{k}\binom{k}{m} \frac{d^{N-k}}{d \tau^{N-k}}\left[e^{-\tau^{2 n}}\right] \frac{d^{m}}{d \tau^{m}}\left[e^{-\varepsilon \tau^{2 n+1}} \psi(\varepsilon \tau)\right] \varepsilon^{k-m} \rho^{(k-m)}(\varepsilon \tau) . \tag{3.4.6}
\end{align*}
$$

Now we must deal with the terms $B(\tau)=e^{-\tau^{2 n}}$ and $C(\tau)=e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)}$. By Lemma $1, B^{(N-k)}(\tau)$ is given by

$$
\begin{equation*}
B^{(N-k)}(\tau)=P_{(N-k)(2 n-1)}(\tau) e^{-\tau^{2 n}} \tag{3.4.7}
\end{equation*}
$$

Now we will study the behavior of the term $C^{(m)}(\tau)$ and derive an estimate for it that can be used in bounding equation (3.4.6).

Lemma 2. The mth derivative $C^{(m)}(\tau)$ takes the form

$$
\begin{equation*}
C^{(m)}(\tau)=P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right) e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)}, \tag{3.4.8}
\end{equation*}
$$

where $P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right)$ is a polynomial of order $m(2 n+1)$ in $\tau$ with smooth and bounded coefficients on the interval $\tau \in \Delta_{\delta, \varepsilon}:=[-\delta / \varepsilon, \delta / \varepsilon]$.

Proof. To deal with the term $C^{(m)}(\tau)$, we use Faà di Bruno's formula for differentiating compositions of functions. Let $g: \Delta_{\delta, \varepsilon} \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}^{+}$be defined by

$$
\begin{equation*}
g(\tau):=-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau) \quad \text { and } \quad f(y):=e^{y} . \tag{3.4.9}
\end{equation*}
$$

Faà di Bruno's formula states that the derivative $(f \circ g)^{(m)}(\tau)$ is given by:

$$
\begin{equation*}
(f \circ g)^{(m)}(\tau)=e^{g(\tau)} \sum_{\substack{\left(l_{1}, \ldots, l_{m}\right) \\ 1 \cdot l_{1}+\ldots+m \cdot l_{m}=m}}\binom{m}{l_{1}, \cdots, l_{m}} \cdot f^{\left(l_{1}+\ldots+l_{m}\right)}(g(\tau)) \cdot \prod_{m=1}^{m}\left(\frac{g^{(m)}(\tau)}{m!}\right)^{l_{m}} \tag{3.4.10}
\end{equation*}
$$

For reference, Faà di Bruno's formula can be found in Chapter 5 of [34]. Now each $g^{(m)}(\tau)$ may be expressed, using the generalized Leibniz rule, as a polynomial in $\tau$ of the form

$$
\begin{align*}
g^{(m)}(\tau) & =\sum_{k=0}^{m}\binom{m}{k} \frac{d^{k}}{d \tau^{k}}\left(\tau^{2 n+1}\right) \frac{d^{m-k}}{d \tau^{m-k}}(\psi(\varepsilon \tau))  \tag{3.4.11}\\
& =\sum_{k=0}^{m}\binom{m}{k} \prod_{l=0}^{k-1}(2 n+1-l) \tau^{2 n+1-k} \varepsilon^{m-k} \psi^{(m-k)}(\varepsilon \tau)  \tag{3.4.12}\\
& =\sum_{k=0}^{m} C_{n, m, k}\left[\varepsilon^{m-k} \psi^{(m-k)}(\varepsilon \tau)\right] \tau^{2 n+1-k}, \tag{3.4.13}
\end{align*}
$$

where $C_{n, m, k}:=\binom{m}{k} \prod_{l=0}^{k-1}(2 n+1-l)$ is a constant for each summand. Each of the coefficients of the polynomial in $\tau$ in (3.4.13) is given by $C_{n, m, k}\left[\varepsilon^{m-k} \psi^{(m-k)}(\varepsilon \tau)\right]$, which is a constant multiple of the function $\psi^{(m-k)}(\varepsilon \tau)$. Furthermore, we know that $\psi^{(l)}(x)$, for $l \in\{0,1,2, \ldots\}$, is bounded on the interval $|x| \leq \delta$, i.e.

$$
\begin{equation*}
\left|\psi^{(l)}(x)\right| \leq M \quad \text { where } x \in[-\delta, \delta], \tag{3.4.14}
\end{equation*}
$$

where $M:=\max \left\{\max _{l} M_{l}, 1\right\}$ with $M_{l}:=\max _{x \in[-\delta, \delta]}\left|\psi^{(l)}(x)\right|<\infty$ for each $l \in$ $\{0,1,2, \ldots, m\}$. Since $x=\varepsilon \tau$, then the above bound holds for $x \in[-\delta, \delta]$ if and only if it holds for $\tau \in \Delta_{\delta, \varepsilon}:=[-\delta / \varepsilon, \delta / \varepsilon]$.

Since (3.4.10) is a product and sum of polynomials of the form (3.4.13), it will also be a polynomial in $\tau$ with coefficients that are sums and products of terms of the form
$C_{n, m, k}\left[\varepsilon^{m-k} \psi^{(m-k)}(\varepsilon \tau)\right]$. By the discussion above, each of the coefficients

$$
\begin{equation*}
C_{n, m, k}\left[\varepsilon^{m-k} \psi^{(m-k)}(\varepsilon \tau)\right] \tag{3.4.15}
\end{equation*}
$$

is smooth and bounded on $\Delta_{\delta, \varepsilon}$, and therefore any finite sum and product of terms of this form will also be smooth and bounded on the same interval. Therefore, (3.4.10) is a polynomial in $\tau$ with smooth and bounded coefficients on the interval $\Delta_{\delta, \varepsilon}$.

Finally, note that the highest order term in $\tau$ occurs when $\left(l_{1}, \ldots, l_{m}\right)=(m, 0, \ldots, 0)$. This yields a term that is order $m(2 n+1)$ in $\tau$, and hence the polynomial in (3.4.10) is order $m(2 n+1)$ in $\tau$.

Combining these results, it follows that the mth derivative $C^{(m)}(\tau)$ takes the form

$$
\begin{equation*}
C^{(m)}(\tau)=P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right) e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \tag{3.4.16}
\end{equation*}
$$

where $P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right)$ is a polynomial of order $m(2 n+1)$ in $\tau$ with smooth and bounded coefficients on the interval $\tau \in \Delta_{\delta, \varepsilon}$.

Working from equation (3.4.6), and using Lemma 2, we have:

$$
\begin{align*}
& \left|\frac{d^{N}}{d \tau^{N}}[A(\tau)]\right| \\
& \leq \sum_{k=0}^{N} \sum_{m=0}^{k}\binom{N}{k}\binom{k}{m}\left|B^{(N-k)}(\tau)\right|\left|C^{(m)}(\tau)\right| \varepsilon^{k-m}\left|\rho^{(k-m)}(\varepsilon \tau)\right|  \tag{3.4.17}\\
& \leq \sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m} e^{-\tau^{2 n}}\left|P_{(N-k)(2 n-1)}(\tau)\right| e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)} \\
& \quad\left|P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right)\right| \varepsilon^{k-m}\left|\rho^{(k-m)}(\varepsilon \tau)\right|  \tag{3.4.18}\\
& =\sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m} e^{-\tau^{2 n}} e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)}|F(\tau)| \varepsilon^{k-m}\left|\rho^{(k-m)}(\varepsilon \tau)\right|, \tag{3.4.19}
\end{align*}
$$

where we define

$$
\begin{equation*}
F(\tau):=P_{(N-k)(2 n-1)}(\tau) P_{m(2 n+1)}\left(\tau, \psi(\varepsilon \tau), \psi^{(1)}(\varepsilon \tau), \ldots, \psi^{(m)}(\varepsilon \tau)\right) . \tag{3.4.20}
\end{equation*}
$$

Now we will estimate the exponential term $e^{-\tau^{2 n}} e^{-\varepsilon \tau^{2 n+1} \psi(\varepsilon \tau)}$. Letting $C:=$ $\max _{x \in[-\delta, \delta]}|\psi(x)|<\infty$ and recalling that $|x| \leq \delta$, we find:

$$
\begin{equation*}
e^{-\lambda\left(x^{2 n}+x^{2 n+1} \psi(x)\right)} \leq e^{-\lambda\left(x^{2 n}+\delta C x^{2 n}\right)}=e^{-\lambda\left((1+\delta C) x^{2 n}\right)}=e^{-\lambda\left(\tilde{C} x^{2 n}\right)}=e^{-\tilde{C} \tau^{2 n}} \tag{3.4.21}
\end{equation*}
$$

where $\tilde{C}=(1+\delta C)>0$. Using this estimate in (3.4.19), yields

$$
\begin{equation*}
\left|\frac{d^{N}}{d \tau^{N}}[A(\tau)]\right| \leq \sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m} e^{-\tilde{C} \tau^{2 n}}|F(\tau)| \varepsilon^{k-m}\left|\rho^{(k-m)}(\varepsilon \tau)\right| . \tag{3.4.22}
\end{equation*}
$$

Now we have

$$
\begin{align*}
& |H(\lambda, y)| \\
& \leq \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \int_{-\infty}^{\infty} \sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m}|F(\tau)| e^{-\tilde{C} \tau^{2 n}} \varepsilon^{k-m}\left|\rho^{(k-m)}(\varepsilon \tau)\right| d \tau  \tag{3.4.23}\\
& =\frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m} \varepsilon^{k-m} \int_{-\infty}^{\infty}\left|F(\tau) \rho^{(k-m)}(\varepsilon \tau)\right| e^{-\tilde{C} \tau^{2 n}} d \tau . \tag{3.4.24}
\end{align*}
$$

By Lemma 1 and Lemma 2, we know that $F(\tau)$ is the product of two polynomials in $\tau$ with coefficients that are smooth and bounded on the interval $\Delta_{\delta, \varepsilon}$. Therefore, $F(\tau)$ is itself a smooth polynomial of finite-order on this interval. Furthermore, $\Delta_{\delta, \varepsilon}$ is also the interval of compact support for $\rho(\varepsilon \tau)$ and hence $\rho^{(k-m)}(\varepsilon \tau)$ as well. Thus, we may restrict our attention to $\tau \in \Delta_{\delta, \varepsilon}$ in order to show that the integral in (3.4.24) is bounded.

Recall that $\varepsilon=\lambda^{-1 / 2 n}$ so that $\varepsilon \rightarrow 0$ as $\lambda \rightarrow \infty$, and hence even restricted to the interval $\Delta_{\delta, \varepsilon}$, we must understand the behavior of the integral in (3.4.24) as $|\tau| \rightarrow \infty$ (recall $\delta$ is fixed). Since $F(\tau)$ is a finite-order polynomial with bounded coefficients on $\Delta_{\delta, \varepsilon}$, the exponential $e^{-\tilde{C} \tau^{2 n}}$ will dominate $F(\tau)$ and the integrand will decay rapidly
as $|\tau| \rightarrow \infty$, and therefore

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left|F(\tau) \rho^{(k-m)}(\varepsilon \tau)\right| e^{-\tilde{C} \tau^{2 n}} d \tau \leq C_{k, m}<\infty \tag{3.4.25}
\end{equation*}
$$

where $C_{k, m}$ is a constant and the above bound holds for all $k, m$ such that $m \leq k \leq N$.
Following from (3.4.24) and using the bound in (3.4.25), we find

$$
\begin{align*}
|H(\lambda, y)| & \leq \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{k=0}^{N} \sum_{m=0}^{k} C_{N, k, m} C_{k, m} \varepsilon^{k-m}  \tag{3.4.26}\\
& =\frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}} \sum_{m=0}^{N} \tilde{C}_{N, m} \varepsilon^{m} \quad[\text { re-index }]  \tag{3.4.27}\\
& \lesssim_{N} \frac{\varepsilon}{|\varepsilon(\lambda-y)|^{N}}\left(\frac{1-\varepsilon^{N+1}}{1-\varepsilon}\right)  \tag{3.4.28}\\
& \lesssim_{\varepsilon, N} \frac{\varepsilon}{1+|\varepsilon(\lambda-y)|^{N}} . \tag{3.4.29}
\end{align*}
$$

This reduces to the same bound that we had for the case $p(x)=-x^{2 n}$, which was covered in Section 3.3. Applying Young's inequality in the same manner gives us

$$
\begin{equation*}
\left\|K_{1} \varphi\right\|=\left\|\int_{-\infty}^{\infty} H(\cdot, y) \hat{\varphi}_{i}(y) d y\right\| \lesssim\|\hat{\varphi}\|=\|\varphi\|, \tag{3.4.30}
\end{equation*}
$$

from which it follows that the map $\varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded for the case $p(x)=-x^{2 n}+$ $x^{2 n+1} \psi(x)$, which is the form of a general analytic concave function up to a constant multiple. This completes the proof of Theorem 1.

### 3.5 Chapter Summary

In this chapter we defined the $\mathcal{F}_{p}$-transform by:

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi \rightarrow \tilde{\varphi}(\lambda)=\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)\left(1+i p^{\prime}(x)\right) d x, \tag{3.5.1}
\end{equation*}
$$

where $\varphi \in L^{2}([-\sigma, \sigma])$, and $p(x)$ is a concave function that is analytic on the interval $[-\sigma, \sigma]$, and satisfies $p(0)=p^{\prime}(0)=0$.

Following this, we proved that the $\mathcal{F}_{p}$-transform is bounded from $L^{2}([-\sigma, \sigma])$ to $L^{2}([0, \infty])$. This is summarized in Theorem 1 which we restate below:

Theorem. The following bound holds for all $\varphi \in L^{2}(\Sigma)$ :

$$
\begin{equation*}
\|\tilde{\varphi}(\lambda)\|_{L^{2}(\Lambda)} \lesssim\|\varphi\|_{L^{2}(\Sigma)} \tag{3.5.2}
\end{equation*}
$$

and therefore the operator $\mathcal{F}_{p}: \varphi \mapsto \tilde{\varphi}(\lambda)$ is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$.

## CHAPTER 4

## Inverse of the $\mathcal{F}_{p}$-transform

Recall, we define the $\mathcal{F}_{p}$-transform by

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi(\lambda) \rightarrow \tilde{\varphi}(\lambda)=\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)\left[1+i p^{\prime}(x)\right] d x, \tag{4.0.1}
\end{equation*}
$$

for $\varphi \in L^{2}(\Sigma)$, where $\Sigma=[-\sigma, \sigma]$. In this chapter we will derive an inverse to the $\mathcal{F}_{p^{\prime}}$-transform. This will be done by performing a spectral analysis of the following differential equation, which is an eigenvalue problem associated with the $\mathcal{F}_{p}$-transform:

$$
\begin{equation*}
\frac{\partial \mu}{\partial x}-i \lambda\left(1+i p^{\prime}(x)\right) \mu=\varphi, \tag{4.0.2}
\end{equation*}
$$

with $\varphi \rightarrow 0$ as $|x| \rightarrow \infty$, and $-\infty<x<\infty$. In performing the spectral analysis, we construct a function that is analytic in specified regions of the complex plane. This function is used to find the inverse of the $\mathcal{F}_{p^{\prime}}$-transform. This technique for constructing an inverse transform follows a method developed by Fokas and Gelfand in [15].

The derivation of the $\mathcal{F}_{P}$-transform is done in several stages, which are outlined as follows. In Section 4.2 we derive three solutions to the differential equation (4.0.2), each of which is analytic in a separate region of the complex plane. Sections 4.2.1-4.2.2 are devoted to finding the regions for which these solutions are analytic. In Section
4.3.1 we pose a generalized Riemann-Hilbert problem that corresponds to the jump of the solutions across the respective domains in which the solutions are defined, and in Section 4.3.2 we appeal to the Cauchy-Pompeiu formula to reconstruct a solution that is sectionally analytic in the complex plane. For an example where the same method is used to derive the inverse Fourier transform and the inverse Radon transform, see Chapter 7 of [1].

Before beginning the derivation of the inverse $\mathcal{F}_{p}$-transform, we will give a brief overview of Riemann-Hilbert problems and their use in use their use in deriving inverses to transforms. The will serve as a guide for the remainder of the chapter.

### 4.1 Background: Riemann-Hilbert Problems

In this section we will introduce the theory for Riemann-Hilbert problems that will be used in this chapter. Following this, we will give an example of how a Riemann-Hilbert problem can be used to derive an inverse to the Fourier transform - this will serve as motivation for the derivation of the $\mathcal{F}_{p}$-transform. The material in this section can be found in standard references including [1] and [36].

Recall, in Section 2.1, we introduced an example of a Riemann-Hilbert problem that was used to construct a representation formula for the Laplace equation on a polygonal domain in the complex plane. More generally, a Riemann-Hilbert problem consists of finding a piecewise analytic function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow \mathbb{C}$ that is discontinuous across a contour $\Gamma$ in the complex plane with a jump condition given by:

$$
\begin{equation*}
\Phi^{+}(z)=\Phi^{-}(z) G(z)+F(z) \quad \text { for } z \in \Gamma, \tag{4.1.1}
\end{equation*}
$$

where $\Phi^{+}$is the limit of $\Phi$ from the right of the contour $\Gamma$, and $\Phi^{-}$is the limit from the left. The functions $G: \Gamma \rightarrow \mathbb{C}$ and $F: \Gamma \rightarrow \mathbb{C}$ are referred to as jump functions.

### 4.1.1 Precise Statement of Riemann-Hilbert Problem

We will now give further definitions and explanations to make the definition of a Riemann-Hilbert problem more precise. We begin by defining the class of contours
that will be considered for the Riemann-Hilbert problems that we will deal with. The definition given below can be found in [36], where it is given as Definition 2.1.

Definition 2 (Complete Contour). A curve $\Gamma$ is said to be a complete contour if $\Gamma$ can be oriented so that $\mathbb{C} \backslash \Gamma$ can be decomposed into left and right components. That is, $\mathbb{C} \backslash \Gamma$ can be written in the form $\mathbb{C} \backslash \Gamma=\Omega^{+} \cup \Omega^{-}$, where $\Omega^{+} \cap \Omega^{-}=\emptyset$.

In the definition below, we specify the class of functions that will be used in the Riemann-Hilbert problems that we consider:

Definition 3 (Sectionally Analytic). Let $\Gamma$ be a complete contour that divides the complex plane into regions $\Omega^{+}$and $\Omega^{-}$which lie to the left and right of the contour $\Gamma$, respectively. A function $\Phi$ defined in the complex plane, except possibly along a complete contour $\Gamma$, is said to be sectionally analytic if the following two properties hold:

- The function $\Phi(z)$ is analytic in each of the regions $\Omega^{+}$and $\Omega^{-}$, except possibly at $z=\infty$.
- The following limits exist:

$$
\begin{align*}
& \Phi^{+}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{+}}} \Phi(z), \quad t \in \Gamma  \tag{4.1.2}\\
& \Phi^{-}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{-}}} \Phi(z), \quad t \in \Gamma, \tag{4.1.3}
\end{align*}
$$

where we assume the limit is taken along a path that is contained entirely in $\Omega^{+}$ or $\Omega^{-}$, respectively, and is not tangent to $\Gamma$ in each case.

By Definition 3 above, a sectionally analytic function is continuous in the closed region $\Omega^{+} \cup \Gamma$ if the value $\Phi^{+}(t)$ is assigned to $\Gamma$ for all $t \in \Gamma$. Similarly, a sectionally analytic function is continuous in the closed region $\Omega^{-} \cup \Gamma$ if the value $\Phi^{-}(t)$ is assigned to $\Gamma$ for all $t \in \Gamma$.

Using the definitions above, we will now give a precise statement of a prototypical Riemann-Hilbert problem of the type that will be considered in this chapter.

Problem Statement 1 (Riemann-Hilbert Problem). Given a complete contour $\Gamma$ in the complex plane, and jump functions $G: \Gamma \rightarrow \mathbb{C}$ and $F: \Gamma \rightarrow \mathbb{C}$, find a function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow \mathbb{C}$ such that:

- $\Phi(z)$ is sectionally analytic, i.e. $\Phi(z)$ is analytic for all $z \in \mathbb{C} \backslash \Gamma$ and the limits $\Phi^{+}(t)$ and $\Phi^{-}(t)$ defined by (4.1.2)-(4.1.3) exist for all $t \in \Gamma$.
- $\Phi(z)$ is bounded at $z=\infty$, and in particular, the following bound holds:

$$
\begin{equation*}
\limsup _{|z| \rightarrow \infty}|z|^{n}|\Phi(z)|<\infty, \tag{4.1.4}
\end{equation*}
$$

for some finite integer $n$.

- $\Phi(z)$ satisfies the following jump condition:

$$
\begin{equation*}
\Phi^{+}(t)=\Phi^{-}(t) G(t)+F(t) \quad \text { for } t \in \Gamma \tag{4.1.5}
\end{equation*}
$$

The integer $n$ from (4.1.4) above is referred to as the degree of $\Phi(z)$ at infinity. Equivalently, we may say that the function $\Phi(z)$ has finite degree $n$ at infinity if

$$
\begin{equation*}
\Phi(z) \sim C_{n} \frac{1}{z^{n}}+O\left(\frac{1}{z^{n-1}}\right), \quad \text { as } z \rightarrow \infty, \tag{4.1.6}
\end{equation*}
$$

where $C_{n}$ is a non-zero constant. The requirement that $\Phi(z)$ have a finite degree at infinity can be thought of as a boundary condition at infinity for the Riemann-Hilbert problem.

We will now introduce an object that will play a fundamental role in the solution
of Riemann-Hilbert problems. Consider the integral:

$$
\begin{equation*}
\mathcal{C}_{\Gamma} \varphi(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau-z} d \tau \tag{4.1.7}
\end{equation*}
$$

where $\Gamma$ is a complete contour, and the function $\varphi: \Gamma \rightarrow \mathbb{C}$ satisfies a Holder condition. That is, for all $z, \tau \in \Gamma$, there exist constants $C$ and $\alpha \in(0,1]$ such that

$$
\begin{equation*}
|f(z)-f(\tau)| \leq C\|z-\tau\|^{\alpha} \tag{4.1.8}
\end{equation*}
$$

We refer to the integral in (4.1.7) as a Cauchy-type integral. It is well-established that integrals of the form (4.1.7) are analytic away from the contour $\Gamma$. Moreover, Cauchy-type integrals are useful in that they map (Holder-continuous) functions on a contour $\Gamma$ to analytic functions in $\mathbb{C} \backslash \Gamma$.

The lemma that follows will be used to show that Cauchy-type integrals provide the solutions to a large class of Riemann-Hilbert problems. This result can be found in [1], where it is given as Lemma 7.2.1.

Lemma 3 (Plemelj Formulae). Let $\Gamma$ be a smooth contour, and let $\varphi(\tau)$ satisfy a Holder condition on $\Gamma$. Then the Cauchy-type integral $\mathcal{C}_{\Gamma} \varphi(z)$ defined in (4.1.7) has the limiting values $\mathcal{C}_{\Gamma}^{+} \varphi(t)$ and $\mathcal{C}_{\Gamma}^{-} \varphi(t)$ as $z$ approaches $t \in \Gamma$ from the left and right, respectively. These limiting values are defined by the following limits:

$$
\begin{align*}
& \mathcal{C}_{\Gamma}^{+} \varphi(t)=\frac{1}{2} \varphi(t)+\frac{1}{2 \pi i} \text { P.V. } \int_{\Gamma} \frac{\varphi(\tau)}{\tau-t} d \tau,  \tag{4.1.9}\\
& \mathcal{C}_{\Gamma}^{-} \varphi(t)=-\frac{1}{2} \varphi(t)+\frac{1}{2 \pi i} \text { P.V. } \int_{\Gamma} \frac{\varphi(\tau)}{\tau-t} d \tau . \tag{4.1.10}
\end{align*}
$$

In equations (4.1.9)-(4.1.10) from Lemma 3 above, the notation P.V. $\int$ denotes the principal value of the respective integral, given by:

$$
\begin{equation*}
\text { P.V. } \int_{\Gamma} \frac{\varphi(\tau)}{\tau-t} d \tau=\lim _{\varepsilon \rightarrow 0} \int_{\Gamma \backslash \Gamma_{\varepsilon}} \frac{\varphi(\tau)}{\tau-t} d \tau \tag{4.1.11}
\end{equation*}
$$

where $\Gamma_{\varepsilon}$ is the segment of $\Gamma$ that is centered around $t$ and has radius $\varepsilon$.
Now consider the following problem which is a sub-case of the more general RiemannHilbert problem specified in Problem Statement 1:

Problem Statement 2. Given a complete contour $\Gamma$ in the complex plane, and jump function $F: \Gamma \rightarrow \mathbb{C}$, find a function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow \mathbb{C}$ such that:

- $\Phi(z)$ is sectionally analytic and has finite degree at infinity,
- $\Phi(z)$ satisfies the following jump condition:

$$
\begin{equation*}
\Phi^{+}(t)-\Phi^{-}(t)=F(t) \quad \text { for } t \in \Gamma . \tag{4.1.12}
\end{equation*}
$$

By applying the Plemelj formulae (Lemma 3), we observe that the unique solution to the Riemann-Hilbert problem given in (4.1.12) can be expressed in terms of a Cauchy-type integral. In particular, if we let $\varphi(\tau)=F(\tau)$ and let $\Phi(z)=\mathcal{C}_{\Gamma} F(z)$ in equation (4.1.7), then we have:

$$
\begin{equation*}
\Phi^{+}(t)-\Phi^{-}(t)=\mathcal{C}_{\Gamma}^{+} F(t)-\mathcal{C}_{\Gamma}^{-} F(t)=F(t) . \tag{4.1.13}
\end{equation*}
$$

The function $\mathcal{C}_{\Gamma} F(z)$ is also sectionally analytic since it is analytic away from $\Gamma$. In addition, by performing a series expansion of $\mathcal{C}_{\Gamma} F(z)$, we find that it is of degree 1 at infinity. Therefore, the solution of a Riemann-Hilbert problem in the form (4.1.12) can be expressed in terms of a Cauchy-type integral. This fact will be used in the following section.

### 4.1.2 Example: Inversion of Fourier Transform

We will now show how an elementary Riemann-Hilbert problem of the form (4.1.12) can be used to derive the inverse Fourier transform. Consider the following linear ODE which is an eigenvalue problem associated to the Fourier transform:

$$
\begin{equation*}
\mu_{x}-i \lambda \mu=\varphi(x), \quad \lambda \in \mathbb{C}, \tag{4.1.14}
\end{equation*}
$$

where $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $x \in(-\infty, \infty)$. We will perform a spectral analysis of this linear differential equation to construct the inverse for the Fourier transform. The method in this section was originally developed in [15] and has been used more recently for more sophisticated problems such as characterizing the Dirichlet-Neumann map for moving initial-boundary value problems in [13].

We begin by rearranging (4.1.14) to get

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[e^{-i \lambda x} \mu\right]=e^{-i \lambda x} \varphi \tag{4.1.15}
\end{equation*}
$$

The following functions are particular solutions to (4.1.15):

$$
\begin{align*}
& \mu^{+}(x, \lambda)=\int_{-\infty}^{x} \varphi(y) e^{i \lambda(x-y)} d y  \tag{4.1.16}\\
& \mu^{-}(x, \lambda)=-\int_{-\infty}^{x} \varphi(y) e^{i \lambda(x-y)} d y \tag{4.1.17}
\end{align*}
$$

We observe that the solutions $\mu^{+}$and $\mu^{-}$are analytic in the upper-half complex $\lambda$ plane (i.e., $x-y>0$ and $\operatorname{Im}(\lambda) \geq 0$ ) and the lower-half complex $\lambda$-plane (i.e., $x-y<0$ and $\operatorname{Im}(\lambda) \leq 0$ ), respectively. We may therefore define the following function which is bounded for all $\lambda \in \mathbb{C}$ :

$$
\mu(x, \lambda)=\left\{\begin{array}{lll}
\mu^{+}(x, \lambda) & \text { if } \quad \operatorname{Im}(\lambda) \geq 0  \tag{4.1.18}\\
\mu^{-}(x, \lambda) & \text { if } \quad \operatorname{Im}(\lambda) \leq 0
\end{array}\right.
$$

In fact, the function $\mu(x, \lambda)$ is sectionally analytic with $\Gamma=\mathbb{R}$ being a complete contour that divides the complex $\lambda$-plane into the upper-half plane $\left(\Omega^{+}\right)$and the lower-half plane ( $\Omega^{-}$).

By subtracting (4.1.17) from (4.1.16), we get the following jump condition across $\Gamma=\mathbb{R}:$

$$
\begin{equation*}
\mu^{+}(x, \lambda)-\mu^{-}(x, \lambda)=e^{i \lambda x} \hat{\varphi}(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.1.19}
\end{equation*}
$$

where $\hat{\varphi}$ is the Fourier transform defined by:

$$
\begin{equation*}
\hat{\varphi}(\lambda)=\int_{-\infty}^{\infty} \varphi(x) e^{-i \lambda x} d x, \quad \lambda \in \mathbb{R} . \tag{4.1.20}
\end{equation*}
$$

Applying integration by parts on equations (4.1.16) and (4.1.17), we find:

$$
\begin{equation*}
\mu=O\left(\frac{1}{\lambda}\right), \quad \text { as } \lambda \rightarrow \infty . \tag{4.1.21}
\end{equation*}
$$

Equations 4.1.19 and 4.1.21 define a Riemann-Hilbert problem of the form given in Problem Statement 2. The unique solution to this problem is given by the following Cauchy-type integral:

$$
\begin{equation*}
\mu(x, \lambda)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{e^{i \zeta x} \hat{\varphi}(\zeta)}{\zeta-\lambda} d \zeta, \quad \lambda \in \mathbb{C} . \tag{4.1.22}
\end{equation*}
$$

By comparing the large- $\lambda$ asymptotics of (4.1.14) and (4.1.22), we will derive an expression for $\varphi(x)$ in terms of $\hat{\varphi}(\lambda)$. For large $\lambda$, equation (4.1.14) implies:

$$
\begin{equation*}
\varphi(x)=-i \lim _{\lambda \rightarrow \infty}(\lambda \mu), \tag{4.1.23}
\end{equation*}
$$

while for large $\lambda$, equation (4.1.22) implies:

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty}(\lambda \mu)=-\frac{1}{2 \pi i} \int_{-\infty}^{\infty} e^{i \lambda x} \hat{\varphi}(\lambda) d \lambda . \tag{4.1.24}
\end{equation*}
$$

Combining equations (4.1.23) and (4.1.24), we arrive at the following formula for $\varphi(x)$ :

$$
\begin{equation*}
\varphi(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i \lambda x} \hat{\varphi}(\lambda) d \lambda . \tag{4.1.25}
\end{equation*}
$$

Equation (4.1.25) is the inverse Fourier transform. We will use the procedure from this section as a template for deriving the inverse of the $\mathcal{F}_{p}$-transform in this chapter.

### 4.1.3 $\bar{\partial}$-Problems

We will now introduce a generalization of the Riemann-Hilbert problem that allows for similar computations with functions that satisfy a generalized definition of analyticity. First, we introduce the following generalized definition of analyticity:

Definition 4 (Generalized Analytic Function). Let $\Phi(z, \bar{z})$ be a function defined in the region $\Omega \subset \mathbb{C}$. The function $\Phi(z, \bar{z})$ is said to be a generalized analytic function if:

$$
\begin{equation*}
\frac{\partial \Phi}{\partial \bar{z}}=f(z, \bar{z}), \quad z \in \Omega . \tag{4.1.26}
\end{equation*}
$$

In the case where $\partial \Phi / \partial \bar{z}=0$, the function $\Phi$ is analytic as a consequence of the Cauchy-Riemann equations.

Correspondingly, we also introduce a generalization of the definition of a sectionally analytic function below.

Definition 5 (Generalized Sectionally Analytic). Let $\Gamma$ be a complete contour that divides the complex plane into regions $\Omega^{+}$and $\Omega^{-}$which lie to the left and right of the contour $\Gamma$, respectively. A function $\Phi(z, \bar{z})$ defined in the complex plane, except possibly along a complete contour $\Gamma$, is referred to as a generalized sectionally analytic if the following two properties hold:

- $\Phi(z, \bar{z})$ is a generalized analytic function in each of the regions $\Omega^{+}$and $\Omega^{-}$, except possibly at $z=\infty$.
- The following limits exist:

$$
\begin{align*}
& \Phi^{+}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{+}}} \Phi(z), \quad t \in \Gamma  \tag{4.1.27}\\
& \Phi^{-}(t):=\lim _{\substack{z \rightarrow t \\
z \in \Omega^{-}}} \Phi(z), \quad t \in \Gamma, \tag{4.1.28}
\end{align*}
$$

where we assume the limit is taken along a non-tangential path that is contained entirely in $\Omega^{+}$or $\Omega^{-}$, respectively.

This leads to a generalization of the Riemann-Hilbert problem, which will be referred to as a so-called $\bar{\partial}$-problem. We make the statement of this problem precise
below:

Problem Statement 3 ( $\overline{\boldsymbol{\partial}}$-Problem). Given a complete contour $\Gamma$ in the complex plane, and jump function $F: \Gamma \rightarrow \mathbb{C}$, find a function $\Phi: \mathbb{C} \backslash \Gamma \rightarrow \mathbb{C}$ such that:

- $\Phi(z)$ is a generlized sectionally analytic function,
- $\Phi(z)$ has finite degree at infinity,
- $\Phi(z)$ satisfies the following jump condition:

$$
\begin{equation*}
\Phi^{+}(t)-\Phi^{-}(t)=F(t) \quad \text { for } t \in \Gamma \tag{4.1.29}
\end{equation*}
$$

We will now introduce a formula that plays a similar role to the Cauchy-type integrals in the solution of $\bar{\partial}$-problems for functions that satisfy the generalized sectionally analytic condition.

Lemma 4 (Cauchy-Pompeiu Formula). Let $f$ be a complex-valued $C^{1}$ function defined on the closure of a region $\Omega \subset \mathbb{C}$, then

$$
\begin{equation*}
f(z, \bar{z})=\frac{1}{2 \pi i} \int_{\partial \Omega} \frac{f(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-z} \tag{4.1.30}
\end{equation*}
$$

For reference, Lemma 4 is given as Lemma 7.4. in [1]. It was also shown in [1] that the Cauchy-Pompeiu formula provides the unique solution to $\bar{\partial}$-problems of the form given in Problem Statement 3. We will make use of the Cauchy-Pompeiu formula in this chapter for solving a $\bar{\partial}$-problem associated with finding the inverse of the $\mathcal{F}_{p}$-transform.

### 4.2 Regions for the $\bar{\partial}$-Problem

Using the derivation of the inverse Fourier transform in Section 4.1.2 as a template, we will now perform a similar spectral analysis for the following linear ODE:

$$
\begin{equation*}
\frac{\partial \mu}{\partial x}-i \lambda\left(1+i p^{\prime}(x)\right) \mu=\varphi \tag{4.2.1}
\end{equation*}
$$

with $\varphi \rightarrow 0$ as $|x| \rightarrow \infty$, and $-\infty<x<\infty$. Rearranging (4.2.1) gives us

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[e^{-i \lambda x+\lambda p(x)} \mu\right]=e^{-i \lambda x+\lambda p(x)} \varphi \tag{4.2.2}
\end{equation*}
$$

The following functions are solutions to (4.2.2):

$$
\begin{align*}
\mu_{1}(x, \lambda) & =e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y, & & \lambda \in \Omega_{1}  \tag{4.2.3a}\\
\mu_{2}(x, \lambda) & =-e^{i \lambda x-\lambda p(x)} \int_{x}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y, & & \lambda \in \Omega_{2}  \tag{4.2.3b}\\
\mu_{3}(x, \lambda, \bar{\lambda}) & =e^{i \lambda x-\lambda p(x)} \int_{x_{3}}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y, & & \lambda \in \Omega_{3} \tag{4.2.3c}
\end{align*}
$$

where $x_{3}=x_{3}(\lambda, \bar{\lambda})$ is defined by $\lambda_{I}+p^{\prime}\left(x_{3}\right) \lambda_{R}=0$. We observe that since $\mu_{3}$ depends on $\bar{\lambda}$, it will not be analytic in $\Omega_{3}$. However, it satisfies the generalized definition of analyticity given in Definition 4 from Section 4.1.3. This fact will be used later in the chapter.

Now we wish to find the regions $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ in the complex plane where each of the respective functions $\mu_{j}(x, \lambda)$ is bounded. This will yield the following generalized sectionally analytic function:

$$
\mu(x, \lambda, \bar{\lambda})= \begin{cases}\mu_{1} & \text { if } \lambda \in \Omega_{1}  \tag{4.2.4}\\ \mu_{2} & \text { if } \lambda \in \Omega_{2} \\ \mu_{3} & \text { if } \lambda \in \Omega_{3}\end{cases}
$$

### 4.2.1 The Regions $\Omega_{1}$ and $\Omega_{2}$

By an application of the Cauchy-Schwarz inequality, in order to show $\mu_{1}$ is bounded, it is sufficient to show $\left|e^{i \lambda(x-y)} e^{\lambda(p(y)-p(x))}\right|$ is bounded. This expression can be rewritten as follows

$$
\begin{align*}
\left|e^{i \lambda(x-y)} e^{\lambda(p(y)-p(x))}\right| & =e^{-\lambda_{I}(x-y)} e^{\lambda_{R}(p(y)-p(x))}  \tag{4.2.5}\\
& =e^{-(x-y)\left[\lambda_{I}+\lambda_{R} R(x, y)\right]} \tag{4.2.6}
\end{align*}
$$

where

$$
\begin{equation*}
\lambda=\lambda_{R}+i \lambda_{I} \quad \text { and } \quad R(x, y):=\frac{p(x)-p(y)}{x-y} \tag{4.2.7}
\end{equation*}
$$

Since $x>y$, in order for (4.2.6) to be bounded, the parameter $\lambda$ must satisfy

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} R(x, y) \geq 0 \tag{4.2.8}
\end{equation*}
$$

By concavity, $R(x, y)$ obeys:

$$
\begin{equation*}
p^{\prime}(x) \leq R(x, y) \leq p^{\prime}(-\sigma) \tag{4.2.9}
\end{equation*}
$$

This can be seen in Figure 4.1 below.


Figure 4.1: The concavity of the curve $p(x)$, demonstrating the inequality $p^{\prime}(x) \leq R(x, y) \leq$ $p^{\prime}(-\sigma)$.

If $\lambda_{R}>0$, then the inequality (4.2.9) implies:

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} p^{\prime}(x) \leq \lambda_{I}+\lambda_{R} R(x, y) \leq \lambda_{I}+\lambda_{R} p^{\prime}(-\sigma) \tag{4.2.10}
\end{equation*}
$$

Further, the condition in (4.2.8) holds if

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} p^{\prime}(x) \geq 0 \quad \Rightarrow \quad \lambda_{I} \geq-\lambda_{R} p^{\prime}(x), \quad \text { for } \lambda_{R}>0 \tag{4.2.11}
\end{equation*}
$$

If $\lambda_{R}<0$, then the inequality (4.2.9) implies:

$$
\begin{equation*}
\lambda_{I}-\left|\lambda_{R}\right| p^{\prime}(-\sigma) \leq \lambda_{I}-\left|\lambda_{R}\right| R(x, y) \leq \lambda_{I}-\left|\lambda_{R}\right| p^{\prime}(x) \tag{4.2.12}
\end{equation*}
$$

Therefore, the condition in (4.2.8) holds if

$$
\begin{equation*}
\lambda_{I}-\left|\lambda_{R}\right| p^{\prime}(-\sigma) \geq 0 \quad \Rightarrow \quad \lambda_{I} \geq\left|\lambda_{R}\right| p^{\prime}(-\sigma), \quad \text { for } \lambda_{R}<0 \tag{4.2.13}
\end{equation*}
$$

The ray for which $\lambda_{I}=-\lambda_{R} p^{\prime}(x)$ can be expressed as a complex number of the form

$$
\begin{align*}
\lambda_{R}+i \lambda_{I} & =\lambda_{R}-i \lambda_{R} p^{\prime}(x)  \tag{4.2.14}\\
& =\lambda_{R}\left[1-i p^{\prime}(x)\right] \tag{4.2.15}
\end{align*}
$$

If $p^{\prime}(x)$ is positive, then this ray is in the fourth quadrant of the complex plane since $\lambda_{R}$ is positive. The ray for which $\lambda_{I}=-\lambda_{R} p^{\prime}(a)$ can be expressed as a complex number of the form

$$
\begin{align*}
\lambda_{R}+i \lambda_{I} & =\lambda_{R}-i \lambda_{R} p^{\prime}(-\sigma)  \tag{4.2.16}\\
& =\lambda_{R}\left[1-i p^{\prime}(-\sigma)\right] \tag{4.2.17}
\end{align*}
$$

Since $\lambda_{R}$ is negative, this ray is in the second quadrant of the complex plane since $p^{\prime}(a)$ is positive.

Putting these facts together, the region $\Omega_{1}$ is defined by:

$$
\Omega_{1}(x)=\left\{\begin{align*}
\lambda_{I} \geq-\lambda_{R} p^{\prime}(x) & \text { if } \lambda_{R}>0  \tag{4.2.18}\\
\lambda_{I} \geq\left|\lambda_{R}\right| p^{\prime}(-\sigma) & \text { if } \lambda_{R}<0
\end{align*}\right.
$$

Repeating a similar analysis for the function $\mu_{2}$, we find that the domain $\Omega_{2}$ is defined by:

$$
\Omega_{2}(x)= \begin{cases}\lambda_{I} \leq-\lambda_{R} p^{\prime}(x) & \text { if } \lambda_{R}>0  \tag{4.2.19}\\ \lambda_{I} \leq\left|\lambda_{R}\right| p^{\prime}(\sigma) & \text { if } \lambda_{R}<0\end{cases}
$$

### 4.2.2 The Region $\Omega_{3}$

Now recall $x_{3}=x_{3}\left(\lambda_{I}, \lambda_{R}\right)=x_{3}(\lambda, \bar{\lambda})$ is defined by

$$
\begin{equation*}
\lambda_{I}+p^{\prime}\left(x_{3}\right) \lambda_{R}=0 \quad \Rightarrow \quad p^{\prime}\left(x_{3}\right)=-\frac{\lambda_{I}}{\lambda_{R}} . \tag{4.2.20}
\end{equation*}
$$

As in the previous cases, in order to show $\mu_{3}(x, \lambda, \bar{\lambda})$ is bounded, it is sufficient to show that the following is bounded:

$$
\begin{equation*}
\left|e^{i \lambda(x-y)} e^{\lambda(p(y)-p(x))}\right|=e^{-(x-y)\left[\lambda_{I}+\lambda_{R} R(x, y)\right]} . \tag{4.2.21}
\end{equation*}
$$

We will consider the cases $x_{3}<x$ and $x_{3}>x$ separately.
If $x_{3}<x$, then $x>y$ and in order for (4.2.21) to be bounded, the parameter $\lambda$ must satisfy

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} R(x, y) \geq 0 . \tag{4.2.22}
\end{equation*}
$$

By convexity, $R(x, y)$ obeys:

$$
\begin{equation*}
p^{\prime}(\sigma) \leq p^{\prime}(x) \leq p^{\prime}\left(x_{3}\right) \leq R(x, y) \leq p^{\prime}(-\sigma) . \tag{4.2.23}
\end{equation*}
$$

Hence, for $\lambda_{R}<0$ we have

$$
\begin{align*}
\lambda_{I}+\lambda_{R} p^{\prime}(\sigma) \geq \lambda_{I}+\lambda_{R} p^{\prime}(x) & \geq \lambda_{I}+\lambda_{R} p^{\prime}\left(x_{3}\right) \\
& \geq \lambda_{I}+\lambda_{R} R(x, y) \geq \lambda_{I}+\lambda_{R} p^{\prime}(-\sigma) . \tag{4.2.24}
\end{align*}
$$

Since $\lambda_{I}+\lambda_{R} p^{\prime}\left(x_{3}\right)=0$, by (4.2.23), we have

$$
\begin{equation*}
0=\lambda_{I}+\lambda_{R} p^{\prime}\left(x_{3}\right) \geq \lambda_{I}+\lambda_{R} R(x, y) \tag{4.2.25}
\end{equation*}
$$

Further, by the inequalities in (4.2.24), we have

$$
\begin{equation*}
0 \geq \lambda_{I}+\lambda_{R} p^{\prime}(-\sigma) \quad \Rightarrow \quad \lambda_{I} \leq\left|\lambda_{R}\right| p^{\prime}(-\sigma) \tag{4.2.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} p^{\prime}(x) \geq 0 \quad \Rightarrow \quad \lambda_{I} \geq\left|\lambda_{R}\right| p^{\prime}(x) \tag{4.2.27}
\end{equation*}
$$

In the case $x_{3}>x$, then $x<y$ and a similar calculation shows that (4.2.22) becomes

$$
\begin{equation*}
\lambda_{I}+\lambda_{R} R(x, y) \leq 0 \tag{4.2.28}
\end{equation*}
$$

and we get the following inequalities:

$$
\begin{array}{ll}
\lambda_{I} \leq\left|\lambda_{R}\right| p^{\prime}(x), & \lambda_{R}<0 \\
\lambda_{I} \geq\left|\lambda_{R}\right| p^{\prime}(\sigma), & \lambda_{R}<0 . \tag{4.2.30}
\end{array}
$$

Putting the cases $x_{3}<x$ and $x_{3}>x$ together we find that the region $\Omega_{3}$ is defined by:

$$
\Omega_{3}(x)=\left\{\begin{align*}
\lambda_{I} \leq\left|\lambda_{R}\right| p^{\prime}(-\sigma) & \text { for } \lambda_{R}<0  \tag{4.2.31}\\
\lambda_{I} \geq\left|\lambda_{R}\right| p^{\prime}(\sigma) & \text { for } \lambda_{R}<0
\end{align*}\right.
$$

Figure 4.2 below shows the regions $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$ in the complex plane.

### 4.3 Inverting the $\mathcal{F}_{p}$-transform

In the previous section we found the regions of the complex plane in which the functions (4.2.3a) - (4.2.3c) are bounded. This allowed us to define the function $\mu(x, \lambda, \bar{\lambda})$ which


Figure 4.2: The regions $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$ are shown in the complex plane.
is analytic in each of the regions $\Omega_{1}$ and $\Omega_{2}$, and generalized analytic in the region $\Omega_{3}$. In this section, we will employ the Cauchy-Pompeiu formula to reconstruct the generalized sectionally analytic function $\mu(x, \lambda, \bar{\lambda})$ from its $\bar{\partial}$-derivatives and jumps across the boundaries of the regions $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$. This will allow us to construct the inverse to the $\mathcal{F}_{p}$-transform.

Recall, from Lemma 4 that the Cauchy-Pompeiu formula states:

$$
\begin{equation*}
f(z, \bar{z})=\frac{1}{2 \pi i} \int_{\partial \Omega} \frac{f(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \iint_{\Omega} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-z}, \tag{4.3.1}
\end{equation*}
$$

where $f$ is a complex-valued $C^{1}$ function defined on the closure of a region $\Omega \subset \mathbb{C}$. In the case where $f$ is a generalized sectionally analytic function in the regions $\Omega_{1}, \ldots, \Omega_{n}$ of the complex plane, the Cauchy-Pompeiu formula becomes

$$
\begin{equation*}
f(z, \bar{z})=\frac{1}{2 \pi i} \sum_{J} \int_{\partial \Omega} \frac{f^{+}(\zeta)-f^{-}(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \sum_{j=1}^{n} \iint_{\Omega_{j}} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-z} \tag{4.3.2}
\end{equation*}
$$

where $J$ is the set of jumps across overlapping regions and $f^{+}$and $f^{-}$are the left and right values of $f$ across the jump, respectively. Therefore, in order to apply the

Cauchy-Pompeiu formula to the function $\mu(x, \lambda, \bar{\lambda})$, we must first compute the jumps across the boundary of each of the regions $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$.

### 4.3.1 The Riemann-Hilbert Problem

We will now compute the jumps across the boundary of each of the regions $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$. This will naturally lead to a $\bar{\partial}$-problem corresponding to each of the jumps between the functions $\mu_{1}, \mu_{2}$, and $\mu_{3}$ over the boundaries of the respective regions. Using the Cauchy-Pompeiu formula, we will be able to solve the $\bar{\partial}$-problem which will allow us to reconstruct the function $\mu(x, \lambda, \bar{\lambda})$.

We begin by computing the jump across the ray $\lambda=\lambda_{R}\left[1-i p^{\prime}(-\sigma)\right]$ :

$$
\begin{align*}
\mu_{1}-\mu_{3} & =e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y-e^{i \lambda x-\lambda p(x)} \int_{x_{3}}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y  \tag{4.3.3}\\
& =e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y+\int_{x}^{x_{3}} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]  \tag{4.3.4}\\
& =e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{x_{3}} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right] . \tag{4.3.5}
\end{align*}
$$

Recall that $x_{3}$ is defined such that $p^{\prime}\left(x_{3}\right)=-\frac{\lambda_{I}}{\lambda_{R}}$, and since $\lambda=\lambda_{R}\left[1-i p^{\prime}(-\sigma)\right]$, we have $p^{\prime}\left(x_{3}\right)=p^{\prime}(-\sigma)$. Moreover, the function $p^{\prime}(x)$ is strictly increasing, and hence it follows that $x_{3}=a$ on the ray $\lambda=\lambda_{R}\left[1-i p^{\prime}(-\sigma)\right]$. Therefore, the final integral (4.3.5) becomes

$$
\begin{equation*}
e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{x_{3}} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]=e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{-\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]=0 . \tag{4.3.6}
\end{equation*}
$$

The jump across the ray $\lambda=\lambda_{R}\left[1-i p^{\prime}(\sigma)\right]$ is given by:

$$
\begin{align*}
\mu_{3}-\mu_{2} & =e^{i \lambda x-\lambda p(x)} \int_{x_{3}}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y+e^{i \lambda x-\lambda p(x)} \int_{x}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y  \tag{4.3.7}\\
& =e^{i \lambda x-\lambda p(x)}\left[\int_{x_{3}}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y+\int_{x}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]  \tag{4.3.8}\\
& =e^{i \lambda x-\lambda p(x)}\left[\int_{x_{3}}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right] . \tag{4.3.9}
\end{align*}
$$

By a similar argument to the one above, $x_{3}=\sigma$ on the ray $\lambda=\lambda_{R}\left[1-i p^{\prime}(\sigma)\right]$, and
therefore

$$
\begin{equation*}
e^{i \lambda x-\lambda p(x)}\left[\int_{x_{3}}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]=e^{i \lambda x-\lambda p(x)}\left[\int_{\sigma}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]=0 . \tag{4.3.10}
\end{equation*}
$$

Finally, the jump across the ray $\lambda=\lambda_{R}\left[1-i p^{\prime}(x)\right]$ is given by:

$$
\begin{align*}
\mu_{2}-\mu_{1} & =-\left[\mu_{1}-\mu_{2}\right]  \tag{4.3.11}\\
& =-e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y+\int_{x}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]  \tag{4.3.12}\\
& =-e^{i \lambda x-\lambda p(x)}\left[\int_{-\sigma}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y\right]  \tag{4.3.13}\\
& =-e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(y) \tag{4.3.14}
\end{align*}
$$

Now returning to the original differential equation (4.0.2), consider $\mu_{1}(x, \lambda)$ which can be written as

$$
\begin{align*}
& \mu_{1}(x, \lambda)=e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y,  \tag{4.3.15}\\
& =e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} \frac{\varphi(y)}{-\lambda\left(i-p^{\prime}(y)\right)} \frac{d}{d y}\left[e^{-i \lambda y+\lambda p(y)}\right] d y  \tag{4.3.16}\\
& =e^{i \lambda x-\lambda p(x)}\left\{\left[\frac{\varphi(y)}{-\lambda\left(i-p^{\prime}(y)\right)} e^{-i \lambda y+\lambda p(y)}\right]_{-\sigma}^{x}-\int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \frac{d}{d y}\left[\frac{\varphi(y)}{-\lambda\left(i-p^{\prime}(y)\right)}\right] d y\right\}
\end{align*}
$$

$$
\begin{equation*}
=\frac{\varphi(x)}{-\lambda\left(i-p^{\prime}(x)\right)}-\frac{\varphi(-\sigma)}{-\lambda\left(i-p^{\prime}(-\sigma)\right)} e^{i \lambda(x-(-\sigma))-\lambda(p(x)-p(-\sigma))} \tag{4.3.17}
\end{equation*}
$$

$$
\begin{equation*}
-e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \frac{d}{d y}\left[\frac{\varphi(y)}{-\lambda\left(i-p^{\prime}(y)\right)}\right] d y \tag{4.3.18}
\end{equation*}
$$

$$
\begin{equation*}
=\frac{\varphi(x)}{-\lambda\left(i-p^{\prime}(x)\right)}+O\left(\frac{1}{\lambda^{2}}\right) . \tag{4.3.19}
\end{equation*}
$$

As a note, in equation (4.3.18), the term

$$
\begin{equation*}
\frac{\varphi(-\sigma)}{-\lambda\left(i-p^{\prime}(-\sigma)\right)} e^{i \lambda(x-(-\sigma))-\lambda(p(x)-p(-\sigma))} \tag{4.3.20}
\end{equation*}
$$

decays exponentially since $(p(x)-p(-\sigma))>0$ by concavity, and therefore $-\lambda(p(x)-$
$p(-\sigma))<0$. Hence,

$$
\begin{equation*}
e^{i \lambda(x-(-\sigma))-\lambda(p(x)-p(-\sigma))} \rightarrow 0 \quad \text { as } \lambda \rightarrow \infty \tag{4.3.21}
\end{equation*}
$$

Furthermore, the final term in equation (4.3.18) is $O\left(1 / \lambda^{2}\right)$. This justifies (4.3.19).
A similar calculation can be performed for $\mu_{2}$ and $\mu_{3}$, and taken together with (4.3.19), we find that

$$
\begin{equation*}
\mu=O\left(\frac{1}{\lambda}\right), \quad \text { as } \lambda \rightarrow \infty . \tag{4.3.22}
\end{equation*}
$$

Equations (4.3.5), (4.3.9), (4.3.14), and (4.3.22) define a $\bar{\partial}$-problem for the generalized sectionally analytic function $\mu(x, \lambda, \bar{\lambda})$. Furthermore, from (4.3.19), we deduce that

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} \lambda \mu_{1}(x, \lambda)=-\frac{\varphi(x)}{i-p^{\prime}(x)} \tag{4.3.23}
\end{equation*}
$$

and similarly for $\mu_{2}(x, \lambda)$ and $\mu_{3}(x, \lambda)$.

### 4.3.2 Solving the $\bar{\partial}$-Problem

Now we wish to solve the $\bar{\partial}$-problem posed in the previous section. Using the CauchyPompeiu formula (i.e., (4.3.2)), the unique solution of the $\bar{\partial}$-problem defined by (4.3.5), (4.3.9), (4.3.14), and (4.3.22) is given by

$$
\begin{align*}
& \mu(\lambda, \bar{\lambda})=\frac{1}{2 \pi i} \int_{\gamma_{1}} \frac{\mu_{1}-\mu_{3}}{\zeta-\lambda} d \zeta+\frac{1}{2 \pi i} \int_{\gamma_{2}} \frac{\mu_{3}-\mu_{2}}{\zeta-\lambda} d \zeta+\frac{1}{2 \pi i} \int_{\gamma_{3}} \frac{\mu_{2}-\mu_{1}}{\zeta-\lambda} d \zeta \\
& +\frac{1}{2 \pi i} \iint_{\Omega_{1}} \frac{\partial \mu_{1}}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-\lambda}+\frac{1}{2 \pi i} \iint_{\Omega_{2}} \frac{\partial \mu_{2}}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-\lambda}+\frac{1}{2 \pi i} \iint_{\Omega_{3}} \frac{\partial \mu_{3}}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-\lambda}, \tag{4.3.24}
\end{align*}
$$

where $\gamma_{1}=\left\{\zeta \in \mathbb{C}: \zeta=\zeta_{R}\left[1-i p^{\prime}(-\sigma)\right], \zeta_{R}<0\right\}, \gamma_{2}=\left\{\zeta \in \mathbb{C}: \zeta=\zeta_{R}[1-\right.$ $\left.\left.i p^{\prime}(\sigma)\right], \zeta_{R}<0\right\}$, and $\gamma_{3}=\left\{\zeta \in \mathbb{C}: \zeta=\zeta_{R}\left[1-i p^{\prime}(x)\right], \zeta_{R}>0\right\}$, and the expressions for $\mu_{1}-\mu_{3}, \mu_{3}-\mu_{2}$, and $\mu_{2}-\mu_{1}$ are given by (4.3.5), (4.3.9), and (4.3.14), respectively. The respective curves are labelled in Figure 4.3 on the following page.


Figure 4.3: The rays $\gamma_{1}, \gamma_{2}$, and $\gamma_{3}$ are shown in the complex plane.

Using the fact that $\mu_{1}-\mu_{3}=0, \mu_{3}-\mu_{2}=0$, as well as the fact that $\partial \mu_{1} / \partial \bar{\zeta}=$ $\partial \mu_{2} / \partial \bar{\zeta}=0$, equation (4.3.24) can be simplified to

$$
\begin{equation*}
\mu(\lambda, \bar{\lambda})=\frac{1}{2 \pi i} \int_{\gamma_{3}} \frac{\mu_{2}-\mu_{1}}{\zeta-\lambda} d \zeta+\frac{1}{2 \pi i} \iint_{\Omega_{3}} \frac{\partial \mu_{3}}{\partial \bar{\zeta}}(\zeta) \frac{d \zeta \wedge d \bar{\zeta}}{\zeta-\lambda} \tag{4.3.25}
\end{equation*}
$$

Now we wish to compare the large- $\lambda$ asymptotics of the original differential equation (4.0.2) with the large- $\lambda$ asymptotics of the expression for $\mu(x, \lambda, \bar{\lambda})$ in equation (4.3.25). For large $\lambda,(4.3 .23)$ and the corresponding expressions for $\mu_{2}$ and $\mu_{3}$ imply:

$$
\begin{align*}
\frac{-\varphi}{\left(i-p^{\prime}(x)\right)} & =\lim _{\lambda \rightarrow \infty}(\lambda \mu)  \tag{4.3.26}\\
& =-\frac{1}{2 \pi i} \int_{\gamma_{3}} \mu_{2}-\mu_{1} d \lambda-\frac{1}{2 \pi i} \iint_{\Omega_{3}} \frac{\partial \mu_{3}}{\partial \bar{\zeta}}(\zeta) d \zeta \wedge d \bar{\zeta}  \tag{4.3.27}\\
& =\frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda-\frac{1}{2 \pi i} \iint_{\Omega_{3}} \frac{\partial \mu_{3}}{\partial \bar{\zeta}}(\zeta) d \zeta \wedge d \bar{\zeta} \tag{4.3.28}
\end{align*}
$$

The final equality above follows from (4.3.14).
Now we will focus our attention on the final term in equation (4.3.28). Using the anti-commutative property of differential forms and the fact that the exterior derivative
of a differential form is given by $d \omega=\sum_{i=1}^{n}\left(d f / d x^{i}\right) d x^{i} \wedge d x^{j}$, equation (4.3.28) can be written as

$$
\begin{equation*}
-\frac{1}{2 \pi i} \iint_{\Omega_{3}} \frac{\partial \mu_{3}}{\partial \bar{\zeta}}(\zeta) d \zeta \wedge d \bar{\zeta}=\frac{1}{2 \pi i} \iint_{\Omega_{3}} d\left(\mu_{3} d \zeta\right) \tag{4.3.29}
\end{equation*}
$$

Applying Stoke's theorem, we have

$$
\begin{align*}
& \frac{1}{2 \pi i} \iint_{\Omega_{3}} d\left(\mu_{3} d \zeta\right)=\frac{1}{2 \pi i} \int_{\partial \Omega_{3}} \mu_{3} d \zeta \\
& \quad=\frac{1}{2 \pi i} \int_{\gamma_{1}} \mu_{3} d \lambda-\frac{1}{2 \pi i} \int_{\gamma_{2}} \mu_{3} d \lambda+\frac{1}{2 \pi i} \lim _{R \rightarrow \infty} \int_{C_{R}} \mu_{3} d \lambda \tag{4.3.30}
\end{align*}
$$

where $C_{R}$ is the arc between $\gamma_{1}$ and $\gamma_{2}$, as $R \rightarrow \infty$. The arc $C_{R}$ is shown in Figure 4.4 below.


Figure 4.4: The $\operatorname{arc} C_{R}$ shown in the complex plane.

On the contours $\gamma_{1}$ and $\gamma_{2}$, we have $\mu_{3}=\mu_{1}$ and $\mu_{3}=\mu_{2}$, respectively, and therefore the above expression may be written as:

$$
\begin{align*}
& \frac{1}{2 \pi i} \iint_{\Omega_{3}} d\left(\mu_{3} d \zeta\right)=\frac{1}{2 \pi i} \int_{\partial \Omega_{3}} \mu_{3} d \zeta \\
& \quad=\frac{1}{2 \pi i} \int_{\gamma_{1}} \mu_{1} d \lambda-\frac{1}{2 \pi i} \int_{\gamma_{2}} \mu_{2} d \lambda+\frac{1}{2 \pi i} \lim _{R \rightarrow \infty} \int_{C_{R}} \mu_{3} d \lambda \tag{4.3.31}
\end{align*}
$$

Collecting terms and returning to equation (4.3.28), we get the following expression:

$$
\begin{align*}
& \frac{-\varphi}{\left(i-p^{\prime}(x)\right)}=\frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x+\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda+\frac{1}{2 \pi i} \int_{\gamma_{1}} \mu_{1} d \lambda \\
&-\frac{1}{2 \pi i} \int_{\gamma_{2}} \mu_{2} d \lambda+\frac{1}{2 \pi i} \lim _{R \rightarrow \infty} \int_{C_{R}} \mu_{3} d \lambda . \tag{4.3.32}
\end{align*}
$$

Now we will focus our attention on dealing with the term $\mu_{3, C_{R}}:=\int_{C_{R}} \mu_{3} d \lambda$. We can parametrize $C_{R}$ by $C_{R}=\left\{\zeta \in \mathbb{C}: \zeta=R e^{i \theta}, \tan ^{-1}\left(-p^{\prime}(-\sigma)\right) \leq \theta \leq\right.$ $\left.\tan ^{-1}\left(-p^{\prime}(\sigma)\right)\right\}$ where $0<R \leq \infty$. Making this substitution, the term $\mu_{3, C_{R}}$ becomes

$$
\begin{equation*}
\int_{C_{R}} \mu_{3} d \zeta=\int_{\theta_{-\sigma}}^{\theta_{\sigma}} \mu_{3}\left(x, R e^{i \theta}, R e^{-i \theta}\right) i R e^{i \theta} d \theta \tag{4.3.33}
\end{equation*}
$$

where $\theta_{-\sigma}=\tan ^{-1}\left(-p^{\prime}(-\sigma)\right)$ and $\theta_{\sigma}=\tan ^{-1}\left(-p^{\prime}(\sigma)\right)$. In equation (4.3.33), the term $\mu_{3}$ can be written as

$$
\begin{equation*}
\mu_{3}\left(x, R e^{i \theta}, R e^{-i \theta}\right)=\frac{1}{R e^{i \theta}}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)+O\left(\frac{1}{R^{2}}\right) . \tag{4.3.34}
\end{equation*}
$$

Now we will evaluate the $\operatorname{limit} \lim _{R \rightarrow \infty} \mu_{3, C_{R}}$ :

$$
\begin{align*}
\lim _{R \rightarrow \infty} \mu_{3, C_{R}} & =\lim _{R \rightarrow \infty} \int_{\theta_{-\sigma}}^{\theta_{\sigma}}\left[\frac{1}{R e^{i \theta}}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)+O\left(\frac{1}{R^{2}}\right)\right] i R e^{i \theta} d \theta  \tag{4.3.35}\\
& =\int_{\theta_{-\sigma}}^{\theta_{\sigma}} \lim _{R \rightarrow \infty}\left[i\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)+O\left(\frac{1}{R}\right)\right] d \theta  \tag{4.3.36}\\
& =\int_{\theta_{-\sigma}}^{\theta_{\sigma}} i\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) d \theta  \tag{4.3.37}\\
& =i\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) \Delta \theta \tag{4.3.38}
\end{align*}
$$

where $\Delta \theta:=\theta_{\sigma}-\theta_{-\sigma}$. Using the above expression, equation (4.3.32) becomes

$$
\begin{align*}
& \frac{-\varphi}{\left(i-p^{\prime}(x)\right)}=\frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda+\frac{1}{2 \pi i} \int_{\gamma_{1}} \mu_{1} d \lambda \\
&-\frac{1}{2 \pi i} \int_{\gamma_{2}} \mu_{2} d \lambda+\frac{1}{2 \pi}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) \Delta \theta . \tag{4.3.39}
\end{align*}
$$

Now we will rotate the integrals over $\gamma_{1}$ and $\gamma_{2}$ so that they are over the positive and negative imaginary axes, respectively, instead. Using Cauchy's integral theorem, we have:

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{\gamma_{1}} \mu_{1} d \lambda=-\frac{1}{2 \pi i} \int_{0}^{+i \infty} \mu_{1} d \lambda+\frac{1}{2 \pi i} \int_{C_{R}^{1}} \mu_{1} d \lambda  \tag{4.3.40a}\\
& \frac{1}{2 \pi i} \int_{\gamma_{2}} \mu_{2} d \lambda=-\frac{1}{2 \pi i} \int_{-i \infty}^{0} \mu_{2} d \lambda-\frac{1}{2 \pi i} \int_{C_{R}^{2}} \mu_{2} d \lambda \tag{4.3.40b}
\end{align*}
$$

where $C_{R}^{1}$ is defined by $C_{R}^{1}=\left\{\lambda \in \mathbb{C}: \lambda=R e^{i \theta}, \pi / 2 \leq \theta \leq \tan ^{-1}\left(-p^{\prime}(-\sigma)\right)\right\}$ where $0<R \leq \infty$ and $C_{R}^{2}$ is defined by $C_{R}^{2}=\left\{\lambda \in \mathbb{C}: \lambda=R e^{i \theta}, \tan ^{-1}\left(-p^{\prime}(\sigma)\right) \leq \theta \leq 3 \pi / 2\right\}$ where $0<R \leq \infty$. In (4.3.40a)- (4.3.40b) above, we have chosen counter-clockwise oriented contours for $C_{R}^{1}$ and $C_{R}^{2}$ and clockwise oriented contours for the positive and negative imaginary axes. The $\operatorname{arcs} C_{R}^{1}$ and $C_{R}^{2}$ are shown in Figure 4.5 below.


Figure 4.5: The $\operatorname{arcs} C_{R}^{1}$ and $C_{R}^{2}$ are shown in the complex plane.

Now we apply Fubini's theorem to the term $\int_{0}^{+i \infty} \mu_{1} d \lambda$ from equation (4.3.40a):

$$
\begin{align*}
\int_{0}^{+i \infty} \mu_{1} d \lambda & =\int_{0}^{+i \infty} e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{x} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y d \lambda  \tag{4.3.41}\\
& =\int_{0}^{+i \infty} \int_{-\sigma}^{x} e^{i \lambda x-\lambda p(x)} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y d \lambda  \tag{4.3.42}\\
& =\int_{-\sigma}^{x}\left[\int_{0}^{+i \infty} e^{i \lambda[x-y]+\lambda[p(y)-p(x)]} d \lambda\right] \varphi(y) d y \tag{4.3.43}
\end{align*}
$$

Focusing on the inner integral we find

$$
\begin{align*}
\int_{0}^{+i \infty} & e^{i \lambda[x-y]+\lambda[p(y)-p(x)]} d \lambda \\
& =\frac{1}{(p(y)-p(x))+i(x-y)} \lim _{A \rightarrow+i \infty}\left[e^{i \lambda[x-y]+\lambda[p(x)-p(y)]}\right]_{\lambda=0}^{\lambda=A}  \tag{4.3.44}\\
& =\frac{1}{(p(y)-p(x))+i(x-y)}\left[\lim _{A \rightarrow+i \infty}\left(e^{i A[x-y]+A[p(y)-p(x)]}\right)-1\right]  \tag{4.3.45}\\
& =\frac{1}{(p(y)-p(x))+i(x-y)}[0-1]  \tag{4.3.46}\\
& =\frac{-1}{(p(y)-p(x))+i(x-y)} \tag{4.3.47}
\end{align*}
$$

The last step, i.e (4.3.45)-(4.3.46), follows from the fact that $x>y$ on the interval $[a, x]$, and hence $|\exp (i A[x-y])| \rightarrow 0$ as $A \rightarrow+i \infty$. Therefore, substituting (4.3.47) back into (4.3.43), we get:

$$
\begin{equation*}
\int_{0}^{+i \infty} \mu_{1} d \lambda=-\int_{-\sigma}^{x} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \tag{4.3.48}
\end{equation*}
$$

We can also do the same procedure for the term $\int_{-i \infty}^{0} \mu_{2} d \lambda$ from equation (4.3.40b) to get:

$$
\begin{equation*}
\int_{-i \infty}^{0} \mu_{2} d \lambda=\int_{x}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \tag{4.3.49}
\end{equation*}
$$

Substituting (4.3.48) back into (4.3.40a) and (4.3.49) back into (4.3.40b) and (4.3.40a)

- (4.3.40b) into (4.3.39) gives us:

$$
\begin{align*}
& \frac{-\varphi}{\left(i-p^{\prime}(x)\right)}=\frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda+\frac{1}{2 \pi i} \int_{-\sigma}^{x} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \\
& +\frac{1}{2 \pi i} \int_{C_{R}^{1}} \mu_{1} d \lambda+\frac{1}{2 \pi i} \int_{x}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y+\frac{1}{2 \pi i} \int_{C_{R}^{2}} \mu_{2} d \lambda \\
& +\frac{1}{2 \pi}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) \Delta \theta, \tag{4.3.50}
\end{align*}
$$

which simplifies to

$$
\begin{align*}
\frac{-\varphi}{\left(i-p^{\prime}(x)\right)} & =\frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda+\frac{1}{2 \pi i} \int_{-\sigma}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \\
& +\frac{1}{2 \pi i} \int_{C_{R}^{1}} \mu_{1} d \lambda+\frac{1}{2 \pi i} \int_{C_{R}^{2}} \mu_{2} d \lambda+\frac{1}{2 \pi}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) \Delta \theta \tag{4.3.51}
\end{align*}
$$

We will now focus on the final term in (4.3.51). A calculation similar to (4.3.35)(4.3.38) shows that

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{C_{R}^{1}} \mu_{1} d \lambda=i\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)\left(\theta_{-\sigma}-\frac{\pi}{2}\right) \tag{4.3.52}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{C_{R}^{2}} \mu_{2} d \lambda=i\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)\left(\frac{3 \pi}{2}-\theta_{\sigma}\right) \tag{4.3.53}
\end{equation*}
$$

Therefore, (4.3.51) becomes:

$$
\begin{align*}
\frac{-\varphi}{\left(i-p^{\prime}(x)\right)}= & \frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda+\frac{1}{2 \pi i} \int_{-\sigma}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \\
& +\frac{1}{2 \pi}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right)(\pi-\Delta \theta)+\frac{1}{2 \pi}\left(\frac{-\varphi(x)}{i-p^{\prime}(x)}\right) \Delta \theta \tag{4.3.54}
\end{align*}
$$

This simplifies to

$$
\begin{align*}
\frac{-\varphi}{\left(i-p^{\prime}(x)\right)}= & \frac{1}{2 \pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda \\
& +\frac{1}{2 \pi i} \int_{-\sigma}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y-\frac{1}{2}\left(\frac{\varphi(x)}{i-p^{\prime}(x)}\right) . \tag{4.3.55}
\end{align*}
$$

Rearranging to get an expression for $\varphi$ gives us:

$$
\begin{align*}
\varphi=-\frac{\left(i-p^{\prime}(x)\right)}{\pi i} \int_{\gamma_{3}} & e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda \\
& \quad-\frac{\left(i-p^{\prime}(x)\right)}{\pi i} \int_{-\sigma}^{\sigma} \frac{\varphi(y)}{(p(y)-p(x))+i(x-y)} d y \tag{4.3.56}
\end{align*}
$$

This can then be written as

$$
\begin{align*}
\varphi-\int_{-\sigma}^{\sigma}\left[\frac{-\left(i-p^{\prime}(x)\right)}{\pi i[(p(y)-p(x))+i(x-y)]}\right] & \varphi(y) d y= \\
& -\frac{\left(i-p^{\prime}(x)\right)}{\pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda \tag{4.3.57}
\end{align*}
$$

The left-hand-side of (4.3.57) can be written as an integral operator acting on $\varphi$. This allows us to reduce the study of $(4.3 .57)$ to the analysis of the given integral operator. We introduce this integral operator in the definition below.

Definition 6. The integral operator $L: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is defined by

$$
\begin{equation*}
L \psi:=\int_{-\sigma}^{\sigma} M(x, y) \psi(y) d y \tag{4.3.58}
\end{equation*}
$$

where the kernel $M(x, y)$ is given by

$$
\begin{equation*}
M(x, y)=\left[\frac{-\left(i-p^{\prime}(x)\right)}{\pi i[(p(y)-p(x))+i(x-y)]}\right] \tag{4.3.59}
\end{equation*}
$$

Using Definition 6, (4.3.57) can be written as:

$$
\begin{equation*}
(I-L) \varphi=-\frac{\left(i-p^{\prime}(x)\right)}{\pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda \tag{4.3.60}
\end{equation*}
$$

In the discussion that follows we will study the operator $L: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ by analyzing the kernel $M(x, y)$. In particular, we will show that the real part of the operator $L$ is compact. This, in turn, will be used to show that $(I-\operatorname{Re}(L))$ is continuously invertible on its range. This allows us to solve (4.3.60) for $\varphi$, thus finding an inverse
for the $\mathcal{F}_{p}$-transform.

### 4.3.3 An Example

We will now give an example where the inverse formula given in (4.3.60) is verified.
Let $p(x)$ be the upper-half of the circle of radius $1 / 4$ that is centered at $x=1 / 2$. The curve $p(x)$ can be written as:

$$
\begin{align*}
p(x) & =\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^{2}}  \tag{4.3.61}\\
& =\sqrt{x-x^{2}} \tag{4.3.62}
\end{align*}
$$

for $x \in[0,1]$. The $\mathcal{F}_{p}$-inverse formula given in (4.3.60) takes the form:

$$
\begin{array}{r}
\varphi(x)-\int_{0}^{1}\left[\frac{-i+\frac{1}{2}\left(\frac{1-2 y}{\sqrt{y-y^{2}}}\right)}{\left.\pi i\left[\sqrt{y-y^{2}}-\sqrt{x-x^{2}}\right)+i(x-y)\right]}\right] \varphi(y) d y \\
=\frac{1}{\pi i}\left(-i+\frac{1}{2}\left(\frac{1-2 y}{\sqrt{y-y^{2}}}\right)\right) \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda \tag{4.3.63}
\end{array}
$$

If we take the real part of the kernel $M(x, y)$ associated to the operator $L: L^{2}([0,1]) \rightarrow$ $L^{2}([0,1])$, we find

$$
\begin{equation*}
\operatorname{Re}(M(x, y))=\frac{1}{2} \frac{1}{\sqrt{y-y^{2}}} \tag{4.3.64}
\end{equation*}
$$

Using this fact, we may take the real part of both sides of (4.3.63) to get:

$$
\begin{equation*}
\varphi(x)-\frac{1}{2 \pi} \int_{0}^{1}\left[\frac{1}{\sqrt{y-y^{2}}}\right] \varphi(y) d y=-\frac{1}{\pi} \operatorname{Re}\left(\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right) \tag{4.3.65}
\end{equation*}
$$

Now, we will verify (4.3.65) for $\varphi(x)=1$. Starting with the left-hand-side, we compute
the following integral:

$$
\begin{align*}
\int_{0}^{1}\left[\frac{1}{\sqrt{y-y^{2}}}\right] 1 d y & =\left[\frac{2 \sqrt{-1+y} \sqrt{y} \log (\sqrt{-1+y}+\sqrt{y})}{\sqrt{y-y^{2}}}\right]_{0}^{1}  \tag{4.3.66}\\
& =\lim _{y \rightarrow 1} \frac{2 \sqrt{-1+y} \sqrt{y} \log (\sqrt{-1+y}+\sqrt{y})}{\sqrt{y-y^{2}}} \\
& =0-(-\pi)  \tag{4.3.67}\\
& =\pi \tag{4.3.68}
\end{align*}
$$

Therefore, the left-hand-side is calculated as:

$$
\begin{equation*}
\varphi(x)-\frac{1}{2 \pi} \int_{0}^{1}\left[\frac{1}{\sqrt{y-y^{2}}}\right] \varphi(y) d y=1-\frac{1}{2}=\frac{1}{2} \tag{4.3.70}
\end{equation*}
$$

Now we will focus on the right-hand-side of (4.3.65). We calculate $\tilde{\varphi}(\lambda)$ as follows:

$$
\begin{align*}
\tilde{\varphi}(\lambda) & =\int_{0}^{1} e^{-i \lambda x+\lambda p(x)} 1\left(1+i p^{\prime}(x)\right) d x  \tag{4.3.71}\\
& =\int_{0}^{1} e^{-i \lambda x+\lambda p(x)} \frac{i}{\lambda} \frac{d}{d x}(-i \lambda x+\lambda p(x)) d x  \tag{4.3.72}\\
& =\frac{i}{\lambda} \int_{0}^{1} \frac{d}{d x}\left(e^{-i \lambda x+\lambda p(x)}\right) d x  \tag{4.3.73}\\
& =\frac{i}{\lambda}\left[e^{-i \lambda}-1\right] \tag{4.3.74}
\end{align*}
$$

Recall, $\gamma_{3}$ is defined by:

$$
\begin{equation*}
\gamma_{3}=\left\{\lambda \in \mathbb{C}: \lambda=\lambda_{R}\left[1-i p^{\prime}(x)\right], \lambda_{R}>0\right\} \tag{4.3.75}
\end{equation*}
$$

We will parametrize $\gamma_{3}$ with the parameter $t \in(0, \infty)$ so that:

$$
\begin{equation*}
\gamma_{3}=\left\{\lambda \in \mathbb{C}: \lambda=t\left[1-i p^{\prime}(x)\right], t \in(0, \infty)\right\} \tag{4.3.76}
\end{equation*}
$$

With this parametrization of $\gamma_{3}$, the integral on the right-hand-side of (4.3.65) can be
written as:

$$
\begin{align*}
& -\frac{1}{\pi} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda  \tag{4.3.77}\\
& =-\frac{1}{\pi} \int_{0}^{\infty} e^{i t\left[1-i p^{\prime}(x)\right] x-t\left[1-i p^{\prime}(x)\right] p(x)} \frac{i\left(e^{-i t\left[1-i p^{\prime}(x)\right]}-1\right)}{\left[1-i p^{\prime}(x)\right] t}\left[1-i p^{\prime}(x)\right] d t  \tag{4.3.78}\\
& =\frac{1}{\pi i} \int_{0}^{\infty} e^{t\left(x p^{\prime}(x)-p(x)\right)} e^{i t\left(x+p(x) p^{\prime}(x)\right)}\left(e^{-t p^{\prime}(x)} e^{-i t}-1\right) \frac{d t}{t}  \tag{4.3.79}\\
& =\frac{1}{\pi i} \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} e^{-t\left[\frac{1}{2} \frac{x}{\sqrt{x-x^{2}}}\right]} e^{i \frac{t}{2}}\left[e^{-t\left[\frac{1}{2} \frac{1-2 x}{\sqrt{x-x^{2}}}\right]} e^{-i t}-1\right] \frac{d t}{t}  \tag{4.3.80}\\
& =\frac{1}{\pi i} \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} e^{i \frac{t}{2}} e^{-t\left[\frac{1}{2} \frac{1-x}{\sqrt{x-x^{2}}}\right]} \frac{d t}{t}-\frac{1}{\pi i} \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} e^{-i \frac{t}{2}} e^{-t\left[\frac{1}{2} \frac{x}{\sqrt{x-x^{2}}}\right]} \frac{d t}{t} \tag{4.3.81}
\end{align*}
$$

Taking the real part of the integral above, we can write the right-hand-side of (4.3.65) as:

$$
\begin{align*}
-\frac{1}{\pi} & \operatorname{Re}\left(\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right)  \tag{4.3.82}\\
& =\frac{1}{\pi} \int_{0}^{\infty} \sin (t) e^{-t\left[\frac{1-x}{\sqrt{x-x^{2}}}\right]} \frac{d t}{t}+\frac{1}{\pi} \int_{0}^{\infty} \sin (t) e^{-t}\left[\frac{x}{\sqrt{x-x^{2}}}\right] \frac{d t}{t}  \tag{4.3.83}\\
& =\frac{1}{\pi} \arctan \left[\frac{\sqrt{x-x^{2}}}{1-x}\right]+\frac{1}{\pi} \arctan \left[\frac{\sqrt{x-x^{2}}}{x}\right] \tag{4.3.84}
\end{align*}
$$

Now define the function:

$$
\begin{equation*}
f(x)=\arctan \left[\frac{\sqrt{x-x^{2}}}{1-x}\right]+\arctan \left[\frac{\sqrt{x-x^{2}}}{x}\right] \tag{4.3.85}
\end{equation*}
$$

and observe that the derivative $f^{\prime}(x)$ is zero:

$$
\begin{align*}
f^{\prime}(x) & =\frac{1}{2} \frac{1}{\sqrt{x-x^{2}}}-\frac{1}{2} \frac{1}{\sqrt{x-x^{2}}}  \tag{4.3.86}\\
& =0 \tag{4.3.87}
\end{align*}
$$

Therefore, $f(x)$ is constant. Evaluating $f(x)$ at $x=1 / 2$, we find:

$$
\begin{align*}
f\left(\frac{1}{2}\right) & =\arctan \left[\frac{\sqrt{\frac{1}{2}-\left(\frac{1}{2}\right)^{2}}}{1-\frac{1}{2}}\right]+\arctan \left[\frac{\sqrt{\frac{1}{2}-\left(\frac{1}{2}\right)^{2}}}{\frac{1}{2}}\right]  \tag{4.3.88}\\
& =\frac{\pi}{2} \tag{4.3.89}
\end{align*}
$$

Substituting this back into equation (4.3.84), we get:

$$
\begin{equation*}
-\frac{1}{\pi} \operatorname{Re}\left(\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right)=\frac{1}{\pi}\left(\frac{\pi}{2}\right)=\frac{1}{2} \tag{4.3.90}
\end{equation*}
$$

This agrees with (4.3.70), which is the left hand side of the relation given in (4.3.65), thus verifying the relation.

### 4.3.4 Verification of Inverse Formula

We will now verify the inverse formula given in (4.3.60) for all $\varphi \in L^{2}(\Sigma)$. We will start by rewriting (4.3.60) as:

$$
\begin{align*}
\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda & =-\pi\left(\frac{\varphi(x)}{1+i p^{\prime}(x)}\right) \\
& -\int_{-\sigma}^{\sigma}\left[\frac{1}{[(p(y)-p(x))+i(x-y)]}\right] \varphi(y) d y \tag{4.3.91}
\end{align*}
$$

We will verify this form of the inverse formula.
The integral on the left hand side of (4.3.91) can be written as follows:

$$
\begin{align*}
\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda & =\int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \int_{-\sigma}^{\sigma} e^{-i \lambda y+\lambda p(y)} \varphi(y) d y d \lambda  \tag{4.3.92}\\
& =\int_{\gamma_{3}} \int_{-\sigma}^{\sigma} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y) d y d \lambda \tag{4.3.93}
\end{align*}
$$

We will now write the integral in (4.3.93) as:

$$
\begin{align*}
& \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda=\int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y) d y d \lambda \\
&+\int_{\gamma_{3}} \int_{[-\sigma, \sigma] \backslash B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y) d y d \lambda, \tag{4.3.94}
\end{align*}
$$

where $B_{\delta}(x)=\{y:|x-y|<\delta\}$. The first integral on the right hand side of (4.3.94) can be written as:

$$
\begin{align*}
& \int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y) d y d \lambda \\
& =\int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))}\left(\frac{\varphi(y)}{1+i p^{\prime}(y)}-\frac{\varphi(x)}{1+i p^{\prime}(x)}\right)\left(1+i p^{\prime}(y)\right) d y d \lambda \\
& \quad+\frac{\varphi(x) e^{i \lambda x-\lambda p(x)}}{1+i p^{\prime}(x)} \int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda y+\lambda p(y)}\left(1+i p^{\prime}(y)\right) d y d \lambda . \tag{4.3.95}
\end{align*}
$$

We will deal with each of the integrals in (4.3.95) separately. In each case we will take the limit as $\delta \rightarrow 0$. For the first integral in (4.3.95), we assume $\varphi \in C([-\sigma, \sigma])$. [Insert proof to arrive at the following result]:
$\lim _{\delta \rightarrow 0} \int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))}\left(\frac{\varphi(y)}{1+i p^{\prime}(y)}-\frac{\varphi(x)}{1+i p^{\prime}(x)}\right)\left(1+i p^{\prime}(y)\right) d y d \lambda=0$.
Using a density argument, this can be extend to all $\varphi \in L^{2}([-\sigma, \sigma])$.
Next, we deal with the second integral in (4.3.95) which we write as follows:

$$
\begin{align*}
& \frac{\varphi(x) e^{i \lambda x-\lambda p(x)}}{1+i p^{\prime}(x)} \int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda y+\lambda p(y)}\left(1+i p^{\prime}(y)\right) d y d \lambda \\
& =-\frac{1}{i \lambda} \frac{\varphi(x)}{1+i p^{\prime}(x)} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \int_{B_{\delta}(x)} \frac{d}{d y}\left(e^{-i \lambda y+\lambda p(y)}\right) d y d \lambda  \tag{4.3.97}\\
& =\frac{\varphi(x)}{1+i p^{\prime}(x)} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)}\left[\frac{e^{-i \lambda(x+\delta)+\lambda p(x+\delta)}-e^{-i \lambda(x-\delta)+\lambda p(x-\delta)}}{i \lambda}\right] d \lambda  \tag{4.3.98}\\
& =\frac{\varphi(x)}{1+i p^{\prime}(x)} \int_{\gamma_{3}} \frac{1}{i \lambda}\left[e^{-i \lambda \delta+\lambda(p(x+\delta)-p(x))}-e^{-i \lambda \delta+\lambda(p(x-\delta)-p(x))}\right] d \lambda . \tag{4.3.99}
\end{align*}
$$

We will parametrize the ray $\gamma_{3}$ by:

$$
\begin{equation*}
\gamma_{3}=\left\{\lambda \in \mathbb{C}: \lambda=t\left[1-i p^{\prime}(x)\right], t \in(0, \infty)\right\} . \tag{4.3.100}
\end{equation*}
$$

Using this parametrization, we may write the integrand of (4.3.99) as:

$$
\begin{align*}
& e^{-i \lambda \delta+\lambda(p(x+\delta)-p(x))}-e^{-i \lambda \delta+\lambda(p(x-\delta)-p(x))} \\
& =e^{-i \delta t} e^{-\delta p^{\prime}(x) t} e^{(p(x+\delta)-p(x)) t} e^{-i p^{\prime}(x)(p(x+\delta)-p(x)) t} \\
& \quad-e^{i \delta t} e^{\delta p^{\prime}(x) t} e^{(p(x-\delta)-p(x)) t} e^{-i p^{\prime}(x)(p(x-\delta)-p(x)) t}  \tag{4.3.101}\\
& =e^{-i \delta t} e^{-i p^{\prime}(x)(p(x+\delta)-p(x)) t} e^{\delta t\left[R(x+\delta, x)-p^{\prime}(x)\right]} \\
& \quad-e^{i \delta t} e^{-i p^{\prime}(x)(p(x-\delta)-p(x)) t} e^{-\delta t\left[R(x-\delta, x)-p^{\prime}(x)\right]}, \tag{4.3.102}
\end{align*}
$$

where

$$
\begin{equation*}
R(x+\delta, x)=\frac{p(x+\delta)-p(x)}{\delta} \quad \text { and } \quad R(x-\delta, x)=\frac{p(x-\delta)-p(x)}{-\delta} \tag{4.3.103}
\end{equation*}
$$

We note by concavity that for any $\delta>0$, the following inequalities hold:

$$
\begin{equation*}
R(x+\delta, x)-p^{\prime}(x)<0 \quad \text { and } \quad R(x-\delta, x)-p^{\prime}(x)>0 \tag{4.3.104}
\end{equation*}
$$

Now we will make the substitution $\tau=\delta t$ in (4.3.102) which gives us:

$$
\begin{align*}
& e^{-i \delta t} e^{-i p^{\prime}(x)(p(x+\delta)-p(x)) t} e^{\delta t\left[R(x+\delta, x)-p^{\prime}(x)\right]} \\
& \quad-e^{i \delta t} e^{-i p^{\prime}(x)(p(x-\delta)-p(x)) t} e^{-\delta t\left[R(x-\delta, x)-p^{\prime}(x)\right]} \\
& =e^{-i \tau} e^{-i p^{\prime}(x) R(x+\delta, x) \tau} e^{\tau\left[R(x+\delta, x)-p^{\prime}(x)\right]} \\
& \quad-e^{i \tau} e^{i p^{\prime}(x) R(x-\delta, x) \tau} e^{-\tau\left[R(x-\delta, x)-p^{\prime}(x)\right]} . \tag{4.3.105}
\end{align*}
$$

Substituting (4.3.105) back into (4.3.99), we evaluate the following limit:

$$
\begin{align*}
& \lim _{\delta \rightarrow 0} \int_{0}^{\infty} \frac{1}{i \tau}\left[e^{-i \tau} e^{-i p^{\prime}(x) R(x+\delta, x) \tau} e^{\tau\left[R(x+\delta, x)-p^{\prime}(x)\right]}\right. \\
&\left.-e^{i \tau} e^{i p^{\prime}(x) R(x-\delta, x) \tau} e^{-\tau\left[R(x-\delta, x)-p^{\prime}(x)\right]}\right] d \tau \\
&=\int_{0}^{\infty} \frac{1}{i \tau}\left[e^{-i \tau} e^{-i\left[p^{\prime}(x)\right]^{2} \tau}-e^{i \tau} e^{i\left[p^{\prime}(x)\right]^{2} \tau}\right] d \tau \tag{4.3.106}
\end{align*}
$$

Now we make the substitution $y=\left(1+\left[p^{\prime}(x)\right]^{2}\right) \tau$ and compute the integral in (4.3.106) as follows:

$$
\begin{align*}
\int_{0}^{\infty} \frac{1}{i \tau}\left[e^{-i \tau} e^{-i\left[p^{\prime}(x)\right]^{2} \tau}-e^{i \tau} e^{i\left[p^{\prime}(x)\right]^{2} \tau}\right] d \tau & =\int_{0}^{\infty} \frac{1}{i y}\left[e^{-i y}-e^{i y}\right] d y  \tag{4.3.107}\\
& =-2 \int_{0}^{\infty} \frac{\sin (y)}{y} d y  \tag{4.3.108}\\
& =-\pi \tag{4.3.109}
\end{align*}
$$

Substituting (4.3.109) back into (4.3.99), we get:

$$
\begin{gather*}
\frac{\varphi(x) e^{i \lambda x-\lambda p(x)}}{1+i p^{\prime}(x)} \int_{\gamma_{3}} \int_{B_{\delta}(x)} e^{-i \lambda y+\lambda p(y)}\left(1+i p^{\prime}(y)\right) d y d \lambda \\
=-\pi\left(\frac{\varphi(x)}{1+i p^{\prime}(x)}\right) \tag{4.3.110}
\end{gather*}
$$

Substituting (4.3.110) back into (4.3.94), we get:

$$
\begin{align*}
& \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda=-\pi\left(\frac{\varphi(x)}{1+i p^{\prime}(x)}\right) \\
& +\lim _{\delta \rightarrow 0} \int_{\gamma_{3}} \int_{[-\sigma, \sigma] \backslash B_{\delta}(x)} e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y)\left(1+i p^{\prime}(y)\right) d y d \lambda \tag{4.3.111}
\end{align*}
$$

When $|x-y|>\delta$, the double-integral in the final term of (4.3.111) converges absolutely. We may therefore apply Fubini's theorem to switch the order of integration. Doing so and following a procedure similar to that in Section 4.3.2, we find that the final
integral in (4.3.111) becomes:

$$
\begin{align*}
\lim _{\delta \rightarrow 0} \int_{\gamma_{3}} \int_{[-\sigma, \sigma] \backslash B_{\delta}(x)} & e^{-i \lambda(x-y)+\lambda(p(y)-p(x))} \varphi(y)\left(1+i p^{\prime}(y)\right) d y d \lambda \\
& =- \text { P.V. } \int_{-\sigma}^{\sigma}\left[\frac{1}{[(p(y)-p(x))+i(x-y)]}\right] \varphi(y) d y \tag{4.3.112}
\end{align*}
$$

Therefore, we have:

$$
\begin{align*}
& \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda=-\pi\left(\frac{\varphi(x)}{1+i p^{\prime}(x)}\right) \\
&- \text { P.V. } \int_{-\sigma}^{\sigma}\left[\frac{1}{[(p(y)-p(x))+i(x-y)]}\right] \varphi(y) d y \tag{4.3.113}
\end{align*}
$$

This verifies the inverse formula given in (4.3.91).

### 4.4 Analysis of the Kernel $M(x, y)$

Now we turn our attention to studying the operator $L$ by analyzing the kernel $M(x, y)$. It is clear from (4.3.59) in Definition 6 that $L$ is a singular integral operator since the kernel $M(x, y)$ is singular when $x=y$. Although $L$ is a singular integral operator, we will show that the real part of $M(x, y)$ is continuous, and therefore non-singular.

In the discussion that follows, we will assume that $\varphi$ is real-valued. We begin by finding the real and imaginary parts of the kernel $M(x, y)$. Note that the kernel $M(x, y)$ can be expressed as follows:

$$
\begin{align*}
M(x, y) & =\frac{-\left(i-p^{\prime}(x)\right)}{\pi i[(p(y)-p(x))+i(x-y)]}  \tag{4.4.1}\\
& =\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]+i\left([x-y]+p^{\prime}(x)[p(x)-p(y)]\right)}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}  \tag{4.4.2}\\
& =\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}+i \frac{[x-y]+p^{\prime}(x)[p(x)-p(y)]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)} \tag{4.4.3}
\end{align*}
$$

Therefore, the real and imaginary parts of $M(x, y)$ are given by:

$$
\begin{align*}
& \operatorname{Re}(M)=\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}  \tag{4.4.4}\\
& \operatorname{Im}(M)=\frac{[x-y]+p^{\prime}(x)[p(x)-p(y)]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)^{2}} \tag{4.4.5}
\end{align*}
$$

Now we test to see if $\operatorname{Re}(M)$ is singular by computing $\lim _{x \rightarrow y} \operatorname{Re}(M)$ :

$$
\begin{align*}
\lim _{x \rightarrow y} \operatorname{Re}(M) & =\lim _{h \rightarrow 0} \frac{[p(y+h)-p(y)]-p^{\prime}(y+h)[(y+h)-y]}{\pi\left([(y+h)-y]^{2}+[p(y+h)-p(y)]^{2}\right)}  \tag{4.4.6}\\
& =\lim _{h \rightarrow 0} \frac{-\frac{h^{2}}{2} p^{\prime \prime}(y)+O\left(h^{3}\right)}{\pi h^{2}\left(1+\left[p^{\prime}(y)+\frac{h}{2} p^{\prime \prime}(y)+O\left(h^{2}\right)\right]^{2}\right)}  \tag{4.4.7}\\
& =\lim _{h \rightarrow 0} \frac{-\frac{1}{2} p^{\prime \prime}(y)+O(h)}{\pi\left(1+\left[p^{\prime}(y)+\frac{h}{2} p^{\prime \prime}(y)+O\left(h^{2}\right)\right]^{2}\right)}  \tag{4.4.8}\\
& =\frac{-\frac{1}{2} p^{\prime \prime}(y)}{\pi\left(1+\left[p^{\prime}(y)\right]^{2}\right)} . \tag{4.4.9}
\end{align*}
$$

Note that the final expression in (4.4.9) is never singular. Since the singularity in the kernel $M(x, y)$ occurs only when $x=y$, it follows that $\operatorname{Re}(M)$ is non-singular, and is in fact continuous.

Now we will compute $\lim _{x \rightarrow y} \operatorname{Im}(M)$ :

$$
\begin{align*}
\lim _{x \rightarrow y} \operatorname{Im}(M) & =\lim _{h \rightarrow 0} \frac{[(y+h)-y]-p^{\prime}(y+h)[p(y+h)-p(y)]}{\pi\left([(y+h)-y]^{2}+[p(y+h)-p(y)]^{2}\right)}  \tag{4.4.10}\\
& =\lim _{h \rightarrow 0} \frac{h-h\left[p^{\prime}(y)\right]^{2}+\frac{3 h^{2}}{2} p^{\prime \prime}(y) p^{\prime}(y)+O\left(h^{3}\right)}{\pi h^{2}\left(1+\left[p^{\prime}(y)+\frac{h}{2} p^{\prime \prime}(y)+O\left(h^{2}\right)\right]^{2}\right)}  \tag{4.4.11}\\
& =\lim _{h \rightarrow 0} \frac{1-\left[p^{\prime}(y)\right]^{2}+\frac{3 h}{2} p^{\prime \prime}(y) p^{\prime}(y)+O\left(h^{2}\right)}{\pi h\left(1+\left[p^{\prime}(y)+\frac{h}{2} p^{\prime \prime}(y)+O\left(h^{2}\right)\right]^{2}\right)} . \tag{4.4.12}
\end{align*}
$$

In the final step above, we see that the imaginary part of the kernel $M(x, y)$ is singular, since the numerator is $O(1)$, while the denominator is $O(h)$.

We now make the following definition:

Definition 7. The integral operator $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is defined by

$$
\begin{equation*}
L_{\operatorname{Re}} \varphi:=\int_{-\sigma}^{\sigma} M_{\operatorname{Re}}(x, y) \varphi(y) d y, \tag{4.4.13}
\end{equation*}
$$

where the kernel $M_{\mathrm{Re}}(x, y)$ is given by

$$
\begin{equation*}
M_{\operatorname{Re}}(x, y)=\operatorname{Re}[M(x, y)]=\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)} . \tag{4.4.14}
\end{equation*}
$$

As we showed above, $\operatorname{Re}(M(x, y))$ is non-singular at $x=y$ and is therefore continuous in both variables $x$ and $y$. Since $\operatorname{Re}(M(x, y))$ is a continuous function on a compact set, i.e. $\Sigma=[-\sigma, \sigma]$, it must attain a maximum. Therefore, we conclude that $M_{\operatorname{Re}}(x, y) \in L^{2}(\Sigma \times \Sigma)$. Furthermore, it follows that $L_{\operatorname{Re}}(x, y): L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is a Hilbert-Schmidt operator, and hence compact, [30].

Now we wish to show that there exists a solution to the eigenvalue problem $L_{\mathrm{Re}} u=$ $\lambda u$. We do this by showing the operator $L_{\mathrm{Re}}$ is 'positive' and apply the Krein-Rutman Theorem, which shows that the spectral radius $r\left(L_{\mathrm{Re}}\right)$ is an eigenvalue with a corresponding eigenvector. First, we recall the definition of a positive operator (this definition is taken from [2]):

Definition 8 (Positive Operator). An operator $T: X \rightarrow Y$ between two ordered vector spaces is called positive if $T x \geq 0$ for all $x \geq 0$.

In addition, we recall the Krein-Rutman theorem (this is given as Theorem 19.2 of [7]):

Theorem (Krein-Rutman Theorem). Let $X$ be a Banach space, $C \subset X$ a total cone and $T: X \rightarrow X$ a compact linear operator that is positive with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in C \backslash\{0\}$ such that $T u=$ $r(T) u$.

We use the Krein-Rutman Theorem to prove the following result.

Proposition 1. Let $r\left(L_{\mathrm{Re}}\right)$ be the spectral radius of the operator $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow$ $L^{2}(\Sigma)$, then $r\left(L_{\mathrm{Re}}\right)$ is an eigenvalue with an eigenvector $u \in L^{2}(\Sigma) \backslash\{0\}$ such that $L_{\mathrm{Re}} u=r\left(L_{\mathrm{Re}}\right) u$.

Proof. Since the operator $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is compact, if we show that $L_{\mathrm{Re}}$ is positive, then the conditions for the Krein-Rutman Theorem will be satisfied, and the result will follow.

In order to show that $L_{\mathrm{Re}}$ is positive, we will show that the kernel $M_{\mathrm{Re}}$ is positive, and therefore $L_{\mathrm{Re}} f \geq 0$ for all $f \geq 0$. Recall that the kernel $L_{\mathrm{Re}}$ can be written as

$$
\begin{equation*}
M_{\mathrm{Re}}=\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)} . \tag{4.4.15}
\end{equation*}
$$

We note that the derivative $d / d x[\arctan (R(x, y))]$ is given by:

$$
\begin{align*}
\frac{d}{d x}[\arctan (R(x, y))] & =\frac{1}{1+\left(\frac{p(x)-p(y)}{x-y}\right)^{2}} \frac{p^{\prime}(x)[x-y]-[p(x)-p(y)]}{(x-y)^{2}}  \tag{4.4.16}\\
& =\frac{p^{\prime}(x)[x-y]-[p(x)-p(y)]}{(x-y)^{2}+(p(x)-p(y))^{2}}  \tag{4.4.17}\\
& =-\pi M_{\mathrm{Re}} . \tag{4.4.18}
\end{align*}
$$

Hence, $M_{\mathrm{Re}}=-\frac{1}{\pi} \frac{d}{d x}[\arctan (R(x, y))]$. For fixed $y$, the function $\arctan (R(x, y))$ measures the slope of the line between the point $(x, p(x))$ and $(y, p(y))$. By concavity, this is a decreasing function with respect to $x$, and therefore $\frac{d}{d x}[\arctan (R(x, y))]$ is negative, which implies $M_{\mathrm{Re}}$ is positive. Thus, $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is positive, and hence the proposition follows.

Proposition 1 establishes that the spectral radius, $r\left(L_{\mathrm{Re}}\right)$, is an eigenvalue of the operator $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$. Now we will show that the spectral radius is less than 1. This will be used to show that the operator $\left(I-L_{\mathrm{Re}}\right)$ is invertible on its range.

Proposition 2. $r\left(L_{\mathrm{Re}}\right)<1$.

Proof. Let $f$ be a positive function such that

$$
\begin{equation*}
L_{\mathrm{Re}} f=\lambda f \tag{4.4.19}
\end{equation*}
$$

where $\lambda=r\left(L_{\mathrm{Re}}\right)$. Without loss of generality, assume $\int_{-\sigma}^{\sigma} f d x=1$. Then,

$$
\begin{equation*}
\int_{-\sigma}^{\sigma}[\lambda f] d x=\lambda \int_{-\sigma}^{\sigma} f d x=\lambda \tag{4.4.20}
\end{equation*}
$$

and hence we have

$$
\begin{equation*}
\lambda=\int_{-\sigma}^{\sigma}[\lambda f] d x=\int_{-\sigma}^{\sigma}\left[L_{\operatorname{Re}} f\right] d x=\int_{-\sigma}^{\sigma} \int_{-\sigma}^{\sigma} M_{\operatorname{Re}}(x, y) f(y) d y d x \tag{4.4.21}
\end{equation*}
$$

Using Fubini's Theorem, we switch the order of integration in equation (4.4.21) to get

$$
\begin{equation*}
\int_{-\sigma}^{\sigma} \int_{-\sigma}^{\sigma} M_{\operatorname{Re}}(x, y) f(y) d y d x=\int_{-\sigma}^{\sigma}\left[\int_{-\sigma}^{\sigma} M_{\operatorname{Re}}(x, y) d x\right] f(y) d y \tag{4.4.22}
\end{equation*}
$$

Recalling that $L_{\mathrm{Re}}=-\frac{1}{\pi} \frac{d}{d x}[\arctan (R(x, y))]$, we have

$$
\begin{align*}
\int_{-\sigma}^{\sigma} & {\left[\int_{-\sigma}^{\sigma} M_{\operatorname{Re}}(x, y) d x\right] f(y) d y } \\
& =\frac{1}{\pi} \int_{-\sigma}^{\sigma}\left[\int_{-\sigma}^{\sigma}-\frac{d}{d x}[\arctan (R(x, y))] d x\right] f(y) d y  \tag{4.4.23}\\
& =\frac{1}{\pi} \int_{-\sigma}^{\sigma}\left[\int_{-\sigma}^{\sigma}-[\arctan (R(x, y))]_{x=-\sigma}^{\sigma}\right] f(y) d y  \tag{4.4.24}\\
& =\frac{1}{\pi} \int_{-\sigma}^{\sigma}\left[\arctan \left(\frac{p(-\sigma)-p(y)}{-\sigma-y}\right)-\arctan \left(\frac{p(\sigma)-p(y)}{\sigma-y}\right)\right] f(y) d y \tag{4.4.25}
\end{align*}
$$

Now since we assume $p(x)$ is analytic, the gradient is never infinite and therefore
$-\pi / 2<\arctan (\theta)<\pi / 2$, with strict inequality. Thus we have:

$$
\begin{equation*}
\arctan \left(\frac{p(-\sigma)-p(y)}{-\sigma-y}\right)-\arctan \left(\frac{p(\sigma)-p(y)}{\sigma-y}\right)<\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi, \tag{4.4.26}
\end{equation*}
$$

and hence

$$
\begin{gather*}
\frac{1}{\pi} \int_{-\sigma}^{\sigma}\left[\arctan \left(\frac{p(-\sigma)-p(y)}{-\sigma-y}\right)-\arctan \left(\frac{p(\sigma)-p(y)}{\sigma-y}\right)\right] f(y) d y  \tag{4.4.27}\\
\quad<\frac{1}{\pi} \int_{-\sigma}^{\sigma}[\pi] f(y) d y=\int_{-\sigma}^{\sigma} f(y) d y=1 . \tag{4.4.28}
\end{gather*}
$$

Therefore,

$$
\begin{equation*}
\lambda=\int_{-\sigma}^{\sigma}\left[\int_{-\sigma}^{\sigma} M_{\mathrm{Re}}(x, y) d x\right] f(y) d y<\int_{-\sigma}^{\sigma} f(y) d y=1, \tag{4.4.29}
\end{equation*}
$$

as desired.
Since $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is compact, it follows that the operator $\left(I-L_{\mathrm{Re}}\right)$ is Fredholm as it is a compact perturbation of $I$. Moreover, the Fredholm alternative states that precisely one of the following must hold for the operator $\left(I-L_{\mathrm{Re}}\right)$ :

- Either: A solution of the form $\left(I-L_{\mathrm{Re}}\right) f=0$ exists
- Or: The inverse $\left(I-L_{\mathrm{Re}}\right)^{-1}$ exists.

Proposition 2 tells us that $r\left(L_{\mathrm{Re}}\right)<1$, and this implies that $\left\|L_{\mathrm{Re}}\right\|<1$. Hence, by the Fredholm alternative, the inverse $\left(I-L_{\mathrm{Re}}\right)^{-1}$ exists. Furthermore, the inverse $\left(I-L_{\mathrm{Re}}\right)^{-1}$ can be computed via successive approximation. To see this, we state Theorem 10.16 of [25] by R. Kress:

Theorem. Let $A: X \rightarrow X$ be a bounded linear operator in a Banach space $X$ with spectral radius $r(A)<1$. Then the successive approximations:

$$
\begin{equation*}
\varphi_{n+1}:=A \varphi_{n}+f, \quad n=0,1,2, \ldots, \tag{4.4.30}
\end{equation*}
$$

converge for each $f \in X$ and each $\varphi_{0} \in X$ to the unique solution of $\varphi-A \varphi=f$.
As an aside, the successive approximations in the theorem above are computed via the Neumann series, which is given by:

$$
\begin{equation*}
\left(I_{X}-L_{\mathrm{Re}}\right)^{-1}=\sum_{j=1}^{\infty} L_{\mathrm{Re}}^{j}, \tag{4.4.31}
\end{equation*}
$$

where $I_{X}$ is the identity on $X$.
We summarize the results from this chapter in the following theorem:

Theorem 2. For $\varphi \in L^{2}(\Sigma)$, the inverse of the $\mathcal{F}_{p}$-transform exists and is computed by:

$$
\begin{equation*}
\varphi(x)=-\left(I-L_{\operatorname{Re}}\right)^{-1}\left\{\operatorname{Re}\left[\frac{i-p^{\prime}(x)}{\pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right]\right\}, \tag{4.4.32}
\end{equation*}
$$

where the integral operator $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is defined by

$$
\begin{equation*}
L_{\operatorname{Re}} \varphi=\int_{-\sigma}^{\sigma}\left[\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}\right] \varphi(y) d y . \tag{4.4.33}
\end{equation*}
$$

Moreover, $\left(I-L_{\mathrm{Re}}\right)^{-1}$ exists, is unique, and can be computed via successive approximation using the corresponding Neumann series.

For future reference, we will denote the inverse of the $\mathcal{F}_{p}$-transform by $\mathcal{F}_{p}^{-1}$. In Chapter 3 we proved that the $\mathcal{F}_{p}$-transform is bounded from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$. It follows directly from the Banach bounded inverse theorem that $\mathcal{F}_{p}^{-1}$ is bounded as well. We state this in the following corollary below.

Corollary 1. The inverse of the $\mathcal{F}_{p}$-transform, denoted by $\mathcal{F}_{p}^{-1}$, is bounded.
We will make use of Corollary 1 in Chapter 5 .

### 4.5 Chapter Summary

In this chapter we performed a spectral analysis of the differential equation:

$$
\begin{equation*}
\frac{\partial \mu}{\partial x}-i \lambda\left(1+i p^{\prime}(x)\right) \mu=\varphi, \tag{4.5.1}
\end{equation*}
$$

with $\varphi \rightarrow 0$ as $|x| \rightarrow \infty$, and $-\infty<x<\infty$. This is an eigenvalue problem associated with the $\mathcal{F}_{p}$-transform. In performing this spectral analysis, we constructed a function that is analytic in specified regions of the complex plane. We then found the domains where this function was bounded and posed a $\bar{\partial}$-problem that corresponded to the jumps of this solution across the respective domains. By appealing to the CauchyPompeiu formula, we reconstructed a solution that is a generalized sectionally analytic function in the complex plane. This allowed us to construct an inverse to the $\mathcal{F}_{p^{-}}$ transform. The inverse is computed by:

$$
\begin{equation*}
\varphi(x)=-\left(I-L_{\operatorname{Re}}\right)^{-1}\left\{\operatorname{Re}\left[\frac{i-p^{\prime}(x)}{\pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right]\right\} \tag{4.5.2}
\end{equation*}
$$

where $L_{\mathrm{Re}}: L^{2}(\Sigma) \rightarrow L^{2}(\Sigma)$ is the integral operator defined by

$$
\begin{equation*}
L_{\operatorname{Re}} \varphi=\int_{-\sigma}^{\sigma}\left[\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}\right] \varphi(y) d y . \tag{4.5.3}
\end{equation*}
$$

Furthermore, the inverse $\mathcal{F}_{p}$-transform, denoted by $\mathcal{F}_{p}^{-1}$, is bounded as a consequence of the boundedness of the $\mathcal{F}_{p}$-transform.

## CHAPTER 5

## The Global Relation for Analytic $\partial \Omega$

In this chapter we will further study the global relation for the Dirichlet problem given by

$$
\begin{cases}\Delta u(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega  \tag{5.0.1}\\ u(\mathbf{x})=f(\mathbf{x}) & \text { for } \mathbf{x} \in \Gamma,\end{cases}
$$

where $\Omega \subset \mathbb{C} \cong \mathbb{R}^{2}$ is a bounded convex region with an analytic boundary $\Gamma=\partial \Omega$, and $f \in C^{1}(\Gamma)$ is given (Dirichlet) boundary data on $\Gamma$. In our analysis we will view the boundary of the domain $\Omega$ as inscribed in a polygon. This will allow us to construct a framework for extending previous results from the polygon to the case of a domain with analytic boundary. By inscribing $\Omega$ in a polygon, this creates a partition of $\partial \Omega$, and we may consider the boundary data on each of the corresponding segments. In the first part of the chapter we use this construction to derive an operator $T$ that allows us to express the global relation in the form

$$
\begin{equation*}
T(\mathcal{N}-i \mathcal{D})=0, \tag{5.0.2}
\end{equation*}
$$

where $\mathcal{D}$ is a vector that contains the transformed Dirichlet data on each segment of the partition, and $\boldsymbol{\mathcal { N }}$ is a vector that contains the corresponding transformed Neumann data. This reduces the study of the global relation to the study of the operator $T$. In
the remainder of the chapter we establish properties of the operator $T$ that allow us to show that the global relation can be solved to find the Neumann data, given the Dirichlet data, for a specified BVP.

### 5.1 Derivation of the operator $T$ for domains with ana-

## lytic boundary

We begin this section by defining the framework that will be used for analyzing the operator $T$. Let $\left\{z_{j}\right\}_{j=1}^{n}$ be a collection of $n$ points on the curve $\Gamma$ such that $j$ increases as the boundary is traversed counter-clockwise, and denote the segment of the boundary connecting $z_{j}$ to $z_{j+1}$ by $\Gamma_{j}$. The collection of curves $\left\{\Gamma_{j}\right\}_{j=1}^{n}$ forms a partition of the boundary $\Gamma$.

Now we will inscribe the region $\Omega$ in a polygon which allows us to establish an analogy with the case of polygonal domains that was discussed in Chapter 2. At the same time, this provides a convenient framework for later calculations. Given the $j$ th component of the boundary, $\Gamma_{j}$, we define the $j$ th edge of the corresponding polygon so that it is tangent to $\Gamma_{j}$ at the midpoint. We will call this edge $l_{j}$. This is shown in Figure 5.1 b below. The intersection of the respective lines form the vertices of the polygon. The region $\Omega$ and inscribing polygon are shown in Figure 5.1a.

(a) An analytic region $\Omega$ in red inscribed (b) An edge of the polygon tangent to the in a 6 -sided polygon. The segments $\Gamma_{1}$ and analytic region at the midpoint of segment $\Gamma_{n}$ of the analytic region and sides $l_{1}$ and $\Gamma_{j}$. $l_{n}$ of the polygon are labelled.

Figure 5.1: A convex region $\Omega$ with analytic boundary inscribed in a polygon.

In addition, we assume that the region $\Omega$ is embedded in $\mathbb{C}$ such that the edge of the polygon corresponding to $\Gamma_{1}$ is aligned with the real-axis, and the midpoint of $\Gamma_{1}$ is at the origin. This allows us to write the global relation in terms of the $\mathcal{F}_{p}$-transform in the discussion that follows.

### 5.1.1 Derivation of the Operator $T$

Recall that the spectral functions for a polygonal domain were defined in equation (2.1.8) from Chapter 2. In this case the integral for each spectral function was taken over the respective edge of the polygon, i.e. a line segment of length $2 \sigma_{j}$. Analogously, In the case of a domain with analytic boundary, the integral for each spectral function is taken over the curved segment $\Gamma_{j}$. The spectral functions $\left\{\rho_{j}\right\}_{j=1}^{n}$ are given by:

$$
\begin{equation*}
\rho_{j}(\lambda)=\int_{\Gamma_{j}} e^{-i \lambda z} \frac{\partial u}{\partial z} d z . \tag{5.1.1}
\end{equation*}
$$

Stoke's theorem may be applied again, in a manner directly analogous to the polygonal domain case, to show that the spectral functions satisfy the global relation:

$$
\begin{equation*}
\sum_{j=1}^{n} \rho_{j}(\lambda)=0 . \tag{5.1.2}
\end{equation*}
$$

We will now show how the operator $T$ is constructed for the given BVP on the domain $\Omega$. First, we introduce the local parametrization $\psi_{j}:\left[-\sigma_{j}, \sigma_{j}\right] \rightarrow \Gamma_{j}$ given by

$$
\begin{equation*}
\psi_{j}(\tau)=m_{j}+\tau e^{i \alpha_{j}(\tau)} \tag{5.1.3}
\end{equation*}
$$

where $m_{j}$ is the midpoint with respect to the arclength parametrization of the curve $\Gamma_{j}$, and $\tau \in\left[-\sigma_{j}, \sigma_{j}\right]$, where the length of the curve segment $\Gamma_{j}$ is $2 \sigma_{j}$. The function $\alpha_{j}(\tau)$ is the angle that the tangent vector makes with the real-axis at the point $\tau$. Under this parametrization, the angle of the tangent vector at the midpoint of the curve $\Gamma$ is given by $\alpha_{j}(0)$.

For a function $f: \Gamma_{j} \rightarrow \mathbb{C}$, the pullback by $\psi_{j}$ is written $\psi_{j}^{*}(f)(\tau)=f\left(\psi_{j}(\tau)\right)$.

Using (5.1.1), the spectral functions $\left\{\rho_{j}(\lambda)\right\}_{j=1}^{n}$ can be written as

$$
\begin{equation*}
\rho_{j}(\lambda)=\int_{\Gamma_{j}} e^{-i \lambda\left(m_{j}+\tau e^{i \alpha_{j}(\tau)}\right)} \psi_{j}^{*}\left(\frac{\partial u}{\partial z}\right)(\tau) e^{i \alpha_{j}(\tau)}\left(1+i \tau \alpha_{j}^{\prime}(\tau)\right) d \tau \tag{5.1.4}
\end{equation*}
$$

Above we used equation (5.1.1) where we set $z=m_{j}+\tau e^{i \alpha_{j}(\tau)}$ and therefore

$$
\begin{equation*}
d z=e^{i \alpha_{j}(\tau)}\left(1+i \tau \alpha_{j}^{\prime}(\tau)\right) d \tau \tag{5.1.5}
\end{equation*}
$$

The following (pointwise) expression may be used for the directional derivative

$$
\begin{equation*}
\left.\frac{\partial u}{\partial z}\right|_{\Gamma_{j}}=\left.\frac{1}{2} e^{-i \alpha_{j}(\tau)}\left(\frac{\partial u}{\partial \boldsymbol{t}}+i \frac{\partial u}{\partial \boldsymbol{n}}\right)\right|_{\Gamma_{j}} \tag{5.1.6}
\end{equation*}
$$

where $\partial u / \partial \boldsymbol{t}$ denotes the tangential derivative of $u$ (which is the derivative of the Dirichlet data) and $\partial u / \partial \boldsymbol{n}$ denotes the normal derivative of $u$ (which is the Neumann data) at each point of $\Gamma_{j}$. Hence, the directional derivative contains information about both the Dirichlet and Neumann data on the curve $\Gamma_{j}$. Now define

$$
\begin{equation*}
\varphi_{j}(\tau)=\psi_{j}^{*}(\partial u / \partial z)(\tau) \tag{5.1.7}
\end{equation*}
$$

In this context, the $\mathcal{F}_{p}$-transform takes the form:

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi_{j} \rightarrow \tilde{\varphi}_{j}(\lambda)=\int_{-\sigma_{j}}^{\sigma_{j}} e^{-i \lambda e^{i \alpha_{j}(\tau)} \tau} \varphi_{j}(\tau)\left(1+i \tau \alpha_{j}^{\prime}(\tau)\right) d \tau \tag{5.1.8}
\end{equation*}
$$

Let $\varphi_{j}^{t}$ denote the tangential component of $\varphi_{j}$ and let $\varphi_{j}^{n}$ denote the normal component of $\varphi_{j}$, i.e.

$$
\begin{equation*}
\varphi_{j}^{t}=\psi_{j}^{*}\left(\frac{\partial u}{\partial \boldsymbol{t}}\right) \quad \text { and } \quad \varphi_{j}^{n}=\psi_{j}^{*}\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) \tag{5.1.9}
\end{equation*}
$$

Using (5.1.6), (5.1.9), and the $\mathcal{F}_{p}$-transform in the form of (5.1.8), the spectral functions $\rho_{j}(\lambda)$ may be expressed as

$$
\begin{equation*}
\rho_{j}(\lambda)=\frac{e^{-i \lambda m_{j}}}{2}\left[\tilde{\varphi}_{j}^{t}\left(e^{i \alpha_{j}(0)} \lambda\right)+i \tilde{\varphi}_{j}^{n}\left(e^{i \alpha_{j}(0)} \lambda\right)\right] \tag{5.1.10}
\end{equation*}
$$

Hence using (5.1.2), the global relation may be written as a sum of the form:

$$
\begin{equation*}
\sum_{j=1}^{n} e^{-i \lambda m_{j}}\left[\tilde{\varphi}_{j}^{t}\left(e^{i \alpha_{j}(0)} \lambda\right)+i \tilde{\varphi}_{j}^{n}\left(e^{i \alpha_{j}(0)} \lambda\right)\right]=0 \tag{5.1.11}
\end{equation*}
$$

Now if we multiply the above equation by $e^{i \lambda m_{j}}$ for fixed $j \in\{1, \ldots, n\}$, this has the effect of shifting the region so that the midpoint of the $j$ th side is at the origin of the complex plane. If we make the substitution $\lambda \mapsto \lambda e^{-i \alpha_{j}(0)}$, this has the effect of rotating the region so that $p_{j}(0)=p_{j}^{\prime}(0)=0$. This is shown in Figure 5.2 below.


Figure 5.2: Rotation of the region $\Omega$, and corresponding polygon, by $e^{i \lambda m_{n}}$, where $m_{n}$ is the midpoint of $\Gamma_{n}$. The rotation is performed such that the midpoint of $\Gamma_{n}$ is tangent to the real-axis and the edge of the polygon $l_{n}$ overlaps with the real-axis.

Multiplying equation (5.1.11) by $e^{i \lambda m_{j}}$ for fixed $j \in\{1, \ldots, n\}$ and replacing $\lambda$ with $\lambda e^{-i \alpha_{j}(0)}$, gives us the following relation:

$$
\begin{equation*}
\tilde{\varphi}_{j}\left(e^{-i \alpha_{j}(0)} \lambda\right)+\sum_{k \neq j} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \tilde{\varphi}_{k}\left(e^{-i \alpha_{j}(0)} \lambda\right)=0 \tag{5.1.12}
\end{equation*}
$$

Set $\Psi_{j}(\lambda)=\tilde{\varphi}_{j}\left(e^{-i \alpha_{j}(0)} \lambda\right)$ so that the above expression is now written as

$$
\begin{equation*}
\Psi_{j}(\lambda)+\sum_{k \neq j} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right)=0, \tag{5.1.13}
\end{equation*}
$$

where $\Delta_{j k}(0):=\alpha_{j}(0)-\alpha_{k}(0)$. This transformation has the effect of rotating the
region $\Omega$ so that the midpoint of the segment $\Gamma_{j}$ is tangent to the real-axis of the complex plane, while fixing the $j$ th component of the equation.

Setting $\Psi=\left(\Psi_{1}, \ldots, \Psi_{n}\right)^{T}$, we define the operator $T=I+K$, where $I$ is the identity, and $K$ is defined by

$$
\begin{equation*}
(K \Psi)_{j}(\lambda):=\sum_{k \neq j} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right), \quad 1 \leq j \leq n . \tag{5.1.14}
\end{equation*}
$$

The operator $T$ can now be expressed as

$$
\begin{align*}
(T \Psi)_{j}(\lambda) & =\Psi_{j}(\lambda)+\sum_{k \neq j} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right)  \tag{5.1.15}\\
& =\Psi_{j}(\lambda)+(K \Psi)_{j}(\lambda) . \tag{5.1.16}
\end{align*}
$$

Using (5.1.16), we may concisely express the global relation as:

$$
\begin{equation*}
T \Psi=0 . \tag{5.1.17}
\end{equation*}
$$

Expressing $\Psi$ in terms of the normal and tangential components in the equation above and multiplying by $-i$ gives us the following expression for the global relation:

$$
\begin{equation*}
T\left(\Psi^{n}-i \Psi^{t}\right)=0, \tag{5.1.18}
\end{equation*}
$$

where $\Psi^{n}=\left(\Psi_{1}^{n}, \ldots, \Psi_{n}^{n}\right)^{T}$ are the normal components of the $\Psi_{j}$ and therefore contains the $\mathcal{F}_{p}$-transform of the Neumann data, and $\Psi^{t}=\left(\Psi_{1}^{t}, \ldots, \Psi_{n}^{t}\right)^{T}$ are the tangential components and therefore contains the $\mathcal{F}_{p}$-transform of the Dirichlet data. Equation (5.1.18) gives a concise form for expressing the global relation and allows us to analyze the global relation (in particular as a map between the Dirichlet and Neumann data) by understanding properties of the operator $T$.

### 5.1.2 Comparison with the Case of Polygonal Domains

We will now briefly compare the derivation of the operator $T$ in Section 5.1 with the derivation of the operator $T$ for the case of convex polygonal domains, as given in [5]. We will do so by outlining several of the similarities and differences in each of the components of the derivation in the list that follows.

- Spectral Functions: In both cases, the spectral functions for each component of a domain with analytic boundary and a polygonal domain take the form:

$$
\begin{equation*}
\rho_{j}(\lambda)=\int_{\Gamma_{j}} e^{-i \lambda z} \frac{\partial u}{\partial z} d z \tag{5.1.19}
\end{equation*}
$$

In the case of a polygonal domain, the variable $z$ lies along a straight segment $\Gamma_{j}$, and as a result, the derivative $\partial u / \partial z$ is a constant and can be computed in a straight-forward manner by performing a fixed rotation to the real axis. In the case of a domain with analytic boundary, however, the derivative $\partial u / \partial z$ takes the form:

$$
\begin{equation*}
\left.\frac{\partial u}{\partial z}\right|_{\Gamma_{j}}=\left.\frac{1}{2}\left(\frac{\partial u}{\partial x}-i \frac{\partial u}{\partial y}\right)\right|_{\Gamma_{j}} \tag{5.1.20}
\end{equation*}
$$

where $z$ is parametrized by $z(\tau)=\tau+i p(\tau)$ for some analytic convex $p$ and $\tau \in[-\sigma, \sigma]$. As a result, the derivative $\partial u / \partial z$ is no longer a constant and can no longer be computed in such a straight-forward manner.

- Parametrization: In the case of a polygonal domain, the parametrization of each edge is given by:

$$
\begin{equation*}
\psi_{j}(\tau)=m_{j}+\tau e^{i \alpha_{j}} \tag{5.1.21}
\end{equation*}
$$

where $\alpha_{j}$ is a constant ${ }^{1}$ that is fixed for a given edge $\Gamma_{j}$. If we contrast this with the parametrization given in (5.1.3), we find that the angle $\alpha_{j}$ now depends on the parameter $\tau \in[-\sigma, \sigma]$. The result of this is that the derivative $\partial u / \partial z$ can no longer be expressed as a simple rotation in the complex plane, as was the case for a polygonal domain with straight edges. Instead, the derivative $\partial u / \partial z$ must be specified at each point along $\Gamma_{j}$, as is done in (5.1.6). This makes the derivation

[^8]of the operator $T$ more complex than was the case for a polygonal domain. In particular, the Fourier transform (and classical results pertaining to the Fourier transform) can no longer be used since the edges are no longer straight. Instead, we must use the $\mathcal{F}_{p}$-transform.

- $\mathcal{F}_{p}$-transform: As mentioned previously, the $\mathcal{F}_{p}$-transform must now be used in the case of domains with analytic boundary instead of the Fourier transform, which was used in the analysis for polygonal domains. Since the $\mathcal{F}_{p}$-transform is a new transform that was introduced in this thesis, many of the basic properties need to be derived (as was done in Chapters 3 and 4) instead of relying on previously established results (as was the case when using the Fourier transform). In particular, we need to show that the $\mathcal{F}_{p}$-transform is bounded as an integral operator from $L^{2}(\Sigma)$ to $L^{2}(\Lambda)$ (as was shown in Chapter 3) and that the inverse of the $\mathcal{F}_{p}$-transform exists and is well-defined (as was shown in Chapter 4). The necessity of these requirements will become apparent in the remainder of this chapter. In addition, the results that we establish in Section 5.3 is done in a more general analysis framework than for the polygonal case.
- The Operator T: In both cases, the operator $T$ for polygonal domains and for domains with analytic boundary have a similar structure. The $T$ operator for polygonal domains takes the form $T=I+K$, where $K$ is given by:

$$
\begin{equation*}
K \Phi_{j}=\sum_{k \neq j} e^{i e^{-i \alpha_{j}}\left(m_{j}-m_{k}\right) \lambda} \Phi_{k}\left(e^{-i\left(\alpha_{j}-\alpha_{k}\right)} \lambda\right), \quad 1 \leq j \leq n \tag{5.1.22}
\end{equation*}
$$

where $\Phi$ is an $n$-dimensional vector that contains the Fourier transform of either the Dirichlet or Neumann boundary data. Comparing this with the definition of the operator $K$ for domains with analytic boundary (given in equation (5.1.14)), we note that the two have a similar structure. The primary difference is that the operator $K$ for domains with analytic boundary acts on vectors that contain the $\mathcal{F}_{p}$-transform of boundary data, and therefore the results from Chapters 3 and 4 are needed for the analysis of the operators $K$ and $T$.

We will now introduce the function spaces that will be used for the remainder of the chapter in the analysis of the operator $T$.

### 5.1.3 Function Spaces

In our analysis of the operator $T$ it will be useful to define the function spaces that will be used. Let $\boldsymbol{\varphi}$ denote the vector

$$
\begin{equation*}
\varphi=\left(\varphi_{1}, \ldots, \varphi_{n}\right) . \tag{5.1.23}
\end{equation*}
$$

Let $\Sigma_{j}=\left[-\sigma_{j}, \sigma_{j}\right]$ and define

$$
\begin{equation*}
\boldsymbol{\Sigma}:=L^{2}\left(\Sigma_{1}\right) \times \ldots \times L^{2}\left(\Sigma_{n}\right), \tag{5.1.24}
\end{equation*}
$$

so that $\boldsymbol{\varphi} \in \boldsymbol{\Sigma}$. Also let $\Lambda=[0, \infty]$ and define

$$
\begin{equation*}
\Lambda:=\underbrace{L^{2}(\Lambda) \times \ldots \times L^{2}(\Lambda)}_{n \text {-copies }}, \tag{5.1.25}
\end{equation*}
$$

where there are $n$-copies of $L^{2}(\Lambda)$ in the product. Let $\boldsymbol{\xi} \in \Sigma$, we define the following norm on the space $\boldsymbol{\Sigma}$ :

$$
\begin{equation*}
\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}:=\left(\sum_{j=1}^{n}\left\|\xi_{j}\right\|_{L^{2}\left(\Sigma_{j}\right)}^{2}\right)^{1 / 2} . \tag{5.1.26}
\end{equation*}
$$

Similarly, let $\Phi \in \boldsymbol{\Lambda}$, we define the following norm on the space $\boldsymbol{\Lambda}$ :

$$
\begin{equation*}
\|\boldsymbol{\Phi}\|_{\boldsymbol{\Lambda}}:=\left(\sum_{j=1}^{n}\left\|\Phi_{j}\right\|_{L^{2}(\Lambda)}^{2}\right)^{1 / 2} \tag{5.1.27}
\end{equation*}
$$

Using this notation, the map $\boldsymbol{\varphi} \mapsto \Psi$ is a map from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ while the maps $\Psi \mapsto K \Psi$ and $\Psi \mapsto T \Psi$ are both maps from $\boldsymbol{\Lambda}$ to $\boldsymbol{\Lambda}$. In the next two sections we will explore these maps further.

### 5.2 Compactness of the Map $\Psi \mapsto K \Psi$

We will now show that the map $\Psi \mapsto K \Psi$ is compact by showing that it takes the form of a Hilbert-Schmidt operator. This will directly imply that the operator $T=I+K$ is a Fredholm operator of index zero. From this fact we will derive further results in Section 5.3.

Lemma 5. The map $\Psi \mapsto K \Psi$ is compact.

Proof. Fix $j \in\{1, \ldots, n\}$ and assume $k \neq j$. We may express the $K_{j k}$ component of $K$ as follows:

$$
\begin{align*}
K_{j k} \Psi_{k}(\lambda) & =e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right)  \tag{5.2.1}\\
& =e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \tilde{\varphi}_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right)  \tag{5.2.2}\\
& =e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \int_{-\sigma_{k}}^{\sigma_{k}} e^{-i \lambda\left(\tau e^{-i \Delta_{j k}(0)}\right)} \varphi_{k}(\tau)\left(1+i \tau \alpha_{k}^{\prime}(\tau)\right) d \tau  \tag{5.2.3}\\
& =\int_{-\sigma_{k}}^{\sigma_{k}} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} e^{-i \lambda\left(\tau e^{-i \Delta_{j k}(0)}\right)} \varphi_{k}(\tau)\left(1+i \tau \alpha_{k}^{\prime}(\tau)\right) d \tau  \tag{5.2.4}\\
& =\int_{-\sigma_{k}}^{\sigma_{k}} e^{-i \lambda\left[e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right)\right]} \varphi_{k}(\tau)\left(1+i \tau \alpha_{k}^{\prime}(\tau)\right) d \tau . \tag{5.2.5}
\end{align*}
$$

Now we define the integral operator $M_{j k}: L^{2}\left(\Sigma_{k}\right) \rightarrow L^{2}(\Lambda)$ by:

$$
\begin{equation*}
M_{j k} \varphi(\lambda)=\int_{-\sigma_{k}}^{\sigma_{k}} M_{j k}(\lambda, \tau) \varphi_{k}(\tau) d \tau \tag{5.2.6}
\end{equation*}
$$

where $M_{j k}(\lambda, \tau)$ is the kernel given by:

$$
\begin{equation*}
M_{j k}(\lambda, \tau)=e^{-i \lambda\left[e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right)\right]}\left(1+i \tau \alpha_{k}^{\prime}(\tau)\right) \tag{5.2.7}
\end{equation*}
$$

We will now focus our attention on the kernel $M_{j k}(\lambda, \tau)$.

Let

$$
\begin{equation*}
z(\tau)=e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right) \tag{5.2.8}
\end{equation*}
$$

We can express $z(\tau)$ in complex exponential form as

$$
\begin{equation*}
z=|z| e^{i \theta}=|z|(\cos (\theta)+i \sin (\theta)) \tag{5.2.9}
\end{equation*}
$$

where $\theta$ is given by

$$
\begin{equation*}
\theta=\arg \left(e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right)\right) \tag{5.2.10}
\end{equation*}
$$

Using the concavity of $\Gamma_{j}$ and the convexity of the domain, we see that for $k \neq j$ :

$$
\begin{equation*}
-\pi+\varepsilon<\underbrace{\arg \left(e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right)\right)}_{\theta} \leq-\varepsilon \tag{5.2.11}
\end{equation*}
$$

for some $\varepsilon>0$. See Figure 5.3 below for a schematic of the setup for this argument.


Figure 5.3: The $j$ th partition of the convex domain with the corresponding $j$ th polygon edge in red.

Now we will show that the kernel $M_{j k}(\lambda, \tau)$ is in $L^{2}\left(\Sigma_{k}\right) \times L^{2}(\Lambda)$. In order to do
this we first establish the following estimate using (5.2.11):

$$
\begin{align*}
\left|e^{-i \lambda z(\tau)}\right| & =\left|e^{-i \lambda|z|(\cos (\theta)+i \sin (\theta))}\right|  \tag{5.2.12}\\
& =\left|e^{-i \lambda|z| \cos (\theta)}\right|\left|e^{\lambda|z| \sin (\theta)}\right|  \tag{5.2.13}\\
& =e^{\lambda|z| \sin (\theta)}  \tag{5.2.14}\\
& \lesssim e^{-\lambda \sin (\varepsilon)} \tag{5.2.15}
\end{align*}
$$

Now we will show $\left\|M_{j k}(\lambda, \tau)\right\|_{L^{\left(\Sigma_{k}\right) \times L^{2}(\Lambda)}}<\infty$, and hence that $M_{j k}(\lambda, \tau) \in L^{2}\left(\Sigma_{k}\right) \times$ $L^{2}(\Lambda):$

$$
\begin{align*}
\left\|M_{j k}(\lambda, \tau)\right\|_{L^{2}\left(\Sigma_{k}\right) \times L^{2}(\Lambda)} & \lesssim \int_{0}^{\infty} \int_{-\sigma_{k}}^{\sigma_{k}}\left|e^{-i \lambda\left[e^{-i \alpha_{j}(0)}\left(\tau e^{i \alpha_{k}(0)}+\left(m_{k}-m_{j}\right)\right)\right]}\right|^{2} d \tau d \lambda  \tag{5.2.16}\\
& \lesssim \int_{0}^{\infty} \int_{-\sigma_{k}}^{\sigma_{k}} e^{-2 \lambda \sin (\varepsilon)} d \tau d \lambda  \tag{5.2.17}\\
& \lesssim \int_{0}^{\infty} e^{-2 \lambda \sin (\varepsilon)} d \lambda  \tag{5.2.18}\\
& <\infty . \tag{5.2.19}
\end{align*}
$$

The inequality in (5.2.17) follows from the estimate derived in (5.2.15), and the inequality in (5.2.16) follows from the fact that $\alpha_{k}^{\prime}(\tau)$ is continuous and therefore $\left|1+i \tau \alpha_{k}^{\prime}(\tau)\right|$ is bounded.

Since $M_{j k}(\lambda, \tau) \in L^{2}\left(\Sigma_{k}\right) \times L^{2}(\Lambda)$, it follows that $M_{j k}: L^{2}\left(\Sigma_{k}\right) \rightarrow L^{2}(\Lambda)$ is a Hilbert-Schmidt operator. Therefore, $M_{j k}$ is a compact operator since every HilbertSchmidt operator is compact, [30]. Now we observe that we can express the integral operator $K_{j k}: L^{2}(\Lambda) \rightarrow L^{2}(\Lambda)$ in terms of the operator $M_{j k}: L^{2}\left(\Sigma_{k}\right) \rightarrow L^{2}(\Lambda)$ as
follows:

$$
\begin{align*}
M_{j k} \varphi_{k} & =M_{j k} \mathcal{F}_{p}^{-1} \mathcal{F}_{p} \varphi_{k}  \tag{5.2.20}\\
& =M_{j k} \mathcal{F}_{p}^{-1} \Psi_{k} . \tag{5.2.21}
\end{align*}
$$

That is,

$$
\begin{equation*}
K_{j k} \Psi_{k}=M_{j k} \mathcal{F}_{p}^{-1} \Psi_{k} \tag{5.2.22}
\end{equation*}
$$

Since $M_{j k}: L^{2}\left(\Sigma_{k}\right) \rightarrow L^{2}(\Lambda)$ is compact and $\mathcal{F}_{p}^{-1}: L^{2}(\Lambda) \rightarrow L^{2}\left(\Sigma_{k}\right)$ is bounded, this implies that $K_{j k}=M_{j k} \mathcal{F}_{p}^{-1}$ is also compact. Hence, the sum

$$
\begin{equation*}
(K \Psi)_{j}(\lambda)=\sum_{k \neq j} K_{j k} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right) \tag{5.2.23}
\end{equation*}
$$

is also compact. Therefore, the operator $K: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is compact, and hence the map $\Psi \mapsto K \Psi$ is compact.

It now follows directly from Lemma 5 that the operator $T=I+K$ is a Fredholm operator of index zero. We summarize this in the theorem below.

Theorem 3. The operator $T$ takes the form $T=I+K$, where $K$ is compact. Therefore, $T$ is a Fredholm operator of index zero.

Proof. Using (5.1.15) and (5.2.23), we may express the operator $T$ as

$$
\begin{align*}
(T \Psi)_{j}(\lambda) & =\Psi_{j}(\lambda)+\sum_{k \neq j} K_{j k} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right)  \tag{5.2.24}\\
& =\Psi_{j}(\lambda)+(K \Psi)_{j}(\lambda) \tag{5.2.25}
\end{align*}
$$

This takes the form $T \Psi=(I+K) \Psi$. Since $K: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is compact, the operator $T=I+K$ is a Fredholm operator of index zero, [26].

As a consequence of Theorem 3, we have that $T$ is a continuous linear operator. By express equation (5.1.18) as:

$$
\begin{equation*}
T \Psi^{n}=i T \Psi^{t}, \tag{5.2.26}
\end{equation*}
$$

we see that the global relation defines a continuous linear map between the transformed Dirichlet data and the transformed Neumann data, i.e. a Dirichlet-Neumann map. We will now study the operator further and use properties of it to show that the global relation can be solved to recover the Neumann data, given the Dirichlet data, for a specified BVP.

### 5.3 Summary of Results

In Section 5.1 we constructed the map $T=I+K$ which establishes a relation between the (transformed) Dirichlet data and the (transformed) Neumann data. In Section 5.2 we proved that the map $\Psi \mapsto K \Psi$ is compact. As a direct consequence of this fact, we were able to conclude that $T$ is a Fredholm operator of index zero. We will now summarize the results that were derived in Chapters 3 and 4. Following this, we will combine these results to show that the operator $T$ is bounded below and therefore continuously invertible on its range $\operatorname{Ran}(T)$.

We consider the following sequence of maps:

$$
\begin{equation*}
\varphi \xrightarrow{A} \Psi \xrightarrow{B} T \Psi, \tag{5.3.1}
\end{equation*}
$$

where $\boldsymbol{\varphi}=\left(\varphi_{1}, \ldots, \varphi_{n}\right)$, and $\boldsymbol{\Psi}=\left(\tilde{\varphi}_{1}\left(e^{-i \alpha_{1}(0)} \lambda\right), \ldots, \tilde{\varphi}_{n}\left(e^{-i \alpha_{n}(0)} \lambda\right)\right)$, as defined previously.

Our study of the operator $T$ is based on the study of the maps $A: \boldsymbol{\Sigma} \rightarrow \boldsymbol{\Lambda}$ and $B: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$. Throughout Chapters 3 and 4 and the present chapter we have explored several properties of these maps. These properties are summarized below:

1. The Map $A$ - Our analysis of the map $A$ takes place in two parts:
(a) In Chapter 3 we showed that the map $\varphi_{j} \mapsto \Psi_{j}$ is bounded from $L^{2}\left(\Sigma_{j}\right)$ to $L^{2}(\Lambda)$ for each $j \in\{1, \ldots, n\}$. This was done by proving:

$$
\begin{equation*}
\int_{\Lambda}\left|\Psi_{j}(\lambda)\right|^{2} d \lambda \lesssim \int_{\Sigma_{j}}\left|\varphi_{j}(\tau)\right|^{2} d \tau \tag{5.3.2}
\end{equation*}
$$

This estimate was proved in Sections 3.1-3.4. This establishes that the $\operatorname{map} A: \boldsymbol{\Sigma} \rightarrow \boldsymbol{\Lambda}$ is bounded, i.e.

$$
\begin{equation*}
\|\boldsymbol{\Psi}\|_{\Lambda} \lesssim\|\varphi\|_{\boldsymbol{\Sigma}} \tag{5.3.3}
\end{equation*}
$$

(b) In Chapter 4 we constructed an inverse, denoted by $\mathcal{F}_{p}^{-1}$, to the $\mathcal{F}_{p}$-operator. This shows that the map $A: \boldsymbol{\Sigma} \rightarrow \boldsymbol{\Lambda}$ is injective and surjective on its range, and therefore is an isomorphism.

As an aside, the properties of the map $A: \boldsymbol{\Sigma} \rightarrow \boldsymbol{\Lambda}$ allow us to show that $\mathcal{F}_{p}$ is bounded below and therefore has closed range. This is done as follows. Since $\mathcal{F}_{p}$ is bounded, it follows from the Banach bounded inverse theorem that $\mathcal{F}_{p}^{-1}$ is also bounded. Therefore, the following inequality holds:

$$
\begin{equation*}
\|x\|=\left\|\mathcal{F}_{p}^{-1} \mathcal{F}_{p}(x)\right\| \lesssim_{p}\left\|\mathcal{F}_{p}(x)\right\| . \tag{5.3.4}
\end{equation*}
$$

This implies that $\mathcal{F}_{p}$ is bounded below.
2. The Map $B$ - Our analysis of the map $B$ relies on the following properties, which we state below:
(a) In [4], Ashton proved that the solution to the global relation is unique. This implies that the operator $T$ is injective.
(b) In Section 5.2, we showed that the map $\Psi \mapsto K \Psi$ is compact.
(c) Since $\Psi \mapsto K \Psi$ is compact, it directly follows that $T=I+K$ is a Fredholm operator of index zero.

From the results given in the outline above, we derive the following theorem:

Theorem 4. The operator $T: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is a bounded injective linear operator with closed range.

Proof. Since $T$ is a Fredholm operator of index zero, it is bounded and its range, $\operatorname{Ran}(T)$, is closed. The fact that $T$ is injective is a consequence property $2(\mathrm{a})$ of the map $B$, as stated above.

The following corollary is a direct consequence of Theorem 4:

Corollary 2. The operator $T: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is bounded below, i.e.

$$
\begin{equation*}
\|\Psi\|_{\boldsymbol{\Lambda}} \lesssim\|T \Psi\|_{\boldsymbol{\Lambda}} . \tag{5.3.5}
\end{equation*}
$$

Proof. Every bounded injective linear operator with closed range in bounded below, [2].

Since the operator $T$ is bounded below, it is continuously invertible on its range $\operatorname{Ran}(T)$. This means that, given the Dirichlet data, the global relation can be solved to find the Neumann data.

Recall the following theorem from [4] that was introduced in Chapter 2:

Theorem. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded, convex domain with analytic boundary $\Gamma$. Suppose there exists a function $g \in C(\Gamma)$ such that

$$
\begin{equation*}
\int_{\Gamma} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left[g(\boldsymbol{x})+i\left(\boldsymbol{\lambda} \cdot \boldsymbol{n}_{\boldsymbol{x}}\right) f(\boldsymbol{x})\right] d \Gamma_{\boldsymbol{x}}=0, \quad \boldsymbol{\lambda} \in Z_{p} \tag{5.3.6}
\end{equation*}
$$

for a given function $f \in C(\Gamma)$. Then, there exists a solution to the corresponding BVP for the Laplace equation and $g$ corresponds to the unknown Neumann boundary value.

The converse to this theorem also holds ${ }^{2}$, namely, if a solution exists to a given BVP for the Laplace equation, then a solution also exists to the corresponding global relation. Therefore, if a given BVP for the Laplace equation is well-posed, then there exists a solution to the corresponding global relation.

As an aside, we mention that an alternative proof of the existence of the solution to the global relation can be constructed using properties of the operator $T$. In particular, since the range of $T$ is closed, we may apply Banach's closed-range theorem to get:

$$
\begin{equation*}
\operatorname{Ran}(T)=\operatorname{Ker}\left(T^{*}\right)^{\perp}, \tag{5.3.7}
\end{equation*}
$$

where $T^{*}$ denotes the adjoint of $T$, and $\operatorname{Ker}\left(T^{*}\right)^{\perp}$ denotes the orthogonal compliment of $\operatorname{Ker}\left(T^{*}\right)$. Furthermore, it can be shown that the known quantity $T \Psi^{t}$ from the global relation (5.2.26) is in $\operatorname{Ker}\left(T^{*}\right)^{\perp}$, thus proving existence of a solution.

From the results in this section, the following theorem can be derived:

Theorem 5. The solution of the global relation (5.1.18) corresponding to the BVP (5.0.1) exists, is unique, and depends continuously on the Dirichlet data. Moreover, the global relation can be solved to find the (unknown) Neumann data.

Proof. The existence of a solution was discussed above, and the uniqueness of the solution is a consequence of the fact that the operator $T$ is injective. Since the operator $T: \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is bounded (and therefore continuous), this implies that the solution of the global relation depends continuously on the Dirichlet data. Furthermore, since the operator $T$ is continuously invertible on its range $\operatorname{Ran}(T)$, the global relation can be solved to find the unknown Neumann data.

In the chapter that follows we construct a numerical method that uses the global relation to find the Neumann data, given the Dirichlet data. This work implicitly uses the fact that a solution to the global relation exists and that the global relation can be solved to find the Neumann data.

[^9]
### 5.4 Chapter Summary

In this chapter we derived the operator $T$ that allows us to express the global relation for a domain, $\Omega$, with analytic boundary, $\Gamma=\partial \Omega$, in the form:

$$
\begin{equation*}
T(\mathcal{N}-i \mathcal{D})=0 \tag{5.4.1}
\end{equation*}
$$

where $\mathcal{D}$ is a vector that contains the transformed Dirichlet data on each segment of $\Gamma$, and $\mathcal{N}$ is a vector that contains the Neumann data on each segment of $\Gamma$. Further, we showed that $T$ takes the form $T=I+K$ where $I$ is the identity and $K$ is an operator of the form:

$$
\begin{equation*}
(K \Psi)_{j}(\lambda):=\sum_{k \neq j} e^{i e^{-i \alpha_{j}(0)}\left(m_{j}-m_{k}\right) \lambda} \Psi_{k}\left(e^{-i \Delta_{j k}(0)} \lambda\right) \tag{5.4.2}
\end{equation*}
$$

and $\Psi$ is an $n$-dimensional vector that contains the $\mathcal{F}_{p}$-transform of the Dirichlet and Neumann data.

By showing that the operator $K$ takes the form of a Hilbert-Schmidt operator, we showed that the map $\Psi \mapsto K \Psi$ is compact. Since the operator $T$ takes the form $T=I+K$, this implies that $T$ is a Fredholm operator of index zero. Using the fact that $T$ is a Fredholm operator along with properties previously established in the thesis (i.e., that the map $\varphi_{j} \mapsto \Psi_{j}$ is bounded from $L^{2}\left(\Sigma_{j}\right)$ to $L^{2}(\Lambda)$ for each $j \in\{1, \ldots, n\}$ and the invertibility of the $\mathcal{F}_{p}$-transform), we showed that the global relation defines a continuous map between the Dirichlet and Neumann data, and moreover, that if the Dirichlet data is given, then this relation can be solved to find the Neumann data.

## CHAPTER 6

## Numerical Implementation

In the previous chapter we showed that the global relation defines a continuously invertible map between the Dirichlet and Neumann data for a specified BVP for the Laplace equation on a domain with analytic boundary. In this chapter we will verify this fact by constructing a numerical method for determining the Neumann data, given the Dirichlet data, for a specified BVP. This will be done by extending a previous method that was developed by Fornberg and Flyer, [16], for the Laplace equation on polygonal domains. The purpose of extending this method is to verify that previous methods that use the global relation can readily be extended from polygonal domains to domains with analytic boundary using the ideas developed in this thesis. This is the first numerical method for domains with analytic boundary that uses the global relation. As such, it is intended as a preliminary study, and we make more extensive comments on the implementation of the method as well as future directions for research, so as to serve as a reference for future work in this area.

The chapter is organized as follows. First, we review some previously developed numerical methods that have been constructed for polygonal domains. Following this, we construct a numerical method for domains with analytic boundary and show how it performs using three test problems. Finally, we discuss the performance of the method and show that it is particularly well-suited for domains where the boundary has low-
curvature. In addition, we show that the method can also be used to solve BVPs where mixed Dirichlet and Neumann boundary data are given.

### 6.1 Background

We will now briefly outline three approaches that use the global relation to find the unknown boundary data for the Laplace equation. In each case, the method was developed for the Laplace equation on a polygonal domain.

- In [17], Fulton et. al. express the global relation as

$$
\begin{equation*}
\sum_{j=1}^{n} \rho_{j}(\lambda)=0, \quad \lambda \in \mathbb{C} \tag{6.1.1}
\end{equation*}
$$

where the $\rho_{j}(\lambda)$ are defined by

$$
\begin{equation*}
\rho_{j}(\lambda)=\int_{S_{j}} e^{-i \lambda z} u_{z}(z) d z, \quad \text { for } j=1, \ldots, N \tag{6.1.2}
\end{equation*}
$$

where $S_{j}$ denotes the $j$ th side of the polygonal domain, and $u_{z}$ denotes the derivative of the solution, $u$, with respect to the complex variable $z$. The derivative $u_{z}$ can be expressed in terms of the tangential and normal derivatives as

$$
\begin{equation*}
u_{z}=\frac{1}{2} e^{-i \alpha_{j}}\left(u_{\boldsymbol{t}}^{(j)}+i u_{\boldsymbol{n}}^{(j)}\right) \tag{6.1.3}
\end{equation*}
$$

where $\left.u_{\boldsymbol{t}}^{( } j\right)$ denotes the tangential derivative of $u$ on side $j$ and $\left.u_{\boldsymbol{n}}^{( } j\right)$ denotes the normal derivative on side j , and $\alpha_{j}$ is the angle the $j$ th side makes with the real axis. This representation of $u_{z}$ allows the global relation to be expressed in terms of the Dirichlet data, via the tangential derivative, and the Neumann data, via the normal derivative.

The boundary data on each side is expanded in a sine-basis. Since the global relation holds for any value of $\lambda \in \mathbb{C}$, this provides an infinite number of linear relations that the boundary data must satisfy. By specifying that the global relation be satisfied at a finite number of $\lambda$ values, a linear system can be written
in terms of the known and unknown boundary data. In particular, if each side is approximated with $M$ basis functions (where $M$ is taken to be even), then $M / 2$ values of $\lambda$ are chosen along each of the rays $\hat{l}_{j}$ defined by ${ }^{1}$

$$
\begin{equation*}
\hat{l}_{j}=\left\{\lambda \in \mathbb{C}: \arg (\lambda)=\pi-\alpha_{j}\right\}, \quad j \in\{1, \ldots, N\} . \tag{6.1.4}
\end{equation*}
$$

It has been shown that by selecting $\lambda$ values this way, the resulting matrix in the linear system is nearly diagonal.

Choosing $M / 2$ values of $\lambda$ along each of the rays $\hat{l}_{j}$ for $j \in\{1, \ldots, N\}$ yields $N \times M / 2$ complex equations. It is further specified that the real and imaginary parts of these equations must be satisfied. This produces a system of $N \times M$ (real) equations with $N \times M$ unknowns. The system can then be solved for the coefficients of the unknown boundary data in the given sine-basis. This method produces an accuracy of order $10^{-3}$ to $10^{-4}$ with approximately 70 basis functions per side. In addition, numerical tests suggest that the approximation becomes better as the number of sides of the polygon increases.

- In [16], Fornberg and Flyer specify that the solution of the Laplace equation, $u(x, y)$, for a given BVP must satisfy the global relation and Schwarz-conjugate of the global relation. That is, the following equations must be satisfied:

$$
\begin{gather*}
\oint_{\partial \Omega} e^{-i \lambda z}\left(\lambda \frac{d z}{d s} u(s)+u_{\boldsymbol{n}}(s)\right) d s=0  \tag{6.1.5}\\
\oint_{\partial \Omega} e^{i \lambda \bar{z}}\left(\lambda \frac{d \bar{z}}{d s} u(s)-u_{\boldsymbol{n}}(s)\right) d s=0 \tag{6.1.6}
\end{gather*}
$$

where $s$ is the arclength. Again, (6.1.5)-(6.1.6) hold for any value of $\lambda \in \mathbb{C}$, which provides an infinite number of (linear) relations that the boundary data must satisfy. Fornberg and Flyer choose values of $\lambda$ by using Halton nodes ${ }^{2}$, which are semi-random points in $\mathbb{R}^{2}$ that avoid clustering. This ensures that the relations obtained from substituting the $\lambda$ values into (6.1.5)-(6.1.6) will be

[^10]relatively independent of each other.
The known boundary data is then expanded using the Legendre polynomials as a basis. If $M$ basis functions are chosen for each of the $N$ sides of the polygon, then the method entails choosing $K \geq\left\lceil\frac{N}{2}\right\rceil \times M$ values for $\lambda$, with a choice of $K=(N+1) \times M$ suggested. This creates a system that is over-determined, which has the effect of regularizing the solution when the system is solved. The system is then solved for the coefficients of the unknown boundary data in the Legendre basis.

Conveniently, the relevant integrals in the global relation can be computed in closed-form. They take the form:

$$
\begin{equation*}
\int_{-1}^{1} e^{\alpha t} P_{m}(t) d t=\frac{\sqrt{2 \pi \alpha}}{\alpha} I_{m+\frac{1}{2}}(\alpha) \tag{6.1.7}
\end{equation*}
$$

where each side is parametrized in terms of a new variable $t \in[-1,1], P_{m}(t)$ denotes the $m$ th Legendre polynomial, and $I$ denotes the Bessel $I$ function. Since the integrals in (6.1.7) can be found in closed-form, the corresponding matrix system can be quickly assembled. Furthermore, the choice of a Legendre basis causes the numerical method to exhibit spectral convergence.

- In [5], Ashton showed that the global relation for the Laplace equation on a convex polygonal domain can be written in the form

$$
\begin{equation*}
T\left(\Phi^{n}-i \Phi^{t}\right)=0, \quad \lambda \in \mathbb{C} \tag{6.1.8}
\end{equation*}
$$

where $\Phi^{n}$ and $\Phi^{t}$ contain the transformed Neumann and Dirichlet data. This leads to a Galerkin method which is formulated as follows: Given $\Phi^{t} \in X$, for an appropriate space ${ }^{3} X$, find $\Phi \in X$ such that the $j$ th component of $T$ satisfies

$$
\begin{equation*}
T\left(\Phi-i \Phi^{t}\right)_{j}(\lambda)=0, \quad \lambda \in D, \quad j=1, \ldots, N \tag{6.1.9}
\end{equation*}
$$

[^11]where $D$ is any set in $\mathbb{C}$ that contains an accumulation point. This formulation leads to the following functional which is to be minimized:
\[

$$
\begin{equation*}
I[\Phi]=\sum_{j=1}^{N} \int_{\gamma_{j}}\left|T\left(\Phi-i \Phi^{t}\right)_{j}(\lambda)\right|^{2} d s(\lambda), \tag{6.1.10}
\end{equation*}
$$

\]

where the $\gamma_{j}$ are locally finite and semi-infinite curves in $\mathbb{C}$ that cross the negative real-axis, and $d s(\lambda)$ is the Lebesgue measure of arclength on each of the respective curves. Further, it was shown that $\Phi \in X$ is a minimizer of the functional $I[\Phi]$ if and only if

$$
\begin{equation*}
a(\Phi, \Psi)=l(\Psi), \quad \forall \Psi \in X \tag{6.1.11}
\end{equation*}
$$

where $a: X \times X$ and $l: X$ are the bilinear and linear forms given by

$$
\begin{align*}
a(\Phi, \Psi) & =\operatorname{Re} \sum_{j=1}^{N} \int_{\gamma_{j}}(T \Phi)_{j}(\lambda) \overline{(T \Psi)_{j}(\lambda)} d s(\lambda),  \tag{6.1.12}\\
l(\Psi) & =-\operatorname{Im} \sum_{j=1}^{N} \int_{\gamma_{j}}\left(T \Phi^{t}\right)_{j}(\lambda) \overline{(T \Psi)_{j}(\lambda)} d s(\lambda) . \tag{6.1.13}
\end{align*}
$$

A numerical method is created from these results as follows. First, each of the $\Phi_{j}(\lambda)$ is expanded in an appropriate basis $\left\{e_{J}^{(j)}\right\}_{J=0}^{M-1}$ as

$$
\begin{equation*}
\Phi_{j}(\lambda) \approx \sum_{J=0}^{M-1} C_{J}^{(j)} e_{J}^{(j)}(\lambda) \tag{6.1.14}
\end{equation*}
$$

Using $a: X \times X$ and $l: X$ and (6.1.11), the following linear system results:

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{I=0}^{M-1} C_{J}^{(j)} a\left(e_{i, I}, e_{j, J}\right)=l\left(e_{j, J}\right) . \tag{6.1.15}
\end{equation*}
$$

This produces a $N M \times N M$ linear system of equations which can be solved computationally to recover the transformed Neumann data.

In [6], Crooks implemented a Galerkin method based on this formulation using a Legendre basis. Similar to the method of Fornberg and Flyer, this method also
exhibits spectral convergence.

In this chapter we will extend the method of Fornberg and Flyer to the case of a domain with analytic boundary and discuss the resulting convergence properties. In the next section we will go into more detail about how the method is constructed for the case of domains with analytic boundary.

### 6.2 A Collocation Method for domains with Analytic Bound-

## ary

We will now construct a numerical method for domains with analytic boundary by extending the ideas of Fornberg and Flyer. In particular, we will use the idea of oversampling values of $\lambda$ using Halton nodes, and we will also use a similar formulation for the global relation. However, we will use the $\mathcal{F}_{p}$-transform instead of the Fourier transform, and we will also try a Chebyshev polynomial basis and compare the results to the Legendre basis. Since we are not able to directly find the $\mathcal{F}_{p}$-transform of the Legendre polynomials we will need to use numerical integration to compute the relevant integrals.

Equations (6.1.5)-(6.1.6) are a consequence of Green's second identity and also hold for domains with analytic boundary. Therefore, we will use this form of the global relation in our calculations for the numerical method. By reparametrizing with respect to a new variable $t \in[-1,1]$, we may write equations (6.1.5)-(6.1.6) as:

$$
\begin{align*}
& \sum_{j=1}^{N}\left(\int_{-1}^{1} e^{-i \lambda z_{j}}\left(\lambda \frac{d z_{j}}{d t} u^{(j)}(t)+\frac{d s_{j}}{d t} u_{\boldsymbol{n}}^{(j)}(t)\right) d t\right)=0  \tag{6.2.1}\\
& \sum_{j=1}^{N}\left(\int_{-1}^{1} e^{i \lambda \bar{z}_{j}}\left(\lambda \frac{d \bar{z}_{j}}{d t} u^{(j)}(t)+\frac{d s_{j}}{d t} u_{\boldsymbol{n}}^{(j)}(t)\right) d t\right)=0 \tag{6.2.2}
\end{align*}
$$

where $j \in\{1, \ldots, N\}$ denotes the side number of the domain $\Omega$. The Dirichlet and Neumann data must satisfy the above equations for every value of $\lambda$, and hence this gives us an infinite number of (linear) relations that couple the boundary data. By choosing a finite number of $\lambda$ 's for which (6.2.1)-(6.2.2) must be satisfied, we may form
a linear system in terms of the known boundary data and the unknown boundary data.

### 6.2.1 Outline of Numerical Method

We will now outline the method in detail. For the sake of presentation, we assume that the Dirichlet data $u(t)$ is given on each side and that the Neumann data $u_{\boldsymbol{n}}(t)$ is unknown - the method is implemented analogously for other combinations of known and unknown boundary data on each side (see Section 6.6 for examples). The method proceeds as follows:

1. Parametrize each side of the domain with a variable $t \in[-1,1]$. Following this, use the parametrization to compute $d z / d t$ and $d s / d t$ for each side of the domain $\Omega$. For the case of a polygon, the quantities $d z / d t$ and $d s / d t$ are constants. In the case of a domain with analytic boundary, these will be non-constant functions of the variable $t$.
2. Specify the number of basis functions for each side in the approximation. We will denote this number by $M$.
3. Expand the known and unknown boundary data in the specified basis. We will denote the specified basis by $\left\{E_{m}(t)\right\}_{m=1}^{\infty}$.
(a) For the known boundary data, the coefficients can be computed directly, and the expansion will take the form of the sum given below:

$$
\begin{equation*}
u^{(j)}(t) \approx \sum_{m=1}^{M} \alpha_{m}^{(j)} E_{m}(t), \quad j \in\{1, \ldots, N\}, \tag{6.2.3}
\end{equation*}
$$

where $u^{(j)}(t)$ denotes the (known) Dirichlet data on the $j$ th side, and $\left\{\alpha_{m}^{(j)}\right\}_{m=1}^{M}$ are the coefficients for the known boundary data on the $j$ th side.
(b) The unknown boundary data is expanded in the form of the sum given below:

$$
\begin{equation*}
u_{n}^{(j)}(t) \approx \sum_{m=1}^{M} \beta_{m}^{(j)} E_{m}(t), \quad j \in\{1, \ldots, N\}, \tag{6.2.4}
\end{equation*}
$$

where $u_{n}^{(j)}(t)$ denotes the (unknown) Neumann data on the $j$ th side, and $\left\{\beta_{m}^{(j)}\right\}_{m=1}^{M}$ are the (unknown) coefficients for the unknown boundary data.
4. Setup the linear system - this is done in several steps outlined below.
(a) Choose the values of $\lambda$, denoted $\left\{\lambda_{k}\right\}_{k=1}^{K}$, where $K \geq\left\lceil\frac{N}{2}\right\rceil \times M$. The values of $\lambda$ are chosen as shifted and scaled Halton nodes. The standard Halton sequence is defined in $[0,1] \times[0,1] \subset \mathbb{R}^{2}$. We shift the sequence by $-1 / 2$ so as to center it at the origin. The points that lie outside of the unit circle are excluded, and the remainder of the points are indexed by $k \in\{1, \ldots, K\}$. We will denote the $k$ th Halton node inside the unit circle by $H_{k}=\left(a_{k}, b_{k}\right)$. Each of the $H_{k}$ values is then scaled by a parameter $R$ as follows:

$$
\begin{equation*}
H_{k, R}=R\left(a_{k}, b_{k}\right) \tag{6.2.5}
\end{equation*}
$$

The Halton nodes are scaled so as to avoid clustering around the origin (clustering of $\lambda$ values generally leads to more linear dependence in the system which decreases the performance of the method). ${ }^{4}$
(b) Let $\mathrm{RD}^{(j)}$ denote the matrix that contains the integrals corresponding to the Dirichlet data, on the $j$ th side, for the regular global relation. Similarly, $\mathrm{RN}^{(j)}$ is the matrix that contains the integrals corresponding to the Neumann data, on the $j$ th side, for the regular global relation; $\mathrm{SD}^{(j)}$ is the matrix that contains the integrals corresponding to the Dirichlet data, on the $j$ th side, for the Schwarz-conjugate of the global relation; and $\mathrm{SN}^{(j)}$ is the matrix that contains the integrals corresponding to the Neumann data, on the $j$ th side, for the Schwarz-conjugate of the global relation.

Form the matrices $\mathrm{RD}^{(1)}, \ldots, \mathrm{RD}^{(N)}, \mathrm{RN}^{(1)}, \ldots, \mathrm{RN}^{(N)}, \mathrm{SD}^{(1)}, \ldots, \mathrm{SD}^{(N)}$,

[^12]and $\mathrm{SN}^{(1)}, \ldots, \mathrm{SN}^{(N)}$ where each of the components is computed by:
\[

$$
\begin{align*}
\mathrm{RD}_{k, m}^{(j)} & =\int_{-1}^{1} \lambda_{k} e^{-i \lambda_{k} z_{j}} \frac{d z_{j}}{d t} E_{m}(t) d t  \tag{6.2.6}\\
\mathrm{RN}_{k, m}^{(j)} & =\int_{-1}^{1} e^{-i \lambda_{k} z_{j}} \frac{d s_{j}}{d t} E_{m}(t) d t  \tag{6.2.7}\\
\mathrm{SD}_{k, m}^{(j)} & =\int_{-1}^{1} \lambda_{k} e^{i \lambda_{k} \bar{z}_{j}} \frac{d \bar{z}_{j}}{d t} E_{m}(t) d t  \tag{6.2.8}\\
\mathrm{SN}_{k, m}^{(j)} & =\int_{-1}^{1} e^{i \lambda_{k} \bar{z}_{j}} \frac{d s_{j}}{d t} E_{m}(t) d t \tag{6.2.9}
\end{align*}
$$
\]

The dimension of each of the matrices is $K \times M$. The components of these matrices are integrals from the global relation and Schwarz-conjugate of the global relation. In the case of a polygonal domain, each of the integrals above is the Fourier transform of the respective basis function. If the basis functions are chosen to be the Legendre polynomials, the corresponding Fourier transform can be expressed in terms of the Bessel I function using (6.1.7). In the case of a domain with analytic boundary, we must use the $\mathcal{F}_{p}$-transform, and the relation given in (6.1.7) no longer holds.
(c) Write out the linear system in the form:

$$
\boldsymbol{B} \times\left[\begin{array}{c}
\beta_{1}^{(1)}  \tag{6.2.10}\\
\vdots \\
\beta_{M}^{(1)} \\
\vdots \\
\beta_{1}^{(N)} \\
\vdots \\
\beta_{M}^{(N)}
\end{array}\right]=-\boldsymbol{A} \times\left[\begin{array}{c}
\alpha_{1}^{(1)} \\
\vdots \\
\alpha_{M}^{(1)} \\
\vdots \\
\alpha_{1}^{(N)} \\
\vdots \\
\alpha_{M}^{(N)}
\end{array}\right]
$$

where $\boldsymbol{A}$ is the $2 K \times(M \cdot N)$ matrix given by:

$$
\boldsymbol{A}=\left[\begin{array}{ccc}
\mathrm{RD}^{(1)} & \ldots & \mathrm{RD}^{(N)}  \tag{6.2.11}\\
\mathrm{SD}^{(1)} & \ldots & \mathrm{SD}^{(N)}
\end{array}\right]
$$

and $\boldsymbol{B}$ is the $2 K \times(M \cdot N)$ matrix given by:

$$
\boldsymbol{B}=\left[\begin{array}{lll}
\mathrm{RN}^{(1)} & \ldots & \mathrm{RN}^{(N)}  \tag{6.2.12}\\
\mathrm{SN}^{(1)} & \ldots & \mathrm{SN}^{(N)}
\end{array}\right]
$$

5. Solve the (over-determined) linear system for the coefficients of the unknown boundary data. In Matlab this can be done using the back-slash operator. In the case of an over-determined system, the back-slash operator computes the solution to the system by performing a least-squares fit, [27].
6. Expand the unknown boundary data in the specified basis using the computed coefficients - in this case $\left\{\beta_{m}^{(j)}\right\}_{m=1}^{M}$, for each $j \in\{1, \ldots, N\}$.

In the discussion above we assumed that the Dirichlet data, $u(t)$, is given on each side and that the Neumann data $u_{\boldsymbol{n}}(t)$ is unknown. The method proceeds analogously if other combinations of boundary data are given, e.g. the Dirichlet data is given on even sides and the Neumann data is given on odd sides. We will now further discuss the points that will be adapted for the case of domains with analytic boundary.

In the case of a domain with analytic boundary, the complex variable $z$ takes the form:

$$
\begin{equation*}
z=x+i p_{j}(x) \tag{6.2.13}
\end{equation*}
$$

for each side of the boundary in the global relation. In the case where $p_{j}(x)$ is a line, the corresponding integrals in the global relation can be computed directly in closed-form using the Fourier transform. In the case where $p_{j}(x)$ is an analytic concave function, we must use the $\mathcal{F}_{p}$-transform instead of the Fourier transform. As an example, if $p_{j}(x)=$ $x^{2}-1$ for a given domain, and the $j$ th side is parametrized by $z_{j}(t)=t+i\left(t^{2}-1\right)$, then the corresponding integral for $\mathrm{RD}_{k, m}^{(j)}$ is given by:

$$
\begin{equation*}
\operatorname{RD}_{k, m}^{(j)}=\int_{-1}^{1} \lambda_{k} e^{-i \lambda_{k}\left(t+i\left(t^{2}-1\right)\right)}(1+2 t) P_{m}(t) d t \tag{6.2.14}
\end{equation*}
$$

where $P_{m}(t)$ denotes the $m$ th Legendre polynomial. The integral in (6.2.14) is the $\mathcal{F}_{p}$-transform of $P_{m}(t)$, and is not integrable in closed-form. This is generally the case when $p_{j}(x)$ is not a linear function. As a result, the integrals in the global relation
can no longer be computed in closed-form for the Legendre basis, as was the case for a polygonal domain, and we must rely on numerical integration instead.

Since each of the integrals in (6.2.6)-(6.2.9) can no longer be performed explicitly with respect to the Legendre basis, we have more freedom to try other bases as the Legendre polynomials are no longer necessarily the preferred basis. In our implementation we will also use the Chebyshev polynomials as a basis. This will give us a point of comparison for the performance of the Legendre polynomials. The Legendre and Chebyshev polynomials are often used in numerical analysis and can be constructed by orthonormalizing the set of polynomials $\left\{1, x, x^{2}, \ldots\right\}$ on the interval $x \in[-1,1]$ with respect to the $L^{2}$-norm and $L^{1}$-norm, respectively. Hence, the Legendre and Chebyshev polynomials define an orthonormal basis on the interval $[-1,1]$.

For reference, the Legendre polynomials are computed using the following the recursion formula:

$$
\begin{equation*}
(k+1) P_{k+1}(x)=(2 k+1) x P_{k}(x)-k P_{k-1}(x), \quad \text { for } k=1,2, \ldots, \tag{6.2.15}
\end{equation*}
$$

where $P_{0}(x)=1$ and $P_{1}(x)=x$. The coefficients for expansion of a function, $f(x)$, in the Legendre basis are given by:

$$
\begin{equation*}
a_{k}=\frac{2 k+1}{2} \int_{-1}^{1} f(x) P_{k}(x) d x, \quad \text { for } k=0,1,2, \ldots \tag{6.2.16}
\end{equation*}
$$

The Chebyshev polynomials are computed using the recursion formula

$$
\begin{equation*}
T_{k+1}(x)=2 x T_{k}(x)-T_{k-1}(x), \quad k=1,2,3, \ldots, \tag{6.2.17}
\end{equation*}
$$

where $T_{0}(x)=1$ and $T_{1}(x)=x$. The coefficients for expansion of a function, $f(x)$, in the Legendre basis are given by:

$$
\begin{align*}
& a_{0}=\frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^{2}}} d x,  \tag{6.2.18}\\
& a_{k}=\frac{2}{\pi} \int_{-1}^{1} f(x) \frac{T_{k}(x)}{\sqrt{1-x^{2}}} d x, \quad \text { for } k=1,2, \ldots \tag{6.2.19}
\end{align*}
$$

We will use the following three metrics in our discussion for comparing the performance of the method with the Chebyshev and Legendre bases:

- Convergence rate - The convergence rate will be used as a measure of the efficiency of the method with the specified basis.
- $L^{1}$-norm - The $L^{1}$-norm error will be used as a measure of the closeness of fit of the approximation to the true solution along the boundary.
- Sup-norm - The sup-norm error will be used as a measure of the overall accuracy of the method.

In addition to using Chebyshev and Legendre polynomials, we also ran numerical tests with step-functions. The benefit of a step-function basis is that many of the integrals in the global relation can be computed in closed-form, which results in a faster implementation. However, the numerical results for the step-function basis often exhibited a high amount of oscillation that decreased the accuracy of the approximation. This is discussed further in Section 6.4.4, along with a comparison of the Chebyshev and Legendre bases.

### 6.2.2 Notes on Implementation

We will now make several notes on the implementation of the numerical method in Matlab.

- For the given choices of basis functions, i.e.. Legendre and Chebyshev polynomials, the $\mathcal{F}_{p}$-transform cannot be computed directly. As a result we must rely on numerical integration. For our calculations we use the built-in 'integral()' function which performs a numerical integration on a specified function handle. Although the 'integral()' function is efficient, numerical integration is computationally more costly than if the $\mathcal{F}_{p}$-transform of the respective basis functions can be computed directly. Conveniently, the linear system is structured so that each element can be computed independently. As a result, when the method is being implemented, the numerical integration calculations can be distributed over several processors so that they can be performed in parallel. It is straightforward to implement this using the
built-in 'parfor' loop in Matlab. Below is a segment of sample code where this is implemented.

```
for j=1:NumLambdas
    lam = Lambda(j) ;
    parfor m=0:M-1
            RD1(j,m+1)= integral(@(t) lam.*Exp1(t,lam) ...
                .*dzdt1(t).*legendreP(m,t ), -1,1) ;
        RD2(j,m+1)= integral(@(t) -lam.*Exp2(t,lam)...
                .* dzdt2(t) .* legendreP(m,t ), -1,1);
            %Repeat for RN, SD, and SN
        end
end
```

In the code above, the inner loop is executed using the 'parfor' loop in Matlab, which distributes the calculations over the number of processors available.

- The code runs (significantly) more efficiently if the Legendre and Chebyshev polynomials are entered as function handles in Matlab, instead of using the built-in 'legendreP( $\mathrm{m}, \mathrm{t}$ )' and 'chebyshevT( $\mathrm{m}, \mathrm{t}$ )' functions, respectively. The following code uses the recursive formula given in (6.2.15) to construct the appropriate function handles in Matlab, which are then stored in a function array:

```
Legendre=cell(1,M);
syms t;
Legendre{1}=@(t) ones(size(t));
Legendre{2}=@(t) t;
for j=3:M
    Legendre{j}=@(t) (( 2* j + 1)*t.* Legendre{j - 1} (t ) - j *
            Legendre{j-2}(t) )/(j+1);
            Legendre{j}=matlabFunction(simplify(Legendre{j}(t)));
end
```

The analogous code for the Chebyshev polynomials can be written in a similar manner by using the recursive relation given in (6.2.17). In Section 6.4.5 we run numerical tests using these function handles in Matlab to show the increase in efficiency.

As a note, numerical tests suggest that after $M=20$, calculations involving the Matlab function handles defined in this way begin to loose accuracy. This is a result of the fact that computations with polynomials that are in expanded form often loose accuracy due to the fact that calculations are sensitive to perturbations in the coefficients, [19], [37]. For computations that require more than $M=20$ basis functions, the built-in Matlab functions 'legendreP(m,t)' and 'chebyshevT(m,t)' can still be used as they retain the desired accuracy, though the calculations are performed more slowly.

- Each of the rows of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in the linear system need to be scaled ${ }^{5}$. This needs to be done since when solving the system with a non-square matrix with $L^{2}$-minimization, the rows are not automatically scaled as would be the case in methods for solving systems with square matrices, e.g. Gaussian elimination. This significantly reduces the condition number for the matrices in the linear system and is essential for getting accurate results with this method. The code below scales each of the rows of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ by the $l^{1}$-norm in Matlab.

```
scale=1./sum(abs ([A,B]),2);
B = bsxfun(@times,scale,B);
A = bsxfun(@times,scale,A);
```


### 6.3 Test Cases

We will now present three problems on which the numerical method will be tested. In each test case we will use the harmonic function

$$
\begin{equation*}
u(x, y)=e^{x} \cos (y) \tag{6.3.1}
\end{equation*}
$$

[^13]to specify the boundary values for the BVP. The harmonic function $u(x, y)$ given in (6.3.1) is the real part of the entire function $\Phi(z)=e^{z}$, where $z=x+i y$. By using the same harmonic function for each of the test cases, this allows us to restrict our attention to how the performance of the method depends on other factors such as the geometry of the domain. The results of additional tests with different boundary values are given in Appendix A for comparison. In our test cases we will specify that the Dirichlet data is given and that the Neumann data is to be determined. The test cases are presented below.

1. Consider the domain $\Omega_{1}$ enclosed by the curves $\Gamma=\Gamma_{1} \cup \Gamma_{2}$, where

$$
\begin{align*}
& \Gamma_{1}=\left\{z \in \mathbb{C}: z=x+i p_{1}(x), x \in[-1,1]\right\}  \tag{6.3.2}\\
& \Gamma_{2}=\left\{z \in \mathbb{C}: z=x+i p_{2}(x), x \in[-1,1]\right\}, \tag{6.3.3}
\end{align*}
$$

with

$$
\begin{array}{ll}
p_{1}(x)=x^{2}-1, & x \in[-1,1] \\
p_{2}(x)=-x^{2}+1, & x \in[-1,1] . \tag{6.3.5}
\end{array}
$$

The domain $\Omega_{1}$ is shown in figure 6.1 below.


Figure 6.1: Graph of test domain 1.

The parametrization of the domain $\Omega_{1}$ is given by:

$$
\begin{align*}
& z_{1}(t)=t+i\left(t^{2}-1\right)  \tag{6.3.6}\\
& z_{2}(t)=-t+i\left(-t^{2}+1\right), \tag{6.3.7}
\end{align*}
$$

where $t \in[-1,1]$. This parametrization ensures that the boundary of the domain is oriented counter-clockwise. The derivatives $d z / d t$ and $d s / d t$ are given by:

$$
\begin{array}{lll}
\text { Side 1: } & \frac{d z_{1}}{d t}=1+i 2 t & \frac{d s_{1}}{d t}=\sqrt{1+4 t^{2}} \\
\text { Side 2: } & \frac{d z_{2}}{d t}=-1-i 2 t & \frac{d s_{2}}{d t}=\sqrt{1+4 t^{2}} \tag{6.3.9}
\end{array}
$$

Using the parametrization given in (6.3.6)-(6.3.7), the Dirichlet data on each side is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u\left(z_{1}(t)\right)=e^{t} \cos \left(t^{2}-1\right) \\
\text { Side 2: } & u\left(z_{2}(t)\right)=e^{-t} \cos \left(-t^{2}+1\right) \tag{6.3.11}
\end{array}
$$

The Neumann data is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{\mathrm{e}^{t}\left(\sin \left(t^{2}-1\right)+2 t \cos \left(t^{2}-1\right)\right)}{\sqrt{4 t^{2}+1}} \\
\text { Side 2: } & u_{\boldsymbol{n}}\left(z_{2}(t)\right)=\frac{\mathrm{e}^{t}\left(\sin \left(t^{2}-1\right)+2 t \cos \left(t^{2}-1\right)\right)}{\sqrt{4 t^{2}+1}} \tag{6.3.13}
\end{array}
$$

The results for Test Case 1 are presented in Section 6.4.1.
2. Consider the domain $\Omega_{2}$ defined by $\Gamma=\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3} \cup \Gamma_{4}$, where

$$
\begin{equation*}
\Gamma_{j}=\left\{z \in \mathbb{C}: z=x+i p_{j}(x), x \in\left[a_{j}, b_{j}\right]\right\}, \quad j=1, \ldots, 4, \tag{6.3.14}
\end{equation*}
$$

with the functions $\left\{p_{j}(x)\right\}_{j=1}^{4}$ defined by:

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & \\
p_{2}(x)=\sqrt{\frac{25}{4}-x^{2}}-\frac{3}{2}, & \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & \\
p_{4}(x)=[-2,0]  \tag{6.3.18}\\
p_{4}\left(x \in \sqrt{\frac{25}{4}-x^{2}}+\frac{3}{2},\right. & \\
x \in[0,2] .
\end{array}
$$

The domain $\Omega_{2}$ is shown in figure 6.2 below.


Figure 6.2: Graph of test domain 2.

We will use the following (counter-clockwise oriented) parametrization for the domain $\Omega_{2}$ :

$$
\begin{align*}
& z_{1}(t)=(1-t)+i\left(-\frac{1}{4}(1-t)^{2}+1\right)  \tag{6.3.19}\\
& z_{2}(t)=-(1+t)+i\left(\sqrt{\frac{25}{4}-(1+t)^{2}}-\frac{3}{2}\right)  \tag{6.3.20}\\
& z_{3}(t)=(t-1)+i\left(\frac{1}{4}(t-1)^{2}-1\right)  \tag{6.3.21}\\
& z_{4}(t)=(t+1)+i\left(-\sqrt{\frac{25}{4}-(t+1)^{2}}+\frac{3}{2}\right) \tag{6.3.22}
\end{align*}
$$

where $t \in[-1,1]$ for each side. The derivatives $d z / d t$ and $d s / d t$ are given by:

$$
\begin{align*}
\frac{d z_{1}}{d t} & =-1+\mathrm{i} \frac{1}{2}(1-t) & \frac{d s_{1}}{d t} & =\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}  \tag{6.3.23}\\
\frac{d z_{2}}{d t} & =-1-\mathrm{i}\left(\frac{t+1}{\sqrt{\frac{25}{4}-(t+1)^{2}}}\right) & \frac{d s_{2}}{d t} & =\sqrt{1-\frac{(2 t+2)^{2}}{4\left((t+1)^{2}-\frac{25}{4}\right)}}  \tag{6.3.24}\\
\frac{d z_{3}}{d t} & =1+\mathrm{i} \frac{1}{2}(t-1) & \frac{d s_{3}}{d t} & =\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}  \tag{6.3.25}\\
\frac{d z_{4}}{d t} & =1-\mathrm{i}\left(\frac{t+1}{\sqrt{\frac{25}{4}-(t+1)^{2}}}\right) & \frac{d s_{4}}{d t} & =\sqrt{1-\frac{(2 t+2)^{2}}{4\left((t+1)^{2}-\frac{25}{4}\right)}} \tag{6.3.26}
\end{align*}
$$

Using the parametrization given in (6.3.19)-(6.3.22), the Dirichlet data on each side is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u\left(z_{1}(t)\right)=e^{1-t} \cos \left(\frac{(t-1)^{2}}{4}-1\right) \\
\text { Side 2: } & u\left(z_{2}(t)\right)=e^{-(1+t)} \cos \left(\sqrt{\frac{25}{4}-(t+1)^{2}}-\frac{3}{2}\right) \\
\text { Side 3: } & u\left(z_{3}(t)\right)=e^{t-1} \cos \left(\frac{(t-1)^{2}}{4}-1\right) \\
\text { Side 4: } & u\left(z_{4}(t)\right)=e^{t+1} \cos \left(\sqrt{\frac{25}{4}-(t+1)^{2}}+\frac{3}{2}\right) \tag{6.3.30}
\end{array}
$$

The Neumann data is given by:

$$
\begin{align*}
& u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{\mathrm{e}^{t} \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}-\frac{\mathrm{e}^{t} \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}  \tag{6.3.31}\\
& u_{\boldsymbol{n}}\left(z_{2}(t)\right)=\frac{1}{5} e^{t}\left(\sqrt{-4 t^{2}-8 t+21} \sin \left(p_{2}(t)\right)-2(t+1) \cos \left(p_{2}(t)\right)\right),  \tag{6.3.32}\\
& u_{\boldsymbol{n}}\left(z_{3}(t)\right)=\frac{\mathrm{e}^{t} \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}+\frac{\mathrm{e}^{t} \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}  \tag{6.3.33}\\
& u_{\boldsymbol{n}}\left(z_{4}(t)\right)=\frac{1}{5} e^{t}\left(\sqrt{-4 t^{2}-8 t+21} \sin \left(p_{4}(t)\right)-2(t+1) \cos \left(p_{4}(t)\right)\right) \tag{6.3.34}
\end{align*}
$$

where $p_{2}(t)$ and $p_{4}(t)$ are the imaginary components of $z_{2}(t)$ and $z_{4}(t)$, respectively,
i.e.

$$
\begin{equation*}
p_{2}(t)=\sqrt{\frac{25}{4}-(t+1)^{2}}-\frac{3}{2} \quad \text { and } \quad p_{4}(t)=\sqrt{\frac{25}{4}-(t+1)^{2}}+\frac{3}{2} . \tag{6.3.35}
\end{equation*}
$$

3. Consider the domain defined by $\Gamma=\Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3} \cup \Gamma_{4}$, where

$$
\begin{equation*}
\Gamma_{j}=\left\{z \in \mathbb{C}: z=x+i p_{j}(x), x \in\left[a_{j}, b_{j}\right]\right\}, \quad j=1, \ldots, 4, \tag{6.3.36}
\end{equation*}
$$

with

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\frac{1}{2} x+1, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=\frac{1}{2} x-1, & x \in[0,2] . \tag{6.3.40}
\end{array}
$$

The domain $\Omega_{3}$ is shown in figure 6.3 below.


Figure 6.3: Graph of test domain 3.

We will use the following (counter-clockwise oriented) parametrization for the domain $\Omega_{3}$ :

$$
\begin{align*}
& z_{1}(t)=(1-t)+i\left(-\frac{1}{4}(1-t)^{2}+1\right)  \tag{6.3.41}\\
& z_{2}(t)=-(1+t)+i\left(-\frac{1}{2}(1+t)+1\right)  \tag{6.3.42}\\
& z_{3}(t)=(t-1)+i\left(\frac{1}{4}(t-1)^{2}-1\right)  \tag{6.3.43}\\
& z_{4}(t)=(t+1)+i\left(\frac{1}{2}(t+1)-1\right) \tag{6.3.44}
\end{align*}
$$

where $t \in[-1,1]$ for each side. The derivatives $d z / d t$ and $d s / d t$ are given by:

$$
\begin{align*}
\frac{d z_{1}}{d t} & =-1+\mathrm{i} \frac{1}{2}(1-t) & \frac{d s_{1}}{d t} & =\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}  \tag{6.3.45}\\
\frac{d z_{2}}{d t} & =-1-\frac{1}{2} \mathrm{i} & \frac{d s_{2}}{d t} & =\sqrt{\frac{5}{4}}  \tag{6.3.46}\\
\frac{d z_{3}}{d t} & =1+\mathrm{i} \frac{1}{2}(t-1) & \frac{d s_{3}}{d t} & =\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}  \tag{6.3.47}\\
\frac{d z_{4}}{d t} & =1+\frac{1}{2} \mathrm{i} & \frac{d s_{4}}{d t} & =\sqrt{\frac{5}{4}} .
\end{align*}
$$

Using the parametrization given in (6.3.41)-(6.3.44), the Dirichlet data on each side is given by:

Side 1: $\quad u\left(z_{1}(t)\right)=e^{1-t} \cos \left(\frac{(t-1)^{2}}{4}-1\right)$,
Side $2: \quad u\left(z_{2}(t)\right)=e^{-(1+t)} \cos \left(\frac{t}{2}-\frac{1}{2}\right)$,
Side 3: $\quad u\left(z_{3}(t)\right)=e^{t-1} \cos \left(\frac{(t-1)^{2}}{4}-1\right)$,
Side 4: $\quad u\left(z_{4}(t)\right)=e^{t+1} \cos \left(\frac{t}{2}-\frac{1}{2}\right)$.

The Neumann data is given by:

$$
\begin{align*}
& u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{\mathrm{e}^{t} \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}-\frac{\mathrm{e}^{t} \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}  \tag{6.3.53}\\
& u_{\boldsymbol{n}}\left(z_{2}(t)\right)=-\mathrm{e}^{t} \sin \left(\frac{t}{2}-C\right),  \tag{6.3.54}\\
& u_{\boldsymbol{n}}\left(z_{3}(t)\right)=\frac{\mathrm{e}^{t} \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}+\frac{\mathrm{e}^{t} \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}  \tag{6.3.55}\\
& u_{\boldsymbol{n}}\left(z_{4}(t)\right)=\mathrm{e}^{t} \cos \left(\frac{t}{2}-C\right), \tag{6.3.56}
\end{align*}
$$

where $C$ is the constant given by:

$$
\begin{equation*}
C=\frac{17359532051249999}{18014398509481984} \tag{6.3.57}
\end{equation*}
$$

Since two of the sides for $\Omega_{3}$ (i.e., $p_{2}(t)$ and $\left.p_{4}(t)\right)$ are linear, we are able to compute the corresponding integrals in the system directly. The relevant integrals from the global relation are computed by:

$$
\begin{equation*}
\int_{-1}^{1} e^{\alpha_{j} t} P_{m}(t) d t=\frac{\sqrt{2 \pi \alpha_{j}}}{\alpha_{j}} I_{m+\frac{1}{2}}\left(\alpha_{j}\right), \quad j=2,4 \tag{6.3.58}
\end{equation*}
$$

where $\alpha_{2}$ and $\alpha_{4}$ are given by:

$$
\begin{align*}
& \alpha_{2}=\lambda_{k}\left[-\frac{1}{2}+i\right]  \tag{6.3.59}\\
& \alpha_{4}=\lambda_{k}\left[\frac{1}{2}-i\right] \tag{6.3.60}
\end{align*}
$$

In the next section, we will run the numerical method on the three test cases that we have introduced here and discuss the corresponding convergence results.

### 6.4 Results and Discussion

We will now present the results for each of the test cases from the previous section. For each test case we specify that the Dirichlet data is given and that the Neumann data is unknown. In each case we show a convergence plot and discuss the performance of the method.

In our convergence plots and discussion, we use the relative sup-norm error defined by:

$$
\begin{equation*}
E_{\infty}(M)=\frac{\left\|f-f_{M}\right\|_{\infty}}{\|f\|_{\infty}} \tag{6.4.1}
\end{equation*}
$$

where $f$ is the true solution and $f_{M}$ is the $M$ th approximation in the specified basis, and the $\|\cdot\|_{\infty}$-norm is given by:

$$
\begin{equation*}
\|g\|_{\infty}=\max _{1 \leq j \leq N}\left\{\max _{-1 \leq t \leq 1}\left|g^{(j)}(t)\right|\right\} \tag{6.4.2}
\end{equation*}
$$

where $g^{(j)}(t)$ is the specified function on the $j$ th side.

### 6.4.1 Test Case 1

We will now present the numerical results for Test Case 1 from Section 6.3. Recall, for Test Case 1, the boundary of the domain is given by the curves:

$$
\begin{array}{ll}
p_{1}(x)=x^{2}-1, & x \in[-1,1] \\
p_{2}(x)=-x^{2}+1, & x \in[-1,1] \tag{6.4.4}
\end{array}
$$

In order to run the method, we must specify $K$, the number of values of $\lambda$ to be chosen in the system, and $R$, the radius of the circle within which the $\lambda$ values are taken. We choose these parameters by running numerical tests for a range of $R$ and $K$ values. In each test we compute the condition number for the matrix $\boldsymbol{B}$. We use this as a measure of numerical sensitivity since the system is solved by inverting the matrix $\boldsymbol{B}$, and the condition number is a measure of the invertibility of a matrix. The table below shows the condition number of the matrix $\boldsymbol{B}$ for values of $R$ ranging from
$R=10$ to $R=40$ and values of $K$ ranging from $K=25$ to $K=50$. Each test is run with $M=15$ basis functions.

|  | $R=10$ | $R=15$ | $R=20$ | $R=25$ | $R=30$ | $R=35$ | $R=40$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $K=30$ | 544.4730 | 544.4730 | 544.4730 | 544.4730 | 544.4730 | 544.4730 | 544.4730 |
| $K=35$ | 40.2908 | 40.2908 | 40.2908 | 40.2908 | 40.2908 | 40.2908 | 40.2908 |
| $K=40$ | 129.4977 | 129.4977 | 129.4977 | 129.4977 | 129.4977 | 129.4977 | 129.4977 |
| $K=45$ | 114.2604 | 114.2604 | 114.2604 | 114.2604 | 114.2604 | 114.2604 | 114.2604 |
| $K=50$ | 318.0930 | 318.0930 | 318.0930 | 318.0930 | 318.0930 | 318.0930 | 318.0930 |
| $K=55$ | 507.3613 | 507.3613 | 507.3613 | 507.3613 | 507.3613 | 507.3613 | 507.3613 |

Table 6.1: Condition Number of the matrix $\boldsymbol{B}$ for a range of $R$ and $K$ values with $M=15$ basis functions per side

Table 6.1 shows that there is little variation in the condition number of $\boldsymbol{B}$ for varying values of $R$, so we choose $R=30$. There is, however, variation with respect to $K$, and in practice we choose a value of $K=45$ for $M=15$ basis functions as it corresponds with the suggested choice of $K=(N+1) \times M$, and the corresponding condition number is relatively lower than for other values of $K$. In general, for Test Case 1, we use $K=(N+1) \times M$ for all the values of $M$ tested.

Using these parameters, we run the method to find the coefficients of the unknown boundary data for Test Case 1. A plot of the Neumann boundary data on each side is shown below, along with the Legendre approximations for $M=5$ and $M=8$.


Figure 6.4: Plot of solution for Test Case 1.

We note from the plot in Figure 6.4 that the approximation is quite accurate even for relatively low values of $M$. In order to quantify this, the convergence plot for Test Case 1 is shown below.


Figure 6.5: Plot of error for Test Case 1.

In Figure 6.5 we plot the convergence for the method using both a Legendre and Chebyshev basis. We will now make several observations about the results for Test Case 1.

- The error for both the Legendre and Chebyshev bases decreases linearly on a loglinear plot. This indicates that the method exhibits spectral convergence. For a numerical method that exhibits spectral convergence, the error can be expressed as a function of the number of basis functions, $M$, in the form:

$$
\begin{equation*}
E_{\infty}(M)=e^{-C M}, \tag{6.4.5}
\end{equation*}
$$

where $C$ is the rate of convergence. We compute $C$ by finding the slope of the best-fit line for the Legendre and Chebyshev error on the log-linear plot. The
values are given by:

$$
\begin{align*}
C_{\mathrm{Leg}} & =0.1811  \tag{6.4.6}\\
C_{\mathrm{Cheb}} & =0.1827, \tag{6.4.7}
\end{align*}
$$

where $C_{\text {Leg }}$ is the convergence rate for the Legendre basis and $C_{\text {Cheb }}$ is the convergence rate for the Chebyshev basis. We note that the values of $C_{\text {Leg }}$ and $C_{\text {Cheb }}$ are quite similar, with the relative difference between the values being only $\sim .5 \%$.

- Despite the fact that $C_{\text {Leg }}$ and $C_{\text {Cheb }}$ are quite close in value, the Chebyshev basis produces a noticeably more accurate approximation for every value of $M$ that is tested for this problem. This suggests that the Chebyshev basis is a better choice for this test case. We further compare the Chebyshev and Legendre bases in Section 6.4.4.
- In this test case the method exhibits spectral accuracy and produces an error of order $10^{-5}$ with $M=30$ basis functions. We will show later, in Section 6.5 , that for domains with lower curvature, even better convergence rates and accuracy can be obtained. For a point of comparison, in Appendix B, we run the numerical method on a sequence of domains defined by:

$$
\begin{equation*}
p_{n}^{(j)}=\frac{(-1)^{j}}{n} x^{2}+\frac{(-1)^{j+1}}{n}, \quad n \in\{1, \ldots, 4\}, j=1,2 \tag{6.4.8}
\end{equation*}
$$

These domains are analogous to the one for this test case and have the property that each successive domain has lower curvature. For each of the successive domains, the convergence rate improves. A similar set of tests is carried out in Section 6.5 to show a relation between convergence rate and the curvature of the domain.

- For the final value of $M=30$ in Figure 6.5 , the error for the Chebyshev approximation is $E_{\infty}(30)=2.140 \times 10^{-6}$, and the error for the Legendre approximation is $E_{\infty}(30)=4.472 \times 10^{-6}$.

We will now plot the error against $t \in[-1,1]$ for $M=10$ in Figure 6.6.


Figure 6.6: Plot of error for Test Case 1.

We observe that the method produces a good approximation for the majority of the $t$-interval with the error being noticeably higher at the endpoints $t=-1$ and $t=1$. For larger values of $M$, the largest errors also tend to be at the endpoints of each segment of the boundary. This results in a higher overall sup-norm error for both the Chebyshev and Legendre bases, as reported on the convergence plot in Figure 6.5. In general, the largest errors tend to be at the endpoints of the $t$-intervals in all of the tests that we have run.

We also observe that the error for the Chebyshev and Legendre bases is quite similar for most of the $t$-interval, away from the endpoints. Near the endpoints, there is a more significant difference between the approximations for the two bases, with the Chebyshev basis giving more accurate results. This results in a lower sup-norm error for the Chebyshev basis. Additional tests show that for larger values of $M$, i.e. $M \geq 10$, the Chebyshev basis continues to give a more accurate approximation at the endpoints of the boundary. The convergence plot in Figure 6.5 reflects this fact, showing that the Chebyshev basis is roughly a third of an order of magnitude more
accurate for $M \geq 10$.

### 6.4.2 Test Case 2

We will now present the numerical results for Test Case 2 from Section 6.3. Recall, for Test Case 2, the boundary of the domain is given by the curves:

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\sqrt{\frac{25}{4}-x^{2}}-\frac{3}{2}, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=-\sqrt{\frac{25}{4}-x^{2}}+\frac{3}{2}, & x \in[0,2] . \tag{6.4.12}
\end{array}
$$

Performing numerical tests similar to those in Table 6.1 for a range of $M$ values, we find again that $R=30$ and $K=(N+1) \times M$ are good choices for the parameters $R$ and $K$. Using these choices for $R$ and $K$, we run the method for Test Case 2 with the Legendre and Chebyshev bases on a range of $M$ values and show the corresponding error in the convergence plot below.


Figure 6.7: Plot of error for Test Case 2.

We make the following observations about the error plot in Figure 6.7:

- The method again exhibits spectral convergence for Test Case 2. The convergence rates for the Legendre and Chebyshev bases are given by:

$$
\begin{align*}
C_{\mathrm{Leg}} & =0.3714  \tag{6.4.13}\\
C_{\text {Cheb }} & =0.3024 . \tag{6.4.14}
\end{align*}
$$

- There is a significant difference between the convergence rates for the Chebyshev and Legendre bases for this test case, i.e. $\sim 18.58 \%$ relative difference. This is due to the fact that the Chebyshev basis exhibits noticeably higher errors for low values of $M$, i.e., $M<10$. Despite this, for larger values of $M$, i.e $M \geq 10$, the Chebyshev basis gives a better approximation.
- The convergence rates for both the Legendre and Chebyshev bases are greater than in Test Case 1, indicating faster convergence. In addition, the error for each value of $M$ is less than that for Test Case 1. This is observed in the fact that the method reaches $O\left(10^{-5}\right)$ accuracy with $M=30$ basis functions in Test Case 1 , while it only requires $M=20$ basis functions to reach the same order of accuracy for Test Case 2. This is due to two factors:
- First, the geometry of the domain is such that each side has lower curvature than the domain for Test Case 1. The dependence of the method on curvature is further explored in Section 6.5. In general, the method tends to perform better for domains with lower curvature.
- Second, there are more components that form the boundary of the domain (i.e., four sides in Test Case 2 versus two sides in Test Case 1). It was observed in [17] that for polygonal domains the numerical method gave more accurate results for polygons with more sides. A similar phenomenon also appears to apply with this numerical method.
- For the final value of $M=20$ in Figure 6.9, the error for the Chebyshev approximation is $E_{\infty}(20)=1.8917 \times 10^{-6}$ and the error for the Legendre approximation is $E_{\infty}(20)=2.8022 \times 10^{-6}$.

We will now plot the error against $t \in[-1,1]$ for $M=10$ in Figure 6.8.


Figure 6.8: Plot of error for Test Case 2.

We observe that for the majority of the domain, the error for the Legendre and Chebyshev bases is quite similar. We will use the $L^{1}$-norm as a measure of the closeness of fit of the approximations to the true solution. This, in turn, will allow us to quantify how close the approximations for the two bases are to each other along the $t$-interval. We define the $L^{1}$-norm error by:

$$
\begin{equation*}
E_{L^{1}}(M)=\frac{1}{N} \sum_{j=1}^{N}\left(\int_{-1}^{1}\left|f^{(j)}(t)-f_{M}^{(j)}(t)\right| d t\right) \tag{6.4.15}
\end{equation*}
$$

where $f^{(j)}(t)$ is the true solution on the $j$ th side, and $f_{M}^{(j)}(t)$ is the $M$ th approximation on the $j$ th side. For $M=10, E_{L^{1}}(10)=1.1319 \times 10^{-4}$ for the Legendre basis and $E_{L^{1}}(10)=1.1045 \times 10^{-4}$ for the Chebyshev basis. The $L^{1}$-norm for the two bases is also close for larger values of $M$. This indicates that both bases give similar approximations. As a notable exception to this, for larger values of $M$, i.e. particularly for $M \geq 15$, the approximation in the Chebyshev basis exhibits lower errors at the endpoints of each of the segments of the boundary. This results in a lower error in the sup-norm,
as exhibited in the convergence plot in Figure 6.7.

### 6.4.3 Test Case 3

In this section we present the numerical results for Test Case 3 from Section 6.3. Recall, for Test Case 3 , the boundary of the domain is given by the curves:

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\frac{1}{2} x+1, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=\frac{1}{2} x-1, & x \in[0,2] \tag{6.4.19}
\end{array}
$$

We again perform numerical tests similar to those in Table 6.1 for a range of $M$ values and find that $R=30$ and $K=(N+1) \times M$ are good choices for the parameters $R$ and $K$. Using these choices for $R$ and $K$, we run the method for Test Case 3 with the Legendre and Chebyshev bases on a range of $M$ values and show the corresponding error in the convergence plot below.


Figure 6.9: Plot of error for Test Case 3.

We make the following observations about the error plot in Figure 6.9:

- Similar to Test Cases 1 and 2, the error for both the Legendre and Chebyshev bases decreases linearly on a log-linear plot, indicating that the method also exhibits spectral convergence for Test Case 3. The convergence rates for the Legendre and Chebyshev bases are given by:

$$
\begin{align*}
C_{\text {Leg }} & =0.7917  \tag{6.4.20}\\
C_{\text {Cheb }} & =0.7438 . \tag{6.4.21}
\end{align*}
$$

- The convergence rates for the Chebyshev and Legendre bases are similar, i.e. $\sim 6.05 \%$ relative difference, however, the convergence plot indicates that for small $M$, i.e. $M<7$, the Legendre basis gives a better approximation. For larger values of $M$, i.e $M \geq 7$, the error for the Chebyshev and Legendre approximations are quite close.
- For this domain two of the sides (i.e., $p_{2}(x)$ and $\left.p_{4}(x)\right)$ are linear which allows the corresponding integrals in the global relation to be computed directly. This allows the computations for the Legendre bases to be performed more efficiently. Since the Chebyshev and Legendre bases give similar errors for $M \geq 7$, the Legendre basis is a better choice for this test case.
- We observe that the error plot for Test Case 1 exhibits a more consistent pattern of convergence than for Test Cases 2 and 3 . We conjecture that this is due to the fact that the geometry of the domain is more symmetric for Test Case 1 with $p_{2}(x)$ being the reflection of $p_{1}(x)$ across the real-axis. In Appendix A we run additional tests on the same domains with different boundary values and find similar patterns of convergence for each domain.
- For the final value of $M=15$ in Figure 6.9, the error for the Chebyshev approximation is $E_{\infty}(15)=5.7614 \times 10^{-9}$ and the error for the Legendre approximation is $E_{\infty}(15)=8.0837 \times 10^{-9}$

We will now plot the error against $t \in[-1,1]$ for $M=10$ in Figure 6.10.


Figure 6.10: Plot of error for Test Case 3.

Similar to Test Case 2, the error is similar for both the Chebyshev and Legendre bases along the $t$-interval. Indeed, for the plots in Figure 6.10, it is difficult to distinguish between the errors for the two bases. Computing the $L^{1}$-norm errors, we find that for the Legendre basis $E_{L^{1}}(10)=7.4549 \times 10^{-6}$, and for the Chebyshev basis $E_{L^{1}}(10)=7.3345 \times 10^{-6}$. There is a $\sim 1.62 \%$ relative difference between the $L^{1}$-norm errors for the two bases, which confirms that the Chebyshev and Legendre polynomials give a similar approximation along the $t$-interval. For larger values of $M$, this trend continues and the relative difference between the $L^{1}$-norm errors is consistently less than $3 \%$.

We observe from Figure 6.10 above that for $M=10$, the Chebyshev and Legendre bases also give a similar approximation at the endpoints of the $t$-interval. This is also the case for larger values of $M$ as well. The convergence plot in Figure 6.9 reflects this fact, showing that the sup-norm errors are close for $M \geq 7$. As a notable exception to this, for the final value $M=15$, the Chebyshev basis gives a better approximation at the endpoints, which is reflected in the sup-norm error as shown in the convergence
plot in Figure 6.9.

### 6.4.4 Comparison of Basis Functions

Throughout Sections 6.4.1-6.4.3 we used the Chebyshev and Legendre polynomials as a basis for our numerical method. To date, Legendre polynomials have given the best approximations in numerical methods involving the global relation, [6], [16], [23]. Our tests suggest that the Chebyshev polynomials provide a good point of comparison for the Legendre basis. In general, the convergence properties and errors were similar for both bases, with the Chebyshev polynomials performing better in some cases. This indicates that the Chebyshev polynomials are a good alternative to the Legendre basis in the case of domains with analytic boundary, and are perhaps even preferable. We will now justify this further by summarizing and comparing the convergence data for the Chebyshev and Legendre bases for Test Cases 1-3.

Recall, in Section 6.2.1 we specified that the Chebyshev and Legendre bases would be compared using three metrics: the convergence rate, the $L^{1}$-norm, and the supnorm. In Table 6.2, we summarize the data for each of these metrics. In the table we record $\log _{10}\left(E_{\infty}(M)\right)$ and $\log _{10}\left(E_{L^{1}}(M)\right)$ for the final value of $M$ tested in each case.

|  | Test Case 1 |  | Test Case 2 |  | Test Case 3 |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Leg. | Cheb. | Leg. | Cheb. | Leg. | Cheb. |
| Conv. | 0.1811 | 0.1827 | 0.3714 | 0.3024 | 0.7917 | 0.7438 |
| Rate |  |  |  |  |  |  |

Table 6.2: Convergence data for the Chebyshev and Legendre bases.

We will now briefly discuss each of the metrics using the data from Table 6.2, along with previous observations.

- Convergence Rate - We use the convergence rate as a measure of the efficiency of the method with the specified basis. For Test Case 1, the relative difference between the convergence rates for the two bases is less than $1 \%$, indicating similar convergence. However, as we noted from Figures 6.5 and 6.6 , the Chebyshev basis gives a better approximation near the endpoints of the boundary segments, $\Gamma_{j}$, which results in
consistently lower errors as measured by the sup-norm. Therefore, the sup-norm is a better metric for this test case.

The difference between the convergence rates for the Chebyshev and Legendre bases for Test Cases 2-3 is considerably greater than for Test Case 1. This is accounted for by the fact that the Chebyshev basis has noticeably higher errors for low values of $M$ (i.e., $M<10$ for Test Case 2 and $M<7$ for Test Case 3). If we adjust for this fact and compute the convergence rates for larger values of $M$, then we find that there is a higher amount of agreement between the bases for Test Case 3 while the Chebyshev basis performs better for Test Case 2. For $M \geq 7$ in Test Case 3, the Legendre basis has a convergence rate of $C_{\mathrm{Leg}}=0.6923$, and the Chebyshev basis has a convergence rate of $C_{\text {Cheb }}=0.6998$. This gives a relative difference of $\sim 1.07 \%$. A similar calculation for Test Case 2 shows that there is a $\sim 6.96 \%$ relative difference between the convergence rates for $M \geq 10$, with the method being more efficient with the Chebyshev basis for these values of $M$. This can be observed in the convergence plot in Figure 6.7.

We conclude that the Chebyshev and Legendre bases are comparable for this metric, with the Chebyshev basis sometimes performing better for larger values of $M$.

- $L^{1}$-norm - We use the $L^{1}$-norm as a measure of the closeness of fit of the approximation to the true solution. This, in turn, allows us to quantify how close the approximations for the two bases are to each other. As we saw in Figures 6.6, 6.8, and 6.10, the error for the two bases exhibited a high amount of agreement (to each other) for the majority of the $t$-interval for each $\Gamma_{j}$, often overlapping so as to be indistinguishable on the graph. Correspondingly, we consistently found that for each test case, the $L^{1}$-errors were quite close in value for the two bases (e.g., less that $3 \%$ relative difference in Test Case 3 for $M \geq 10$ ). This indicates that the two bases give a similar approximation for most of the $t$-interval.

To further demonstrate this, we show a convergence plot with the $L^{1}$-errors for Test Case 2 in Figure 6.11. We choose Test Case 2 because for lower values of $M$, the Chebyshev and Legendre bases exhibited the greatest differences in the sup-norm among all the test cases.


Figure 6.11: Plot of $L^{1}$-error for Test Case 2.

Figure 6.11 shows that the $L^{1}$-errors for the two bases are quite close in value, even for $M \leq 10$. This indicates that there is a high amount of agreement between the two approximations along the interval $t \in[-1,1]$.

We conclude that the Chebyshev and Legendre polynomials are comparable for this metric, and both give an equally accurate approximation for the majority of the $t$-interval for each $\Gamma_{j}$.

- Sup-norm - We use the sup-norm as a measure of the overall accuracy of the method. Of the three metrics, the most noticeable differences between the two bases are exhibited with the sup-norm error. In Test Case 1 the difference between the supnorm errors for the two bases was particularly pronounced. As we saw in Figure 6.6, the Chebyshev polynomials give a considerably more accurate approximation at the endpoints of each $\Gamma_{j}$ for $M=10$. This resulted in a lower sup-norm error for this value of $M$. The same trend is also observed for larger values of $M$, and as a result, the sup-norm error is consistently a third of an order of magnitude more accurate for the Chebyshev basis for this test case. This can also be observed for the sup-norm values given in Table 6.2.

For Test Cases 2-3, the Legendre basis produced lower sup-norm errors for smaller values of $M$. However, for larger values of $M$, the sup-norm errors for the two bases
were closer with the Chebyshev basis exhibiting lower errors in both test cases. Again, this was a result of the fact that the Chebyshev basis gave a better approximation at the endpoints of each $\Gamma_{j}$. This is also reflected in the sup-norm values in Table 6.2. We conclude that the Chebyshev basis generally gives more accurate results for larger values of $M$, and is therefore a better choice based on this metric

Combining our analysis of the three metrics, we find that the two bases give equally good approximations for the majority of the $t$-interval on each $\Gamma_{j}$. However, the Chebyshev basis generally gives a more accurate approximation at the endpoints of each segment of the boundary, which gives more accurate results as measured by the sup-norm. We therefore propose that the Chebyshev polynomials be used as the default basis for computations on domains with analytic boundary. A notable exception to this is when one or more of the sides of the domain is linear, such as in Test Case 3. In such cases, the Legendre basis is preferred as the relevant integrals can be computed in closed-form which makes the method more efficient.

Among the bases that we have experimented with, the Chebyshev and Legendre polynomials give the best convergence rates and lowest errors. In the remainder of this section, we will briefly mention two other bases that have the potential to be useful in future implementations.

In addition to the Chebyshev and Legendre polynomials, we also performed preliminary tests with a step function basis. The benefit of using a step function basis is that many of the integrals in the global relation can be done in closed-form. To see this, we will approximate $u^{(j)}(t)$ with step functions as follows:

$$
\begin{equation*}
u^{(j)}(t) \approx \sum_{m=1}^{M} \alpha_{m}^{(j)} 1_{A_{m}}, \tag{6.4.22}
\end{equation*}
$$

Using the approximation given in (6.4.22), the corresponding integral for the Dirichlet component of the global relation is approximated by the following integrals:

$$
\begin{align*}
\int_{-1}^{1} \lambda e^{-i \lambda z_{j}} \frac{d z_{j}}{d t} u^{(j)}(t) d t & \approx \int_{-1}^{1} \lambda e^{-i \lambda z_{j}} \frac{d z_{j}}{d t}\left(\sum_{m=1}^{M} \alpha_{m}^{(j)} 1_{A_{m}}\right) d t  \tag{6.4.23}\\
& =\sum_{m=1}^{M}\left(\int_{-1}^{1} \lambda e^{-i \lambda z_{j}} \frac{d z_{j}}{d t} \alpha_{m}^{(j)} 1_{A_{m}} d t\right) . \tag{6.4.24}
\end{align*}
$$

Now the relevant integrals in the sum given in (6.4.24) can be computed in closed-form as follows:

$$
\begin{align*}
\int_{-1}^{1} \lambda e^{-i \lambda z_{j}} \frac{d z_{j}}{d t} \alpha_{m}^{(j)} 1_{A_{m}} d t & =\alpha_{m}^{(j)} \int_{a_{m}}^{b_{m}} \lambda e^{-i \lambda z_{j}} \frac{d z_{j}}{d t} d t  \tag{6.4.25}\\
& =i \alpha_{m}^{(j)}\left[e^{-i \lambda z_{j}\left(b_{m}\right)}-e^{-i \lambda z_{j}\left(a_{m}\right)}\right] \tag{6.4.26}
\end{align*}
$$

As a note, the corresponding integrals for the Neumann boundary data are not similarly integrable in closed-form, due to the presence of the $d s_{j} / d t$ term. Nevertheless, the numerical method runs significantly faster with a step function basis since the integrals for the Dirichlet data can be computed directly in closed-form.

Unfortunately, there are several issues with the accuracy of the numerical method with a step function basis. For low values of $M$ the method gives a good approximation, however, for higher values of $M$ there is a high amount of oscillation with the frequency and amplitude of the oscillation increasing as $M$ increases. In Figure 6.12 below we plot the $M=8$ and $M=10$ approximations in the step function basis along with a standard interpolation.

## Solution for Side 1



Solution for Side 2


Figure 6.12: Plot of solution for step function basis with $M=8$ and $M=10$.

In this numerical test we used the harmonic function

$$
\begin{equation*}
u(x, y)=x y \tag{6.4.27}
\end{equation*}
$$

so that there is less oscillation in the true solution $\partial u / \partial \boldsymbol{n}$, thus allowing us to focus on the oscillations in the approximation that are an artifact of the step function basis. We observe from Figure 6.12 that for $M=8$ the approximation is relatively close to the true solution (given in blue). The approximation for $M=10$, on the other hand, has a higher amount of oscillation with an increased amplitude. For larger values of $M$, the amplitude and frequency of the oscillation increases. This is typical of the numerical tests that we have run with a step function basis. We note, however, that it may be possible to increase the accuracy of the method by choosing the evaluation points at the zeros of the Chebyshev polynomials (i.e., Chebyshev nodes) or other points that will minimize the oscillation. Chebyshev nodes are often used to reduce oscillations in approximations to continuous functions by polynomial interpolation, [18], and may also help to reduce the oscillation in approximating the solution $\partial u / \partial \boldsymbol{n}$. We will experiment with this further in future work.

In addition, we also performed initial tests with the basis $\left\{Q_{m}^{(j)}(t)\right\}_{m=1}^{M}$ which is defined on the $j$ th side of the boundary $\partial \Omega$ and is given by:

$$
\begin{align*}
Q_{m}^{(j)}(t) & =e^{i z_{j}(t)} \frac{d}{d t}\left[t^{m} e^{-i z_{j}(t)}\right]  \tag{6.4.28}\\
& =m t^{m-1}-i \frac{d z_{j}}{d t} t^{m} \tag{6.4.29}
\end{align*}
$$

where $j \in\{1, \ldots, N\}$. This basis has the feature that the $\mathcal{F}_{p}$-transform of each of the basis functions can be computed in closed-form. However, there are additional terms in the global relation, i.e. $d z_{j} / d t$ for the Dirichlet part and $d s_{j} / d t$ for the Neumann part, that do not allow the relevant integrals in the global relation to be integrated directly. Tests using numerical integration suggest that the method also exhibits spectral convergence with this basis, although the convergence rate is less than for Chebyshev and Legendre polynomials. We mention this basis because it will likely prove useful for other applications that involve the $\mathcal{F}_{p}$-transform.

### 6.4.5 Runtime of Numerical Method

In Section 6.2.2 we mentioned two adaptations that make the numerical method more efficient. Recall, these adaptations were a parallel computing implementation and the use of polynomial function-handles instead of the corresponding built-in Matlab functions. In this section we will show how each these adaptations affects the runtime of the numerical method. This will be done by running the numerical method under three configurations which are listed below:

- Configuration 1: Numerical integration of the basis functions without the use of parallel computing and using the built-in Matlab function 'legendreP(m,t)'.
- Configuration 2: Numerical integration of the basis functions with the parallel computing implementation on 4 processors and using the built-in Matlab function 'legendreP (m,t)'.
- Configuration 3: Numerical integration with the parallel computing implementation on 4 processors and using polynomials for Legendre $\{m\}(t)$ as functionhandles in Matlab.

We will test how each of the adaptations mentioned above affects the runtime of the numerical method by running the program under Configurations 1-3 on the domain and boundary conditions given in Test Case 2. In Figure 6.13 on the next page, the runtime is plotted on a regular graph as well as a log-linear graph for each of the configurations. Each of the computations are performed with the Legendre basis. Similar runtimes are also exhibited for the Chebyshev basis. Each of the tests is run on a standard desktop computer with a 2.41 GHz Intel Pentium quad-core processor.


Plot of runtime vs. number of basis functions.


Plot of runtime vs. number of basis functions on a log-linear graph.
Figure 6.13: Plot of runtime vs. number of basis functions for Test Case 2 under three configurations, using a Legendre basis.

In Table 6.3 below we list the runtimes for $M=20$ basis functions for each of the configurations.

| Configuration | Time (sec.) | Time (hrs. $/ \mathrm{min})$. | $\log _{10}($ Time $(\mathrm{sec}))$. |
| :---: | :---: | :---: | :---: |
| 1 | 9763.01 | 3 hrs .12 min. | 4.06 |
| 2 | 3148.00 | 61 min .32 sec. | 3.57 |
| 3 | 92.48 | 1 min .53 sec. | 2.05 |

Table 6.3: Runtime for Configurations 1-3

The increase in efficiency for each of the successive configurations is apparent from Figure 6.13, as well as the data in Table 6.3. Indeed, the runtime for Configuration 1 is two orders of magnitude faster than that for Configuration 3. In future implementations we will experiment with using Gauss-Legendre and Clenshaw-Curtis integration. These numerical integration techniques use Legendre and Chebyshev polynomials, respectively, to approximate the integrand of the relevant integral, and often perform faster than standard numerical integration methods. Furthermore, since the integrals from the global relation have smooth integrands (in each case the integrand is an exponential multiplied by a Chebyshev or Legendre polynomial), the Gauss-Legendre and Clenshaw-Curtis integration methods will converge quickly.

Finally, we note that in our implementation, 4 processors were used in Configurations 2-3. For real-world applications in Engineering and Physics, significantly more computational resources will likely be available which will increase the performance even further.

### 6.5 Dependence on Curvature

In this section we will test the hypothesis that the curvature of the domain affects the rate of convergence of the numerical method. This will be done by running the method on a series of domains with decreasing curvature. Each of the domains that we consider will be formed from two circular arcs, one forming the top half of the domain and the other forming the bottom half of the domain. Since each of the segments that forms the boundary of these domains has constant curvature, this will allow us to more precisely understand the relationship between the curvature of the boundary and the convergence of the numerical method. In particular, we will use these experiments to
derive a relation between the curvature of the domain and the convergence rate of the method. For comparison, we also include additional tests on a series of domains with non-constant curvature in Appendix B.

We will now define five domains on which the numerical method will be tested. For each domain we also give the radius of curvature and curvature. Recall the relation:

$$
\begin{equation*}
R=\frac{1}{\kappa}, \tag{6.5.1}
\end{equation*}
$$

where $R$ is the radius of curvature and $\kappa$ is the curvature.

- Domain A: The boundary of Domain A is given by the curves:

$$
\begin{array}{ll}
p_{1}^{A}(x)=-\sqrt{\frac{25}{16}-x^{2}}+\frac{3}{4}, & x \in[-1,1], \\
p_{2}^{A}(x)=\sqrt{\frac{25}{16}-x^{2}}-\frac{3}{4}, & x \in[-1,1] . \tag{6.5.3}
\end{array}
$$

The radius of curvature for Domain A is $R=5 / 4$, and the curvature is $\kappa=4 / 5$.

- Domain B: The boundary of Domain B is given by the curves:

$$
\begin{array}{ll}
p_{1}^{B}(x)=-\sqrt{4-x^{2}}+\sqrt{3}, & x \in[-1,1], \\
p_{2}^{B}(x)=\sqrt{4-x^{2}}-\sqrt{3}, & x \in[-1,1] . \tag{6.5.5}
\end{array}
$$

The radius of curvature for Domain B is $R=2$, and the curvature is $\kappa=1 / 2$.

- Domain C: The boundary of Domain C is given by the curves:

$$
\begin{array}{ll}
p_{1}^{C}(x)=-\sqrt{9-x^{2}}+\sqrt{8}, & x \in[-1,1], \\
p_{2}^{C}(x)=\sqrt{9-x^{2}}-\sqrt{8}, & x \in[-1,1] . \tag{6.5.7}
\end{array}
$$

The radius of curvature for Domain C is $R=3$, and the curvature is $\kappa=1 / 3$.

- Domain D: The boundary of Domain D is given by the curves:

$$
\begin{array}{ll}
p_{1}^{D}(x)=-\sqrt{16-x^{2}}+\sqrt{15}, & x \in[-1,1], \\
p_{2}^{D}(x)=\sqrt{16-x^{2}}-\sqrt{15}, & x \in[-1,1] . \tag{6.5.9}
\end{array}
$$

The radius of curvature for Domain D is $R=4$, and the curvature is $\kappa=1 / 4$.

- Domain E: The boundary of Domain E is given by the curves:

$$
\begin{array}{ll}
p_{1}^{E}(x)=-\sqrt{25-x^{2}}+\sqrt{24}, & x \in[-1,1], \\
p_{2}^{E}(x)=\sqrt{25-x^{2}}-\sqrt{24}, & x \in[-1,1] . \tag{6.5.11}
\end{array}
$$

The radius of curvature for Domain E is $R=5$, and the curvature is $\kappa=1 / 5$.

In Figure 6.14 we plot the error for the numerical method for each of the test domains given above and note the trend that results. In each case we will assign boundary values using the harmonic function $u(x, y)=e^{x} \cos (y)$, and we use a Chebyshev basis.


Figure 6.14: Plot of error for Domains A-E.

It is apparent from Figure 6.14 that the convergence rate for each of the successive test cases improves. A similar trend is also observed in Appendix B for domains with non-constant (but also decreasing) curvature. In order to quantify these observations, we record the convergence rate and error for the final value of $M$ tested, denoted by $M_{\text {final }}$, for each of the test cases in this section. These values are given in Table 6.4 below. ${ }^{6}$

| Domain | $R$ | $\kappa$ | Convergence Rate | $\log _{10}\left(E_{\infty}\left(M_{\text {final }}\right)\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| A | $5 / 4$ | $4 / 5$ | -0.1981 | -5.70 |
| B | 2 | $1 / 2$ | -0.5575 | -10.15 |
| C | 3 | $1 / 3$ | -0.7923 | -11.81 |
| D | 4 | $1 / 4$ | -0.8923 | -12.23 |
| E | 5 | $1 / 5$ | -1.046 | -13.02 |

Table 6.4: Curvature data for Domains A-E

In Figure 6.15 below we plot the convergence rate against the curvature $(\kappa)$ and observe that there is a linear relation.


Figure 6.15: Plot of convergence rate vs. curvature for Domains A-E.

By performing a least-squares regression, we find the following relation between

[^14]the convergence rate and the curvature of the domain:
\[

$$
\begin{equation*}
C=1.354 \kappa-1.261 \tag{6.5.12}
\end{equation*}
$$

\]

where $C$ denotes the convergence rate. The $R$-squared value for this regression is $98.75 \%$, which indicates that the linear model is a good fit for the data.

This gives a good heuristic for understanding how the curvature of the domain affects the convergence rate of the method. Furthermore, we conclude that the numerical method is particularly well-suited for domains for which the boundary has low curvature. Indeed, even for a domain for which the curvature is $\kappa=1 / 2$, the method produces an accuracy of order $10^{-10}$ with less than $M=20$ basis functions.

It is important to note that there are a number of other factors that will also affect the convergence rate of the method. Among these are the number of components in the domain, the symmetry of the domain, and the regularity of the given boundary data. In addition, if the curvature of the domain is non-constant then this will also alter the relation between curvature and convergence (though, in general, we expect that the convergence rate will continue to improve as the curvature decreases, for an example see Appendix B).

### 6.6 Mixed Boundary Conditions

Thus far we have shown that the numerical method developed in this chapter works for the Dirichlet problem on domains with analytic boundary - that is, given the Dirichlet data on each side of a domain, we have used the method to find the (unknown) Neumann data. We will now show that the numerical method can also be used for boundary value problems where mixed boundary data is given. That is, if the Dirichlet data is given on a subset of the sides, and the Neumann data is given on the remainder of the sides, then the numerical method can be used to find the unknown boundary data for each side. We will demonstrate this by solving two boundary value problems where mixed boundary conditions are given.

In our tests we will use the domain for Test Case 2 for the first boundary value problem and the domain from Test Case 3 for the second boundary value problem. We
specify the boundary problems below and show the corresponding convergence plots. In each case, the harmonic function $u(x, y)=e^{x} \cos (y)$ is used to specify the boundary values.

- BVP I Recall, the domain for Test 2 was given by:

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\sqrt{\frac{25}{4}-x^{2}}-\frac{3}{2}, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=-\sqrt{\frac{25}{4}-x^{2}}+\frac{3}{2}, & x \in[0,2] . \tag{6.6.4}
\end{array}
$$

We will solve the Laplace equation on this domain with Dirichlet data specified on the odd sides and Neumann data specified on even sides. This is shown in Figure 6.16 below.


Figure 6.16: Plot of domain for BVP I.

The corresponding convergence plot for BVP I is given in Figure 6.17 on the following page. We use the Chebyshev polynomials as a basis since they produce a more accurate approximation (as measured by the sup-norm) for larger values of $M$.


Figure 6.17: Error plot for BVP I.

Comparing the convergence plot in Figure 6.17 with Figure 6.7 from Section 6.4.2, we find that the method produces a similar pattern of convergence to the case where the Dirichlet data is specified on all sides for this domain. In fact, a further comparison of the data shows that for low values of $M$ (i.e., $M \leq 10$ ), the mixed BVP produces lower errors than the analogous Dirichlet problem. This results in a faster convergence rate for the mixed BVP. The convergence rate as well as the error for the final value of $M=20$ are given in Table 6.5.

- BVP II Recall, the domain for Test 3 was given by:

$$
\begin{array}{ll}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\frac{1}{2} x+1, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=\frac{1}{2} x-1, & x \in[0,2] \tag{6.6.8}
\end{array}
$$

We will solve the Laplace equation on this domain with Dirichlet data specified on the even sides and Neumann data specified on odd sides. This is shown in Figure 6.18 on the following page.


Figure 6.18: Plot of domain for BVP II.

The corresponding convergence plot for BVP II is given in Figure 6.19 below. Again, we use the Chebyshev polynomials as a basis for the method since they produce a more accurate sup-norm approximation for larger values of $M$.


Figure 6.19: Error plot for BVP II.

Comparing the convergence plot in Figure 6.19 with Figure 6.9 from Section 6.4.3, we find that the method produces a similar convergence pattern to the case where the Dirichlet data is specified on all sides for this domain. Similar to BVP I, we find that for this test the errors are as accurate as for the Dirichlet problem. Indeed, for the final value of $M=15$, the error is lower for the mixed BVP than for the Dirichlet problem, as shown in Table 6.5.

For comparison, the convergence rates and sup-norm errors for the two tests above, as well as for the corresponding Dirichlet problems from Sections 6.4.2 and 6.4.3, are given in Table 6.5 below. In the table, $M_{\text {final }}$ denotes the final value of $M$ that was tested in each case. The data for the Chebyshev basis is reported for each entry. Domain I denotes the domain for BVP I, and similarly, Domain II denotes the domain for BVP II.

|  | Conv. Rate |  | $\log _{10}\left(E_{\infty}\left(M_{\text {final }}\right)\right)$ |  |
| :--- | :---: | :---: | :---: | :---: |
|  | Dirich. | Mixed | Dirich. | Mixed |
| Domain | 0.3024 | 0.3493 | -5.7231 | -5.7372 |
| I |  |  |  |  |
| Domain <br> II | 0.7438 | 0.7349 | -8.2395 | -8.5674 |
|  |  |  |  |  |

Table 6.5: Convergence data for mixed boundary value tests.

We observe from the data in Table 6.5 that for mixed boundary value problems, the method exhibits similar convergence properties to the analogous Dirichlet problem on the specified domain. For both BVP I and BVP II, the sup-norm error for $M_{\text {final }}$ is the same order of magnitude for both the Dirichlet and mixed BVPs. Indeed, in both cases, the error is lower for the mixed BVP than for the Dirichlet problem. For BVP I, we note that for lower values of $M$, the method gives more accurate results for the mixed BVP than for the analogous Dirichlet problem on the same domain. This is reflected in the fact that the convergence rate for the mixed BVP is higher than for the Dirichlet BVP. Hence, the method actually performs better for the mixed BVP problem on this domain.

We conclude that the method is also effective for solving boundary value problems where mixed data is given, and the convergence properties are similar to those for the analogous Dirichlet problem.

### 6.7 Summary and Discussion

In this chapter we have constructed a numerical method that uses the global relation to find the unknown boundary data for a specified BVP for the Laplace equation on a domain with analytic boundary. This was done by extending the method of Fornberg and Flyer, found in [16], which was developed for polygonal domains. Similar to the original method, we observed spectral convergence when using the Legendre polynomials as a basis.

Unlike the case of polygonal domains, the relevant integrals in the global relation cannot be computed in closed-form with the Legendre basis. This is due to the fact that the $\mathcal{F}_{p}$-transform of the Legendre polynomials, in general, cannot be computed in closed-form when $p(x)$ is not a linear function. As a result, we had more freedom to experiment with other bases. In particular, we found that the Chebyshev polynomials were a good point of comparison. We found that the Chebyshev polynomials gave an approximation that was very similar to the Legendre basis (as measured by the $L^{1}$-norm) for each segment of the boundary $\Gamma_{j}$. However, in every case we found that for larger values of $M$, the Chebyshev polynomials gave a better approximation at the endpoints of each $\Gamma_{j}$. This resulted in a more accurate approximation as measured in the sup-norm. We therefore recommend that the Chebyshev polynomials be used as the default basis when computing on domains with analytic boundary. A notable exception to this is when one or more of the sides is linear. In this case, the Legendre polynomials are the preferred basis as the relevant integrals in the global relation can be computed directly, resulting in a more efficient implementation.

In order to counter the additional computational cost of numerical integration, we devised several strategies for decreasing the runtime of the method. In particular, by exploiting the structure of the system, we were able to use parallel computing to reduce the computational runtime. We also note that since the system can be constructed in parallel, this is an ideal method for applications in engineering and physics where a larger amount of computational resources are likely to be available. In Section 6.4.5, we observed that our implementation was able to run on Test Case 2 in $\sim 2$ minutes with a standard desktop computer. It is likely that with even marginally more computational resources, the method will be able to run in a matter of seconds
on problems similar to those in this chapter.
Additionally, we found that the method is more efficient for domains with lower curvature. This was shown in Section 6.5 by comparing the convergence rates for a series of domains with decreasing curvature. Since each of the domains was formed from circular arcs (which have constant curvature), we were able to derive a relation between the curvature of the domain and the convergence rate for the corresponding Dirichlet problem. We found for this case that the convergence rate increased linearly as the curvature of the domain decreased. This gives us a rough heuristic for understanding the relation between curvature and the convergence rate of the numerical method. Hence, we conclude that the method is particularly effective for domains with low curvature.

The method developed in this chapter gives a practical demonstration of the fact that the global relation for the Laplace equation on domains with analytic boundary can be solved to find the Neumann data, given the Dirichlet data. Hence, the tests in this chapter confirm Theorem 5 from Chapter 5. To the author's knowledge, this is the first numerical method for domains with analytic boundary that uses the global relation. As we have seen, it is an effective method for computation on such domains, exhibiting spectral convergence with the use of Chebyshev and Legendre polynomials. We anticipate the method will be useful in practical applications due to the potential for parallel implementation. Further research is required to adapt the method so that it is equally efficient on domains with higher curvature.

## CHAPTER 7

## Conclusion

In this thesis we studied the boundary value problem given by:

$$
\begin{cases}\Delta u(\mathbf{x})=0 & \text { for } \mathbf{x} \in \Omega  \tag{7.0.1}\\ u(\mathbf{x})=f(\mathbf{x}) & \text { for } \mathbf{x} \in \partial \Omega\end{cases}
$$

where $u \in C^{2}(\Omega) \cap C^{1}(\bar{\Omega})$ and $f \in C^{1}(\Gamma)$ is given (Dirichlet) boundary data on $\partial \Omega$. If both the Dirichlet and Neumann data associated to (7.0.1) are known, then the solution to the BVP can be written using a standard representation formula. Therefore, we focused our attention on developing a method to find the unknown Neumann data. This was done by means of the global relation, which couples the Dirichlet and Neumann data for the Laplace equation, and is given by:

$$
\begin{equation*}
\int_{\partial \Omega} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left(i(\boldsymbol{\lambda} \cdot \boldsymbol{n}) u(\boldsymbol{x})+\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x})\right) d \sigma(\boldsymbol{x})=0, \quad \text { for } \boldsymbol{\lambda} \in Z_{p}, \tag{7.0.2}
\end{equation*}
$$

where $Z_{p}=\left\{\boldsymbol{\lambda} \in \mathbb{R}^{2}: \Delta(\boldsymbol{\lambda})=0\right\}$. To date, in the context of elliptic PDEs in $\mathbb{C} \cong \mathbb{R}^{2}$, the global relation has primarily been used for the analysis and solution of boundary value problems on polygonal domains [5], [12], [17]. To extend the global relation to more general domains with analytic boundary, we introduced the $\mathcal{F}_{p}$-transform,
defined by:

$$
\begin{equation*}
\mathcal{F}_{p}: \varphi \mapsto \mathcal{F}_{p}(\varphi)(\lambda)=\int_{-\sigma}^{\sigma} e^{-i \lambda x+\lambda p(x)} \varphi(x)\left(1+i p^{\prime}(x)\right) d x \tag{7.0.3}
\end{equation*}
$$

where $p(x)$ is an analytic concave function that satisfies $p(0)=p^{\prime}(0)=0$.
In the first part of the thesis, we focused on the analysis of the $\mathcal{F}_{p}$-transform and proved the following the properties:

- The map $\varphi \rightarrow \mathcal{F}_{p}(\varphi)(\lambda)$ is bounded from $L^{2}([-\sigma, \sigma])$ to $L^{2}([0, \infty])$. This was shown in Chapter 3 by proving the estimate:

$$
\begin{equation*}
\int_{0}^{\infty}|\tilde{\varphi}(\lambda)|^{2} d \lambda \lesssim \int_{-\sigma}^{\sigma}|\varphi(x)|^{2} d x . \tag{7.0.4}
\end{equation*}
$$

- For $\varphi \in L^{2}([-\sigma, \sigma])$, there exists an inverse to the $\mathcal{F}_{p}$-transform, and it is given by:

$$
\begin{equation*}
\varphi(x)=-\left(I-L_{\operatorname{Re}}\right)^{-1}\left\{\operatorname{Re}\left[\frac{i-p^{\prime}(x)}{\pi i} \int_{\gamma_{3}} e^{i \lambda x-\lambda p(x)} \tilde{\varphi}(\lambda) d \lambda\right]\right\}, \tag{7.0.5}
\end{equation*}
$$

where $\gamma_{3}=\{\lambda \in \mathbb{C}: \lambda=t[1-i p(x)], t \in(0, \infty)\}$, and the integral operator $L_{\mathrm{Re}}: L^{2}([-\sigma, \sigma]) \rightarrow L^{2}([-\sigma, \sigma])$ is defined by:

$$
\begin{equation*}
L_{\operatorname{Re}} \varphi=\int_{-\sigma}^{\sigma}\left[\frac{[p(x)-p(y)]-p^{\prime}(x)[x-y]}{\pi\left([x-y]^{2}+[p(x)-p(y)]^{2}\right)}\right] \varphi(y) d y . \tag{7.0.6}
\end{equation*}
$$

Moreover, by the Banach bounded inverse theorem, since the $\mathcal{F}_{p}$-transform is bounded, it follows that $\mathcal{F}_{p}^{-1}$ is also bounded. The derivation of the inverse formula was done in Chapter 4.

Following our study of the $\mathcal{F}_{p}$-transform, in Chapter 5 we derived an operator $T$ that allowed us to express the global relation in the form:

$$
\begin{equation*}
T\left(\Psi^{n}-i \Psi^{t}\right)=0, \tag{7.0.7}
\end{equation*}
$$

where $\Psi^{n}$ is a vector that contains the $\mathcal{F}_{p}$-transform of the Neumann data, and $\Psi^{t}$ is
a vector that contains the $\mathcal{F}_{p}$-transform of the Dirichlet data. Using results from the literature and the first part of the thesis, we established the following properties for the operator $T$ :

- The operator $T$ takes the form $T=I+K$, where $K$ is a compact operator. This implied that $T$ is a Fredholm operator of index zero. The proof of this result followed from the convexity of the domain $\Omega$ and the fact that $\mathcal{F}_{p}^{-1}$ is bounded.
- The operator $T$ is bounded below. This followed from the fact that $T$ is a Fredholm operator of index zero along with the injectivity of the operator $T$.

From these properties, we derived the following result:

- A solution of the global relation (7.0.7) corresponding to the BVP (7.0.1) exists, is unique, and depends continuously on the Dirichlet data. Moreover, the global relation can be solved to find the (unknown) Neumann data.

Thus, the work presented in this thesis establishes that the global relation can be used to recover the Neumann data for a specified BVP for the Laplace equation on a domain with analytic boundary. More generally, this work extends the Fokas method and global relation to more general domains than those that have been considered to date. In so doing, it opens a new approach for the study of boundary value problems for the Laplace equation, and other elliptic PDEs, on domains with analytic boundary.

Finally, in Chapter 6, we constructed a numerical method by extending the method of Fornberg and Flyer found in [16]. Through our study of the numerical method, we observed the following:

- The method exhibits spectral convergence when using either the Chebyshev or Legendre polynomials as a basis. Furthermore, we found that the method was particular effective on domains for which the boundary has low curvature.
- The Chebyshev basis tends to produce lower errors in the sup-norm due to the fact that it gives a more accurate approximation at the endpoints of each segment of the domain. We therefore recomend that the Chebyshev polynomials be used as the default basis for future implementations.
- The method is equally effective for solving BVPs where mixed Dirichlet and Neumann boundary data is given.
- Through the use of parallel computing, the runtime of the numerical method can be improved significantly.

To the author's knowledge, this is the first numerical method that uses the global relation on domains with analytic boundary. Furthermore, the work done in Chapter 6 demonstrates that the ideas developed in this thesis allow previously constructed numerical methods, that use the global relation, to readily be extended to more general domains.

We conclude by mentioning an area for future research. In this thesis we studied the Laplace equation on domains with analytic boundary. The next step would be to extend the results derived here to more general elliptic PDEs. We mention the following BVP as an example:

$$
\begin{align*}
-\Delta u(\boldsymbol{x})+\beta^{2} u(\boldsymbol{x}) & =0, & & \boldsymbol{x} \in \Omega  \tag{7.0.8}\\
u(\boldsymbol{x}) & =f(\boldsymbol{x}), & & \boldsymbol{x} \in \partial \Omega, \tag{7.0.9}
\end{align*}
$$

which corresponds to the modified Helmholtz equation if $\beta$ is real, and the Helmholtz equation if $\beta$ is imaginary. Similar to the Laplace equation, the global relation for this BVP is given by:

$$
\begin{equation*}
\int_{\partial \Omega} e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}\left(i(\boldsymbol{\lambda} \cdot \boldsymbol{n}) u(\boldsymbol{x})+\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x})\right) d \sigma(\boldsymbol{x})=0, \quad \text { for } \boldsymbol{\lambda} \in Z_{p}, \tag{7.0.10}
\end{equation*}
$$

where $Z_{p}=\left\{\boldsymbol{\lambda} \in \mathbb{R}^{2}: P(\boldsymbol{\lambda})=0\right\}$, and $P(D)$ is the symbol of the operator corresponding to the PDE given in (7.0.8), i.e. $P(D)=D^{2}+\beta^{2}$, where $D=-i \partial$. The set $Z_{p}$ can be described by a local parametrization as:

$$
\begin{equation*}
\lambda_{1}=\frac{\beta}{2}\left(k-\frac{1}{k}\right) \quad \text { and } \quad \lambda_{2}=i \frac{\beta}{2}\left(k+\frac{1}{k}\right) \quad \text { for } k \in \mathbb{C}, \tag{7.0.11}
\end{equation*}
$$

where $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}\right)$. The corresponding exponential term in the $\mathcal{F}_{p}$-transform is written as:

$$
\begin{equation*}
e^{-i \boldsymbol{\lambda} \cdot \boldsymbol{x}}=e^{-i\left(\lambda_{1} x+\lambda_{2} p(x)\right)}, \tag{7.0.12}
\end{equation*}
$$

and the analysis proceeds in a similar manner. The corresponding operator $T$, denoted
by $T(\beta)$, can be written in the form:

$$
\begin{equation*}
T(\beta)=T(0)+K(\beta), \tag{7.0.13}
\end{equation*}
$$

where $T(0)$ is the operator for the Laplace equation, and $K(\beta)=T(\beta)-T(0)$. By showing that $K(\beta): \boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$ is compact, this will imply that $T(\beta)$ is a compact perturbation of $T(0)$, and a similar procedure to the analysis for the Laplace equation can be applied. We therefore propose that the methods developed in this thesis be used to study the Helmholtz and modified Helmholtz equations in future research.

## APPENDIX A

## Additional Test Cases

In this Appendix we run the numerical method on the domains for Test Cases 1 and 3 from Chapter 6 using two different sets of boundary data. These additional tests are included so that the performance of the numerical method can be compared for boundary values other than those used in Chapter 6. Recall, the domains for Test Cases 1 and 3 are given by:

- Domain 1: The boundary of Domain 1 is given by the curves:

$$
\begin{align*}
& p_{1}(x)=x^{2}-1  \tag{A.0.1}\\
& p_{2}(x)=-x^{2}+1 \tag{A.0.2}
\end{align*}
$$

- Domain 3: The boundary of Domain 3 is given by the curves:

$$
\begin{array}{lr}
p_{1}(x)=-\frac{1}{4} x^{2}+1, & x \in[0,2] \\
p_{2}(x)=\frac{1}{2} x+1, & x \in[-2,0] \\
p_{3}(x)=\frac{1}{4} x^{2}-1, & x \in[-2,0] \\
p_{4}(x)=\frac{1}{2} x-1, & x \in[0,2] . \tag{A.0.6}
\end{array}
$$

We will run tests by assigning boundary data using the two harmonic functions given below.

- Function A: The harmonic function A is given by:

$$
\begin{equation*}
u_{A}(x, y)=x y \tag{A.0.7}
\end{equation*}
$$

- Function B: The harmonic function B is given by:

$$
\begin{equation*}
u_{B}(x, y)=\cosh (x) \cos (y) \tag{A.0.8}
\end{equation*}
$$

In each of the numerical tests that follows, we will specify that the Dirichlet data is given and that the Neumann data is to be determined. We will run each of the tests with both the Chebyshev and Legendre bases.

## A. 1 Domain 1

We will now present the convergence results for Domain 1 with the boundary data specified by the harmonic functions $u_{A}(x, y)$ and $u_{B}(x, y)$.

## A.1.1 Test 1A

In this test case, we will assign boundary data on Domain 1 using the harmonic function $u_{A}(x, y)$. The Dirichlet data on each side for this test case is given by:

Side 1: $\quad u\left(z_{1}(t)\right)=t\left(t^{2}-1\right)$
Side 2: $\quad u\left(z_{2}(t)\right)=-t\left(-t^{2}+1\right)$.

The Neumann data is given by:

Side 1: $\quad u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{2 t\left(t^{2}-1\right)}{\sqrt{4 t^{2}+1}}-\frac{t}{\sqrt{4 t^{2}+1}}$
Side 2: $\quad u_{\boldsymbol{n}}\left(z_{2}(t)\right)=\frac{t}{\sqrt{4 t^{2}+1}}-\frac{2 t\left(t^{2}-1\right)}{\sqrt{4 t^{2}+1}}$.

Below is a convergence plot for the harmonic function $u_{A}(x, y)$ on Domain 1.


Figure A.1: Plot of error for Test Case 1A.

## A.1.2 Test 1B

In this test case, we will assign boundary data on Domain 1 using the harmonic function $u_{B}(x, y)$. The Dirichlet data on each side for this test case is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u\left(z_{1}(t)\right)=\cosh (t) \cos \left(t^{2}-1\right) \\
\text { Side 2: } & u\left(z_{2}(t)\right)=\cosh (-t) \cos \left(-t^{2}+1\right) \tag{A.1.6}
\end{array}
$$

The Neumann data is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{\sin \left(t^{2}-1\right) \cosh (t)}{\sqrt{4 t^{2}+1}}+\frac{2 t \sinh (t) \cos \left(t^{2}-1\right)}{\sqrt{4 t^{2}+1}} \\
\text { Side 2: } & u_{\boldsymbol{n}}\left(z_{2}(t)\right)=\frac{\sin \left(t^{2}-1\right) \cosh (t)}{\sqrt{4 t^{2}+1}}+\frac{2 t \sinh (t) \cos \left(t^{2}-1\right)}{\sqrt{4 t^{2}+1}} \tag{A.1.8}
\end{array}
$$

Below is a convergence plot for the harmonic function $u_{B}(x, y)$ on Domain 1.


Figure A.2: Plot of error for Test Case 1B.

In both Test Case 1A and Test Case 1B, we observe a similar convergence pattern to that found in Test Case 1 in Chapter 6 where the harmonic function $u(x, y)=e^{x} \cos (y)$ was used to assign boundary data.

## A. 2 Domain 3

We will now present the convergence results for Domain 3 with the boundary data specified by the harmonic functions $u_{A}(x, y)$ and $u_{B}(x, y)$.

## A.2.1 Test 3A

In this test case, we will assign boundary data on Domain 3 using the harmonic function $u_{A}(x, y)$. The Dirichlet data on each side for this test case is given by:

$$
\begin{array}{ll}
\text { Side 1: } & u\left(z_{1}(t)\right)=(1-t)\left(\frac{1}{4}(t-1)^{2}-1\right), \\
\text { Side 2: } & u\left(z_{2}(t)\right)=-(1+t)\left(\frac{t}{2}-\frac{1}{2}\right), \\
\text { Side 3: } & u\left(z_{3}(t)\right)=(t-1)\left(\frac{1}{4}(t-1)^{2}-1\right), \\
\text { Side 4: } & u\left(z_{4}(t)\right)=(t+1)\left(\frac{t}{2}-\frac{1}{2}\right) . \tag{A.2.4}
\end{array}
$$

The Neumann data is given by:

$$
\begin{align*}
& u_{\boldsymbol{n}}\left(z_{1}(t)\right)=\frac{4 t^{3}-3 t^{2}+3 t+11}{8 \sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}},  \tag{A.2.5}\\
& u_{\boldsymbol{n}}\left(z_{2}(t)\right)=\frac{2 \sqrt{5}(t+1)}{5}-\frac{\sqrt{5}\left(\frac{t}{2}-\frac{1}{2}\right)}{5},  \tag{A.2.6}\\
& u_{\boldsymbol{n}}\left(z_{3}(t)\right)=\frac{-t^{3}+3 t^{2}+9 t-11}{8 \sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}},  \tag{А.2.7}\\
& u_{\boldsymbol{n}}\left(z_{4}(t)\right)=\frac{\sqrt{5}\left(\frac{t}{2}-\frac{1}{2}\right)}{5}-\frac{2 \sqrt{5}(t+1)}{5} . \tag{A.2.8}
\end{align*}
$$

Below is a convergence plot for the harmonic function $u_{A}(x, y)$ on Domain 3.


Figure A.3: Plot of error for Test Case 3A.

## A.2.2 Test 3B

In this test case, we will assign boundary data on Domain 3 using the harmonic function $u_{B}(x, y)$. The Dirichlet data on each side for this test case is given by:

Side 1: $\quad u\left(z_{1}(t)\right)=\cosh (1-t) \cos \left(\frac{1}{4}(t-1)^{2}-1\right)$,
Side 2: $\quad u\left(z_{2}(t)\right)=\cosh (-1-t) \cos \left(\frac{t}{2}-\frac{1}{2}\right)$,
Side 3: $\quad u\left(z_{3}(t)\right)=\cosh (t-1) \cos \left(\frac{1}{4}(t-1)^{2}-1\right)$,
Side 4: $\quad u\left(z_{4}(t)\right)=\cosh (t+1) \cos \left(\frac{t}{2}-\frac{1}{2}\right)$.

The Neumann data is given by:

$$
\begin{align*}
& u_{\boldsymbol{n}}\left(z_{1}(t)\right)=-\frac{\cosh (t-1) \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}-\frac{\sinh (t-1) \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}},  \tag{A.2.13}\\
& u_{\boldsymbol{n}}\left(z_{2}(t)\right)=-\frac{2 \sqrt{5} \sin \left(\frac{t}{2}-\frac{1}{2}\right) \cosh (t+1)}{5}-\frac{\sqrt{5} \sinh (t+1) \cos \left(\frac{t}{2}-\frac{1}{2}\right)}{5},  \tag{A.2.14}\\
& u_{\boldsymbol{n}}\left(z_{3}(t)\right)=\frac{\cosh (t-1) \sin \left(\frac{(t-1)^{2}}{4}-1\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}}+\frac{\sinh (t-1) \cos \left(\frac{(t-1)^{2}}{4}-1\right)\left(\frac{t}{2}-\frac{1}{2}\right)}{\sqrt{\left(\frac{t}{2}-\frac{1}{2}\right)^{2}+1}},  \tag{A.2.15}\\
& u_{\boldsymbol{n}}\left(z_{4}(t)\right)=\frac{2 \sqrt{5} \sin \left(\frac{t}{2}-\frac{1}{2}\right) \cosh (t+1)}{5}+\frac{\sqrt{5} \sinh (t+1) \cos \left(\frac{t}{2}-\frac{1}{2}\right)}{5} . \tag{A.2.16}
\end{align*}
$$

Below is a convergence plot for the harmonic function $u_{B}(x, y)$ on Domain 3 .


Figure A.4: Plot of error for Test Case 3B.

Again, in both Test Case 3A and Test Case 3B, we observe a similar convergence pattern to that found in Test Case 3 in Chapter 6 where the harmonic function $u(x, y)=e^{x} \cos (y)$ was used to assign boundary data.

## APPENDIX B

## Curvature Test Cases

In Section 6.5 of Chapter 6 we studied the relation between the curvature of the boundary of the domain and the convergence rate of the numerical method. In our discussion, we used a set of domains that were formed from circular arcs so that the boundary of each domain would have constant curvature. In this appendix, we perform a similar numerical test with domains that have non-constant curvature. We give a series of four domains below for which the curvature of the boundary decreases for each successive domain.

- Domain A: The boundary of Domain A is given by the curves:

$$
\begin{array}{ll}
p_{1}^{A}(x)=x^{2}-1, & x \in[-1,1] \\
p_{2}^{A}(x)=-x^{2}+1, & x \in[-1,1] . \tag{B.0.2}
\end{array}
$$

- Domain B: The boundary of Domain B is given by the curves:

$$
\begin{array}{ll}
p_{1}^{B}(x)=\frac{1}{2} x^{2}-\frac{1}{2}, & x \in[-1,1], \\
p_{2}^{B}(x)=-\frac{1}{2} x^{2}+\frac{1}{2}, & x \in[-1,1] . \tag{B.0.4}
\end{array}
$$

- Domain C: The boundary of Domain C is given by the curves:

$$
\begin{array}{ll}
p_{1}^{C}(x)=\frac{1}{3} x^{2}-\frac{1}{3}, & x \in[-1,1], \\
p_{2}^{C}(x)=-\frac{1}{3} x^{2}+\frac{1}{3}, & x \in[-1,1] . \tag{B.0.6}
\end{array}
$$

- Domain D: The boundary of Domain D is given by the curves:

$$
\begin{array}{ll}
p_{1}^{D}(x)=\frac{1}{4} x^{2}-\frac{1}{4}, & x \in[-1,1], \\
p_{2}^{D}(x)=-\frac{1}{4} x^{2}+\frac{1}{4}, & x \in[-1,1] . \tag{B.0.8}
\end{array}
$$

In Figure B. 1 below we show the convergence plot for the numerical method on each of the Domains A-D.


Figure B.1: Plot of error for Domains A,B,C, and D.

We observe that for each of the successive domains the convergence rate improves. This confirms the results from Section 6.5 on domains with non-constant curvature.

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[^0]:    ${ }^{1}$ Here $\|\cdot\|_{C^{1+\alpha}}$ denotes the norm on the Holder space $C^{1+\alpha}(\Gamma)=\left\{f \in C^{1}(\Gamma):\|f\|_{C^{1+\alpha}(\Gamma)}<\infty\right\}$. The norm $\|\cdot\|_{C^{1+\alpha}}$ is defined by:

    $$
    \begin{equation*}
    \|f\|_{C^{1+\alpha}}:=\sup _{\boldsymbol{x} \in \Gamma}|\partial f(\boldsymbol{x})|+\sup _{\boldsymbol{x}, \boldsymbol{y} \in \Gamma, \boldsymbol{x} \neq \boldsymbol{y}} \frac{|\partial f(\boldsymbol{x})-\partial f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha}}, \quad \partial=\frac{d}{d \sigma} . \tag{1.2.23}
    \end{equation*}
    $$

[^1]:    ${ }^{1}$ In general, a Riemann-Hilbert problem consists of constructing a piecewise analytic function in the complex plane, given the behavior of the function on its discontinuities. A more detailed introduction to Riemann-Hilbert problems is given in Chapter 4.

[^2]:    ${ }^{2}$ Note: The derivation of the operator $T$ is given in more detail in Chapter 5 for the case of a domain with analytic boundary.

[^3]:    ${ }^{3}$ In terms of the notation of equation (2.3.1), $\boldsymbol{\Psi}=\mathcal{D}+i \mathcal{N}$.

[^4]:    ${ }^{1}$ The notation $\lesssim$ is used to denote that a function is asymptotically less than or equal to another function. Specifically, let $f$ and $g$ be functions parametrized by a variable $x \in X$, then $f \lesssim g$ if and only if there exists a constant $C>0$ such that $f(x)>C g(x)$ for all $x \in X$.

[^5]:    ${ }^{2}$ Without loss of generality, we will prove the results in this chapter for functions of the form $\varphi(x)=\left[1+i p^{\prime}(x)\right] \psi(x)$, where $\psi \in L^{2}(\Sigma)$. This is done for the sake of having a more compact notation.

[^6]:    ${ }^{3}$ The symbol $\lesssim A$ is used to denote that a function is asymptotically less than or equal to another function, with a constant depending on $A$. Specifically, let $f$ and $g$ be functions parametrized by a variable $x \in X$, then $f \lesssim g$ if and only if there exists a constant $C_{A}>0$, which depends on $A$, such that $f(x)>C_{A} g(x)$ for all $x \in X$.

[^7]:    ${ }^{4}$ A function $f(x)$ is analytic on an interval $\Omega \subset \mathbb{R}$ if for every $x_{0} \in \mathbb{R}$ a convergent power series can be written in the form:

    $$
    f(x)=\sum_{j=0}^{\infty} a_{j}\left(x-x_{0}\right)^{j},
    $$

    where each of the coefficients $\left\{a_{j}\right\}_{j=1}^{\infty}$ are real numbers. This implies the existence of a Taylor series. Moreover, if the function $f(x)$ is concave or convex, then the index of the first non-zero (and nonconstant) term will be even, i.e. $f(x) \sim a_{2 n}\left(x-x_{0}\right)^{2 n}$ for some $n \in\{1,2, \ldots\}$.

[^8]:    ${ }^{1}$ Recall, $\alpha_{j}$ is the angle that the tangent to the edge $\Gamma_{j}$ makes with the real-axis.

[^9]:    ${ }^{2}$ The converse holds as a consequence of Green's second theorem.

[^10]:    ${ }^{1}$ This choice of $\lambda \in \hat{l}_{j}$ allows the unknowns in the system to be strongly coupled by the global relation.
    ${ }^{2}$ The Halton nodes are a quasi-random deterministic sequence that is formed from coprimes and is often used in Monte Carlo sampling, [29].

[^11]:    ${ }^{3}$ For the case of polygonal domains a natural choice is $X=X_{\text {sym }}$, where $X_{\text {sym }}:=P W_{\text {sym }}^{\sigma_{1}} \times$ $\cdots P W_{\text {sym }}^{\sigma_{N}}$ and $P W_{\text {sym }}^{\sigma_{j}}$ is the symmetric part of the Paley-Wiener space $P W^{\sigma_{j}}=\mathcal{F} L^{2}\left(\left[-\sigma_{j}, \sigma_{j}\right]\right)$, which is the space containing the Fourier transform of functions in $L^{2}\left(\left[-\sigma_{j}, \sigma_{j}\right]\right)$.

[^12]:    ${ }^{4}$ Despite the fact that the Halton nodes are distributed semi-randomly to avoid clustering in $\mathbb{R}^{2} \cong \mathbb{C}$, it is possible that there will still be linear dependence in the system. The number of $\lambda$ values is oversampled in order to counter this. The intuition for oversampling is as follows. A square system contains the least number of equations that can produce a unique solution for a linear system (fewer equations would result in infinitely many solutions or no solutions). In fact, if there is any linear dependence in the system (such as due to clustering), a square system will result in a singularity. In contrast, by including more equations than unknowns, this can only produce a more stable system. To demonstrate this, consider the worse case in which one of the additional equations is a linear combination of previous equations. This will not adversely affect the overall stability of the system since by including an additional equation we are only providing more information from the case of a square system. In practice, redundant (or mostly redundant) equations automatically end up disregarded, while those that happen to be independent of each other are utilized when the system is solved. Hence, oversampling values of $\lambda$ serves to stabilize the system.

[^13]:    ${ }^{5}$ The author would like to thank Dr. Natasha Flyer for this suggestion.

[^14]:    ${ }^{6}$ In this section we will express the error as $E_{\infty}(M)=\exp (C M)$ so that the convergence rate, $C$, is negative. This is done so that the graph of convergence rate vs. curvature will have a positive slope.

