**Abstract**

Recent empirical research by Mark Taylor and coauthors has found evidence of hybrid dynamics for real exchange rates. While there is a random walk near equilibrium, for real exchange rates some distance from equilibrium there is mean-reversion which increases with the degree of misalignment. An interesting question is whether this nonlinear mean-reversion might be policy-induced. John Williamson (1998), for example, has proposed a "monitoring band" in which there is no intervention near equilibrium but there is substantial intervention triggered by exchange rate deviations outside a preset band. In this paper we develop a theoretical model of such a monitoring band to see whether it can generate patterns of nonlinear mean-reversion akin to those reported in empirical research.

JEL: D52, F31, G12.
Key-Words: Monitoring Band, Non-linear Mean-Reversion, Near Random Walk Dynamics.

# Exchange Rate Monitoring Bands: Theory and Policy*

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## 1 Introduction

Until recently, most econometric studies of the behaviour of floating rates came to the conclusion that a random walk describes the evolution of the exchange rate better than more sophisticated models. Isard (1995) for example concludes a survey of these studies with the observation:

> "In short, neither the behavioural relationships suggested by theory, nor the information obtained through autoregression, provided a model that could forecast better than a random-walk. And furthermore, while the random walk model performed at least as well as other models, it predicted very poorly".

However, the random walk view of exchange rates has been effectively challenged in the last few years. By using long historical data series, Lothian and Taylor (1996), for example, found econometric evidence of mean-reversion in real exchange rates. Thus, even if the random walk outperforms any of the structural models of exchange rate determination within a time horizon of less than a year, Rogoff (1996) acknowledges that "there is now pretty conclusive evidence that a floating rate will revert slowly toward relative purchasing power parity (PPP), with half the adjustment being completed in something under 5 years".

In further exploring the behaviour of real exchange rates, several other econometric studies (O'Connell, 1998) have applied a nonlinear analysis to deviations from

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PPP and have found empirical evidence of nonlinear mean-reversion[1].

In an earlier study of nominal exchange rates between major European currencies during the 1980s, Krager and Kluger (1993) had also confirmed the presence of such nonlinearities. In examining the movements against the US dollar, they identified three different regimes: the first containing appreciation of the currencies considered (German mark, French franc, Italian lira and Swiss franc); the second containing moderate depreciations and the third region strong depreciations. They show that in the first and third regimes there is strong exchange rate autocorrelation and a tendency to mean-reversion, whereas in the second region there is no autocorrelation and the exchange rate moves as a random walk[2].

In a more recent study of the dollar-sterling and dollar-mark exchange rates over the recent floating rate period, Taylor and Peel (1999) estimate nonlinear time-series models of deviations of the nominal exchange rates from the levels suggested by simple monetary fundamentals. The estimated parameters imply mean-reversion towards the monetary fundamental equilibrium, where the speed of mean-reversion increases with the size of the deviation from equilibrium. In (as yet unpublished) research in a Leverhulme project at the University of Warwick, Mark Taylor and Chris Kubelec have found evidence of hybrid dynamics for the real exchange rate with random walk behaviour near equilibrium, and mean-reversion which increases with the degree of misalignment, for real exchange rates some distance from equilibrium. On examining the determinants of mean-reversion, they find evidence for the role of state-contingent foreign currency intervention.

In policy arena, state-contingent foreign currency intervention has been advocated as part of a strategy for maintaining a 'monitoring band', see the report of the Tarapore Committee, 1997. In common with canonical 'target zone' models, there is no intervention within the band — which the Tarapore Committee recommended should be set $\pm 5\%$ around the 'neutral' rate representing the official and announced estimate of the equilibrium exchange rate. Unlike target zone models of exchange rates, however, there is no obligation to defend the edge of the band *per se*: intervention only takes place when the exchange rate is outside the band. In his discussion of exchange rate policy in South-East Asia, Rajan (2000, p. 17) suggests the Monetary Authority of Singapore (MAS) may already have adopted a monitoring band:

[1] A theoretical basis for such nonlinearity was provided by Dumas (1992) assuming proportional transaction costs in spatially separated markets. Outside the transaction band deviations from PPP are shown to follow a nonlinear process that is mean-reverting, with the speed of adjustment toward equilibrium varying directly with the extent of the deviation from PPP: within the transaction band, however, no trade takes place and the exchange rate follows a random walk.

[2] Theoretical support for such nonlinear behaviour of nominal exchange rates was provided by Hsieh (1992), in a model where the exchange rate switches between two linear processes, one where the intervention is present and the central bank "leans against the wind when the wind is blowing hard" (Hsieh, 1992 p. 236) and the other where intervention is absent.

| | KTZ | WMB | WMB+BSB |
|---|---|---|---|
| Nonlinear Mean-Reversion | Only close to the edge | Weak inside and increasing outside the band | Weak inside and increasing outside the band |
| Bubbles | No | Yes | No |

Table 1: Comparative Properties of Exchange Rate Models. KTZ=Krugman's Target Zone;WMB=Williamson's Monitoring Band;BSB=Back-Stop Band.

> "(the) MAS manages the Singapore dollar against a basket of currencies of Singapore's main trading partners and competitors. The basket is composed of the currencies of those countries that are the main sources of imported inflation and competition in export markets. The trade-weighted Singapore dollar is allowed to float within an undisclosed target band. The level and width of the band are reviewed periodically to ensure that they are consistent with economic fundamentals and market conditions. [But] the MAS intervenes in the foreign exchange rate market from time to time to ensure that movements of the (Singapore dollar) exchange rate are orderly and consistent with the exchange rate policy".[3].

In Krugman's canonical target zone model there is some nonlinear mean-reversion but only very close to the edge of the band, where the marginal intervention takes place; elsewhere inside the band the exchange rate behaves as a random walk. A key feature of the model is that fully credible marginal intervention rules out intrinsic bubbles, see Table 1 column 1. In a monitoring band, however, the mean-reversion is weak inside the band but grows more powerful as the exchange rate deviates from equilibrium, see column 2. As Williamson (1998) argues, the advantage of such an exchange rate policy is that it exploits the stabilising properties of what he refers to as the 'restoration rule', without committing the authorities to defend a 'Maginot line' as in traditional target zone regimes. A drawback, however, is that models with soft-buffers admit "intrinsic bubbles". To rule these out we propose that the monitoring band be supplemented with back-stop intervention, see column 3.

Recently released data on Japanese interventions[4], displayed in Figure (1), provide several examples of such back stop interventions being used to limit severe misalignments. For example the Japanese minister of finance, Mr Sakakibara, authorised massive intervention on the exchange rate market at 148 yen per dollar in 1998, i.e. about 30% below its long-run equilibrium rate of 125 yen per dollar.[5] For an econometric analysis of intervention rules in Japan and their effects on the exchange rate,

[3]MAS website: www.mas.gov.sg.

[4]Data on the amounts of intervention are available at the Japanese Ministry of Finance (MOF) webpage: www.mof.go.jp/english/e1c021.htm.

[5]In similar fashion the European Central Bank intervened decisively to support the euro when it fell from $1.17 to 85 cents.

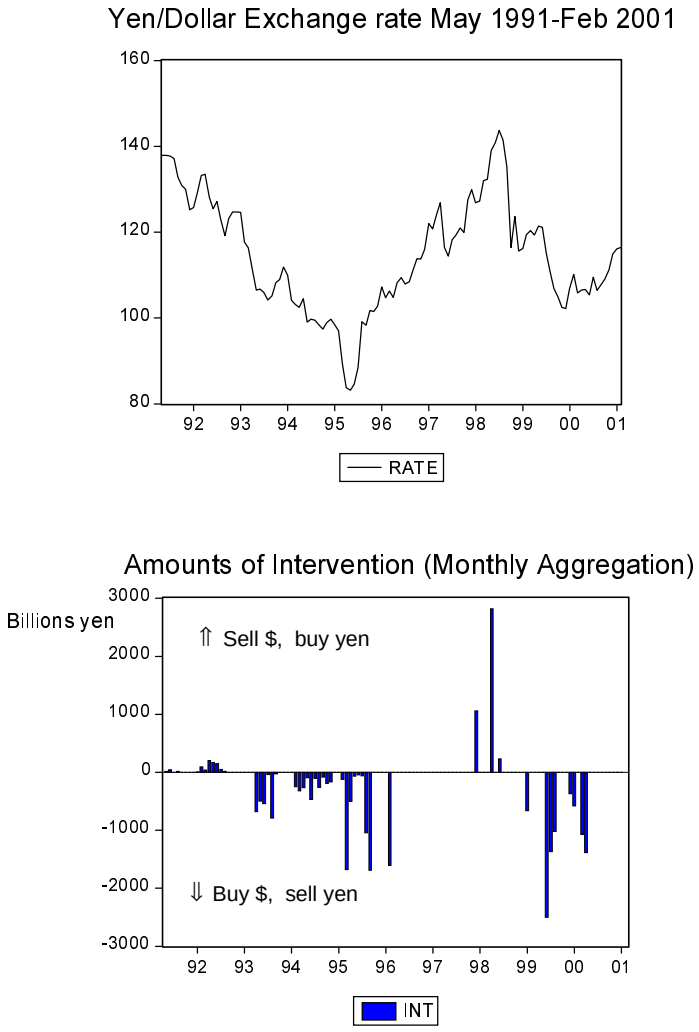


Figure 1: Yen/Dollar Exchange Rate and Amounts of Intervention.

see Ito (2001).

The plan in this paper is to develop a theoretical model for a stylised monitoring band in which there is no intervention near equilibrium but substantial intervention triggered by exchange rate deviations outside a preset band. (Note that, in line with theoretical literature on target zones, we will work with nominal not real bands and consider only unsterilised intervention). One of the objectives of developing such models is to see whether they can generate patterns of nonlinear mean-reversion akin to those reported in empirical research. If so, this could provide theoretical support for the empirical findings discussed above.

The paper is organised as follows: the next section examines target zone models from the perspective of recent empirical findings on mean-reversion. Section 3 builds a monitoring band model for the nominal exchange rate with interventions linked to the degree of fundamental misalignment. Section 4 discusses existence of "intrinsic bubbles" (Froot and Obstfeld, 1991); and how they may be ruled out by "backstop" intervention. It also contains a simple model of Fundamentalists and Chartists where endogenous market composition generates a self-fulfilling bubble and discusses policies to rule them out. Section 5 analyses the — more realistic but more complicated — case of a monitoring band with intervention triggered by exchange rate misalignment. Section 6 concludes.

## 2 Varieties of Intervention and Exchange Rate Dynamics

The canonical target zone model developed by Krugman (1991) assumes a fully credible exchange rate band supported by infinitesimal interventions at the margin. He highlights the stabilizing effect of a target zone on the exchange rate, due to market expectations of monetary interventions if the exchange rate hits the band. These market expectations generate a nonlinear S-shaped relationship between the exchange rate and economic fundamentals (money demand and a stochastic velocity shock). The implication of the model is that a credible band will generate mean-reversion in exchange rates, which increases as the rate moves towards the edge of the band.

However, empirical studies (Svensson, 1991) suggest that, at least until 1987, the target zone regime adopted among the European currencies was not credible, given the large number of realignments occurred during the EMS (up to 12 for Italy).[6] As Obstfeld (1995) stresses:

> "One drawback of target zones is that they may not exert a stabilising effect unless markets are confident that their edges will be defended suc-

[6] At the end of the August 1993 turmoil most of the surviving hard ERM was replaced by a much weaker scheme. After a series of realignment, the size of the bands was widened from 2.25% up to 15% on either side of the unaltered central parities.

> cessfully... If markets can figure out the fragility of the edges and perform the requisite backward induction, a target zone loses much of its stabilising power. It may even become destabilising".

It may even be the case, as Williamson (1998) observes, that edges to the band are providing the market with targets to attack, rather than assuring the market that the rate will not move further.

This has stimulated further studies of partially credible target zones. We can group this research into: models with endogenous realignment risks, models with intramarginal interventions and those (including the present work) which focus on state-contingent interventions (linked to the fundamentals and/or exchange rate misalignment).

Advocates of *models with endogenous realignments risks* (Bertola and Caballero, 1992; Bertola and Svensson, 1993) argue that a high probability of realignment can cancel out the stabilising effect that the target zone has on the exchange rate (honeymoon effect) and can reverse the S-shape solution for the exchange rate.

In the second group, various authors (Froot and Obstfeld, 1991; Lindberg and Söderlind 1994) have combined marginal (i.e., at the edge of the band) and *intramarginal intervention* (within the band) in order to explain the hump-shaped distribution of the exchange rate. Svensson (1992) argues that intramarginal intervention can better describe real world central bank intervention strategy. Svensson assumes that, as fundamentals deviate from equilibrium, the authorities implement mean-reverting interventions which drive the exchange rate back towards the central parity. However, interventions in the real world are more likely to be driven by the degree of misalignment in the exchange rate itself rather than in the underlying fundamentals.

What we plan to do in this paper, is to enrich the stream of literature on imperfectly credible target zone models by studying alternative hybrid regimes, including a monitoring band, where intervention is triggered when degree of misalignment of the exchange rate from equilibrium is greater than a certain threshold, with no intervention otherwise. While this intervention is not designed to keep the exchange rate inside the band, it grows ever more determined as the misalignment increases.

As we shall see in section four, such contingent interventions may introduce an additional feedback (from the exchange rate to fundamentals) not present in previous models of mean-reverting fundamentals (Froot and Obstfeld, 1991; Lindberg and Söderlind 1994).[7]

Assume the following reduced form for the exchange rate:

$$s_t = x_t + \beta \frac{E ds_t}{dt} \tag{1}$$

[7]The effect of state-contingent interventions based on the degree of misalignment has been explored in Corrado (1996).

where $x$ is the velocity-adjusted money (in log form) and $s$ is the deviation of the exchange rate from its equilibrium (normalised to zero).[8]

In the absence of intervention, $x$ follows the stochastic process:

$$dx_t = \sigma \, dz_t \tag{2}$$

where $z$ is a standard Brownian motion and $\sigma$ is a parameter measuring the volatility of the fundamental.

Integration of (1) yields:

$$s_t = E_t \left\{ \frac{1}{\beta} \int_t^{\tau} x_y e^{-\frac{1}{\beta}\,(y-t)} dy + e^{-\frac{1}{\beta}\,(\tau-t)} s_{\tau} \right\} \tag{3}$$

The transversality condition is given by:

$$\lim_{\tau \to \infty} E_t \left( e^{-\frac{1}{\beta}\,(\tau-t)} s_{\tau} \right) = 0 \tag{4}$$

Under (4), the relationship (3) represents $s_t$ as present value of future fundamentals:

$$s_t = E_t \left\{ \frac{1}{\beta} \int_t^{\infty} x_y e^{-\frac{1}{\beta}\,(y-t)} dy \right\} \tag{5}$$

For $x$ given in (2) the exchange rate is as follows:

$$s_f = E_t \left\{ \frac{1}{\beta} \int_t^{\infty} x_y e^{-\frac{1}{\beta}\,(y-t)} dy \right\} = \; x_t \tag{6}$$

where $s_f$ denotes the solution with a free-floating. Hence, in the absence of any barrier or controls, the exchange rate simply tracks the fundamentals.

## 2.1 Bounded Fundamentals

Given the process for the fundamental as described in (2), applying Ito's lemma to (1) yields the following differential equation:

$$\frac{1}{2}\sigma^2 s''(x) + \frac{1}{\beta}(x - s) = 0 \tag{7}$$

[8]The nominal exchange rate can be expressed as follows:

$$\varphi_t = s_t + c_t \tag{N.1}$$

The relationship (N.1) defines the normalized exchange rate $s_t$ as the (log) deviation of the nominal exchange rate $\varphi_t$, from the equilibrium level $c_t$. The interest parity condition can also expressed as:

$$i_t dt = E[ds_t + dc_t] = E[ds_t] + E[dc_t] \tag{N.2}$$

So long as there is not a constant probability of a jump at every moment of time, the second term on the right-hand side of (N.2), $E[dc_t]$, is zero and we can express the reduced form of the exchange rate as in (1).

which has the following general solution:

$$s_n(x) = x + A_1 \exp[\lambda x] + A \exp[-\lambda x] \tag{8}$$

where $s_n(x)$ denotes the solution for the exchange rate with no-intervention, $A_1$ and $A$ are two arbitrary constants and the parameter $\lambda$ is defined by $\lambda = \sqrt{2/\beta\sigma^2}$.

To keep the exchange rate within the symmetric band $[-\bar{s}, \bar{s}]$ it is sufficient to confine the fundamental process to $[-\bar{x}, \bar{x}]$ by means of reflecting barrier at both ends. Assuming the symmetry of the solution at the two reflecting barriers, $s(-\bar{x}) = -s(\bar{x})$, implies that $A_1 = A$. Hence:

$$s_k(x) = \frac{A}{2}(e^{\lambda x} - e^{-\lambda x}) + x = A \sinh(\lambda x) + x \tag{9}$$

$s_k(x)$ denotes the canonical Krugman solution and $A = -\frac{1}{\lambda \cosh(\lambda \bar{x})} < 0$ is derived by imposing the smooth pasting condition $s'(-\bar{x}) = s'(\bar{x}) = 0$.

As (9) shows, the standard target zone solution implies that the authorities intervene at the edge of the band by means of marginal interventions. However, this may not be true in reality.

### 2.2 Mean-Reverting Fundamentals

In practice, as Svensson (1992) argues, most interventions are 'leaning against the wind' or intra-marginal, i.e., they aim at returning the exchange rate to a specified target within the band. This may be captured by mean-reverting interventions modelled as follows:

$$dx = -\gamma \; x \; dt + \sigma dz \tag{10}$$

where the expected rate of change of the fundamental, its 'drift', is proportional to its deviation from equilibrium, assumed for simplicity to be zero.

By applying Ito's lemma to the reduced form of the monetary model (1) subject to the mean-reverting process for $x$ represented by (10), we get an ordinary differential equation for the exchange rate:

$$\frac{1}{2}\sigma^2 s''(x) - \gamma \; x \; s'(x) \; dt + \frac{1}{\beta}(x - s) = 0 \tag{11}$$

As is shown in Appendix A, the general solution to (11) is given by:

$$s_i(x) = \frac{1}{1 + \beta\gamma} x + C \; H \left[ \frac{1}{2\beta\gamma}; \frac{1}{2}; \frac{\gamma}{\sigma^2} x^2 \right] + B \; U \left[ \frac{1}{2\beta\gamma}; \frac{1}{2}; \frac{\gamma}{\sigma^2} x^2 \right] \tag{12}$$

where $H[\cdot]$ and $U[\cdot]$ are two Kummer's functions, $B$ and $C$ are constants that have to be determined by boundary behaviour.

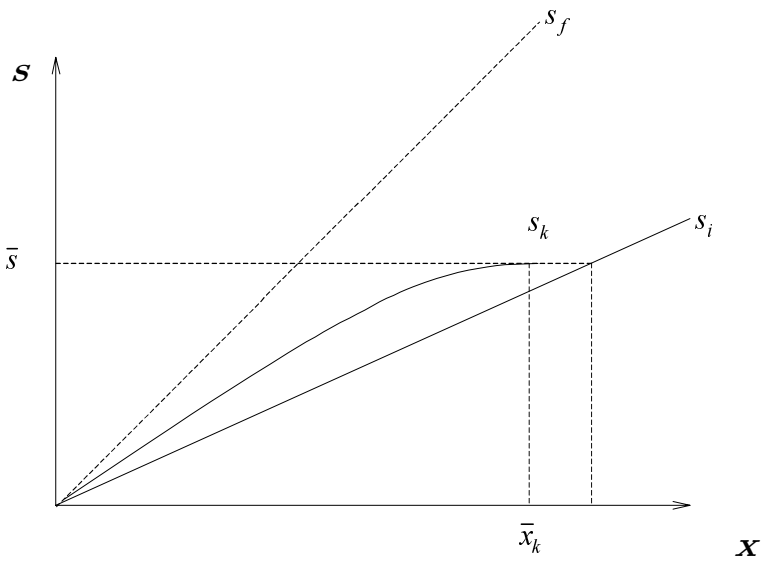


Figure 2: Free-float Solution, Mean-Reverting Solution and Krugman's Solution.

The 'free-float' solution is given by the first term of (12) and depends on the degree of mean-reversion. The higher is the degree of mean-reversion, $\gamma$, the flatter is the 'free-float' solution relative to the free-float benchmark solution given by the 45° line.

Figure 2 illustrates the key differences between the solution of the model with mean-reverting interventions, $s_i(x)$, and the ordinary Krugman's target zone model, $s_k(x)$, with marginal interventions only. By assuming a symmetric band for the nominal exchange rate, the $s$-axis is defined in terms of the deviation of the nominal exchange rate from the central parity while the $x$-axis is defined in terms of a composite fundamental which embodies a stochastic velocity shock and the money stock.

In the fully credible target zone solution, $s_k(x)$, expectations of intervention at the edge of the band of $\bar{s}$ stabilise the exchange rate as shown by the nonlinear S-shaped solution (where the reduction of the slope below one at the origin is referred to as the 'honeymoon effect'). If interventions are mean-reverting towards a central parity, as suggested by Svensson (1992) and there is no specified band for marginal interventions, then the solution $s_i(x)$ shows that a similar stabilising effect also operates in this regime (whether the slope at the origin is greater or less than for the canonical target zone solution depends of course on the size of $\gamma$).

# 3 Modelling Monitoring Bands

As the ERM crisis of 1992-1993 demonstrated, nominal exchange rate bands with hard edges are difficult to defend. Nor does allowing the centre of the band to crawl solve the problem. As Velasco (2000) notes:

> "if exchange rate bands crawl, so that their centre remains close to an estimate of the 'equilibrium' exchange rate, then medium term misalignment can be avoided. Avoided, that is, to the extent that the edges of the band are defensible — and, in the aftermath of the Asian, Brazilian, Mexican and Russian crises — the consensus in the profession seems to be that they cannot be. Bands with hard edges eventually fall prey to the pressure of the market-place"( p.12).

In order to avoid defending the indefensible, Williamson (1998) proposes a 'monitoring band' with soft edges for the real exchange rate. The key difference with respect to the crawling peg is that there is no obligation to defend the edge of the band. The obligation instead is to take action to stabilise the exchange rate when it goes outside the band.

In this section we study Williamson's proposal with a formal model of a stylised monitoring band. For simplicity, we use a monitoring band for the nominal exchange

rate and assume that the size of intervention depends on the deviation of the fundamental from its equilibrium value. This has the advantage that it allows a direct comparison with Krugman's canonical target zone model and Svensson's alternative including intra-marginal intervention. In contrast to the latter, where mean-reverting intervention takes place *inside* an exchange rate band, in the monitoring band mean-reverting intervention takes place only when the exchange rate is *outside* the preset band $[-\bar{s}, \bar{s}]$.

Since we are looking at the symmetric solution we will consider the case when the exchange rate and the fundamental vary between 0 and $\infty$. The intervention policy implies that there is a threshold level of fundamental $\bar{x}$, in the interior of the range $[\bar{s}, (1+\beta\gamma)\bar{s}]$, such that fundamental follows a pure Brownian motion without drift for $x \leq \bar{x}$, and switches to a mean-reverting process for $x > \bar{x}$. Thus, while in the canonical target zone model the threshold, $\bar{x}$, is a reflecting barrier, it is in this case a transitional threshold where the process which drives the fundamental undergoes a change.

The stochastic process including such transitions is defined as:

$$dx = -\gamma \; x \; I \; dt + \sigma dz \tag{13}$$

where the indicator $I$, which defines whether there is intervention or not, can be either 0 or 1 with:

$$\begin{cases} I = 0 & \text{if} \quad 0 < s \leq \bar{s} \\ I = 1 & \text{if} \quad s \geq \bar{s} \end{cases} \tag{14}$$

To represent the exchange rate dynamics under this hybrid regime, we can use appropriate general solutions on each side of $\bar{s}$, namely:

$$\begin{cases} s_n(x) & = x + A\sinh(\lambda \; x) & for \quad 0 < s \leq \bar{s} \\ s_i(x) & = \frac{x}{1+\beta\gamma} + CH\left[\frac{1}{2\beta\gamma}; \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2\right] + BU\left[\frac{1}{2\beta\gamma}; \frac{1}{2}; \frac{\gamma}{\sigma^2}x^2\right] & for \quad s \geq \bar{s} \end{cases} \tag{15}$$

where $s_n(x)$ and $s_i(x)$ define the solution without intervention and with intervention respectively, and the authorities intervene to stabilise the exchange rate only when it is outside the band.

Since the switching at $s = \bar{s}$ is reversible, the values for $A, B, C$ and $\bar{x}$ are derived (see Appendix C for details) by both imposing the smooth-pasting and value-matching conditions at $\bar{s}$ (see Dixit and Pindyck, 1994):

$$s_n(\bar{x}) = \bar{s} \tag{16}$$

$$s_i(\bar{x}) = \bar{s} \tag{17}$$

$$s_n^{'}(\bar{x}) = s_i^{'}(\bar{x}) \tag{18}$$

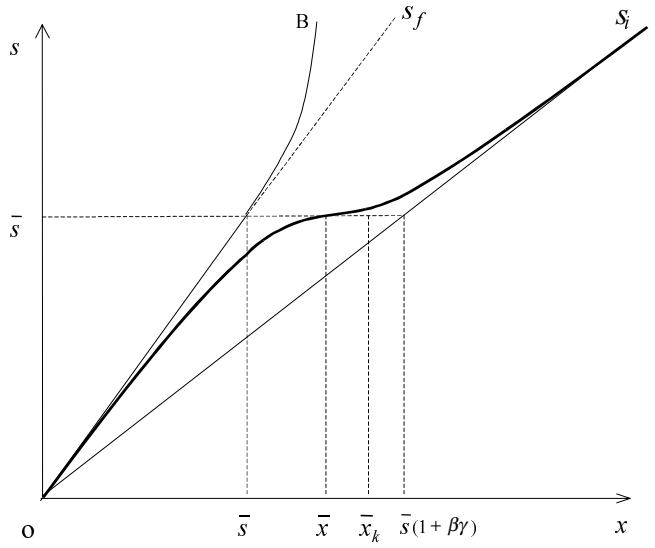


Figure 3: A Monitoring Band with Interventions driven by the Deviation of the Fundamental from Equilibrium.

and also the transversality condition for the exchange rate (see Appendix B for details). The latter ensures that the solution (12) converges asymptotically to the linear mean-reverting solution shown in Figure 3, i.e.,

$$\lim_{x\to\infty} s_i(x) \to \frac{1}{1+\beta\gamma}x \tag{19}$$

This is because for $x \to \infty$, the probability of switching back to the no-intervention regime goes to zero; so it is as if the mean-reverting fundamentals apply everywhere.

In the no-intervention zone, as Figure 3 shows, the exchange rate lies close to the free-float solution so it tends to track the fundamental. Given the presumption that the authorities will intervene if the "exchange rate strays too far" (Williamson, 1998), the solution for the exchange rate in the no-intervention zone starts to bend before the limit for the fundamental, $\bar{x}$, is reached. This is like the honeymoon effect in a fully credible target zone, but here it is driven by expectations of non-infinitesimal intervention aimed at discouraging the exchange rate from straying too far outside of the band. [9]

It can be shown, however, that the stabilising effect of expected non-marginal interventions must be weaker than Krugman's honeymoon effect.

[9]Note that in this hybrid regime the authorities "have a whole extra degree of flexibility in deciding the tactics they will employ to achieve this" (Williamson, 1998).

**Proposition 1** *As $\gamma \uparrow \infty$ then $\bar{x} \uparrow \bar{x}_k$, where $\bar{x}_k$ is the reflecting barrier for the canonical target zone solution with the same band halfwidth $[0, \bar{s}]$.*

PROOF: See Appendix D.

The intuitive explanation for this result is that when the degree of mean-reversion, $\gamma$, tends to infinity the strength of the intervention is such that the exchange rate cannot drift even momentarily from the given exchange rate band.

In terms of Figure 3 this implies the solution inside the band tends toward Krugman's solution only as $\gamma$ tends to infinity, otherwise the exchange rate is closer to a random walk and $s'(\bar{x}) > 0$, i.e., there is no smooth-pasting against the edge of the band. The absence of significant mean-reversion close to equilibrium is consistent with empirical facts discussed above. The detailed pattern of mean-reversion both inside and outside the band is analysed next.

### 3.1 Measuring the Degree of Nonlinear Mean-Reversion

As a preliminary, we represent the dynamics of the exchange rate as:

$$ds = -\tilde{\gamma}(s)\ sdt + g(s)dz \tag{20}$$

where $\tilde{\gamma}(s)$ is the degree of mean-reversion for the exchange rate and $g(s)$ represents exchange rate volatility and both can vary with $s$. Given that in the monetary model the exchange rate, $s$, is a function of the fundamental, $x$, which follows the process (13), Ito's lemma implies:

$$ds = \left[\frac{1}{2}\sigma^2 s''(x) - \gamma\ x\ I\ s'(x)\right] dt + \sigma\ s'(x)\ dz \tag{21}$$

where $s(x)$ is the functional form of the solution. It is clear from (21) that the first term represents mean-reversion and the second term is the Brownian motion process.

Before deriving asymptotic properties of the nonlinear mean-reversion displayed by the exchange rate, it is interesting to note that despite the switch of stochastic regime the mean-reversion is a continuous function of $s$ and $x$. Comparing (20) and (21) it is evident that the degree of local mean-reversion can be defined as:

$$\tilde{\gamma}(s) = -\frac{E(ds)/dt}{s} \tag{22}$$

But from the reduced money demand relationship (1) it follows that:

$$E(ds)/dt = \frac{1}{\beta}\left(s(x) - x\right)$$

which implies that for $s = s(x)$:

$$\tilde{\gamma}(x) = \frac{1}{\beta}\left(\frac{x}{s(x)} - 1\right) \tag{23}$$

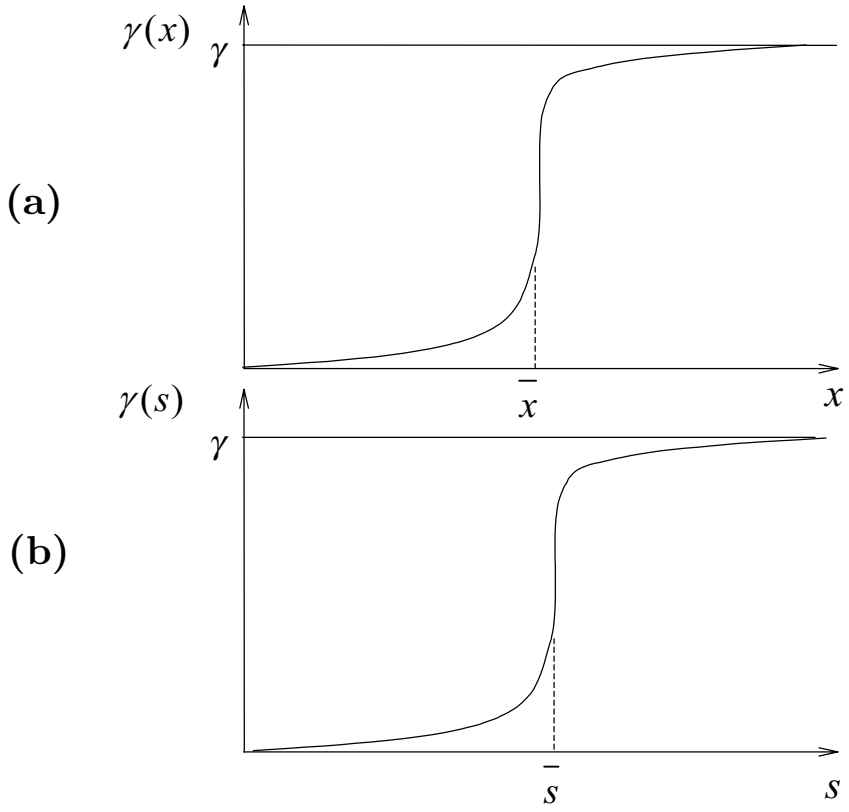


Figure 4: Degree of Local Mean-Reversion

Mean-reversion scaled by $-s$ is the opportunity cost of holding (velocity adjusted) real balances, defined as $(x - s(x))$; since neither the exchange rate nor the fundamental are discontinuous across the regime transition, neither is mean-reversion.

Formally we can establish the following propositions:

**Proposition 2** *(a)* $\lim_{x \downarrow 0} \tilde{\gamma}(x) = 0$ *and* $\lim_{x \uparrow \infty} \tilde{\gamma}(x) = \gamma$

*(b)* $\tilde{\gamma}(x)$ *is continuous at* $x = \bar{x}$.

*(c)* $\tilde{\gamma}$ $(x)$ *is increasing and convex for* $0 < x < \bar{x}$ *and increasing and concave for* $x > \bar{x}$.

Proof: See Appendix E.

As part (a) of Figure 4 illustrates, mean-reversion is very weak near equilibrium, so the exchange rate exhibits near-random walk behaviour much like that highlighted in the empirical literature on nonlinear mean-reversion (Taylor, 2001; Taylor and Peel,

1999; Kilien and Taylor, 2000). But as the fundamental moves toward the switch point $\bar{s}$ the degree of mean-reversion increases due to the expectation of intervention beyond the threshold $\bar{s}$. The degree of mean-reversion is, however, less than in the credible target zone as we show in the proposition below.

**Proposition 3** $\lim_{\gamma\uparrow\infty}\tilde{\gamma}(x) = \tilde{\gamma}_k(x)$, *where* $\tilde{\gamma}_k(x)$ *defines the degree of local mean-reversion in Krugman's case.*

PROOF: See Appendix F.

The proposition is straightforward to explain. As the degree of mean-reversion in the fundamental, $\gamma$, approaches infinity, the effect of the monitoring band converges to that of the fully credible target zone: so the mean-reversion in the exchange rate replicates that of Krugman's canonical case.

As suggested by Part (b) of Figure 4, mean-reversion in exchange rate plotted against rate itself starts at zero at equilibrium and rises to $\gamma$ as $s$ tends to infinity, i.e., the pattern of mean-reversion with respect to the exchange rate has the same characteristics as with respect to the fundamental. This can be summarised in the proposition:

**Proposition 4** *The degree of local mean-reversion with respect to the exchange rate,* $\tilde{\gamma}(s)$*, has the same qualitative properties of* $\tilde{\gamma}(x)$*.*

PROOF: See Appendix G.

This similarity is a consequence of the regularity of the solution linking the exchange rate to the fundamental.

# 4 Intrinsic Bubbles and a Back-stop Band

The properties analysed in propositions 1 to 3 pertain to the fundamental solution which satisfy the transversality condition that the exchange rate is the present discounted value of expected future fundamentals. But stochastic models of this type posses other solutions which do not satisfy this condition. Thus even in the presence of a monitoring band with unsterilised intervention, there are an infinity of what Froot and Obstfeld (1991) have dubbed 'intrinsic bubbles'.

To see this, consider a path inside the band which lies above the fundamental solution, e.g., the 45° line. Smooth-pasting and value-matching to the Kummer's function at $\bar{s}$ will define an intrinsic bubble solution, similar to $OB$ in Figure 3. The same logic will generate a whole family of intrinsic bubbles, all consistent with the announced intervention policy.

In this case the second term in (3) is different from zero and the market may fail in the long-run since in this case it does not posses the global rationality assumption.

In principle of course such bubbles can be ruled out by imposing a transversality condition, similar to (19); this is equivalent to assume all the non-economic factors affecting the exchange rate, embedded in the second term in (3), should converge asymptotically to zero. This may be sufficient to ensure that the solution (12) converges asymptotically to the linear mean-reverting solution represented in Figure 3. Although the imposition of transversality conditions can be a convenient analytical device to rule out such bubble solutions, this may be in practice not feasible. In fact, short-horizon on the part of asset holders may limit the practical relevance of this restriction.

In the absence of transversality condition, policy action to restore global rationality may be required to rule out these intrinsic bubbles. One such policy might be that of having a much wider 'target zone' as a back stop, as, for example, the $\pm$ 15% wide bands adopted by the ERM after the speculative attacks on the much narrower bands in 1992/3. So the policy response would be sequenced –with no intervention inside the monitoring band, followed by 'leaning against the wind' and ending with 'back stop' intervention. In Svensson (1994) it is suggested that, with mean reverting intervention, a back stop target zone will be play no useful role: but this is because transversality was being assumed. If agents are not globally rational then the wide zone can play the complementary role of checking intrinsic bubbles.

It is true that Buiter and Pesenti (1990) have shown that bubbles can persist even within a target zone: but this requires that intervention be 'bubble friendly' in the sense that it takes the fundamental from the bubble path to the stable path without bursting the bubble (for divergent paths above the 45 degree line in the target zone, for example, this would involve selling and not buying domestic currency at the weak edge of the band.). Absent such perverse intervention, bubbles will violate arbitrage and can be ruled out without appealing to asymptotic arbitrage conditions.[10]

### 4.1 Self-fulfilling bubbles with Fundamentalists and Chartists

In their analysis of foreign exchange markets, Frenkel and Froot (1986) distinguish between the trading strategies of Fundamentalists and Chartists, where the former treat the exchange rate as the present discounted value of future fundamentals, while the latter follow technical trading rules based on price movements. The econometric work of Ito (2001) suggests the presence of these two influences in the market; it shows that the yen-dollar exchange rate is partly driven by extrapolating past price changes,

[10] More generally, it seems that a mix of sterilised and non-sterilised intervention is used.

> "Evidence to date....suggests that central banks appear to use largely a policy of leaning against the wind, to react to both changes of the exchange rate from the target and to exchange rate movements, and to sterilise — at least partially — their intervention operations" (Sarno and Taylor, 2001, p25)

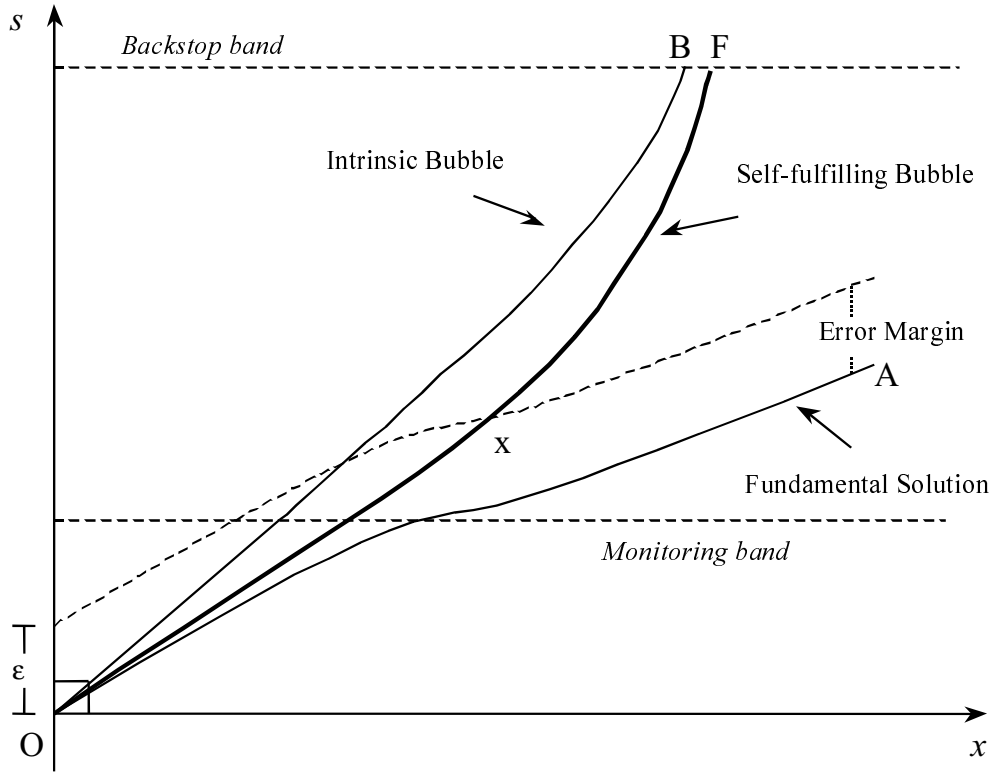


Figure 5: Switching between Fundamentalists and Chartists.

which might characterise the chartists' behaviour, and partly by the deviation of the nominal exchange rate from its long-run equilibrium, which might capture the influence of fundamentalists.

The dynamics of the exchange rate would also reflect changes in the composition of traders which would evolve endogenously depending on which group makes more profits. No attempt is made here to model this 'evolutionary game' in any detail: but we use the framework of a monitoring band to show how endogenous market composition can generate self-fulfilling bubbles: and discuss how these may be eliminated by back-stop intervention.

The argument is illustrated in Figure 5, where $OA$ indicates the fundamental (no-bubbles) solution described earlier and $OB$ represents the intrinsic bubble solution which begins on the 45 degree line inside the monitoring band and becomes convex when the edge of the band is reached and the authorities start 'leaning against the

wind'.

To see how the rate might evolve when the composition of traders is endogenous, assume that the views of fundamentalists will dominate unless their forecasts go seriously wrong. Specifically we assume that their views will be discounted completely when the actual rates deviates by $\epsilon$ from their forecasts. At this point, Chartists' views would dominate where these are represented by intrinsic bubble $OB$. If the switch is reversible, the actual exchange rate will follow the path shown as $OxF$ in the figure where $x$ marks the spot where fundamentalists get fired — or re-hired as the case may be. (We assume that when the rates move inside the error margin shown, the views of fundamentalists will once again dominate, i.e., the regime switch is "reversible".) As can be seen from the figure, $OxF$ "hugs" the fundamental solution fairly closely until it reaches point $x$ where it deviates sharply as it asymptotically approaches the intrinsic bubble $OB$. Note that the rate only deviates from $OA$ because of expectations that the Fundamentalists will be fired: but the deviation of the rate outside the error margin ensures that this occurs. So this is an example of self-fulfilling bubble.

If only the market believed the Fundamentalists will *not* be fired, there will be no bubble. This can be achieved if the path approaching $OB$ violates the no-profitable-arbitrage condition. But in the presence of "back-stop" band, there is no smooth pasting when the solution reaches $F$, the point of intervention designed to ensure that the fundamentals can no longer increase, so arbitrarily large profits can be made by betting on the currency strengthening. The perception that the authorities will strengthen the hand of the Fundamentalists by defending the backup band means that the self-fulfilling bubble $OF$ can be ruled out ex ante.

If the policy prerequisite is to enforce losses on those who ignore fundamentals, this may presumably be achieved by public actions, such as sterilised interventions. The effort to prevent a self-fulfilling plunge in the Euro, for example, took the form of coordinated sterilised intervention. Wadhawani (2000) discusses the effectiveness of such action as follows:

> "Sterilised intervention is no magic weapon to wheel out generally.....and it should only be used when the chances of success are relatively high, e.g., during periods of significant misalignment" (p.17).

Perhaps sterilised interventions at point $F$ in Figure 5 — which inflict losses on those buying foreign currency — would satisfy Wadhwani's prescription as the bubble is pricked only when there is significant misalignment. Wadhwani (2000) does, however, warn against delaying intervention for too long: "Allowing an overshoot to continue can, of itself, begin to affect the 'fundamentals', or at least, the market's perceptions of them. For example, allowing the Euro to fall indefinitely might, rightly or wrongly, increase perceptions of the political risk associated with holding that

currency. It is, therefore, a mistake to assume that the fundamentals are independent of the precise time-path of the moves of a currency. Intervention can, therefore, even affect the fundamentals by altering the time-path of a currency's movements" (p.18). In the next section therefore, we consider a model in which the exchange rate itself affects evolution of fundamentals.

# 5 Monitoring Band with Intervention Proportional to the Exchange Rate Misalignment

In the hybrid regime studied above, intervention is triggered by the exchange rate reaching the edge of the band, but the strength of intervention is strictly proportional to the level of fundamentals. While this is mathematically convenient, it is not as plausible as assuming that the intervention is proportional to the degree of exchange rate misalignment, which corresponds more closely to the models of nonlinear mean-reversion investigated empirically by Taylor and Kubelec for example, where fundamentals play no explicit role.

Where the strength of the intervention is linked to misalignment, the process driving the fundamental outside the band becomes:

$$dx = -\gamma\ s\ dt + \sigma dz \tag{24}$$

In the presence of explicit feedback from the exchange rate to the fundamental, the model no longer has the recursive structure of Krugman's target zone model or of Svensson's model with mean-reversion: now the exchange rate is a function of the fundamental *and* the fundamental is a function of the exchange rate, as (24) shows. We begin by analysing solutions for the special case of continuous feedback, i.e. as if the monitoring band is equal to zero ($\bar{s} = 0$), so:

$$\begin{bmatrix} dx \\ E(ds_i) \end{bmatrix} = \begin{bmatrix} 0 & -\gamma \\ -\frac{1}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} xdt \\ s_i dt \end{bmatrix} + \begin{bmatrix} \sigma dz \\ 0 \end{bmatrix} \tag{25}$$

where $dx$ is given by (24) and the expected change in the exchange rate, $E(ds_i)$, is derived from (1).

Since the determinant of the matrix of the coefficients $A = \begin{bmatrix} 0 & -\gamma \\ -\frac{1}{\beta} & \frac{1}{\beta} \end{bmatrix}$ is negative, the system is saddle-point stable. The slopes of the stable and unstable eigenvectors, denoted $\theta_s$ and $\theta_u$ respectively, are given by (see Appendix H):

$$\theta_s = \frac{-1 + \sqrt{1 + 4\gamma\beta}}{2\gamma\beta} > 0 \tag{26}$$

$$\theta_u = \frac{-1 - \sqrt{1 + 4\gamma\beta}}{2\gamma\beta} < 0 \tag{27}$$

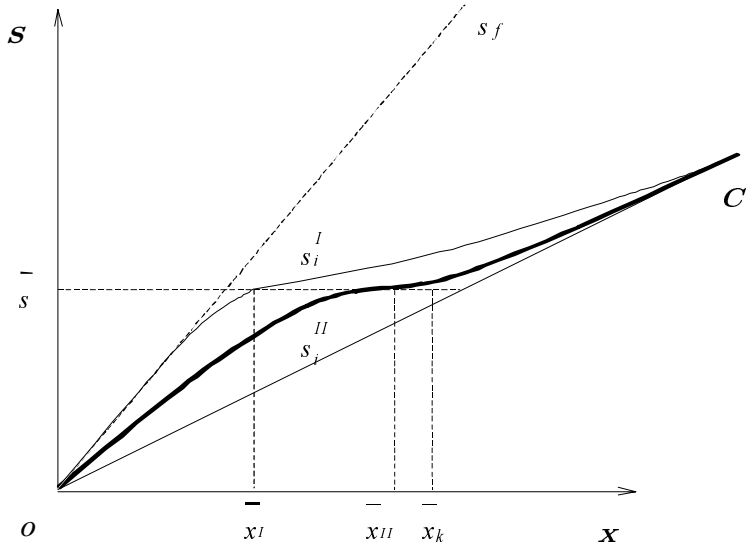


Figure 6: Monitoring Bands with Interventions driven by Fundamentals, $s_i^I$, *versus* Monitoring Bands with Interventions driven by the Exchange Rate Misalignment, $s_i^{II}$.

By appropriate choice of the feedback coefficient on the exchange rate (type II intervention) it is possible to make the stable eigenvector coincide with Svensson's mean-reverting solution when the intervention is linked to the fundamental (type I intervention).

The resulting linear solution is labelled $OC$ in Figure 6.

Leaving aside the trivial case of a monitoring band of zero width we consider the solution for $\bar{s} > 0$ with type II intervention. As was the case for type I intervention, this will be composed of two segments. First for $s < \bar{s}$ we get the hyperbolic solution reported in (8); for $s > \bar{s}$ no analytical solution exists, but qualitatively it is clear from Ito's lemma that the solution must be convex to the horizontal axis and converge to the stable eigenvector. As before these two segments will smooth-past and value-match at $s = \bar{s}$. While the solution under intervention of type II will closely resemble the explicit solution described above under intervention of type I, they will not be identical. This can be seen from the following argument. Assume counterfactually

that $s_i^{II}$ did coincide with $s_i^I$ shown by the solid lighter line in the diagram and consider the degree of intervention at the switch point $\bar{x}_I$. As can be seen from the Figure, type II intervention at that point is proportional to $\bar{s}$ and exceeds type I intervention which is proportional to $\bar{x}_I$ (in fact, $\bar{x}_I < \bar{s}$). The same logic applies for all the values of $x > \bar{x}_I$. Since the feedback rule always leads to more powerful intervention, this drags the solution $s_i^{II}$ below $s_i^I$ as shown in the Figure.

### 5.1 Intrinsic Bubble Solutions: a Tentative Analysis

The fundamental solution with type II intervention evidently resembles the earlier monitoring band solution under type I intervention in converging to the stable eigenvector. Likewise the intervention policy will generate intrinsic bubbles in this case too. In the absence of analytical representation of these bubbles outside the band can anything be said about them (in the context of type II intervention)?

Firstly they must value-match and smooth-paste to the hyperbolic solution at the edge of the band, as before. Secondly they must satisfy the fundamental differential equation for the system (25) (see Appendix H). While numerical methods could be used to determine the nature of such intrinsic bubbles, at least one bubble path can be characterised using the arguments of Miller and Weller (1995). They provide a qualitative description of all the solutions of the system (25), i.e. with continuous intervention and a zero monitoring band. The following logic suggests that one of those solutions will characterise a bubble for the hybrid system with a non trivial monitoring band.

It is well known from the target zone literature that the slope of the hyperbolic function at the edge of the band falls from infinity to zero as the fundamental moves over the range $[0, \bar{x}_K]$. Given the opposing slopes of stable and unstable eigenvectors described above, the qualitative intrinsic bubble solutions described by Miller and Weller (1995) will have slopes which vary from negative to positive over the same range. It follows that there must be a solution that satisfies smooth-pasting and value-matching; this is illustrated in Figure 7 by the path $OAU'$ which asymptotically converges to the unstable manifold, $U'U'$ which has slope given by (27). We have already described in the previous section the properties of the solution $OES'$, which converges asymptotically to the stable manifold $S'S'$ which has slope given by (26).

This is only one example, but it is sufficient to confirm that imposing feedback rules corresponding to a type II intervention it is not in itself enough to stabilise exchange rate. How can such policy-induced intrinsic bubbles be ruled out? Once again one may appeal to transversality.

In addition, it appears that credibly announcing the monitoring band will be suspended if it proves clearly counterproductive (e.g. when $s > \bar{s}$ and $x = 0$), would be sufficient. (As can be seen from Figure 7 such a policy switch will violate the arbitrage condition underlying intrinsic bubble solutions.) Finally, of course, this

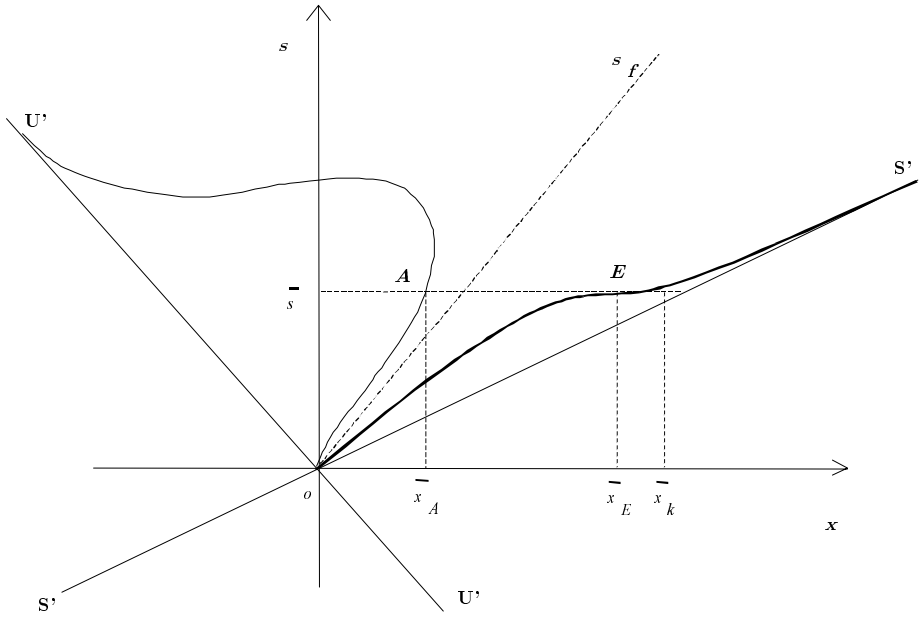


Figure 7: "Intrinsic" Bubbles

may be achieved by a wider back-stop band as a trigger for intervention (possibly sterilised) as discussed in the previous section.

# 6 Conclusions

The empirical evidence of local random walk behaviour for the exchange rate has led some authors (Taylor and Peel, 1999) to reach the following conclusion:

> "Intuitively (the nonlinear adjustment process of the exchange rate) may arise because small deviation from 'fundamental equilibrium' may be considered unimportant by the market and by policy makers, and perhaps of secondary importance to the influence of market forces not strictly governed by economic fundamentals, such as technical or chart analysis (Taylor and Allen, 1992). As the exchange rate becomes increasingly misaligned with the economic fundamentals, however, one might expect that the pressure both from the market and from policy makers to return the exchange rate to the neighbourhood of fundamental equilibrium would become increasingly strong" (p.4).

The paper has illustrated how a monitoring band for the exchange rate, with no intervention near equilibrium but substantial intervention triggered by exchange rate or fundamental deviations outside a preset band, is able to reproduce the patterns of nonlinear mean-reversion found in many recent empirical works cited above (among others see Taylor and Peel, 1999; Kilien and Taylor, 2000).

We have shown the effect on the exchange rate dynamics of imposing a monitoring band where the intervention is triggered by the degree of fundamental misalignment *versus* the case where the intervention is triggered by the degree of exchange rate misalignment *per se*.

One desirable feature of the solution is that a monitoring band could offer an attractive intermediate regime for developing countries to manage their currencies. One of the main advantages of such a solution, as Velasco (2000) notes, is that if the authorities "decide the market pressure is overwhelming, they can choose to allow the rate to take the strain even if this involves the rate going outside the band".

But the paper has also shown that "leaning against wind" is not sufficient to rule out 'intrinsic' bubbles. Although the imposition of transversality conditions may be a convenient analytical device to rule out such bubble solutions, this may not translate in practice. So we suggest a role for a wide back-stop band to rule out such bubbles. How a wide band could prevent self-fulfilling bubbles was illustrated using a simple model of Fundamentalists and Chartists with non-sterilised intervention. Given Ito's evidence of the success of sterilised interventions, future work needs to incorporate

a role for foreign exchange rate interventions that do not involve changes in monetary policy. In Sarno and Taylor (2001) portfolio balance and signaling effects are considered. Abreu and Brunnermeier (2001) also discuss the role of public intervention. The framework they use is one of asymmetric information and private signals similar to that of Morris and Shin (1998) and the bubble ends only when enough traders believe that others believe that the bubble will end. In this setting, sterilised intervention could be effective as a coordination device.

# Appendix

## A General Solution with Mean-Reverting Fundamentals

By making the substitution $x = \left(\frac{\sigma^2 z}{\gamma}\right)^{\frac{1}{2}}$, we can transform (11) into a standard Kummer's equation. Since $\frac{dz}{dx} = 2\left(\frac{\gamma}{\sigma^2}\right)^{\frac{1}{2}} z^{\frac{1}{2}}$, then:

$$\begin{aligned} \frac{\partial s}{\partial z} &= \frac{\partial s}{\partial z}\frac{dz}{dx} = 2\left(\frac{\gamma}{\sigma^2}\right)^{\frac{1}{2}} z^{\frac{1}{2}} s' \\ \frac{\partial^2 s}{\partial z^2} &= \frac{4\gamma}{\sigma^2}\left[\frac{1}{2}s' + z^{\frac{1}{2}} s''\right] \end{aligned} \tag{A.1}$$

where $s' = \frac{\partial s}{\partial z}$ and $s'' = \frac{\partial^2 s}{\partial z^2}$. Substituting (A.1) into (11) gives:

$$z s'' + (b - z)\, s' - a s = 0 \tag{A.2}$$

where $b = \frac{1}{2}$ and $a = \frac{1}{2\gamma\beta}$. The general solution associated to (A.2) is given by one of the two forms below:

$$\begin{aligned} s(z) &= C' H[a; b; z] + B'\, z^{\frac{1}{2}} H[1 + a - b; 2 - b; z] \\ &= C H[a; b; z] + B U[a; b; z] \end{aligned} \tag{A.3}$$

where $H[a; b; z]$ and $U[a; b; z]$are Kummer's functions, $z = \gamma x^2/\sigma^2$ and:

$$U[a; b; x] = \frac{\Gamma(1-b) H[a; b; x]}{\Gamma(1 + a - b)} + \frac{\Gamma(b-1) x^{1-b} H[1 + a - b; 2 - b; x]}{\Gamma(a)}$$

(see Slater, 1960, p. 5). By incorporating the particular solution and rearranging we arrive at equation (12) in the text.

## B Asymptotic Properties of Kummer's Function

As $z \to \infty$ the two Kummer's function converge to (Slater, 1960, p.60, 4.1.7 and 4.1.12):

$$\begin{aligned} &\lim_{z\to\infty} H[a; b; z] \sim \tfrac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \\ &\lim_{z\to\infty} U[a; b; z] \sim z^{-a} \end{aligned} \tag{B.4}$$

where $\Gamma(.)$ indicates the Gamma function. By applying (B.4):

$$\begin{aligned} &\lim_{x\to\infty} H[\tfrac{1}{2\beta\gamma}; \tfrac{1}{2}; \tfrac{\gamma}{\sigma^2}x^2] = \lim_{x\to\infty} \tfrac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2\beta\gamma})} \left(e^{\frac{\gamma x^2}{\sigma^2}}\right) \left(\tfrac{\gamma}{\sigma^2}x^2\right)^{\frac{1}{2\beta\gamma} - \frac{1}{2}} \to \infty \\ &\lim_{x\to\infty} U[\tfrac{1}{2\beta\gamma}; \tfrac{1}{2}; \tfrac{\gamma}{\sigma^2}x^2] = \lim_{x\to\infty} \left(\tfrac{\gamma}{\sigma^2}x^2\right)^{-\frac{1}{2\beta\gamma}} = 0 \end{aligned} \tag{B.5}$$

Asymptotically the solution (12) must converge to the corresponding free-float solution which implies that the non-linear part of (12) must be zero:

$$C\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2\beta\gamma})}\left(e^{\frac{\gamma x^2}{\sigma^2}}\right)\left(\frac{\gamma}{\sigma^2}x^2\right)^{\frac{1}{2\beta\gamma}-\frac{1}{2}}+B\left(\frac{\gamma}{\sigma^2}x^2\right)^{-\frac{1}{2\beta\gamma}}=0 \tag{B.6}$$

Therefore applying (19) to (12) yields $C=0$.

# C Boundary Conditions at $\bar{x}$ and Equation for $\bar{x}$

We use the three boundary conditions (16),(17),(18)to solve for the three unknowns $A, B$ and $\bar{x}$ in (9) and (12).

Solving (16) yields:

$$A=\frac{\bar{s}-\bar{x}}{\sinh(\lambda\bar{x})} \tag{C.7}$$

Note that (16) permits solution for $A$ only if $\bar{x}<\bar{x}_k$ where $\bar{x}_k$ is the marginal intervention point for a credible target zone for $\bar{s}$, i.e.:

$$\bar{x}_k=\{\bar{x}:1+\lambda(\bar{s}-\bar{x})\coth(\lambda\bar{x})=0\} \tag{C.8}$$

The above solution is unique and $\bar{x}_k>\bar{s}$. (For $\bar{x}>\bar{x}_k$, $s_i(x)$ would intersect $\bar{s}$ with a negative slope. This implies $s_i(x)$ goes beyond $\bar{s}$ for $\bar{x}<\bar{x}_k$.)

Given this natural restriction, we only need to consider the solution for $\bar{x}$ when $\bar{x}\in[0,\bar{x}_k]$. Solving (17) for $B$ yields:

$$B=\frac{\bar{s}-\bar{x}/(1+\beta\gamma)}{U[\frac{1}{2\beta\gamma};\frac{1}{2};\frac{\gamma}{\sigma^2}\bar{x}^2]} \tag{C.9}$$

The smooth-pasting condition (18) implies:

$$1+\lambda A\cosh(\lambda\bar{x})=\frac{1}{(1+\beta\gamma)}+B\;U^{'}[\frac{1}{2\beta\gamma};\frac{1}{2};\frac{\gamma}{\sigma^2}\bar{x}^2] \tag{C.10}$$

Using the differential properties of Kummer's function $U$ (Slater, p.16, 2.1.14), we derive:

$$\begin{aligned}1+\lambda A\cosh(\lambda\bar{x}) &= \tfrac{1}{1+\beta\gamma}-\tfrac{1}{2\beta\gamma}B\;U\left[\tfrac{1}{2\beta\gamma}+1;\tfrac{1}{2}+1;\tfrac{\gamma}{\sigma^2}\bar{x}^2\right]\tfrac{2\gamma}{\sigma^2}\bar{x}\\ &= \tfrac{1}{1+\beta\gamma}-\tfrac{\bar{x}}{\beta\sigma^2}B\;U\left[\tfrac{1}{2\beta\gamma}+1;\tfrac{3}{2};\tfrac{\gamma}{\sigma^2}\bar{x}^2\right]\end{aligned} \tag{C.11}$$

Substituting (C.7) and (C.9) into (C.11) yields:

$$\begin{aligned}1+\lambda(\bar{s}-\bar{x})\coth(\lambda\bar{x}) &= \tfrac{1}{(1+\beta\gamma)}-\tfrac{\bar{x}}{\beta\sigma^2}\left(\bar{s}-\tfrac{\bar{x}}{(1+\beta\gamma)}\right)\tfrac{U[\frac{1}{2\beta\gamma}+1;\frac{3}{2};\frac{\gamma}{\sigma^2}\bar{x}^2]}{U[\frac{1}{2\beta\gamma};\frac{1}{2};\frac{\gamma}{\sigma^2}\bar{x}^2]}\\ &= \tfrac{1}{1+\beta\gamma}-\tfrac{1}{\beta}\left(\tfrac{\gamma}{\sigma^2}\right)^{1/2}\left(\bar{s}-\tfrac{\bar{x}}{1+\beta\gamma}\right)\tfrac{Z^{1/2}U[\frac{1}{2\beta\gamma}+1;\frac{3}{2};z]}{U[\frac{1}{2\beta\gamma};\frac{1}{2};z]}\end{aligned} \tag{C.12}$$

Define the LHS of (C.12) as $f(\bar{x})$ and the RHS of (C.12) as $g(\bar{x})$ where $z=\frac{\gamma}{\sigma^2}\bar{x}^2$.

**Proposition 5** ***Lemma 6*** *(C.12) has a unique solution* $\bar{x}^* \in (\bar{s}, \min\{\bar{x}_k, (1+\beta\gamma)\,\bar{s}\})$

**Proof.**

(i) $f(\bar{x})$ is a strictly decreasing function.

To show that $f(\bar{x})$ is strictly decreasing, differentiating $f(\bar{x})$ w.r.t. $\bar{x}$ :

$$f'(\bar{x}) = -\lambda \coth(\lambda\bar{x}) + \lambda^2\,(\bar{s}-\bar{x})\left(1-\coth^2(\lambda\bar{x})\right) \tag{C.13}$$

- for $\bar{x} < \bar{s}$ since $\left(1-\coth^2(\lambda\bar{x})\right) < 0$ then $f'(\bar{x}) < 0$. If $(1+\beta\gamma)\,\bar{s} \le \bar{x}_k$, $f\left[(1+\beta\gamma)\,\bar{s}\right] \le \frac{1}{1+\beta\gamma}$
- for $\bar{x} \in (\bar{s}, \bar{x}_k)$ we first show that $f'(\bar{x}) > 0$ for $(0, \bar{x}_k)$

  We reverse the following procedure for the proof: $1+\lambda(\bar{s}-\bar{x})\coth(\lambda\bar{x}) > 0 \Leftrightarrow g_1 = \frac{1}{\lambda}\tanh(\lambda\bar{x}) + (\bar{s}-\bar{x}) > 0$. Since $g_1(0) = \bar{s}$, $l_1(\bar{x}) = \frac{1}{\lambda}\left(1-\tanh^2(\lambda\bar{x})\right)\lambda - 1 = -\tanh^2(\lambda\bar{x}) < 0$. So $g_1(\bar{x}) > 0$ for $\bar{x} \in (0, \bar{x}_k)$. Rewrite (C.13) as:

$$f'(\bar{x}) = -\lambda\coth(\lambda\bar{x})\left[\lambda\,(\bar{s}-\bar{x})\coth(\lambda\bar{x})\right] + \lambda^2\,(\bar{s}-\bar{x}) < 0 \tag{C.14}$$

So $f(\bar{x})$ is strictly increasing for $\bar{x} \in (0, \bar{x}_k)$ .

To show $f\left[(1+\beta\gamma)\,\bar{s}\right] \le \frac{1}{1+\beta\gamma}$, we do the following. If $(1+\beta\gamma)\,\bar{s} < \bar{x}_k$ then $f\left[(1+\beta\gamma)\,\bar{s}\right] = 1 - \lambda\beta\gamma\bar{s}\coth\left[\lambda\,(1+\beta\gamma)\,\bar{s}\right] < \frac{1}{1+\beta\gamma} \Leftrightarrow 1 - x\coth(x) < 0$ for $x = \lambda\,(1+\beta\gamma)\,\bar{s} > 0$. We can show that $l_2(\bar{x}) = \tanh(x) - x < 0$ with $l_2(0) = 0$, $l_2(\infty) \to -\infty, l_2'(\bar{x}) = -\tanh^2(x) < 0 \Rightarrow l_2(\bar{x}) < 0, \bar{x} > 0$.

# D Proof of Proposition 1

**Proof.** From the Krugman's solution we know that at the intervention point $\bar{x}_k$, with $\bar{x}_k > \bar{x}$, the following condition is satisfied[11]:

$$\lambda\,(\bar{x}_k - \bar{s})\coth(\lambda\bar{x}_k) = 1 \tag{D.19}$$

Taking the limit of (C.12) $\gamma \to \infty$ (which also implies $\bar{x} \to \bar{x}_k$) we get (D.19).

[11] The value matching and smooth pasting conditions at $\bar{x}_k$ are:

$$\bar{s} = \bar{x}_k + C_1\sinh(\lambda\bar{x}_k) \tag{D.15}$$

$$1 + C_1\lambda\cosh(\lambda\bar{x}_k) = 0 \tag{D.16}$$

which imply the following values for $C_1$ and $\bar{x}_k$ :

$$C_1 = -\frac{1}{\lambda}\cosh(\lambda\bar{x}_k) \tag{D.17}$$

# E Proof of Proposition 2

**Proof.** Proposition (a)

- For $0 < s \leq \bar{s}$

$$\lim_{x\to 0} \frac{1}{\beta}\left(\frac{x}{s_n(x)} - 1\right) = \frac{1}{\beta}\left(\frac{x}{x + \frac{\bar{s}-\bar{x}}{\sinh(\lambda\bar{x})}\sinh(\lambda x)} - 1\right) = 0 \tag{E.20}$$

- For $s \geq \bar{s}$

$$\lim_{x\to\infty} \frac{1}{\beta}\left(\frac{x}{s_i(x)} - 1\right) = \gamma \tag{E.21}$$

where to get the result we use the asymptotic condition:

$$\lim_{x\to\infty} s_i(x) = \frac{1}{1+\gamma\beta} \tag{E.22}$$

**Proof.** Proposition (b)

Since $\widetilde{\gamma}$ depends only on $s$ and $x$ and given that the solutions for $s$ are continuous at $\bar{x}$ it follows that:

$$\lim_{x\uparrow\bar{x}} \frac{1}{\beta}\left(\frac{x}{s_n(x)} - 1\right) = \lim_{x\downarrow\bar{x}} \frac{1}{\beta}\left(\frac{x}{s_i(x)} - 1\right) \tag{E.23}$$

**Proof.** Proposition (c)

**Lemma 1** $s(x) > xs'(x)$ for $0 < s \leq \bar{s}$ and $s \geq \bar{s}$.

- Given Lemma 1:

$$\frac{\partial\widetilde{\gamma}(x)}{\partial\, x} = \frac{1}{\beta s(x)^2}\left[s(x) - xs'(x)\right] > 0 \qquad \text{for} \qquad 0 < s \leq \bar{s} \quad \textit{and} \quad s > \bar{s}. \tag{E.24}$$

- The general result for $\frac{\partial\widetilde{\gamma}(x)}{\partial x^2}$ is:

$$\frac{\partial\widetilde{\gamma}(x)}{\partial\, x^2} = \frac{1}{\beta s(x)^3}\left[2\; s'(x)\left\{xs'(x) - s(x)\right\} - xs''(x)\; s(x)\right] \tag{E.25}$$

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$$\lambda\left(\bar{x}_k - \bar{s}\right)\coth(\lambda\bar{x}_k) = 1 \tag{D.18}$$

where (D.18) corresponds to (D.19) in the text.

- – $0 < s \leq \bar{s}$ .

$$\frac{\partial \widetilde{\gamma}(x)}{\partial x^2} = \frac{C_1}{3}\left[(7C_1\lambda - 1)\,(\lambda x)^3\right] > 0 \tag{E.26}$$

– $s \geq \bar{s}$. In this region $s''(x) > 0$ and given Lemma 1:

$$\frac{\partial^2 \widetilde{\gamma}(x)}{\partial x^2} < 0 \tag{E.27}$$

# F Proof of Proposition 3

**Proof.** In Krugman's model the degree of local mean reversion at $\bar{x}_k$ is given by:

$$\frac{1}{\beta}\left(\frac{\bar{x}_k}{\bar{s}} - 1\right) \tag{F.28}$$

Evaluating (12) as $\gamma \to \infty$ and $\bar{x} \to \bar{x}_k$ with $B$ replaced by (C.9) we get:

$$\lim_{\bar{x}\to\bar{x}_k} \frac{1}{\beta}\left(\frac{\bar{x}_k}{s_i(\bar{x}_k)} - 1\right) = \frac{1}{\beta}\left(\frac{\bar{x}_k}{\bar{s}} - 1\right) \tag{F.29}$$

which demonstrates the proposition.

# G Proof of Proposition 5

**Proof.** We first define:

$$\frac{d\widetilde{\gamma}(x)}{ds} = \frac{\partial \widetilde{\gamma}(x)}{\partial x}\frac{dx}{ds} = \widetilde{\gamma}'(x)\frac{dx}{ds} \tag{G.30}$$

$$\frac{d^2\widetilde{\gamma}(x)}{ds^2} = \widetilde{\gamma}''(x)\left(\frac{dx}{ds}\right)^2 + \widetilde{\gamma}'(x)\frac{d^2x}{ds^2} \tag{G.31}$$

- $0 < s \leq \bar{s}$. In this region $\frac{d\widetilde{\gamma}(x)}{dx} > 0$ and $\frac{d^2\widetilde{\gamma}(x)}{dx^2} > 0$. Hence:

$$\frac{d\widetilde{\gamma}(x)}{ds} > 0 \qquad \frac{d^2\widetilde{\gamma}(x)}{ds^2} > 0 \tag{G.32}$$

For the property of the inverse function if $\frac{ds}{dx} > 0$ and $\frac{d^2s}{dx^2} > 0$ then $\frac{dx}{ds} > 0$ and $\frac{d^2x}{ds^2} > 0$.

- $s \geq \bar{s}$. In this region $\frac{d\widetilde{\gamma}(x)}{dx} > 0$ and $\frac{d^2\widetilde{\gamma}(x)}{dx^2} < 0$. Hence:

$$\frac{d\widetilde{\gamma}(x)}{ds} > 0 \qquad \frac{d^2\widetilde{\gamma}(x)}{ds^2} < 0 \tag{G.33}$$

For the property of the inverse function if $\frac{ds}{dx} > 0$ and $\frac{d^2s}{dx^2} < 0$ then $\frac{dx}{ds} > 0$ and $\frac{d^2x}{ds^2} < 0$.

# H The Solution with Type II Interventions

Solutions for linear systems with saddle-point dynamics and feedback effects have been qualitatively analysed by Miller and Weller (1995). The solution method assumes a mapping between the exchange rate and the composite fundamental. Thus, to derive the general solution it is necessary to postulate a deterministic functional relationship between $s_i$ and $x$:

$$s_i(x) = f(x) \tag{H.34}$$

If the function $f(x)$ is twice differentiable and the second derivative is continuous, by applying Ito's lemma:

$$ds_i = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 \tag{H.35}$$

Since $(dx)^2 = \sigma^2 dt$, the previous expression can be written as:

$$E[ds_i] = f'(x)dx + \frac{1}{2}\sigma^2 f''(x)dt \tag{H.36}$$

Since in addition:

$$E[ds_i] = \frac{1}{\beta}(s_i - x)\,dt \tag{H.37}$$

the fundamental differential equation can be derived by equating (H.36) and (H.37) with $s_i = f(x)$:

$$\frac{1}{2}\sigma^2 f''(x)\ dt - \gamma\ \ f(x)f'(x)\ dt + \frac{1}{\beta}(x - f(x))\ \ dt = 0 \tag{H.38}$$

The relationship (H.38) is a second-order non-linear equation which does not in general admit closed form solutions.

In the deterministic case where the variance of the exchange rate with respect to the fundamental $\sigma$ is zero, the two eigenvectors $\theta_s$ and $\theta_u$ are the only paths connected to the origin. To determine the slopes of the two linear manifolds, substitute the linear relationship $f(x) = a + \theta\ x$ in (H.38). By applying the method of undetermined coefficient, the slopes of the stable and unstable eigenvectors, denoted $\theta_s$ and $\theta_u$, respectively, are given by:

$$\theta_s = \frac{-1 + \sqrt{1 + 4\gamma\beta}}{2\gamma\beta} > 0 \tag{H.39}$$

$$\theta_u = \frac{-1 - \sqrt{1 + 4\gamma\beta}}{2\gamma\beta} < 0 \tag{H.40}$$

If the system is subject to stochastic shocks there turns out to be an infinity of other non-linear solutions connected to the origin. To see this consider (H.38) rearranged as follows:

$$\frac{1}{2}\sigma^2 f''(x)dt = \gamma \;\; f(x)f'(x)\; dt - \frac{1}{\beta}\left(x - f(x)\right)\; dt \tag{H.41}$$

It represents the fundamental differential equation to be satisfied by any solution to the linear stochastic system for which $\sigma^2 \neq 0$. Thus, all the non-linear solutions can be thought of as bubble paths, along which the log-deviation of the official exchange rate deviates from its fundamental value. For a qualitative analysis of these solutions see Miller and Weller (1995).

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