Fabular: Regression Formulas as Probabilistic Programming

Johannes Borgström
Uppsala University
Sweden

Andrew D. Gordon
Microsoft Research and
University of Edinburgh
UK

Long Ouyang
Stanford University
USA

Claudio Russo
Microsoft Research
UK

Adam ´Scibior
University of Cambridge and
MPI Tübingen
Germany

Marcin Szymczak
University of Edinburgh
UK

Abstract
Regression formulas are a domain-specific language adopted by several R packages for describing an important and useful class of statistical models: hierarchical linear regressions. Formulas are succinct, expressive, and clearly popular, so are they a useful addition to probabilistic programming languages? And what do they mean? We propose a core calculus of hierarchical linear regression, in which regression coefficients are themselves defined by nested regressions (unlike in R). We explain how our calculus captures the essence of the formula DSL found in R. We describe the design and implementation of Fabular, a version of the Tabular schema-driven probabilistic programming language, enriched with formulas based on our regression calculus. To the best of our knowledge, this is the first formal description of the core ideas of R’s formula notation, the first development of a calculus of regression formulas, and the first demonstration of the benefits of composing regression formulas and latent variables in a probabilistic programming language.

Categories and Subject Descriptors D.3.2 [Programming Languages]: Language Classifications—Specialized application languages; I.2.6 [Artificial Intelligence]: Learning—Parameter Learning

Keywords Bayesian inference; linear regression; probabilistic programming; relational data; hierarchical models

1. Introduction
Our goal is to embrace and extend R’s hugely popular regression formulas to get better probabilistic programming languages.

1.1 Background: R’s Regression Formulas
The R statistical programming language allows notation of the form \( y \sim x \) to express linear regression models. If \( x_1, y \) are the data in row \( i \) of a table, this model expresses that each \( y_i = \alpha + \beta x_i + e_i \), where \( e_i \) is an error term. Given this data and model, the regression task is to learn the global parameters \( \alpha \) and \( \beta \), the intercept and the slope of the line, by statistical inference.

While this example is an elementary univariate regression, the domain-specific languages of R formulas, as implemented by several different inference packages, support a wide range of classes of regressions (including multivariate, hierarchical, and generalized). The notation anonymises the parameters, such as \( \alpha \) and \( \beta \), has useful defaults, such as including the intercept and error terms automatically, and hence is extremely succinct.

Still, the published descriptions of R formulas are informal and non-compositional. If we are to transplant R formulas to other languages, the first problem is to obtain precise syntax and semantics.

1.2 Background: Probabilistic Programming
A system for probabilistic programming (Goodman 2013; Gordon et al. 2014b) asks the user to provide a probabilistic model as a piece of code, and provides a compiler to generate efficient code for statistical inference. Following the earliest system BUGS (Gilks et al. 1994; Lunn et al. 2013), there are many systems, including BLOG (Milch et al. 2007), Infer.NET (Minka et al. 2009), Church (Goodman et al. 2008), Figaro (Pfeffer 2009), HANSEI (Kiselyov and Shan 2009), Fun (Borgström et al. 2013), Stan (Stan Development Team 2014a), R2 (Nori et al. 2014), Probabilistic C (Paige and Wood 2014), Anglican (Wood et al. 2014), Probabilistic C (Paige and Wood 2014), Venture (Mansinghka et al. 2014), and Wolfe (Riedel et al. 2014).

From the start, BUGS, Stan, and other languages have been applied to hierarchical models, but written as explicit nested loops over the data. To the best of our knowledge, no previous probabilistic programming language has adopted R’s formula notation.

1.3 Part 1: Regression Calculus for Hierarchical Models
In their classic textbook, Gelman and Hill (2007) define a hierarchical/multilevel model to be “a regression (a linear or generalized linear model) in which the parameters—the regression coefficients—are given a probability model.” Their textbook uses R formulas for simple regressions, but since there is no R notation for defining priors on coefficients or for directly describing hierarchical models, Gelman and Hill use probabilistic programs (in BUGS) when describing hierarchical models with priors.

Calculus of Hierarchical Regression The purpose of our regression calculus is to be a precise notation for hierarchical models with explicit priors for coefficients, and with default choices of priors to retain the succinctness of R formulas. Our calculus is inspired by R formulas and translates to probabilistic programs (in Fun, but
easily adapted to BUGS). A unique feature in our calculus is the coefficient regression \( v(\alpha \sim r) \), which introduces a coefficient \( \alpha \) together with its nested probability model \( r \), directly corresponding to Gelman and Hill’s definition of a hierarchical model.

Section 3 presents the standard Bayesian interpretation of regression via our calculus. We recall Fun as a syntax for interpreting regressions as typed probabilistic programs, themselves formalised in measure theory. Theorem 1 (Type Preservation) guarantees that each well-typed regression maps to a well-typed Fun expression, and hence is interpreted as a measure over the measurable space for its type. Hence, for any well-typed regression \( y \sim r \) we define the prior distribution on its parameters and the output column of data for \( y \); and conditioned on observed data \( V \), possibly with missing values, we define the posterior distribution on its parameters and output, which yields predictions for the missing values.

Our calculus has new features beyond R’s regression formulas:

1. The coefficient construct \( v(\alpha \sim r) \) is a nested syntax for hierarchical linear models. R has no nested syntax for models.
2. We can set priors for coefficients such as slopes, intercepts, or the precision of error terms.
3. Input and output data and parameters are all typed.

**Flattening Multilevel Formulas to Single Level** A hierarchical regression such as \( \{ (\alpha \sim r_\alpha) \mid s \} \times (\beta \sim r_\beta) \) includes sub-expressions \( r_\alpha \) and \( r_\beta \) that model the parameters \( \alpha \) and \( \beta \) (here, the \( s \) syntax after \( (\alpha \sim r_\alpha) \) indicates that we will have one \( \alpha \) for every value of the categorical variable \( s \)). The regressions \( r_\alpha \) and \( r_\beta \) may themselves include nested coefficients on group-level input data. In Section 5 we recall Gelman and Hill’s discussion of how a hierarchical regression may be re-arranged so there are no nested coefficients, and by accessing group-level data from the top-level. Theorem 2 establishes that any well-typed regression has an equivalent single-level counterpart. Still, the advantage of our calculus over flattened formulas in R is that hierarchical syntax better reflects the intended structure of the model.

Explaining R’s Regression Formulas We developed our calculus to be a core language to explain the regression formulas of R. Section 6 describes a semantics for the formula dialects implemented by \( \text{lm} \), \( \text{mer} \), and \( \text{blmer} \) by mapping to the regression calculus.

1.4 Part 2: Fabular = Tabular + Regression Calculus

Tabular (Gordon et al. 2014a, 2015) is a schema-driven probabilistic programming language embedded in a spreadsheet, with inference by Infer.NET (Minka et al. 2009). In Section 7, we extend Tabular with columns defined by regressions from the regression calculus. Hence, we can express hierarchical models. In addition, since Tabular, like most probabilistic programming languages, supports latent variables defined by a model, we can use formulas to express models based on latent variables, such as clustering or ranking models. Moreover, we adopt a vectorized interpretation of regressions, where coefficients and outputs are vectors; hence, we obtain a formula notation for vector-based models. We develop in detail the bilinear recommender model Matchbox (Stern et al. 2009). We report a list of Fabular syntax and informal semantics of our regression calculus via a series of examples. Section 4 completes the exposition by explaining more complex examples.

Section 8 concludes the paper. Technical report (Borgström et al. 2015) contains more details, definitions and proofs.

1.5 Contributions of the Paper

We propose the first formal calculus of regressions, with a unique recursive syntax for hierarchical models (based on the coefficient construct \( v(\alpha \sim r) \)), and a rigorous typed semantics. We develop a semantic equivalence for regressions, which we apply to transform multilevel regressions to equivalent single-level regressions. We explain the essence of the popular formula notation in R’s \( \text{lm} \), \( \text{mer} \), and \( \text{blmer} \) by converting formulas to terms of the regression calculus. To our knowledge, this is the first formal description of the core ideas of R’s formula notation. We design and implement Fabular, a version of the Tabular schema-driven probabilistic programming language that is enriched with formulas from our regression calculus.

1.6 Related Work

We discussed probabilistic programming systems in Section 1.2. Morandat et al. (2012) conduct a careful analysis of the design of the R programming language, but do not consider its formula notation for regression. There are informal descriptions of R formulas such as (Hahn) and in the documentation for R packages. For example, Bates et al. (2014) provide a careful description of the semantics of \( \text{mer} \) formulas in terms of matrix representations and algorithms. In addition, they detail instructive examples of a variety of \( \text{mer} \) formulas but do not provide a grammar for this language or discuss the precise semantics of the syntax.

2. A Core Calculus of Regression, by Example

We give the syntax and informal semantics of the regression calculus, together with a series of examples. The formal typing rules and semantics are in Section 3.

2.1 Syntax and Informal Semantics

The types of the calculus are \texttt{real}, bounded naturals \( \texttt{mod}(n) \) for \( n \geq 0 \), and sized array types. Let \( \tau \) range over real constants.

**Variables, Naming Conventions, and Types:**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ), ( U )</td>
<td>( \text{real} )</td>
<td>type ( (n \geq 0) )</td>
</tr>
<tr>
<td>( x, y )</td>
<td>( \text{real} )</td>
<td>(continuous)</td>
</tr>
<tr>
<td>( c, d )</td>
<td>( \text{category} )</td>
<td>(category)</td>
</tr>
<tr>
<td>( \alpha, \beta )</td>
<td>( \text{parameter} )</td>
<td>(parameter)</td>
</tr>
</tbody>
</table>

A core idea of the calculus is that expressions denote probability distributions over multidimensional arrays of data, referred to as cubes. A cube may be a column of predicted data, a single parameter (a zero-dimensional array), an array or a doubly-indexed array of parameters. (Higher dimensions are possible.)

**Dimensions and Cube-Expressions:**

Let a dimension, \( \vec{e} \) or \( \vec{f} \), be a finite list of natural numbers.

Let a cube-expression with dimension \( \vec{e} = [e_1; \ldots; e_\ell] \) be a phrase that denotes a multi-dimensional array of some type \( T[e_1; \ldots; e_\ell] \).

An index for \( \vec{e} \) is a list \([i_1; \ldots; i_\ell]\) with \( 0 \leq i_j < e_j \) for each \( j \).

A predictor \( v \) is a (deterministic) cube-expression made up of constants, variables, interactions, and paths.

**Syntax of Predictors:**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u, v )</td>
<td>predictor</td>
</tr>
<tr>
<td>( s )</td>
<td>scalar (common cases are 1 and 0)</td>
</tr>
<tr>
<td>( x )</td>
<td>variable (categorical or continuous)</td>
</tr>
<tr>
<td>( u : v )</td>
<td>interaction (multiplication)</td>
</tr>
<tr>
<td>( (u_1, \ldots, u_\ell).v )</td>
<td>path</td>
</tr>
</tbody>
</table>
A scalar $s$ returns the cube with $s$ at each index. A variable $x$ returns the cube denoted by $x$. An interaction $u v$ is the pairwise multiplication of $u$ and $v$; it returns the cube with $v^i u^j$ at each index $i$. Finally, a path $(u_1 \ldots u_n) v$ computes an intermediate $[f_1 \ldots f_n]$-cube for $v$, computes cubes $u_1, \ldots, u_n$ containing indexes for dimensions $f_1, \ldots, f_n$, and returns the cube obtained by applying the indexes from $u_1, \ldots, u_n$ to $v$. For instance, the form $(\cdot, v)$ allows a $[\cdot]$-dimensional cube $v$ (that is, a scalar) to be mapped to a cube of arbitrary dimension. We write $u_1 v$ for $(u_1) v$.

A regression $r$ is a probabilistic cube-expression that returns a tuple of static parameters alongside a cube of outputs.

**Syntax of Regressions:**

$$ r := \begin{cases} \text{regression} & D(v_1, \ldots, v_n) \\ \text{noise with distribution} & \nu(\alpha \sim r) \\ \text{predictor with coefficient} & r + r' \\ \text{grouping} & (v \alpha) r \\ \text{restriction (scope of } \alpha \text{ is } r) & (u_n \alpha) r \end{cases} $$

A noise term $D(v_1, \ldots, v_n)$ returns a cube with an independent random draw from the distribution $D(v_1[i], \ldots, v_n[i])$ at each index $i$. We assume the following families $D$ of distributions.

**Distributions:**

$$ D : \langle v_1 : U_1, \ldots, v_n : U_n \rangle \rightarrow T $$

- **Dirac:** $(\text{point} : T) \rightarrow T$
- **Gaussian:** $(\text{mean} : \text{real}, \text{variance} : \text{real}) \rightarrow \text{real}$
- **Gamma:** $(\text{shape} : \text{real}, \text{rate} : \text{real}) \rightarrow \text{real}$

As indicated by the type signatures, the basic parameterization of a Gaussian is in terms of the mean and variance, and for a Gamma in terms of shape and scale. We write $\text{Gaussian}(m, s^2)$ for the normal distribution with mean $m$ and standard deviation $s$; its variance is $s^2$ and its precision is $1/s^2$. We allow some inverted parameterizations such as $\text{Gaussian}(u, 1/v)$ for a Gaussian with mean and precision (the inverse of variance) given by $u$ and $v$, and $\text{Gamma}(u, 1/v)$ for a Gamma distribution with shape and rate (the inverse of scale) given by $u$ and $v$. The following is a useful special case of noise: the distribution $\text{Dirac}(v)$ has all its mass on $v$.

**Derived Form of Regression:**

$$ \delta v \triangleq \text{Dirac}(v) $$

In the simple case, a predictor with coefficient $\nu(\alpha \sim r)$ defines parameter $\alpha$ by the $[\cdot]$-dimensional cube of $r$ (that is, a scalar), and returns the parameters of $r$ together with $\alpha$, alongside a cube of the same dimension as $v$, with each component of $v$ multiplied by $\alpha$. A more complex case arises in hierarchical models, where $\alpha$ denotes not a single scalar coefficient, but a whole array or even multi-dimensional array of coefficients.

A sum $r_1 + r_2$ returns the concatenation of the parameters of $r_1$ and $r_2$ alongside the pairwise sum of their cubes.

A grouping $r$ introduces a hierarchical model; in the simple case, where $r$ itself contains no grouping and $v$ is a cube of indexes of bounded type $\text{mod}(f)$, the regression $r$ is the same as $r$ except it generates arrays of parameters of dimension $[f]$, and uses $v$ to choose which parameter to select. In the more complex case, the parameters form an arbitrary cube.

A restriction $(v \alpha) r$ is the same as $r$, except that the parameter $\alpha$ returned by $r$ is hidden.

We write $f v(v)$ and $f v(r)$ for the sets of variables free in $v$ and $r$. We identify phrases of syntax up to alpha-conversion, the consistent renaming of bound variables. We write $\{/\phi/\}$ for syntactic substitution of phrase $\phi$ for variable $x$, avoiding capture of bound variables. For example, $(v \alpha) r = (v \alpha') (r^{\{\alpha/\alpha\}})$ if $\alpha' \notin f v(r)$.

Let the domain of regression $r$ be the list of names of parameters defined by $r$; we define $\text{dom}(r)$ below. We write lists as $[x_1, \ldots, x_n]$ or $[x_1 \ldots x_n]$ for ordered index set $I$, and $\otimes$ is list concatenation. If $\alpha = [\alpha^I \otimes I]$ and $j \in I$ then $\alpha \mid j = [\alpha^I \otimes j]$. We write $\nu(\alpha \sim r)$ for syntactic substitution of $\nu(\alpha)$ for $\nu$.

**Domain of a Regression:** $\text{dom}(r)$

$$ \begin{align*} \text{dom}(D(v_1, \ldots, v_n)) & = [] \\ \text{dom}(\nu(\alpha \sim r)) & = \text{dom}(r) \otimes \alpha \\ \text{dom}(r + r') & = \text{dom}(r) \otimes \text{dom}(r') \\ \text{dom}(r \mid v) & = \text{dom}(r) \\ \text{dom}(\nu(\alpha) r) & = \text{dom}(r) \setminus \alpha \end{align*} $$

**2.2 Setting: Predicting Students’ Grades**

To introduce the calculus, we consider a series of example regressions for a dataset corresponding to the opening example of Gelman and Hill (2007). The dataset consists of tables of schools and students, containing schools and students rows. (Throughout we assume that the rows in a table $t$ of size $\mathcal{I}$ have primary keys numbered $0, \ldots, \mathcal{I} - 1$, and hence we treat a table as a set of arrays of the same size.) Each student $i$ has a property $x[i]$, and a school $s[i]$, while each school $j$ has a level property $u[j]$ (such as average parents’ incomes). We refer to the data via variables in the type environment $\Gamma$ given below.

$$ \Gamma = \begin{cases} u : \text{real}[\text{schools}], \\ s : \text{mod}[\text{schools}][\text{students}], \\ x : \text{real}[\text{students}] \end{cases} $$

Moreover we have a column $y = v$, of type $\text{real}[\text{students}]$ of test-grades, possibly with missing values. We consider a series of regressions that model $y$, that is, they return a cube of dimension $\text{students}$. Each regression defines a joint distribution over its parameters and its output column $y$. If we condition this prior distribution on the observed data $V$, we obtain predictions for the missing test scores, and a posterior distribution for the model’s parameters.

The task defined by a regression $r$ plus input data matching $\Gamma$ and observed output column $V$, is to compute (approximations of) these conditional distributions. (We formalize in Section 3.5.)

We now consider a series of regressions for this data.

**2.3 Pure Intercept:** $r_1 = 1 (\alpha \sim \text{Gaussian}(0, s_{\text{large}}^2))$

Our first regression $r_1$ is a flat baseline set by a parameter $\alpha$ with an uninformative prior, that is, a very wide Gaussian distribution. Our semantics for $r_1$ is a probabilistic program: the Fun expression below.

$$ \text{let } \alpha = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } (\alpha, [z < \text{students} \rightarrow 1 \times \alpha]) $$

(Fun is a probabilistic dialect of ML. We introduce its formal syntax in Section 3.1. A for-loop expression $\{\text{for } x < n \rightarrow F\}$ produces an array $\{F[0]/x, \ldots, F[n-1]/x\]}$.)

The expression defines $\alpha$ by a draw from a Gaussian with a large standard deviation $s_{\text{large}}$, and returns $\alpha$ alongside an array $y = [z < \text{students} \rightarrow 1 \times \alpha]$ that sets each entry to $\alpha$. A plot of each input $x[i]$ versus $y[i]$ for each student $i$ is a flat line that intercepts the $Y$-axis at $y = \alpha$, so we refer to it as the intercept.

In practice, choosing $s_{\text{large}}$ is a balance between being $\alpha$ biased toward small numbers, and being so large as to trigger overflows. The subject of configuring priors in detail is a statistical question beyond the scope of this paper.

In general, a regression $\nu(\alpha \sim \text{Gaussian}(0, s_{\text{large}}^2))$ chooses an uninformative prior for a coefficient $\alpha$ for a predictor $v$. This is a common pattern, so we allow the following abbreviations.

$$ \begin{align*} \nu(\alpha) & \triangleq \nu(\alpha \sim \text{Gaussian}(0, s_{\text{large}}^2)) \\ \nu & \triangleq (\nu(\alpha) \nu(\alpha)) \text{ for } \alpha \notin f v(v) \end{align*} $$
For instance, $1\{\alpha\}$ is the same as $r$ above, while 1 on its own is the same as $r$ except the parameter $\alpha$ is hidden.

2.4 Pure Noise: $r_2 = ?$

A model such as $r_2$ does not fit data unless all the points fall exactly on the intercept; the model allows the intercept to be learnt, but allows no per-point variation from the intercept. In practice, all data is noisy in that there is deviation from the line and so we need to include noise, also known as an error term. The regression written $r_2 = ?$ is a pure noise model. Its semantics is the following.

$$ \text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{() \mid \text{for } z < \text{students } \rightarrow \text{Gaussian}(0, 1/\pi)\} $$

Each item in the output array is a draw from Gaussian$(0, 1/\pi)$, a zero-mean Gaussian with precision $\pi$. The smaller the precision the greater the variance of the noise. The precision $\pi$ is drawn from distribution Gamma$(1, 1/\lambda_{\text{large}})$ with a small rate parameter $1/\lambda_{\text{large}}$ to achieve a non-informative prior on the precision. The effect is that the precision of the noise is determined by the observed data.

The syntax $? = \pi$ is not primitive in the calculus but is derived from other constructs as follows.

$$ ?(\pi \sim r) \triangleq \{0(\pi \sim r) + \text{Gaussian}(0, 1/().\pi)\} $$
$$ ?(\forall \pi)?(\pi \sim \text{Gamma}(1, 1/\lambda_{\text{large}})) $$

The coefficient $0(\pi \sim r)$ is a coding trick that defines the parameter $\pi$ by $r$, but makes no contribution to the output column. The literal semantics of $?$ is the following, though the term $0 \times \pi$ may be cancelled out.

$$ \text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{() \mid \text{for } z < \text{students } \rightarrow 0 \times \pi + \text{Gaussian}(0, 1/\pi)\} $$

2.5 Intercept (with Noise): $r_3 = 1\{\alpha\} + ?$

Combining an intercept and noise term yields the following model, which learns the intercept $\alpha$ while allowing for noise.

$$ \text{let } \alpha = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{() \mid \text{for } z < \text{students } \rightarrow \alpha + \text{Gaussian}(0, 1/\pi)\} $$

2.6 Slope and Intercept (with Noise): $r_4 = 1\{\alpha\} + x\{\beta\} + ?$

By including a slope $x\{\beta\}$, we obtain a regression equivalent to the R formula $y \sim x$ from Section 1.1, except that our notation includes priors on the parameters.

$$ \text{let } \alpha = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } \beta = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{() \mid \text{for } z < \text{students } \rightarrow \alpha + x\{\beta\} + \text{Gaussian}(0, 1/\pi)\} $$

2.7 Varying Intercept per School: $r_5 = 1\{\alpha\} | x \}$

The regression $r_5$ groups the intercept on the school $s$, so that we learn an array $\alpha$ of parameters, one per school.

$$ \text{let } \alpha = \{\text{for } z < \text{schools } \rightarrow \text{Gaussian}(0, s_{\text{large}}^2)\} \text{ in } \text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{() \mid \text{for } z < \text{students } \rightarrow \alpha[z] + \text{Gaussian}(0, 1/\pi)\} $$

The regression $1\{\alpha\} | s + x\{\beta\} + ?$ is the same, but has slope $\beta$.

2.8 Hierarchical (Varying-Intercept, Fixed-Slope): $r_6$

To take into account the school-level data $u$, we construct a nested regression $r_6$ with slope $b$ to predict each school-level parameter $\alpha$. (The model $r_6$ is similar to $r_4$ but at school not student level.)

$$ r_6 = 1\{\alpha\} + u\{b\} + ? $$

The meaning of $r_6$ is the following:

$$ \text{let } a = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } b = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } \pi' = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{(a, b) \mid \text{for } z < \text{schools } \rightarrow a + u[z] \times b + \text{Gaussian}(0, 1/\pi')\} $$

Hence, we assemble the meaning $E_6$ of the whole model $r_6$:

$$ \text{let } a = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } b = \text{Gaussian}(0, s_{\text{large}}^2) \text{ in } \text{let } \pi' = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \{(a, b, \alpha, \beta) \mid \text{for } z < \text{students } \rightarrow \alpha[z] + x[z] \times \beta + \text{Gaussian}(0, 1/\pi')\} $$

This is the first hierarchical model of Gelman and Hill (2007).

3. Type System and Semantics of Regression

3.1 Fun: Probabilistic Expressions (Review)

Syntax of Fun We use a version of the core calculus Fun (Borgström et al. 2013) as presented by Gordon et al. (2013) with arrays of deterministic size, but without a conditioning operation within expressions.

We assume a collection of total deterministic functions $g$, including arithmetic and logical operators.

Expressions of Fun:

| $E, F ::= | $x$ | variable $s$ | constant (real, unit, int, Boolean) $g(E_1, \ldots, E_n)$ | deterministic primitive $g$ $D(F_1, \ldots, F_n)$ | random draw from distribution $D$ if $E_1$ then $E_2$ else $E_3$ | if-then-else $[E_1, \ldots, E_n] \mid [E, F]$ | array literal, lookup $\text{for } x < n \rightarrow F$ | for loop (scope of index $x$ is $F$) $\text{let } x = E \text{ in } F$ | let (scope of $x$ is $F$) $\text{pair } (E, F)$ $\text{fst}(E) \text{ snd}(E)$ | projections |

Type system of Fun We here recall the type system of Fun without zero-probability observations (Bhat et al. 2013). The syntax of types is as in Section 2, with the addition of unit, bool, int, and pair types $T_1 \times T_2$. We write $\Gamma : E, T$ to mean that in type environment $\Gamma = \tau_1 : T_1, \ldots, \tau_q : T_n$ (with distinct) expression $E$ has type $T$. Let $\text{det}(E)$ mean that $E$ contains no occurrence of $D(\ldots)$. The typing rules for Fun are standard for a first-order functional language.

Semantics of Fun Intuitively, an expression $E$ defines a probability distribution over its return type. For each type $T$, we define a measurable space $T[\pi]$; probability measures on that space formalize distributions over values of the type. A valuation $\rho = [\{x_i \mapsto V_i[\pi, i].\pi\}]$ is a map from variables to values. For each expression $E$ and valuation $\rho$ for its free variables, we define its semantics as $P[E][\pi][\rho]$.

Lemma 1. If $\Gamma \vdash E : T$ and $\Gamma \vdash \rho$ then $P[E][\pi][\rho]$ is a probability measure on $T[\pi]$. |
The semantics has a corresponding notion of equivalence.

**Definition 1.** Let $\Gamma \vdash E_1 \equiv E_2 : T$ if and only if both $\Gamma \vdash E_1 : T$ and $\Gamma \vdash E_2 : T$, and for all $\rho$, $\Gamma \vdash \rho$ implies that $P[E_1]_\rho = P[E_2]_\rho$.

Finally, we define notation for conditioning the distribution defined by a whole expression. (We have no operators for conditioning within the syntax of expressions.) If $\Gamma \vdash E : T_1 \times \cdots \times T_n$ and $\Gamma \vdash \rho$, and for $i \in 1..n$ we have $\ominus \vdash V_i : U_i$ and $\det(F_i)$ and $x_1 : T_1, \ldots, x_n : T_n \vdash F_i : U_i$, we write

$$P[E]_\rho[x_1, \ldots, x_n] \mid F_i = V_1 \land \cdots \land F_m =ting for a version of the conditional probability distribution of $P[E]_\rho$ given that the random variable $f(x_1, \ldots, x_n) \triangleq (F_1, \ldots, F_m)$ equals $(V_1, \ldots, V_m)$.

### 3.2 Typing the Regression Calculus

Let a type environment $\Gamma$ be of the form $x_1 : T_1, \ldots, x_n : T_n$ where the variables $x_i$ are distinct, and let $\dom(\Gamma) = \{x_1, \ldots, x_n\}$.

**Judgments of the Type System:**

- $\Gamma; \vec{e} \vdash:\vdash T$ predicts $\nu$ yields an $\vec{e}$-cube of $T$.
- $\Gamma; \vec{e}; \vec{f} \vdash \nu \Gamma$ regression $\rho$ yields $\vec{e}$-cube with parameter $\vec{f}$-cubes.

We type-check a regression $\rho$ with output dimension $\vec{e}$ and parameter dimension $\vec{f}$. The effect of the regression is to introduce parameters described by the Fun context $\Pi$.

The judgment for predictors ensures that their cubes are accessible, constructible or reachable from the ambient dimensions $\vec{e}$:

**Typing Rules for Predictors:**

- **(SCALAR)**
  - $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$ 
- **(VAR)**
  - $\Gamma \vdash \nu : T[\vec{e}]$ 
- **(INTERACT)**
  - $\Gamma \vdash \nu \vdash \nu : \mathbb{R}T[\vec{e}]$ 
- **(PATH)**
  - $\Gamma; \vec{e} \vdash u : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$ 
- **(COEFF)**
  - $\Gamma; \vec{e} \vdash u : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$ 
- **(NOISE)**
  - $D : (x_1 : U_1, \ldots, x_n : U_n) \rightarrow \mathbb{R}T[\vec{e}]$ 
- **(SUM)**
  - $\Gamma; \vec{e} \vdash \nu : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$ 
- **(GROUP)**
  - $\Gamma; \vec{e} \vdash \nu : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$ 
- **(RES)**
  - $\Gamma; \vec{e} \vdash \nu : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$

The rules for typing regressions check all subregressions are real-valued in the current dimension $\vec{e}$ and parameter dimension $\vec{f}$. Rule (COEFF) changes dimension from $\vec{e}$ to $\vec{f}$, entering the nested regression; Rule (GROUP) adds a categorical dimension to $\vec{f}$. All rules additionally accumulate or drop parameters introduced by the regression or its subregressions, threading an output context $\Pi$.

For example, we can derive the following. (Recall $\gamma(\langle x, y, z \rangle) = \langle 0 \pi, \pi \rangle$ is short for $\gamma(\langle x, y, z \rangle) = \langle 0 \pi, \pi \rangle$.)

$$\vdash \nu : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$$

### 3.3 Translation to Pure Fun

**Translation of Predictors to Fun:**

$$\vdash \nu : \mathbb{R}T[\vec{e}] \\forall i \in 1..n$$

**Translation of Regressions to Fun:**

If $\Gamma; \vec{e} \vdash \nu : T$ and $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$ for all $i \in I$ then $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$.

**Lemma 2.** If $\Gamma; (e_i)_i \vdash \nu : T$ and $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$ for all $i \in I$ then $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$.

**Theorem 1 (Type Preservation).** If $\Gamma; \vec{e} \vdash \nu : T$ and $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$ for all $i \in I$, then $\Gamma \vdash \nu : \mathbb{R}T[\vec{e}]$.

**Proof:** By induction on the derivations of $\Gamma; \vec{e} \vdash \nu : T$.

### 3.4 Data for Regression: Column-Oriented Databases

We consider regression in the context of a column-oriented database, where the columns consist of arrays of values, and columns of the same size are grouped into tables. Let $t$ range over table names and $c$ range over column names. We consider a database to be a pair $DB = (\delta_t, \rho_c)$ consisting of a record of tables $\delta_t = [\{x_j \rightarrow \text{inst}(V_j)_{j \in 1..m}\}]$ and a record of columns $\rho_c = [\{x_j \rightarrow \text{inst}(V_j)_{j \in 1..m}\}]$ defining the number of rows $t_j \in N$ in each column $V_j$ of table $t_j$. To name the columns in a database, we flatten it into an environment with an array-type variable for each column. Consider an environment $\Gamma$ and a valuation $\rho$. We say that $\Gamma$ and
\( \rho \) match DB to mean that \( \Gamma = (c_{i,j} : T_j \rho_{x_i}(t_i))^\in\mathbb{N},j\in\mathbb{M} \) and \( \rho = [(c_{i,j} \mapsto V_{ij})]_{\in\mathbb{N},j\in\mathbb{M}} \) and that \( \Gamma \vdash \rho \). The latter implies that the column names \( c_{i,j} \) are pairwise distinct. We write \( \Gamma \vdash \rho \) to mean that the values in \( \rho \) match the types in \( \Gamma \), that is, \( \Gamma \vdash [x_i \mapsto V_{ij}]_{\in\mathbb{N}} \) if and only if \( \Gamma = (x_i : T_j)_{\in\mathbb{N}} \) and \( \rho \vdash V_{ij} : T_j \) for all \( i \in \mathbb{N} \).

For example, consider a database for our schools example:

\[
\begin{align*}
DB &= (\delta_{\text{in}}, \rho_{\text{c}}) \\
\rho_{\text{c}} &= \{\text{schools} \mapsto 20, \text{students} \mapsto 200\} \\
\delta_{\text{in}} &= \{\text{schools} \mapsto \tau_{\text{schools}}, \text{students} \mapsto \tau_{\text{students}}\} \\
\tau_{\text{schools}} &= [u \mapsto \text{inst}(V_{1,1})] \\
\tau_{\text{students}} &= [x \mapsto \text{inst}(V_{2,1}), x \mapsto \text{inst}(V_{2,2})]
\end{align*}
\]

We have that \( \Gamma \) and \( \rho \) match \( DB \), where \( \Gamma \) is as in Section 2.2 and \( \rho = [u \mapsto V_{1,1}, x \mapsto V_{2,1}, y \mapsto V_{2,2}] \).

### 3.5 Semantics of Regression

Consider \( \Gamma \) and \( \rho \) that match the input database \( DB \), so that \( \Gamma \vdash \rho \). We wish to use the regression \( r \), as model for a column \( y \) on a table \( t \) of size \( l \), where \( y \) does not appear in \( DB \). We then define the prior distribution \( \mu \) of the tuple \( (\text{dom}(r), y) \) of coefficients and \( y \) as

\[
\mu \triangleq P(\Gamma \vdash \rho) \quad \text{where} \quad E_{\text{y}} = \{e \mid \forall \}
\]

**Lemma 3.** Suppose that \( \Gamma \vdash \rho \) and \( \Gamma ; [\vdash r \vdash \Pi] \). Then \( \mu \) is a probability measure on the measurable space \( T(\text{tuple}(\Pi) \times \text{real}(\Pi)) \). 

**Proof:** By Theorem 1 (Type Preservation), \( \Gamma ; [\vdash r \vdash \Pi] \) implies that \( \Gamma ; E_{\text{y}} : \text{tuple}(\Pi) \times \text{real}(\Pi) \) where \( E_{\text{y}} = \{e \mid \forall \} \). By Lemma 1 and inversion of typing, \( \Gamma ; E_{\text{y}} : \text{tuple}(\Pi) \times \text{real}(\Pi) \) and \( \Gamma \vdash \rho \) imply that \( P(\Gamma \vdash \rho) \) is a probability measure on \( T(\text{tuple}(\Pi) \times \text{real}(\Pi)) \). 

To apply a regression, we need observations of some or all of the items in the column \( y \) predicted by \( r \). Consider \( O \) a subset of the indexes of \( y \), that is, \( O \subseteq \{i \mid 0 \leq i \leq l\} \). Let an \( O \)-observation \( o \) be an array of observed values for the indexes in \( O \), that is, an array \( V_{y} = \{d_{0}, \ldots, d_{l-1}\} \) such that \( d_{i} \in \mathbb{R} \) if \( i \in O \), and otherwise \( d_{i} = \_ \) represents a missing value.

The posterior distribution of \( (\text{dom}(r), y) \) given an \( O \)-observation \( V_{y} \) on \( r \) is the conditional distribution

\[
\mu(\bar{\alpha}, y | y) = V_{y}[\alpha] \quad \text{for all} \quad i \in O
\]

which is \( \mu \) conditioned on the outputs \( y[i] \) being equal to the observed values \( d_{i} \) where \( i \in O \). Marginalizing this distribution yields the posterior for each parameter in \( \bar{\alpha} \) and the posterior prediction for each unobserved output \( y[i] \) for \( i \notin O \).

Semantic equivalence for regressions is given by semantic equivalence of the corresponding Fun terms.

**Definition 2.** Let \( \Gamma ; \bar{\epsilon}(f_{j})^{\in\mathbb{E}} \vdash r_{1} \equiv r_{2} \vdash \Pi \) if and only if both \( \Gamma ; \bar{\epsilon}(f_{j})^{\in\mathbb{E}} \vdash r_{1} \vdash \Pi \) for \( i \in 1,2 \) and for all \( (F_{j})^{\in\mathbb{E}} \) such that \( \Gamma \vdash F_{j} : \text{mod}(\Pi) \cdot c \). If \( E_{\bar{\epsilon}} = \{e_{1}\} \), \( \bar{\epsilon}(f_{j})^{\in\mathbb{E}}(F_{j})^{\in\mathbb{E}} \) and then \( \Gamma ; E_{\bar{\epsilon}} \equiv E_{2} : \text{tuple}(\Pi) \times \text{real}(\Pi) \).

### 4. Examples: Radon and InstEval

We now resume the exposition of models in the regression calculus from Section 2. by explaining how our calculus captures typical multi-level models and techniques such as partial pooling and multiple grouping, as illustrated by the radon example from Gelman and Hill (2007), and the InstEval example from Bates et al. (2014).

#### 4.1 Example of Partial Pooling: Radon

Gelman and Hill (2007) consider a specific data set that contains radiation measurements taken in houses across different counties in Minnesota. Each measurement includes the floor \( f \) (either 0 or 1, represented by a real) and the radon activity activity and the county \( c \) of the house. For each county, we have a background level of uranium radiation \( u \).

\[
\Gamma = u \cdot \text{real}[c \cdot \text{counties}], \quad c \cdot \text{mod}(\text{counties}) \cdot \text{houses}, \quad f \cdot \text{real} \cdot \text{houses}
\]

We treat floor as a continuous predictor and assume that the county-level intercept depends on the uranium level in each county—in particular, this intercept is a linear function of uranium level. Our model \( r_{1} \) defines radiation activity in terms of the floor the measurement was taken on (basement, first floor, ... , and the county: \( \alpha_{c} = 1[a] + u(b) + \{\pi \sim \delta \nu\} \), \( r_{2} = \{1(\alpha \sim r_{1}) \cdot c\} + f(\beta) + \? \), \( \Pi = a, b, \pi \cdot \text{real}, \alpha : \text{real}[\text{counties}], f : \text{real} \).

(This concrete dataset is in fact isomorphic to the hypothetical schools example. The model is isomorphic to \( r_{0} \) except that we are exposing the precision parameter \( \tau \) for the group-level noise.)

Our purpose with this example is to discuss a critical feature: partial pooling of data across different groups of observations. Partial pooling is an interpolation between no pooling, where observations from different groups are treated completely separately, and complete pooling, where observations from different groups are treated identically.

For example, in the radon data set, suppose that we wish to model radon activity in a house simply as a function of the county that the house is located in. This means that \( y[i] = \alpha[c[i]] + \text{noise} \), where \( y[i] \) is the measured radon activity in house \( i \) and \( \alpha[c[i]] \) is an intercept for county \( c[i] \), or as a regression calculus formula, \( 1(\alpha \sim r_{0}) \cdot c \cdot + \?
\)

In complete pooling, we set a single \( \alpha \) for all counties, i.e., \( \forall i, j \alpha[c[i]] = \alpha[c[j]] \), which is equivalent to leaving off the grouping \( c \) from the regression calculus formula above. In no pooling, we independently fit the \( \alpha[c] \) (in particular, we sample these values from a distribution with known parameters so that the individual \( \alpha[c] \) are conditionally i.i.d.). Partial pooling is a compromise between these two extremes; we sample the \( \alpha[c] \) from some distribution with uncertain parameters, e.g., \( 1(\alpha \sim \text{Gaussian}(\mu_{c}, \sigma_{c})) \cdot c \).

\[ r_{\text{county}} = 1[a] + u[b] + \{\eta \sim \delta \eta\} \]
\[ r_{\text{house}} = 1(\alpha \sim r_{\text{county}}) \cdot c + \]

Here, \( r_{\text{county}} \) is the county-level regression, and \( r_{\text{house}} \) is the house-level regression. To get complete pooling, we set \( r_{1} \) to \( \delta \cdot 0 \). This maximal precision for the county-level noise will result in all \( \alpha \) of having the same value across counties. To get no pooling, we set \( r_{1} \) to \( \delta \cdot 0 \). This minimal precision for the county-level noise will result in the \( \alpha \) essentially being free parameters. To get partial pooling,
we set \( r_\eta \) to Gamma\((1, 1/\lambda_{\text{large}})\). By placing uncertainty over the
precision \( \eta \), we allow information to “flow” between counties—
information about one county informs estimates about others. For
concreteness, here is a simplified Fun translation of the above
model (with \( r_\eta \) left unspecified, and assuming that \( \text{dom}(r_\eta) = \emptyset \)):

\[
\text{let } a = \text{Gaussian}(0, \sigma_{\text{large}}^2) \text{ in } \\
\text{let } ((\cdot, \eta) = [\eta]) \text{ in } \\
\text{let } \alpha = \text{for } z < \text{counties} \rightarrow a \text{ Gaussian}(0, 1/\eta) \text{ in } \\
\text{let } \pi = \text{Gamma}(1, 1/\lambda_{\text{large}}) \text{ in } \\
((\cdot, \eta), \forall z < \text{houses} \rightarrow \alpha[c[z]][\text{Gaussian}(0, 1/\pi)])
\]

4.2 Example of Multiple Grouping: InstEval

One formula pattern that is possible with hierarchical models is
what we call multiple grouping: grouping on the Cartesian product
of multiple variables. For example, consider an analysis of the
InstEval dataset (from the lme4 package by Bates et al. (2014)) which
contains ratings that ETH Zurich students gave their professors:

\[
\text{rating} \sim (1|\text{student}) + (1|\text{professor}) + \\
(1|\text{department:service})
\]

Here, department indicates the department that the course was
taught in and service indicates whether the course was a service
course taught to students outside the department. We model
the rating as depending on three random effects intercepts: one that
varies by student (e.g., some students may tend to give high ratings,
another that varies by professor (e.g., some professors may be
particularly well-liked), and another that varies by department
service status (e.g., some departments might put less effort into their
course services than others and thus receive lower ratings). We obtain the same behavior in our
calculus by composing grouping operators:

\[
\Gamma = \text{student} : \text{mod}(\text{students})[\text{ratings}], \\
\text{course} : \text{mod}(\text{courses})[\text{ratings}], \\
\text{professor} : \text{mod}(\text{professors})[\text{courses}], \\
\text{department} : \text{mod}(\text{departments})[\text{courses}], \\
\text{service} : \text{mod}(2)[\text{courses}]
\]

\[
\text{Rating} = (1|\alpha) \text{ student} + (1|b) \text{ course.professor} + \\
((1|\epsilon) \text{ course.department}) \text{ course.service} + ?
\]

\[
\Pi = a : \text{real}(\text{students}), b : \text{real}(\text{professors}), \\
c : \text{real}(2)[\text{departments}]
\]

5. Reducing Multilevel to Classical Regression

5.1 Equivalent Formulations of the Radon Example

Hierarchical linear models can be written in several equivalent
forms, as demonstrated in section 12.5 of Gelman and Hill (2007).

The essence of such equivalence is that predictors can be placed
at different levels of a regression and it can be useful both for
understanding the model and for computational reasons. In this
section we use the Radon model as an example to present such
an equivalence and then we develop an equational theory for our
regression calculus. We use it to prove that every regression can be
reduced to a certain normal form.

Recall the Radon model with no or complete pooling from
Section 4.1 (where we omit the known precision \( \pi = 1/s \)):

\[
r_\alpha = (1|\alpha \sim \alpha) + u(b) + \text{Gaussian}(0, s) \\
r_\text{activity} = (1|\alpha \sim r_\alpha) c + f(\beta) + ? \\
\Pi = a, b, \alpha : \text{real}(\text{counties}), \beta : \text{real}
\]

We can write it using a single formula in three equivalent forms:

\[
r_\alpha = (1|\alpha \sim (1|\alpha \sim a) + u(b) + \text{Gaussian}(0, s)) c + f(\beta) + ? \\
r_\alpha = (c.u)(b) + (1|\alpha_\text{activity} \sim \text{activity}) c + f(\beta) + ? \\
r_\alpha = (1|\alpha) + (c.u)(b) + (1|\alpha_\text{activity} \sim \text{activity}) c + f(\beta) + ?
\]

The regressions are equivalent in the sense of producing
the same predictions and the same posteriors for parameters \( a, b, \beta \),
but their original parameters are slightly different. They are related by:

\[
\alpha_\Pi = \alpha_\text{activity} = a_\alpha[c] + c.a \times b = \alpha_\text{activity}[c] + c.a \times b.
\]

The \( r_\alpha \) form is particularly interesting in that the county level
contains no inner coefficients. Below we show that every multilevel
regression can be put in such a form.

5.2 Every Regression Has Single-Level Counterpart

We here give an algorithm that normalizes a regression to an equiva-
 lent single-level regression. We first define specific classes of
terms that help state the normal form, the algorithm, and its cor-
rectness theorem.

We use the path notation \( \mod{u,v} \) as short for \((u_1 \ldots u_n).v\). We write \( \mod{\mod{u,v}^n} \) for the regression \( r_1 + \ldots + r_n \), letting \( \mod{\emptyset} = \emptyset \). We also write \((\mod{v,\beta})r \) for \((v,\beta)\mod{r} \) and \((r)\mod{v} \) for \( r \).

Classes of Terms: \( N, P \)

\[
\begin{align*}
N & : \mod{\sum_{i=1}^{n} \mod{u_i}} \mid \mod{\sum_{i=1}^{n} \mod{v_i}} \\
P & : \mod{\sum_{i=1}^{n} \mod{w_i}} \mid \mod{\sum_{i=1}^{n} \mod{v_i}}
\end{align*}
\]

Every regression \( r \) normalizes to a single-level regression of the form \((\mod{\mod{v,\beta}})(P+N)\). Here \( \mod{\beta} \) contains all restrictions in \( r \), as well as the auxiliary coefficients that can be used to reconstruct
original coefficients, as seen above in Section 5.1. The term \( N \) describes the noise. \( P \) intuitively contains two different kinds of
terms: coefficients \( c(\mod{v,\beta}) \mod{w} \) and post-processing terms \( 0(\mod{\beta}) \mod{w} \) that compute the original coefficient \( \beta \) in terms of \( \beta \) (which appear as part of \( N \)).

The algorithm applies constant folding, writing \( c(u) \), to predictors:
paths ending in scalars simplify to the scalar, and interactions
with the scalars 0 and 1 are simplified (0 is an absorbing element
for interaction, 1 is a unit).

Normalization of regressions: \( r \mid \mod{w} \mod{(\mod{\mod{v,\beta}})(P+N)} \)

\[
\begin{align*}
\text{(NORM NOISE)} & : D(v_1, \ldots, v_n) \mid \mod{w} \mod{\emptyset + D(v_1, \ldots, v_n)} \\
\text{(NORM RES)} & : r \mid \mod{\mod{w}} \mod{\mod{\mod{v,\beta}}(P+N)} \mod{\beta} \mod{\mod{\mod{\beta}}(P+N)} \\
& \mod{(\mod{\mod{v,\beta}}(P+N)} \mod{\beta} \mod{\mod{\mod{\beta}}(P+N)} \\
\text{(NORM PLUS)} & : r_1 \mid \mod{\mod{\mod{\mod{v,\beta}}}}(P+N) \mod{\mod{\mod{\beta}}(P+N)} \\
& \mod{\mod{\mod{\mod{\beta}}(P+N)} \mod{\mod{\mod{\beta}}(P+N)} \\
\text{(NORM COEFF NOISE)} & : r \mid \mod{\mod{\mod{\mod{v,\beta}}}}(P+N) \mod{\mod{\mod{\beta}}(P+N)} \\
& \mod{\mod{\mod{\mod{\beta}}(P+N)} \mod{\mod{\mod{\beta}}(P+N)} \\
\text{(NORM COEFF)} & : r \mid \mod{\mod{\mod{\mod{v,\beta}}}}(P+N) \mod{\mod{\mod{\beta}}(P+N)} \\
& \mod{\mod{\mod{\mod{\beta}}(P+N)} \mod{\mod{\mod{\beta}}(P+N)} \\
\end{align*}
\]
Noise terms lose any grouping $\vec{w}$. Restrictions and groupings are simply recurred into. The normal form of a sum is the sum of the normal forms, rearranged to match the desired format. The interesting case is a predictor $v$ whose coefficient is given by an inner regression with normal form $(\vec{v}\vec{a})(P + N)$. If there are no inner coefficients (i.e., $P = \emptyset$), we simply rearrange the term to fit the format. Otherwise, we create a fresh coefficient $\vec{\beta}$ for the noise term $N$. Each term $u_i(\vec{\beta} \sim t_i) | \nu_i$ of $P$ is normalised to an interaction between the top-level predictor $v$ and the inner predictor $u_i$, where $u_i$ is obtained through the path $\vec{w}$. The normal form of a sum is the sum of interactions between the predictors $u_i$ in $P$ and their coefficients $\vec{\beta}_i$ (obtained via the path $\vec{v}_i$), plus the fresh coefficient $\vec{\beta}'$.

The special case treated by (NORM COEFF NOISE) ensures that the common patterns $v(t)$ and $?([\pi \sim N])$ normalise to themselves, i.e., $v[\alpha] | e \Downarrow v[t]$ and $?([\pi \sim N]) | e \Downarrow ?([\pi \sim N])$.

The normal form always exists, and is unique.

**Lemma 4.** For all $r, w$ there exist $\hat{\alpha}, P, N$ such that $r \Downarrow w \Downarrow (v\vec{a})(P + N)$, Moreover, if $r \Downarrow e \Downarrow (v\vec{a}')(P' + N')$ then $(v\vec{a}')(P + N) = (v\vec{a}')(P' + N')$.

The normal form has the same type as the original regression formula, and represents the same prior probability distribution.

**Theorem 2.** If $\Gamma; \bar{v} \models_r P \models e \Downarrow (v\vec{a})(P + N)$ then $\Gamma; \bar{v} \models e \Downarrow (v\vec{a})(P + N) ! P$.

**Proof:** The proof is via a typed equational theory for regressions, which is proven sound with respect to $\equiv$ via a typed equational theory for Fun.

In the Radon example at the beginning of this section, we have

$$r_1 \Downarrow (v\vec{a}_3)(\{a\} + c.a(b)) + 1\{\alpha_3 \sim Gaussian(0, s)\} | c) + \{0\{\alpha_3 \sim \delta a + \delta u : b + \delta \alpha_3\} | c) + f(\vec{\beta}) + ?$$

$$r_2 \Downarrow (v\vec{a}_3)(c.a[b] + 1\{a\} + 1\{\alpha_3 \sim Gaussian(0, s)\} | c) + \{0\{\alpha_3 \sim \delta a + \delta \alpha_3\} | c) + f(\vec{\beta}) + ?$$

Thus, the normal forms of both $r_1$ and $r_2$ have the same terms as $r_3$, apart from one additional term from which we immediately can read off the relationship between $\alpha_3$ and $\alpha_1$ (resp. $\alpha_2$).

When $r \Downarrow (v\vec{a})(P + N)$, the single-level regression $P + N$ is a flattened form of $r$, and can be solved by many methods. The multilevel regression $r$ directly expresses the modeller’s intent in terms of group-level coefficients that are modelled, and should be easier to read and understand as its structure follows the structure of the schema.

6. The Essence of R’s Regression Formulas

Here, we show that our regression calculus explains the formula languages recognized by three R methods: lm, lmer, and blmer.

6.1 lm

Base R (R Core Team 2015) provides a method `lm` for fitting linear regressions. The core syntax of formulas recognised by `lm` is:

$$v_1 + \cdots + v_n + 0$$

In addition, the following constructs recognised by `lm` can be seen as macros that expand to formulas in the core syntax: `^` (used for controlling the arity of interaction terms), `*` (xy is shorthand for x + y + x:y), `I` (used to indicate nesting of variable levels within each other, e.g., c/d desugars to c + c:d), and `I` (used to temporarily supplant the regression meanings of +, *, and ^ with their traditional arithmetic meanings). Furthermore, formulas can include transformations of variables, e.g., log(area). We omit these features because they can all be captured by +, *, and appropriate pre-processing of predictors.
on the coefficient values. $lmer$ formulas can be translated\(^2\) to our
calculus by extending our translation for $lm$ with the following rule
to handle random effects:
\[
[x_1 \ldots x_n | c_1 \ldots c_m | d_1 \ldots d_j] = (1 | x_1 | \ldots | x_n | c_1 | \ldots | c_m | d_1 | \ldots | d_j)
\]
Continuing the nutrition example introduced above, one possible
$lmer$ formula is:
```
bar.in.ravens.z ~ treatment + initial.age + (1|school)
```
Here, we fit a separate baseline for each school, expressing the
belief that baseline measures of cognition may differ across schools.
(e.g., one school may be in a wealthier area where parents can
afford after-school tutors).

It is worth noting that actually running this model in $lmer$
returns a degenerate result where all schools have the same baseline
(no variation across schools); this is because $lmer$ uses maximum
likelihood estimation, which can hit this boundary condition when
only small amounts of data are available. This problem can be
avoided by setting a priors on the fixed/random effects terms so as
to avoid the boundary, but $lmer$ does not support this.

6.3 $blmer$

The $blmer$ method, provided in the blme package (Dorie 2014),
uses the same syntax of regressions as $lmer$, but additionally sup-
ports setting priors on the fixed effects (Gaussian and $t$ priors only)
and random effects (arbitrary priors). To round out the nutrition ex-
ample, we can express partial pooling using $blmer$ like so:
```
blmer(ravens.z ~ treatment + initial.age + (1|school),
cov.prior = gamma(2.5, 0))
```
Here, we place a Gamma(2.5, 0) prior on the covariance matrix for
the terms specified by $(1|school)$. This essentially expresses the
belief that the variation in baseline cognitive scores across schools
should be non-zero. In our calculus, we can write the same model
using the regression formula
\[
treatment + initial.age + (1|\alpha \sim \pi|\pi \sim \text{Gamma}(2.5, 0)) | \text{school}
\]
Note that $blmer$ does not define a language for setting priors but
rather enables this through method arguments (e.g., the cov.prior
argument set above). By contrast, our calculus allows to set priors
compactly and compositionally. To our knowledge, our regression
calculus admits a superset of the models expressible in $blmer$.

7. Fabular = Tabular + Regression Formulas

Tabular (Gordon et al. 2014a, 2015) is a table-oriented probabilistic
programming language embedded in Excel. A Tabular program is a
schema that lists a sequence of named tables. In turn, each table is
described by sequence of named attribute declarations. A static
attribute declares a value that is shared amongst all rows of the table.
An inst (instance) attribute, on the other hand, declares a column of
values in that table. The definition of an attribute is a Fun expres-
sion that may refer to the value of any previously declared attribute,
whether static or, for an instance level attribute, the value of any
previous instance attribute (belonging to the same row). Tabular
expressions may dereference attributes of other tables using a dot-
notation like syntax that, for instance level columns, corresponds
to array indexing. Attributes declared as input take their (deter-
ministic) values from an input database (in Excel, the database is
the collection of Excel tables). Attributes marked as output have a
probabilistic definition—a Fun expression $E$—denoting a random
variable. If an output is present in the input database, then its deter-
imistic value is used to condition the static or instance level value
of the corresponding random variable. If an output is missing (that
is, null) in the database, then its output is given as the (marginal)
distribution of its definition. Output attributes may be missing from
all, some or none of the rows in a column. If the attribute name
is not present in the database, then the attribute is itself a latent
variable or table-sized collection of latent variables. Finally, local
attributes are similar to output attributes but statically scoped to
the current table. Unlike inputs and outputs, local attributes cannot
be referenced from other tables (even via links).

A Tabular schema defines a generative model of the database.
Conditioning the model on the database allow us to infer the distri-
butions of missing values and latent columns. Tabular is compiled
to a lower level Infer.NET model that performs message-passing
inference, yielding approximations to the marginal distributions of
all missing values.

In this section, we extend Tabular with linguistic support for
regressions. In Tabular, an attribute defined by a regression is ex-
pressed as a sequence of attributes defining the static parameters
of the regression and an eponymous body defined as a sum of prod-
ucts and noise. Schemas may contain multiple regressions. In the
full Tabular language, regressions may also occur within Tabular
functions, supporting useful abstraction.

7.1 Core Tabular (Review)

Databases Tabular acts on databases of the following form, a
slight generalization of the databases from Section 3.1 to include
singleton values (static($V$)) for static columns and the null value
(_$) denoting a missing value of any type.

Databases, Tables, Attributes, and Values:

\[
\delta_m := [t_1 \mapsto \pi \in \text{Gamma}(2.5, 0)] \quad \text{whole database}
\]
\[
\tau := [c_j \mapsto a_j \in \text{Gamma}(2.5, 0)] \quad \text{table in database}
\]
\[
a := \ell(V) \quad \text{attribute value: } V \text{ with level } \ell
\]
\[
V := \_ | \{V_0, \ldots, V_{n-1}\} \quad \text{nullable value}
\]
\[
\ell, pc := \text{static} | \text{inst} \quad \text{level (static < inst)}
\]

Schemas We use the Fun types $T$ from Section 3.1 in Tabular.
We write link($t$) as a shorthand for mod($t$), for foreign keys to
the table. Tabular expressions are just Fun expressions extended with
a construct $E : t . c$ to dereference columns of other tables. In $E : t . c$
we expect that $E : \text{link}(t)$ and $c$ is a column of table $t$.

A Tabular schema is a sequential declaration of named tables.
Table $S$ are sequential declarations of named static or inst level
attributes. The definition of a input attribute must be the empty model
$\epsilon$; other attributes must have a model that is a proper expression $E$.

Tabular Schemas:

\[
S ::= [(t_1 = T_1); \ldots; (t_n = T_n)] \quad \text{(database schema)}
\]
\[
T ::= [c_1; \ldots; c_m] \quad \text{(table or function)}
\]
\[
col ::= (c : T \mid \text{viz } M) \quad \text{attribute } c \text{ declaration}
\]
\[
viz ::= \text{input} | \text{local} | \text{output} \quad \text{visibility}
\]
\[
M ::= \epsilon | E \quad \text{model expression}
\]

7.2 Fabular: Extending Tabular with Formulas

We endow Tabular with regression syntax by extending the syntax
of model expressions with regression formulas (and predictors):

Fabular model expressions:

\[
M ::= \ldots | \sim r \quad \text{model expression}
\]

\(^2\) However, there is one subtle point of difference. In $lmer$, all
coefficients in a single random effects term are assumed to be correlated,
e.g., in $(1 + x + y|c)$, it is assumed that, a priori, the intercept 1 and slope
terms for $x$ and $y$ have some correlation (Bates et al. 2014). This default behavior

can be overridden using a double bar, e.g., $(1 + x + y||c)$. Thus, the single
bar in our calculus actually corresponds to the double bar in $lmer$. \_]
The meaning of a well-typed regression attribute is given by a simple translation to Tabular:

**Translation of Regressions to Tabular:** \[
[[r]] \text{ } \vec{F} \hat{=} (\vec{T}, \vec{E})
\]

where \((\vec{T}, \vec{E}) = [[r]] \vec{F}\) and

\[
\begin{align*}
\langle D(u_1, \ldots, u_n) \rangle \vec{F} \hat{=} ((), D([u_1], \ldots, [u_n]) \vec{z}) \\
\langle \mu (\alpha \sim r) \rangle \vec{F} \hat{=} \text{let } (\vec{T}, \vec{E}) = [[r]] \vec{F} \text{ in } \\
(\vec{T} \bowtie (\alpha : \text{real} / \text{static output } E)), \\
\langle \vec{v} \parallel \vec{z} \times \alpha(\vec{F}) \rangle
\end{align*}
\]

- \([r + r'] \vec{F} \hat{=} \text{let } (\vec{T}, \vec{E}) = [[r]] \vec{F} \text{ in } \\
\text{let } (\vec{T}', \vec{E}') = [[r']] \vec{F} \text{ in } (\vec{T} \bowtie \vec{T}', \vec{E} + \vec{E}')

- \((\forall \alpha r) \vec{F} \hat{=} \text{let } (\vec{T}, \vec{E}) = [[r]] \vec{F} \text{ in } \\
\text{let } \vec{T}' = \llbracket \text{for } (\beta : T \ell \bowtie viz \vec{F}) \text{ in } \vec{T} \rightarrow \\
(\beta : T \ell \text{ if } \beta = \alpha \text{ then local else viz } F) \rrbracket \text{ in } (\vec{T}', \vec{E})
\]

**Attribute \(c\) Defined by Regression Formula \(r\):**

\[
\langle (c : \text{real inst } viz \sim \sim r) \rangle \vec{F} \hat{=} \text{let } (\vec{T}, \vec{E}) = [[c : \text{real inst } viz E]) \text{ in } (\vec{T}, \vec{E})
\]

The Fabular translation function \([r] \vec{F}\) mirrors our earlier translation to Fun, but now returns a pair of a Tabular \(\vec{T}\) and an expression \(E\). The table binds the parameters introduced by the regression into static attributes; the expression \(E\) is the body of the regression that is used as the model of the inst-level attribute \(c\). Though syntactically different, this translation is semantically equivalent to our previous semantics—it is re-factored to introduce a single nested loop per regression rather than one loop per regression term. The local function \([r]\) constructs the body of this loop, for fixed loop variables \(\vec{z}\) with bounds \(\vec{e}\). This scheme yields more legible code but does not affect (nor improve) the computational complexity of the model. One subtlety is that setting \(\vec{e}\) to the empty list in the initial call to \([r] [\List] [\List] [\List]\) ensures that local predictors are referenced directly and not inappropriately indexed.

For Fabular, the translation of variable predictors must also be slightly adjusted. The translation is induced by the Tabular context — it is the identity on locally declared attributes, introduces a static reference \((t.c)\) for a predictor \(c\) declared statically in table \(t\), and an instance reference \((E : c)\) for a predictor \(c\) declared at instance level in table \(t\). We elide the details.

### 7.3 Implementation of Fabular

We have an implementation of Fabular that includes support for vectorized regressions and syntactic sugar (not shown here) to omit parameter names and default priors on coefficients and noise. We have successfully run Fabular on a variety of models, including all of the ones mentioned in the paper. The add-in allows the user to selectively reduce Fabular regression to equivalent Tabular programs as well as extract auto-generated C# code to construct the Infer.NET model. The following table shows the code expansions (measured in LOC) from Fabular to Tabular to C# Infer.NET modeling code on a selection of models, together with the runtime for inference. We use the auto-generated code as a proxy for the length of the manually coded, equivalent Infer.NET model.

<table>
<thead>
<tr>
<th>Model</th>
<th>Fabular</th>
<th>Tabular</th>
<th>C#</th>
<th>Runtime</th>
</tr>
</thead>
<tbody>
<tr>
<td>MatchBox</td>
<td>30</td>
<td>37</td>
<td>522</td>
<td>5s</td>
</tr>
<tr>
<td>LinearClassifier</td>
<td>6</td>
<td>9</td>
<td>130</td>
<td>1.5s</td>
</tr>
<tr>
<td>Radon</td>
<td>6</td>
<td>13</td>
<td>117</td>
<td>1.5s</td>
</tr>
<tr>
<td>Cheese</td>
<td>9</td>
<td>13</td>
<td>99</td>
<td>6.7s</td>
</tr>
<tr>
<td>Elections</td>
<td>31</td>
<td>53</td>
<td>373</td>
<td>2.5s</td>
</tr>
</tbody>
</table>

We compared Fabular, which performs inference with expectation propagation (EP), with three other systems: STAN (Stan Development Team 2014b), which implements an HMC-based method called NUTS; lmer (Bates et al. 2014), an R package that performs maximum likelihood estimation for hierarchical linear models; and bmer (Dorie 2014), which augments lmer with Bayesian priors.

We compared these in modelling a data set taken from Gelman and Hill (2007) that contains radiation measurements taken in houses across different counties in Minnesota. We use the hierarchical model \(r_7\) from Section 4.1, which defines radiation activity in terms of the floor of the measurement (basement versus first floor) and the county. Not directly runnable in lmer or bmer, we reduced the model to activity \(\sim floor + (1 | county) + uranium\).

The floor coefficients in the Fabular/STAN and classical forms are directly comparable. We compared the 1\(\{x \sim 1 + u + v\}\) for Fabular/STAN with the uramium term of lmer/bmer by comparing the county-intercept-versus-uranium slope for Fabular/STAN with the activity-versus-uranium slope for lmer/bmer. Estimates of model coefficients are shown in Figure 1. Observe that Fabular produces similar results to the other three systems. This indicates that the expectation propagation algorithm of Infer.NET gives viable results for inference in hierarchical models.

### 7.4 Example: Generalized Linear Models

A **generalized** linear model passes its continuous output through a **link** function; for example, in logistic regression the output passes through a non-linear **logistic** function to produce a sigmoidal output. The regression calculus does not include link-functions on outputs, although it would be a simple extension.

Adding regressions to Tabular extends our reach to such generalized linear models. Take for example:

<table>
<thead>
<tr>
<th>table Data</th>
<th>X1</th>
<th>real</th>
<th>input</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X6</td>
<td></td>
<td>real</td>
<td>input</td>
</tr>
<tr>
<td>Z</td>
<td>real</td>
<td>output</td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>bool</td>
<td>output</td>
<td></td>
</tr>
</tbody>
</table>

stores in 46 cities belonging to 50 chains. The Elections model is a classic hierarchical regression discussed by Gelman and Hill (2007) and predicts the outcome of a US election given 561 past state election outcomes across 50 states and 12 years.
Table Data consists of six real-valued clinical measurements $X_1$ through $X_6$ and a Boolean label $Y$ to be predicted (given some labelled training points). This model is an instance of the Bayes Point Machine (Minka 2001), a boolean classifier, in which the prior over an implicit weight vector is drawn from a vector of Gaussian priors, and the label $Y$ is generated by thresholding a latent score $Z$ (the inner product of the weight and input vectors). Intuitively, $Z$ is the distance of the point $X_1,...X_6$ from the hyperplane defined by the weight vector; $Z$'s sign determines the side of the hyperplane occupied by the point. Here, the weight vector is implicitly defined by the parameters of the regression; thresholding is the link function. (Gordon et al. 2015) describe a more verbose variant of this model.

7.5 Example: Latent Variable Mixture Model

Fabular goes beyond traditional regression. Predictors are no longer restricted to deterministic input data: they can be partially or even completely unobserved random variables. For example, consider the task of regressing points belonging to a mixture of various lines. If each point is labelled as belonging to a given class then we can use the label as the predictor that groups a family of linear regressions and infer the parameters of each line.

However, we can also generalize this model by assuming that only a subset of the points have been labelled. In Figure 2, we explicitly impose a discrete distribution on each point’s class. The figure contains both the Fabular code and its expansion to Core Tabular. The result is a supervised classifier, that infers the most likely class that each unlabelled point belongs to. (The operator $|$ has lower precedence than $+$ so the entire sum is grouped by Class.)

For example, here is a plot we generated using synthetic data from a mixture of three lines (training data points in light blue).

The blue, purple and green points are correctly classified, but the red ones were incorrectly classified. The red points all lie close to the intersections of the lines and are thus harder to separate.

If we furthermore assume that none of the points are labelled, then we obtain an unsupervised clustering algorithm, that partitions the points into three distinct sets of similar points.

7.6 Extension: Vectorized Regression

Until now our predictors have had scalar types, but our notation extends naturally to vectorized regressions, whose predictors have vector types and contain arrays of scalars. The vectorized notation is convenient shorthand for the simultaneous definition of a family of regressions. Our Matchbox example, presented in the sequel, illustrates the feature. To generalize the notation, we index our regression judgments by the dimensionality $d$ of the regression. The scalar dimensionality $\bullet$ indicates a regression or predictor producing a single scalar value (as before). The vector dimensionality $\bullet[n]$ indicates a regression or predictor producing an $n$-vector of scalar values. We also define the operation of dimensioning a type, written $d(T)$: $\bullet(T) \triangleq T$ is just the identity on types while $\bullet[n](T) \triangleq T[n]$, vectorizing its type argument.

### Dimensionalities of Types:

$$d ::= \bullet \mid \bullet[n] \quad \text{scalar or vector dimensionality}$$

The typing rules for regressions are indexed by an additional dimensionality $d$ and enforce that the dimensions of sub-regressions and coefficients are invariant and thus consistent.

#### Judgments of the Vectorized Type System:

$$\Gamma ; \vec{e} \vdash_d v : T \quad \text{d-dimensional predictor has type } T$$

$$\Gamma ; \vec{e} ; f \vdash_d r ! \Pi \quad \text{d-dimensional regression } r \text{ exports } \Pi$$

Most rules merely propagate the invariant into sub-expressions. Rule (COEFF-VEC) generalizes rule (COEFF): it requires the type of the predictor expression to match the expected dimension $d$ and vectorizes the parameters $\alpha$ when $d$ is a vector. Rule (SUM-VEC) restricts its terms to have the same dimensionality.

#### Typing Rules for Vectorized Regressions (extract):

(COEFF-VEC)

$$\Gamma ; \vec{e} \vdash_d v : d(\text{real}) \quad \Gamma ; \vec{f} ; [\|] \vdash_r r ! \Pi \quad \alpha \not\in \text{dom}(\Gamma, \Pi)$$

$$\Gamma ; \vec{e} ; f \vdash_d v(\alpha \sim r) ! (\Pi, \alpha: d(\text{real})[\vec{f}])$$

(SUM-VEC)

$$\Gamma ; \vec{e} ; \vec{f} \vdash_d r ! \Pi \quad (\Gamma, \Pi) ; \vec{e} ; f \vdash_d r' ! \Pi'$$

$$\Gamma ; \vec{e} ; \vec{f} \vdash_d r + r' ! (\Pi, \Pi')$$

Full typing judgments are presented in (Borgström et al. 2015).

The details of the vectorizing translation are straightforward but omitted for the sake of brevity. Constant predictors are replicated as vectors of constants; vectorized coefficient terms translate to vectors of component-wise products and sums of vectorized regressions translate to component-wise sums of vectors.

In our Fabular implementation, the decision to vectorize a regression is driven by the type of the attribute defined by that regression. For added convenience, scalar predictors appearing in a context expecting a vector are implicitly cast into the appropriately sized vector (by replication).

7.7 Example: Matchbox

Matchbox (Stern et al. 2009) is a recommender system that works by integrating metadata about users and items. The system represents users and items as trait vectors in a common latent space and defines the affinity between users and items as the dot product of their vectors. Matchbox then uses this affinity to predict the ratings that a user would give to an item. We demonstrate that, to our surprise, this matrix model of recommendations can be concisely captured as a Fabular vectorized regression.

In Matchbox, $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{m} \in \mathbb{R}^m$ are sparse binary vectors representing metadata for users and items, respectively. For illustration, we might have $n=3$ users and $m=2$ movies as items:

$$\mathbf{u} = \begin{pmatrix} \text{userId}_0 \\ \text{userId}_1 \\ \text{userId}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{m} = \begin{pmatrix} \text{movielId}_0 \\ \text{movielId}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, rows are labelled by values for dimensions of the metadata, e.g., $\mathbf{u}$ has one row per user id, while $\mathbf{m}$ has one row per movie id. We project $\mathbf{u}$ and $\mathbf{m}$ into a common trait space by left-multiplying them by random matrices $UT$ and $MT$, which have dimensions $k \times n$ and $k \times m$, yielding $\mathbf{Ut} = UT\mathbf{u}$ and $\mathbf{Mt} = MT\mathbf{m}$, e.g., for $k=2$ and trait space $\mathbb{R}^2$:

$$\mathbf{Ut} = \begin{pmatrix} UT_{00} & UT_{01} & UT_{02} \\ UT_{10} & UT_{11} & UT_{12} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} UT_{02} \\ UT_{12} \end{pmatrix}$$

Where the individual $UT_{ij}$ and $MT_{ij}$ components are drawn from independent Gaussians (note that we sample a single $UT$ and $MT$
and use this for all users and movies, respectively). Because \( \mathbf{u} \) and \( \mathbf{m} \) are sparse binary vectors, \( \mathbf{ut} \) and \( \mathbf{mt} \) each denote a single column of \( \mathbf{UT} \) and \( \mathbf{MT} \). Thus, we can view \( \mathbf{ut} \) and \( \mathbf{mt} \) as vector intercepts, selected by the encoded values, \( \mathbf{u} \) and \( \mathbf{m} \), of the user and movie ids.

For a generic user \( \mathbf{u} \), we can express the trait vector componentwise using \( k \) scalar regressions or, more succinctly, as a single \( k \)-dimensional vectorized regression:

\[
\mathbf{ut} = \begin{pmatrix} u_{t0} \\ \vdots \\ u_{tk} \end{pmatrix} = \begin{pmatrix} (1 \{ U_{t0} \sim \text{Gaussian}(0, 1) \}) | \text{userId} \\ \vdots \\ (1 \{ U_{tk} \sim \text{Gaussian}(0, 1) \}) | \text{userId} \end{pmatrix} = (1 \{ UT \sim \text{Gaussian}(0, 1) \}) | \text{userId}
\]

Trait vectors, the crucial data representation in Matchbox, are surprisingly compact in vectorized Fabular.

A more realistic model exploits features of the entities, such as the gender of users or the genre of movies. The features are used to impose pooled priors on the Gaussian trait vectors, grouped by feature. The pooled prior addresses the cold-start problem by assuming that new users behave like other users of the same gender:

\[
\mathbf{utm} = (1 \{ \mathbf{um} \sim \text{Gaussian}(0, 1) \}) | \text{gender}
\]

\[
\mathbf{ut} = (1 \{ UT \sim \delta \mathbf{utm} \}) | \text{userId}
\]

See Figure 3 for the full model, which exploits features in this way but also includes bias terms for individuals users and movies. The model also adds an ordinal regression response that thresholds ratings to a 5 point scale — illustrating a link function.

8. Conclusions

We set out to enrich probabilistic programming languages with the regression formulas from R. Our regression calculus amounts to an explicit description of the underlying syntax and (probabilistic) semantics of R formulas. We embed the calculus in Tabular, and hence allow users of a particular probabilistic programming system to write models with formulas. They go beyond R in setting priors for coefficients, using latent variables, tolerating missing input values, and getting the benefits of efficient inference using Infer.NET.

Tabular seeks to empower spreadsheet users with probabilistic models. By incorporating R’s popular formula notation, we hope Fabular makes it easier for spreadsheet users to get started with modelling, as they need only specify a single formula. In future work, we aim to develop a graphical user interface to ease the construction of Fabular formulas, and to help configure priors.

And formulas would make a great feature for other languages!

Acknowledgments We are grateful for useful discussions with Aditya Nori and Tom Minka. Adam Ścibior received travel support from the DARPA PPAML programme. Marcin Szymczak was supported by Microsoft Research through its PhD Scholarship Programme.
References


