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Abstract

We study the representative consumer's risk attitude and efficient risk-sharing rules in a single-period, single-good economy in which consumers have homogeneous probabilistic beliefs but heterogeneous risk attitudes. We prove that if all consumers have convex absolute risk tolerance, so must the representative consumer. We also identify a relationship between the curvature of an individual consumer's individual risk sharing rule and his absolute cautiousness, the first derivative of absolute risk-tolerance. Some consequences of these results and refinements of these results for the class of HARA utility functions are discussed.

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1 Introduction

We consider an exchange economy under uncertainty with a single good and a single consumption period, in which all consumers hold common probability assessments over the state space and yet differing expected utility functions. Two well known properties hold for each Pareto efficient allocation in such an economy. First, every consumer's consumption level is uniquely determined by the aggregate consumption level. Hence every consumer's state-contingent consumption levels can be specified as a function, called the *risk sharing rule*, of aggregate consumption levels. Second, there exists a *representative consumer*, having an expected utility function, in the sense that the support price of the single-consumer economy consisting solely of the representative consumer is also the support price for the Pareto efficient allocation of the original, multi-consumer economy. Hence, knowing the representative consumer's risk attitude is sufficient to price all assets in financial markets.

The benchmark result on this subject matter is the *mutual fund theorem*. Define *absolute risk tolerance* as the reciprocal of the Arrow-Pratt measure of absolute risk aversion, and call its first derivative *absolute cautiousness*. Then, hyperbolic absolute risk aversion, linear (or, to be more precise, affine) absolute risk tolerance, and constant absolute cautiousness are all equivalent properties of an expected utility function u , and mathematically boil down to the existence of a $\tau \in \mathbb{R}$ and a $\gamma \in \mathbb{R}$ such that

$$-\frac{u'(x)}{u''(x)} = \tau + \gamma x$$

for every x . In particular, this property is met if u exhibits constant absolute or relative risk aversion. The mutual fund theorem states that if all consumers have a constant, common absolute cautiousness γ , then the representative consumer also has the same constant absolute cautiousness γ and all individuals' risk-sharing rules are linear (affine). In this paper, we drop the assumption of a constant, common absolute cautiousness and analyze the implication of heterogeneous absolute cautiousness on the risk-sharing rules and the representative consumer's risk attitude. As can be inferred from existing results dispersed in the wide range of literature, the mutual fund theorem would not hold without the assumption. The contribution of this paper is, in short, to provide a detailed description of the way in which the representative consumer's absolute cautiousness is *not* constant and the risk-sharing rules are *not* linear in this environment.

It has been well perceived in the literature that the assumptions for the mutual fund

theorem are so stringent that the applicability of the theorem is questionable in both economics and finance. While there have been many contributions dealing with cases in which the assumptions are not met,¹ they tend to concentrate on rather special cases with regards to consumers' risk attitudes, the number of consumers in the economy, wealth distributions across consumers, and probabilistic distributions of initial endowments and asset returns. Moreover, they often appeal to numerical, as opposed to analytical, methods, without fully clarifying the principles behind their results.

We find this situation rather unsatisfactory. The reason is that while the assumptions that the mutual fund theorem imposes on consumers' risk attitudes are stringent, the theorem does not require any additional assumption on the number of consumers, wealth distributions, or asset returns. In this paper, we obtain qualitative results concerning the risk sharing rules and the representative consumer's risk attitude that do not depend on these characteristics of the economy. We do not obtain any calibration results or closed-form solutions, but we believe that this paper is an important theoretical contribution to the literature, because it uncovers some important phenomena arising exclusively from the nature (in particular, heterogeneity) of the consumers' risk attitude. Let us also remark that should the financial markets be complete, the equilibrium allocations are Pareto efficient, and our results are therefore true for all equilibrium allocations.

Throughout the paper we establish our results for the static, one-period model. It can be shown (Hara, in preparation) that all the results can be extended to the multi-period case provided all consumers have time-homogeneous and time-separable expected utility functions and the same time-discount rate. Hence, our results are directly comparable with dynamic models such as those of Mehra and Prescott (1985), Dumas (1989), Campbell and Cochrane (1999), Wang (1996), Benninga and Mayshar (2000), and Chan and Kogan (2002), where there are multiple (possibly continuous and infinite) consumption periods and a common discount rate is assumed.

In Section 3, we establish results on the effect of heterogeneity of consumers' risk attitudes on the absolute cautiousness of the representative consumer. The formula of Theorem 5 expresses the derivative of the absolute cautiousness of the representative consumer as the sum of two components. The first one is a weighted sum of the derivatives of the individual consumers' absolute cautiousness, and the second is a positive multiple of the weighted vari-

¹We refer to some of these contributions in the rest of this introduction and Section 7.

ance of the absolute cautiousness across all individual consumers. A corollary to this theorem (Corollary 6) is that if every consumer exhibits convex risk tolerance (non-increasing cautiousness), then so does the representative consumer; and that any heterogeneity in absolute cautiousness leads to strictly convex risk tolerance. In Propositions 11 to 13, we show that the representative consumer's absolute cautiousness tends to the absolute cautiousness of the most absolutely cautious individual consumer in the economy as the aggregate consumption level tends to its upper bound (which may be finite or infinite); and that it tends to that of the least absolutely cautious individual consumer as the aggregate consumption level tends to its lower bound (which may be finite or negative infinite).

Our results indicate that the risk attitude of the representative consumer may well be qualitatively different from the risk attitude of any individual in the economy. Implications of this fact on asset pricing will be discussed in Section 7.

The crucial result about consumers' risk-sharing rules (Proposition 4) builds on results of Wilson (1968). It relates the curvature of an individual consumer's risk-sharing rule to how the individual's cautiousness compares to the cautiousness of the representative consumer. More specifically, a risk-sharing rule is locally convex, concave, or linear if and only if the individual's cautiousness is locally greater than, smaller than, or equal to the representative consumer's cautiousness. The result also allows us to rank the curvature of the individual consumers' risk-sharing rules according to their cautiousness.

The behavior of the risk-sharing rules as the aggregate consumption level tends to the upper or lower bounds is described by Propositions 11 and 13. The results state that as the aggregate consumption level tends to the upper bound, the most absolutely cautious consumers' share of consumption as well as their marginal increment in consumption converge to one; and that as the aggregate consumption level tends to the lower bound, the same is true for the least absolutely cautious consumers. Hence the distribution of the individual consumers' consumption levels are more biased when the realization of the aggregate endowment is very large or very small than when it is of a modest value.

Much stronger results can be obtained when all individual consumers exhibit constant cautiousness, and the constants differ across them. We show (Theorem 18) that an individual consumer's risk-sharing rule can then be only of three types, depending on the individual's absolute cautiousness. Each least absolutely cautious consumer has an everywhere strictly concave risk-sharing rule. Each most absolutely cautious consumer has an everywhere strictly

convex risk-sharing rule. Any of the other consumers has a risk-sharing rule that is initially convex up to a unique inflection point and concave thereafter. The inflection points, furthermore, are ordered according to the consumers' absolute cautiousness, so that the more absolutely cautious the consumer the lower the inflection point of his risk-sharing rule.² This is illustrated in Figure 1 in Section 6. Implications to the literature on portfolio insurance are mentioned in Section 7.

This paper is organized as follows. Section 2 states the model and gives a few preliminary results on the representative consumer's risk attitude. In Section 3 the curvature of an individual's risk-sharing rule is related to the difference between his absolute cautiousness and the representative consumer's counterpart. Section 4 gives the formula expressing the derivative of absolute cautiousness of the representative consumer in terms of those of the individual consumers. Section 5 investigates the limiting behavior of the representative consumer's risk attitude and of the risk-sharing rules when aggregate consumption tends to the upper or lower bounds. In Section 6 refinements of the previous results are obtained for the case when all individual consumers exhibit linear absolute risk tolerance. Much of the discussion of the consequences and implications of our results is deferred to Section 7 which also concludes.

2 Model

There are I consumers, $i \in \{1, \dots, I\}$. Consumer i has a von-Neumann Morgenstern (also known as Bernoulli) utility function $u_i : (\underline{d}_i, \bar{d}_i) \rightarrow \mathbb{R}$, where $\underline{d}_i \in \mathbb{R} \cup \{-\infty\}$, $\bar{d}_i \in \mathbb{R} \cup \{\infty\}$, and u_i is infinitely many times differentiable and satisfies $u'_i(x_i) > 0$ and $u''_i(x_i) < 0$ for every $x_i \in (\underline{d}_i, \bar{d}_i)$.

The uncertainty of the economy is described by a probability measure space (Ω, \mathcal{F}, P) . The probability measure P specifies the common (objective) belief on the likelihood of the states. Denote by E the expectation with respect to P . The aggregate endowment of the economy and each consumer's consumption are both random variables on the probability measure space.

For each consumer i , we define his *consumption set* Z_i to be

²We should also add that Kurosaki (2001) claimed that if all consumers exhibit constant relative risk aversion, then the *logarithmic* risk-sharing rule, which assigns the mean of the logs of the consumers' consumption levels to each individual consumer's consumption level, is linear with a slope proportional to his own relative risk tolerance.

$\{\zeta_i \in L^1(\Omega, \mathcal{F}, P) \mid \underline{d}_i < \zeta_i < \bar{d}_i \text{ almost surely}\}$. Define $Z_i^* = \{\zeta_i \in Z_i \mid u_i(\zeta_i) \in L^1(\Omega, \mathcal{F}, P)\}$. Then Z_i^* is the set of random variables ζ_i for which the expected utility $E(u_i(\zeta_i))$ is finite. Note that since u_i is strictly concave, Z_i^* is a convex set. Moreover, for every $x_i \in (\underline{d}_i, \bar{d}_i)$, $u_i(\zeta_i) \leq u'_i(x_i)(\zeta_i - x_i) + u_i(x_i)$. The right hand side of this inequality is integrable, and hence the positive part $u_i(\zeta_i)^+$ of $u_i(\zeta_i)$ is integrable for every $\zeta_i \in Z_i$. Hence, $\zeta_i \in Z_i^*$ if and only if the negative part $u_i(\zeta_i)^-$ is integrable.

Define a binary relation \succsim_i on Z_i by letting, for each $\zeta_i \in Z_i$ and $\eta_i \in Z_i$, $\zeta_i \succsim_i \eta_i$ if and only if either of the following two conditions is met: $\eta_i \notin Z_i^*$; or $\zeta_i \in Z_i^*$, $\eta_i \in Z_i^*$, and $E(u_i(\zeta_i)) \geq E(u_i(\eta_i))$. Then \succsim_i is reflexive, transitive, and complete. Denote its strict part by \succ_i and symmetric part by \sim_i , then $\zeta_i \succ_i \eta_i$ for every $\zeta_i \in Z_i^*$ and every $\eta_i \notin Z_i^*$, and $\zeta_i \sim_i \eta_i$ for every $\zeta_i \notin Z_i^*$ and every $\eta_i \notin Z_i^*$. Thus the random variables ζ_i for which $u_i(\zeta_i)$ is not integrable are the least preferable ones. Since $u_i(\zeta_i)$ is integrable if and only if the negative part $u_i(\zeta_i)^-$ is integrable, the way we have defined \succsim_i is intuitively consistent with the expected utility calculation.

A consumption allocation $(\zeta_1, \dots, \zeta_I) \in Z_1 \times \dots \times Z_I$ is *feasible* for an aggregate endowment ζ if $\sum \zeta_i = \zeta$ almost surely. A feasible consumption allocation $(\zeta_1^*, \dots, \zeta_I^*) \in Z_1 \times \dots \times Z_I$ is *efficient* (in the sense of Pareto) for an aggregate endowment ζ if there is no other feasible consumption allocation $(\zeta_1, \dots, \zeta_I) \in Z_1 \times \dots \times Z_I$ for ζ such that $\zeta_i \succsim_i \zeta_i^*$ for every i , and $\zeta_i \succ_i \zeta_i^*$ for some i . While we shall not give a formal proof, it is easy to check that, for every aggregate endowment ζ , if there exists a feasible allocation $(\zeta_1, \dots, \zeta_I)$ for ζ such that $\zeta_i \in Z_i^*$ for some i and if $(\zeta_1^*, \dots, \zeta_I^*)$ is an efficient allocation of ζ , then $\zeta_i^* \in Z_i^*$ for every i . In short, if the aggregate endowment is sufficiently far away from the lower bound \underline{d} so that some consumer can attain a finite utility level, then every consumer attains a finite utility level at every efficient allocation. This, in particular, implies that when the aggregate endowment is sufficiently far away from the lower bound \underline{d} , an allocation is efficient if and only if it is efficient when the comparison is restricted to Z_i^* .

It follows from the separating hyperplane theorem that a feasible allocation $(\zeta_1^*, \dots, \zeta_I^*) \in Z_1^* \times \dots \times Z_I^*$ is efficient if and only if there exists a $\lambda \in \mathbb{R}_{++}^I$ such that it is a solution to the maximization problem

$$\begin{aligned} & \max_{(\zeta_1, \dots, \zeta_I) \in Z_1^* \times \dots \times Z_I^*} \sum \lambda_i E(u_i(\zeta_i)), \\ & \text{subject to} \quad \sum \zeta_i = \zeta \text{ almost surely.} \end{aligned} \tag{1}$$

Furthermore, the assumption of a common probabilistic belief and expected utility allows the

efficient allocations to be represented in terms of risk-sharing rules. Write $\underline{d} = \sum \underline{d}_i$ and $\bar{d} = \sum \bar{d}_i$. A *risk-sharing rule* is an infinitely many times differentiable function $f : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_1, \bar{d}_1) \times \cdots \times (\underline{d}_I, \bar{d}_I)$ that satisfies $\sum f_i(x) = x$ for every $x \in (\underline{d}, \bar{d})$, where f_i is the i -th coordinate function of f . Note that if ζ is an aggregate endowment with $\underline{d} < \zeta < \bar{d}$ almost surely and if $f_i(\zeta) \in L^1(\Omega, \mathcal{F}, P)$ for every i , then $(f_1(\zeta), \dots, f_I(\zeta))$ is a feasible consumption allocation for ζ .

For each $\lambda = (\lambda_1, \dots, \lambda_I) \in \mathbb{R}_{++}^I$ and each $x \in (\underline{d}, \bar{d})$, consider the following maximization problem:

$$\begin{aligned} \max_{(x_1, \dots, x_I) \in (\underline{d}_1, \bar{d}_1) \times \cdots \times (\underline{d}_I, \bar{d}_I)} \quad & \sum \lambda_i u_i(x_i), \\ \text{subject to} \quad & \sum x_i = x. \end{aligned} \tag{2}$$

By strict concavity for each x , there exists at most one solution to this problem, which we denote by $f_\lambda(x)$. In general, there may not be any solution for some values of x and λ , because the intervals $(\underline{d}_i, \bar{d}_i)$ are open. In particular, it is possible that for every $\lambda \in \mathbb{R}_{++}^I$ there exist some x for which the maximization problem has no solution. In such a case, there may not exist any efficient allocation at all. However, if the u_i satisfy the Inada condition, that is, $u'_i(x_i) \rightarrow \infty$ as $x_i \rightarrow \underline{d}_i$ and $u'_i(x_i) \rightarrow 0$ as $x_i \rightarrow \bar{d}_i$, then, for every λ and x , there exists a solution. This is proved in Appendix A. Then, for every λ , the mapping $f_\lambda : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_1, \bar{d}_1) \times \cdots \times (\underline{d}_I, \bar{d}_I)$ is well defined. We shall assume this throughout the paper. Since f_λ is smooth by the implicit function theorem, it is a risk-sharing rule. It is straightforward to show that $(\zeta_1^*, \dots, \zeta_I^*) \in Z_1^* \times \cdots \times Z_I^*$ is a solution to the maximization problem (1) if and only if $\zeta_i^* = f_{\lambda i}(\zeta)$ for every i . This argument establishes the following lemma, which can be traced back to Borch (1962, p. 428) and Wilson (1968), and is nicely explained in Kreps (1990, Section 5.4).

Lemma 1 *If $(\zeta_1^*, \dots, \zeta_I^*) \in Z_1^* \times \cdots \times Z_I^*$ is an efficient allocation of the aggregate endowment ζ , then there exists a $\lambda \in \mathbb{R}_{++}^I$ such that $\zeta_i^* = f_{\lambda i}(\zeta)$ for every i . Conversely, for every $\lambda \in \mathbb{R}_{++}^I$, if $f_{\lambda i}(\zeta) \in Z_i^*$ for every i , then $(f_{\lambda 1}(\zeta), \dots, f_{\lambda I}(\zeta))$ is an efficient allocation of ζ .*

As pointed out earlier, if the aggregate endowment ζ is sufficiently far away from the lower bound \underline{d} , then the conditions $\zeta_i^* \in Z_i^*$ and $f_{\lambda i}(\zeta) \in Z_i^*$ are redundant.³ By virtue of this lemma, we say that a risk-sharing rule f is *efficient* if there exists a $\lambda \in \mathbb{R}_{++}^I$ such that $f = f_\lambda$.

³Dumas (1989) also investigated necessary and sufficient conditions for the welfare maximization problem of the type (1) to have a solution in a dynamic model.

Let f be an efficient risk-sharing rule. Denote the maximum attained in the problem (2), with the same λ as corresponds to f , by $u(x)$. We are thereby defining a function $u : (\underline{d}, \bar{d}) \rightarrow \mathbb{R}$, which is the value function of the problem. Since

$$\sum \lambda_i E(u_i(f_i(\zeta))) = E\left(\sum \lambda_i u_i(f_i(\zeta))\right) = E(u(\zeta))$$

if $f_{\lambda i}(\zeta) \in Z_i^*$ for every i , the function u can be interpreted as the von-Neumann Morgenstern utility function of the representative consumer corresponding to the efficient risk-sharing rule f . Note that the assumption of the common probabilistic belief is crucial for this interpretation of u . By the implicit function theorem, u is smooth. To contrast with the representative consumer, we sometimes refer to the I consumers as *individual* consumers.

The Arrow-Pratt measure of absolute risk aversion of consumer i is defined as

$$a_i(x_i) = -\frac{u_i''(x_i)}{u_i'(x_i)} > 0.$$

The reciprocal of the absolute risk aversion, $1/a_i(x_i)$, is the *absolute risk tolerance* and denoted by $t_i(x_i)$. The Arrow-Pratt measure of relative risk aversion of consumer i is defined, for $x_i > 0$, as

$$b_i(x_i) = -\frac{u_i''(x_i)x_i}{u_i'(x_i)} > 0.$$

The reciprocal of the relative risk aversion, $1/b_i(x_i)$, is the *relative risk tolerance* and denoted by $s_i(x_i)$. All of these are smooth functions.

Wilson (1968, page 129) referred to the first derivative of the absolute risk tolerance, $t_i'(x_i)$, as cautiousness, but we shall call it the *absolute cautiousness*, to distinguish it from the *relative cautiousness*, which is $s_i'(x_i)$. According to this terminology, if two consumers exhibit constant but differing relative risk aversion, then they are equally relatively cautious but the one with the smaller relative risk aversion is more absolutely cautious. This might sound a bit confusing, but we follow the path paved by Wilson.

The absolute risk aversion $a(x)$, absolute risk tolerance $t(x)$, relative risk aversion $b(x)$, relative risk tolerance $s(x)$, absolute cautiousness $t'(x)$, and relative cautiousness $s'(x)$ are similarly defined for the representative consumer's utility function u . Bear in mind that they depend on the choice of an efficient risk-sharing rule f and hence on the choice of the weights λ , although none of our analytical results depends on the choice of λ . In particular, if markets are complete, then the first welfare theorem implies that every equilibrium allocation is efficient. Hence our results are applicable to equilibrium allocations. The values of λ are

then determined by the individual consumers' initial endowments as well as the choice of an equilibrium in case there is more than one, but our analytical results always hold regardless of the specification of initial endowments or the choice of an equilibrium.

The following lemma is due to Wilson (1968, Theorems 4 and 5).

Lemma 2 (Wilson (1968)) *Let f be an efficient risk-sharing rule and t be the representative consumer's absolute risk tolerance corresponding to f , then, for every i and $x \in (\underline{d}, \bar{d})$,*

$$t(x) = \frac{1}{f'_i(x)} t_i(f_i(x)), \quad (3)$$

$$t(x) = \sum t_i(f_i(x)), \quad (4)$$

$$t'(x) = \sum f'_i(x) t'_i(f_i(x)). \quad (5)$$

Here are some implications of this lemma. First, by (3), $f'_i(x) > 0$, so that f_i is strictly increasing for every x . This property is called *comonotonicity*. Also note that $\sum f'_i(x) = 1$ and hence that $f'_i(x)$ can be interpreted as a probability mass function over the set of individual consumers. Equation (5) then states that the representative consumer's absolute cautiousness is the expected absolute cautiousness of the individual consumers with respect to the this probability mass function. Third, both the absolute risk tolerance and absolute cautiousness are bounded by the individual consumers' counterpart via

$$\max \left\{ \max_i t_i(f_i(x)), I \min_i t_i(f_i(x)) \right\} \leq t(x) \leq I \max_i t_i(f_i(x)), \quad (6)$$

$$\min_i t'_i(f_i(x)) \leq t'(x) \leq \max_i t'_i(f_i(x)). \quad (7)$$

An immediate corollary of inequality (7) is a sufficient condition for the monotonicity of t , and hence of a .

Corollary 3 1. *If t_i is non-decreasing for every i , then so is t .*

2. *If a_i is non-increasing for every i , then so is a .*

3. *If t_i is non-increasing for every i , then so is t .*

4. *If a_i is non-decreasing for every i , then so is a .*

3 Curvature of the Efficient Risk-Sharing Rules

The following proposition is rich in interpretations.

Proposition 4 For every i and $x \in (\underline{d}, \bar{d})$,

$$\frac{f_i''(x)}{f_i'(x)} = \frac{1}{t(x)} (t_i'(f_i(x)) - t'(x)). \quad (8)$$

Proof of Proposition 4 By equality (3),

$$t_i(f_i(x)) = t(x)f_i'(x) \quad (9)$$

for every $x \in (\underline{d}, \bar{d})$. Differentiating both sides with respect to x , we obtain

$$t_i'(f_i(x)) f_i'(x) = t'(x)f_i'(x) + t(x)f_i''(x). \quad (10)$$

Rearranging this, we complete the proof. ■

The first implication of Proposition 4 is that for every $x \in (\underline{d}, \bar{d})$ and every i , $f_i''(x) > 0$ if $t_i'(f_i(x)) > t'(x)$; $f_i''(x) = 0$ if $t_i'(f_i(x)) = t'(x)$; and $f_i''(x) < 0$ if $t_i'(f_i(x)) < t'(x)$. This seems similar to Proposition II of Leland (1980) but in fact differs crucially from it in that the absolute risk tolerance t is derived from the efficient risk-sharing rule f rather than exogenously given.⁴ Its message is otherwise the same: an individual consumer's risk-sharing rule is (locally) convex if he is more absolutely cautious than the representative consumer; (locally) concave if he is less so; and (infinitesimally) linear if they are equally absolutely cautious. In the context of portfolio insurance, as in Leland (1980) and Brennan and Solanki (1981), it implies that only those who are more absolutely cautious than the representative consumer at every level x of aggregate consumption would purchase portfolio insurances.

The second, finer, implication of the proposition is that for every $x \in (\underline{d}, \bar{d})$ and all i and j ,

$$t_i'(f_i(x)) \geq t_j'(f_j(x))$$

if and only if

$$\frac{f_i''(x)}{f_i'(x)} \geq \frac{f_j''(x)}{f_j'(x)}.$$

To appreciate this, recall that the ratios of the first and second derivatives, such as $f_i''(x)/f_i'(x)$ and $f_j''(x)/f_j'(x)$, often appear in expected utility theory. They measure the curvatures of the individual risk-sharing rules f_i and f_j . For example, $f_i''(x)/f_i'(x) \geq f_j''(x)/f_j'(x)$ for every x if and only if f_i is a convex function of f_j . Proposition 4 therefore implies that the degree of convexity of f_i is positively related to consumer i 's absolute cautiousness. That is, the

⁴See section 7 for a more detailed discussion.

marginal consumption that consumer i receives as the aggregate endowment increases grows at a rate higher than its counterpart for consumer j if consumer i is more absolutely cautious than consumer j . What this means in the context of portfolio insurance is that consumer i purchases more portfolio insurance (or options) relative to the size of the reference portfolio he holds than consumer j does. Although both Leland (1980) and Brennan and Solanki (1981) were concerned with the second derivatives $f_i''(x)$ and $f_j''(x)$, rather than the ratios $f_i''(x)/f_i'(x)$ and $f_j''(x)/f_j'(x)$, we believe that the latter is a better notion of convexity, as it allows comparisons of convexity which are unaffected by linear transformations of the risk-sharing rules. Our result also shows that the *levels* of risk tolerance do not matter for the curvatures of the risk-sharing rules, although they do matter for the slopes.⁵ This is an important point, especially in the analysis of background risk, which was a topic included in earlier version of this paper but is to be dealt with in a separate paper in preparation.⁶

4 Representative Consumer's Risk Tolerance

Throughout this section, we let f be an efficient risk-sharing rule and denote by a , t , b , and s the representative consumer's absolute risk aversion, absolute risk tolerance, relative risk aversion, and relative risk tolerance, corresponding to f .

We show that if every consumer exhibits convex absolute risk-tolerance (non-decreasing absolute cautiousness), then so does the representative consumer. Moreover, even the slightest heterogeneity in consumers' absolute cautiousness would cause the representative consumer's absolute risk-tolerance to be strictly convex (that is, the representative consumer's cautiousness would be strictly increasing). The following formula establishes these conclusions.

Theorem 5 *For every $x \in (\underline{d}, \bar{d})$,*

$$t''(x) = \sum (f_i'(x))^2 t_i''(f_i(x)) + \frac{1}{t(x)} \sum f_i'(x) (t_i'(f_i(x)) - t'(x))^2. \quad (11)$$

Recall that, by equality (5), the mean of the individual consumers' absolute cautiousness $t_i'(f_i(x))$ with respect to the probability mass function $f_i'(x)$ equals the representative consumer's cautiousness $t'(x)$. The sum of the second term on the right hand side of (11) is thus

⁵We thank Christian Gollier for clarifying this point.

⁶A drawback of equality (8), pointed out by Jan Werner, is that the absolute cautiousness $t_i'(f_i(x))$ depends in general on the consumption level $f_i(x)$ at which it is evaluated, but which may, in turn, be difficult to identify. However, in the case of constant cautiousness, to be covered in Section 6, it is not necessary to identify it.

the variance of the $t'_i(f_i(x))$ with respect to the same probability mass function. It represents the contribution of heterogeneity in absolute cautiousness to the derivative of the representative consumer's absolute cautiousness. As we will see in the subsequent analysis, this theorem has many implications, but its proof is surprisingly simple.

Proof of Theorem 5 Differentiate both sides of equality (5), then we obtain

$$t''(x) = \sum f''_i(x) t'_i(f_i(x)) + \sum (f'_i(x))^2 t''_i(f_i(x)). \quad (12)$$

By $\sum f''_i(x) = 0$ and equality (10),

$$\begin{aligned} & \sum f''_i(x) t'_i(f_i(x)) \\ &= \sum f''_i(x) (t'_i(f_i(x)) - t'(x)) \\ &= \sum f'_i(x) \frac{f''_i(x)}{f'_i(x)} (t'_i(f_i(x)) - t'(x)) \\ &= \frac{1}{t(x)} \sum f'_i(x) (t'_i(f_i(x)) - t'(x))^2. \end{aligned}$$

Plug this result into equality (12), then we obtain (11). ■

A corollary of this theorem, in terms of the absolute risk tolerance, is:

Corollary 6 *If t_i is a convex function for every i , then so is t . If, moreover, the individual consumers' absolute cautiousness are not completely equal at any aggregate consumption level (that is, for every $x \in (\underline{d}, \bar{d})$, there exist two consumers i and j such that $t'_i(f_i(x)) \neq t'_j(f_j(x))$), then t is strictly convex.*

Formula (11) suggests that even if all consumers exhibit concave, rather than convex, risk tolerance, the representative consumer may exhibit convex risk tolerance. We can therefore say that the aggregation over heterogeneous consumers tends to induce the representative consumer to exhibit convex risk tolerance.

Calvet, Grandmont, and Lemaire (1999) gave a similar result for the representative consumer's *relative* risk tolerance. Specifically, denote by $s_i(x_i)$ consumer i 's relative risk tolerance $t_i(x_i)/x_i$ and by $s(x)$ the representative consumer's relative risk tolerance $t(x)/x$. Rewriting their equality (6.10), multiplying $x/s(x)$ to both sides, and rearranging the terms, we obtain the following formula.⁷

⁷We owe this proof to an anonymous referee

Proposition 7 *For every $x \in (\underline{d}, \bar{d})$, if $f_i(x) > 0$ for every i , then*

$$s'(x) = \sum \frac{f_i(x)}{x} f'_i(x) s'_i(f_i(x)) + \frac{1}{s(x)x} \sum \frac{f_i(x)}{x} (s_i(f_i(x)) - s(x))^2. \quad (13)$$

It can be derived from equality (4) that the mean of the individual consumers' relative risk tolerance $s_i(f_i(x))$ with respect to the probability mass function $f_i(x)/x$ equals the representative consumer's relative risk tolerance $s(x)$. The sum in the second term on the right hand side of (11) is thus the variance of the $s_i(f_i(x))$ with respect to this probability mass function. It represents the contribution of heterogeneity in relative risk tolerance to the representative consumer's relative cautiousness $s'(x)$.

Denote the relative risk *aversions* by $b_i(x_i) = \frac{1}{s_i(x_i)}$ and $b(x) = \frac{1}{s(x)}$. A corollary to Proposition 7, which is analogous to Corollary 6 is the following.

Corollary 8 *Assume that $\underline{d}_i \geq 0$ for every i .*

1. *If s_i is a non-decreasing function for every i , then so is s . If, moreover, the individual consumers' relative risk tolerances are not completely equal at any aggregate consumption level (that is, for every $x \in (\underline{d}, \bar{d})$, there exist two consumers i and j such that $s_i(f_i(x)) \neq s_j(f_j(x))$), then s is strictly increasing.*
2. *If b_i is a non-increasing function for every i , then so is b . If, moreover, the individual consumers' relative risk aversions are not completely equal at any aggregate consumption level (that is, for every $x \in (\underline{d}, \bar{d})$, there exist two consumers i and j such that $b_i(f_i(x)) \neq b_j(f_j(x))$), then b is strictly decreasing.*

The symmetry between formulas (11) and (13) is remarkable. The first derivative of the representative consumer's relative risk tolerance and absolute cautiousness are increased by heterogeneity of individual consumers' risk attitudes. Neither of the two formulas is strictly more general than the other, as either accommodates some cases that the other cannot. However, when all individual consumers exhibit constant relative risk aversion, (11) provides a finer restriction on the representative consumer's risk attitude. We shall come back to this point in Section 6.

5 Limit Behavior

In this section, we investigate the limit behavior of the representative consumer's absolute cautiousness, relative risk tolerance (and hence relative risk aversion), and the risk-sharing

rules. Roughly speaking, we show that the representative consumer's absolute cautiousness tends to the limit of the most absolutely cautious consumers' counterpart as the aggregate consumption level tends to its upper bound \bar{d} (which may be infinite); and these consumers' share of both the consumption levels, out of the aggregate consumption level, and of marginal consumptions, converges to one. This result is particularly relevant in the analysis of a dynamic growing economy. We also provide an analogous result when the aggregate consumption level tends to its lower bound \underline{d} (which may be negative infinite), but the dominant consumers are then the least absolutely cautious ones. This result is relevant in the analysis of a dynamic *contracting* economy.⁸ We also make statements of the limit behavior of the representative consumer's relative risk tolerance (and hence relative risk aversion). All of these results will be applied to the case where all consumers exhibit linear absolute risk tolerance in the next section.

As a convention of this paper, we allow \lim to be ∞ or $-\infty$; \max and \min may be ∞ or $-\infty$ accordingly. From the outset, we impose the following assumption.

Assumption 9 For every consumer i , both $\lim_{x_i \rightarrow \underline{d}_i} t'_i(x_i)$ and $\lim_{x_i \rightarrow \bar{d}_i} t'_i(x_i)$ exist.

It is possible to generalize the following propositions by replacing \lim by \limsup or \liminf , if the limits do not exist.

5.1 Absolute Cautiousness and Risk-Sharing Rules

We first consider the following additional condition. It is intended to cover the case of increasing absolute risk tolerance (and hence decreasing absolute risk aversion).

Assumption 10 For every consumer i , $\underline{d}_i > -\infty$, $\bar{d}_i = \infty$, and $\lim_{x_i \rightarrow \underline{d}_i} t_i(x_i) = 0$.

Define \bar{I} as the set of consumers i such that $\lim_{x_i \rightarrow \infty} t'_i(x_i) \geq \lim_{x_j \rightarrow \infty} t'_j(x_j)$ for every j , and \underline{I} as the set of consumers i such that $\lim_{x_i \rightarrow \underline{d}_i} t'_i(x_i) \leq \lim_{x_j \rightarrow \underline{d}_j} t'_j(x_j)$ for every j . The following proposition states that the share of consumers in \bar{I} in the aggregate consumption level, as well as in the marginal consumptions, converges to one as the aggregate consumption level diverges to infinity, and that the representative consumer's absolute cautiousness eventually equals these consumers' absolute cautiousness. It also states that the share of extra consumption

⁸Dumas (1989) gave an analysis of this kind in a dynamic economy with two consumers exhibiting constant relative risk aversion.

in excess of the lower bound which is consumed by consumers in \underline{I} converges to one as the aggregate consumption level converges to the lower bound. Also the representative consumer's absolute cautiousness eventually equals these consumers' absolute cautiousness.

Proposition 11 *Under Assumptions 9 and 10,*

1. $\lim_{x \rightarrow \infty} \sum_{i \in \bar{I}} \frac{f_i(x)}{x} = \lim_{x \rightarrow \infty} \sum_{i \in \bar{I}} f'_i(x) = 1.$
2. $\lim_{x \rightarrow \infty} t'(x) = \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} t'_i(x_i).$
3. $\lim_{x \rightarrow \underline{d}} \frac{\sum_{i \in \underline{I}} (f_i(x) - \underline{d}_i)}{x - \underline{d}} = \lim_{x \rightarrow \underline{d}} \sum_{i \in \underline{I}} f'_i(x) = 1.$
4. $\lim_{x \rightarrow \underline{d}} t'(x) = \min_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \underline{d}_i} t'_i(x_i).$

The proof of this proposition is given in Appendix B.

We next consider the following additional condition. It is intended to cover the case of decreasing absolute risk tolerance (and hence increasing absolute risk aversion), such as quadratic utility functions.

Assumption 12 For every consumer i , $\underline{d}_i = -\infty$, $\bar{d}_i < \infty$, and $\lim_{x_i \rightarrow \bar{d}_i} t_i(x_i) = 0$.

Define \bar{H} as the set of consumers i such that $\lim_{x_i \rightarrow \bar{d}_i} t'_i(x_i) \geq \lim_{x_j \rightarrow \bar{d}_i} t'_j(x_j)$ for every j , and \underline{H} as the set of consumers i such that $\lim_{x_i \rightarrow -\infty} t'_i(x_i) \leq \lim_{x_j \rightarrow -\infty} t'_j(x_j)$ for every j .

Proposition 13 *Under Assumptions 9 and 12,*

1. $\lim_{x \rightarrow \bar{d}} \frac{\sum_{i \in \bar{H}} (\bar{d}_i - f_i(x))}{\bar{d} - x} = \sum_{i \in \bar{H}} f'_i(x) = 1.$
2. $\lim_{x \rightarrow \bar{d}} t'(x) = \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \bar{d}_i} t'_i(x_i).$
3. $\lim_{x \rightarrow -\infty} \sum_{i \in \underline{H}} \frac{f_i(x)}{x} = \lim_{x \rightarrow -\infty} \sum_{i \in \underline{H}} f'_i(x) = 1.$
4. $\lim_{x \rightarrow -\infty} t'(x) = \min_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow -\infty} t'_i(x_i).$

The proof of this proposition is analogous to that of Proposition 11. We thus omit it.

5.2 Relative Risk Tolerance and Relative Risk Aversion

The key observation for the analysis of the limit behavior of the representative consumer's relative risk tolerance and relative risk aversion is that under suitable assumptions, $\lim_{x_i \rightarrow \infty} s_i(x_i) = \lim_{x_i \rightarrow \infty} t_i(x_i)/x_i = \lim_{x_i \rightarrow \infty} t'_i(x_i)$ and $\lim_{x_i \rightarrow 0} s_i(x_i) = \lim_{x_i \rightarrow 0} t_i(x_i)/x_i = \lim_{x_i \rightarrow 0} t'_i(x_i)$ by L'Hôpital's rule. This allows us to apply Proposition 11 to the relative risk aversion. The additional assumption we need for this argument is the following.

Assumption 14 For every consumer i , $\underline{d}_i = 0$, and t_i is a convex function.

This assumption can be satisfied by utility functions exhibiting constant relative risk aversion. Along with other assumptions, it implies that t'_i is a strictly positive, non-decreasing function. Thus $t_i(x_i) \rightarrow \infty$ as $x_i \rightarrow \infty$ and $\lim_{x_i \rightarrow \infty} s_i(x_i) = \lim_{x_i \rightarrow \infty} t'_i(x_i)$ and $\lim_{x_i \rightarrow 0} s_i(x_i) = \lim_{x_i \rightarrow 0} t'_i(x_i)$.

The following proposition generalizes Proposition 3 of Benninga and Mayshar (2000).

Proposition 15 Under Assumptions 9, 10, and 14,

1. $\lim_{x \rightarrow \infty} s(x) = \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} s_i(x_i)$ and $\lim_{x \rightarrow 0} s(x) = \min_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow 0} s_i(x_i)$.
2. $\lim_{x \rightarrow \infty} b(x) = \min_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} b_i(x_i)$ and $\lim_{x \rightarrow 0} b(x) = \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow 0} b_i(x_i)$.

Proof of Proposition 15 1. By Proposition 11, $\lim_{x \rightarrow \infty} t'(x)$ exists and, by L'Hôpital's rule, equals $\lim_{x \rightarrow \infty} s(x)$. By the same proposition, $\lim_{x \rightarrow \infty} t'(x)$ equals $\lim_{x_i \rightarrow \infty} t'_i(x_i)$ for every $i \in \bar{I}$, which equals $\max_i \lim_{x_i \rightarrow \infty} s_i(x_i)$. Hence $\lim_{x \rightarrow \infty} s(x) = \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} s_i(x_i)$.

As for the limit as $x \rightarrow 0$, note that as $x \rightarrow 0$, $f_i(x) \rightarrow 0$ and hence $t_i(f_i(x)) \rightarrow 0$. Thus $t(x) = \sum t_i(f_i(x)) \rightarrow 0$. This shows that L'Hôpital's rule is applicable and the rest of the argument is as before.

2. This follows from part 1 and the definition of b and s . ■

Now define \bar{J} as the set of consumers i such that $\lim_{x_i \rightarrow \infty} s_i(x_i) \geq \lim_{x_j \rightarrow \infty} s_j(x_j)$ for every j , which is equivalent to $\lim_{x_i \rightarrow \infty} b_i(x_i) \leq \lim_{x_j \rightarrow \infty} b_j(x_j)$ for every j . Analogously, define \underline{J} as the set of consumers i such that $\lim_{x_i \rightarrow 0} s_i(x_i) \leq \lim_{x_j \rightarrow 0} s_j(x_j)$ for every j , which is equivalent to $\lim_{x_i \rightarrow 0} b_i(x_i) \geq \lim_{x_j \rightarrow 0} b_j(x_j)$ for every j . We have already seen that $\bar{J} = \bar{I}$ and $\underline{J} = \underline{I}$ under Assumption 14. Proposition 11 thus implies the following:

Proposition 16 Under Assumptions 9, 10, and 14,

1. $\lim_{x \rightarrow \infty} \sum_{i \in \bar{J}} \frac{f_i(x)}{x} = \lim_{x \rightarrow \infty} \sum_{i \in \bar{J}} f'_i(x) = 1$.

$$2. \lim_{x \rightarrow 0} \sum_{i \in \underline{J}} \frac{f_i(x)}{x} = \lim_{x \rightarrow 0} \sum_{i \in \underline{J}} f'_i(x) = 1.$$

6 Linear Absolute Risk Tolerance

Combining the preceding results and assuming that all consumers' utility functions exhibit linear absolute risk tolerance, we show in this section that an individual consumer's risk-sharing rule is either everywhere concave, everywhere convex, or has a unique inflection point below which it is convex and above which it is concave.

Mathematically, a utility function $u_i : (\underline{d}_i, \bar{d}_i) \rightarrow \mathbb{R}$ exhibits *linear absolute risk tolerance* if, for the corresponding absolute risk tolerance t_i , there exist two numbers τ_i and γ_i such that

$$t_i(x_i) = \tau_i + \gamma_i x_i. \quad (14)$$

for every $x_i \in (\underline{d}_i, \bar{d}_i)$. This is equivalent to *hyperbolic absolute risk aversion* $a_i(x_i) = \frac{1}{\tau_i + \gamma_i x_i}$ and *constant absolute cautiousness* $t'_i(x_i) = \gamma_i$.

Note that the right hand side of equality (14) is of course positive for every $x_i \in (\underline{d}_i, \bar{d}_i)$ but τ_i and γ_i may be positive, zero, or negative. However, if $\gamma_i = 0$, then $\tau_i > 0$ and we take $\underline{d}_i = -\infty$ and $\bar{d}_i = \infty$. On the other hand, if $\gamma_i > 0$ then we take $\underline{d}_i = -\tau_i/\gamma_i$ and $\bar{d}_i = \infty$ and hence $t_i(x_i) = \gamma_i(x_i - \underline{d}_i)$ and $t_i(x_i) \rightarrow 0$ as $x_i \rightarrow \underline{d}_i$. If $\gamma_i < 0$, then $\underline{d}_i = -\infty$ and $\bar{d}_i = -\tau_i/\gamma_i$ and hence $t_i(x_i) = -\gamma_i(\bar{d}_i - x_i)$ and $t_i(x_i) \rightarrow 0$ as $x_i \rightarrow \bar{d}_i$. Indeed, although we do not provide the proof here, these choices of \underline{d}_i and \bar{d}_i are the only ones that allows u_i to satisfy the Inada condition.

As in the previous sections, let $f : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_1, \bar{d}_1) \times \cdots \times (\underline{d}_I, \bar{d}_I)$ be an efficient risk-sharing rule, and denote the representative consumer's absolute risk aversion, absolute risk tolerance, and relative risk aversion by a , t , and b , all corresponding to f .

The celebrated mutual fund theorem is documented in, for example, Wilson (1968), Huang and Litzenberger (1988, Sections 5.15 and 5.26), Magill and Quinzii (1996, Proposition 16.3), Gollier (2001a, Section 21.3.3), and LeRoy and Werner (2001, Section 15.6)). We do not reproduce the statement of the theorem here. We just point out that if all consumers have the same absolute cautiousness, that is, $\gamma_1 = \cdots = \gamma_I$, then the risk-sharing rule is affine and the representative consumer has the same absolute cautiousness as the individual consumers.

Denote $\bar{\gamma} = \max\{\gamma_1, \dots, \gamma_I\}$ and $\underline{\gamma} = \min\{\gamma_1, \dots, \gamma_I\}$. Then, according to the notation in the previous section, $\bar{I} = \{i \mid \gamma_i = \bar{\gamma}\}$ and $\underline{I} = \{i \mid \gamma_i = \underline{\gamma}\}$. Then \bar{I} is the set of the most absolutely cautious consumers and \underline{I} is the set of the least absolutely cautious consumers. All

consumers are equally cautious if and only if $\bar{\gamma} = \underline{\gamma}$. Of course, this case has been dealt with by the mutual fund theorem, and we thus assume in the remainder of this section that $\bar{\gamma} > \underline{\gamma}$.

The first result of this section is concerned with the representative consumer's absolute risk tolerance.

Proposition 17 *Assume that $\bar{\gamma} > \underline{\gamma}$. Then $t''(x) > 0$ for every $x \in (\underline{d}, \bar{d})$, $\lim_{x \rightarrow \bar{d}} t'(x) = \bar{\gamma}$, and $\lim_{x \rightarrow \underline{d}} t'(x) = \underline{\gamma}$.*

Proof of Proposition 17 The first part of this proposition follows from Theorem 6. The second and third parts follow from Corollary 11 or 13.

The main result of this section is the following classification of risk-sharing rules.

Theorem 18 *Assume that $\bar{\gamma} > \underline{\gamma}$.*

1. $f_i''(x) > 0$ for every $i \in \bar{I}$ and $x \in (\underline{d}, \bar{d})$.
2. $f_i''(x) < 0$ for every $i \in \underline{I}$ and $x \in (\underline{d}, \bar{d})$.
3. For every $i \notin \bar{I} \cup \underline{I}$, there exists a unique $y_i \in (\underline{d}_i, \bar{d}_i)$ such that $f_i''(x) > 0$ for every $x < y_i$ and $f_i''(x) < 0$ for every $x > y_i$.
4. For the y_i defined as in part 3, $y_i < y_j$ if $\gamma_i < \gamma_j$; $y_i = y_j$ if $\gamma_i = \gamma_j$; and $y_i > y_j$ if $\gamma_i > \gamma_j$.

Proof of Theorem 18 By Proposition 17, $\underline{\gamma} < t'(x) < \bar{\gamma}$ for every $x \in (\underline{d}, \bar{d})$. Parts 1 and 2 then follow from Proposition 4. As for part 3, note that Proposition 17 implies that $t' : (\underline{d}, \bar{d}) \rightarrow (\underline{\gamma}, \bar{\gamma})$ is strictly increasing and onto. Hence, for every $i \notin \bar{I} \cup \underline{I}$, there exists a unique $y_i \in (\underline{d}_i, \bar{d}_i)$ such that $\gamma_i = t'(y_i)$. Since $\gamma_i = t'_i(f_i(x))$ for every x , Proposition 4 implies that y_i has the property of part 3. Part 4 also follows from this property of y_i and the fact that t' is strictly increasing. ■

The next proposition is concerned with the total proportion of consumption levels consumed by those consumers with the largest or smallest absolute cautiousness. It immediately follows from Propositions 11 and 13. We thus omit the proof.

Proposition 19

1. If $\underline{\gamma} > 0$, then $\lim_{x \rightarrow \infty} \sum_{i \in \bar{I}} \frac{f_i(x)}{x} = 1$ and $\lim_{x \rightarrow \underline{d}} \sum_{i \in \underline{I}} \frac{f_i(x) - \underline{d}_i}{x - \underline{d}} = 1$.

$$2. \text{ If } \bar{\gamma} < 0, \text{ then } \lim_{x \rightarrow -\infty} \sum_{i \in I} \frac{f_i(x)}{x} = 1 \text{ and } \lim_{x \rightarrow \bar{d}} \sum_{i \in \bar{I}} \frac{\bar{d}_i - f_i(x)}{\bar{d} - x} = 1.$$

If we further assume that $\underline{d}_i = 0$, $\tau_i = 0$, and $\gamma_i > 0$ for every i , then $b_i(x_i) = 1/\gamma_i$ and hence u_i exhibits constant relative risk aversion $1/\gamma_i$. The following result, which follows from Proposition 17, is concerned with this case.

Proposition 20 *Assume that $\underline{d}_i = 0$, $\tau_i = 0$, and $\gamma_i > 0$ for every i , and that $\bar{\gamma} > \underline{\gamma}$.*

1. $\lim_{x \rightarrow \infty} s(x) = \bar{\gamma}$ and $\lim_{x \rightarrow 0} s(x) = \underline{\gamma}$.
2. $\lim_{x \rightarrow \infty} b(x) = 1/\bar{\gamma}$ and $\lim_{x \rightarrow 0} b(x) = 1/\underline{\gamma}$.

Let us now come back to the point we made at the end of Section 4, that when the individual consumers exhibit constant relative risk aversion, formula (11) provides a finer restriction on the representative consumer's risk attitude than (13) does. To see this, note first that an immediate implication of the latter formula is that his relative risk aversion b is strictly decreasing. On the other hand, an immediate implication of formula (11) is that his absolute risk tolerance t is strictly convex. Since $\lim_{x \rightarrow 0} t(x) = 0$ by equality (4) or (6), this strict convexity implies that the elasticity of t is strictly greater than one, which is equivalent to saying that b is strictly decreasing. Note however that the strict convexity of t is a strictly stronger property than its elasticity being greater than one. This argument therefore tells us that although the heterogeneous constant relative risk aversion of the individual consumers leads to strictly decreasing relative risk aversion for the representative consumer, not all strictly decreasing relative risk aversion functions can be generated for the representative consumer by such individual consumers. An additional necessary condition is that his absolute risk tolerance be strictly convex.

Theorem 18 is illustrated in Figure 1, which shows the risk-sharing rules in a four-consumer economy. Their first and second derivatives are also given. Consumers differ with respect to their constant relative risk aversion. The risk-sharing rules of the most and least risk averse consumers are concave and convex, respectively. Intermediate consumers have sharing rules which turn from convex for lower aggregate consumption levels to concave for higher ones. The inflection point of the individual risk sharing rule is higher for the less risk-averse intermediate consumer. This is better seen in the graphs of the two derivatives of the risk-sharing rules. Re-scaling the individual utility functions or choosing a different set of weights λ_i , here set to one, would change the quantitative results but nothing of the qualitative results, except

that the individual risk-sharing rules do not in general all intersect at exactly the same point. The figures are numerically calculated and then plotted using the constrained optimization package in GAUSS. The values of the relative risk aversion and weights are chosen to enhance graphical effects, not to fit to empirical findings.

7 Discussion

We have presented detailed properties of the efficient risk-sharing rules and the representative consumer's risk attitude in an economy under uncertainty where individual consumers have homogeneous probabilistic beliefs over the state space but heterogeneous risk attitudes. In particular, we showed that heterogeneity in the consumers' absolute cautiousness, which is the derivative of the reciprocal of the Arrow-Pratt measure of absolute risk aversion, is a key factor for the curvature of the risk-sharing rules. We also showed that the heterogeneity in the individual consumers' risk attitudes has a convexifying effect on the representative consumer's absolute risk tolerance. We now turn to a discussion of the consequences of our results.

7.1 Convex Absolute Risk Tolerance

Based on recent data on Italian households, Guiso and Paiella (2000) found that individual consumers exhibit concave risk tolerance and that there is some heterogeneity in their risk attitudes. Hence, by Theorem 5, the representative consumer may well exhibit convex absolute risk tolerance. Now suppose that this is indeed the case, and yet we erroneously assumed that the economy were to consist of individual consumers having the same risk attitude as the representative consumer. We would then conclude that individual consumers exhibit convex absolute risk tolerance, which has a few testable implications. One is that, according to Gollier and Zeckhauser (2002), younger individual consumers invest more in risky assets than wealth-equivalent older counterparts, but this contradicts the empirical findings of, for example, Guiso, Jappelli, and Terlizzese (1996). While this contradiction would constitute a puzzle under the erroneous assumption, it does not do so if the heterogeneity in absolute cautiousness and their convexifying effect are taken into consideration, as exemplified by formula (11).

Another implication of convex absolute risk tolerance for the representative consumer is given by Gollier (2001b), who showed that if all consumers have the same utility function, then wealth inequality (which would correspond to the biases in the utility weights λ_i in our

maximization problem (2)) increases the equilibrium price of the aggregate endowment ζ if and only if the absolute risk tolerance (of every consumer in this case) is convex. The effect of wealth inequality in a model of consumers with heterogeneous risk attitudes is, however, yet to be explored.

7.2 Risk-Sharing Rules

Parts 1 and 2 of Theorem 18, which dealt with the risk sharing rules for the most and least cautious consumer, have been obtained by Leland (1980) and Brennan and Solanki (1981), who considered the expected utility maximization problem of a consumer who chooses over state-contingent claims of a reference portfolio. Holding the underlying asset and a put option is equivalent to holding cash and a call option of the same exercise price, but these are also equivalent to having a portfolio insurance as well. In all of these cases, the generated return is a convex function of the values of the portfolio. They were thus led to identify conditions on the consumer's utility function for his optimal choice of return to be a convex function of the value of the portfolio.

The most important differences between this work and theirs is that they took the representative consumer's risk aversion as given, while we *derive* it as a result of efficient risk-sharing among heterogeneous consumers. In fact, our result shows that the case Leland (1980) analyzed on page 589, where the individual and the representative consumers exhibit constant but differing relative risk aversion, is in fact impossible, if all the other consumers also exhibit constant relative risk aversion.

Also, the importance of part 3 of Theorem 18, i.e. the fact that risk-sharing rules for intermediate linearly risk tolerant consumers are initially convex and eventually concave, cannot be overemphasized. It is exactly the point that is not present in the analysis of Leland (1980) and Brennan and Solanki (1981). When individual consumers have differing degrees of absolute cautiousness, the representative consumer's absolute cautiousness is strictly increasing, ranging from the smallest to the largest. If an individual consumer has neither the smallest nor the largest absolute cautiousness, then his absolute cautiousness must be caught up with by the representative consumer's counterpart at some aggregate consumption level. Below this level, his risk-sharing rule is convex, and, above this level, it is concave. An important implication of this result in the context of portfolio insurance is that only consumers with the smallest relative risk aversion (the largest absolute cautiousness) would buy portfolio insur-

ance, as the others' risk-sharing rules would eventually become concave. This significantly undermines the applicability of the results of Leland (1980) and Brennan and Solanki (1981). They are valid in a two-consumer economy, but do not generalize to an economy with a large number of consumers with diverse levels of relative risk aversion. This confirms a conjecture by Dumas (1989), who concentrated on a two-consumer economy but concluded by suggesting that the equilibrium behavior of a three-consumer economy may be critically different from that in his two-consumer economy.

Part 3 of Theorem 18 can be partially extended to the general case. Call an intermediate consumer a consumer whose absolute cautiousness is neither the largest nor the smallest when aggregate endowment tends to either of its limits. Then this intermediate consumer's risk-sharing rule must be initially convex and eventually concave. Given smoothness of all utility functions, this consumer's risk-sharing rule must have at least one inflection point.

7.3 Asset Pricing

As is well known, any positive multiple of the representative consumer's marginal utility is a state price deflator (also known as the state price density and as the pricing kernel). This state-price deflator can be expressed as a function of the representative consumer's absolute risk tolerance. We now explore how assets may be mis-priced if a modeler ignores the issue of aggregation and erroneously assumes that the representative agent behaves just as an individual consumer in the economy.

To illustrate our first example of mis-pricing, assume that each consumer exhibits linear absolute risk tolerance, but its first derivative, the absolute cautiousness, differs across them. We then know from Corollary 6 that the representative consumer's absolute risk tolerance is a strictly convex function of aggregate consumptions. Yet, suppose that a modeler erroneously assumed that the representative consumer would also exhibit linear absolute risk tolerance. It can then be shown that even if the absolute risk tolerance and cautiousness are chosen to match the true values at some aggregate consumption level,⁹ the price of any asset with an increasing payoff function (of aggregate endowment) would be under-estimated. Since the aggregate endowment is an increasing function of itself, this implies that the equity premium

⁹This is equivalent to saying that the absolute risk tolerance of the hypothetical representative consumer is a linear approximation of that of the true representative consumer at some aggregate consumption level. Since the absolute risk tolerance of the hypothetical representative consumer need not be equal to zero at zero consumption, the resulting relative risk aversion need not be constant.

is under-estimated. Hence a modeler would find it more difficult to match the observed equity premium with reasonable risk preferences if she ignores the convexifying effect of aggregation on the representative consumer's absolute risk tolerance.

For our second example of mis-pricing, assume that each consumer exhibits constant relative risk aversion, but the constants differ across consumers. We then know from Corollary 8 that the representative consumer's relative risk aversion is a strictly decreasing function of aggregate consumption. Yet, suppose that a modeler erroneously assumed that the representative consumer would also exhibit constant relative risk aversion. Franke, Stapleton, and Subrahmanyam (1999) showed that even if the relative risk aversion is chosen such that the theoretical equity premium is matched to the true equity premium (of the aggregate endowment), the price of any asset with a convex payoff function (of aggregate endowment), such as call and put options, is under-estimated. The consistency with empirical findings should be noted: Aït-Sahalia and Lo (2000) derived the representative consumer's relative risk aversion from option prices in a non-parametric, non-linear way. They find that it is decreasing (almost) everywhere. This is exactly what we would expect and is not necessarily in contradiction to individual consumers having constant relative risk aversion.

A Existence of a Solution to the Maximization Problem (2)

In this appendix, we prove that for every λ and x , there exists a solution to the maximization problem (2), that is, $f_\lambda : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_1, \bar{d}_1) \times \cdots \times (\underline{d}_I, \bar{d}_I)$ is well defined.

Indeed, for each i , the function $\lambda_i u'_i : (\underline{d}_i, \bar{d}_i) \rightarrow \mathbb{R}_{++}$ is strictly decreasing and onto. Hence it has an inverse, which we denote by $\varphi_i : \mathbb{R}_{++} \rightarrow (\underline{d}_i, \bar{d}_i)$. Then φ_i is also strictly decreasing and onto. Define $\varphi : \mathbb{R}_{++} \rightarrow (\underline{d}, \bar{d})$ by $\varphi = \sum \varphi_i$. Then φ is also strictly decreasing and onto. Then the composite mapping $\varphi \circ (\lambda_i u'_i) : (\underline{d}_i, \bar{d}_i) \rightarrow (\underline{d}, \bar{d})$ is well defined. It is easy to check that the inverse of this mapping equals f_{λ_i} . Note that we have also shown that $f'_{\lambda_i}(x) > 0$ for every x and $f_{\lambda_i}(x) \rightarrow \underline{d}$ as $x \rightarrow \underline{d}$ and $f_{\lambda_i}(x) \rightarrow \bar{d}$ as $x \rightarrow \bar{d}$.

B Proof of Proposition 11

To prove Proposition 11, we need two lemmas. The first one is concerned with the ratio of two individual consumers' risk-sharing rules and their derivatives.

Lemma 21 Under Assumptions 9 and 10, if $\lim_{x_i \rightarrow \infty} t'_i(x_i) < \lim_{x_j \rightarrow \infty} t'_j(x_j)$, then $\lim_{x \rightarrow \infty} \frac{f_i(x)}{f_j(x)} = \lim_{x \rightarrow \infty} \frac{f'_i(x)}{f'_j(x)} = 0$.

Proof of Lemma 21 Let two real numbers δ_i and δ_j be such that

$$\lim_{x_i \rightarrow \infty} t'_i(x_i) < \delta_i < \delta_j < \lim_{x_j \rightarrow \infty} t'_j(x_j).$$

Since $\bar{d}_i = \infty$ and $t_i(x_i) > 0$ for every x_i , $\lim_{x_i \rightarrow \infty} t'_i(x_i) \geq 0$. Hence $\delta_i > 0$ and $\delta_j > 0$. Then let $\underline{x} > \underline{d}$ be such that $t'_i(x_i) < \delta_i < \delta_j < t'_j(x_j)$ for every $x_i \geq f_i(\underline{x})$ and $x_j \geq f_j(\underline{x})$. Then, for such x_i and x_j ,

$$\begin{aligned} t_i(x_i) &< \delta_i (x_i - f_i(\underline{x})) + t_i(f_i(\underline{x})), \\ t_j(x_j) &> \delta_j (x_j - f_j(\underline{x})) + t_j(f_j(\underline{x})). \end{aligned}$$

By equality (3) and the fact that a consumer's absolute risk aversion $a_i(\cdot)$ is the reciprocal of his absolute risk tolerance $t_i(\cdot)$,

$$\int_{\underline{x}}^x a_i(f_i(z)) f'_i(z) dz = \int_{\underline{x}}^x a_j(f_j(z)) f'_j(z) dz$$

for every $x \geq \underline{x}$. By integration by parts, this is equivalent to

$$\int_{f_i(\underline{x})}^{f_i(x)} a_i(z) dz = \int_{f_j(\underline{x})}^{f_j(x)} a_j(z) dz. \quad (15)$$

Thus

$$\int_{f_i(\underline{x})}^{f_i(x)} \frac{dz}{\delta_i (z - f_i(\underline{x})) + t_i(f_i(\underline{x}))} < \int_{f_j(\underline{x})}^{f_j(x)} \frac{dz}{\delta_j (z - f_j(\underline{x})) + t_j(f_j(\underline{x}))}.$$

Take the integral and then the exponential of both sides, then we obtain

$$\left(\frac{\delta_i (f_i(x) - f_i(\underline{x})) + t_i(f_i(\underline{x}))}{t_i(f_i(\underline{x}))} \right)^{1/\delta_i} < \left(\frac{\delta_j (f_j(x) - f_j(\underline{x})) + t_j(f_j(\underline{x}))}{t_j(f_j(\underline{x}))} \right)^{1/\delta_j},$$

because $0 < \delta_i < \delta_j$. Thus

$$f_i(x) - f_i(\underline{x}) + \frac{t_i(f_i(\underline{x}))}{\delta_i} < k \left(f_j(x) - f_j(\underline{x}) + \frac{t_j(f_j(\underline{x}))}{\delta_j} \right)^{\delta_i/\delta_j},$$

where

$$k = \frac{t_i(f_i(\underline{x}))}{\delta_i} \left(\frac{\delta_j}{t_j(f_j(\underline{x}))} \right)^{\delta_i/\delta_j} > 0.$$

Since $0 < \delta_i/\delta_j < 1$,

$$\frac{f_i(x) - f_i(\underline{x}) + \frac{t_i(f_i(\underline{x}))}{\delta_i}}{f_j(x) - f_j(\underline{x}) + \frac{t_j(f_j(\underline{x}))}{\delta_j}} \rightarrow 0 \quad (16)$$

as $x \rightarrow \infty$. Hence $f_i(x)/f_j(x) \rightarrow 0$ as $x \rightarrow \infty$.

By equality (2),

$$\frac{f'_i(x)}{f'_j(x)} = \frac{t_i(f_i(x))}{t_j(f_j(x))} < \frac{\delta_i}{\delta_j} \frac{f_i(x) - f_i(\underline{x}) + \frac{t_i(f_i(\underline{x}))}{\delta_i}}{f_j(x) - f_j(\underline{x}) + \frac{t_j(f_j(\underline{x}))}{\delta_j}}.$$

By (16), the far right hand side converges to 0. Hence $f'_i(x)/f'_j(x) \rightarrow 0$. ■

The next lemma is concerned with the limit behavior of the risk-sharing rules when the aggregate consumption levels converge to the lower bound.

Lemma 22 *Under Assumptions 9 and 10, if $\lim_{x_j \rightarrow \underline{d}_j} t'_j(x_j) < \lim_{x_i \rightarrow \underline{d}_i} t'_i(x_i)$, then $\lim_{x \rightarrow \infty} \frac{f_i(x)}{f_j(x)} = \lim_{x \rightarrow \infty} \frac{f'_i(x)}{f'_j(x)} = 0$.*

Proof of Lemma 22 Let two real numbers δ_i and δ_j be such that

$$\lim_{x_j \rightarrow \underline{d}_j} t'_j(x_j) < \delta_j < \delta_i < \lim_{x_i \rightarrow \underline{d}_i} t'_i(x_i).$$

Since $t_j(x_j) \geq 0$ for every x_j and $t_j(x_j) \rightarrow 0$ as $x_j \rightarrow \underline{d}_j$, we have $\limsup_{x_j \rightarrow \underline{d}_j} t'_j(x_j) \geq 0$. Hence $\delta_j > 0$ and $\delta_i > 0$. Then let $\bar{x} > \underline{d}$ be such that $t'_j(x_j) < \delta_j < \delta_i < t'_i(x_i)$ for every $x_i \leq f_i(\bar{x})$ and $x_j \leq f_j(\bar{x})$. Thus, for such x_i and x_j , $t_i(x_i) > \delta_i(x_i - \underline{d}_i)$ and $t_j(x_j) < \delta_j(x_j - \underline{d}_j)$. Since, for every $x \in (\underline{d}, \bar{x})$,

$$\int_{f_j(x)}^{f_j(\bar{x})} \frac{dz}{t_j(z)} = \int_{f_i(x)}^{f_i(\bar{x})} \frac{dz}{t_i(z)},$$

we have

$$\int_{f_j(x)}^{f_j(\bar{x})} \frac{dz}{\delta_j(z - \underline{d}_j)} < \int_{f_i(x)}^{f_i(\bar{x})} \frac{dz}{\delta_i(z - \underline{d}_i)} \quad (17)$$

Thus

$$\left(\frac{f_j(\bar{x}) - \underline{d}_j}{f_j(x) - \underline{d}_j} \right)^{1/\delta_j} < \left(\frac{f_i(\bar{x}) - \underline{d}_i}{f_i(x) - \underline{d}_i} \right)^{1/\delta_i}.$$

Hence there exists a positive number k such that

$$f_i(x) - \underline{d}_i < k(f_j(x) - \underline{d}_j)^{\delta_i/\delta_j}. \quad (18)$$

Recall that both $f_i : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_i, \bar{d}_i)$ and $f_j : (\underline{d}, \bar{d}) \rightarrow (\underline{d}_j, \bar{d}_j)$ are smooth, one-to-one, and onto, and have strictly positive derivatives. Hence there exists a $\varphi : (0, \bar{d}_j - \underline{d}_j) \rightarrow (0, \bar{d}_i - \underline{d}_i)$ that is smooth, one-to-one, and onto, has strictly positive derivatives, and satisfies $f_i(x) - \underline{d}_i = \varphi(f_j(x) - \underline{d}_j)$. Thus, also by inequality (18), $0 < \varphi(z) < kz^{\delta_i/\delta_j}$ for every $z \in (0, \bar{d}_j - \underline{d}_j)$.

Hence, by $\delta_j/\delta_i > 1$, $\varphi(z)/z \rightarrow 0$ and $\varphi'(z) \rightarrow 0$ as $z \rightarrow 0$. If z and x satisfy $z = f_i(x) - \underline{d}_i$, then $z \rightarrow 0$ if and only if $x \rightarrow \underline{d}$. Hence $(f_i(x) - \underline{d}_i) / (f_j(x) - \underline{d}_j) \rightarrow 0$ as $x \rightarrow \underline{d}$. Moreover, since $\varphi'(z) = f'_j(x)/f'_i(x)$, $f'_j(x)/f'_i(x) \rightarrow 0$ as $x \rightarrow \underline{d}$. ■

We can now turn to the proof of Proposition 11.

Proof of Proposition 11 To show the first two parts, let $i \in \bar{I}$ and $j \notin \bar{I}$. Since $\limsup_{x \rightarrow \infty} f_i(x)/x \leq 1$,

$$0 \leq \liminf_{x \rightarrow \infty} \frac{f_j(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{f_j(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{f_j(x)}{f_i(x)} \limsup_{x \rightarrow \infty} \frac{f_i(x)}{x} \leq \limsup_{x \rightarrow \infty} \frac{f_j(x)}{f_i(x)}.$$

By Lemma 21, the far right hand side equals zero. Thus $f_j(x)/x \rightarrow 0$. Since this is true for every $j \notin \bar{I}$ and $\sum_{i=1}^I f_i(x)/x = 1$, we must have $\sum_{i \in \bar{I}} f_i(x)/x \rightarrow 1$ as $x \rightarrow \infty$.

Also, since $0 < f'_i(x) < 1$,

$$0 < f'_j(x) < \frac{f'_j(x)}{f'_i(x)}$$

and, for such i and j as in the preceding paragraph, the far right hand side converges to zero as $x \rightarrow \infty$. Hence $f'_j(x) \rightarrow 0$ as $x \rightarrow \infty$. We must have $\sum_{i \in \bar{I}} f'_i(x) \rightarrow 1$ as $x \rightarrow \infty$. Since $\lim_{x_j \rightarrow \infty} t'_j(x_j) < \lim_{x_i \rightarrow \infty} t'_i(x_i) \leq \infty$ for every $i \in \bar{I}$ and $j \notin \bar{I}$, $t'_j(f_j(x)) f'_j(x) \rightarrow 0$ as $x \rightarrow \infty$ for every $j \notin \bar{I}$. Thus, by Lemma 2 and $0 < \sum_{i \in \bar{I}} f'_i(x) \leq 1$, we have

$$\begin{aligned} & \limsup_{x \rightarrow \infty} t'(x) \\ &= \limsup_{x \rightarrow \infty} \sum_{i=1}^I t'_i(f_i(x)) f'_i(x) \\ &= \limsup_{x \rightarrow \infty} \sum_{i \in \bar{I}} t'_i(f_i(x)) f'_i(x) \\ &\leq \limsup_{x \rightarrow \infty} \max_{i \in \bar{I}} t'_i(f_i(x)) \\ &\leq \max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} t'_i(x_i). \end{aligned}$$

The other inequality,

$$\max_{i \in \{1, \dots, I\}} \lim_{x_i \rightarrow \infty} t'_i(x_i) \leq \liminf_{x \rightarrow \infty} t'(x),$$

can be shown analogously. This proves the first two parts of this proposition.

To prove part 3, let $i \in \underline{I}$ and $j \notin \underline{I}$. Since $0 < \frac{f_i(x) - \underline{d}_i}{x - \underline{d}} < 1$ for every x ,

$$0 \leq \liminf_{x \rightarrow \underline{d}} \frac{f_j(x) - \underline{d}_j}{x - \underline{d}} \leq \limsup_{x \rightarrow \underline{d}} \frac{f_j(x) - \underline{d}_j}{x - \underline{d}} \leq \limsup_{x \rightarrow \underline{d}} \frac{f_j(x) - \underline{d}_j}{f_i(x) - \underline{d}_i} \limsup_{x \rightarrow \underline{d}} \frac{f_i(x) - \underline{d}_i}{x - \underline{d}} \leq \limsup_{x \rightarrow \underline{d}} \frac{f_j(x) - \underline{d}_j}{f_i(x) - \underline{d}_i}.$$

By Lemma 22, the far right hand side equals zero. Hence $\frac{\sum_{i \in I} (f_i(x) - \underline{d}_i)}{x - \underline{d}} \rightarrow 1$ as $x \rightarrow \underline{d}$.

Part 4 can be shown in the same manner as for part 2. ■

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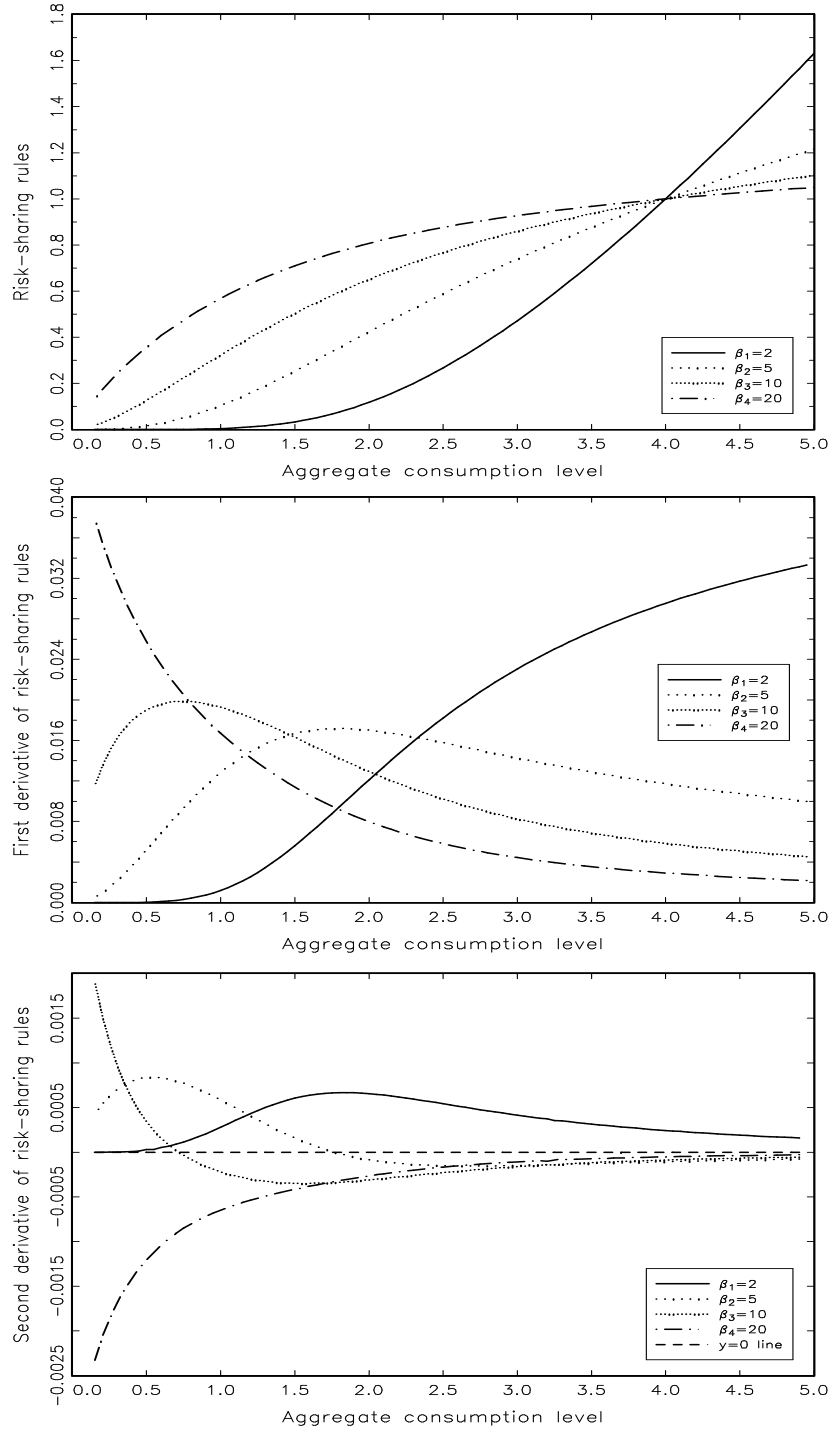


Figure 1: The risk-sharing rules and their first and second derivatives in a four-consumer economy. Consumers have differing constant coefficients of relative risk aversion $\beta_i = 1/\gamma_i > 0$. The weights λ_i in the maximization problem (2) are all set equal to one.