Homological stability of spaces of manifolds via E_k-algebras

Ismael Sierra

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

Trinity College August 2023

I would like to dedicate this thesis to my brother and my parents.

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted before for any degree or other qualification except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the Mathematics Degree Committee.

Ismael Sierra
August 2023

Acknowledgements

I would like to thank first and foremost to my supervisor Oscar Randal-Williams for his immense support, inspiration, and exceptional dedication and guidance throughout the course of my PhD.

I am very grateful to Nathalie Wahl for her invitations to Copenhagen, her hospitality, and for all the helpful maths conversations. Additionally, I would like to extend my thanks to Alexander Kupers and Manuel Krannich for their suggestions and mathematical advice. I would also like to express my gratitude to my examiners for their comments and suggestions. I am supported by an EPSRC PhD Studentship, grant no. 2261123, and by O. Randal-Williams' Philip Leverhulme Prize from the Leverhulme Trust.

I am also thankful to Celia Rubio for her constant help with LaTeX. Moreover, I want to thank to all my friends and colleagues, for their support and all the fun lunches and activities; especially to אדוות, Samuel, Guillem (for all the unvaluable conversations), Andreas, Alexis, Laura, Eve, Macarena and Jenny.

My special thanks go to my family, especially to my parents, Alberto and Olga, and my brother Efrén for all their unconditional support and love.

Abstract

In this thesis we study homological stability properties of different families of spaces using the technique of cellular E_k-algebras. Firstly, we will consider spin mapping class groups of surfaces, and their algebraic analogue —quadratic symplectic groups— using cellular E_2-algebras. We will obtain improvements in their stability results, which for the spin mapping class groups we will show to be optimal away from the prime 2. We will also prove that in both cases the $\mathbb{F}_2$-homology satisfies secondary homological stability. Finally, we will give full descriptions of the first homology groups of the spin mapping class groups and of the quadratic symplectic groups. Secondly, we will study the classifying spaces of the diffeomorphism groups of the manifolds $W_{g,1} := D^{2n}\#(S^n \times S^n)^{\#g}$. We will get new improvements in the stability results, especially when working with rational coefficients. Moreover, we will prove a new type of stability result —quantised homological stability— which says that either the best integral stability result is a linear bound of slope $1/2$ or the stability is at least as good as a line of slope $2/3$.

Table of contents

Chapter 1

Introduction

1.1 Homological stability and cellular E_k-algebras

We say that a sequence of spaces $\mathcal{M}_1 \xrightarrow{s_1} \mathcal{M}_2 \xrightarrow{s_2} \mathcal{M}_3 \xrightarrow{s_3} \cdots$ satisfies *homological stability* if the induced maps $(s_{n-1})_* : H_d(\mathcal{M}_{n-1}) \to H_d(\mathcal{M}_n)$ are isomorphisms for $d < f(n)$, for a divergent function f. One alternative way of thinking about homological stability, which will be more convenient in this thesis, is to say that the relative homology groups $H_d(\mathcal{M}_n, \mathcal{M}_{n-1})$ vanish in a range of the form $d < f(n)$. In most of the known examples, $f(n) = \lambda n + c$ for λ, c constants, and then we say that the sequence of spaces satisfies homological stability of *slope* λ.

We are most interested in homological stability for families of *moduli spaces*, and in many classical examples they are classifying spaces of groups, i.e. $\mathcal{M}_n = BG_n$ for some sequence of groups $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$. The first example of such homological stability result was due to Quillen (in his unpublished notes) for the case $G_n = GL_n(\mathbb{F}_\ell)$. Many other families of groups have been shown to have homological stability since then, for example most classical groups (such as general linear groups, symplectic groups or different flavours of orthogonal and unitary groups) over fields, see [SW20]. General linear groups over many rings also satisfy homological stability, see [Maa79] for the case of PID's and [vdK80] for finite-dimensional Noetherian rings and more general cases. Indefinite orthogonal groups over many rings are also examples by [Vog81].

In geometry, the first example of a family of groups satisfying homological stability is mapping class groups of orientable surfaces, due to Harer in [Har85]. Some other examples include mapping class groups of non-orientable surfaces by Wahl, see [Wah08], and spin mapping class groups of (orientable) surfaces by Harer, see [Har90]. More generally, many families of moduli spaces of surfaces with tangential structures satisfy homological stability by Randal-Williams in [RW14]. More recently, the result of Galatius–Randal-Williams in

[GRW18] gives many families of moduli spaces of high dimensional manifolds which satisfy homological stability.

In many cases where homological stability holds, one can access the *stable homology*, i.e. $\mathrm{colim}_n H_d(\mathcal{M}_n)$, whereas computing the unstable homology is not usually possible. For arithmetic groups some examples of stable computations are due to Borel [Bor74], and Quillen [Qui72]; and for moduli spaces of manifolds the work of Galatius–Randal-Williams [GRW14] gives explicit descriptions of the stable homology of many such families.

Thus, the problem of optimizing the homological stability ranges has become important: improvements in the stability results directly lead to new homology computations. Since the spaces $\mathcal{M}_n$ are some type of moduli space, their cohomology groups $H^*(\mathcal{M}_n)$ are characteristic classes of some geometric structure, and hence being able to compute them has lots of potential applications.

Most of the homological stability results in the literature are proved using an approach similar to the original one of Quillen in his notes. However, in the last years, a new technique for proving homological stability has appeared, called the *cellular E_k-algebra approach*, due to Galatius–Kupers–Randal-Williams in [GKRW18]. The basic idea is that in many cases the whole family $\bigsqcup_n \mathcal{M}_n$ has the rich structure of a (graded) E_k-algebra, and moreover the *stabilization maps* $\mathcal{M}_{n-1} \to \mathcal{M}_n$ are induced by the E_k-multiplication. One can then use this richer structure, and not just the stabilisation maps, to gain new information.

In this thesis we will use the cellular E_k-algebra approach and ideas from its application to the homological stability of mapping class groups in [GKRW19], to improve the knowledge of homological stability for three families of moduli spaces: quadratic symplectic groups, spin mapping class groups of surfaces, and diffeomorphism groups of certain high-dimensional manifolds.

One of the advantages of the cellular E_k-algebra method is that it can also prove results beyond homological stability. One such type of results is *secondary homological stability*, which is a concept first appearing in [GKRW19] for the family of mapping class groups of surfaces. Homological stability is the statement that the relative homology groups $H_d(\mathcal{M}_n, \mathcal{M}_{n-1})$ vanish in a range, whereas secondary homological stability says that these relative homology groups are themselves stable (in a range greater than their vanishing range). We will prove some such results in this thesis.

1.2 Statement of results

The results of this thesis are about homological stability of three families of groups: spin mapping class groups, quadratic symplectic groups and diffeomorphism groups of certain high-dimensional manifolds. The first two of these families are closely related and the results we get for both of them are very similar in content and method of proof. The third family is also related but exhibits some extra complications.

1.2.1 Spin mapping class groups and quadratic symplectic groups

We denote by $\Sigma_{g,1}$ the orientable surface of genus g with one boundary component, and by $\Gamma_{g,1} = \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1}))$ its *mapping class group*, defined to be the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing pointwise a neighbourhood of its boundary. We will define the spin mapping class groups using the approach of [Har90], which is based on the notion of quadratic refinements.

Given an integer-valued skew-symmetric bilinear form (M,λ) on a finitely generated free $\mathbb{Z}$-module M, a *quadratic refinement* is a function $q : M \to \mathbb{Z}/2$ such that $q(x+y) \equiv q(x)+q(y)+\lambda(x,y) (\mathrm{mod}\, 2)$ for all $x,y \in M$. There are $2^{\mathrm{rk}(M)}$ quadratic refinements since a quadratic refinement is uniquely determined by its values on a basis of M, and any set of values is possible.

The set of quadratic refinements of $(H_1(\Sigma_{g,1};\mathbb{Z}),\cdot)$ has a right $\Gamma_{g,1}$-action by precomposition. By [Joh80, Corollary 2] this action has precisely two orbits for $g \geq 1$, distinguished by the *Arf invariant*, which is a $\mathbb{Z}/2$-valued function on the set of quadratic refinements, see Definition 3.2.1. For $\varepsilon \in \{0,1\}$ we will denote by $\Gamma^{1/2}_{g,1}[\varepsilon] := \mathrm{Stab}_{\Gamma_{g,1}}(q)$ where q is a choice of quadratic refinement of Arf invariant ε, and call this the *spin mapping class group in genus g and Arf invariant* ε.

Similarly, for $g \geq 1$ the group $Sp_{2g}(\mathbb{Z})$ acts on the set of quadratic refinements of the *standard symplectic form* $(\mathbb{Z}^{2g},\Omega_g)$, where Ω_g is the block diagonal sum of g copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, with precisely two orbits, also distinguished by the Arf invariant. Thus, for $\varepsilon \in \{0,1\}$ we can define the *quadratic symplectic group in genus g and Arf invariant* ε to be $Sp^{\varepsilon}_{2g}(\mathbb{Z}) := \mathrm{Stab}_{Sp_{2g}(\mathbb{Z})}(q)$ for a fixed quadratic refinement q of Arf invariant ε.

Before stating the results let us explain in detail what *stabilization maps* mean in this context. We begin by fixing quadratic refinements q_0, q_1 of $(H_1(\Sigma_{1,1};\mathbb{Z}),\cdot) \cong (\mathbb{Z}^2,\Omega_1)$ of Arf invariants $0,1$ respectively. Then, given any quadratic refinement q of $(H_1(\Sigma_{g-1,1};\mathbb{Z}),\cdot) \cong (\mathbb{Z}^{2(g-1)},\Omega_{g-1})$ we get a quadratic refinement $q \oplus q_\varepsilon$ of $(H_1(\Sigma_{g,1};\mathbb{Z}),\cdot) \cong (\mathbb{Z}^{2g},\Omega_g)$. Moreover, the Arf invariant is additive, see Definition 3.2.1, so $\mathrm{Arf}(q \oplus q_\varepsilon) = \mathrm{Arf}(q)+\varepsilon$.

Thus, using the inclusions $\Gamma_{g-1,1} \subset \Gamma_{g,1}$ and $Sp_{2(g-1)}(\mathbb{Z}) \subset Sp_{2g}(\mathbb{Z})$ we get *stabilization maps*

$$\Gamma^{1/2}_{g-1,1}[\delta-\varepsilon] \to \Gamma^{1/2}_{g,1}[\delta]$$

and

$$Sp^{\delta-\varepsilon}_{2(g-1)}(\mathbb{Z}) \to Sp^{\delta}_{2g}(\mathbb{Z}).$$

More generally, additivity of the Arf invariant under direct sum of quadratic refinements also allows us to define products

$$\Gamma^{1/2}_{g,1}[\varepsilon] \times \Gamma^{1/2}_{g',1}[\varepsilon'] \to \Gamma^{1/2}_{g+g',1}[\varepsilon+\varepsilon']$$

and

$$Sp^{\varepsilon}_{2g}(\mathbb{Z}) \times Sp^{\varepsilon'}_{2g'}(\mathbb{Z}) \to Sp^{\varepsilon+\varepsilon'}_{2(g+g')}(\mathbb{Z})$$

which contain the stabilisation maps as particular cases. As we will see in Chapter 3 these products allow us to endow $\bigsqcup_{g,\varepsilon} B\Gamma^{1/2}_{g,1}[\varepsilon]$ and $\bigsqcup_{g,\varepsilon} BSp^{\varepsilon}_{2g}(\mathbb{Z})$ with E_2-algebra structures, with grading in $\{0\} \sqcup (\mathbb{N}_{>0} \times \mathbb{Z}/2)$ instead of $\mathbb{N}$ (as these families are indexed by both the genus and the Arf invariant).

It is known since [Har90, Theorem 3.1] that spin mapping class groups satisfy homological stability in the range $d \lesssim g/4$, and their stable homology can be understood by [Gal06, Section 1]. The previously known best homological stability bounds can be found in [RW14, Theorem 2.14], where a range of the form $d \lesssim 2g/5$ was shown. The first main result of this thesis improves this stability range.

Theorem A. *Consider the stabilization map*

$$H_d(\Gamma^{1/2}_{g-1,1}[\delta-\varepsilon];\Bbbk) \to H_d(\Gamma^{1/2}_{g,1}[\delta];\Bbbk).$$

Then:

(i) If $\Bbbk = \mathbb{Z}$, it is surjective for $2d \leq g-2$ and an isomorphism for $2d \leq g-4$.

(ii) If $\Bbbk = \mathbb{Z}[\frac{1}{2}]$, it is surjective for $3d \leq 2g-4$ and an isomorphism for $3d \leq 2g-7$.

Moreover, there is a homology class $\theta \in H_2(\Gamma^{1/2}_{4,1}[0];\mathbb{F}_2)$ such that

$$\theta \cdot - : H_{d-2}(\Gamma^{1/2}_{g-4,1}[\delta], \Gamma^{1/2}_{g-5,1}[\delta-\varepsilon];\mathbb{F}_2) \to H_d(\Gamma^{1/2}_{g,1}[\delta], \Gamma^{1/2}_{g-1,1}[\delta-\varepsilon];\mathbb{F}_2)$$

is surjective for $3d \leq 2g-5$ and an isomorphism for $3d \leq 2g-8$.

The result with $\mathbb{Z}[1/2]$-coefficients is essentially optimal by Lemma 3.3.2, and in particular the slope $2/3$ cannot be improved. The last part of this theorem is an example of secondary homological stability. In Corollary 2.2.5 we will show that a consequence of the above theorem is that the $\mathbb{F}_2$-homology satisfies a $2/3$ slope stability if and only if

$\theta^3 \in H_6(\Gamma^{1/2}_{12,1}[0];\mathbb{F}_2)$ comes from stabilising a class in $H_6(\Gamma^{1/2}_{11,1}[\varepsilon];\mathbb{F}_2)$; and otherwise the slope $1/2$ of part (i) would be optimal with $\mathbb{F}_2$-coefficients, and hence integrally. We do not know which of these two alternatives holds.

The second main result of this thesis is about homological stability of quadratic symplectic groups. The previously known best homological stability result, [Fri17, Theorem 5.2], is of the form $d \lesssim g/2$.

Theorem B. *Consider the stabilization map*

$$H_d(Sp^{\delta-\varepsilon}_{2(g-1)}(\mathbb{Z});\mathbb{k}) \to H_d(Sp^{\delta}_{2g}(\mathbb{Z});\mathbb{k}).$$

Then:

(i) If $\mathbb{k}=\mathbb{Z}$, it is surjective for $2d \le g-2$ and an isomorphism for $2d \le g-4$.

(ii) If $\mathbb{k}=\mathbb{Z}[\frac{1}{2}]$, it is surjective for $3d \le 2g-4$ and an isomorphism for $3d \le 2g-7$.

Moreover, there is a homology class $\theta \in H_2(Sp^0_8(\mathbb{Z});\mathbb{F}_2)$ such that

$$\theta\cdot - : H_{d-2}(Sp^{\delta}_{2(g-4)}(\mathbb{Z}), Sp^{\delta-\varepsilon}_{2(g-5)}(\mathbb{Z});\mathbb{F}_2) \to H_d(Sp^{\delta}_{2g}(\mathbb{Z}), Sp^{\delta-\varepsilon}_{2(g-1)}(\mathbb{Z});\mathbb{F}_2)$$

is surjective for $3d \le 2g-5$ and an isomorphism for $3d \le 2g-8$.

Part (i) recovers the previously known bound, but its proof uses a different method. However, the improvement to $d \lesssim 2g/3$ in part (ii) is new. As before, the last part is a secondary stability result which implies that either the $\mathbb{F}_2$-homology also has slope $2/3$ stability (if θ^3 destabilises) or the optimal slope of the $\mathbb{F}_2$-homology is $1/2$ (otherwise).

The groups $Sp^0_{2g}(\mathbb{Z})$ have appeared in the literature under the name of *theta subgroups of the symplectic groups*, and are sometimes denoted by $Sp^q_{2g}(\mathbb{Z})$. These groups are of importance in number theory, see [Lu92] for example, and in the study of manifolds, as in [KRW21b, Section 4]. The groups $Sp^1_{2g}(\mathbb{Z})$ are less common but have appeared recently in the study of manifolds in [KRW21b, Section 4], where they are denoted by $Sp^a_{2g}(\mathbb{Z})$.

One of the insights of Chapter 3 is that it is easier to prove homological stability for $Sp^q_{2g}(\mathbb{Z})$ and $Sp^a_{2g}(\mathbb{Z})$ at the same time than studying them separately. This is because understanding the E_2-homology of the whole family will be quite easy by Theorem 3.1.4, whereas the E_2-homology of the subalgebra $\bigsqcup_g Sp^q_{2g}(\mathbb{Z})$ has been inaccessible so far. Thus, even if one is only interested in the theta-subgroups of the symplectic groups it is a good idea to study all the quadratic symplectic groups.

Let us remark that the class θ appearing in Theorems A and B is not uniquely defined, but its indeterminacy is understood by Remarks 2.2.4 and 2.2.6.

1.2.2 Moduli spaces of high dimensional manifolds

For a d-dimensional manifold with boundary W let us denote the group of diffeomorphisms of W fixing pointwise a neighbourhood of the boundary by $\mathrm{Diff}_\partial(W)$. The classifying space $B\mathrm{Diff}_\partial(W)$ of this group is of geometric importance since it is the moduli space which classifies smooth fibre bundles with fibre W and a trivialization over ∂W.

We will focus on a class of manifolds which generalize orientable surfaces with one boundary component to high dimensions: for a fixed $n \geq 1$ we define the following $2n$-dimensional manifold for each $g \geq 0$

$$W_{g,1} := D^{2n} \# (S^n \times S^n)^{\# g}.$$

These manifolds have already appeared in many papers in the literature, see [GRW18], [GRW14], [KRW21a], [KRW21b] for example. In particular, the study of their diffeomorphism groups is of great importance due to its potential applications, as in [KRW20].

We can view $W_{g,1}$ as the boundary connected sum of $W_{g-1,1}$ with $W_{1,1}$, and hence extension by the identity gives inclusions $\mathrm{Diff}_\partial(W_{g-1,1}) \hookrightarrow \mathrm{Diff}_\partial(W_{g,1})$. The stable homology of this family is known by [GRW14, Theorems 1.1, 1.2]. Furthermore, by [GRW18, Theorem 1.2], for $n \geq 3$ the maps

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1})) \to H_d(B\mathrm{Diff}_\partial(W_{g,1}))$$

are isomorphisms for $2d \leq g-4$. Thus, one can access the homology groups $H_d(B\mathrm{Diff}_\partial(W_{g,1}))$ in the range $d \lesssim g/2$.

The remaining results of this thesis are improvements on the homological stability bound of the family of groups $\{\mathrm{Diff}_\partial(W_{g,1})\}_{g\geq 1}$, and hence on the range in which their homology groups can be computed. The third main result of the thesis is

Theorem C. *For $n \geq 3$ odd, consider the stabilization map*

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1}); \Bbbk) \to H_d(B\mathrm{Diff}_\partial(W_{g,1}); \Bbbk).$$

Then:

(i) If $n = 3$ or 7 and $\Bbbk = \mathbb{Z}$, it is surjective for $3d \leq 2g-1$ and an isomorphism for $3d \leq 2g-4$.

(ii) If $n \neq 3$ or 7 and $\Bbbk = \mathbb{Z}$, it is surjective for $2d \leq g-2$ and an isomorphism for $2d \leq g-4$.

(iii) If $n \neq 3$ or 7 and $\Bbbk = \mathbb{Z}[\frac{1}{2}]$, it is surjective for $3d \leq 2g-4$ and an isomorphism for $3d \leq 2g-7$.

(iv) If $n = 3$ or 7 and $\Bbbk = \mathbb{Q}$, it is surjective for $d < \frac{3n-6}{3n-5}g$ and an isomorphism for $d < \frac{3n-6}{3n-5}g - 1$.

(v) If $n \neq 3$ or 7 and $\Bbbk = \mathbb{Q}$, it is surjective for $d < \frac{3n-6}{3n-5}(g-1)$ and an isomorphism for $d < \frac{3n-6}{3n-5}(g-1)-1$.

Let us briefly comment on the different parts of this result: part (ii) is the same stability result as found in [GRW18, Theorem 1.2], however our proof is done using different methods. Part (i) gives an improvement of the previously known stability result for $n = 3$ or 7. Part (iii) improves the previous stability results when $n \neq 3$ or 7 but only away from the prime 2. Finally, parts (iv) and (v) give a significant improvement in homological stability when restricting to rational coefficients.

Remark 1.2.1. *In fact, the stability result of part (ii) of the previous theorem can be improved to: surjective for $2d \leq g-1$ and isomorphism for $2d \leq g-3$. This will be done in Theorem 4.2.17, and the reason to include it as a separate result is that its proof makes use of the weaker version stated above.*

One of the main novelties of Theorem C is that it deals with the topological groups $\mathrm{Diff}_\partial(W_{g,1})$ rather than discrete groups, being the first example of an application of the cellular E_k-algebra method in this context.

One of the features of the cellular E_k-algebra method is that it can give different homological stability bounds for different coefficients. In our example, the stability range we get with $\mathbb{Q}$-coefficients, $d \lessapprox \frac{3n-6}{3n-5}g$, is significantly better than the previously known one $d \lessapprox g/2$. It suggests the conjecture that the optimal stability bound should have slope at least 1, meaning that it should be of the form $d \leq \lambda g + c$ for $\lambda \geq 1$ and c a constant.

The restriction to n odd is due to a technical step explained in Remark 4.4.10. It has to do with the fact that the intersection form on $W_{g,1}$ is either symmetric or skew-symmetric depending on the parity of n, and these two types of forms behave very differently. Most of the results of Chapter 4 can be carried out for the case n even, and in fact it is likely that one could eliminate the restriction to n odd by using a different "arc complex" to the one of Section 4.4. We will analyse in full detail the case n even and what we know about it in Chapter 5.

The final main result of this thesis is the following "quantisation stability" result, which, as explained in Remark 4.2.16, differs from other similar results in the literature since it is not a direct corollary of a secondary stability result (as it was the case for Theorems A and B).

Theorem D. *Let $n \geq 5$ be odd, $n \neq 7$, then one of the following two options holds*

(i) The relative homology groups $H_{2k}(B\mathrm{Diff}_\partial(W_{4k,1}), B\mathrm{Diff}_\partial(W_{4k-1,1}); \mathbb{Z})$ are non-zero for all $k \geq 1$.

(ii) The relative homology groups $H_d(B\mathrm{Diff}_\partial(W_{g,1}), B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{Z})$ vanish for $3d \leq 2g-6$.

In other words, it says that either part (ii) of Theorem C is essentially optimal or the true stability result has slope at least $2/3$. We do not know which of the two alternatives holds.

Finally, let us mention that one of the insights of Chapter 4 is to not use the "obvious E_k-algebra" $\bigsqcup_g B\mathrm{Diff}_\partial(W_{g,1})$, which would be the analogue of what one does for surfaces (i.e. the case $n = 1$). Instead, we "enlarge" this E_k-algebra by allowing manifolds W which are not the $W_{g,1}$'s and by slightly modifying the moduli spaces under study. In some sense, the relationship between the algebra $\mathbf{R}$ we construct in Definition 4.1.6 and $\bigsqcup_g B\mathrm{Diff}_\partial(W_{g,1})$ is the same as the relationship between the algebra of theta subgroups of the symplectic groups and the algebra of all quadratic symplectic groups. This new enlarged algebra $\mathbf{R}$ will have the advantage that its E_k-homology is more accessible, as we will discuss in detail in Chapter 4.

1.3 Overview of cellular E_k-algebras

The purpose of this section is to explain the methods from [GKRW18] used in this thesis: we aim for an informal discussion and refer to [GKRW18] for the details. We will begin by introducing the technical language of cellular E_k-algebras and some basic results and then we will explain the outline of how the cellular E_k-algebra approach is used in practice to produce homological stability results.

In the E_k-algebras part of the thesis, mainly Chapter 2, we will work in the category $\mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ of G-graded simplicial $\Bbbk$-modules, for $\Bbbk$ a commutative ring and G a discrete symmetric monoid. Formally, $\mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ denotes the category of functors from G, viewed as a category with objects the elements of G and only identity morphisms, to $\mathsf{sMod}_{\Bbbk}$. This means that each object M consists of a simplicial $\Bbbk$-module $M(x) = M_\bullet(x)$ for each $x \in \mathsf{G}$. The tensor product $\otimes$ in this category is given by Day convolution, i.e.

$$(M \otimes N)_p(x) = \bigoplus_{y+z=x} M_p(y) \otimes_{\Bbbk} N_p(z)$$

where $+$ denotes the monoidal structure of G.

In a similar way one can define the category of G-graded spaces, denoted by $\mathsf{Top}^{\mathsf{G}}$, and endow it with a monoidal structure by Day convolution using cartesian product of spaces.

In many examples that have appeared in the literature $\mathsf{G} = \mathbb{N}$, but in this thesis we will see examples where other monoids appear.

The *little k-cubes* operad in Top has n-ary operations given by rectilinear embeddings $I^k \times \{1,\cdots,n\} \hookrightarrow I^k$ such that the interiors of the images of the cubes are disjoint. (The space of 0-ary operations is empty.) We define the little k-cubes operad in $\mathsf{sMod}_{\Bbbk}$ by applying the symmetric monoidal functor $(-)_{\Bbbk} : \mathsf{Top} \to \mathsf{sMod}_{\Bbbk}$ given by the free $\Bbbk$-module on the singular simplicial set of a space. Moreover, $(-)_{\Bbbk}$ can be promoted to a functor $(-)_{\Bbbk} : \mathsf{Top}^{\mathsf{G}} \to \mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ between the graded categories, and we define the little k-cubes operad in these by concentrating it in grading 0, where $0 \in \mathsf{G}$ denotes the identity of the monoid. We shall denote this operad by $\mathcal{C}_k$ in all the categories $\mathsf{Top}, \mathsf{Top}^{\mathsf{G}}, \mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ which we use, and define an *E_k-algebra* to mean an algebra over this operad.

The *E_k-indecomposables* $Q^{E_k}(\mathbf{R})$ of an E_k-algebra $\mathbf{R}$ in $\mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ is defined by the exact sequence of graded simplicial $\Bbbk$-modules

$$\bigoplus_{n\geq 2} \mathcal{C}_k(n) \otimes \mathbf{R}^{\otimes n} \to \mathbf{R} \to Q^{E_k}(\mathbf{R}) \to 0.$$

The functor $\mathbf{R} \mapsto Q^{E_k}(\mathbf{R})$ is not homotopy-invariant but has a derived functor $Q_{\mathbb{L}}^{E_k}(-)$ which is. See [GKRW18, Section 13] for details and how to define it in more general categories such as E_k-algebras in Top or $\mathsf{Top}^{\mathsf{G}}$. The *$E_k$-homology* groups of $\mathbf{R}$ are defined to be

$$H_{x,d}^{E_k}(\mathbf{R}) := H_d(Q_{\mathbb{L}}^{E_k}(\mathbf{R})(x))$$

for $x \in \mathsf{G}$ and $d \in \mathbb{N}$.

One important property of E_k-indecomposables is that they are computable in terms of (generalised) bar constructions as explained in [GKRW18, Section 13]. Let us briefly explain the most relevant part of the theory of bar constructions for the purposes of this thesis: following [GKRW18, Section 12.2], there is a strictly associative algebra $\overline{\mathbf{R}}$ which is equivalent to the unitalization $\mathbf{R}^+ := \mathbb{1} \oplus \mathbf{R}$, where $\mathbb{1}$ is the monoidal unit in simplicial modules. For $\mathbf{M}$ a right $\overline{\mathbf{R}}$-module and $\mathbf{N}$ a left $\overline{\mathbf{R}}$-module we define the *bar construction* $B(\mathbf{M}, \overline{\mathbf{R}}, \mathbf{N})$ to be the geometric realization of the semisimplicial object $B_\bullet(\mathbf{M}, \overline{\mathbf{R}}, \mathbf{N})$ with p-simplices $\mathbf{M} \otimes \overline{\mathbf{R}}^{\otimes p} \otimes \mathbf{N}$, and face maps given by using the $\overline{\mathbf{R}}$-module structures and multiplication. By [GKRW18, Lemma 9.16], $B(\mathbf{M}, \overline{\mathbf{R}}, \mathbf{N})$ computes the derived tensor product $\mathbf{M} \otimes_{\overline{\mathbf{R}}}^{\mathbb{L}} \mathbf{N}$, and by [GKRW18, Corollary 9.17, Theorem 13.7] we can compute E_1-indecomposables using bar constructions via

$$Q_{\mathbb{L}}^{E_1}(\mathbf{R}) \simeq B(\mathbb{1}, \overline{\mathbf{R}}, \mathbf{R}) = \mathbb{1} \otimes_{\overline{\mathbf{R}}}^{\mathbb{L}} \mathbf{R}.$$

If one can understand the E_1-homology then there are spectral sequences computing the E_k-homology for $k \geq 2$ as explained in [GKRW18, Section 14]. We will also use bar constructions to do some base-change constructions which are important in some of the proofs.

In [GKRW18, Section 6] the notion of a *CW E_k-algebra* is defined, built in terms of free E_k-algebras by iteratively attaching cells in the category of E_k-algebras in order of dimension.

Let $\Delta^{x,d} \in \mathsf{sSet}^{\mathsf{G}}$ be the standard d-simplex placed in grading x and let $\partial\Delta^{x,d} \in \mathsf{sSet}^{\mathsf{G}}$ be its boundary. By applying the free $\Bbbk$-module functor we get objects $\Delta^{x,d}_{\Bbbk}, \partial\Delta^{x,d}_{\Bbbk} \in \mathsf{sMod}^{\mathsf{G}}_{\Bbbk}$. We then define the graded spheres in $\mathsf{sMod}^{\mathsf{G}}_{\Bbbk}$ via $S^{x,d}_{\Bbbk} := \Delta^{x,d}_{\Bbbk}/\partial\Delta^{x,d}_{\Bbbk}$, where the quotient denotes the cofibre of the inclusion of the boundary into the disc. In $\mathsf{sMod}^{\mathsf{G}}_{\Bbbk}$, the data for a cell attachment to an E_k-algebra $\mathbf{R}$ is an *attaching map* $e : \partial\Delta^{x,d}_{\Bbbk} \to \mathbf{R}$, which is the same as a map $\partial\Delta^{d}_{\Bbbk} \to \mathbf{R}(x)$ of simplicial $\Bbbk$-modules. To attach the cell we form the pushout in $\mathrm{Alg}_{E_k}(\mathsf{sMod}^{\mathsf{G}}_{\Bbbk})$

$$\begin{tikzcd}
\mathbf{E_k}(\partial\Delta^{x,d}_{\Bbbk}) \arrow[r] \arrow[d] & \mathbf{R} \arrow[d] \\
\mathbf{E_k}(\Delta^{x,d}_{\Bbbk}) \arrow[r] & \mathbf{R} \cup^{E_k}_{e} D^{x,d}_{\Bbbk}
\end{tikzcd}$$

where $\mathbf{E_k}(-)$ denotes the free E_k-algebra functor. One important property of cell attachments is that $Q^{E_k}_{\mathbb{L}}(\mathbf{R} \cup^{E_k}_{e} D^{x,d}_{\Bbbk}) = Q^{E_k}_{\mathbb{L}}(\mathbf{R}) \cup_{Q^{E_k}_{\mathbb{L}}(e)} D^{x,d}_{\Bbbk}$, provided that $\mathbf{R}$ is cofibrant. Thus, the E_k-homology of CW E_k-algebras is accessible.

A weak equivalence $\mathbf{C} \xrightarrow{\sim} \mathbf{R}$ from a CW E_k-algebra is called a *CW-approximation* to $\mathbf{R}$, and a key result, [GKRW18, Theorem 11.21], is that if $\mathbf{R}(0) \simeq 0$ then $\mathbf{R}$ admits a CW-approximation. Moreover, whenever $\Bbbk$ is a field, we can construct a CW-approximation in which the number of (x,d)-cells needed is precisely the dimension of $H^{E_k}_{x,d}(\mathbf{R})$. By "giving the d-cells filtration d", see [GKRW18, Section 11] for a more precise discussion of what this means, one gets a skeletal filtration of this E_k-algebra and a spectral sequence computing the homology of $\mathbf{R}$.

In order to discuss homological stability of E_k-algebras we will need some preparation. Let us start by explaining what stability means for algebras in graded simplicial modules and then generalise to graded topological spaces. Let $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}^{\mathsf{G}}_{\Bbbk})$, where G is equipped with a symmetric monoidal functor $\mathrm{rk} : \mathsf{G} \to \mathbb{N}$; and suppose we are given a homology class $\sigma \in H_{x,0}(\mathbf{R})$ with $\mathrm{rk}(x) = 1$. By definition σ is a homotopy class of maps $\sigma : S^{x,0}_{\Bbbk} \to \mathbf{R}$, so it gives a map $\sigma \cdot - : S^{x,0}_{\Bbbk} \otimes \overline{\mathbf{R}} \to \overline{\mathbf{R}}$ by using the associative product of $\overline{\mathbf{R}}$. We then define $\overline{\mathbf{R}}/\sigma$ to be the cofibre of this map.

Observe that a-priori $\sigma \cdot -$ is not a (left) $\overline{\mathbf{R}}$-module map, so the cofibre $\overline{\mathbf{R}}/\sigma$ is not a (left) $\overline{\mathbf{R}}$-module. When $\overline{\mathbf{R}}$ is strictly commutative this is not an issue since the natural right $\overline{\mathbf{R}}$-module structure on $\overline{\mathbf{R}}/\sigma$ induces a left module structure. However, if $k \geq 2$ and $\mathbf{R}$ is only assumed to be an E_k-algebra, there is already enough commutativity to view $\overline{\mathbf{R}}/\sigma$ as a left $\overline{\mathbf{R}}$-module: by the "adapters construction" in [GKRW18, Section 12.2], there is a way of defining a cofibre sequence $S^{x,0}_{\Bbbk} \otimes \overline{\mathbf{R}} \xrightarrow{\sigma\cdot-} \overline{\mathbf{R}} \to \overline{\mathbf{R}}/\sigma$ in the category of left $\overline{\mathbf{R}}$-modules in such a way that forgetting the $\overline{\mathbf{R}}$-module structure recovers the construction of the previous paragraph. We will make use of this extra (left) $\overline{\mathbf{R}}$-module structure in several places in this thesis.

By construction $\sigma \cdot -$ induces maps $\mathbf{R}(y) \to \mathbf{R}(x+y)$ between the different graded components of $\mathbf{R}$ and the homology of the object $\overline{\mathbf{R}}/\sigma$ captures the relative homology of these. Thus, homological stability results of $\mathbf{R}$ using σ to stabilize can be reformulated as vanishing ranges for $H_{z,d}(\overline{\mathbf{R}}/\sigma)$. In other words, to prove homological stability we need to find a function f such that $H_{z,d}(\overline{\mathbf{R}}/\sigma) = 0$ for $d < f(\mathrm{rk}(z))$. The advantage of this formulation is that skeletal filtrations of $\mathbf{R}$ give rise to filtrations of $\overline{\mathbf{R}}/\sigma$ and hence to spectral sequences capable of detecting vanishing ranges.

Moreover, this language is also very convenient to discuss secondary stability results. In this thesis secondary stability results start with a class σ as above and another class $\theta \in H_{y,d}(\mathbf{R})$, and will give a vanishing in the homology of the iterated cofibre construction $\overline{\mathbf{R}}/(\sigma,\theta) := (\overline{\mathbf{R}}/\sigma)/\theta$, which is defined to be the cofibre of the composition

$$\theta \cdot - : S^{y,d}_{\Bbbk} \otimes \overline{\mathbf{R}}/\sigma \xrightarrow{\theta \otimes \mathrm{id}} \overline{\mathbf{R}} \otimes \overline{\mathbf{R}}/\sigma \to \overline{\mathbf{R}}/\sigma$$

where the last map uses the (left) $\overline{\mathbf{R}}$-module structure of $\overline{\mathbf{R}}/\sigma$, defined using the adapters construction mentioned earlier. Observe that $\theta \cdot -$ gives maps between the relative homology groups of $\mathbf{R}$ with respect to σ, so vanishing ranges on $H_{*,*}(\overline{\mathbf{R}}/(\sigma,\theta))$ have the interpretation of stability for the relative homology groups, i.e. of secondary homological stability. This formulation of secondary stability makes clear than one could use this formalism to define higher stability phenomena (for example tertiary stability) by taking further cofibres. However, we will not investigate such results in this thesis since the knowledge of homology classes in the E_k-algebras under study is very limited.

If we started with a graded E_k-algebra in spaces, $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathsf{G}})$, then its homological stability with $\Bbbk$-coefficients with respect to a point $\sigma \in \mathbf{R}(x)$ for some $x \in \mathsf{G}$ with $\mathrm{rk}(x) = 1$ is studied as follows: we consider $\mathbf{R}_{\Bbbk} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}^{\mathsf{G}}_{\Bbbk})$, and the corresponding homology class $\sigma \in H_{x,0}(\mathbf{R}_{\Bbbk})$. Then a vanishing in the homology of $\overline{\mathbf{R}_{\Bbbk}}/\sigma$ in the sense of the previous paragraphs is equivalent to a vanishing of the relative homology groups $H_d(\mathbf{R}(x+y), \mathbf{R}(y); \Bbbk)$ in a range of the form $d < f(\mathrm{rk}(x+y))$. We can then do a similar analysis for the secondary stability.

One very useful formal property of the derived indecomposables that allows us to carry out the above paragraph is that it commutes with $(-)_{\Bbbk}$, see [GKRW18, Lemma 18.2]. Thus, for $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathsf{G}})$ its E_k-homology with $\Bbbk$ coefficients is the same as the E_k-homology of $\mathbf{R}_{\Bbbk}$. Hence we can indeed study homological stability of $\mathbf{R}$ with different coefficients by working with the E_k-algebras $\mathbf{R}_{\Bbbk}$ instead, which enjoy better properties as they are cofibrant and the category of graded simplicial $\Bbbk$-modules offers some technical advantages as explained in [GKRW18, Section 11]. However, at the same time, we can do computations in Top of the homology or E_k-homology of $\mathbf{R}$ and then transfer them to $\mathsf{sMod}_{\Bbbk}$.

Now let us explain how to put the above technical work in practice and study homological stability results of a family of (moduli) spaces using the cellular E_k-algebra method. This method is explained in [GKRW18] and has been applied in some situations to improve homological stability ranges, such as in [GKRW19], [GKRW20]. The method consists of four main steps.

(i) We build the "total moduli space" given by the disjoint union of the moduli spaces we want to study, and we give it an E_k-algebra structure (in Top) for some $k \geq 2$. Then we identify the correct grading for the algebra, which is precisely the discrete monoid of its path-components with respect to the E_k-algebra product. This gives $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathsf{G}})$.

(ii) We want to have some control on the E_k-cells of $\mathbf{R}$ in order to understand the spectral sequences computing $H_{*,*}(\overline{\mathbf{R}_{\Bbbk}}/\sigma)$. The place to start is an "a-priori vanishing line", i.e. we need to show that $H^{E_k}_{x,d}(\mathbf{R}) = 0$ for $d < A\,\mathrm{rk}(x) + B$ for some constants A, B. By the methods of [GKRW18, Section 14] it suffices to find an a-priori vanishing line on the E_1-homology. Using the formula for E_1-homology as a bar construction one produces certain semisimplicial spaces/sets whose high-connectivities imply the required vanishing line. This is done in detail in [GKRW18, Section 17] for moduli spaces coming from discrete groups, and we will do it in detail for some classifying spaces of diffeomorphism groups in Section 4.3.7.

(iii) We need information about the E_k-cells of $\mathbf{R}$ in small bidegrees. This is done by computing $H_{x,d}(\mathbf{R};\Bbbk)$ manually for small x, d and the action on these groups of certain homology operations universally defined on E_k-algebras.

(iv) We then run a spectral sequence argument as follows: we pick a CW-approximation of $\mathbf{R}_{\Bbbk}$ in which we understand the E_k-cells in a range using step (iii) and then we get a spectral sequence converging to $H_{*,*}(\overline{\mathbf{R}_{\Bbbk}}/\sigma)$. We then use the vanishing line of step (ii) to "filter away" the information contained in the cells which we don't know, so that we can forget about lots of terms in the spectral sequence. Finally, we use the fact that homology of free E_k-algebras is known, see [GKRW18, Section 16], to understand the spectral sequence and get a vanishing range on some page of it.

To finish this discussion, let us mention that step (ii) is usually the hardest as it requires showing that certain complexes are highly connected. In this thesis the whole of Sections 4.3 and 4.4 is devoted to checking such a result for one concrete E_k-algebra. On the other hand, step (iii) is what limits the power of the E_k-algebra method in many cases: accessing unstable homology groups is very hard in general, especially in degrees larger than one; and if one does not know enough groups then the results one can get are weaker. Step (iv) is highly technical but more computation-based, and in this thesis Chapter 2 contains all the required spectral sequence computations.

1.4 Outline of the thesis

This thesis is naturally divided into three parts. The first part is Chapter 2, which contains some technical results about homological stability of E_k-algebras. The second part is Chapter 3, in which we study homological stability of the spin mapping class groups and quadratic symplectic groups. This is based on [Sie22b]. The third part consists of the last two chapters. Chapter 4 is based on [Sie22a] and devoted to the study of the diffeomorphism groups of the manifolds $W_{g,1}^{2n}$ for n odd. Finally, Chapter 5 contains some partial results about homological stability of diffeomorphism groups of the manifolds $W_{g,1}^{2n}$ when n is even.

Both Chapters 3 and 4 can be seen as generalizations of the ideas of [GKRW19] to mapping class groups with spin structures and to high-dimensional analogues respectively. Each of them will follow the outline of the cellular E_k-algebra method explained at the end of Section 1.3.

Chapter 3 begins with Section 3.1 in which we give a precise construction of the E_k-algebras of spin mapping class groups and quadratic symplectic groups. Section 3.4 is devoted to completely describing the first homology groups of both the spin mapping class groups and the quadratic symplectic groups. This is done using the program GAP by explicitly computing presentations of the groups involved. Finally, we prove Theorem B in Section 3.2 and Theorem A in Section 3.3.

Chapter 4 begins with Section 4.1 where we give a precise definition of the E_k-algebras $\mathbf{R}$ (there is one such algebra for each dimension n) needed to prove Theorems C and D. Moreover, we will state the following result

Theorem E. *For $n \geq 3$ odd the E_{2n-1}-algebra $\boldsymbol{R}$ satisfies $H_{x,d}^{E_1}(\boldsymbol{R}) = 0$ for $x \in \mathsf{G}_n$ with $d < \mathrm{rk}(x) - 1$.*

This will be essential to get the a-priori bounds on E_k-cells needed for the cellular E_k-algebra method to work. The above theorem is also a main result of this thesis, although of a more technical nature than the previously stated ones. In Section 4.2 we will use the cellular E_k-algebra method to prove Theorems C and D assuming the above vanishing on E_k-cells for $\mathbf{R}$. Finally, Sections 4.3 and 4.4 are devoted to the proof of Theorem E.

Chapter 5 begins by explaining how most of Chapter 4 applies when the dimension is $2n$ for n even. Then, assuming that Theorem E is true for n even we prove a (rational) homological stability result similar to Theorem C(iv) valid for $n \geq 3$ even. In doing so we will need to show a new generic homological stability result for E_k-algebras, Theorem 5.4.1, similar to the ones in Chapter 2, but whose proof is of a different nature and involves new ideas and techniques.

Chapter 2

Generic homological stability results

2.1 Statements

In this section we will state the four generic homological stability results for E_k-algebras that will be applied later to prove the main theorems of this thesis. The first two of these are inspired by the generic homological stability theorem [GKRW18, Theorem 18.1], in the sense that they input a vanishing line on the E_2-homology of an E_2-algebra along with some information about the homology in small bidegrees, and they output homological stability results for the algebra. The third one is a secondary stability result, which is inspired by [GKRW19, Lemma 5.6, Theorem 5.12]. The forth one is a different type of result, which says that under suitable conditions a vanishing line on E_k-cells plus some partial homological stability result, i.e. stability up to some homological degree, leads to a global homological stability result valid for all degrees.

In addition, we also have two corollaries of Theorem 2.1.3. Corollary 2.2.5 says that E_2-algebras satisfying the assumptions of Theorem 2.1.3 have homological stability of slope either exactly $1/2$ or at least $2/3$ depending on the value of a certain homology class; and Corollary 2.2.7 gives a constant term improvement for the homological stability such E_2-algebras. The precise statement of these corollaries can be found after the proof of Theorem 2.1.3.

Before stating the results let us define the grading category that will be relevant:

Notation. For the rest of this thesis, let H be the discrete monoid $\{0\} \sqcup (\mathbb{N}_{>0} \times \mathbb{Z}/2)$, where the monoidal structure $+$ is given by addition in both coordinates. We denote by $\mathrm{rk} : \mathsf{H} \to \mathbb{N}$ the monoidal functor given by projection to the first coordinate.

Also, let us recall that on $\mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{k}}^{\mathbb{N}})$ there is a homology operation $Q^1_{\mathbb{k}}(-) : H_{*,0}(-) \to H_{2*,1}(-)$ defined in [GKRW18, Page 199]. This operation satisfies that $-2Q^1_{\mathbb{k}}(-) = [-,-]$, where $[-,-]$ is the Browder bracket (see [GKRW18, Section 16.1.1] for details). By using the canonical rank functor $\mathrm{rk}:\mathsf{H}\to\mathbb{N}$ we can view any H-graded E_2-algebra as $\mathbb{N}$-graded and hence make sense of this operation on $\mathrm{Alg}_{E_2}(\mathsf{sMod}_{\Bbbk}^{\mathsf{H}})$ too.

Theorem 2.1.1. *Let $\Bbbk$ be a commutative ring and let $\mathbf{X}\in\mathrm{Alg}_{E_2}(\mathsf{sMod}_{\Bbbk}^{\mathsf{H}})$ be such that $H_{0,0}(\mathbf{X})=0$, $H^{E_2}_{x,d}(\mathbf{X})=0$ for $d<\mathrm{rk}(x)-1$, and $H_{*,0}(\overline{\mathbf{X}})=\frac{\Bbbk[\sigma_0,\sigma_1]}{(\sigma_1^2-\sigma_0^2)}$ as a ring, for some classes $\sigma_\varepsilon\in H_{(1,\varepsilon),0}(\mathbf{X})$. Then, for any $\varepsilon\in\{0,1\}$ and any $x\in\mathsf{H}$ we have $H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon)=0$ for $2d\leq\mathrm{rk}(x)-2$.*

Theorem 2.1.2. *Let $\Bbbk$ be a commutative $\mathbb{Z}[1/2]$-algebra, let $\boldsymbol{X}\in\mathrm{Alg}_{E_2}(\mathsf{sMod}_{\Bbbk}^{\mathsf{H}})$ be such that $H_{0,0}(\mathbf{X})=0$, $H^{E_2}_{x,d}(\boldsymbol{X})=0$ for $d<\mathrm{rk}(x)-1$, and $H_{*,0}(\overline{\mathbf{X}})=\frac{\Bbbk[\sigma_0,\sigma_1]}{(\sigma_1^2-\sigma_0^2)}$ as a ring, for some classes $\sigma_\varepsilon\in H_{(1,\varepsilon),0}(\boldsymbol{X})$. Suppose in addition that for some $\varepsilon\in\{0,1\}$ we have:*

(i) $\sigma_\varepsilon\cdot-:H_{(1,1-\varepsilon),1}(\boldsymbol{X})\to H_{(2,1),1}(\boldsymbol{X})$ *is surjective.*

(ii) $\mathrm{coker}(\sigma_\varepsilon\cdot-:H_{(1,\varepsilon),1}(\boldsymbol{X})\to H_{(2,0),1}(\boldsymbol{X}))$ *is generated by $Q^1_{\Bbbk}(\sigma_0)$ as a $\mathbb{Z}$-module.*

(iii) $\sigma_{1-\varepsilon}\cdot Q^1_{\Bbbk}(\sigma_0)\in H_{(3,1-\varepsilon),1}(\boldsymbol{X})$ *lies in the image of $\sigma_\varepsilon^2\cdot-:H_{(1,1-\varepsilon),1}(\boldsymbol{X})\to H_{(3,1-\varepsilon),1}(\boldsymbol{X})$.*

Then $H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon)=0$ for $3d\leq 2\,\mathrm{rk}(x)-4$.

Theorem 2.1.3. *Let $\boldsymbol{X}\in\mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{F}_2}^{\mathsf{H}})$ be such that $H_{0,0}(\mathbf{X})=0$, $H^{E_2}_{x,d}(\boldsymbol{X})=0$ for $d<\mathrm{rk}(x)-1$, and $H_{*,0}(\overline{\mathbf{X}})=\frac{\mathbb{F}_2[\sigma_0,\sigma_1]}{(\sigma_1^2-\sigma_0^2)}$ as a ring, for some classes $\sigma_\varepsilon\in H_{(1,\varepsilon),0}(\boldsymbol{X})$. Suppose in addition that for some $\varepsilon\in\{0,1\}$ we have:*

(i) $\sigma_\varepsilon\cdot-:H_{(1,1-\varepsilon),1}(\boldsymbol{X})\to H_{(2,1),1}(\boldsymbol{X})$ *is surjective.*

(ii) $\mathrm{coker}(\sigma_\varepsilon\cdot-:H_{(1,\varepsilon),1}(\boldsymbol{X})\to H_{(2,0),1}(\boldsymbol{X}))$ *is generated by $Q^1_{\mathbb{F}_2}(\sigma_0)$.*

(iii) $\sigma_{1-\varepsilon}\cdot Q^1_{\mathbb{F}_2}(\sigma_0)\in H_{(3,1-\varepsilon),1}(\boldsymbol{X})$ *lies in the image of $\sigma_\varepsilon^2\cdot-:H_{(1,1-\varepsilon),1}(\boldsymbol{X})\to H_{(3,1-\varepsilon),1}(\boldsymbol{X})$.*

(iv) $\sigma_0\cdot Q^1_{\mathbb{F}_2}(\sigma_0)\in H_{(3,0),1}(\boldsymbol{X})$ *lies in the image of $\sigma_\varepsilon^2\cdot-:H_{(1,0),1}(\boldsymbol{X})\to H_{(3,0),1}(\boldsymbol{X})$.*

Then there is a class $\theta\in H_{(4,0),2}(\mathbf{X})$ such that $H_{x,d}(\overline{\mathbf{X}}/(\sigma_\varepsilon,\theta))=0$ for $3d\leq 2\,\mathrm{rk}(x)-5$.

Remark 2.1.4. *The appearance of the ring $\Bbbk[\sigma_0,\sigma_1]/(\sigma_1^2-\sigma_0^2)$ in all the above results will become clear after Lemma 3.2.2 and Proposition 4.1.4. A geometric interpretation of this fact will be explained after Remark 4.1.5, and is related to the fact that $K\natural K\cong W_{2,1}=W_{1,1}\natural W_{1,1}$ where K is the Kervaire manifold (see Remark 4.1.5 for details).*

Before properly stating the final result, let us give some motivation for some of the assumptions that will appear on the theorem.

If $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathbb{N}})$ satisfies that $\mathbf{R}(0) = \varnothing$ and that $\mathbf{R}(g)$ is path-connected for $g > 0$ then $H_{1,0}(\mathbf{R}_{\mathbb{Q}}) = \mathbb{Q}\{\sigma_0\}$ for σ_0 represented by a point in $\mathbf{R}(1)$. This is represented by a map $S^{1,0}_{\mathbb{Q}} \to \mathbf{R}_{\mathbb{Q}}$ and hence we get an E_k-algebra map $\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0) \to \mathbf{R}_{\mathbb{Q}}$. The long exact sequence in homology of the map $\sigma_0 \cdot -$ gives $H_{1,0}(\overline{\mathbf{R}}_{\mathbb{Q}}/\sigma_0) = 0$.

On the other hand, let $\mathbf{R} \in \mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathsf{H}})$ be such that $\mathbf{R}(0) = \varnothing$ and $\mathbf{R}(g,\varepsilon)$ is path-connected for any $(g,\varepsilon) \in \mathbb{N}_{>0} \times \mathbb{Z}/2$. We can also view $\mathbf{R}$ as an object in $\mathrm{Alg}_{E_k}(\mathsf{Top}^{\mathbb{N}})$ via $R(g) = R(g,0) \sqcup R(g,1)$, see [GKRW18, Section 3] for details. Then $H_{1,0}(\mathbf{R}_{\mathbb{Q}}) = \mathbb{Q}\{\sigma_0,\sigma_1\}$ for σ_ε represented by a point in $\mathbf{R}(1,\varepsilon)$, so we get an E_k-algebra map $\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0}_{\mathbb{Q}}\sigma_1) \to \mathbf{R}_{\mathbb{Q}}$. Moreover, by the monoidal structure of H both σ_0^2 and σ_1^2 lie in the same path-component $\mathbf{R}(2,0)$, so $\sigma_1^2 - \sigma_0^2 = 0 \in H_{2,0}(\mathbf{R}_{\mathbb{Q}})$. Hence, we can use the cell attachment construction explained in Section 1.3, or [GKRW18, Section 6] for further details, to extend the previous map to an E_k-algebra map $\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0}_{\mathbb{Q}}\sigma_1) \cup^{E_k}_{\sigma_1^2-\sigma_0^2} D^{2,1}_{\mathbb{Q}}\rho \to \mathbf{R}_{\mathbb{Q}}$. The long exact sequence in homology of the map $\sigma_0 \cdot -$ gives $H_{1,0}(\overline{\mathbf{R}}_{\mathbb{Q}}/\sigma_0) = \mathbb{Q}\{\sigma_1\}$.

Remark 2.1.5. *The above two cases are of special interest for us since they will apply to the E_{2n-1}-algebras $\mathbf{R}$ of Definition 4.1.6: the first case will apply for $n = 3,7$ and the second one in all the other dimensions. On each of the two cases we get a map from a certain cellular E_k-algebra to the rationalization $\mathbf{R}_{\mathbb{Q}}$. Moreover, the above discussion gives information about $H_{1,0}(\overline{\mathbf{R}}_{\mathbb{Q}}/\sigma_0)$ in each of the two cases which will be useful later.*

Theorem 2.1.6. *Let $k \geq 3$, $\mathbf{A},\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}^{\mathbb{N}}_{\mathbb{Q}})$ such that $H_{0,0}(X) = 0$, and $\mathbf{X}$ satisfies $H^{E_k}_{g,d}(X) = 0$ for $d < g-1$. Suppose that there is an E_k-algebra map $\mathbf{A} \to \mathbf{X}$ and a $D \in \mathbb{Z}_{>0} \cup \{\infty\}$ for which one of the following two cases holds.*

(i) $\mathbf{A} = \mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)$ *and* $H_{g,d}(\overline{\mathbf{X}}/\sigma_0) = 0$ *for* $d < \min\{g,D\}$

(ii) $\mathbf{A} = \mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0}_{\mathbb{Q}}\sigma_1) \cup^{E_k}_{\sigma_1^2-\sigma_0^2} D^{2,1}_{\mathbb{Q}}\rho$ *and* $H_{g,d}(\overline{\mathbf{X}}/\sigma_0) = 0$ *for* $d < \min\{g,D\}$ *except for* $(g,d) = (1,0)$ *where* $H_{1,0}(\overline{\mathbf{X}}/\sigma_0) = \mathbb{Q}\{\sigma_1\}$.

Then in case (i) we have $H_{g,d}(\overline{\mathbf{X}}/\sigma_0) = 0$ for $d < \frac{D}{D+1}g$ and in case (ii) we have $H_{g,d}(\overline{\mathbf{X}}/\sigma_0) = 0$ for $d < \frac{D}{D+1}(g-1)$.

2.2 Proving Theorems 2.1.1, 2.1.2 and 2.1.3

We will need some preparation. For $\Bbbk$ a commutative ring we define

$$\mathbf{A}_{\Bbbk} := \mathbf{E_2}(S^{(1,0),0}_{\Bbbk}\sigma_0 \oplus S^{(1,1),0}_{\Bbbk}\sigma_1) \cup^{E_2}_{\sigma_1^2-\sigma_0^2} D^{(2,0),1}_{\Bbbk}\rho \in \mathrm{Alg}_{E_2}(\mathsf{sMod}^{\mathsf{H}}_{\Bbbk}).$$

This should be thought of as a "universal example" in a sense that will be clear in Sections 2.2.1 and 2.2.3.

Remark 2.2.1. *To attach the* $((2,0),1)$*-cell* ρ *we need a map* $\partial\Delta_{\Bbbk}^{(2,0),1} \to \mathbf{E_2}(S_{\Bbbk}^{(1,0),0}\sigma_0 \oplus S_{\Bbbk}^{(1,1),0}\sigma_1)$*, whereas* $\sigma_1^2 - \sigma_0^2$ *is a-priori a map* $S_{\Bbbk}^{(2,0),0} \to \mathbf{E_2}(S_{\Bbbk}^{(1,0),0}\sigma_0 \oplus S_{\Bbbk}^{(1,1),0}\sigma_1)$.

The precise construction is as follows: for any d *there is a canonical map* $\partial\Delta_{\Bbbk}^{d+1} \to S_{\Bbbk}^{d}$ *which extends to the graded categories, hence allowing to make sense of cell attachments along maps defined on spheres. We will always use this slight abuse of notation without further mention.*

Proposition 2.2.2. *The* E_2*-algebra* $\mathbf{A}_{\Bbbk}$ *satisfies the assumptions of Theorem 2.1.1, i.e* $H_{0,0}(\mathbf{A}_{\Bbbk}) = 0$, $H_{x,d}^{E_2}(\mathbf{A}_{\Bbbk}) = 0$ *for* $d < \mathrm{rk}(x) - 1$ *and* $H_{*,0}(\overline{\mathbf{A}_{\Bbbk}}) = \Bbbk[\sigma_0, \sigma_1]/(\sigma_1^2 - \sigma_0^2)$ *as a ring.*

Proof. Since $\mathbf{A}_{\Bbbk}$ is built using cells then $Q_{\mathbb{L}}^{E_2}(\mathbf{A}_{\Bbbk}) = S_{\Bbbk}^{(1,0),0}\sigma_0 \oplus S_{\Bbbk}^{(1,1),0}\sigma_1 \oplus S_{\Bbbk}^{(2,0),1}\rho$, see [GKRW18, Sections 6.1.3 and 8.2.1] for details. Thus $H_{x,d}^{E_2}(\mathbf{A}_{\Bbbk}) = 0$ for $d < \mathrm{rk}(x) - 1$.

For the homology computations it suffices to consider the case $\Bbbk = \mathbb{Z}$ by the following argument:

Let us write $-\otimes\Bbbk : \mathsf{sMod}_{\mathbb{Z}} \to \mathsf{sMod}_{\Bbbk}$ for the base-change functor and for the corresponding functor between H-graded categories. Base-change is symmetric monoidal, preserves colimits and satisfies $S_{\mathbb{Z}}^{x,d} \otimes \Bbbk = S_{\Bbbk}^{x,d}$, $\Delta_{\mathbb{Z}}^{x,d} \otimes \Bbbk = \Delta_{\Bbbk}^{x,d}$, and hence we recognize $\mathbf{A}_{\Bbbk} = \mathbf{A}_{\mathbb{Z}} \otimes \Bbbk$. Thus, the universal coefficient theorem in homological degree 0 gives that $H_{x,0}(\mathbf{A}_{\mathbb{Z}}) \otimes \Bbbk \xrightarrow{\cong} H_{x,0}(\mathbf{A}_{\Bbbk})$, implying the claimed reduction. We are implicitly using that $\mathbf{A}_{\mathbb{Z}}$ is a cofibrant simplicial module, but it is a general fact that any cellular E_k-algebra is cofibrant by [GKRW18, Section 8.2.1]. We will use this without mention throughout the rest of this thesis.

To simplify notation we will not write ${}_{\mathbb{Z}}$ for the rest of this proof since we will only treat the case $\Bbbk = \mathbb{Z}$. Consider the cell-attachment filtration $\mathbf{fA} \in \mathrm{Alg}_{E_2}((\mathsf{sMod}_{\mathbb{Z}}^{\mathsf{H}})^{\mathbb{Z}_{\leq}})$, which by [GKRW18, Section 6.2.1] is given by

$$\mathbf{fA} = \mathbf{E_2}(S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1) \cup^{E_2}_{\sigma_1^2-\sigma_0^2} D^{(2,0),1,1}\rho,$$

where the last grading represents the filtration stage. By [GKRW18, Corollary 10.17] there is a spectral sequence

$$E^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathbf{E_2}(S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1 \oplus S^{(2,0),1,1}\rho)}) \Rightarrow H_{x,p+q}(\overline{\mathbf{A}}).$$

The first page of this spectral sequence can be accessed by [GKRW18, Theorems 16.4, 16.7] and the description of the homology operation $Q^1_{\mathbb{Z}}(-)$ in [GKRW18, Page 199]. In homological degrees $p+q \leq 1$ the full answer is given by

- $E^1_{x,p,-p}$ vanishes for $p \neq 0$, and $\bigoplus_{x\in\mathsf{H}} E^1_{x,0,0}$ is the free $\mathbb{Z}$-module on the set of generators $\{\sigma_0^a \cdot \sigma_1^b : a+b \geq 0\}$, where $\sigma_0^0 = \sigma_1^0 = 1$.
- The only elements in homological degree $p+q=1$ are stabilizations by powers of σ_0 and σ_1 of one of the classes ρ, $Q^1_{\mathbb{Z}}(\sigma_0)$, $Q^1_{\mathbb{Z}}(\sigma_1)$, or $[\sigma_0,\sigma_1]$. Thus they have filtration $p \leq 1$.

By [GKRW18, Section 16.6] the spectral sequence is multiplicative and its differential satisfies $d^1(\sigma_0)=0$, $d^1(\sigma_1)=0$ and $d^1(\rho)=\sigma_1^2-\sigma_0^2$. Thus, $d^1(Q^1_{\mathbb{Z}}(\sigma_0)) = 0 = d^1(Q^1_{\mathbb{Z}}(\sigma_0))$, $d^1([\sigma_0,\sigma_1])=0$, by [GKRW18, Theorem 16.8], and hence $\bigoplus_{x\in\mathsf{H}} E^2_{x,0,0}$ is given as a ring by $\mathbb{Z}[\sigma_0,\sigma_1]/(\sigma_1^2-\sigma_0^2)$. Hence to finish the proof it suffices to show that $E^2_{x,0,0} = E^\infty_{x,0,0}$ for any $x \in \mathsf{H}$. This holds because d^r decreases filtration by r and homological degree by 1 so d^r vanishes on all the elements of homological degree 1 for $r \geq 2$. □

2.2.1 Proof of Theorem 2.1.1

Proof. We will do a series of reductions to get to the case $\mathbf{X} = \mathbf{A}_{\mathbb{k}}$ for some appropriate coefficients $\mathbb{k}$, and then we will do a direct computation.

Step 1. The aim of this step is to reduce to the particular case $\mathbf{X} = \mathbf{A}_{\mathbb{k}}$.

Each class $\sigma_\varepsilon \in H_{(1,\varepsilon),0}(\mathbf{X})$ is represented by a homotopy class $\sigma_\varepsilon : S^{(1,\varepsilon),0}_{\mathbb{k}} \to \mathbf{X}$. Thus there is an E_2-algebra map $\mathbf{E_2}(S^{(1,0),0}_{\mathbb{k}}\sigma_0 \oplus S^{(1,1),0}_{\mathbb{k}}\sigma_1) \to \mathbf{X}$ sending σ_0,σ_1 to the corresponding homology classes of $\mathbf{X}$. By assumption $\sigma_1^2 - \sigma_0^2 = 0 \in H_{(2,0),0}(\mathbf{X})$, so picking a nullhomotopy gives an extension to an E_2-algebra map $c : \mathbf{A}_{\mathbb{k}} \to \mathbf{X}$.

Now we claim that for the map c we have $H^{E_2}_{x,d}(\mathbf{X},\mathbf{A}_{\mathbb{k}}) = 0$ for $d < \mathrm{rk}(x)/2$. Indeed, by assumption $H^{E_2}_{x,d}(\mathbf{X}) = 0$ for $d < \mathrm{rk}(x)-1$ and by Proposition 2.2.2 $H^{E_2}_{x,d}(\mathbf{A}_{\mathbb{k}}) = 0$ for $d < \mathrm{rk}(x)-1$ too. Thus, it suffices to show the claim for $(\mathrm{rk}(x)=1, d=0)$. Both $\mathbf{X}$ and $\mathbf{A}_{\mathbb{k}}$ are reduced, i.e. $H_{0,0}(-)$ vanishes on both of them, and hence by [GKRW18, Corollary 11.12] it suffices to show that $H_{x,0}(\mathbf{X},\mathbf{A}_{\mathbb{k}}) = 0$ for $\mathrm{rk}(x)=1$, which holds by our assumption about the zero-th homology of $\mathbf{X}$ and Proposition 2.2.2.

Now let us suppose that the theorem holds for $\mathbf{X} = \mathbf{A}_{\mathbb{k}}$. By [GKRW18, Corollary 15.10] with $\rho(x) = \mathrm{rk}(x)/2$, $\mu(x) = (\mathrm{rk}(x)-1)/2$ and $\mathbf{M} = \overline{\mathbf{A}_{\mathbb{k}}}/\sigma_\varepsilon$ we find that $H_{x,d}(B(\overline{\mathbf{X}},\overline{\mathbf{A}_{\mathbb{k}}},\mathbf{M})) = 0$ for $d < \mu(x)$. But, by [GKRW18, Section 12.2.4], $B(\overline{\mathbf{X}},\overline{\mathbf{A}_{\mathbb{k}}},\mathbf{M}) \simeq \overline{\mathbf{X}}/\sigma_\varepsilon$, giving the required reduction. Note that we use the “adapters construction” of [GKRW18, Section 12.2] to view $\mathbf{M}$ as a left $\overline{\mathbf{A}_{\mathbb{k}}}$-module. We will also use this construction in the rest of this proof without explicit mention.

Step 2. Now we will further reduce to the case $\mathbf{A}_{\mathbb{F}_\ell}$, for ℓ a prime number or 0, where $\mathbb{F}_0 := \mathbb{Q}$.

Recall from the proof of Proposition 2.2.2 that $\mathbf{A}_{\mathbb{k}} = \mathbf{A}_{\mathbb{Z}} \otimes \mathbb{k}$. Since base-change preserves colimits, the cofibration sequence $S^{(1,\varepsilon),0}_{\mathbb{k}} \otimes \overline{\mathbf{A}_{\mathbb{k}}} \xrightarrow{\sigma_\varepsilon\cdot -} \overline{\mathbf{A}_{\mathbb{k}}} \to \overline{\mathbf{A}_{\mathbb{k}}}/\sigma_\varepsilon$ shows that $\overline{\mathbf{A}_{\mathbb{k}}}/\sigma_\varepsilon \cong \overline{\mathbf{A}_{\mathbb{Z}}}/\sigma_\varepsilon \otimes \mathbb{k}$. Thus, by the universal coefficient theorem it suffices to prove the case $\mathbb{k} = \mathbb{Z}$.

We claim that the homology groups of $\overline{\mathbf{A}_{\mathbb{Z}}}$ are finitely generated. Indeed, by [GKRW18, Theorem 16.4] each entry on the first page of the spectral sequence of the proof of Proposition 2.2.2 is finitely generated. Observe that this is a quite general fact that holds for any cellular E_2-algebra with finitely many cells by considering the analogous cell attachment filtration. We will use this again in the proof of Theorem 2.1.2.

Hence, by the homology long exact sequence of $\sigma_\varepsilon \cdot -$, the homology groups of $\overline{\mathbf{A}_{\mathbb{Z}}}/\sigma_\varepsilon$ are also finitely generated.

Thus, it suffices to check the cases $\Bbbk = \mathbb{F}_\ell$ for ℓ a prime number or 0, by another application of the universal coefficient theorem and using the finite generation of the homology groups.

Step 3. We will prove the theorem for a fixed ℓ, $\Bbbk = \mathbb{F}_\ell$ and $\mathbf{X} = \mathbf{A}_\Bbbk$. To simplify notation we will not write the subscripts $\mathbb{F}_\ell$ for the rest of this proof. Consider the cell attachment filtration $\mathbf{fA} \in \mathrm{Alg}_{E_2}((\mathsf{sMod}_{\mathbb{F}_\ell}^{\mathsf{H}})^{\mathbb{Z}_\leq})$ as in the proof of Proposition 2.2.2. The filtration 0 part is given by $\overline{\mathbf{fA}}(0) = \overline{\mathbf{E_2}(S^{(1,0),0}\sigma_0 \oplus S^{(1,1),0}\sigma_1)}$ so we can lift the maps σ_ε to filtered maps $\sigma_\varepsilon : S^{(1,\varepsilon),0,0} \to \overline{\mathbf{fA}}$. Thus, using adapters we can form the left $\overline{\mathbf{fA}}$-module $\overline{\mathbf{fA}}/\sigma_\varepsilon$ filtering $\overline{\mathbf{A}}/\sigma_\varepsilon$.

Since $\mathrm{gr}(-)$ commutes with pushouts and with $\overline{(-)}$ by [GKRW18, Lemma 12.7], we get two spectral sequences

(i) $F^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathbf{E_2}(S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1 \oplus S^{(2,0),1,1}\rho)}) \Rightarrow H_{x,p+q}(\overline{\mathbf{A}})$

(ii) $E^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathbf{E_2}(S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1 \oplus S^{(2,0),1,1}\rho)}/\sigma_\varepsilon) \Rightarrow H_{x,p+q}(\overline{\mathbf{A}}/\sigma_\varepsilon)$.

In order to prove the theorem it suffices to show the following claim

Claim. $E^2_{x,p,q} = 0$ for $p+q < (\mathrm{rk}(x)-1)/2$.

We will need some preparation. As in the proof of Proposition 2.2.2, the first spectral sequence is multiplicative and its differential satisfies $d^1(\sigma_0) = 0$, $d^1(\sigma_1) = 0$ and $d^1(\rho) = \sigma_1^2 - \sigma_0^2$. Moreover, by [GKRW18, Theorem 16.4, Section 16.2], its first page is given by $\Lambda(L)$ where $\Lambda(-)$ denotes the free graded-commutative algebra, and L is the $\mathbb{F}_\ell$-vector space with basis $Q^I_\ell(y)$ such that y is a basic Lie word in $\{\sigma_0, \sigma_1, \rho\}$ and I is admissible, in the sense of [GKRW18, Section 16.2].

The second spectral sequence is a module over the first one, and we can identify $E^1 = F^1/(\sigma_\varepsilon)$ because $\sigma_\varepsilon \cdot -$ is injective in F^1, by the above description of F^1, and its image is the ideal (σ_ε). Therefore $E^1 = \Lambda(L/\mathbb{F}_\ell\{\sigma_\varepsilon\})$ and hence the d^1 differential in F^1 completely determines the d^1 differential in E^1, making it into a CDGA.

Proof of Claim. E^2 is given by the homology of the CDGA (E^1, d^1), and to prove the result we will introduce a "computational filtration" in this CDGA that has the virtue of filtering away most of the d^1 differential.

We let $\mathcal{F}^\bullet E^1$ be the filtration in which $\sigma_{1-\varepsilon}$ and ρ are given filtration 0, the remaining elements of a homogeneous basis of $L/\mathbb{F}_\ell\{\sigma_\varepsilon\}$ extending these are given filtration equal to their homological degree, and then we extend the filtration to $\Lambda(L/\mathbb{F}_\ell\{\sigma_\varepsilon\})$ multiplicatively.

Since d^1 preserves this filtration we get a spectral sequence converging to E^2 whose first page is the homology of the associated graded $\mathrm{gr}(\mathcal{F}^\bullet E^1)$. Thus, it suffices to show that $H_*(\mathrm{gr}(\mathcal{F}^\bullet E^1))$ already has the required vanishing line.

Let us denote by D the corresponding differential on this computational spectral sequence. Since d^1 lowers homological degree by 1 we can decompose $(\mathrm{gr}(\mathcal{F}^\bullet E^1), D)$ as a tensor product

$$(\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon},\rho\}),D)\otimes_{\mathbb{F}_\ell}(\Lambda(L/\mathbb{F}_\ell\{\sigma_0,\sigma_1,\rho\}),0),$$

where D satisfies $D(\sigma_{1-\varepsilon})=0$ and $D(\rho)=(-1)^\varepsilon\sigma_{1-\varepsilon}^2$. By the Künneth theorem the homology of this tensor product is $\mathbb{F}_\ell\{1,\sigma_{1-\varepsilon}\}\otimes_{\mathbb{F}_\ell}\Lambda(L/\mathbb{F}_\ell\{\sigma_0,\sigma_1,\rho\})$ when $\ell\neq 2$, because graded-commutativity forces $\rho^2=0$; and $\mathbb{F}_2\{1,\sigma_{1-\varepsilon}\}\otimes_{\mathbb{F}_2}\mathbb{F}_2[\rho^2]\otimes_{\mathbb{F}_2}\Lambda(L/\mathbb{F}_2\{\sigma_0,\sigma_1,\rho\})$ if $\ell=2$.

By the *slope* of an element we shall mean the ratio between its homological degree and the rank of its H-valued grading. Since ρ^2 has slope $1/2$ and $\sigma_{1-\varepsilon}$ has homological degree 0 and rank 1, in order to prove the required vanishing line it suffices to show that all the elements in $L/\mathbb{F}_\ell\{\sigma_0,\sigma_1,\rho\}$ have slope $\geq 1/2$. Since the slope of $Q_\ell^I(y)$ is always larger than or equal to the one of y, and the slope of the Browder bracket of two elements is always greater than the minimum of their slopes, the only elements in L that have slope less than $1/2$ are those in the span of σ_0,σ_1, giving the result. □

□

2.2.2 Proof of Theorem 2.1.2

Proof. The idea of the proof is identical to the previous one, so we will not spell out all the details, but we will focus instead in the extra complications that arise in the computations, mainly in the latter steps.

Step 1. We will construct a certain cellular E_2-algebra $\mathbf{S}_{\Bbbk}$ and show that it suffices to prove that $H_{x,d}(\overline{\mathbf{S}_{\Bbbk}}/\sigma_\varepsilon)=0$ for $3d\leq 2\,\mathrm{rk}(x)-4$.

The assumptions of the statement imply that $[\sigma_0,\sigma_1]=\sigma_\varepsilon\cdot y$ for some $y\in H_{(1,1-\varepsilon),1}(\mathbf{X})$, that $Q_{\Bbbk}^1(\sigma_1)=\sigma_\varepsilon\cdot x+tQ_{\Bbbk}^1(\sigma_0)$ for some $x\in H_{(1,\varepsilon),1}(\mathbf{X})$ and some $t\in\mathbb{Z}$, and that $\sigma_{1-\varepsilon}\cdot Q_{\Bbbk}^1(\sigma_0)=\sigma_\varepsilon^2\cdot z\in H_{(3,1-\varepsilon),1}(\mathbf{X})$ for some $z\in H_{(1,1-\varepsilon),1}(\mathbf{X})$.

Let

$$\begin{aligned}\mathbf{S}_{\Bbbk} := \mathbf{E_2}(S^{(1,0),0}_{\Bbbk}\sigma_0 \oplus S^{(1,1),0}_{\Bbbk}\sigma_1 \oplus S^{(1,\varepsilon),1}_{\Bbbk}x \oplus S^{(1,1-\varepsilon),1}_{\Bbbk}y \oplus S^{(1,1-\varepsilon),1}_{\Bbbk}z)\\ \cup^{E_2}_{\sigma_1^2-\sigma_0^2} D^{(2,0),1}_{\Bbbk}\rho \cup^{E_2}_{Q^1_{\Bbbk}(\sigma_1)-\sigma_\varepsilon\cdot x - tQ^1_{\Bbbk}(\sigma_0)} D^{(2,0),2}_{\Bbbk}X \cup^{E_2}_{[\sigma_0,\sigma_1]-\sigma_\varepsilon\cdot y} D^{(2,1),2}_{\Bbbk}Y\\ \cup^{E_2}_{\sigma_{1-\varepsilon}\cdot Q^1_{\Bbbk}(\sigma_0)-\sigma_\varepsilon^2\cdot z} D^{(3,1-\varepsilon),2}_{\Bbbk}Z \in \mathrm{Alg}_{E_2}(\mathsf{sMod}^{\mathsf{H}}_{\Bbbk})\end{aligned}$$

By proceeding as in Step 1 of the proof of Theorem 2.1.1 there is an E_2-algebra map $f : \mathbf{S}_{\Bbbk} \to \mathbf{X}$ sending each of $\sigma_0, \sigma_1, x, y, z$ to the corresponding homology classes in $\mathbf{X}$ with the same name.

Claim. $H^{E_2}_{x,d}(\mathbf{X}, \mathbf{S}_{\Bbbk}) = 0$ for $d < 2\,\mathrm{rk}(x)/3$.

Assuming the claim, we can apply [GKRW18, Corollary 15.10] with $\rho(x) = 2\,\mathrm{rk}(x)/3$, $\mu(x) = (2\,\mathrm{rk}(x) - 3)/3$ and $\mathbf{M} = \overline{\mathbf{S}_{\Bbbk}}/\sigma_\varepsilon$ to obtain the required reduction. Thus, to finish this step we just need to show the claim.

Proof of Claim. Proceeding as in the proof of Proposition 2.2.2 one can compute $Q^{E_2}_{\mathbb{L}}(\mathbf{S}_{\Bbbk})$ and check that $H^{E_2}_{x,d}(\mathbf{S}_{\Bbbk}) = 0$ for $d < \mathrm{rk}(x) - 1$. Since $\mathbf{X}$ has the same vanishing line on its E_2-homology it suffices to check that $H^{E_2}_{x,d}(\mathbf{X}, \mathbf{S}_{\Bbbk}) = 0$ for $(\mathrm{rk}(x) = 1, d = 0)$ and $(\mathrm{rk}(x) = 2, d = 1)$.

For $(\mathrm{rk}(x) = 1, d = 0)$ we use [GKRW18, Corollary 11.12] to reduce it to showing that $H_{x,0}(\mathbf{X}, \mathbf{S}_{\Bbbk}) = 0$, as in Step 1 in the proof of Theorem 2.1.1. This holds because the zero-th homology of $\mathbf{X}$ in rank 1 is generated by σ_0, σ_1, which factor trough f by construction.

For $(\mathrm{rk}(x) = 2, d = 1)$ the argument will be more elaborate but use the same ideas. Pick sets of $\Bbbk$-module generators $\{u_a\}_{a\in A}$ for $H_{0,1}(\mathbf{X})$ and $\{v_b\}_{b\in B}$ for $H_{(1,0),1}(\mathbf{X}) \oplus H_{(1,1),1}(\mathbf{X})$, where each v_b has H-grading $(1, \varepsilon_b)$ for some $\varepsilon_b \in \{0, 1\}$. Consider

$$\tilde{\mathbf{S}}_{\Bbbk} := \mathbf{S}_{\Bbbk} \oplus^{E_2} \mathbf{E_2}(\bigoplus_{a\in A} S^{0,1}_{\Bbbk}u_a \oplus \bigoplus_{b\in B} S^{(1,\varepsilon_b),1}_{\Bbbk}v_b).$$

The map $f : \mathbf{S}_{\Bbbk} \to \mathbf{X}$ factors trough the canonical map $\mathbf{S}_{\Bbbk} \to \tilde{\mathbf{S}}_{\Bbbk}$ in the obvious way, so we get a long exact sequence in E_2-homology for the triple $\mathbf{S}_{\Bbbk} \to \tilde{\mathbf{S}}_{\Bbbk} \to \mathbf{X}$:

$$\cdots \to H^{E_2}_{x,1}(\tilde{\mathbf{S}}_{\Bbbk}, \mathbf{S}_{\Bbbk}) \to H^{E_2}_{x,1}(\mathbf{X}, \mathbf{S}_{\Bbbk}) \to H^{E_2}_{x,1}(\mathbf{X}, \tilde{\mathbf{S}}_{\Bbbk}) \to \cdots.$$

The first term vanishes by direct computation of $Q^{E_2}_{\mathbb{L}}(\tilde{\mathbf{S}}_{\Bbbk})$ because $\mathrm{rk}(x) = 2$, so it suffices to show that the third term vanishes too. Using [GKRW18, Corollary 11.12] it suffices to show that $H_{x',d'}(\mathbf{X}, \tilde{\mathbf{S}}_{\Bbbk}) = 0$ for $d' \le 1$ and $\mathrm{rk}(x') \le 2$.

For a given $x' \in \mathsf{H}$ with $\mathrm{rk}(x') \le 2$ we have an exact sequence

$$\cdots \to H_{x',1}(\tilde{\mathbf{S}}_{\Bbbk}) \to H_{x',1}(\mathbf{X}) \to H_{x',1}(\mathbf{X}, \tilde{\mathbf{S}}_{\Bbbk}) \to H_{x',0}(\tilde{\mathbf{S}}_{\Bbbk}) \to H_{x',0}(\mathbf{X}) \to H_{x',0}(\mathbf{X}, \tilde{\mathbf{S}}_{\Bbbk}) \to 0,$$

so it suffices to show that $H_{x',1}(\tilde{\mathbf{S}}_{\Bbbk}) \to H_{x',1}(\mathbf{X})$ is surjective and that $H_{x',0}(\tilde{\mathbf{S}}_{\Bbbk}) \to H_{x',0}(\mathbf{X})$ is an isomorphism.

The isomorphism in degree 0 holds because $H_{*,0}(\tilde{\mathbf{S}}_{\Bbbk}) = \Bbbk[\sigma_0,\sigma_1]/(\sigma_1^2-\sigma_0^2)$ as a ring: the proof is analogous to the computation of the zero-th homology of $\mathbf{A}_{\Bbbk}$ in the proof of Proposition 2.2.2 because the extra cells that we have in $\tilde{\mathbf{S}}_{\Bbbk}$ are either in degree ≥ 2 or in degree 1 but attached trivially, so they have no effect in the homological degree 0 part of the spectral sequence.

The surjectivity in degree 1 holds by construction if $\mathrm{rk}(x') \leq 1$. When $\mathrm{rk}(x') = 2$ it holds by assumptions (i) and (ii) in the statement of the theorem plus the surjectivity in ranks ≤ 1. $\square$

Step 2. Now we will further reduce it to the case $\mathbf{S}_{\mathbb{F}_\ell}$ for ℓ an odd prime or 0.

By proceeding as in Proposition 2.2.2 we find that $\mathbf{S}_{\Bbbk} = \mathbf{S}_{\mathbb{Z}[1/2]} \otimes_{\mathbb{Z}[1/2]} \Bbbk$, so it suffices to consider the case $\Bbbk = \mathbb{Z}[1/2]$ by the universal coefficient theorem.

By reasoning as in Step 2 in the proof of Theorem 2.1.1 we get that the homology groups of $\mathbf{S}_{\mathbb{Z}[1/2]}/\sigma_\varepsilon$ are finitely generated $\mathbb{Z}[1/2]$-modules because $\mathbf{S}_{\mathbb{Z}[1/2]}$ only has finitely many E_2-cells. Thus, another application of the universal coefficient theorem allows us to reduce to the case $\Bbbk = \mathbb{F}_\ell$ with ℓ either an odd prime or 0.

Step 3. Since we are working with $\mathbb{F}_\ell$-coefficients for a fixed ℓ we will drop the ℓ and $\mathbb{F}_\ell$ subscripts from now on. Let us begin by considering the cellular attachment filtration of $\mathbf{S}$, see [GKRW18, Section 6.2.1] for details, where the last grading denotes the filtration.

$$\begin{aligned}\mathbf{fS} := \mathbf{E_2}(&S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1 \oplus S^{(1,\varepsilon),1,0}x \oplus S^{(1,1-\varepsilon),1,0}y \oplus S^{(1,1-\varepsilon),1,0}z)\\ &\cup^{E_2}_{\sigma_1^2-\sigma_0^2} D^{(2,0),1,1}\rho \cup^{E_2}_{Q^1(\sigma_1)-\sigma_\varepsilon\cdot x - tQ^1(\sigma_0)} D^{(2,0),2,1}X \cup^{E_2}_{[\sigma_0,\sigma_1]-\sigma_\varepsilon\cdot y} D^{(2,1),2,1}Y\\ &\cup^{E_2}_{\sigma_{1-\varepsilon}\cdot Q^1(\sigma_0)-\sigma_\varepsilon^2\cdot z} D^{(3,1-\varepsilon),2,1}Z \in \mathrm{Alg}_{E_2}((\mathsf{sMod}^{\mathsf{H}}_{\mathbb{F}_\ell})^{\mathbb{Z}_\leq})\end{aligned}$$

This gives two spectral sequences as in Step 3 of the proof of Theorem 2.1.1:

(i) $F^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathrm{gr}(\mathbf{fS})}) \Rightarrow H_{x,p+q}(\overline{\mathbf{S}})$

(ii) $E^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathrm{gr}(\mathbf{fS})}/\sigma_\varepsilon) \Rightarrow H_{x,p+q}(\overline{\mathbf{S}}/\sigma_\varepsilon)$.

The first spectral sequence is multiplicative, its first page is $\Lambda(L)$ where L is the $\mathbb{F}_\ell$-vector space with basis $Q^I(u)$ such that u a basic Lie word in $\{\sigma_0,\sigma_1,x,y,z,\rho,X,Y,Z\}$ and I is admissible; and its d^1-differential satisfies $d^1(\sigma_0)=0$, $d^1(\sigma_1)=0$, $d^1(x)=0$, $d^1(y)=0$, $d^1(z)=0$, $d^1(\rho)=\sigma_1^2-\sigma_0^2$, $d^1(X)=Q^1(\sigma_1)-\sigma_\varepsilon\cdot x - tQ^1(\sigma_0)$, $d^1(Y)=[\sigma_0,\sigma_1]-\sigma_\varepsilon\cdot y$ and $d^1(Z)=\sigma_{1-\varepsilon}\cdot Q^1(\sigma_0)-\sigma_\varepsilon^2\cdot z$.

The second spectral sequence has the structure of a module over the first one, and its first page is $E^1 = \Lambda(L/\mathbb{F}_\ell\{\sigma_\varepsilon\})$, so (E^1,d^1) has the structure of a CDGA.

Thus, in order to finish the proof it suffices to show that $E^2_{x,p,q} = 0$ for $p+q < (2\,\mathrm{rk}(x) - 3)/3$.

We will show the required vanishing line on E^2 by introducing a filtration on the CDGA (E^1, d^1), similar to the one in Step 3 of the proof of Theorem 2.1.1. We let $\mathcal{F}^\bullet E^1$ be the filtration in which $\sigma_{1-\varepsilon}$, x, y, z, ρ, $Q^1(\sigma_0)$, $Q^1(\sigma_1)$, $[\sigma_0, \sigma_1]$, X, Y, Z are given filtration 0, the remaining elements of a homogeneous basis of $L/\mathbb{F}_\ell\{\sigma_\varepsilon\}$ extending these are given filtration equal to their homological degree, and we extend the filtration to $\Lambda(L/\mathbb{F}_\ell\{\sigma_\varepsilon\})$ multiplicatively.

This gives a spectral sequence converging to E^2 whose first page is the homology of the associated graded of the filtration $\mathcal{F}^\bullet E^1$. We will show the vanishing line on the first page of this spectral sequence.

Applying [GKRW18, Theorems 16.7 and 16.8] gives that $d^1([\sigma_0, \sigma_1]) = 0$, $d^1(Q^1(\sigma_0)) = 0$ and $d^1(Q^1(\sigma_1)) = 0$. This allows to split the associated graded as a tensor product

$$\begin{aligned}(\mathrm{gr}(\mathcal{F}^\bullet E^1), D) = {} & (\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z, Q^1(\sigma_1), X\}), D) \otimes_{\mathbb{F}_\ell} \\ & (\Lambda(\mathbb{F}_\ell\{[\sigma_0,\sigma_1], Y\}), D) \otimes_{\mathbb{F}_\ell} (\Lambda(\mathbb{F}_\ell\{x,y,z\}), 0) \otimes_{\mathbb{F}_\ell} \\ & (\Lambda(L/\mathbb{F}_\ell\{\sigma_0, \sigma_1, x, y, z, \rho, Q^1(\sigma_0), Q^1(\sigma_1), [\sigma_0,\sigma_1], X, Y, Z\}), 0)\end{aligned}$$

where D is the induced differential and satisfies $D(\sigma_{1-\varepsilon}) = 0$, $D(\rho) = (-1)^\varepsilon \sigma^2_{1-\varepsilon}$, $D(Q^1(\sigma_0)) = 0$, $D(Z) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)$, $D(Q^1(\sigma_1)) = 0$, $D(X) = Q^1(\sigma_1) - tQ^1(\sigma_0)$, $D([\sigma_0,\sigma_1]) = 0$, $D(Y) = [\sigma_0, \sigma_1]$. By the Künneth theorem it suffices to compute the homology of each of the factors separately.

By direct computation we see that

- Elements in $\Lambda(L/\mathbb{F}_\ell\{\sigma_0, \sigma_1, x, y, z, \rho, Q^1(\sigma_0), Q^1(\sigma_1), [\sigma_0,\sigma_1], X, Y, Z\})$ have slope $\geq 2/3$.
- Elements in $\Lambda(\mathbb{F}_\ell\{x,y,z\})$ have slope $\geq 1 \geq 2/3$.
- Since $\ell \neq 2$ we have $[\sigma_0,\sigma_1]^2 = 0$ so the homology of $(\Lambda(\mathbb{F}_\ell\{[\sigma_0,\sigma_1], Y\}), D)$ is $\mathbb{F}_\ell[Y^\ell] + [\sigma_0,\sigma_1] \cdot \mathbb{F}_\ell\{Y^j : \ell | j+1\}$. Since $[\sigma_0,\sigma_1]$ has bidegree $(\mathrm{rk} = 2, d = 1)$, Y has bidegree $(\mathrm{rk} = 2, d = 2)$ and $\ell \geq 3$ then all these elements have slope $\geq 5/6 \geq 2/3$.

Thus, it suffices to check that $H_*(\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z, Q^1(\sigma_1), X\}), D)$ vanishes for $3d < 2\,\mathrm{rk} - 3$, where d denotes the homological degree. The remaining of the proof will be about studying this CDGA. We will separate this as an extra step because it will require some additional filtrations and work.

Step 4. We firstly claim that it suffices to consider $t = 0$: $\sigma_{1-\varepsilon}$, ρ, $Q^1(\sigma_0)$, $Q^1(\sigma_1)$, X, Z are now just the generators of a certain CDGA. Since both $Q^1(\sigma_0)$ and $Q^1(\sigma_1)$ lie in

$\ker(D)$ and have the same homological degree and rank, the change of variables $Q^1(\sigma_1) \mapsto Q^1(\sigma_1) - tQ^1(\sigma_0)$ reparameterises $t \mapsto 0$.

Secondly, once we are in the case $t = 0$, we can further split the CDGA as a tensor product

$$(\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z\}), D) \otimes_{\mathbb{F}_\ell} (\Lambda(X, Q^1(\sigma_1)), D)$$

and the homology of the second factor is $\mathbb{F}_\ell[X^\ell] + Q^1(\sigma_1) \cdot \mathbb{F}_\ell\{X^j : \ell | j+1\}$ (since $\ell \neq 2$), so all its elements have slope $\geq 5/6 \geq 2/3$. Thus, we have reduced it to proving that $H_*(\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z\}), D)$ vanishes for $3d < 2\mathrm{rk} - 3$. For this, we will introduce an additional filtration by giving $Q^1(\sigma_0)$ filtration 0, $\sigma_{1-\varepsilon}$ filtration 1 and ρ, Z filtration 2, and then extending the filtration multiplicatively to the whole CDGA.

The differential D preserves this filtration and the associated graded splits as a tensor product

$$(\Lambda(\sigma_{1-\varepsilon}, \rho), D(\sigma_{1-\varepsilon}) = 0, D(\rho) = (-1)^\varepsilon \sigma_{1-\varepsilon}^2) \otimes_{\mathbb{F}_\ell} (\Lambda(Q^1(\sigma_0), Z), 0)$$

so, using that $\ell \neq 2$ to compute the homology of the first factor, we get a multiplicative spectral sequence of the form

$$\mathcal{E}^1 = \mathbb{F}_\ell[\sigma_{1-\varepsilon}]/(\sigma_{1-\varepsilon}^2) \otimes_{\mathbb{F}_\ell} \Lambda(Q^1(\sigma_0), Z) \Rightarrow H_*(\Lambda(\mathbb{F}_\ell\{\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z\}), D)$$

whose first differential satisfies $D^1(Z) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)$, $D^1(\sigma_{1-\varepsilon}) = 0$ and $D^1(Q^1(\sigma_0)) = 0$.

To finish the proof we will establish the required vanishing range on $\mathcal{E}^2$. To do so, we write $\mathcal{E}^1 = \mathbb{F}_\ell\{1, \sigma_{1-\varepsilon}, Q^1(\sigma_0), \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)\} \otimes \mathbb{F}_\ell[Z]$ as a $\mathbb{F}_\ell$-vector space, and then compute $\ker(D^1), \mathrm{im}(D^1)$ explicitly as $\mathbb{F}_\ell$-vector spaces, where $()$ denotes the ideal generated by an element:

$$\ker(D^1) = (\sigma_{1-\varepsilon}) + (Q^1(\sigma_0)) + \mathbb{F}_\ell[Z^\ell]$$

and

$$\mathrm{im}(D^1) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0) \cdot \mathbb{F}_\ell\{Z^i : \ell \nmid i+1\}.$$

Thus, we get that $\mathcal{E}^2 = \ker(D^1)/\mathrm{im}(D^1)$ is, as a $\mathbb{F}_\ell$-vector space, given by

$$\mathcal{E}^2 = \mathbb{F}_\ell[Z^l] + \sigma_{1-\varepsilon} \cdot \mathbb{F}_\ell[Z] + Q^1(\sigma_0) \cdot \mathbb{F}_\ell[Z] + \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0) \cdot \mathbb{F}_\ell\{Z^i : \ell \mid i+1\}.$$

Using the bidegrees of the generators we find that the first summand vanishes for $d < 2\mathrm{rk}/3$, the second vanishes for $d < 2(\mathrm{rk}-1)/3$, the third one for $d < (2\mathrm{rk}-1)/3$, and the forth one for $d < (2\mathrm{rk}-3)/3$, as required. $\square$

Let us mention that one could prove a weaker version of Theorem 2.1.2 (same stability slope but worse constant term) in a simpler way by taking $\mathbf{S}_{\Bbbk} = \mathbf{A}_{\Bbbk}$ directly. Step 1 of the above proof can be easily adapted by noting that all the cells in the algebra $\mathbf{S}_{\Bbbk}$ are either in $\mathbf{A}_{\Bbbk}$ or have slope $\geq 2/3$. Thus, $H^{E_2}_{x,d}(\mathbf{X}, \mathbf{A}_{\Bbbk}) = 0$ for $d < 2\,\mathrm{rk}(x)/3$. Step 2 follows identically so we reduce the theorem to studying the homological stability of $\mathbf{A}_{\Bbbk}$, for $\Bbbk = \mathbb{F}_\ell$ with ℓ an odd prime or 0. The final part (Steps 3 and 4) can be done by considering the cell attachment spectral sequence for $\mathbf{A}_{\Bbbk}$ and using that any element in bidegree (x,d) with d odd squares to zero by skew-symmetry.

However, the proof presented above gives a better constant term and generalises better to studying $\mathbb{F}_2$ coefficients, which will be useful in the following sections to prove Theorem 2.1.3.

2.2.3 Construction of the class θ

In this section we will explain how the class $\theta \in H_{(4,0),2}(\mathbf{X})$ of Theorem 2.1.3 is defined.

The first step will be to define $\theta \in H_{(4,0),2}(\mathbf{A}_{\mathbb{F}_2})$. Since we will only work with $\mathbb{F}_2$-coefficients for now, we will drop all the $\mathbb{F}_2$-indices. Consider the spectral sequence (i) of the proof of Theorem 2.1.1:

$$F^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathbf{E_2}(S^{(1,0),0,0}\sigma_0 \oplus S^{(1,1),0,0}\sigma_1 \oplus S^{(2,0),1,1}\rho)}) \Rightarrow H_{x,p+q}(\overline{\mathbf{A}}).$$

As we said, this is a multiplicative spectral sequence whose first page is given by $\mathbb{F}_2[L]$, where L is the $\mathbb{F}_2$-vector space with basis $Q^I(y)$ such that y is a basic Lie word in $\{\sigma_0, \sigma_1, \rho\}$ and I is admissible. (Note that this time we get a free commutative algebra instead of graded-commutative as we work with $\mathbb{F}_2$-coefficients.) Thus we have $F^1_{(4,0),2,0} = \mathbb{F}_2\{\rho^2\}$.

Claim. ρ^2 survives to F^∞.

Proof. Since $F^1_{(4,0),2+r,1-r} = 0$ for $r \geq 1$ then ρ^2 cannot be a boundary of any d^r-differential. Moreover, $d^r : F^r_{(4,0),2,0} \to F^r_{(4,0),2-r,r-1}$ vanishes for $r > 2$ since $\mathbf{fA}$ vanishes on negative filtration. Thus, it suffices to show that both $d^1(\rho^2)$ and $d^2(\rho^2)$ vanish. By Leibniz rule we have $d^1(\rho^2) = 0$, so we only need to show that $d^2(\rho^2) = 0$.

Since $\rho^2 = Q^1(\rho)$ and $d^1(\rho) = \sigma_1^2 - \sigma_0^2$ then [GKRW18, Theorem 16.8 (i)] gives that $d^2(\rho^2)$ is represented by $Q^1(\sigma_1^2 - \sigma_0^2)$. (As a technical note let us mention that the result we just quoted is stated for E_∞-algebras, but the same result holds for E_2-algebras as explained in [GKRW18, Page 184].) Finally, $Q^1(\sigma_1^2 - \sigma_0^2)$ vanishes by the properties of Q^1 shown in [GKRW18, Section 16.2.2]. $\square$

Definition 2.2.3. *The class $\theta \in H_{(4,0),2}(\mathbf{A}_{\mathbb{F}_2})$ is defined to be any lift of the class $[\rho^2] \in F^{\infty}_{(4,0),2,0}$.*

Given $\mathbf{X}$ satisfying the assumptions of Theorem 2.1.3 we define $\theta \in H_{(4,0),2}(\mathbf{X})$ as follows: we pick an E_2-map $\mathbf{A} \xrightarrow{c} \mathbf{X}$ as in Step 1 in the proof of Theorem 2.1.1, and set $\theta := c_*(\theta) \in H_{(4,0),2}(\mathbf{X})$.

Remark 2.2.4. *There is not a unique choice of class θ, however the statement of Theorem 2.1.3 will be true for any choice of class θ with the property of Definition 2.2.3. In fact, θ is well-defined up to adding any linear combination of $Q^1(\sigma_0)^2$, $Q^1(\sigma_0)\cdot Q^1(\sigma_1)$, $Q^1(\sigma_1)^2$ and $[\sigma_0,\sigma_1]^2$, or multiples of $\sigma_0^2 = \sigma_1^2$ to it. (We will not use this fact but we added an explanation below.)*

In order to show the above remark one can use the same spectral sequence and check that

1. $F^1_{(4,0),0,2}$ is generated by $Q^1(\sigma_0)^2$, $Q^1(\sigma_0)\cdot Q^1(\sigma_1)$, $Q^1(\sigma_1)^2$ and $[\sigma_0,\sigma_1]^2$, and all these terms are permanent cycles and no boundaries.

2. $d^1 : F^1_{(4,0),1,1} \to F^1_{(4,0),0,1}$ is injective, and hence $F^2_{(4,0),1,1} = 0$.

Thus, $\theta \in H_{(4,0),2}(\mathbf{A}_{\mathbb{F}_2})$ is well-defined up to a linear combination of $Q^1(\sigma_0)^2$, $Q^1(\sigma_0)\cdot Q^1(\sigma_1)$, $Q^1(\sigma_1)^2$ and $[\sigma_0,\sigma_1]^2$.

The definition of the map c is not unique as we need to choose a nullhomotopy of $\sigma_1^2 - \sigma_0^2$ in $\mathbf{X}$, and the set of such choices is a $H_{(2,0),1}(\mathbf{X})$-torsor. In particular, by assumptions (i) and (ii) about $\mathbf{X}$ any new choice of ρ differs by a class in $\mathrm{im}(\sigma_\varepsilon \cdot -)$ or by a multiple of $Q^1(\sigma_0)$, giving the result.

Finally, let us mention that by analysing the cell attachment spectral sequence for $\mathbf{A}_{\mathbb{F}_2}$ one can show that the classes $\theta^k \in H_{(4k,0),2k}(\overline{\mathbf{A}_{\mathbb{F}_2}}/\sigma_\varepsilon)$ do not vanish for any $k \geq 0$. This explains why one cannot simply adapt the proof of Theorem 2.1.2 to work with $\mathbb{F}_2$ coefficients.

2.2.4 Proof of Theorem 2.1.3

Before proving the theorem let us briefly recall the construction of $\overline{\mathbf{X}}/(\sigma_\varepsilon,\theta)$. We start by viewing θ as a homotopy class of maps $S^{(4,0),2} \to \mathbf{X}$. Then, using the adapters construction, see [GKRW18, Section 12.3] we get an $\overline{\mathbf{X}}$-module map $S^{(4,0),2} \otimes \overline{\mathbf{X}}/\sigma_\varepsilon \xrightarrow{\theta\cdot -} \overline{\mathbf{X}}/\sigma_\varepsilon$ and we define $\overline{\mathbf{X}}/(\sigma_\varepsilon,\theta)$ to be its cofibre (in the category of left $\overline{\mathbf{X}}$-modules).

Proof. The proof will be very similar to that of Theorem 2.1.2, so we will focus on the parts that are different and skip details.

Step 1. We will construct a certain cellular E_2-algebra $\mathbf{S}$ and show that it suffices to prove that $H_{x,d}(\overline{\mathbf{S}}/(\sigma_\varepsilon,\theta)) = 0$ for $3d < 2\,\mathrm{rk}(x) - 4$.

The assumptions of the statement imply that $[\sigma_0, \sigma_1] = \sigma_\varepsilon \cdot y$ for some $y \in H_{(1,1-\varepsilon),1}(\mathbf{X})$, that $Q^1(\sigma_1) = \sigma_\varepsilon \cdot x + tQ^1(\sigma_0)$ for some $x \in H_{(1,\varepsilon),1}(\mathbf{X})$ and some $t \in \mathbb{F}_2$, and that $\sigma_{1-\varepsilon} \cdot Q^1(\sigma_0) = \sigma_\varepsilon^2 \cdot z \in H_{(3,1-\varepsilon),1}(\mathbf{X})$ for some $z \in H_{(1,1-\varepsilon),1}(\mathbf{X})$.

Moreover, we claim that there is $u \in H_{(4,0),3}(\mathbf{X})$ such that $Q^1(\sigma_0)^3 = \sigma_\varepsilon^2 \cdot u$.

Indeed, condition (iv) says that $\sigma_0 \cdot Q^1(\sigma_0) = \sigma_\varepsilon^2 \cdot \tau$ for some $\tau \in H_{(1,0),1}(\mathbf{X})$, and then we can apply $Q^2(-)$ to both sides and use the formulae in [GKRW18, Section 16.2.2] to find $Q^1(\sigma_0)^3 + \sigma_0^2 \cdot Q^2(Q^1(\sigma_0)) + \sigma_0[\sigma_0, Q^1(\sigma_0)]Q^1(\sigma_0) = \sigma_\varepsilon^2 \cdot [\sigma_\varepsilon, \sigma_\varepsilon] \cdot Q^1(\tau) + \sigma_\varepsilon^4 \cdot Q^2(\tau) + \sigma_\varepsilon^2 \cdot [\sigma_\varepsilon^2, \tau] \cdot \tau$, hence the result as $\sigma_\varepsilon^2 = \sigma_0^2$ and as $[\sigma_0, Q^1(\sigma_0)] = [\sigma_0, [\sigma_0, \sigma_0]] = 0$ (by [GKRW18, Section 16.2.2] again).

Let

$$\begin{aligned}\mathbf{S} := \mathbf{E_2}(&S^{(1,0),0}\sigma_0 \oplus S^{(1,1),0}\sigma_1 \oplus S^{(1,\varepsilon),1}x \oplus S^{(1,1-\varepsilon),1}y \oplus S^{(1,1-\varepsilon),1}z \oplus S^{(4,0),3}u)\\ &\cup^{E_2}_{\sigma_1^2-\sigma_0^2} D^{(2,0),1}\rho \cup^{E_2}_{Q^1(\sigma_1)-\sigma_\varepsilon\cdot x - tQ^1(\sigma_0)} D^{(2,0),2}X \cup^{E_2}_{[\sigma_0,\sigma_1]-\sigma_\varepsilon\cdot y} D^{(2,1),2}Y\\ &\cup^{E_2}_{\sigma_{1-\varepsilon}\cdot Q^1(\sigma_0)-\sigma_\varepsilon^2\cdot z} D^{(3,1-\varepsilon),2}Z \cup^{E_2}_{Q^1(\sigma_0)^3-\sigma_\varepsilon^2\cdot u} D^{(6,0),4}U \in \mathrm{Alg}_{E_2}(\mathsf{sMod}^{\mathsf{H}}_{\mathbb{F}_2})\end{aligned}$$

By proceeding as in Step 1 of the proof of Theorem 2.1.1 there is an E_2-algebra map $f : \mathbf{S} \to \mathbf{X}$ sending each of $\sigma_0, \sigma_1, x, y, z, u$ to the corresponding homology classes in $\mathbf{X}$ with the same name. Moreover, we can assume that f extends any given map $\mathbf{A} \to \mathbf{X}$ and hence that it sends $\theta \mapsto \theta$.

Claim. $H^{E_2}_{x,d}(\mathbf{X}, \mathbf{S}) = 0$ for $d < 2\mathrm{rk}(x)/3$.

The proof is identical to the corresponding claim in Step 1 in the proof of Theorem 2.1.2. The only difference now is that $\mathbf{S}$ has a cell U below the "critical line" $d = \mathrm{rk} - 1$. However, it causes no trouble since it has bidegree $(\mathrm{rk} = 6, d = 4)$, so it lies on the region $3d \geq 2\mathrm{rk}$.

Assuming the claim, we can apply [GKRW18, Corollary 15.10] with $\rho(x) = 2\mathrm{rk}(x)/3$, $\mu(x) = (2\mathrm{rk}(x) - 4)/3$ and $\mathbf{M} = \overline{\mathbf{S}}/(\sigma_\varepsilon, \theta)$ to obtain the required reduction.

Step 2. We proceed as in Step 3 in the proof of Theorem 2.1.2 to get a cell attachment filtration $\mathbf{fS} \in \mathrm{Alg}_{E_2}((\mathsf{sMod}^{\mathsf{H}}_{\mathbb{F}_2})^{\mathbb{Z}_\leq})$. The key now is to observe that $\theta \in H_{(4,0),2}(\mathbf{S})$ lifts to a filtered map $\theta : S^{(4,0),2,2} \to \mathbf{fS}$ which maps to $\rho^2 \in H_{*,*,*}(\mathrm{gr}(\mathbf{fS}))$.

Indeed, $\theta \in H_{(4,0),2}(\mathbf{A}) = H_{(4,0),2}(\mathrm{colim}(\mathbf{fA})) = \mathrm{colim}_f(H_{(4,0),2,f}(\mathbf{fA}))$, so it can be represented by a class $\theta \in H_{(4,0),2,f}(\mathbf{fA})$ for some f large. In fact, $f = 2$ is the smallest possible such value since the obstruction to lift the class $\theta \in H_{(4,0),2,f}(\mathbf{fA})$ to a class in $H_{(4,0),2,f-1}(\mathbf{fA})$ is precisely the image of θ in $H_{(4,0),2,f}(\mathrm{gr}(\mathbf{fA}))$ which is ρ^2 by definition, giving the result. Finally observe that there is a canonical map of filtered E_2-algebras $\mathbf{fA} \to \mathbf{fS}$. Thus, we get spectral sequences

(i) $F^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathrm{gr}(\mathbf{fS})}) \Rightarrow H_{x,p+q}(\overline{\mathbf{S}})$

(ii) $E^1_{x,p,q} = H_{x,p+q,p}(\overline{\mathrm{gr}(\mathbf{fS})}/(\sigma_\varepsilon,\rho^2)) \Rightarrow H_{x,p+q}(\overline{\mathbf{S}}/(\sigma_\varepsilon,\theta))$.

The first spectral sequence is multiplicative, its first page is $\mathbb{F}_2[L]$ where L is the $\mathbb{F}_2$-vector space with basis $Q^I(\alpha)$ such that α a basic Lie word in $\{\sigma_0,\sigma_1,x,y,z,u,\rho,X,Y,Z,U\}$ and I is admissible; and its d^1-differential satisfies $d^1(\sigma_0)=0$, $d^1(\sigma_1)=0$, $d^1(x)=0$, $d^1(y)=0$, $d^1(z)=0$, $d^1(u)=0$, $d^1(\rho)=\sigma_1^2-\sigma_0^2$, $d^1(X)=Q^1(\sigma_1)-\sigma_\varepsilon\cdot x - tQ^1(\sigma_0)$, $d^1(Y)=[\sigma_0,\sigma_1]-\sigma_\varepsilon\cdot y$, $d^1(Z)=\sigma_{1-\varepsilon}\cdot Q^1(\sigma_0)-\sigma_\varepsilon^2\cdot z$ and $d^1(U)=Q^1(\sigma_0)^3-\sigma_\varepsilon^2\cdot u$.

The second spectral sequence has the structure of a module over the first one, and its first page is given by $E^1=\mathbb{F}_2[L/\mathbb{F}_2\{\sigma_\varepsilon\}]/(\rho^2)$ because $\sigma_\varepsilon\cdot -$ is injective on $\mathbb{F}_2[L]$ and $\rho^2\cdot -$ is injective on $\mathbb{F}_2[L]/(\sigma_\varepsilon)=\mathbb{F}_2[L/\mathbb{F}_2\{\sigma_\varepsilon\}]$. Thus (E^1,d^1) has the structure of a CDGA.

Thus, in order to finish the proof it suffices to show that $E^2_{x,p,q}=0$ for $p+q<(2\,\mathrm{rk}(x)-4)/3$.

Step 3. Now we will introduce additional filtrations to simplify the CDGA until we get the required result. The first filtration is similar to the one of Step 3 in the proof of Theorem 2.1.2: we give $\sigma_{1-\varepsilon}$, x, y, z, u, ρ, $Q^1(\sigma_0)$, $Q^1(\sigma_1)$, $[\sigma_0,\sigma_1]$, X, Y, Z, U filtration 0, we give the remaining elements of a homogeneous basis of $L/\mathbb{F}_2\{\sigma_\varepsilon\}$ extending these filtration equal to their homological degree, and we extend the filtration to $\mathbb{F}_2[L/\mathbb{F}_2\{\sigma_\varepsilon\}]/(\rho^2)$ multiplicatively (which we can as ρ is in filtration 0).

This allows us to split the associated graded as a tensor product and all the factors are concentrated in the region $3d\geq 2\,\mathrm{rk}$ except possibly the one given by

$$(\mathbb{F}_2[\sigma_{1-\varepsilon},\rho,Q^1(\sigma_0),Q^1(\sigma_1),X,Z,U]/(\rho^2),D)$$

where the non-zero part of D is characterized by $D(X)=Q^1(\sigma_1)-tQ^1(\sigma_0)$, $D(Z)=\sigma_{1-\varepsilon}\cdot Q^1(\sigma_0)$ and $D(U)=Q^1(\sigma_0)^3$. (This computation is easier than the one of the proof of Theorem 2.1.2 since $\ell=2$ simplifies the homology of the other factors.)

Then, we can proceed as in Step 4 in the proof of Theorem 2.1.2 to go to the case $t=0$ and hence split the CDGA further to simplify it to

$$(\mathbb{F}_2[\sigma_{1-\varepsilon},\rho,Q^1(\sigma_0),Z,U]/(\rho^2),D).$$

Next we introduce a new filtration by giving $\sigma_{1-\varepsilon},\rho,Q^1(\sigma_0)$ filtration 0, and Z,U filtration 1 and then extending multiplicatively. The associated graded of this splits as a tensor product

$$(\mathbb{F}_2[\sigma_{1-\varepsilon},\rho]/(\rho^2),D(\rho)=\sigma_{1-\varepsilon}^2)\otimes_{\mathbb{F}_2}(\mathbb{F}_2[Q^1(\sigma_0),Z,U],0)$$

and the homology of the first factor is precisely $\mathbb{F}_2[\sigma_{1-\varepsilon}]/(\sigma_{1-\varepsilon}^2)$, yielding a spectral sequence of the form

$$\mathcal{E}^1 = \mathbb{F}_2[\sigma_{1-\varepsilon}]/(\sigma_{1-\varepsilon}^2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[Q^1(\sigma_0), Z, U] \Rightarrow H_*(\mathbb{F}_2[\sigma_{1-\varepsilon}, \rho, Q^1(\sigma_0), Z, U]/(\rho^2), D)$$

whose first differential D^1 satisfies $D^1(Z) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)$ and $D^1(U) = Q^1(\sigma_0)^3$. We will establish the required vanishing line on $\mathcal{E}^2$ of this spectral sequence. For that we will introduce yet another filtration by letting $\sigma_{1-\varepsilon}, Q^1(\sigma_0), U$ have filtration 0 and Z have filtration 1.

The associated graded is given by

$$(\mathbb{F}_2[\sigma_{1-\varepsilon}, Z]/(\sigma_{1-\varepsilon}^2), 0) \otimes_{\mathbb{F}_2} (\mathbb{F}_2[Q^1(\sigma_0), U], \delta(U) = Q^1(\sigma_0)^3)$$

where δ is the new differential. Thus, its homology is given by

$$\mathbb{F}_2[\sigma_{1-\varepsilon}, Q^1(\sigma_0), Z, U^2]/(\sigma_{1-\varepsilon}^2, Q^1(\sigma_0)^3)$$

and the δ^1-differential satisfies $\delta^1(Z) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)$. Since U has slope $2/3$ itself, in order to prove the required vanishing line we can just focus on the remaining part

$$(\mathbb{F}_2[\sigma_{1-\varepsilon}, Q^1(\sigma_0), Z]/(\sigma_{1-\varepsilon}^2, Q^1(\sigma_0)^3), \delta^1(Z) = \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)).$$

For that we explicitly compute $\ker(\delta^1)$, $\operatorname{im}(\delta^1)$ as $\mathbb{F}_2$-vector spaces (similar to the last CDGA of the proof of Theorem 2.1.2).

$$\ker(\delta^1) = (\sigma_{1-\varepsilon}) + (Q^1(\sigma_0)^2) + \mathbb{F}_2\{1, Q^1(\sigma_0)\} \cdot \mathbb{F}_2[Z^2]$$

and

$$\operatorname{im}(\delta^1) = \mathbb{F}_2\{\sigma_{1-\varepsilon} \cdot Q^1(\sigma_0), \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)^2\} \cdot \mathbb{F}_2[Z^2].$$

Thus we get

$$\begin{aligned} \ker(\delta^1)/\operatorname{im}(\delta^1) = \sigma_{1-\varepsilon} \cdot \mathbb{F}_2[Z] + \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0) \cdot \mathbb{F}_2\{Z^i : 2 \nmid i\} + \\ \sigma_{1-\varepsilon} \cdot Q^1(\sigma_0)^2 \cdot \mathbb{F}_2\{Z^i : 2 \nmid i\} + \mathbb{F}_2\{1, Q^1(\sigma_0), Q(\sigma_0)^2\} \cdot \mathbb{F}_2[Z]. \end{aligned}$$

Using the bidegrees of the generators it is immediate that all vanish for $3d < 2\,\mathrm{rk} - 4$, hence the result. □

Finally we will finish the section by giving two corollaries of Theorem 2.1.3.

Corollary 2.2.5. *Let* **X** *be as in Theorem 2.1.3 then* *(i) If $\theta^3 \in H_{(12,0),6}(\mathbf{X})$ does not destabilize by σ_ε then $H_{(4k,0),2k}(\overline{\mathbf{X}}/\sigma_\varepsilon) \neq 0$ for all $k \geq 1$, and in particular the optimal slope for the stability is $1/2$.*

(ii) If $\theta^3 \in H_{(12,0),6}(\mathbf{X})$ destabilizes by either σ_0 or σ_1 then $H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon) = 0$ for $3d \leq 2\operatorname{rk}(x) - 6$, so $\mathbf{X}$ satisfies homological stability of slope at least $2/3$ with respect to σ_ε.

Proof. By definition (using the adapters construction as in Section 2.2.4) there is a cofibration of left $\overline{\mathbf{X}}$-modules

$$S^{(4,0),2} \otimes \overline{\mathbf{X}}/\sigma_\varepsilon \xrightarrow{\theta\cdot -} \overline{\mathbf{X}}/\sigma_\varepsilon \to \overline{\mathbf{X}}/(\sigma_\varepsilon, \theta),$$

and hence a corresponding long exact sequence in homology groups which implies that $\theta \cdot - : H_{x-(4,0),d-2}(\overline{\mathbf{X}}/\sigma_\varepsilon) \to H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon)$ is surjective for $3d \leq 2\operatorname{rk}(x) - 5$ and an isomorphism for $3d \leq 2\operatorname{rk}(x) - 8$.

Similarly, the cofibration of left $\overline{\mathbf{X}}$-modules $S^{(1,\varepsilon),0} \otimes \overline{\mathbf{X}} \xrightarrow{\sigma_\varepsilon\cdot -} \overline{\mathbf{X}} \to \overline{\mathbf{X}}/\sigma_\varepsilon$ gives another long exact sequence in homology groups.

Proof of (i). If θ^3 does not destabilise by σ_ε then the second long exact sequence gives $\theta^3 \neq 0 \in H_{(12,0),6}(\overline{\mathbf{X}}/\sigma_\varepsilon)$. But

$$\theta \cdot - : H_{4(k-1),2(k-1)}(\overline{\mathbf{X}}/\sigma_\varepsilon) \to H_{4k,2k}(\overline{\mathbf{X}}/\sigma_\varepsilon)$$

is an isomorphism for $k \geq 4$, so $\theta^k \neq 0 \in H_{4k,2k}(\overline{\mathbf{X}}/\sigma_\varepsilon)$ for $k \geq 4$ (hence for $k \geq 1$).

Proof of (ii). If $3d \leq 2\operatorname{rk}(x) - 6$ then $3(d+2k) \leq 2(\operatorname{rk}(x) + 4k) - 8$ for any $k \geq 1$ and hence the map $\theta^k \cdot - : H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon) \to H_{x+(4k,0),d+2k}(\overline{\mathbf{X}}/\sigma_\varepsilon)$ is an isomorphism for any $k \geq 1$. Thus, it suffices to find some k for which $\theta^k \cdot -$ is the 0 map. We will show that in fact $k = 6$ works.

Since θ^3 destabilises by either σ_0 or σ_1 then (using that $\sigma_0^2 = \sigma_1^2$) $\theta^6 = \alpha \cdot \sigma_\varepsilon^2$ for some $\alpha \in H_{(22,0),12}(\mathbf{X})$. Thus, $\theta^6 \cdot - = \alpha \cdot (\sigma_\varepsilon^2 \cdot -)$ as (homotopy classes of) maps $S^{(24,0),12} \otimes \overline{\mathbf{X}}/\sigma_\varepsilon \to \overline{\mathbf{X}}/\sigma_\varepsilon$. Thus, it suffices to show that $\sigma_\varepsilon^2 \cdot - : S^{(2,0),0} \otimes \overline{\mathbf{X}}/\sigma_\varepsilon \to \overline{\mathbf{X}}/\sigma_\varepsilon$ is nullhomotopic. This is a special case of the following general fact, see [Ram, Proposition 2.3], that if $\mathbf{X}$ is an object in a stable ∞-category ($\mathsf{sMod}^{\mathsf{H}}_{\mathbb{F}_2}$ in our case) and $f : \mathbf{X} \to \mathbf{X}$ is a self-map then (any) induced morphism $\overline{f} : \mathbf{X}/f \to \mathbf{X}/f$ on the cofibre satisfies that $\overline{f}^2$ is nullhomotopic. □

Remark 2.2.6. *By Remark 2.2.4 we know that θ itself is not well-defined. However, the map $\theta \cdot - : H_{x-(4,0),d-2}(\overline{\mathbf{X}}/\sigma_\varepsilon) \to H_{x,d}(\overline{\mathbf{X}}/\sigma_\varepsilon)$ is well-defined up to adding $Q^1(\sigma_0)^2 \cdot -$ and $Q^1(\sigma_0) \cdot Q^1(\sigma_1) \cdot -$, and the map $\theta^2 \cdot -$ is well-defined.*

This can shown by using Remark 2.2.4 and the assumptions on $\mathbf{X}$ about the classes $Q^1(\sigma_0)$, $Q^1(\sigma_1)$, $[\sigma_0, \sigma_1]$ plus the fact that $Q^1(\sigma_0)^3$ destabilises twice as explained in Step 1 of the proof of Theorem 2.1.3, and using [Ram, Proposition 2.3] again.

Finally, we also have the following corollary of Theorem 2.1.3.

Corollary 2.2.7. *Let* $\mathbf{X} \in \mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{F}_2}^{\mathsf{H}})$ *satisfy the assumptions of Theorem 2.1.3 with* $\varepsilon = 0$. *Then* $H_{(n,0),d}(\overline{\mathbf{X}}/\sigma_0) = 0$ *for* $2d \leq n-1$.

Proof. By Theorem 2.1.1 we know that $H_{(n,0),d}(\overline{\mathbf{X}}/\sigma_0) = 0$ for $2d \leq n-2$, so it suffices to check the cases $n = 2d+1$ for each $d \geq 0$. As we said in the proof of Corollary 2.2.5, a consequence of Theorem 2.1.3 is that $\theta \cdot - : H_{x-(4,0),d-2}(\overline{\mathbf{X}}/\sigma_0) \to H_{x,d}(\overline{\mathbf{X}}/\sigma_0)$ is surjective for $3d \leq 2\,\mathrm{rk}(x) - 5$. Thus, $\theta \cdot - : H_{(2d+1,0),d}(\overline{\mathbf{X}}/\sigma_0) \to H_{(2(d+2)+1,0),d+2}(\overline{\mathbf{X}}/\sigma_0)$ is surjective for $d \geq 1$. Hence, it suffices to check $H_{(2d+1,0),d}(\overline{\mathbf{X}}/\sigma_0) = 0$ for $d = 0,1,2$.

The case $d = 0$ is immediate by our assumption about $H_{*,0}(\overline{\mathbf{X}})$.

For $d = 1,2$ we need a different argument: let $\mathbf{S}$ be the cellular E_2-algebra appearing in Step 1 in the proof of Theorem 2.1.3. Since the class σ_0 factors through $\mathbf{S}$ then base-change gives $\overline{\mathbf{X}}/\sigma_0 \simeq \overline{\mathbf{X}} \otimes^{\mathbb{L}}_{\overline{\mathbf{S}}} \overline{\mathbf{S}}/\sigma_0 = B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \overline{\mathbf{S}}/\sigma_0)$.

If $\mathbf{M}$ is an $\overline{\mathbf{S}}$-module then we can descendently filter it by its $\mathbb{N}$-grading (i.e. by rank of its H-grading), giving an associated graded $\mathrm{gr}(\mathbf{M})$ which is isomorphic to $\mathbf{M}$ itself in $\mathsf{sMod}_{\mathbb{F}_2}^{\mathsf{H}}$, but which has the trivial $\overline{\mathbf{S}}$-module structure induced by the augmentation $\overline{\mathbf{S}} \to \mathbb{1}$, c.f. [GKRW18, Remark 19.3]. This induces a filtration on $B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \mathbf{M})$ whose associated graded is $B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \mathbb{1}) \otimes \mathrm{gr}(\mathbf{M})$. Therefore, letting $\mathbf{M} = \overline{\mathbf{S}}/\sigma_0$ we get a spectral sequence

$$H_{*,*}(B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \mathbb{1})) \otimes H_{*,*}(\overline{\mathbf{S}}/\sigma_0) \Rightarrow H_{*,*}(\overline{\mathbf{X}}/\sigma_0)$$

where we are suppressing the internal grading.

By the claim in Step 1 in the proof of Theorem 2.1.3 we have $H^{E_2}_{x,d}(\mathbf{X}, \mathbf{S}) = 0$ for $d < 2\,\mathrm{rk}(x)/3$. Moreover, by construction all the cells of $\mathbf{S}$ lie on the region $d \geq 2\,\mathrm{rk}(x)/3 - 1$. Thus, we can apply [GKRW18, Theorem 15.9] with $\rho(x) = \sigma(x) = 2\,\mathrm{rk}(x)/3$ to get that $H^{\overline{\mathbf{S}}}_{x,d}(\overline{\mathbf{X}}, \overline{\mathbf{S}}) = 0$ for $d < 2\,\mathrm{rk}(x)/3$. Therefore, $H_{x,d}(B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \mathbb{1})) = H^{\overline{\mathbf{S}}}_{x,d}(\overline{\mathbf{X}}) = 0$ for $d < 2\,\mathrm{rk}(x)/3$ too. Moreover, $H_{0,0}(B(\overline{\mathbf{X}}, \overline{\mathbf{S}}, \mathbb{1})) = H^{\overline{\mathbf{S}}}_{0,0}(\overline{\mathbf{X}}) = \mathbb{1}$ because the Hurewicz theorem, [GKRW18, Corollary 11.12], implies that $H^{\overline{\mathbf{S}}}_{0,0}(\overline{\mathbf{X}}, \overline{\mathbf{S}}) = 0$.

We can apply Theorem 2.1.1 to $\mathbf{S}$ to get $H_{x,d}(\overline{\mathbf{S}}/\sigma_0) = 0$ for $2d \leq \mathrm{rk}(x) - 2$. To be precise $\mathbf{S}$ does not satisfy one of the assumptions of Theorem 2.1.1 because it has a cell U in bidegree $((6,0),4)$. However, it causes no trouble because it has slope $2/3 \geq 1/2$, so we could repeat Step 1 in the proof of Theorem 2.1.1 to get the exact same conclusion and then the remaining steps are unchanged.

Using the previous two paragraphs and the above spectral sequence we find that for $d = 1,2$ each of the groups $H_{(2d+1,0),d}(\overline{\mathbf{X}}/\sigma_0)$ must be a subquotient of the corresponding group $H_{(2d+1,0),d}(\overline{\mathbf{S}}/\sigma_0)$, as the latter agrees with the first page of the spectral sequence of the correct bidegree.

Finally, we use the spectral sequence computing $H_{*,*}(\overline{\mathbf{S}}/\sigma_0)$ coming from the cellular filtration of $\mathbf{S}$ (analogous to the ones used in Step 2 in the proof of Theorem 2.1.3) to check that $H_{(2d+1,0),d}(\overline{\mathbf{S}}/\sigma_0) = 0$ for $d = 1,2$. In fact, the second page of the spectral sequence already vanishes on those two bidegrees by direct inspection (one can use the computational filtrations of Step 3 of the proof of Theorem 2.1.3 to avoid computing the differentials again). □

2.3 Proof of Theorem 2.1.6

Proof. The proof is divided into several steps: we will firstly check an universal example and then reduce the general statement to that one.

Step 0. Consider the case $\mathbf{X} = \mathbf{A}$, $\mathbf{A} \to \mathbf{X}$ given by the identity. We want to show the conclusion of the theorem for $D = \infty$.

Consider case (i): by [GKRW18, Section 16] $H_{*,*}(\overline{\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)}) = \Lambda(L)$ is the free graded-commutative algebra on L, where L is the free $(k-1)$-Lie algebra on σ_0. Explicitly, L is generated as a graded $\mathbb{Q}$-vector space by σ_0 in grading $(1,0)$ and by $[\sigma_0,\sigma_0]$ in grading $(2,k-1)$. Thus, the map $\sigma_0 \cdot -$ acts injectively on homology, so the homology long exact sequence of the cofibration

$$S^{1,0}_{\mathbb{Q}} \otimes \overline{\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)} \xrightarrow{\sigma_0 \cdot} \overline{\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)} \to \overline{\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)}/\sigma_0$$

gives that $H_{*,*}(\overline{\mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0)}/\sigma_0) = \Lambda(\sigma_0,[\sigma_0,\sigma_0])/\sigma_0 = \Lambda([\sigma_0,\sigma_0])$ is supported in bidegrees in the line of slope $(k-1)/2$ through the origin, and hence it satisfies the required range as $k \geq 3$.

For the case (ii) we let $\mathbf{fA} \in \mathrm{Alg}_{E_k}((\mathsf{sMod}^{\mathbb{N}}_{\mathbb{Q}})^{\mathbb{Z}_\leq})$ be the cell attachment filtration of $\mathbf{A}$, i.e.

$$\mathbf{fA} = \mathbf{E_k}(S^{1,0,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0,0}_{\mathbb{Q}}\sigma_1) \cup^{E_k}_{\sigma_1^2-\sigma_0^2} D^{2,1,1}_{\mathbb{Q}}\rho$$

where the last grading is the filtration degree. This satisfies $\operatorname{colim}(\mathbf{fA}) = \mathbf{A}$, and by [GKRW18, Theorem 6.4] $\operatorname{gr}(\mathbf{fA}) = \mathbf{E_k}(S^{1,0,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0,0}_{\mathbb{Q}}\sigma_1 \oplus S^{2,1,1}_{\mathbb{Q}}\rho)$.

The map $\sigma_0 : S^{1,0}_{\mathbb{Q}} \to \mathbf{A}$ lifts to a filtered map $\sigma_0 : S^{1,0,0}_{\mathbb{Q}} \to \mathbf{fA}$, obtained from $\sigma_0 : S^{1,0}_{\mathbb{Q}} \to \mathbf{fA}(0) = \mathbf{E_k}(S^{1,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0}_{\mathbb{Q}}\sigma_1)$. Thus, using the adapters construction in [GKRW18, Section 12.2], we can form the $\overline{\mathbf{fA}}$-module $\overline{\mathbf{fA}}/\sigma_0$ whose colimit is $\overline{\mathbf{A}}/\sigma_0$.

(i) By [GKRW18, Corollary 10.7, Section 16.6] there is a multiplicative spectral sequence

$$E^1_{g,p,q} = H_{g,p+q,p}(\overline{\mathbf{E_k}(S^{1,0,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0,0}_{\mathbb{Q}}\sigma_1 \oplus S^{2,1,1}_{\mathbb{Q}}\rho)}) \Rightarrow H_{g,p+q}(\overline{\mathbf{A}})$$

whose d^1 differential satisfies $d^1(\rho) = \sigma_1^2 - \sigma_0^2$.

(ii) By [GKRW18, Theorem 10.10] there is a spectral sequence

$$F^1_{g,p,q} = H_{g,p+q,p}(\overline{\mathbf{E_k}(S^{1,0,0}_{\mathbb{Q}}\sigma_0 \oplus S^{1,0,0}_{\mathbb{Q}}\sigma_1 \oplus S^{2,1,1}_{\mathbb{Q}}\rho)}/\sigma_0) \Rightarrow H_{g,p+q}(\overline{\mathbf{A}}/\sigma_0).$$

The $\overline{\mathbf{fA}}$-module structure on $\overline{\mathbf{fA}}/\sigma_0$ gives the second spectral sequence $F^t_{*,*,*}$ the structure of a module over the first one $E^t_{*,*,*}$.

By [GKRW18, Theorem 16.4] $E^1_{*,*,*}$ is given as a $\mathbb{Q}$-algebra by the free graded-commutative algebra $\Lambda(L)$, where L is the free $(k-1)$-Lie algebra on generators σ_0, σ_1, ρ. Since $\sigma_0 \cdot -$ is injective on $E^1_{*,*,*}$ then $F^1_{*,*,*} = \Lambda(L/\mathbb{Q}\{\sigma_0\})$ inherits the structure of a CDGA because its d^1 differential is determined by the one on the first spectral sequence. We can split $L = \mathbb{Q}\{\sigma_0\} \oplus \mathbb{Q}\{\sigma_1\} \oplus \mathbb{Q}\{\rho\} \oplus L'$, where L' consists of elements whose word-length is at least 2, and satisfies that every element of L' has slope ≥ 1 (because $k \geq 3$).

Let us introduce a filtration on the chain complex $(F^1_{*,*,*}, d^1)$ by giving σ_1 and ρ filtration degree 0, elements of L' filtration equal to their homological degree, and then extending the filtration multiplicatively. The differential d^1 preserves this filtration, and its associated graded is given by a tensor product of chain complexes

$$(\Lambda(\sigma_1, \rho), \delta) \otimes (\Lambda(L'), 0)$$

where $\delta(\sigma_1) = 0$ and $\delta(\rho) = \sigma_1^2$. By the Künneth theorem the homology of this chain complex is $\mathbb{Q}[\sigma_1]/(\sigma_1^2) \otimes \Lambda(L')$, and so there is a spectral sequence of the form

$$\mathbb{Q}[\sigma_1]/(\sigma_1^2) \otimes \Lambda(L') \Rightarrow H(F^1_{*,*,*}, d^1) = F^2_{*,*,*}.$$

Thus, $F^2_{g,p,q} = 0$ for $p+q < g-1$ because $\mathbb{Q}[\sigma_1]/(\sigma_1^2) \otimes \Lambda(L')$ already has this vanishing line, and hence $H_{g,d}(\overline{\mathbf{A}}/\sigma_0) = 0$ for $d < g-1$, as required.

Step 1. We will show that under the hypotheses in the statement of the theorem $H^{\overline{\mathbf{A}}}_{g,d}(\overline{\mathbf{X}}/\sigma_0) = 0$ for $d < \min\{g, D\}$.

The composition $S^{1,0}_{\mathbb{Q}} \otimes \overline{\mathbf{A}} \xrightarrow{\sigma_0 \cdot -} \overline{\mathbf{A}} \to \overline{\mathbf{X}} \to \overline{\mathbf{X}}/\sigma_0$ is nullhomotopic as an $\overline{\mathbf{A}}$-module map because it agrees with $S^{1,0}_{\mathbb{Q}} \otimes \overline{\mathbf{A}} \to S^{1,0}_{\mathbb{Q}} \otimes \overline{\mathbf{X}} \xrightarrow{\sigma_0 \cdot -} \overline{\mathbf{X}} \to \overline{\mathbf{X}}/\boldsymbol{\sigma_0}$ and the last two maps form a cofibration sequence. Thus there is an $\overline{\mathbf{A}}$-module map $\overline{\mathbf{A}}/\sigma_0 \to \overline{\mathbf{X}}/\sigma_0$. (We are using adapters throughout so all the previous constructions are in the category of left $\overline{\mathbf{A}}$-modules.)

If $\mathbf{M}$ is an $\overline{\mathbf{A}}$-module then we can descendently filter it by its $\mathbb{N}$-grading, as we did in the proof of Corollary 2.2.7. This induces a filtration on $B(\mathbb{1}, \overline{\mathbf{A}}, \mathbf{M})$ whose associated graded is

$B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1})\otimes \mathrm{gr}(\mathbf{M})$ and hence there is a spectral sequence

$$H_{*,*}(B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1}))\otimes H_{*,*}(\mathbf{M})\Rightarrow H^{\overline{\mathbf{A}}}_{*,*}(\mathbf{M})$$

where we are suppressing the internal grading. Applying this to both $\mathbf{M}=\overline{\mathbf{A}}/\sigma_0$ and $\mathbf{M}=\overline{\mathbf{X}}/\sigma_0$ gives spectral sequences

$$\mathcal{E}^1_{*,*}=H_{*,*}(B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1}))\otimes H_{*,*}(\overline{\mathbf{A}}/\sigma_0)\Rightarrow H^{\overline{\mathbf{A}}}_{*,*}(\overline{\mathbf{A}}/\sigma_0)$$

and

$$\mathcal{F}^1_{*,*}=H_{*,*}(B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1}))\otimes H_{*,*}(\overline{\mathbf{X}}/\sigma_0)\Rightarrow H^{\overline{\mathbf{A}}}_{*,*}(\overline{\mathbf{X}}/\sigma_0).$$

Moreover, the $\overline{\mathbf{A}}$-module map $\overline{\mathbf{A}}/\sigma_0\to\overline{\mathbf{X}}/\sigma_0$ induces a morphism of spectral sequences $\mathcal{E}^t_{*,*}\to\mathcal{F}^t_{*,*}$.

It follows from [GKRW18, Lemmas 13.3, 13.5, Theorem 13.7] that $B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1})\simeq\mathbb{1}\oplus S^{0,1}_{\mathbb{Q}}\otimes Q^{E_1}_{\mathbb{L}}(\mathbf{A})$. By definition of $\mathbf{A}$ and the transferring vanishing lines down theorem, [GKRW18, Theorem 14.6], $H^{E_1}_{g,d}(\mathbf{A})=0$ for $d<g-1$ in cases (i) and (ii); and thus $H_{g,d}(B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1}))=0$ for $d<g$. In other words, in both spectral sequences the first factor of the tensor product vanishes below slope 1.

Thus, by assumption there cannot be elements in $\mathcal{F}^1_{*,*}$ with $d<\min\{g,D\}$ except in case (ii), in which case they are of the form $\zeta\otimes\sigma_1$ for some $\zeta\in H_{*,*}(B(\mathbb{1},\overline{\mathbf{A}},\mathbb{1}))$ lying on the line $d=g$. Thus, all such elements lift to $\mathcal{E}^1_{*,*}$ as σ_1 does. Moreover, $\zeta\otimes\sigma_1\in\mathcal{E}^1_{*,*}$ is a permanent cycle because it has bidegree of the form $(d+1,d)$ for some $d<D$, and so $d^r(\zeta\otimes\sigma_1)$ has bidegree $(d+1,d-1)$, but $\mathcal{E}^1_{d+1,d-1}=0$. Since $Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\sigma_0\cdot -)$ is nullhomotopic and $S^{1,0}_{\mathbb{Q}}\otimes\overline{\mathbf{A}}\xrightarrow{\sigma_0\cdot -}\overline{\mathbf{A}}\to\overline{\mathbf{A}}/\sigma_0$ is a cofibration sequence of $\overline{\mathbf{A}}$-modules, $Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{A}}/\sigma_0)\simeq\mathbb{1}\oplus S^{1,1}_{\mathbb{Q}}$. Thus $H^{\overline{\mathbf{A}}}_{*,*}(\overline{\mathbf{A}}/\sigma_0)=\mathbb{Q}[0,0]\oplus\mathbb{Q}[1,1]$, so $\mathcal{E}^\infty_{d+1,d}=0$ and hence all elements $\zeta\otimes\sigma_1\in\mathcal{E}^1_{*,*}$ are boundaries. Using the morphism of spectral sequences and our assumptions on $H_{*,*}(\overline{\mathbf{X}}/\sigma_0)$ it follows that $\zeta\otimes\sigma_1\in\mathcal{F}^1_{*,*}$ is also a permanent cycle and a boundary. Therefore, $\mathcal{F}^\infty_{g,d}=0$ for $d<\min\{g,D\}$, giving the result.

Step 2. We will show that $H^{\overline{\mathbf{A}}}_{g,d}(\overline{\mathbf{X}})=0$ for $d<\min\{g,D\}$.

By [GKRW18, Section 12.2.4], $\overline{\mathbf{X}}/\sigma_0\simeq\overline{\mathbf{A}}/\sigma_0\otimes^{\mathbb{L}}_{\overline{\mathbf{A}}}\overline{\mathbf{X}}$ as left $\overline{\mathbf{A}}$-modules. Also, in the proof of Step 1 we showed that $Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{A}}/\sigma_0)\simeq\mathbb{1}\oplus S^{1,1}_{\mathbb{Q}}$. By [GKRW18, Corollary 9.17] $Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{X}}/\sigma_0)=\mathbb{1}\otimes^{\mathbb{L}}_{\overline{\mathbf{A}}}\overline{\mathbf{X}}/\sigma_0$ and $Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{X}})=\mathbb{1}\otimes^{\mathbb{L}}_{\overline{\mathbf{A}}}\overline{\mathbf{X}}$. Thus,

$$Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{X}}/\sigma_0)=(\mathbb{1}\oplus S^{1,1}_{\mathbb{Q}})\otimes Q^{\overline{\mathbf{A}}}_{\mathbb{L}}(\overline{\mathbf{X}})$$

which together with Step 1 implies the required result.

Step 3. We will show that $H^{E_k}_{g,d}(\mathbf{X},\mathbf{A}) = 0$ for $d < \min\{g, D\}$.

Let $\delta := \min\{d : H^{E_k}_{d+1,d}(\mathbf{X},\mathbf{A}) \neq 0\} \in \mathbb{N} \cup \{\infty\}$. Our goal is to show that $\delta \geq D$.

Since $H^{E_k}_{g,d}(\mathbf{X}) = 0 = H^{E_k}_{g,d}(\mathbf{A})$ for $d < g-1$ it follows that $H^{E_k}_{g,d}(\mathbf{X},\mathbf{A}) = 0$ for $d < g-1$ and $d < \frac{\delta}{\delta+1}g$. By [GKRW18, Theorem 15.9] with $\rho(g) = \frac{\delta}{\delta+1}g = \sigma(g)$, the map $H^{\overline{\mathbf{A}}}_{g,d}(\overline{\mathbf{X}},\overline{\mathbf{A}}) \to H^{E_k}_{g,d}(\mathbf{X},\mathbf{A})$ is surjective for $d < \frac{\delta}{\delta+1}g + 1$. Thus, $H^{\overline{\mathbf{A}}}_{\delta+1,\delta}(\overline{\mathbf{X}}) \xrightarrow{\cong} H^{\overline{\mathbf{A}}}_{\delta+1,\delta}(\overline{\mathbf{X}},\overline{\mathbf{A}}) \to H^{E_k}_{\delta+1,\delta}(\mathbf{X},\mathbf{A}) \neq 0$ is a surjection, and hence $H^{\overline{\mathbf{A}}}_{\delta+1,\delta}(\overline{\mathbf{X}}) \neq 0$. By Step 2 we have $\delta \geq D$, as required.

Step 4. Now we can finish the proof.

By Step 3 and the vanishing lines in E_k-homology for both $\mathbf{A}$ and $\mathbf{X}$ we get $H^{E_k}_{g,d}(\mathbf{X},\mathbf{A}) = 0$ for $d < \frac{D}{D+1}g$. Applying [GKRW18, Corollary 15.10] with $\rho(g) = \frac{D}{D+1}g$, $\mathbf{M} = \overline{\mathbf{A}}/\sigma_0$, and $\mu(g)$ either $\frac{D}{D+1}g$ in case (i) or $\frac{D}{D+1}(g-1)$ in case (ii); we find that $H_{g,d}(B(\overline{\mathbf{A}}/\sigma_0, \overline{\mathbf{A}}, \overline{\mathbf{X}})) = 0$ for $d < \mu(g)$. Finally, [GKRW18, Section 12.2.4, Lemma 9.16] gives that $B(\overline{\mathbf{A}}/\sigma_0, \overline{\mathbf{A}}, \overline{\mathbf{X}}) \simeq \overline{\mathbf{X}}/\sigma_0$. $\square$

Chapter 3

Quadratic symplectic groups and spin mapping class groups

In this chapter we will prove Theorems A and B. To do so, we will firstly construct the relevant E_2-algebras in Section 3.1, and then we will apply the generic homological stability results of Chapter 2. In order to apply those results we will need to know some first homology computations for the spin mapping class groups and quadratic symplectic groups, which will be delayed to Section 3.4, after Theorems A and B have already been shown.

3.1 E_2-algebras from quadratic data

There is a general framework of how to get an E_2-algebra from a braided monoidal groupoid, see [GKRW18, Section 17.1]. In this section we will consider braided monoidal groupoids with the extra data of a strong braided monoidal functor to Set, and we will observe that the "Grothendieck construction" yields another braided monoidal groupoid, called the "*associated quadratic groupoid"*, and hence another E_2-algebra.

This construction will generalize the way quadratic symplectic groups are constructed from symplectic groups and the way that spin mapping class groups are related to mapping class groups, if we let the extra data be the set of quadratic refinements (hence the use of the term "quadratic").

We will also study the relationship between the E_2-algebra of the original groupoid and the one of the associated quadratic groupoid; in particular Theorem 3.1.4 and Corollary 3.1.5 allow us to get some vanishing lines in the E_2-homology of the associated quadratic groupoid from knowledge of the original groupoid.

3.1.1 Definition and construction of the E_2-algebras

Let us start by introducing some notation based on the one in [GKRW18, Section 17]. All the categories for the rest of this section are discrete. A *braided monoidal groupoid* is a triple $(\mathsf{G},\oplus,\mathbb{1})$, where G is a groupoid, $\oplus$ a braided monoidal structure on G and $\mathbb{1}$ the monoidal unit. For an object $x\in\mathsf{G}$ we write $\mathsf{G}_x := \mathsf{G}(x,x) = \mathrm{Aut}_{\mathsf{G}}(x)$. We can view any (discrete) monoid as an example of a monoidal groupoid where the only morphisms are the identity; for example $\mathbb{N}$ is naturally a symmetric monoidal groupoid, so in particular braided.

Definition 3.1.1. *A quadratic data consists of a triple* $(\mathsf{G},\mathrm{rk},Q)$ *where*

(i) $\mathsf{G}=(\mathsf{G},\oplus,\mathbb{1})$ *is a braided monoidal groupoid such that* $\mathsf{G}_{\mathbb{1}}$ *is trivial and for any objects* $x,y\in\mathsf{G}$ *the map* $-\oplus-:\mathsf{G}_x\times\mathsf{G}_y\to\mathsf{G}_{x\oplus y}$ *is injective.*

(ii) $\mathrm{rk}:\mathsf{G}\to\mathbb{N}$ *is a braided monoidal functor such that* $\mathrm{rk}^{-1}(0)$ *consists precisely of those objects isomorphic to* $\mathbb{1}$.

(iii) $Q:\mathsf{G}^{\mathsf{op}}\to\mathsf{Set}$ *is a strong braided monoidal functor.*

Parts (i) and (ii) are precisely the assumptions needed to apply all the constructions of [GKRW18, Section 17], and part (iii) is the extra "quadratic" data. One should think of $Q(x)$ as the set of "quadratic refinements" of the object x; and strong monoidality implies in particular that $Q(\mathbb{1})$ is a one element set.

Definition 3.1.2. *Given a quadratic data* $(\mathsf{G},\mathrm{rk},Q)$*, its associated quadratic groupoid is the braided monoidal groupoid* $(\mathsf{G}^{\mathsf{q}},\oplus^{\mathsf{q}},\mathbb{1}^{\mathsf{q}})$ *given by the Grothendieck construction* $\mathsf{G}\wr Q$*, i.e.*

(i) The set of objects of G^{q} *is* $\bigsqcup_{x\in\mathsf{G}}Q(x)$.

(ii) The sets of morphisms are given as follows: for $q\in Q(x)$ *and* $q'\in Q(x')$*,* $\mathsf{G}^{\mathsf{q}}(q,q') = \{\phi\in\mathsf{G}(x,x'):Q(\phi)(q')=q\}$.

(iii) The braided monoidal structure $\oplus^{\mathsf{q}}$ *is induced by the strong braided monoidality of* Q*, and the monoidal unit* $\mathbb{1}^{\mathsf{q}}$ *is given by the only element in* $Q(\mathbb{1})$.

Let us denote by $\mathrm{rk}^{\mathsf{q}}:\mathsf{G}^{\mathsf{q}}\to\mathbb{N}$ the braided monoidal functor given by $q\in Q(x)\mapsto\mathrm{rk}(x)$. By construction the group $\mathsf{G}^{\mathsf{q}}{}_{\mathbb{1}^{\mathsf{q}}}$ is trivial and for any objects $q,q'\in\mathsf{G}^{\mathsf{q}}$ the map $-\oplus^{\mathsf{q}}-:\mathsf{G}^{\mathsf{q}}{}_q\times\mathsf{G}^{\mathsf{q}}{}_{q'}\to\mathsf{G}^{\mathsf{q}}{}_{q\oplus^{\mathsf{q}}q'}$ is injective. Also, $(\mathrm{rk}^{\mathsf{q}})^{-1}(0)$ consists precisely of those objects isomorphic to $\mathbb{1}^{\mathsf{q}}$. Thus, $(\mathsf{G}^{\mathsf{q}},\oplus^{\mathsf{q}},\mathbb{1}^{\mathsf{q}},\mathrm{rk}^{\mathsf{q}})$ satisfies all the assumptions of [GKRW18, Section 17], so by [GKRW18, Section 17.1] there is $\mathbf{R}^{\mathsf{q}}\in\mathrm{Alg}_{E_2}(\mathrm{sSet}^{\mathbb{N}})$ such that

$$\mathsf{R}^{\mathsf{q}}(n)\simeq\begin{cases}\varnothing & if \quad n=0\\ \displaystyle\bigsqcup_{[q]\in\pi_0(\mathsf{G}^{\mathsf{q}}):\,\mathrm{rk}^{\mathsf{q}}(q)=n} B\mathsf{G}^{\mathsf{q}}{}_q & if \quad n>0.\end{cases}$$

We shall call $\mathbf{R}^{\mathfrak{q}}$ the *quadratic E_2-algebra* associated to a quadratic data. Alternatively, in the explicit construction of $\mathbf{R}^{\mathfrak{q}}$ in [GKRW18, Section 17.1] we can perform the Kan extension along the projection $\mathsf{G}^{\mathfrak{q}} \to \pi_0(\mathsf{G}^{\mathfrak{q}})$ instead of along $\mathsf{G}^{\mathfrak{q}} \xrightarrow{\mathrm{rk}^{\mathfrak{q}}} \mathbb{N}$, and hence we can view $\mathbf{R}^{\mathfrak{q}} \in \mathrm{Alg}_{E_2}(\mathsf{sSet}^{\pi_0(\mathsf{G}^{\mathfrak{q}})})$ such that

$$\mathsf{R}^{\mathfrak{q}}([q]) \simeq \begin{cases} \varnothing & if\ q \cong \mathbb{1}^{\mathfrak{q}} \\ B\mathsf{G}^{\mathfrak{q}}{}_q & otherwise. \end{cases}$$

We will not distinguish between these two, as sometimes it will be more convenient to think of $\mathbf{R}^{\mathfrak{q}}$ as being $\mathbb{N}$-graded and other times as $\pi_0(\mathsf{G}^{\mathfrak{q}})$-graded.

Remark 3.1.3. *When we view $\mathbf{R}^{\mathfrak{q}}$ as $\pi_0(\mathsf{G}^{\mathfrak{q}})$-graded we have that $\mathbf{R}^{\mathfrak{q}}([q])$ is path-connected for any $[q] \neq 0 \in \pi_0(\mathsf{G}^{\mathfrak{q}})$. Thus, the strictly associative algebra $\overline{\mathbf{R}^{\mathfrak{q}}}$ satisfies that $\pi_0(\overline{\mathbf{R}^{\mathfrak{q}}}) \cong \pi_0(\mathsf{G}^{\mathfrak{q}})$ as monoids, where the monoid structure on the left-hand-side is induced by the product. In particular, the ring $H_{*,0}(\overline{\mathbf{R}^{\mathfrak{q}}})$ is determined by the monoidal structure of $\pi_0(\mathsf{G}^{\mathfrak{q}})$.*

Similarly, we can apply the construction of [GKRW18, Section 17.1] to $(\mathsf{G}, \oplus, \mathbb{1}, \mathrm{rk})$ to get $\mathbf{R} \in \mathrm{Alg}_{E_2}(\mathsf{sSet}^{\mathbb{N}})$ such that

$$\mathsf{R}(n) \simeq \begin{cases} \varnothing & if \quad n = 0 \\ \displaystyle\bigsqcup_{[x] \in \pi_0(\mathsf{G}) : \mathrm{rk}(x) = n} B\mathsf{G}_x & if \quad n > 0. \end{cases}$$

We will refer to $\mathbf{R}$ as the *non-quadratic E_2-algebra*. The obvious braided monoidal functor $\mathsf{G}^{\mathfrak{q}} \to \mathsf{G}$ then induces an E_2-algebra map $\mathbf{R}^{\mathfrak{q}} \to \mathbf{R}$.

3.1.2 E_1-splitting complexes of quadratic groupoids

Recall [GKRW18, Definition 17.9] that given a monoidal groupoid G with a rank functor $\mathrm{rk} : \mathsf{G} \to \mathbb{N}$ satisfying properties (i) and (ii) of Definition 3.1.1 and an element $x \in \mathsf{G}$, the *E_1-splitting complex $S_\bullet^{E_1,\mathsf{G}}(x)$* is the semisimplicial set with p-simplices given by

$$S_p^{E_1,\mathsf{G}}(x) := \operatorname*{colim}_{(x_0,\cdots,x_{p+1}) \in \mathsf{G}_{\mathrm{rk}>0}^{p+2}} \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x)$$

and face maps given by the monoidal structure. (Where $\mathsf{G}_{\mathrm{rk}>0}$ denotes the full subgroupoid of G on those objects x with $\mathrm{rk}(x) > 0$, i.e. on those objects not isomorphic to $\mathbb{1}$.)

The main result of this subsection is the following result which will allow us to understand splitting complexes of quadratic groupoids.

Theorem 3.1.4. *Let* $(\mathsf{G}, \mathrm{rk}, Q)$ *be a quadratic data, then for any* $q \in Q(x)$ *there is an isomorphism of semisimplicial sets* $S_\bullet^{E_1,\mathsf{G}^\mathfrak{q}}(q) \cong S_\bullet^{E_1,\mathsf{G}}(x)$.

Proof. By definition

$$S_p^{E_1,\mathsf{G}}(x) = \operatorname*{colim}_{(x_0,\cdots,x_{p+1})\in \mathsf{G}^{p+2}_{\mathrm{rk}>0}} \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x)$$

and

$$S_p^{E_1,\mathsf{G}^\mathfrak{q}}(q) = \operatorname*{colim}_{(q_0,\cdots,q_{p+1})\in {\mathsf{G}^\mathfrak{q}}^{p+2}_{\mathrm{rk}^\mathfrak{q}>0}} \mathsf{G}^\mathfrak{q}(q_0 \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}, q)$$

The inclusions $\mathsf{G}^\mathfrak{q}(q_0 \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}, q) \subset \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x)$, for each $(q_0, \cdots, q_{p+1}) \in {\mathsf{G}^\mathfrak{q}}^{p+2}_{\mathrm{rk}^\mathfrak{q}>0}$ with $q_i \in Q(x_i)$ and $q \in Q(x)$, assemble into canonical maps

$$S_p^{E_1,\mathsf{G}^\mathfrak{q}}(q) = \operatorname*{colim}_{(q_0,\cdots,q_{p+1})\in {\mathsf{G}^\mathfrak{q}}^{p+2}_{\mathrm{rk}^\mathfrak{q}>0}} \mathsf{G}^\mathfrak{q}(q_0 \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}, q) \to \operatorname*{colim}_{(x_0,\cdots,x_{p+1})\in \mathsf{G}^{p+2}_{\mathrm{rk}>0}} \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x) = S_p^{E_1,\mathsf{G}}(x)$$

which are compatible with the face maps of both semisimplicial sets because the natural functor $\mathsf{G}^\mathfrak{q} \to \mathsf{G}$ is monoidal. Thus, it suffices to show that $S_p^{E_1,\mathsf{G}^\mathfrak{q}}(q) \to S_p^{E_1,\mathsf{G}}(x)$ is a bijection of sets for all p.

Surjectivity: any element on the right hand side is represented by some $\phi \in \mathsf{G}(x_0 \oplus \cdots \oplus x_{p+1}, x)$ which is an isomorphism since G is a groupoid. Since Q is strong monoidal, $Q(\phi) : Q(x) \xrightarrow{\cong} Q(x_0) \times \cdots \times Q(x_{p+1})$ is an isomorphism. Let $q_i := \mathrm{proj}_i(Q(\phi)(q)) \in Q(x_i)$, then $\phi \in \mathsf{G}^\mathfrak{q}(q_0 \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}, q)$ defines an element on the left hand side mapping to the required element.

Injectivity: suppose that two elements on the left hand side have the same image on the right hand side. Represent them by $\phi^i \in \mathsf{G}^\mathfrak{q}(q_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}^i, q)$, where $Q(\phi^i)(q) = q_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}^i$ and $i \in \{1, 2\}$ is an index.

Since ϕ^1 and ϕ^2 agree on the colimit of the right hand side then there is an element $\phi \in \mathsf{G}(x_0 \oplus \cdots + x_{p+1}, x)$ and morphisms $(\psi_0^i, \cdots, \psi_{p+1}^i) : (x_0^i, \cdots, x_{p+1}^i) \to (x_0, \cdots, x_{p+1})$ in $\mathsf{G}^{p+2}_{\mathrm{rk}>0}$ such that $\phi^i \circ (\psi_0^i, \cdots, \psi_{p+1}^i)^{-1} = \phi$ for $i \in \{1, 2\}$.

Let ${q'}_a^i := Q({\psi_a^i}^{-1})(q_a^i) \in Q(x_a)$, we claim that ${q'}_a^1 = {q'}_a^2$ for $i \in \{1, 2\}$: $Q(\psi_0^i \oplus \cdots \oplus \psi_{p+1}^i)({q'}_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} {q'}_{p+1}^i) = (q_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}^i) = Q(\phi^i)(q)$, so ${q'}_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} {q'}_{p+1}^i = Q(\phi^i \circ (\psi_0^i \oplus \cdots \oplus \psi_{p+1}^i)^{-1})(q) = Q(\phi)(q)$. Since $Q(\phi)(q)$ is independent of $i \in \{1, 2\}$ then the claim follows by the strong monoidality Q because ${q'}_a^1, {q'}_a^2 \in Q(x_a)$ for all a.

Now let $q_a := {q'}_a^1 = {q'}_a^2$, then $Q(\psi_0^i \oplus \cdots \oplus \psi_{p+1}^i)(q_0 \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}) = (q_0^i \oplus^\mathfrak{q} \cdots \oplus^\mathfrak{q} q_{p+1}^i)$ and hence $(\psi_0^i, \cdots, \psi_{p+1}^i) \in {\mathsf{G}^\mathfrak{q}}^{p+2}_{\mathrm{rk}^\mathfrak{q}>0}$. Since $\phi^i \circ (\psi_0^i, \cdots, \psi_{p+1}^i)^{-1} = \phi$ for $i \in \{1, 2\}$ by construction, then ϕ^1 and ϕ^2 agree on the left hand side colimit, as required. $\square$

Recall [GKRW18, Definition 17.6, Lemma 17.10]: we say that $(\mathsf{G},\oplus,\mathbb{1},\mathrm{rk})$ *satisfies the standard connectivity estimate* if for any $x\in\mathsf{G}$, the reduced homology of $S^{E_1,\mathsf{G}}(x) := \|S_\bullet^{E_1,\mathsf{G}}\|$ is concentrated in degree $\mathrm{rk}(x)-2$. As explained in [GKRW18, Page 188] the standard connectivity estimate implies that $H^{E_1}_{n,d}(\mathbf{R}) = 0$ for $d < n-1$, where $\mathbf{R}$ is the E_2-algebra defined in Section 3.1.1. The following corollary says that the standard connectivity estimate on the underlying braided groupoid of a quadratic data also gives a vanishing line on the E_2-homology of the corresponding quadratic E_2-algebra.

Corollary 3.1.5. *If* $(\mathsf{G},\mathrm{rk},Q)$ *is a quadratic data such that* (G,rk) *satisfies the standard connectivity estimate then* $H^{E_2}_{x,d}(\mathbf{R}^{\mathfrak{q}}) = 0$ *for* $d < \mathrm{rk}(x)-1$.

Proof. By Theorem 3.1.4 and the standard connectivity estimate, the reduced homology of $S^{E_1,\mathsf{G}^{\mathfrak{q}}}(q)$ is concentrated in degree $\mathrm{rk}^{\mathfrak{q}}(q)-2$ for any $q\in\mathsf{G}^{\mathfrak{q}}$. Thus, by [GKRW18, Page 188] we have $H^{E_1}_{x,d}(\mathbf{R}^{\mathfrak{q}}) = 0$ for $d<\mathrm{rk}(x)-1$. Finally, the transferring vanishing lines up theorem, [GKRW18, Theorem 14.4], implies the result. □

3.2 Quadratic symplectic groups

3.2.1 Construction of the E_2-algebra

For a given $g\geq 0$ we let the *standard symplectic form* on $\mathbb{Z}^{2g}$ be the matrix Ω_g given by the block diagonal sum of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The *genus g symplectic group* is defined by $Sp_{2g}(\mathbb{Z}) := \mathrm{Aut}(\mathbb{Z}^{2g},\Omega_g)$.

Let $(\mathsf{Sp},\oplus,0)$ be the symmetric monoidal groupoid with objects the non-negative integers, morphisms $\mathsf{Sp}(g,h) = \begin{cases} Sp_{2g}(\mathbb{Z}) & if\ g=h \\ \varnothing & otherwise, \end{cases}$ where the (strict) monoidal structure $\oplus$ is given by addition on objects and block diagonal sum on morphisms, 0 is the (strict) monoidal unit and the braiding $\beta_{g,h}: g\oplus h \xrightarrow{\cong} h\oplus g$ is given by the matrix $\begin{pmatrix} 0 & I_{2h} \\ I_{2g} & 0 \end{pmatrix}$, which satisfies $\beta_{h,g}\beta_{g,h} = \mathrm{id}_{g+h}$.

We let $\mathrm{rk}:\mathsf{Sp}\to\mathbb{N}$ be the symmetric monoidal functor given by identity on objects and let $Q:\mathsf{Sp}^{\mathsf{op}}\to\mathsf{Set}$ be the functor given as follows

(i) On objects, $Q(g) := \{q:\mathbb{Z}^{2g}\to\mathbb{Z}/2:\ q(x+y)\equiv q(x)+q(y)+x\cdot y(\mathrm{mod}\,2)\}$, where $\cdot$ represents the skew-symmetric product induced by the standard symplectic form.

(ii) On morphisms, for $\phi\in Sp_{2g}(\mathbb{Z})$ and $q\in Q(g)$ we let $Q(\phi)(q) = q\circ\phi$.

In other words, $Q(g)$ is the set of quadratic refinements on the standard symplectic form $(\mathbb{Z}^{2g},\Omega_g)$. Strong symmetric monoidality of Q follows from the fact that a quadratic refinement $q \in Q(g)$ is the same data as a function of sets from a basis of $\mathbb{Z}^{2g}$ to $\mathbb{Z}/2$. Thus, $(\mathsf{Sp}, \mathrm{rk}, Q)$ is a quadratic data in the sense of Definition 3.1.1.

By Section 3.1.1 we get an associated quadratic groupoid Sp^{q} and a quadratic E_2-algebra $\mathbf{R}^{\mathsf{q}}$, which in this case is actually E_∞ because the groupoid is symmetric and not just braided. Being E_∞ instead of just E_2 has implications about the algebra $\mathbf{R}^{\mathsf{q}}$, such as the vanishing of Browder brackets, but they do not yield an improvement in the stability and secondary stability results that we show in this thesis.

The next goal is to describe $\pi_0(\mathsf{Sp}^{\mathsf{q}})$, which by Remark 3.1.3 gives a computation of $H_{*,0}(\overline{\mathbf{R}^{\mathsf{q}}})$. In order to do so, we need to introduce the so called *Arf invariant*.

Definition 3.2.1. *Given a quadratic refinement $q \in Q(g)$ of the standard symplectic form on $\mathbb{Z}^{2g}$, we define the Arf invariant of q via* $\mathrm{Arf}(q) := \sum_{i=1}^{g} q(e_i)q(f_i) \in \mathbb{Z}/2$, *where $(e_1, f_1, \cdots, e_g, f_g)$ is the standard (ordered) basis of $\mathbb{Z}^{2g}$.*

The key property of this invariant is that for $q, q' \in Q(g)$ we have $\mathrm{Arf}(q) = \mathrm{Arf}(q')$ if and only if there exists $\phi \in Sp_{2g}(\mathbb{Z})$ such that $q' = Q(\phi)(q)$. Moreover, for $g \geq 1$ it is clear that $\mathrm{Arf} : Q(g) \to \mathbb{Z}/2$ is surjective.

Before stating the next result recall the monoid $\mathsf{H} := \{0\} \sqcup (\mathbb{N}_{>0} \times \mathbb{Z}/2)$, where the monoidal structure $+$ is given by addition in both coordinates, considered at the beginning of Chapter 2.

Lemma 3.2.2. *Taking rank and Arf invariant gives an isomorphism of monoids* $(\mathrm{rk}, \mathrm{Arf}) : \pi_0(\mathsf{Sp}^{\mathsf{q}}) \xrightarrow{\cong} \mathsf{H}$.

Proof. The map $(\mathrm{rk}, \mathrm{Arf}) : \pi_0(\mathsf{Sp}^{\mathsf{q}}) \to \mathsf{H}$ is clearly surjective, it is injective and well-defined by the above discussion of the Arf invariant, and it is monoidal because rk is monoidal and Arf is also monoidal by its explicit formula. □

Under this identification of $\pi_0(\mathbf{R}^{\mathsf{q}})$ we have that $\mathbf{R}^{\mathsf{q}}(g, \varepsilon) = BSp_{2g}^{\varepsilon}(\mathbb{Z})$ is the classifying space of a quadratic symplectic group in the sense of Section 1.2.1. Thus, by Section 1.3, Theorem B is equivalent to certain vanishing lines of $H_{*,*}(\overline{\mathbf{R}^{\mathsf{q}}}/\sigma_\varepsilon; \Bbbk)$ and $H_{*,*}(\overline{\mathbf{R}^{\mathsf{q}}}/(\sigma_\varepsilon, \theta); \mathbb{F}_2)$.

3.2.2 Proof of Theorem B

The only additional ingredient that we need to prove Theorem B is to understand the E_1-splitting complex of $(\mathsf{Sp}, \mathrm{rk})$.

Proposition 3.2.3. $(\mathsf{Sp}, \mathrm{rk})$ *satisfies the standard connectivity estimate, i.e. for $g \in \mathbb{N}$ the reduced homology of $S^{E_1, \mathsf{Sp}}(g)$ is concentrated in degree $g-2$.*

Proof. Let $P(g)$ be the poset whose elements are submodules $0 \subsetneq M \subsetneq \mathbb{Z}^{2g}$ such that $(M, \Omega_g|_M)$ is isomorphic to the standard symplectic form $(\mathbb{Z}^{2h}, \Omega_h)$ for some $0 < h < g$, ordered by inclusion. Let $P_\bullet(g)$ be the nerve of the poset, viewed as a semisimplicial set with p-simplices strict chains $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_p$ in $P(g)$, and face maps given by forgetting elements in the chain. (We will discuss nerves of posets in more detail in Section 4.3.3.)

The first step in the proof is about comparing $P_\bullet(g)$ with the E_1-splitting complex.

Claim. There is an isomorphism of semisimplicial sets $S_\bullet^{E_1,\mathsf{Sp}}(g) \to P_\bullet(g)$.

Proof. By [GKRW18, Remark 17.11] we have the following more concrete description of $S_\bullet^{E_1,\mathsf{Sp}}(g)$:

$$S_p^{E_1,\mathsf{Sp}}(g) = \bigsqcup_{(g_0,\cdots,g_{p+1}):\, g_i>0,\, \sum_i g_i = g} \frac{Sp_{2g}(\mathbb{Z})}{Sp_{2g_0}(\mathbb{Z}) \times Sp_{2g_1}(\mathbb{Z}) \times \cdots \times Sp_{2g_{p+1}}(\mathbb{Z})}$$

with the obvious face maps.

For each $0 < n < g$ we let $M_n := \mathbb{Z}^{2n} \oplus 0 \subset \mathbb{Z}^{2g}$, so that we have a chain $M_1 < \cdots < M_{g-1}$ in $P(g)$. For each tuple $(g_0, \cdots, g_{p+1})$ with $g_i > 0$ and $\sum_i g_i = g$ we have a p-simplex $\sigma_{g_0,\cdots,g_{p+1}} := M_{g_0} < M_{g_0+g_1} < \cdots < M_{g_0+\cdots+g_p} \in P_p(g)$.

The group $Sp_{2g}(\mathbb{Z})$ acts simplicially on $P_\bullet(g)$, and under this action the stabilizer of $\sigma_{g_0,\cdots,g_{p+1}}$ is precisely $Sp_{2g_0}(\mathbb{Z}) \times Sp_{2g_1}(\mathbb{Z}) \times \cdots \times Sp_{2g_{p+1}}(\mathbb{Z}) \subset Sp_{2g}(\mathbb{Z})$. Thus, we indeed get a levelwise injective map of semisimplicial sets $S_\bullet^{E_1,\mathsf{Sp}}(g) \to P_\bullet(g)$.

Level-wise surjectivity follows from the fact that for a given $M \in P(g)$, any isomorphism $(M, \Omega_g|_M) \xrightarrow{\cong} (\mathbb{Z}^{\mathrm{rk} M}, \Omega_{\mathrm{rk}(M)/2})$ can be extended to an automorphism of $(\mathbb{Z}^{2g}, \Omega_g)$. This is a consequence of the classification of non-degenerate skew-symmetric forms over finitely generated free $\mathbb{Z}$-modules, that we will state in full detail in Theorem 4.4.7. □

Let us denote $L := (\mathbb{Z}^{2g}, \Omega_g)$. The poset $P(g)$ is then the same as $\mathcal{U}(L)_{0<-<L}$ in the sense of [vdKL11, Section 1]. By [vdKL11, Theorem 1.1] the poset $\mathcal{U}(L)_{0<-<L}$ is $(g-3)$-connected and $(g-2)$-dimensional, giving the result. □

Now we can finally finish the proof of Theorem B, assuming the first homology computations of Section 3.4.

Proof. **Part (i).** By Lemma 3.2.2 and Remark 3.1.3 we have $\mathbf{R}^{\mathfrak{q}} \in \mathrm{Alg}_{E_2}(\mathsf{sSet}^{\mathsf{H}})$ such that $\mathbf{R}^{\mathfrak{q}}(x)$ is path-connected for each $x \in \mathsf{H} \setminus \{0\}$ and $\mathbf{R}^{\mathfrak{q}}(0) = \varnothing$. Thus, $H_{0,0}(\mathbf{R}^{\mathfrak{q}}) = 0$ and $H_{*,0}(\overline{\mathbf{R}^{\mathfrak{q}}}) = \mathbb{Z}[\sigma_0, \sigma_1]/(\sigma_1^2 - \sigma_0^2)$ as a ring, where σ_ε is generated by a point in $\mathbf{R}^{\mathfrak{q}}((1,\varepsilon))$. By Proposition 3.2.3, $(\mathsf{Sp}, \mathrm{rk})$ satisfies the standard connectivity estimate, and thus by Corollary 3.1.5 we get that $H_{x,d}^{E_2}(\mathbf{R}^{\mathfrak{q}}) = 0$ for $d < \mathrm{rk}(x) - 1$.

If we now consider $\mathbf{X} := \mathbf{R}^{\mathsf{q}}{}_{\mathbb{Z}} \in \mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{Z}}^{\mathsf{H}})$ then it satisfies the assumptions of Theorem 2.1.1 by [GKRW18, Lemma 18.2] and the properties of $(-)_{\mathbb{Z}}$ explained in Section 1.3. Thus the claimed homological stability for $\mathbf{R}^{\mathsf{q}}$ follows.

Part (ii). This time let $\mathbf{X} := \mathbf{R}^{\mathsf{q}}{}_{\mathbb{Z}[1/2]} \in \mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{Z}[1/2]}^{\mathsf{H}})$. As before, this algebra satisfies the unnumbered assumptions of Theorem 2.1.2. We will check that it also verifies assumptions (i),(ii) and (iii), and then the required stability will follow from Theorem 2.1.2. By the universal coefficient theorem, to check them it suffices to prove that $H_{x,1}(\mathbf{R}^{\mathsf{q}})$ is 2-torsion for $\mathrm{rk}(x) \in \{2,3\}$, which follows from Theorems 3.4.7 and 3.4.8 in Section 3.4.2.

Secondary stability. Let $\mathbf{X} := \mathbf{R}^{\mathsf{q}}{}_{\mathbb{F}_2} \in \mathrm{Alg}_{E_2}(\mathsf{sMod}_{\mathbb{F}_2}^{\mathsf{H}})$. Then Theorem 2.1.3 applies by Theorems 3.4.7 and 3.4.8 and the universal coefficient theorem. The result then follows by the long exact sequence of the cofibration $S^{(4,0),2} \otimes \overline{\mathbf{X}}/\sigma_\varepsilon \xrightarrow{\theta\cdot -} \overline{\mathbf{X}}/\sigma_\varepsilon \to \overline{\mathbf{X}}/(\sigma_\varepsilon, \theta)$. □

3.3 Spin mapping class groups

Consider the braided monoidal groupoid $(\mathsf{MCG}, \oplus, 0)$ defined in [GKRW19, section 4] whose objects are the non-negative integers and morphisms are given by

$$\mathsf{MCG}(g,h) = \begin{cases} \Gamma_{g,1} & if\ g = h \\ \varnothing & otherwise. \end{cases}$$

The monoidal structure on MCG is given by addition on objects and by "gluing diffeomorphisms" on morphisms, using the decomposition of $\Sigma_{g+h,1}$ as a boundary connected sum $\Sigma_{g,1} \natural \Sigma_{h,1}$. The braiding is induced by the half right-handed Dehn twist along the boundary. Let $\mathrm{rk} : \mathsf{MCG} \to \mathbb{N}$ be the braided monoidal functor given by identity on objects.

Let $Q : \mathsf{MCG}^{\mathsf{op}} \to \mathsf{Set}$ be the functor given as follows

(i) On objects,

$$Q(g) = \{q : H_1(\Sigma_{g,1};\mathbb{Z}) \to \mathbb{Z}/2 : \ q(x+y) \equiv q(x) + q(y) + x \cdot y (\mathrm{mod}\, 2)\},$$

where $\cdot$ is the homology intersection pairing.

(ii) On morphisms, for $\phi \in \Gamma_{g,1}$ and $q \in Q(g)$ we let $Q(\phi)(q) = q \circ \phi_*$.

In other words, $Q(g)$ is the set of quadratic refinements of the intersection product in $H_1(\Sigma_{g,1};\mathbb{Z})$, which is isomorphic to the standard symplectic form of genus g. By the argument of Section 3.2, Q is strong braided monoidal, so $(\mathsf{MCG}, \mathrm{rk}, Q)$ is a quadratic data. Moreover, by mimicking the proof of Lemma 3.2.2 we get that $(\mathrm{rk}, \mathrm{Arf}) : \pi_0(\mathsf{MCG}^{\mathsf{q}}) \xrightarrow{\cong} \mathsf{H}$ is a monoidal isomorphism. (This uses the surjectivity of the map $\Gamma_{g,1} \to Sp_{2g}(\mathbb{Z})$.)

Remark 3.3.1. *Using [RW14, Section 2] one can check that* $\mathbf{R^{\mathfrak{q}}}$ *agrees with the "moduli space of spin surfaces with one boundary component", defined in more geometric terms using tangential structures. Since the spin mapping class groups are finite index subgroups of the usual mapping class groups then the path-components of* $\mathbf{R^{\mathfrak{q}}}$ *are finite degree covers of the usual moduli spaces of surfaces with one boundary component.*

Since the E_2-algebra $\mathbf{R^{\mathfrak{q}}}$ satisfies that $\mathbf{R^{\mathfrak{q}}}(g,\varepsilon) \simeq B\Gamma^{1/2}_{g,1}[\varepsilon]$, Theorem A is equivalent to certain vanishing lines in the homology of $\overline{\mathbf{R^{\mathfrak{q}}}}/\sigma_\varepsilon$ and $\overline{\mathbf{R^{\mathfrak{q}}}}/(\sigma_\varepsilon,\theta)$.

3.3.1 Proof of Theorem A

Proof. In this case the standard connectivity estimate for $(\mathsf{MCG},\mathrm{rk})$ is proven in [GKRW19, Theorem 3.4]. Thus, proceeding as in the proof of Theorem B we can apply Theorem 2.1.1 to $\mathbf{R^{\mathfrak{q}}}_{\mathbb{Z}}$ to get part (i) of the Theorem.

To prove part (ii) we consider $\mathbf{X} := \mathbf{R^{\mathfrak{q}}}_{\mathbb{Z}[\mathbf{1/2}]}$ and apply Theorem 2.1.2. To verify assumptions (i),(ii) and (iii) we use the universal coefficient theorem and Theorems 3.4.1, 3.4.2, 3.4.3 and 3.4.4 in Section 3.4.1.

The secondary stability part follows by considering $\mathbf{X} := \mathbf{R^{\mathfrak{q}}}_{\mathbb{F}_2}$ and applying Theorem 2.1.3, where all assumptions needed hold by Theorems 3.4.1, 3.4.2, 3.4.3 and 3.4.4. □

As we said in Section 1.2 we can also prove that the bound of Theorem A is (almost) optimal.

Lemma 3.3.2. *For all* $k \geq 1$ *and for all* $\varepsilon,\delta \in \{0,1\}$ *the map*

$$\sigma_\varepsilon \cdot - : H_{2k}(B\Gamma^{1/2}_{3k-1,1}[\delta-\varepsilon];\mathbb{Q}) \to H_{2k}(B\Gamma^{1/2}_{3k,1}[\delta];\mathbb{Q})$$

is not surjective.

Proof. Suppose for a contradiction that it was surjective for some $k \geq 1$, $\varepsilon,\delta \in \{0,1\}$. By the transfer, $H_{2k}(B\Gamma^{1/2}_{3k,1}[\delta];\mathbb{Q}) \to H_{2k}(B\Gamma_{3k,1};\mathbb{Q})$ is also surjective since the spin mapping class groups are finite index subgroups of the mapping class groups.

Thus, the stabilisation map $\sigma \cdot - : H_{2k}(B\Gamma_{3k-1,1};\mathbb{Q}) \to H_{2k}(B\Gamma_{3k,1};\mathbb{Q})$ (where we are using the notation of [GKRW19]) must be surjective. By the universal coefficient theorem, $H^{2k}(B\Gamma_{3k,1};\mathbb{Q}) \to H^{2k}(B\Gamma_{3k-1,1};\mathbb{Q})$ is injective; which is false by the computations in [GKRW19, Proof of Corollary 5.8]. □

Thus, the stability bound obtained in Theorem A is optimal up to an additive constant of at most $4/3$.

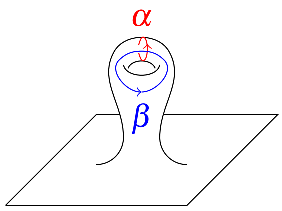
3.4 First homology computations

In this section we will explain the first homology computations of spin mapping class groups and quadratic symplectic groups. These computations are done using GAP and we have included the code that we used. By Chapter 2 and the previous two sections of this thesis, these first homology computations are important in getting homological stability results.

3.4.1 Spin mapping class groups

g = 1

Consider the simple closed curves α, β in $\Sigma_{1,1}$ shown below, with orientations chosen so that $\alpha \cdot \beta = +1$. Let $a, b \in \Gamma_{1,1}$ be the isotopy classes represented by the right-handed Dehn twists along the curves α and β respectively.



The set $Q(1)$ of quadratic refinements of $H_1(\Sigma_{1,1};\mathbb{Z})$ is $\{q_{0,0}, q_{1,0}, q_{0,1}, q_{1,1}\}$, where $q_{i,j}$ satisfies $q_{i,j}(a) = i$ and $q_{i,j}(b) = j$. The first three of them have Arf invariant 0 and the forth one has Arf invariant 1.

Thus, we can get explicit models of $\Gamma_{1,1}^{1/2}[\varepsilon]$ via $\Gamma_{1,1}^{1/2}[0] := \mathrm{Stab}_{\Gamma_{1,1}}(q_{0,0})$ and $\Gamma_{1,1}^{1/2}[1] := \mathrm{Stab}_{\Gamma_{1,1}}(q_{1,1})$.

Theorem 3.4.1. *(i) $H_1(\Gamma_{1,1};\mathbb{Z}) = \mathbb{Z}\{\tau\}$, where τ is represented by both a and b.*

(ii) $H_1(\Gamma_{1,1}^{1/2}[0];\mathbb{Z}) = \mathbb{Z}\{x\} \oplus \mathbb{Z}\{y\}$, where x is represented by a^{-2} and y is represented by aba^{-1}. Moreover, b^{-2} also represents the class x.

(iii) $H_1(\Gamma_{1,1}^{1/2}[1];\mathbb{Z}) = \mathbb{Z}\{z\}$, where z is represented by a.

(iv) Under the inclusion $\Gamma_{1,1}^{1/2}[0] \subset \Gamma_{1,1}$ we have $x \mapsto -2\tau$ and $y \mapsto \tau$.

(v) Under the inclusion $\Gamma_{1,1}^{1/2}[1] \subset \Gamma_{1,1}$ we have $z \mapsto \tau$.

Proof. Parts (i), (ii) and (iii) immediately imply parts (iv) and (v). Moreover, parts (i) and (iii) are equivalent since there is a unique quadratic refinement of Arf invariant 1 so $\Gamma_{1,1}^{1/2}[1] = \Gamma_{1,1}$.

By [Kor02, Page 8] we have the presentation $\Gamma_{1,1} = \langle a,b |\ aba = bab\rangle$.

Abelianizing this presentation gives part (i). We will prove (ii) by finding a presentation for $\Gamma_{1,1}^{1/2}[0]$ and then abelianizing it using GAP.

The right action of a,b on the set of quadratic refinements of Arf invariant 0 is given by: $a^*(q_{0,0}) = q_{0,1}$, $a^*(q_{0,1}) = q_{0,0}$, $a^*(q_{1,0}) = q_{1,0}$, $b^*(q_{0,0}) = q_{1,0}$, $b^*(q_{0,1}) = q_{0,1}$, $b^*(q_{1,0}) = q_{0,0}$. (This is shown by analysing the effect on homology of the corresponding right-handed Dehn twists.)

We will denote $q_{0,0} := 1$, $q_{0,1} := 2$ and $q_{1,0} := 3$, so that a acts on $\{1,2,3\}$ by the permutation (12) and b acts on $\{1,2,3\}$ by the permutation (13).

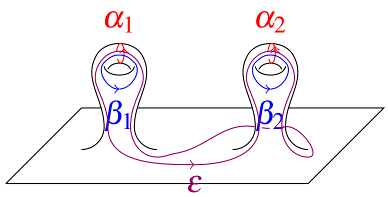
```
gap> F := FreeGroup("a","b");
gap> AssignGeneratorVariables(F);
gap> rel := [a*b*a*b^-1*a^-1*b^-1];
gap> G := F/rel;
% This defines group G=\Gamma_{1,1}
gap> Q := Group((1,2),(1,3));
gap> hom := GroupHomomorphismByImages
(G,Q,GeneratorsOfGroup(G),GeneratorsOfGroup(Q));
[ a, b ] -> [ (1,2), (1,3) ]
% permutation representation of G on the elements of Q(1)
of Arf invariant 0.
gap> S := PreImage(hom,Stabilizer(Q,1));
% S is the spin mapping class group \Gamma_{1,1}{1/2}[0]
gap> genS := GeneratorsOfGroup(S);
[ a^-2, b^-2, a*b*a^-1 ] % explicit generators for S.
gap> iso :=  IsomorphismFpGroupByGenerators(S,genS);
[ a^-2, b^-2, a*b*a^-1 ] -> [ F1, F2, F3 ]
gap> s := ImagesSource(iso);
% explicit fp group isomorphic to S via iso, so that the generators correspond via iso.
gap> RelatorsOfFpGroup(s);
[ F3*F1*F3^-1*F2^-1, F3*F2*F3^-1*F2*F1^-1*F2^-1 ]
% explicit relations for the group S. Thus we have a presentation of S.
gap> AbelianInvariants(s);
[ 0, 0 ]
% Thus, the abelianization of S is isomorphic to Z \oplus Z.
gap> q := MaximalAbelianQuotient(s);
gap> AbS := ImagesSource(q);
gap> GeneratorsOfGroup(AbS);
[ f1, f2, f3 ]
gap> RelatorsOfFpGroup(AbS);
```

```
[ f1^-1*f2^-1*f1*f2, f1^-1*f3^-1*f1*f3, f2^-1*f3^-1*f2*f3, f1 ]
% Description of the abelianization AbS of S as a f.p. group. Thus, f2 and f3 are the free generators of AbS.
gap> Image(q,s.1);
f2
% Thus, s.1 corresponds to f2 under abelianization.
% i.e. a^-2 is one generator.
gap> Image(q,s.3);
f3
% Thus, s.3 corresponds to f3 under abelianization.
% i.e. aba^-1 is the other generator.
gap> Image(q,s.2)=Image(q,s.1);
true
% Thus, b^-2 and a^-2 agree in the abelianization.
```

□

g = 2

Consider the simple closed curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \varepsilon$ as drawn below, and the corresponding right-handed Dehn twists along them, denoted by $a_1, a_2, b_1, b_2, e \in \Gamma_{2,1}$ respectively.



The set of quadratic refinements $Q(2)$ is $\{q_{i_1,j_1,i_2,j_2} : i_1, j_1, i_2, j_2 \in \{0,1\}\}$, where $q = q_{i_1,j_1,i_2,j_2}$ satisfies $q(\alpha_1) = i_1$, $q(\alpha_2) = i_2$, $q(\beta_1) = j_1$ and $q(\beta_2) = j_2$.

Now we fix models of $\Gamma_{2,1}^{1/2}[\varepsilon]$ via $\Gamma_{2,1}^{1/2}[0] := \mathrm{Stab}_{\Gamma_{2,1}}(q_{0,0,0,0})$, $\Gamma_{2,1}^{1/2}[1] := \mathrm{Stab}_{\Gamma_{2,1}}(q_{1,0,1,1})$.

Theorem 3.4.2. *(i)* $H_1(\Gamma_{2,1};\mathbb{Z}) = \mathbb{Z}/10\{\sigma\cdot\tau\}$*, and* $\sigma\cdot\tau$ *is represented by* a_1*.*

(ii) $H_1(\Gamma_{2,1}^{1/2}[0];\mathbb{Z}) = \mathbb{Z}\{A\} \oplus \mathbb{Z}/2\{B\}$*, where A is represented by* $a_1b_1a_1^{-1}b_1b_2e^{-1}$ *and B is represented by* $(a_1b_1a_1)^2eb_2^{-1}b_1^{-1}$*.*

(iii) $H_1(\Gamma_{2,1}^{1/2}[1];\mathbb{Z}) = \mathbb{Z}/80\{C\}$*, where C is represented by* a_1*.*

(iv) Under the inclusion $\Gamma_{2,1}^{1/2}[0] \subset \Gamma_{2,1}$ *we have* $A \mapsto 2\sigma\cdot\tau$ *and* $B \mapsto 5\sigma\cdot\tau$*.*

(v) Under the inclusion $\Gamma_{2,1}^{1/2}[1] \subset \Gamma_{2,1}$ *we have* $C \mapsto \sigma\cdot\tau$*.*

Proof. We say that the pair (u,v) satisfies the *braid relation* if $uvu = vuv$. By [Waj99, Theorem 2] there is a presentation

$$\Gamma_{2,1} = \langle a_1,a_2,b_1,b_2,e | R_1 \sqcup R_2 \sqcup \{(b_1a_1ea_2)^5 = b_2a_2ea_1b_1^2a_1ea_2b_2\}\rangle,$$

where R_1 says that $(a_1,b_1),(a_2,b_2),(a_1,e),(a_2,e)$ satisfy the braid relation, and R_2 says that each of $(a_1,a_2),(b_1,b_2),(a_1,b_2),(a_2,b_1),(b_1,e),(b_2,e)$ commutes.

Part (i) follows from abelianizing the above presentation of $\Gamma_{2,1}$, and it is compatible with [GKRW19, Lemma 3.6].

For part (ii) we will describe the (right) action of a_1,b_1,a_2,b_2,e on the set of 10 quadratic refinements of Arf invariant 0, which we will label as $q_{0,0,0,0} := 1$, $q_{0,0,0,1} := 2$, $q_{1,0,0,0} := 3$, $q_{0,0,1,0} := 4$, $q_{1,0,0,1} := 5$, $q_{1,0,1,0} := 6$, $q_{0,1,0,0} := 7$, $q_{0,1,1,0} := 8$, $q_{0,1,0,1} := 9$, $q_{1,1,1,1} := 10$. The explicit action of each generator as a permutation in S_{10} can be found in the GAP computation below. We use GAP to find presentation of $\Gamma_{2,1}^{1/2}[0]$ and its first homology group as follows.

```
gap> F := FreeGroup("a","x","b","y","c");
% a means a1, x means a2, b means b1, y means a2, c means e.
gap> AssignGeneratorVariables(F);
gap> rel := [ a*b*a*b^-1*a^-1*b^-1, x*y*x*y^-1*x^-1*y^-1,
a*c*a*c^-1*a^-1*c^-1, x*c*x*c^-1*x^-1*c^-1, a*x*a^-1*x^-1,
a*y*a^-1*y^-1, b*x*b^-1*x^-1, b*y*b^-1*y^-1, b*c*b^-1*c^-1,
y*c*y^-1*c^-1, (y*x*c*a*b^2*a*c*x*y)*(b*a*c*x)^-5];
gap> G := F/rel;
% This defines group G=\Gamma_{2,1}
gap> Q := Group((1,7)(2,9)(4,8),(1,2)(3,5)(7,9),(1,3)(2,5)(4,6),
(1,4)(3,6)(7,8),(1,6)(3,4)(9,10));
gap> hom := GroupHomomorphismByImages
(G,Q,GeneratorsOfGroup(G),GeneratorsOfGroup(Q));
[ a, x, b, y, c ] -> [ (1,7)(2,9)(4,8), (1,2)(3,5)(7,9),
(1,3)(2,5)(4,6), (1,4)(3,6)(7,8), (1,6)(3,4)(9,10) ]
% permutation representation of G on the elements of Q(2) of Arf invariant 0.
gap> S := PreImage(hom,Stabilizer(Q,1));
% This is the spin mapping class group \Gamma_{2,1}{1/2}[0]
gap> genS := GeneratorsOfGroup(S);
[ a^-2, x^-2, b^-2, y^-2, c^-2, a*b*a^-1, a*c*a^-1, x*y*x^-1,
x*c*x^-1, b*y*c^-1 ]
% Generators for S: there are 10 of them called s.1,...,s.10.
gap> iso :=  IsomorphismFpGroupByGenerators(S,genS);
gap> s := ImagesSource(iso);
<fp group on the generators [ F1, F2, F3, F4, F5, F6, F7, F8,
F9, F10 ]>
```

```
gap> AbelianInvariants(s);
[ 0, 2 ]
% Thus, the abelianization of S is isomorphic to Z \oplus Z/2.
gap> q := MaximalAbelianQuotient(s);
gap> AbS := ImagesSource(q);
gap> GeneratorsOfGroup(AbS);
[ f1, f2, f3, f4, f5, f6, f7, f8, f9, f10 ]
% this gives a description of the abelianization AbS of S.
gap> RelatorsOfFpGroup(AbS);
[ f1^-1*f2^-1*f1*f2, f1^-1*f3^-1*f1*f3, f1^-1*f4^-1*f1*f4,
  f1^-1*f5^-1*f1*f5, f1^-1*f6^-1*f1*f6, f1^-1*f7^-1*f1*f7,
  f1^-1*f8^-1*f1*f8, f1^-1*f9^-1*f1*f9, f1^-1*f10^-1*f1*f10,
  f2^-1*f3^-1*f2*f3, f2^-1*f4^-1*f2*f4, f2^-1*f5^-1*f2*f5,
  f2^-1*f6^-1*f2*f6, f2^-1*f7^-1*f2*f7, f2^-1*f8^-1*f2*f8,
  f2^-1*f9^-1*f2*f9, f2^-1*f10^-1*f2*f10, f3^-1*f4^-1*f3*f4,
  f3^-1*f5^-1*f3*f5, f3^-1*f6^-1*f3*f6, f3^-1*f7^-1*f3*f7,
  f3^-1*f8^-1*f3*f8, f3^-1*f9^-1*f3*f9, f3^-1*f10^-1*f3*f10,
  f4^-1*f5^-1*f4*f5, f4^-1*f6^-1*f4*f6, f4^-1*f7^-1*f4*f7,
  f4^-1*f8^-1*f4*f8, f4^-1*f9^-1*f4*f9, f4^-1*f10^-1*f4*f10,
  f5^-1*f6^-1*f5*f6, f5^-1*f7^-1*f5*f7, f5^-1*f8^-1*f5*f8,
  f5^-1*f9^-1*f5*f9, f5^-1*f10^-1*f5*f10, f6^-1*f7^-1*f6*f7,
  f6^-1*f8^-1*f6*f8, f6^-1*f9^-1*f6*f9, f6^-1*f10^-1*f6*f10,
  f7^-1*f8^-1*f7*f8, f7^-1*f9^-1*f7*f9, f7^-1*f10^-1*f7*f10,
  f8^-1*f9^-1*f8*f9, f8^-1*f10^-1*f8*f10, f9^-1*f10^-1*f9*f10,
  f1, f2, f3, f4, f5, f6, f7, f8, f9^2 ]
% Thus, f9 generates the Z/2 and f10 generates the Z.
gap> PreImagesRepresentative(q,AbS.9);
(F9*F5^-1)^2*F10^-1
gap> Image(q,(s.9*s.5^-1)^2*s.10^-1)=AbS.9;
true
% Element of S generating the Z/2 summand, where the Fi index the ten generators of S in order. F9 and F6
    agree on abelianization, and so do F5 and F1. Thus, we can replace this generator by
    (s.6*s.1^-1)^2*s.10^-1, which gives B by substituting what s.1,s.6 and s.10 are.
gap> Image(q,s.5^-1*s.10^-1*s.9)=AbS.10;
true
% Generator of the Z summand. This gives A.
```

Part (iii) is done similarly to part (ii): there are 6 quadratic refinements of Arf invariant 1, which we index as: $q_{1,0,1,1} := 1$, $q_{0,0,1,1} := 2$, $q_{0,1,1,1} := 3$, $q_{1,1,0,1} := 4$, $q_{1,1,0,0} := 5$, $q_{1,1,1,0} := 6$. The explicit action of each generator as a permutation on S_6 can be found in the GAP computation below. The group G in the computation represents $\Gamma_{2,1}$ and it is input in the same way as above.

```
gap> Q := Group((2,3),(4,5),(1,2),(5,6),(3,4));
gap> hom := GroupHomomorphismByImages
(G,Q,GeneratorsOfGroup(G),GeneratorsOfGroup(Q));
[ a, x, b, y, c ] -> [ (2,3), (4,5), (1,2), (5,6), (3,4) ]
% permutation representation of G on the elements of Q(2) of Arf invariant 1
gap> S := PreImage(hom,Stabilizer(Q,1));
% This is the spin mapping class group \Gamma_{2,1}{1/2}[1]
gap> genS := GeneratorsOfGroup(S);
[ a, x, b^-2, y, c ]
gap> iso := IsomorphismFpGroupByGenerators(S,genS);
gap> s := ImagesSource(iso);
gap> AbelianInvariants(s);
[ 5, 16 ] % this shows that the abelianization of s is isomorphic to Z/5 \oplus Z/16 \cong Z/80.
gap> q := MaximalAbelianQuotient(s);
gap> AbS := ImagesSource(q);
gap> Image(q,s.1);
f1
gap> Image(q,s.1)=AbS.5;
true
gap> Order(AbS.5);
80
% This gives the required class $C$.
```

Finally, parts (iv) and (v) follow from the explicit description of A, B, C plus using the relations in the abelianization of $\Gamma_{2,1}$. □

Stabilizations, Browder brackets and the $Q^1_{\mathbb{Z}}(-)$-operation

Theorem 3.4.3. *(i)* $[\sigma,\sigma] = 4\sigma\cdot\tau$.

(ii) $Q^1_{\mathbb{Z}}(\sigma) = 3\sigma\cdot\tau$.

(iii) $x\cdot\sigma_0 = 4A$ *and* $y\cdot\sigma_0 = 3A+B$.

(iv) $x\cdot\sigma_1 = 28C$ *and* $y\cdot\sigma_1 = C$.

(v) $z\cdot\sigma_0 = C$.

(vi) $z\cdot\sigma_1 = 3A+B$.

(vii) $[\sigma_0,\sigma_0] = -8A$, $[\sigma_1,\sigma_1] = 72A$ *and* $[\sigma_0,\sigma_1] = 24C$.

(viii) $Q^1_{\mathbb{Z}}(\sigma_0) = 4A+B$ *and* $Q^1_{\mathbb{Z}}(\sigma_1) = -36A+B$.

Proof. Parts (i) and (ii) appear in [GKRW19, Lemma 3.6].

For part (iii) we use the same GAP computation as in Theorem 3.4.2, (ii). Right stabilization by σ_0 sends $q_{0,0} \mapsto q_{0,0,0,0}$ and $a \mapsto a_1$, $b \mapsto b_1$. Therefore, $x \cdot \sigma_0$ is represented by a_1^{-2} and $y \cdot \sigma_0$ is represented by $a_1 b_1 a_1^{-1}$

```
gap> Image(q,s.1);
f1*f9^-2*f10^4
gap> Image(q,s.6);
f6*f9^-3*f10^3
```

This says that under abelianization a_1^{-2} (which is $s.1$) is mapped to $f1 * f9^{-2} * f10^4$, which is $4A$ by the GAP computation in the proof of Theorem 3.4.2, (ii). Also, since $s.6$ means $a_1 b_1 a_1^{-1}$ then we get $y \cdot \sigma_0 = 3A + B$.

Proof of (iv):

Observe that when we stabilize by $- \cdot \sigma_1$ we send $q_{0,0} \mapsto q_{0,0,1,1}$, whereas our choice of quadratic refinement of Arf invariant 1 is $q_{0,1,1,1}$. To fix this we will use conjugation: $b_1 \in \Gamma_{2,1}$ satisfies $b_1^*(q_{1,0,1,1}) = q_{0,0,1,1}$ and so

$$\mathrm{Stab}_{\Gamma_{2,1}}(q_{0,0,1,1}) \xrightarrow{b_1 \cdot - \cdot b_1^{-1}} \mathrm{Stab}_{\Gamma_{2,1}}(q_{1,0,1,1})$$

is an isomorphism.

This is non-canonical, but its action in group homology is canonical: If $u \in \Gamma_{2,1}$ satisfies $u^*(q_{1,0,1,1}) = q_{0,0,1,1}$ then the maps $u \cdot - \cdot u^{-1}$ and $b_1 \cdot - \cdot b_1^{-1}$ differ by conjugation by $b_1 u^{-1} \in \mathrm{Stab}_{\Gamma_{2,1}}(q_{1,0,1,1})$, which acts trivially in group homology. Thus, for homology computations, $x \cdot \sigma_1$ is represented by $b_1 a_1^{-2} b_1^{-1} \in \Gamma_{2,1}^{1/2}[1]$ and $y \cdot \sigma_1$ is represented by $b_1 a_1 b_1 a_1^{-1} b_1^{-1}$.

Now we use GAP computation in the proof of Theorem 3.4.2,(iii) to see where these elements map

```
gap> Image(iso,G.3*G.1^-2*G.3^-1);
F1^-1*F3*F1
% Expression for b1*a1^-1*b1^-1 \in S in terms of its generators.
gap> Image(q,Image(iso,G.3*G.1^-2*G.3^-1))=Image(q,s.1)^28;
true
% This shows that indeed b1*a1^-2*b1^-1 is the element 28 in the abelianization.
gap> Image(iso,G.3*G.1*G.3*G.1^-1*G.3^-1);
F1
gap> Image(q, Image(iso,G.3*G.1*G.3*G.1^-1*G.3^-1))=Image(q,s.1);
true
% This shows that b_1 a_1 b_1 a_1^{-1} b_1^{-1} is the element 1.
```

Part (v) is similar to the previous part: right stabilization by σ_0 sends $q_{1,1} \mapsto q_{1,1,0,0}$, and so we need to conjugate by $b_1a_1ea_2 \cdot - \cdot (b_1a_1ea_2)^{-1}$ to go to $\Gamma^{1/2}_{2,1}[0]$. Also, z is represented by a, so $z \cdot \sigma_0$ is represented by $b_1a_1ea_2a_1(b_1a_1ea_2)^{-1}$.

Using GAP:

```
gap> Image(iso,(G.3*G.1*G.5*G.2)*G.1*(G.3*G.1*G.5*G.2)^-1);
F3^-1*F5*F3
gap> Image(q, Image(iso,(G.3*G.1*G.5*G.2)*G.1*(G.3*G.1*G.5*G.2)^-1))=Image(q,s.1);
true
```

Part (vi) follows from the following GAP computation:

```
gap> Image(iso,(G.1*G.2*G.5)*G.1*(G.1*G.2*G.5)^-1);
F9
gap> Image(q,Image(iso,(G.1*G.2*G.5)*G.1*(G.1*G.2*G.5)^-1))=Image(q,s.9);
true
```

To prove part (vii) we will need the following claim

Claim. The element $-[\sigma,\sigma] \in \Gamma_{2,1}$ is represented by $(b_2a_2ea_1b_1)^6(a_1b_1)^6(a_2b_2)^{-6}$

Proof. By [GKRW19, Lemma 3.6, Figure 4] we can write $-[\sigma,\sigma]$ as $t_wt_u^{-1}t_v^{-1}$, where u,v,w are the curves called "a", "b" and "c" respectively in [GKRW19], and t_w,t_u,t_v are the corresponding right-handed Dehn twists along them. Now we use [Waj99, Lemma 21 (iii)] to write each of t_u,t_v,t_w in terms of the generators, yielding the following: $t_u = (a_1b_1)^6$, $t_v = (a_2b_2)^6$ and $t_w = (b_2a_2ea_1b_1)^6$. □

The above element lies in $\mathrm{Stab}_{\Gamma_{2,1}}(q)$ for any quadratic refinement q because each of the curves u,v,w is disjoint from the curves $\alpha_1,\beta_1,\alpha_2,\beta_2$, and hence t_u,t_v,t_w do not affect the value of q along the standard generators of $H_1(\Sigma_{2,1};\mathbb{Z})$. Thus, $[\sigma_i,\sigma_j] \in H_1(B\Gamma^{1/2}_{2,1}[i+j(\mathrm{mod}\,2)];\mathbb{Z})$ is represented by

$$(b_2a_2ea_1b_1)^6(a_1b_1)^6(a_2b_2)^{-6} \in \mathrm{Stab}_{\Gamma_{2,1}}(q_{i,i,j,j}),$$

and then we conjugate this element so that it lies in our fixed choices of stabilizers.

For $-[\sigma_0,\sigma_0]$ we don't need to conjugate, so we find

```
gap> Image(q, Image(iso,(G.4*G.2*G.5*G.1*G.3)^6*(G.1*G.3)^-6
    *(G.2*G.4)^-6))=AbS.10^8;
true
% this means that abelianization is (8,0) \in Z \oplus Z/2.
```

For $-[\sigma_1,\sigma_1]$ need to conjugate by a_1a_2e, and we find

```
gap> Image(q,Image(iso,G.1*G.2*G.5*(G.4*G.2*G.5*G.1*G.3)^6
    *(G.1*G.3)^-6*(G.2*G.4)^-6*(G.1*G.2*G.5)^-1))=AbS.10^-72;
true
```

For $-[\sigma_0,\sigma_1]$ we need to conjugate by b_1, and we get

```
gap> Image(q,Image(iso,G.3*(G.4*G.2*G.5*G.1*G.3)^6
    *(G.1*G.3)^-6*(G.2*G.4)^-6*G.3^-1))=AbS.5^56;
true
```

Finally, to prove (viii) we use that $2Q^1_{\mathbb{Z}}(\sigma_\varepsilon) = -[\sigma_\varepsilon,\sigma_\varepsilon]$, which follows from the discussion in [GKRW19, Page 9].

For $\varepsilon = 0$, Theorem 3.4.2 together with part (vii) of this theorem say that $Q^1_{\mathbb{Z}}(\sigma_0)$ is either $4A$ or $4A+B$. But, by part (ii) of this theorem we know that it must map to $3\sigma\cdot\tau \in H_1(\Gamma_{2,1};\mathbb{Z})$. Using Theorem 3.4.2,(iv) we get that the answer must be $4A+B$. The computation of the case $\varepsilon = 1$ is similar. □

g = 3

Theorem 3.4.4. *(i)* $H_1(\Gamma^{1/2}_{3,1}[0];\mathbb{Z}) \cong \mathbb{Z}/4$, *where* $A\cdot\sigma_0 = y\cdot\sigma_0^2$ *is a generator and* $B\cdot\sigma_0 = 2A\cdot\sigma_0$.

(ii) $Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_0 = 2y\cdot\sigma_0^2$.

(iii) $H_1(\Gamma^{1/2}_{3,1}[1];\mathbb{Z}) \cong \mathbb{Z}/4$, *where* $y\cdot\sigma_0\cdot\sigma_1 = z\cdot\sigma_0^2$ *is a generator.*

(iv) $Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_1 = 2y\cdot\sigma_0\cdot\sigma_1 = Q^1_{\mathbb{Z}}(\sigma_1)\cdot\sigma_0$ *and* $Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_0 = Q^1_{\mathbb{Z}}(\sigma_1)\cdot\sigma_1$.

Proof. It follows from [Waj99, Theorem 1] that $\Gamma_{3,1}$ has a presentation with generators $a_1,a_2,a_3,b_1,b_2,b_3,e_1,e_2$, where the a_i,b_i are defined as in the cases $g = 1,2$; e_1 is what was called e in the $g = 2$ case using the first two handles, and e_2 is defined analogously, but using the second and third handles instead.

To prove (i) and (ii) we fix our quadratic refinement of Arf invariant 0 to be the one evaluating to 0 on all the α_i's and β_i's. In this case the strategy to find a presentation for $\Gamma^{1/2}_{3,1}[0]$ is different: instead of computing the action on quadratic refinements we will write down elements of $\Gamma^{1/2}_{3,1}[0]$ (inspired by expressions from previous computations) and check that the subgroup they generate has index 36 inside $\Gamma_{3,1}$, and hence that it must agree with $\Gamma^{1/2}_{3,1}[0]$.

```
gap> F := FreeGroup("a1","a2","a3","b1","b2","e1","e2");
gap> AssignGeneratorVariables(F);
gap> rel := [a1*b1*a1*b1^-1*a1^-1*b1^-1, a2*b2*a2*b2^-1*a2^-1*b2^-1,
a1*e1*a1*e1^-1*a1^-1*e1^-1, a2*e1*a2*e1^-1*a2^-1*e1^-1,
a2*e2*a2*e2^-1*a2^-1*e2^-1, a3*e2*a3*e2^-1*a3^-1*e2^-1,
a1*a2*a1^-1*a2^-1, a1*a3*a1^-1*a3^-1, a3*a2*a3^-1*a2^-1,
b1*b2*b1^-1*b2^-1, a1*b2*a1^-1*b2^-1, a2*b1*a2^-1*b1^-1,
a3*b2*a3^-1*b2^-1, a3*b1*a3^-1*b1^-1, b1*e1*b1^-1*e1^-1,
b2*e1*b2^-1*e1^-1, b1*e2*b1^-1*e2^-1, b2*e2*b2^-1*e2^-1,
a1*e2*a1^-1*e2^-1, a3*e1*a3^-1*e1^-1, e1*e2*e1^-1*e2^-1,
(b1*a1*e1*a2)^-5*b2*a2*e1*a1*b1^2*a1*e1*a2*b2, ((b2*a2*e1*b1^-1)*
(e2*a2*a3*e2)* (a2*e1*a1*b1)^-1*b2*(a2*e1*a1*b1) *(e2*a2*a3*e2)^-1*
(b1*e1^-1*a2^-1*b2^-1)*a1*a2*a3)^-1*(a2*e1*a1*b1)^-1*b2*
(a2*e1*a1*b1)* (e2*a2*a3*e2)* (a2*e1*a1*b1)^-1*b2*(a2*e1*a1*b1)
*(e2*a2*a3*e2)^-1*(e1*a1*a2*e1)*(e2*a2*a3*e2)*(a2*e1*a1*b1)^-1*
b2*(a2*e1*a1*b1) *(e2*a2*a3*e2)^-1*(e1*a1*a2*e1)^-1];
% this encodes the presentation.
gap> G := F/rel;
% this is \Gamma_{3,1}
gap> H := Subgroup(G,[G.1^-2,G.2^-2,G.3^-2,G.4^-2,G.5^-2,G.6^-2,
G.7^-2,G.4*G.5*G.6^-1,G.5*G.2*G.5^-1,G.4*G.1*G.4^-1,G.2*G.7*G.2^-1,
G.7*G.3*G.7^-1]);
% H lies inside Stab_{\Gamma_{3,1}}(q_{000000}) as each generator of H fixes q_{000000}.
gap> Index(G,H);
36
% Thus, H=\Gamma_{3,1}^{1/2}[0] as there are 36 elements in Q(3) of Arf invariant 0.
gap> AbelianInvariants(H);
[ 4 ]
% This gives H_1(\Gamma_{3,1}^{1/2}[0]) \cong Z/4.
gap> genH := GeneratorsOfGroup(H);
gap> iso := IsomorphismFpGroupByGenerators(H,genH);
gap> S := ImagesSource(iso);
% f.p. group isomorphic to H.
gap> q := MaximalAbelianQuotient(S);
gap> AbS := ImagesSource(q);
gap> Order(Image(q,Image(iso,G.1*G.4*G.1^-1)));
4  % Thus, y*\sigma_0^2 is a generator of H_1(\Gamma_{3,1}{1/2}[0])
gap> Order( Image(q,Image(iso,(G.1*G.4*G.1)^2*G.6*G.5^-1*G.4^-1)));
2
% Thus B* \sigma_0 =2 mod 4 by definition of B.
```

Now, to finish we need to check two things: The first one is that $A\cdot\sigma_0 = y\cdot\sigma_0^2$ is a generator: By Theorem 3.4.3(ii) we have $y\cdot\sigma_0 = 3A+B$ so using the above GAP computations we find $A\cdot\sigma_0 = y\cdot\sigma_0^2$.

By Theorem 3.4.3 (viii) we have $Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_0 = 4A\cdot\sigma_0 + B\cdot\sigma_0 = B\cdot\sigma_0$, as required.

To prove parts (iii) and (iv) we fix our quadratic refinement of Arf invariant 1 to be the one with value 1 in all the a_i and b_i. Now we use GAP (F,G are as before, so we will not copy that part again) and a similar idea as above to get the result.

```
gap> H := Subgroup(G,[G.1,G.2,G.3,G.4,G.5,G.6^-2,G.7^-2,G.6*G.4*G.6^-1,
G.6*G.5*G.6^-1,G.7*G.5*G.7^-1,(G.6*G.2*G.1)*G.3*(G.6*G.2*G.1)^-1,
(G.6*G.2*G.1)*G.7*(G.6*G.2*G.1)^-1, (G.6*G.2*G.1)*(G.6*G.5*G.4)*
(G.6*G.2*G.1)^-1,(G.7*G.3*G.2)*G.1*( G.7*G.3*G.2)^-1,
( G.7*G.3*G.2)*G.6*( G.7*G.3*G.2)^-1]);
% H lies inside Stab_{\Gamma_{3,1}}(q_{111111}) as each generator of H fixes q_{111111}.
gap> Index(G,H);
28 %Thus, H=\Gamma_{3,1}{1/2}[1] as there are 28 elements in Q(3) of Arf invariant 1.
gap> AbelianInvariants(H);
[ 4 ]
% This gives H_1(\Gamma_{3,1}{1/2}[1])=Z/4.
```

This shows that $H_1(\Gamma^{1/2}_{3,1}[1];\mathbb{Z}) \cong \mathbb{Z}/4$.

By Theorem A (i), the map $\sigma_\varepsilon\cdot - : H_1(\Gamma^{1/2}_{g-1,1}[\delta-\varepsilon];\mathbb{Z}) \to H_1(\Gamma^{1/2}_{g,1}[\delta];\mathbb{Z})$ is surjective for $g\geq 4$ and any ε,δ. (The proof of Theorem A(i) is independent of these first homology computations.)

Moreover, the stable values $H_1(\Gamma^{1/2}_{\infty,1}[\delta];\mathbb{Z})$ are both isomorphic to $\mathbb{Z}/4$ by [RW12, Theorem 1.4] plus [RW14, Theorem 2.14]. Thus the groups $H_1(\Gamma^{1/2}_{g,1}[\delta];\mathbb{Z})$ are stable for any $g\geq 3$ and any $\delta\in\{0,1\}$.

Since $\sigma_0^2=\sigma_1^2$ then $(Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_1)\cdot\sigma_1 = Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_0^2 = 2y\cdot\sigma_0^3 = (2y\sigma_0\cdot\sigma_1)\cdot\sigma_1$, and by the above stability result $Q^1_{\mathbb{Z}}(\sigma_0)\cdot\sigma_1 = 2y\cdot\sigma_0\cdot\sigma_1$.

Also, $y\cdot\sigma_0^3 = (y\cdot\sigma_0\cdot\sigma_1)\cdot\sigma_1$ is a generator of $H_1(\Gamma^{1/2}_{4,1}[0];\mathbb{Z})$ by the stability plus part (i) of this theorem. Thus, $y\cdot\sigma_0\cdot\sigma_1$ is a generator of $H_1(\Gamma^{1/2}_{3,1}[1];\mathbb{Z})$ by applying stability. Using that $z\cdot\sigma_0 = y\cdot\sigma_1$ (by Theorem 3.4.3) we find $z\cdot\sigma_0^2 = y\cdot\sigma_1\cdot\sigma_0$.

Finally, by Theorem 3.4.3, $Q^1_{\mathbb{Z}}(\sigma_1) - Q^1_{\mathbb{Z}}(\sigma_0) = -40A$, so any stabilization of this vanishes because it lives in a 4-torsion group. □

3.4.2 Quadratic symplectic groups

The proofs of this section will be very similar to the ones of Section 3.4.1, but using the explicit presentations of $Sp_{2g}(\mathbb{Z})$ given in [Lu92]. The computation in Theorems 3.4.6, 3.4.7 and 3.4.8 about the first homology of the quadratic symplectic groups of Arf invariant 1 is used in [KRW21b, Section 4.1].

Remark 3.4.5. *In [Lu92] they write matrices using a different basis. We will change the matrices given [Lu92] to our choice of basis of Section 3.2 without further notice in all the following computations.*

g = 1

Theorem 3.4.6. *(i) $H_1(Sp_2(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/12\{t\}$, where t is represented by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$.*

(ii) $H_1(Sp_2^0(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}\{\mu\} \oplus \mathbb{Z}/4\{\lambda\}$, where μ is represented by $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right) \in Sp_2^0(\mathbb{Z})$ and λ is represented by $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) \in Sp_2^0(\mathbb{Z})$.

(iii) $H_1(Sp_2^1(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/12\{t'\}$, where t' is represented by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in Sp_2^1(\mathbb{Z})$.

Proof. By [Lu92, Theorem 1] we have

$$Sp_2(\mathbb{Z}) = \langle L, N \mid (LN)^2 = N^3, N^6 = 1\rangle$$

where $L = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)$.

We will use the same notation as in Section 3.4.1 for the quadratic refinements, where now α, β are the standard hyperbolic basis of $(\mathbb{Z}^2, \Omega_1)$. We let $Sp_2^0(\mathbb{Z}) := \mathrm{Stab}_{Sp_2(\mathbb{Z})}(q_{0,0})$ and $Sp_2^1(\mathbb{Z}) := \mathrm{Stab}_{Sp_2(\mathbb{Z})}(q_{1,1})$. We then compute the action of L, N on the set of quadratic refinements of each invariant (see the GAP formulae below).

Since there is a unique quadratic refinement of Arf invariant 1 then $Sp_2^1(\mathbb{Z}) = Sp_2(\mathbb{Z})$ so parts (i) and (iii) are equivalent. Thus, it suffices to show parts (i) and (ii). To prove (i) we abelianize the presentation of $Sp_2(\mathbb{Z})$ to get $\mathbb{Z}/12\{L\}$. To prove (ii) we use GAP

```
gap> F := FreeGroup("L","N");
gap> AssignGeneratorVariables(F);
gap> rel := [(L*N)^2*N^-3, N^6];
gap> G := F/rel;
gap> Q := Group((1,2),(1,2,3));
gap> hom := GroupHomomorphismByImages
(G,Q,GeneratorsOfGroup(G),GeneratorsOfGroup(Q));
[ L, N ] -> [ (1,2), (1,2,3) ]
```

```
gap> S := PreImage(hom,Stabilizer(Q,1));
% This is Sp_2^0(Z)
gap> AbelianInvariants(S);
[ 0, 4 ]
gap> genS := GeneratorsOfGroup(S);
[ L^-2, N*L^-1 ]
gap> iso := IsomorphismFpGroupByGenerators(S,genS);
gap> s := ImagesSource(iso);
gap> q := MaximalAbelianQuotient(s);
[ F1, F2 ] -> [ f2, f_1^-1*f2 ]
gap> AbS := ImagesSource(q);
gap> GeneratorsOfGroup(AbS);
[ f1, f2 ]
gap> RelatorsOfFpGroup(AbS);
[ f1^-1*f2^-1*f1*f2, f1^4 ]
```

From these we get that L^2 is a generator of the $\mathbb{Z}$ summand. Moreover, $NL^{-1}L^2 = NL$ maps to a generator of the $\mathbb{Z}/4$ summand, and this matrix is precisely the conjugation by Ω_1 of our choice of matrix for λ. $\square$

g=2

Theorem 3.4.7. *(i)* $H_1(Sp_4(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/2\{t\cdot\sigma\}$, *where* σ, t *are as in Theorem 3.4.6.*

(ii) $H_1(Sp_4^0(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/2\{Q_{\mathbb{Z}}^1(\sigma_0)\}\oplus\mathbb{Z}/4\{\lambda\cdot\sigma_0\}$, *and* $Q_{\mathbb{Z}}^1(\sigma_0)$ *is represented by* $\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix} \in Sp_4^0(\mathbb{Z})$.

Moreover, $\mu\cdot\sigma_0 = 0$.

(iii) $H_1(Sp_4^1(\mathbb{Z});\mathbb{Z}) = \mathbb{Z}/4\{t'\cdot\sigma_0\}$, *where* t' *is as in Theorem 3.4.6.*

(iv) $t'\cdot\sigma_1 = \lambda\cdot\sigma_0$, $\mu\cdot\sigma_1 = 0$, $\lambda\cdot\sigma_1 = t'\cdot\sigma_0$, $Q_{\mathbb{Z}}^1(\sigma_0) = Q_{\mathbb{Z}}^1(\sigma_1)$ *and* $[\sigma_0,\sigma_1] = 0$.

Proof. By [Lu92, Theorem 2] $Sp_4(\mathbb{Z})$ has a presentation with two generators L, N (see the GAP computations below for the relations), where L is given by the stabilization of the matrix called L in the proof of Theorem 3.4.6.

To prove (i) we compute

```
gap> F := FreeGroup("L","N");
gap> AssignGeneratorVariables(F);
gap> rel := [N^6, (L * N)^5, (L *N^-1)^10, (L* N^-1* L * N)^6,
L *(N^2*L*N^4)* L^-1 * (N^2*L*N^4)^-1, L *(N^3*L*N^3)* L^-1 *
```

```
(N^3*L*N^3)^-1, L *(L*N^-1)^5* L^-1 * (L*N^-1)^-5];
gap> G := F/rel;
% This is Sp_4(Z)
gap> p := MaximalAbelianQuotient(G);
[ L, N ] -> [ f1, f1 ]
gap> AbG := ImagesSource(p);
<pc group of size 2 with 2 generators>
% This says H_1(Sp_4(Z)) is a group with 2 elements.
gap> Order(Image(p,G.1));
2
%This gives the required generator: L
```

To prove part (ii) we add more GAP computations to the above, using a permutation representation of how L,N act on the 10 quadratic refinements of Arf invariant 0 (we use same indexing as in the proof of Theorem 3.4.2, and action is computed similarly).

```
gap> Q := Group((1,2)(4,6)(5,8),(2,3,4,5,6,7)(8,9,10));
gap> hom := GroupHomomorphismByImages
(G,Q,GeneratorsOfGroup(G),GeneratorsOfGroup(Q));
% Permutation representation of G on the 10 quadratic refinements of Arf invariant 0.
gap> S := PreImage(hom,Stabilizer(Q,1));
% This is the group Sp_4^0(Z)
gap> genS := GeneratorsOfGroup(S);
[ L^-2, N, L*N*L*N^-1*L^-1, L*N^-1*L*N*L^-1 ]
gap> iso :=  IsomorphismFpGroupByGenerators(S,genS);
gap> s := ImagesSource(iso);
gap> q := MaximalAbelianQuotient(s);
gap> AbS := ImagesSource(q);
gap> AbelianInvariants(S);
[ 2, 4 ]
gap> Order(Image(q,s.1));
1
% By definition of \mu it is represented by L^2, so its stabilization vanishes.
gap> Order(Image(q,s.2));
2
gap> Order(Image(q,s.4));
4
gap> Image(q,s.2)=Image(q,s.4)^2;
false
% These last computations say that L*N^-1*L*N*L^-1 generates the Z/4 summand, and that N generates
    the Z/2 summand.
```

By [Lu92, Theorem 2], he matrix N is given by N=$\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Thus, $N^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in Sp_4^0(\mathbb{Z})$ represents $Q^1_{\mathbb{Z}}(\sigma_0)$ because it represents $Q^1_{\mathbb{Z}}(\sigma)$ and it stabilizes the quadratic refinement $q_{0,0,0,0}$, so this generates the $\mathbb{Z}/2$ summand.

Also $LN^{-1}LNL = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which by the last paragraph of the proof of Theorem 3.4.6 is the stabilization of the matrix λ conjugated by Ω_1. Since $\Omega_1 \in Sp_4^0(\mathbb{Z})$ then $LN^{-1}LNL$ represents the homology class $\lambda \cdot \sigma_0$. By the GAP computations $LN^{-1}LNL = LN^{-1}LNL^{-1}L^2$ is a generator of the $\mathbb{Z}/4$ summand, as required.

To prove part (iii) we also use the same GAP program but this time we compute the permutation representation on the quadratic refinements of Arf invariant 1. We will pick our quadratic refinement of Arf invariant 1 to be $q_{1,1,0,0}$.

```
gap> T := Group((3,4),(1,2,3,4,5,6));
gap> homtwo := GroupHomomorphismByImages
(G,T,GeneratorsOfGroup(G),GeneratorsOfGroup(T));
% Permutation representation of G on the 6 quadratic refinements
of Arf invariant 1 indexed so that q_{1,1,0,0}=1.
gap> SS := PreImage(homtwo,Stabilizer(T,1));
% This is Sp_4^1(Z)
gap> genSS := GeneratorsOfGroup(SS);
[ L, N*L*N^-1, N^-1*L*N, N^2*L^-2*N^-2, N^3*L^-1*N^-2 ]
gap> isotwo := IsomorphismFpGroupByGenerators(SS,genSS);
gap> ss := ImagesSource(isotwo);
gap> qq := MaximalAbelianQuotient(ss);
gap> AbSS := ImagesSource(qq);
<pc group of size 4 with 5 generators>
% This says that H_1(Sp_4^1(Z)) is a group of order 4
gap> Order(Image(qq,ss.1));
4
% This shows the group is cyclic and gives the claimed generator.
```

To prove (iv) we use the E_2-algebra map from the E_2-algebra of spin mapping class groups to the one of quadratic symplectic groups, which is induced by the obvious functor $\mathsf{MCG} \to \mathsf{Sp}$ and the fact that the quadratic refinements functor Q is essentially the same in both cases. In more concrete terms, the functor just sends the spin mapping class groups to their actions on first homology, which are quadratic symplectic groups.

The Dehn twist $a \in \Gamma_{1,1}$ maps to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$, and the Dehn twist $b \in \Gamma_{1,1}$ maps to $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) \in Sp_2(\mathbb{Z})$. Thus, $a^{-2} \mapsto \left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$ and $aba^{-1} \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$

By Theorems 3.4.1, 3.4.2 and 3.4.3 we get $x \mapsto -\mu$, $y \mapsto \lambda - \mu$ and $z \mapsto t'$. Also by definition $\sigma_\varepsilon \mapsto \sigma_\varepsilon$ for $\varepsilon \in \{0,1\}$. Thus, by Theorem 3.4.3 we get: $x \cdot \sigma_1 = 28z \cdot \sigma_0$ and so $-\mu \cdot \sigma_1 = 28t' \cdot \sigma_0 = 0$. Also, $y \cdot \sigma_1 = z \cdot \sigma_0$ so $(\lambda - \mu) \cdot \sigma_1 = t' \cdot \sigma_0$, giving the result. Furthermore, $z \cdot \sigma_1 = y \cdot \sigma_0$ so $t' \cdot \sigma_1 = (\lambda - \mu) \cdot \sigma_0$, hence giving the result. Finally, $Q^1_{\mathbb{Z}}(\sigma_1) = Q^1_{\mathbb{Z}}(\sigma_0) - 10x \cdot \sigma_0$, so $Q^1_{\mathbb{Z}}(\sigma_1) = Q^1_{\mathbb{Z}}(\sigma_0) + 10\mu \cdot \sigma_0 = Q^1_{\mathbb{Z}}(\sigma_0)$, and $[\sigma_0, \sigma_1] = 24z \cdot \sigma_0 \mapsto 0$. □

g = 3

Theorem 3.4.8. *(i)* $H_1(Sp^0_6(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/4\{\lambda \cdot \sigma_0^2\}$.

(ii) $Q^1_{\mathbb{Z}}(\sigma_0) \cdot \sigma_0 = 2\lambda \cdot \sigma_0^2$.

(iii) $Q^1_{\mathbb{Z}}(\sigma_0) \cdot \sigma_0 = Q^1_{\mathbb{Z}}(\sigma_1) \cdot \sigma_1$ *and* $Q^1_{\mathbb{Z}}(\sigma_0) \cdot \sigma_1 = Q^1_{\mathbb{Z}}(\sigma_1) \cdot \sigma_0 = 2\lambda \cdot \sigma_0 \cdot \sigma_1$.

(iv) $H_1(Sp^1_6(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/4\{\lambda \cdot \sigma_0 \cdot \sigma_1\}$ *and* $\lambda \cdot \sigma_0 \cdot \sigma_1 = t' \cdot \sigma_0^2$.

Proof. By Theorem 3.4.4(i) $H_1(\Gamma^{1/2}_{3,1}[0]; \mathbb{Z}) = \mathbb{Z}/4\{y \cdot \sigma_0^2\}$. The homomorphism $\Gamma^{1/2}_{3,1}[0] \to Sp^0_6(\mathbb{Z})$ is surjective because $\Gamma_{3,1} \to Sp_6(\mathbb{Z})$ is, and hence the group $\mathbb{Z}/4\{y \cdot \sigma_0^2\}$ surjects onto $H_1(Sp^0_6(\mathbb{Z}); \mathbb{Z})$. Using the E_2-algebra map of the previous section $y \cdot \sigma_0^2 \mapsto \lambda \cdot \sigma_0^2$. This gives part (ii) by Theorem 3.4.4(ii). The rest of part (i) follows from Theorem 1.1 in [JM90, Theorem 1.1], which says that $H_1(Sp^0_6(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/4$.

Part (iii) follows by using the E_2-algebra map again and Theorem 3.4.4.

For part (iv) we use Theorem B, Part (i), to get that all the stabilization maps $\sigma_\varepsilon \cdot - : H_1(Sp^{\delta-\varepsilon}_{2(g-1)}(\mathbb{Z}); \mathbb{Z}) \to H_1(Sp^{\delta}_{2g}(\mathbb{Z}); \mathbb{Z})$ are surjective for $g \geq 4$. (The proof of Theorem B(i) is independent of these computations.)

By [JM90, Theorem 1.1] the stable first homology group of the quadratic symplectic groups of Arf invariant 0 is $\mathbb{Z}/4$. The stable first homology group of the quadratic symplectic groups of Arf invariant 1 must be the same by homological stability. Thus, $H_1(Sp^1_6(\mathbb{Z}); \mathbb{Z})$ surjects onto $\mathbb{Z}/4$. Finally, by a similar reasoning to the one at the beginning of this proof we get that $H_1(\Gamma^{1/2}_{3,1}[1]; \mathbb{Z}) \cong \mathbb{Z}/4$ surjects onto $H_1(Sp^1_6(\mathbb{Z}); \mathbb{Z})$. The expression for the generator follows from Theorem 3.4.4 and the E_2-algebra map. □

Remark 3.4.9. *All the computations of Section 3.4.2 are consistent with the ones of [Kra20a, Appendix A].*

Chapter 4

Moduli spaces of high dimensional manifolds

In this chapter we will prove Theorems C, D and E. The proof of the first two of these results will be done in Section 4.2, and they will be a consequence of the generic stability results of Chapter 2 and Theorem E. The proof of Theorem E is more technical and it will be delayed to Sections 4.3 and 4.4.

4.1 Spaces of manifolds and E_{2n-1}-algebras

4.1.1 Spaces of manifolds

In this subsection we define the moduli spaces of manifolds that we will study. Before doing so let us fix some notation: for each $s \in \mathbb{N}$ let us denote

$$J_s := \partial I^{s+1} \setminus \Big(\mathrm{int}(I^s \times \{1\}) \cup \mathrm{int}(\{1\} \times I^s)\Big) \subset \partial I^{s+1}$$

where I is the unit interval $[0,1]$. This represents the boundary of the standard $(s+1)$-cube once we remove the interiors of the top and right faces.

From now on let n denote an odd natural number with $n \geq 3$.

Definition 4.1.1. *Let $A \subset I^{2n-1} \times \mathbb{R}^\infty$ be a smooth, $(n-2)$-connected, $(2n-1)$-manifold with corners such that $A \cap (\partial I^{2n-1} \times \mathbb{R}^\infty) = \partial A = \partial I^{2n-1} \times \{0\}$, and A agrees with $I^{2n-1} \times \{0\}$ in a neighbourhood of its boundary.*

Then let $\mathcal{M}[A]$ be the set of all $W \subset I^{2n} \times \mathbb{R}^\infty$ which are smooth $2n$-submanifolds with corners such that:

(i) *W is $(n-1)$-connected and s-parallelizable.*

(ii) *$W \cap (\partial I^{2n} \times \mathbb{R}^\infty) = \partial W \supset J_{2n-1}$ and W agrees with $I^{2n} \times \{0\}$ in a neighbourhood of J_{2n-1}.*

(iii) *$\partial W \cap (\{1\} \times I^{2n-1} \times \mathbb{R}^\infty) = \{1\} \times A$.*

(iv) *$\mathcal{D}(W) := \partial W \cap (I^{2n-1} \times \{1\} \times \mathbb{R}^\infty) \subset I^{2n-1} \times \{1\} \times \mathbb{R}^\infty$ is a smooth $(2n-1)$-submanifold with corners such that*

(a) *$\mathcal{D}(W) \cap (\partial I^{2n-1} \times \{1\} \times \mathbb{R}^\infty) = \partial\mathcal{D}(W) = \partial I^{2n-1} \times \{1\} \times \{0\}$ and $\mathcal{D}(W)$ agrees with $I^{2n-1} \times \{1\} \times \{0\}$ in a neighbourhood of its boundary.*

(b) *$\mathcal{D}(W)$ is contractible (hence diffeomorphic to I^{2n-1} by the h-cobordism theorem).*

(v) *We have a decomposition $\partial W = \partial^- W \cup \mathcal{D}(W)$, where $\partial^- W := J_{2n-1} \cup \{1\} \times A$.*

(vi) *W has a product structure near its boundary, in the following sense: for each $(2n-1)$-face F of I^{2n} let $\pi_F : I^{2n} \to F$ be the orthogonal projection onto that face. Then W agrees with $\{(u,x) \in I^{2n} \times \mathbb{R}^\infty : (\pi_F(u), x) \in \partial W\}$ in a neighbourhood of $F \times \mathbb{R}^\infty$.*

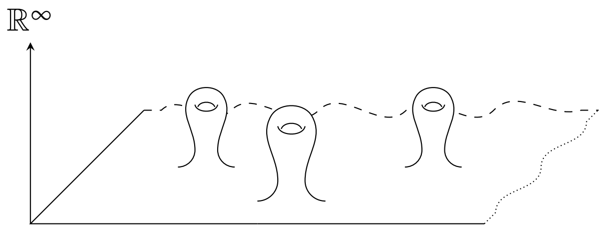


Fig. 4.1 Picture for $n = 1$: the dotted edge represents $A \subset \{1\} \times I^{2n-1} \times \mathbb{R}^\infty$, the dashed edge represents a given $\mathcal{D}(W) \subset I^{2n-1} \times \{1\} \times \mathbb{R}^\infty$, and J_1 consists of the remaining edges together with the vertex where the dashed and dotted edges meet.

The topology of $\mathcal{M}[A]$ is defined in two steps as follows.

Firstly, for each $W \in \mathcal{M}[A]$ we let $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)$ be the space, with the C^∞-topology, of smooth embeddings of W into $I^{2n} \times \mathbb{R}^\infty$, sending the interior of W to $\mathrm{int}(I^{2n}) \times \mathbb{R}^\infty$, fixing pointwise a neighbourhood of $\partial^- W$, and with a product structure near ∂W, in the following sense: for each $\varepsilon \in \mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)$ and each $(2n-1)$-face F of I^{2n} we have

$$\varepsilon(\partial W \cap (F \times \mathbb{R}^\infty)) = \partial\varepsilon(W) \cap (F \times \mathbb{R}^\infty)$$

and there is a small neighbourhood of $\partial W \cap (F \times \mathbb{R}^\infty)$ in W such that

$$\varepsilon(u,x) = (p_1(\varepsilon(\pi_F(u),x)) + u - \pi_F(u), p_2(\varepsilon(\pi_F(u),x)))$$

whenever $(u,x) \in I^{2n} \times \mathbb{R}^\infty$ lies in that neighbourhood, where p_1, p_2 are the projections of $I^{2n} \times \mathbb{R}^\infty$ to each of its factors.

Then we get a function of sets

$$\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathcal{M}[A]$$

via $\varepsilon \mapsto \varepsilon(W)$; and we give $\mathcal{M}[A]$ the finest topology so that all these functions are continuous.

The most natural moduli space to consider is the collection of manifolds inside $I^{2n} \times \mathbb{R}^\infty$ agreeing pointwise with $I^{2n} \times \{0\}$ near their boundary, plus some conditions on the manifolds such as the $(n-1)$-connectivity and the s-parallelizability. Such a moduli space has a natural E_{2n}-algebra structure by “re-scaling and gluing the manifolds inside the cube”.

However, in latter proofs in Section 4.3 we will need to “cut arcs of the manifolds” and study the leftover pieces, so we cannot restrict ourselves to manifolds whose boundary is ∂I^{2n}. Instead, the only control we will have on their boundary is the $(n-2)$-connectivity. This is similar to the fact that [GKRW19, Section 4] introduces surfaces with more boundary components in order to understand the ones with just one boundary component. The need of spaces of manifolds whose boundary is more general than just the standard disc motivates the introduction of the face A in the above definition.

On the other hand, by cutting arcs we can get to manifolds whose boundary is an exotic sphere, and it will be convenient that these form part of the E_k-algebra too: see Remark 4.3.18 for details. Hence, we introduced the further freedom of $\mathcal{D}(W)$, so that even when A is the standard face we allow exotic spheres as boundaries and still get an E_k-algebra. This new freedom must be “movable”, in the sense that $\mathcal{D}(W)$ depends on W itself (instead of being a once-and-for-all chosen disc), in order to get an E_k-algebra structure in a natural way.

In Section 4.1.3 we will give an interpretation of the moduli spaces $\mathcal{M}[A]$ as classifying spaces of diffeomorphism groups fixing some part of the boundary, and in Section 4.2.2 we will relate these to the classifying spaces of diffeomorphism groups relative the full boundary that we are originally interested in. The idea is that the E_k-algebra we will study is (very closely related to) $\mathcal{M}[I^{2n-1}]$, and the relationship between this algebra and the most natural one $\bigsqcup_g B\mathrm{Diff}_\partial(W_{g,1})$ is similar to the relationship between the algebra of all quadratic symplectic groups and the one of quadratic symplectic groups of Arf invariant zero. Making this analogy precise is in some sense the content of Sections 4.1 and 4.2.

4.1.2 $\mathcal{M}[I^{2n-1}]$ as a (graded) E_{2n-1}-algebra

In this subsection we will give an E_{2n-1}-algebra structure to $\mathcal{M}[I^{2n-1}] \in \mathsf{Top}$. This will be, up to a small modification, the E_{2n-1}-algebra that we will study for the rest of this chapter.

To do so recall the explicit model of the E_{2n-1}-operad in Section 1.3 given by the space of rectilinear embeddings of little $(2n-1)$-cubes into the standard I^{2n-1}. The E_{2n-1}-algebra structure on $\mathcal{M}[I^{2n-1}]$ is then given by scaling the manifolds using the rectilinear embeddings, noting that any rectilinear embedding $e : I^{2n-1} \hookrightarrow I^{2n-1}$ gives a canonical embedding $e \times I \times \mathbb{R}^\infty : I^{2n} \times \mathbb{R}^\infty \hookrightarrow I^{2n} \times \mathbb{R}^\infty$, and then re-gluing them inside $I^{2n} \times \mathbb{R}^\infty$; so that outside the little cubes the manifold still agrees with $I^{2n} \times \{0\}$. The resulting manifold will then satisfy all the conditions of Definition 4.1.1, including the s-parallelizability: for each manifold W in $\mathcal{M}[I^{2n-1}]$ we can pick a stable framing which agrees with a given one on a neighbourhood of $\partial^- W \cong D^{2n-1}$, and hence we can glue these stable framings along the above E_{2n-1}-product to get a stable framing.

We want to grade this E_{2n-1}-algebra in order to keep track of the different components, so the natural discrete monoid to take is $\mathsf{G}_n := \pi_0(\mathcal{M}[I^{2n-1}])$, where the (symmetric) monoidal structure is given by the E_{2n-1}-multiplication. Thus we can view

$$\mathcal{M}[I^{2n-1}] \in \mathrm{Alg}_{E_{2n-1}}\left(\mathsf{Top}^{\mathsf{G}_n}\right).$$

We will describe G_n explicitly in Section 4.1.4.

4.1.3 Interpretation of the moduli spaces as classifying spaces of diffeomorphisms

For the rest of this subsection we give all the embedding and diffeomorphism spaces the C^∞-topology. Also, W will always be assumed to be a manifold belonging to one of the spaces $\mathcal{M}[A]$ of Definition 4.1.1.

Let us denote by $\mathrm{Diff}_{\partial^- W}(W)$ the group of diffeomorphisms of W fixing pointwise a neighbourhood of $\partial^- W$. We say that a diffeomorphism $\phi \in \mathrm{Diff}_{\partial^- W}(W)$ has a *product structure* near the boundary if when composed with the inclusion $W \hookrightarrow I^{2n} \times \mathbb{R}^\infty$ it has a product structure in the sense of Definition 4.1.1. We denote by $\mathrm{Diff}^p_{\partial^- W}(W)$ the group of diffeomorphisms of W fixing pointwise a neighbourhood of $\partial^- W$ and with a product structure near ∂W.

Proposition 4.1.2. *The path-component of* $W \in \mathcal{M}[A]$ *is the image of* $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)$ *inside* $\mathcal{M}[A]$*, and moreover it is a model of* $B\mathrm{Diff}^p_{\partial^- W}(W)$*. Thus we can identify*

$$\mathcal{M}[A] = \bigsqcup_{[W]} B\mathrm{Diff}^p_{\partial^- W}(W)$$

where the coproduct is taken over path-components.

Proof. The definition of the topology in $\mathcal{M}[A]$ in Definition 4.1.1 forces the path-component of W to lie in the image of $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathcal{M}[A]$, and since $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)$ is path-connected by the Whitney embedding theorem then the result follows.

By adapting the proof of [Mic80, Chapter 13] insisting that all the maps involved have a product structure, the quotient map

$$\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)/\mathrm{Diff}^p_{\partial^- W}(W)$$

has slices and hence is a principal $\mathrm{Diff}^p_{\partial^- W}(W)$-bundle. Moreover, its base is canonically homeomorphic to the image of $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathcal{M}[A]$. The result follows since $\mathrm{Emb}^p_{\partial^- W}(W, I^{2n} \times \mathbb{R}^\infty)$ is weakly contractible by the Whitney embedding theorem. □

The following lemma says that up to homotopy the product structure plays no role. However, it is convenient to have it in our moduli spaces for technical purposes.

Lemma 4.1.3. *For any* W *the inclusion* $\mathrm{Diff}^p_{\partial^- W}(W) \hookrightarrow \mathrm{Diff}_{\partial^- W}(W)$ *is a homotopy equivalence. Thus*

$$\mathcal{M}[A] \simeq \bigsqcup_{[W]} B\mathrm{Diff}_{\partial^- W}(W)$$

Proof. We can construct a deformation-retraction $\mathrm{Diff}_{\partial^- W}(W) \to \mathrm{Diff}^p_{\partial^- W}(W)$ as follows: firstly observe that any $\phi \in \mathrm{Diff}_{\partial^- W}(W)$ automatically preserves the product structure everywhere except possibly at the top face. Secondly, the product structure of W at its top face gives a collar $c : \mathcal{D}(W) \times [0, \varepsilon] \hookrightarrow W$, and we pick a Riemannian metric on ∂W. Finally, we use geodesic interpolation to modify ϕ inside the collar until it becomes product-preserving. □

For the particular case that $A = I^{2n-1}$ we shall denote $\mathrm{Diff}_{\frac{1}{2}\partial}(W) := \mathrm{Diff}_{\partial^- W}(W)$, because ∂W splits as the union of two contractible pieces: $\partial^- W$ and $\mathcal{D}(W)$. We will think of $\partial^- W$ as the "*fixed half of the boundary*" because it is the standard part of it, whereas the other piece depends on the choice of W. This notation and point of view will be used in the whole Section 4.2.

4.1.4 Identifying the grading

In this subsection we will determine $\mathsf{G}_n = \pi_0(\mathcal{M}[I^{2n-1}])$ explicitly.

Any $W \in \mathcal{M}[I^{2n-1}]$ can be canonically identified with a smooth manifold W^s by canonically smoothing the corners of the standard cube I^{2n}. W and W^s are homeomorphic and have diffeomorphic interiors. W^s is s-parallelizable, $(n-1)$-connected and its boundary is a homotopy sphere. By [Wal62, pages 165-167] we can associate to any such manifold W^s a triple $(H_n(W^s), \lambda_{W^s}, q_{W^s})$, where $H_n(W^s)$ denotes its middle homology; $\lambda_{W^s}: H_n(W^s) \otimes H_n(W^s) \to \mathbb{Z}$ its intersection product; and $q_{W^s}: H_n(W^s) \to \mathbb{Z}/\Lambda_n$, its quadratic refinement, defined in terms of normal bundles using the stable parallelizability of W^s, where $\Lambda_n = \begin{cases} \mathbb{Z} & if\, n = 3,7 \\ 2\mathbb{Z} & \text{otherwise.} \end{cases}$

Since $H_n(W) = H_n(W^s)$ we get an equivalent triple $(H_n(W), \lambda_W, q_W)$. By Poincaré duality and the odd parity of n the bilinear form $(H_n(W), \lambda_W)$ is skew-symmetric and non-degenerate on a free finitely generated $\mathbb{Z}$-module, and hence it is isomorphic to the standard hyperbolic form of genus g for a unique $g = g(W) \in \mathbb{N}$ called the *genus* of W. When $n = 3,7$ the quadratic refinement vanishes identically, whereas when $n \neq 3,7$ it takes values in $\mathbb{Z}/2$. Recall that for $g \geq 1$ there are precisely two isomorphism classes of $\mathbb{Z}/2$-valued quadratic refinements on the standard hyperbolic form of genus g, distinguished by the Arf invariant (c.f. Definition 3.2.1). We define the *Arf invariant* of W, denoted $\mathrm{Arf}(W) \in \{0,1\}$, to the the Arf invariant of the corresponding q_W. When $g = 0$ the Arf invariant is just set to be 0.

Proposition 4.1.4. *For $n = 3,7$ the genus gives an isomorphism of monoids $\mathsf{G}_n \cong \mathbb{N}$. For $n \neq 3,7$ odd, $n \geq 3$, taking genus and Arf invariant gives an isomorphism of monoids $\mathsf{G}_n \cong \mathsf{H} = \{0\} \sqcup (\mathbb{N}_{>0} \times \mathbb{Z}/2)$.*

Proof. Let us denote by G'_n the monoid $\mathbb{N}$ when $n = 3,7$ and the monoid H otherwise. For each n the genus and Arf invariant give a function $\Psi: \mathcal{M}[I^{2n-1}] \to \mathsf{G}'_n$. Since both the genus and the Arf invariant are diffeomorphism invariants then Ψ factors through $\mathsf{G}_n = \pi_0(\mathcal{M}[I^{2n-1}])$ by Proposition 4.1.2. Moreover Ψ is monoidal because both the genus and the Arf invariant are additive under orthogonal direct sums, and the E_{2n-1}-product of manifolds induces orthogonal direct sum on their associated algebraic data.

Surjectivity: given any $x \in \mathsf{G}'_n$ we can find a corresponding quadratic form (H, λ, q) of the correct genus and Arf invariant. By [Wal62, page 168] we can find a smooth, $(n-1)$-connected, s-parallelizable $2n$-manifold W^s with $(H_n(W^s), \lambda_{W^s}, q_{W^s}) \cong (H, \lambda, q)$. The non-degeneracy of H gives by Poincaré duality that ∂W^s is a homotopy sphere because ∂W^s is 1-connected by construction. We claim that any homotopy sphere can be realized as (the canonical smoothing of) a smooth $(2n-1)$-dimensional manifold with corners $\Sigma \subset \partial I^{2n} \times \mathbb{R}^\infty$

such that Σ agrees with $\partial I^{2n} \times \{0\}$ in a neighbourhood of $(J_{2n-1} \cup \{1\} \times I^{2n-1}) \times \mathbb{R}^\infty$ and $\Sigma \cap (I^{2n-1} \times \{1\} \times \mathbb{R}^\infty)$ is contractible. Indeed, this follows from the fact that any exotic sphere can be written as a union of two discs along a diffeomorphism, and we can view one of these discs to be standard, and the other can be embedded in an infinite-dimensional ambient space so that both discs intersect in a standard boundary. Finally we introduce corners by viewing the standard disc as the smoothing of the unit cube with the interior of its top face removed. By the isotopy extension theorem we can then realize W^s as the canonical smoothing of a manifold $W \in \mathcal{M}[I^{2n-1}]$.

Injectivity: suppose $W, W' \in \mathcal{M}[I^{2n-1}]$ satisfy $\Psi(W) = \Psi(W')$, we will show that $[W] = [W'] \in \pi_0(\mathcal{M}[I^{2n-1}])$. By definition both W and W' agree with $I^{2n} \times \{0\}$ on a neighbourhood U of $\partial^-(W) = \partial^-(W') = J_{2n-1} \cup \{1\} \times I^{2n-1}$ in $I^{2n} \times \mathbb{R}^\infty$. We can then write both W and W' as $U \cap (I^{2n} \times \{0\})$ with $2g$ n-handles attached. By the same argument of [Wal62, page 166] we can find an embedding $e : W' \hookrightarrow W$ which is the identity on $U \cap (I^{2n} \times \{0\})$: this uses that both forms $(H_n(W^s), \lambda_{W^s}, q_{W^s})$ and $(H_n(W'^s), \lambda_{W'^s}, q_{W'^s})$ are isomorphic, which holds because $\Psi(W) = \Psi(W')$ and the genus and Arf invariant completely classify non-degenerate skew-symmetric quadratic forms. By Whitehead's theorem the difference $\overline{W \setminus \mathrm{im}(e)}$ is a h-cobordism between the interiors of $\mathcal{D}(W)$ and $\mathcal{D}(W')$, and hence it is trivial by the contractibility of the ends and the h-cobordism theorem. We then fix a trivialization of the h-cobordism relative to the internal boundary and use the trivialization to push this boundary to the external one, giving a diffeomorphism $W' \xrightarrow{e'} W$ which is the identity on a neighbourhood of $\partial^-(W) = \partial^-(W') = J_{2n-1} \cup \{1\} \times I^{2n-1}$ in $I^{2n} \times \mathbb{R}^\infty$. Observe that e' respects the product structure except possibly near $\mathcal{D}(W')$. By an argument similar to the one of Lemma 4.1.3 we can use geodesic interpolation to produce a diffeomorphism $\tilde{e} : W' \to W$ which preserves the product structure near the boundary and is the identity near $\partial^-(W) = \partial^-(W')$.

Thus, $W \in \mathcal{M}[I^{2n-1}]$ lies in the image of $\mathrm{Emb}^p_{\partial^- W'}(W', I^{2n} \times \mathbb{R}^\infty)$. Since the latter space is path-connected by the Whitney embedding theorem then W and W' lie on the same path-component. $\square$

Remark 4.1.5. *We can view the manifold $W_{g,1}$ as (the smoothing of) an element in $\mathcal{M}[I^{2n-1}]$ with genus g. Since $W_{g,1}$ has genus g and Arf invariant 0 (when defined) then:*

(i) When $n = 3,7$ each path-component of $\mathcal{M}[I^{2n-1}]$ is represented by $W_{g,1}$.

(ii) When $n \neq 3,7$ the path-component $(g,0) \in \mathsf{G}_n$ is represented by $W_{g,1}$.

(iii) When $n \neq 3,7$ the path-component $(g,1) \in \mathsf{G}_n$ is represented by $W_{g-1,1} \natural K$, where K is the Kervaire manifold, whose boundary is the Kervaire exotic sphere Σ_K, and whose Arf invariant is 1 by construction.

Thus, a point $\sigma_0 \in R(1,0)$ is represented by (a manifold diffeomorphic to) $W_{1,1}$ and a point $\sigma_1 \in R(1,1)$ is represented by (a manifold diffeomorphic to) K. The geometric fact that $W_{1,1} \natural W_{1,1} = W_{2,1} \cong K \natural K$ (relative half of the boundary), then implies that σ_0^2 and σ_1^2 lie in the same path-component of $\mathcal{M}[I^{2n-1}]$ by Proposition 4.1.2. This is the geometric reason why the ring $\Bbbk[\sigma_0,\sigma_1]/(\sigma_1^2 - \sigma_0^2)$ appears in the homology computations of the E_k-algebras that we study, as we will see in the next subsection.

4.1.5 The graded E_{2n-1}-algebra R

We should think of $\mathcal{M}[I^{2n-1}]$ as the E_{2n-1}-algebra that we study, but the CW-approximation theorem of E_k-algebras, [GKRW18, Theorem 11.21], requires us to work with E_k-algebras in graded simplicial modules which are 0 in grading 0. In order to achieve that, we would like our algebra in spaces to be empty in grading 0. Observe that this modification has no consequences in the homological stability statements because we are not changing the moduli spaces in large genus.

Definition 4.1.6. *We define $\boldsymbol{R} \in \mathrm{Alg}_{E_{2n-1}}(\mathsf{Top}^{\mathsf{G}_n})$ to be $\{W \in \mathcal{M}[I^{2n-1}] : g(W) > 0\}$, i.e. the subspace of manifolds in $\mathcal{M}[I^{2n-1}]$ of strictly positive genus, so that by definition $\boldsymbol{R}(0) = \varnothing$.*

By construction, the associative unital replacement $\overline{\mathbf{R}}(x)$ is path-connected $\forall x \in \mathsf{G}_n$, and then using the monoidal structure of G_n of Proposition 4.1.4 we get

Corollary 4.1.7. *$H_{*,0}(\overline{\mathbf{R}})$ is given (as a a ring) by*

(i) $\mathbb{Z}[\sigma]$, *where $\sigma \in H_0(\mathbf{R}(1);\mathbb{Z})$ is the canonical generator, if $n = 3,7$.*

(ii) $\frac{\mathbb{Z}[\sigma_0,\sigma_1]}{(\sigma_1^2-\sigma_0^2)}$, *where $\sigma_\varepsilon \in H_0(\mathbf{R}(1,\varepsilon);\mathbb{Z})$ are the canonical generators, for $n \neq 3,7$.*

The key property that **R** has and that allows the study of its homological stability properties is an a-priori vanishing line on its E_{2n-1}-cells. We will denote by $\mathrm{rk} : \mathsf{G}_n \to \mathbb{N}$ the obvious monoidal functor taking the genus.

Theorem E. *The E_{2n-1}-algebra $\boldsymbol{R}$ satisfies $H^{E_1}_{x,d}(\boldsymbol{R}) = 0$ for $x \in \mathsf{G}_n$ with $d < \mathrm{rk}(x) - 1$.*

Corollary 4.1.8. *$\boldsymbol{R}$ satisfies the standard connectivity estimate, i.e. that $H^{E_{2n-1}}_{x,d}(\boldsymbol{R}) = 0$ for $x \in \mathsf{G}_n$ with $d < \mathrm{rk}(x) - 1$. Moreover, the algebras $\mathbf{R}_{\Bbbk}$ also satisfy the standard connectivity estimate.*

Proof. It is an immediate consequence of Theorem E and [GKRW18, Theorem 14.4]. The moreover part follows from [GKRW18, Lemma 18.2]. □

The proof of this theorem will be delayed until Sections 4.3 and 4.4, once we have already deduced the main homological stability results from it.

4.2 Homological stability results

The goal of this section is to show the homological stability results of Theorems C and D. In Sections 4.2.1, 4.2.2 we will compare homological stability of diffeomorphism groups relative the full boundary to homological stability of **R**, in Sections 4.2.3, 4.2.4 we will do some homology computations, and finally in Sections 4.2.5, 4.2.6 we put everything together using the results of Chapter 2.

Let us begin by giving the precise definition of *stabilization maps* that will be used for the rest of this section. Given $W_i \in \mathcal{M}[I^{2n-1}]$, $i \in \{0,1\}$ we can use the E_1-algebra structure on $\mathcal{M}[I^{2n-1}]$ and the element $[0,1/2] \sqcup [1/2,1] \in \mathcal{C}_1(2)$ to define the *boundary connected sum* operation, which produces a new manifold $W_0 \natural W_1 \in \mathcal{M}[I^{2n-1}]$.

Definition 4.2.1. *For a fixed $W_0 \in \boldsymbol{R}$ the (left) stabilization maps corresponding to W_0 are*

(i) $B(\mathrm{id}_{W_0} \natural -) : B\mathrm{Diff}_{\frac{1}{2}\partial}(W_1) \to B\mathrm{Diff}_{\frac{1}{2}\partial}(W_0 \natural W_1)$

(ii) $B(\mathrm{id}_{W_0} \natural -) : B\mathrm{Diff}_{\partial}(W_1) \to B\mathrm{Diff}_{\partial}(W_0 \natural W_1)$

defined for any $W_1 \in \mathcal{M}[I^{2n-1}]$.

Remark 4.2.2. *Under the identification $\boldsymbol{R} \simeq \bigsqcup_{[W]} B\mathrm{Diff}_{\frac{1}{2}\partial}(W)$ of Lemma 4.1.3, stabilization map (i) agrees with the stabilization map corresponding to $W_0 \in \boldsymbol{R}$ using the E_{2n-1}-algebra structure in the sense of Section 1.3.*

4.2.1 A technical result

In order to compare the two notions of stabilisation maps of Definition 4.2.1 we need to understand $B\mathrm{Diff}_{\partial}(W) \to B\mathrm{Diff}_{\frac{1}{2}\partial}(W)$ for an arbitrary $W \in \mathcal{M}[I^{2n-1}]$. The following result will be used throughout the whole of Section 4.2.

Theorem 4.2.3. *If $W \in \mathcal{M}[I^{2n-1}]$ then*

$$B\mathrm{Diff}_{\partial}(W) \xrightarrow{B\mathrm{incl}} B\mathrm{Diff}_{\frac{1}{2}\partial}(W) \xrightarrow{B\mathrm{res}} B\mathrm{Diff}_{\partial,0}(\mathcal{D}(W))$$

is a homotopy fibration sequence, where $\mathrm{Diff}_{\partial,0}(\mathcal{D}(W)) \subset \mathrm{Diff}_{\partial}(\mathcal{D}(W))$ *denotes the path-component of the identity.*

Proof. The main step is to show that for $W \in \mathcal{M}[I^{2n-1}]$ the image of

$$\pi_0(\mathrm{res}) : \pi_0(\mathrm{Diff}_{\frac{1}{2}\partial}(W)) \to \pi_0(\mathrm{Diff}_{\partial}(\mathcal{D}(W)))$$

is trivial.

Assuming the main step, $B\mathrm{res} : B\mathrm{Diff}_{\frac{1}{2}\partial}(W) \to B\mathrm{Diff}_{\partial}(\mathcal{D}(W))$ induces a map $B\mathrm{res} : B\mathrm{Diff}_{\frac{1}{2}\partial}(W) \to B\mathrm{Diff}_{\partial,0}(\mathcal{D}(W))$. By the isotopy extension theorem the map $\mathrm{Diff}_{\frac{1}{2}\partial}(W) \xrightarrow{\mathrm{res}} \mathrm{Diff}_{\partial,0}(\mathcal{D}(W))$ is a Serre fibration, and its fibre over id is homotopy equivalent to $\mathrm{Diff}_{\partial}(W)$. Taking classifying spaces on the corresponding fibration of groups gives the result.

Now let us show the main step: given $\phi \in \mathrm{Diff}_{\frac{1}{2}\partial}(W)$, we need to prove that $\phi|_{\mathcal{D}(W)}$ is isotopic to the identity relative to $\partial\mathcal{D}(W)$. Let $M := W \natural W \in \mathcal{M}[I^{2n-1}]$, $\Phi := \phi \natural \mathrm{id}_W \in \mathrm{Diff}(M)$, and $\mathbb{D} := \overline{\partial M \smallsetminus \mathcal{D}(W_0)}$, where W_0 denotes the left copy of W:

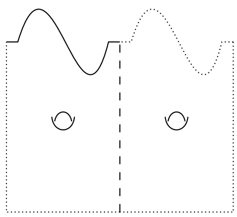


Fig. 4.2 Manifold M, where the dotted part is $\mathbb{D} \subset \partial M$ and the dashed line is where the boundary connected sum is performed

Let $\partial\Phi := \Phi|_{\partial M} \in \mathrm{Diff}(\partial M, \mathbb{D})$, then it suffices to prove that $\partial\Phi$ is isotopic to $\mathrm{id}_{\partial M}$ relative to $\mathbb{D}$. Since $M \in \mathcal{M}[I^{2n-1}]$ has $g(M) = 2g$, where $g = g(W)$, and $\mathrm{Arf}(M) = 2\,\mathrm{Arf}(W) = 0$, then by Proposition 4.1.4 and Remark 4.1.5 there is a diffeomorphism $f : M \xrightarrow{\cong} W_{2g,1}$. (Moreover, f can be taken to be orientation-preserving by Proposition 4.1.2.) Thus, Φ induces a diffeomorphism

$$W_{2g} := W_{2g,1} \cup_{\mathrm{id}_{\partial D^{2n}}} D^{2n} \xrightarrow{f\circ\Phi\circ f^{-1} \cup \mathrm{id}_{D^{2n}}} W_{2g,1} \cup_{\partial f\circ\partial\Phi\circ\partial f^{-1}} D^{2n} = W_{2g} \# \Sigma_{[\partial f\circ\partial\Phi\circ\partial f^{-1}]}.$$

By [Kos67, Theorem 3.1] the inertia group of W_{2g} is trivial, and hence $\partial f \circ \partial\Phi \circ \partial f^{-1}$ is isotopic to $\mathrm{id}_{S^{2n-1}}$, so $\partial\Phi$ is isotopic to $\mathrm{id}_{\partial M}$. It suffices to show that there is an isotopy from $\partial\Phi$ to $\mathrm{id}_{\partial M}$ relative $\mathbb{D}$: fix an isotopy $H : I \to \mathrm{Diff}^+(\partial M)$ such that $H(0) = \mathrm{id}_{\partial M}$ and $H(1) = \partial\Phi$; then we get a commutative square

$$\begin{tikzcd}
\partial I \arrow[r, "\partial H"] \arrow[d] & \mathrm{Diff}(\partial M, \mathbb{D}) \arrow[d, hook] \\
I \arrow[r, "H"] \arrow[ur, dashed] & \mathrm{Diff}^+(\partial M)
\end{tikzcd}$$

The obstruction to compress this map is

$$[(H,\partial H)] \in \pi_1(\mathrm{Diff}^+(\partial M), \mathrm{Diff}(\partial M, \mathbb{D}), \mathrm{id}).$$

Claim. $\pi_1(\mathrm{Diff}^+(\partial M), \mathrm{id}) \to \pi_1(\mathrm{Diff}^+(\partial M), \mathrm{Diff}(\partial M, \mathbb{D}), \mathrm{id})$ is surjective.

Given the claim we can finish the proof: pick a loop $\gamma : I \to \mathrm{Diff}^+(\partial M)$ based at id and mapping to $-[(H,\partial H)] \in \pi_1(\mathrm{Diff}^+(\partial M), \mathrm{Diff}(\partial M, \mathbb{D}), \mathrm{id})$, and then the new isotopy given by concatenation $H' := \gamma * H : I \to \mathrm{Diff}^+(\partial M)$ has a compression, and this will be the required isotopy.

Proof of claim. By isotopy extension $\mathrm{Diff}^+(\partial M) \xrightarrow{\mathrm{res}} \mathrm{Emb}^+(\mathbb{D}, \partial M)$ is a fibration whose fibre over the inclusion $\mathbb{D} \subset \partial M$ is $\mathrm{Diff}(\partial M, \mathbb{D})$. Thus we can identify

$$\pi_1(\mathrm{Diff}^+(\partial M), \mathrm{Diff}(\partial M, \mathbb{D}), \mathrm{id}) \cong \pi_1(\mathrm{Emb}^+(\mathbb{D}, \partial M), \mathrm{incl}).$$

Since $\mathbb{D}$ is diffeomorphic to a disc and $\partial M \cong S^{2n-1}$ via ∂f then we can identify $\mathrm{Emb}^+(\mathbb{D}, \partial M) = \mathrm{Emb}^+(D^{2n-1}, S^{2n-1}) \simeq \mathrm{Fr}^+(TS^{2n-1}) = SO(2n)$.

The composition

$$SO(2n) \hookrightarrow \mathrm{Diff}^+(S^{2n-1}) \cong \mathrm{Diff}^+(\partial M) \to \mathrm{Emb}^+(\mathbb{D}, \partial M) \simeq \mathrm{Fr}^+(TS^{2n-1}) = SO(2n)$$

is a homotopy equivalence, so it induces an isomorphism on $\pi_1(-)$, giving the required surjectivity. □

□

4.2.2 Comparing homological stability of R to diffeomorphism groups

Our ultimate goal is to show homological stability results about $B\mathrm{Diff}_\partial(W_{g,1})$ but as we saw in Lemma 4.1.3 the path-components of $\mathbf{R}$ are models of $B\mathrm{Diff}_{\frac{1}{2}\partial}(W)$. The main result of this subsection says that homological stability of diffeomorphism groups relative the full boundary is equivalent to homological stability of $\mathbf{R}$ itself. Before stating the precise statement let us introduce some notation: a map is called *homologically K-connected* if its relative homology groups vanish in degrees $\leq K$, i.e. if it is a homology isomorphism in degrees smaller than K and a homology surjection in degree K.

Note. Our definition of homologically K-connected is called homologically $(K+1)$-connective in [GKRW18].

Theorem 4.2.4. *Let $W_0 \in \boldsymbol{R}$ and $W \in \mathcal{M}[I^{2n-1}]$, then* *(i) The stabilization map* $B(\mathrm{id}_{W_0} \natural -) : B\mathrm{Diff}_{\frac{1}{2}\partial}(W_1) \to B\mathrm{Diff}_{\frac{1}{2}\partial}(W_0 \natural W_1)$ *is homologically* K*-connected (with* $\Bbbk$ *coefficients) for some* K *if and only if* $B(\mathrm{id}_{W_0} \natural -) : B\mathrm{Diff}_{\partial}(W_1) \to B\mathrm{Diff}_{\partial}(W_0 \natural W_1)$ *is homologically* K*-connected (with* $\Bbbk$ *coefficients).*

(ii) The map

$$\operatorname{coker}\big(H_1(B\mathrm{Diff}_\partial(W)) \to H_1(B\mathrm{Diff}_\partial(W_0 \natural W))\big) \xrightarrow{\cong} \operatorname{coker}\big(H_1(B\mathrm{Diff}_{\frac{1}{2}\partial}(W)) \to H_1(B\mathrm{Diff}_{\frac{1}{2}\partial}(W_0 \natural W))\big)$$

is an isomorphism.

In order to prove the above result we need the following result on fibrations

Lemma 4.2.5. *Suppose we have a homotopy-commutative diagram of (homotopy) fibrations*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

where B*,* B' *are 1-connected and* h *is a homotopy equivalence.*
If $H_*(F',F;\Bbbk) = 0$ *for* $* \le K$ *then* $H_*(E',E;\Bbbk) = 0$ *for* $* \le K$ *and furthermore* $H_{K+1}(F',F;\Bbbk) \xrightarrow{\cong} H_{K+1}(E',E;\Bbbk)$.

Proof. By pulling back the bottom fibration along the map h and using the naturality of the Serre spectral sequence to compare this new one to the original bottom one, we reduce to the special case that $B = B'$ and $h = \mathrm{id}_B$. Since B is simply-connected, there is a relative Serre spectral sequence

$$E^2_{p,q} = H_p(B, H_q(F',F)) \Rightarrow H_{p+q}(E',E)$$

where we have removed the coefficients $\Bbbk$ in the notation as they play no role in this proof.

By assumption, $H_q(F',F) = 0$ for $q \le K$, and so $E^2_{p,q} = 0$ for $p+q \le K$, hence $H_d(E',E) = 0$ for $d \le K$. Moreover, on the line $p+q = K+1$ the only non-vanishing entry on the E^2-page of the spectral sequence is $E^2_{0,K+1} = H_{K+1}(F',F)$, so it suffices to show that this group survives to the E^∞-page: it is immediate that all differentials vanish on this group, and any differential targeting position $(0, K+1)$ must come from a position of the form $(p, K+2-p)$ with $p \ge 2$, but all these entries already vanish in the E^2-page. □

Proof of Theorem 4.2.4. By Theorem 4.2.3 there is a commutative diagram of (homotopy) fibrations with simply-connected base spaces

$$\begin{tikzcd} B\mathrm{Diff}_\partial(W_1) \arrow[r, "B\mathrm{incl}"] \arrow[d, "B(\mathrm{id}_{W_0} \natural -)"] & B\mathrm{Diff}_{\frac{1}{2}\partial}(W_1) \arrow[r, "B\mathrm{res}"] \arrow[d, "B(\mathrm{id}_{W_0} \natural -)"] & B\mathrm{Diff}_{\partial,0}(\mathcal{D}(W_1)) \arrow[d, "B(\mathrm{id}_{\mathcal{D}(W_0)} \natural -)"] \\ B\mathrm{Diff}_\partial(W_0 \natural W_1) \arrow[r, "B\mathrm{incl}"] & B\mathrm{Diff}_{\frac{1}{2}\partial}(W_0 \natural W_1) \arrow[r, "B\mathrm{res}"] & B\mathrm{Diff}_{\partial,0}(\mathcal{D}(W_0 \natural W_1)) \end{tikzcd}$$

By Lemma 4.2.5 it suffices to show that the rightmost vertical map in the diagram is a homotopy equivalence. We will show that in fact

$$B(\mathrm{id}_{\mathcal{D}(W_0)} \natural -)) : B\mathrm{Diff}_\partial(\mathcal{D}(W_1)) \to B\mathrm{Diff}_\partial(\mathcal{D}(W_0 \natural W_1))$$

is a homotopy equivalence: there is a canonical identification $\mathcal{D}(W_0 \natural W_1) = \mathcal{D}(W_0) \natural \mathcal{D}(W_1)$, and both $\mathcal{D}(W_i)$ are abstractly diffeomorphic to D^{2n-1}, say via diffeomorphisms $\phi_i : \mathcal{D}(W_i) \xrightarrow{\cong} D^{2n-1}$. Think of D^{2n-1} as I^{2n-1} and without loss of generality pick ϕ_0 to be the identity in a neighbourhood of the rightmost face $\{1\} \times I^{2n-2}$ and ϕ_1 to be the identity in a neighbourhood of the leftmost face $\{0\} \times I^{2n-2}$. Thus we get a commutative diagram in which the vertical maps are homeomorphisms

$$\begin{tikzcd}
\mathrm{Diff}_\partial(\mathcal{D}(W_1)) \arrow[r, "\mathrm{id}_{\mathcal{D}(W_0)} \natural -"] \arrow[d, "\phi_1 \circ - \circ \phi_1^{-1}"] & \mathrm{Diff}_\partial(\mathcal{D}(W_0 \natural W_1)) \arrow[d, "(\phi_0 \natural \phi_1) \circ - \circ (\phi_0 \natural \phi_1)^{-1}"] & \\
\mathrm{Diff}_\partial(D^{2n-1}) \arrow[r, "\mathrm{id} \natural -"] & \mathrm{Diff}_\partial(D^{2n-1} \natural D^{2n-1}) \arrow[r, "="] & \mathrm{Diff}_\partial(D^{2n-1})
\end{tikzcd}$$

Thus, it suffices to show that $B(\mathrm{id}_{D^{2n-1}} \natural -)$ is a homotopy equivalence as a self map on $B\mathrm{Diff}_\partial(D^{2n-1})$, which is inmediate. □

4.2.3 Mapping class groups and arithmetic groups

In this subsection we study the first homology groups of **R**. To do so, recall that by Lemma 4.1.3 each path-component of **R** is the classifying space of a certain group of diffeomorphisms; and that for any topological group G, $H_1(BG;\mathbb{Z})$ is the abelianization of $\pi_0(G)$. Thus, to understand the first homology of **R** we need to understand some mapping class groups and their abelianizations. We will follow results of [Kre79] to achieve this.

Given $W \in \mathcal{M}[I^{2n-1}]$, its *mapping class group relative to half of the boundary* is $\Gamma_{\frac{1}{2}\partial}(W) := \pi_0(\mathrm{Diff}_{\frac{1}{2}\partial}(W))$. Let $\gamma(W) := \mathrm{Aut}(H_n(W), \lambda_W, q_W)$ and $g = g(W)$ be the genus of W. The arithmetic groups $\gamma(W)$ are well-known groups: in dimensions $n = 3,7$, $\gamma(W) = Sp_{2g}(\mathbb{Z})$ is a symplectic group, and for $n \neq 3,7$, $\gamma(W)$ is a quadratic symplectic group: either $Sp^0_{2g}(\mathbb{Z})$ if $\mathrm{Arf}(W) = 0$ or $Sp^1_{2g}(\mathbb{Z})$ if $\mathrm{Arf}(W) = 1$.

Theorem 4.2.6 (Kreck, Krannich)**.** *For $W \in \mathcal{M}[I^{2n-1}]$ there is a short exact sequence*

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(H_n(W), S\pi_n(SO(n))) \to \Gamma_{\frac{1}{2}\partial}(W) \to \gamma(W) \to 1$$

where $S\pi_n(SO(n)) := \mathrm{im}(\pi_n(SO(n)) \to \pi_n(SO(n+1)))$.

Proof. This proof is just a small generalization of the one given in [Kra20a] and [Kre79] to allow manifolds $W \in \mathbf{R}$ which are not $W_{g,1}$, so we will focus on explaining the differences and refer to the original papers for the details of the argument. By [Kre79, Proposition 3] for $W \in \mathcal{M}[I^{2n-1}]$ there are short exact sequences

(i) $1 \to \mathcal{I}_\partial(W) \to \Gamma_\partial(W) \to \gamma(W) \to 1$

(ii) $1 \to \Theta_{2n+1} \to \mathcal{I}_\partial(W) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(W), S\pi_n(SO(n))) \to 1$

where $\Gamma_\partial(W) := \pi_0(\mathrm{Diff}_\partial(W))$ and $\mathcal{I}_\partial(W)$ is the kernel of the obvious map $\Gamma_\partial(W) \to \gamma(W)$. Let us remark that [Kre79, Proposition 3] assumes that ∂W is the standard sphere, but the same proof works even when ∂W is a homotopy sphere: all the ingredients that the proof uses are the handle structure of W relative to a disc in its boundary, that W is s-parallelizable, and that the inertia group of W vanishes. The vanishing of the inertia group of W is equivalent to the main step in the proof of Theorem 4.2.3 (which in turn is based on the vanishing of the inertia groups of the manifolds $(S^n \times S^n)^{\# g}$ for any g).

Thus we get a short exact sequence

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(H_n(W), S\pi_n(SO(n))) \to \frac{\Gamma_\partial(W)}{\Theta_{2n+1}} \to \gamma(W) \to 1$$

By [Kra20a, Lemma 1.2] the image of $\Theta_{2n+1} \to \Gamma_\partial(W)$ is central and becomes trivial in $\Gamma_{\frac{1}{2}\partial}(W)$. Moreover, the induced map $\frac{\Gamma_\partial(W)}{\Theta_{2n+1}} \to \Gamma_{\frac{1}{2}\partial}(W)$ is an isomorphism. Only the case $W = W_{g,1}$ is treated in [Kra20a], but the proof works in our context too by following the same steps and replacing [Kra20a, equation (1.3)] by Theorem 4.2.3. The result then follows. □

The argument of [Kra20a, Lemma 1.3] applies in our situation too, giving that the action of $\gamma(W)$ on $\mathrm{Hom}_{\mathbb{Z}}(H_n(W), S\pi_n(SO(n)))$ agrees with the standard action of $\gamma(W)$ on $H_n(W)$ and the trivial one on $S\pi_n(SO(n))$. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(H_n(W), S\pi_n(SO(n))) \cong H_n(W) \otimes S\pi_n(SO(n))$ as $\gamma(W)$-modules by Poincaré duality. Thus Theorem 4.2.6 can be re-written in the following way.

Corollary 4.2.7. *There is a short exact sequence*

$$1 \to H_n(W) \otimes S\pi_n(SO(n)) \to \Gamma_{\frac{1}{2}\partial}(W) \to \gamma(W) \to 1$$

which is compatible with the $\gamma(W)$-action on the first group.

The main result of this subsection is

Theorem 4.2.8. *Let $W_0, W \in \boldsymbol{R}$, then the stabilization map by $W_0 \natural -$ induces an isomorphism*

$$\mathrm{coker}(H_1(\Gamma_{\frac{1}{2}\partial}(W)) \to H_1(\Gamma_{\frac{1}{2}\partial}(W_0 \natural W))) \xrightarrow{\cong} \mathrm{coker}(H_1(\gamma(W)) \to H_1(\gamma(W_0 \natural W))).$$

Moreover, if $V \in \mathbf{R}$ has $g(V) \geq 2$ then there is an isomorphism

$$H_1(\Gamma_{\frac{1}{2}\partial}(V)) \xrightarrow{\cong} H_1(\gamma(V)).$$

Proof. By Corollary 4.2.7 we get a commutative diagram of exact sequences

$$\begin{tikzcd}
(H_n(W)\otimes S\pi_n(SO(n)))_{\gamma(W)} \arrow[r] \arrow[d, "s"] & H_1(\Gamma_{\frac{1}{2}\partial}(W)) \arrow[r] \arrow[d, "s"] & H_1(\gamma(W)) \arrow[r] \arrow[d, "s"] & 0 \\
(H_n(W_0 \natural W)\otimes S\pi_n(SO(n)))_{\gamma(W_0\natural W)} \arrow[r] & H_1(\Gamma_{\frac{1}{2}\partial}(W_0\natural W)) \arrow[r] & H_1(\gamma(W_0\natural W)) \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, it suffices to show that the leftmost vertical map is surjective. Since $W, W_0 \in \mathbf{R}$ then $g(W_0 \natural W) \geq 2$. Thus, it suffices to show that for any $V \in \mathbf{R}$ with $g(V) \geq 2$, $(H_n(V)\otimes S\pi_n(SO(n)))_{\gamma(V)} = 0$. In fact, this last statement implies the moreover part of the theorem too. We will show that for any abelian group A and $g \geq 2$ we have $(\mathbb{Z}^{2g}\otimes A)_{\gamma_g} = 0$ for the following three cases

(i) $\gamma_g = Sp_{2g}(\mathbb{Z})$

(ii) $\gamma_g = Sp^q_{2g}(\mathbb{Z}) = Sp^0_{2g}(\mathbb{Z})$

(iii) $\gamma_g = Sp^a_{2g}(\mathbb{Z}) = Sp^1_{2g}(\mathbb{Z})$

where in all the cases γ_g has the standard action on $\mathbb{Z}^{2g}$ and acts trivially on A.

Cases (i) and (ii) follow from [Kra20a, Lemma A2]. For case (iii) fix a hyperbolic basis $e_0, f_0, \cdots, e_{g-1}, f_{g-1}$ of $\mathbb{Z}^{2g}$ with $\lambda(e_i,e_j) = \lambda(f_i,f_j) = 0$, $\lambda(e_i,f_j) = \delta_{i,j}$, and $q(e_i) = q(f_i) = \begin{cases} 1 & if\ i=0 \\ 0 & \text{otherwise} \end{cases}$ $\forall i,j$, and write $[-]$ for the residue class of a given element of $\mathbb{Z}^{2g}\otimes A$ in the coinvariants. The permutations of the $g-1$ hyperbolic summands generated by each pair e_i, f_i with $1 \leq i \leq g-1$ show that $[e_i \otimes a] = [e_j \otimes a]$ and $[f_i \otimes a] = [f_j \otimes a]$ for any $1 \leq i, j \leq g-1$ and any $a \in A$. Also, for each fixed $0 \leq i \leq g-1$ the transformation $e_i \mapsto f_i$, $f_i \mapsto -e_i$ lies in $Sp^a_{2g}(\mathbb{Z})$, and so $[e_i \otimes a] = [f_i \otimes a]$ for any $0 \leq i \leq g-1$ and any $a \in A$. Since $g \geq 2$, the transformation given by $e_0 \mapsto e_1 - f_1$, $f_0 \mapsto f_1 + e_0 + f_0$, $e_1 \mapsto e_0 - e_1 + f_1$, $f_1 \mapsto f_0 + e_1 - f_1$ exists and lies in $Sp^a_{2g}(\mathbb{Z})$, and hence it implies that $[e_0 \otimes a] = [e_1 \otimes a] - [f_1 \otimes a] = 0$ and $[e_1 \otimes a] = [e_0 \otimes a] - [e_1 \otimes a] + [f_1 \otimes a] = [e_0 \otimes a]$ for any $a \in A$. □

Now we will need the following two inputs about the homology of arithmetic groups. The proof of the first result can be found in [Kra20a, Lemma A.1,(i),(ii)], and the second result summarizes the parts of Theorems 3.4.6, 3.4.7 and 3.4.8 which are relevant here.

Theorem 4.2.9 (Krannich)**.** *The stabilization map*

$$H_1(BSp_2(\mathbb{Z});\mathbb{Z}) \xrightarrow{\sigma^{g-1}\cdot -} H_1(BSp_{2g}(\mathbb{Z});\mathbb{Z})$$

is always surjective.

Theorem 4.2.10. *For any $\varepsilon \in \{0,1\}$ the stabilisation map*

$$H_1(BSp_2^{\delta-\varepsilon}(\mathbb{Z});\mathbb{Z}) \xrightarrow{\sigma_\varepsilon\cdot\sigma_0^{g-2}\cdot -} H_1(BSp_{2g}^{\delta}(\mathbb{Z});\mathbb{Z})$$

(i) Has cokernel isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}$ generated by $Q^1_{\mathbb{Z}}(\sigma_0) = Q^1_{\mathbb{Z}}(\sigma_1)$ if $\delta = 0$ and $g = 2$.

(ii) Is surjective if $\delta = 1$ and $g = 2$.

(iii) Is surjective if $g \geq 3$ for any δ.

The above two results and Theorem 4.2.8 imply

Corollary 4.2.11. *(i) If $n = 3,7$ then the stabilization map $H_{1,1}(\boldsymbol{R}) \to H_{g,1}(\boldsymbol{R})$ is surjective for any $g \geq 2$.*

(ii) If n is odd, $n \neq 3,7$ and $\varepsilon \in \{0,1\}$ then the stabilization map

$$H_{(1,\delta-\varepsilon),1}(\boldsymbol{R}) \to H_{(g,\delta),1}(\boldsymbol{R})$$

(i) Has cokernel isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}$ generated by $Q^1_{\mathbb{Z}}(\sigma_0)$ if $\delta = 0$ and $g = 2$.

(ii) Is surjective if $\delta = 1$ and $g = 2$.

(iii) Is surjective if $g \geq 3$ for any δ. In particular, $\sigma_{1-\varepsilon}\cdot Q^1_{\mathbb{Z}}(\sigma_0)$ lies in the image of $\sigma_\varepsilon^2\cdot - : H_{(1,1-\varepsilon),1}(\boldsymbol{R}) \to H_{(3,1-\varepsilon),1}(\boldsymbol{R})$.

Remark 4.2.12. *The moreover part of Theorem 4.2.8 plus the results of Section 3.4.2 actually give full computations of $H_{x,1}(\mathbf{R})$ for $\mathrm{rk}(x) \geq 2$.*

4.2.4 Results on rational homology of diffeomorphism groups

The aim of this subsection is to understand the rational homology of the diffeomorphism groups we are interested in.

Theorem 4.2.13 (Berglund-Madsen, Krannich)**.** *For $W \in \boldsymbol{R}$ the stabilization map*

$$H_d(B\mathrm{Diff}_\partial(W);\mathbb{Q}) \to H_d(B\mathrm{Diff}_\partial(W_{1,1}\natural W);\mathbb{Q})$$

is surjective for $d \le \min\{g(W), 3n-7\}$ *and an isomorphism for* $d \le \min\{g(W)-1, 3n-7\}$.

Let us remark that the proof is just a small generalization of the one given in [Kra20b, Theorem A] to allow manifolds $W \in \mathbf{R}$ which are not $W_{g,1}$, so we will focus on explaining the differences and refer to the original paper for the details of the argument.

Proof. By combining [KRW21a, Proposition 4.3, Theorem 4.1] it follows that for both $V = W$ and $V = W_{1,1} \natural W$, $B\mathrm{Diff}_\partial(D^{2n}) \to B\mathrm{Diff}_\partial(V) \to B\widetilde{\mathrm{Diff}}_\partial(V)$ is a rational fibre sequence in degrees $\le 3n-7$. Note that we need to generalize the above citation slightly to take into account the possibility that $V \in \mathbf{R}$ is not a $W_{g,1}$, but the identical method of proof works in this case too. Moreover, the action of $\pi_1(B\widetilde{\mathrm{Diff}}_\partial(V))$ on $B\mathrm{Diff}_\partial(D^{2n})$ is trivial since any diffeomorphism can be isotoped to fix a disc near the boundary.

Now consider the corresponding diagram of rational fibrations for both $V = W$ and $V = W_{1,1} \natural W$ with stabilization maps between the rows. The map on $B\mathrm{Diff}_\partial(D^{2n})$ is a self-homotopy equivalence. Thus, by the relative Serre spectral sequence with rational coefficients, similar to the one of Lemma 4.2.5, it suffices to show that the stabilisation map

$$H_d(B\widetilde{\mathrm{Diff}}_\partial(W);\mathbb{Q}) \to H_d(B\widetilde{\mathrm{Diff}}_\partial(W_{1,1} \natural W);\mathbb{Q})$$

is surjective for $d \le g(W)$ and an isomorphism for $d \le g(W)-1$.

In our context [Kra20b, Theorem 1.1] also applies, the only difference being that when $n \neq 3,7$ the group called $\mathbf{G_g}$ can be either Sp_{2g}^q or Sp_{2g}^a, depending on the Arf invariant of the manifold. In order to get the surjectivity range in one degree higher than the isomorphism range we need the surjectivity range of [Kra20b, Theorem 1.1] to be one degree higher than the isomorphism range too. We can improve the surjectivity range as follows: by [Tsh19, Theorem 2], it suffices to show that $H_g(Sp_{2g}^\varepsilon(\mathbb{Z});\mathbb{Q}) \to H_g(Sp_{2(g+1)}^\varepsilon(\mathbb{Z});\mathbb{Q})$ is surjective when ε is either q, a or nothing. By transfer $H_g(Sp_{2g}^\varepsilon(\mathbb{Z});\mathbb{Q}) \to H_g(Sp_{2g}(\mathbb{Z});\mathbb{Q})$ is surjective for ε as above, the stable rational homology of the quadratic symplectic groups is isomorphic to the stable rational homology of the usual symplectic groups by the work of Borel, and by [Kra20b, Theorem 1.1] the groups $H_g(Sp_{2(g+1)}^\varepsilon(\mathbb{Z});\mathbb{Q})$ are already stable. Thus, it suffices to show that $H_g(Sp_{2g}(\mathbb{Z});\mathbb{Q}) \to H_g(Sp_{2(g+1)}(\mathbb{Z});\mathbb{Q})$ is surjective, which is precisely [GKT21, Proposition 12].

The whole of [Kra20b, Section 2] also applies in our case: the methods of [BM20] generalize to any $W \in \mathbf{R}$ and give the same expression for the homology of block diffeomorphisms homotopic to the identity as the one for $W_{g(W),1}$: this is because rationally both W and $W_{g(W),1}$ have the same homology, intersection product and boundary map.

Thus, the spectral sequence argument presented in [Kra20b] also applies to our case, and the surjectivity result on arithmetic groups in one degree higher gives the required surjectivity range for Block diffeomorphisms too. □

4.2.5 Proof of Theorem C

Now we will finally prove Theorem C in two parts: one for $\mathbb{Z}$ and $\mathbb{Z}[1/2]$-coefficients, and another one for $\mathbb{Q}$-coefficients.

Theorem 4.2.14. *For $n \geq 3$ odd, consider the stabilization map*

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1});\Bbbk) \to H_d(B\mathrm{Diff}_\partial(W_{g,1});\Bbbk).$$

Then

(i) If $n = 3,7$ and $\Bbbk = \mathbb{Z}$, it is surjective for $3d \leq 2g-1$ and an isomorphism for $3d \leq 2g-4$.

(ii) If $n \neq 3,7$ and $\Bbbk = \mathbb{Z}$, it is surjective for $2d \leq g-2$ and an isomorphism for $2d \leq g-4$.

(iii) If $n \neq 3,7$ and $\Bbbk = \mathbb{Z}[\frac{1}{2}]$, it is surjective for $3d \leq 2g-4$ and an isomorphism for $3d \leq 2g-7$.

Proof of Theorem 4.2.14. By Remarks 4.2.2, 4.1.5 and Theorem 4.2.4 it suffices to prove the corresponding stability results for $\mathbf{R}$. As we explained in Section 1.3, we can show them for $\mathbf{R}_\Bbbk$ instead.

When $n = 3,7$ apply [GKRW18, Theorem 18.1]: the assumptions to verify are the vanishing line in E_2-homology, which holds by Theorem E and Corollary 4.1.8, the surjectivity of the stabilisation map on first homology and grading 2, which holds by Corollary 4.2.11,(i), and the computation of zero-th homology, which holds by Corollary 4.1.7.

When $n \neq 3,7$ we take $\Bbbk = \mathbb{Z}$ for part (ii) or $\Bbbk = \mathbb{Z}[1/2]$ for part (iii). Then apply Theorems 2.1.1, 2.1.2: the vanishing line in E_2-homology holds by Theorem E and Corollary 4.1.8, and the remaining assumptions hold by Corollary 4.1.7, and by the universal coefficient theorem in homology and Corollary 4.2.11. □

Theorem 4.2.15. *For $n \geq 3$ odd, the stabilization maps*

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{Q}) \to H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q})$$

are surjective for $d < \frac{3n-6}{3n-5}(g-c_n)$ and isomorphisms for $d < \frac{3n-6}{3n-5}(g-c_n)-1$, where $c_n = 0$ for $n = 3,7$ and $c_n = 1$ otherwise.

Proof. We proceed as in the above proof to reduce it to verifying the assumptions of Theorem 2.1.6 for $\mathbf{X} = \mathbf{R}_{\mathbb{Q}}$ and $D = 3n-6$. The vanishing line in E_{2n-1}-homology holds by Theorem E and Corollary 4.1.8. The existence of appropriate maps $\mathbf{A} \to \mathbf{R}_{\mathbb{Q}}$ follows from Remark 2.1.5. Now we need to verify the remaining part of assumption (i) if $n = 3, 7$ or assumption (ii) otherwise.

When $g = 1$ only the case $d = 0$ needs to be considered, but this case is fine by Remark 2.1.5. When $g \geq 2$, Remark 4.1.5 says that the class σ_0 is generated by a model of $W_{1,1}$ so by Theorem 4.2.4 it suffices to check that for any $W \in \mathbf{R}$ the stabilization map

$$H_d(B\mathrm{Diff}_\partial(W);\mathbb{Q}) \to H_d(B\mathrm{Diff}_\partial(W_{1,1} \natural W);\mathbb{Q})$$

is surjective for $d \leq \min\{g(W), 3n-7\}$ and an isomorphism for $d \leq \min\{g(W)-1, 3n-7\}$, which follows from Theorem 4.2.13. $\square$

4.2.6 Proof of Theorem D

The proof of Theorem D is based on Corollary 2.2.5.

Proof. We let $\mathbf{X} = \mathbf{R}_{\mathbb{F}_2}$ and $\varepsilon = 0$, then it satisfies all the assumptions needed to apply Corollary 2.2.5 by Theorem E, Corollaries 4.1.8, 4.1.7 and by the universal coefficient theorem and Corollary 4.2.11. Now we have two cases to consider.

Case (i) Suppose $H_{(4k,0),2k}(\overline{\mathbf{R}_{\mathbb{F}_2}}/\sigma_0) \neq 0$ for all $k \geq 1$. By Remark 4.1.5 and Theorem 4.2.4 this implies that for each $k \geq 1$ there is some $d(k) \leq 2k$ for which

$$H_{d(k)}(B\mathrm{Diff}_\partial(W_{4k,1}), B\mathrm{Diff}_\partial(W_{4k-1,1});\mathbb{F}_2) \neq 0.$$

By Theorem C(ii) and the universal coefficient theorem we have $d(k) \geq 2k$, and thus

$$H_{2k}(B\mathrm{Diff}_\partial(W_{4k,1}), B\mathrm{Diff}_\partial(W_{4k-1,1});\mathbb{Z}) \neq 0$$

for all $k \geq 1$ by another application of the universal coefficient theorem.

Case (ii) Suppose $H_{x,d}(\overline{\mathbf{R}_{\mathbb{F}_2}}/\sigma_0) = 0$ for $3d \leq 2\,\mathrm{rk}(x) - 6$. Then by Remark 4.1.5 and Theorem 4.2.4 we have

$$H_d(B\mathrm{Diff}_\partial(W_{g,1}), B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{F}_2) = 0$$

for $3d \leq 2g - 6$. Moreover, by Theorem C(iii),

$$H_d(B\mathrm{Diff}_\partial(W_{g,1}), B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{Z}[1/2]) = 0$$

for $3d \leq 2g-4$. Finally, the required integral vanishing follows from the universal coefficient theorem and the finite generation of $H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Z})$ for any g,d, which follows from [BKK23, Theorem 6.1, Remark 6.2]. □

Remark 4.2.16. *By Corollary 2.2.5 we find "quantisation stability results" in Theorems A and B, but they are consequence of a secondary stability result. However, our case is different: the secondary stabilisation map lives in the algebra* $\mathbf{R}$ *itself, but it is not defined in* $\bigsqcup_g B\mathrm{Diff}_\partial(W_{g,1})$. *However, Theorem 4.2.4 allows us to "pull-back" the quantisation stability from* $\mathbf{R}$ *even if we cannot pull-back the secondary stability.*

Let us also mention that quoting the finite generation of homology groups from [BKK23, Theorem 6.1, Remark 6.2] is not really needed: one can use the ideas of the proofs of Theorems 2.1.1, 2.1.2 and 2.1.3 to remove the hypothesis of finite generation and the expense of working with a more elaborated CW E_2-algebra model.

4.2.7 Improvement of Theorem C(ii)

The goal of this subsection is to improve the constant term of Theorem C(ii).

Theorem 4.2.17. *The map*

$$H_d(B\mathrm{Diff}_\partial(W_{g-1,1});\mathbb{Z}) \to H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Z})$$

is surjective for $2d \leq g-1$ *and an isomorphism for* $2d \leq g-3$.

Proof. By proceeding as in the proofs of Theorems C and D, we can reduce it to prove that $H_{(g,0),d}(\overline{\mathbf{R}_\mathbb{Z}}/\sigma_0) = 0$ for $2g \leq d-1$. By the universal coefficient theorem and the finite generation of the homology groups of $\overline{\mathbf{R}}$ (by [KRW20, Theorem 6.1, Remark 6.2]) it suffices to check that $H_{(g,0),d}(\overline{\mathbf{R}_{\mathbb{Z}[1/2]}}/\sigma_0) = 0$ for $2g \leq d-1$ and $H_{(g,0),d}(\overline{\mathbf{R}_{\mathbb{F}_2}}/\sigma_0) = 0$ for $2g \leq d-1$.

The second of these follows from Corollary 2.2.7, where all the assumptions are verified as we did when proving Theorem C itself. The first one follows from (the proof of) Theorem C(iii) provided $d \geq 2$. For $d = 1$ the result follows from Corollary 4.2.11(iii) and for $d = 0$ the result is trivial. □

4.3 Splitting complexes and the proof of Theorem E

In this section we will prove Theorem E. The overall argument is based on [GKRW19, Section 4] but we will need some additional steps to deal with issues associated to the non-discreteness of the diffeomorphism groups.

4.3.1 The arc complex

We will make use of a high-dimensional analogue of an "arc complex", inspired by [GKRW19, Definition 4.7]. The intuition is to replace the choice of two points in the boundary by an isotopy class of embeddings of S^{n-1}, to replace arcs by embeddings of D^n, and to replace "non-separating" with the assumption that the corresponding "cut manifold" remains $(n-1)$-connected. Observe that when $n = 1$ we basically recover the (unordered) arc complex for surfaces, except that in the surface case arcs are usually taken up to isotopy.

Definition 4.3.1. *A valid geometric data is a pair* (W,Δ) *where* $W \in \mathcal{M}[A]$, *and* Δ *is an isotopy class of embeddings* $S^{n-1} \hookrightarrow \{1\} \times \mathrm{int}(A) \subset \partial W$.

Definition 4.3.2. *Given a valid geometric data* (W,Δ), *we define the arc complex* $\mathcal{A}(W,\Delta)$ *to be the following simplicial complex:*

(1) A vertex is an embedding $a : D^n \hookrightarrow W$ *such that*

- *(i) Its boundary* $\partial a := a|_{\partial D^n}$ *has image contained in* $\mathrm{int}(A) \subset W$ *and lies in the isotopy class* Δ.
- *(ii)* a *intersects* ∂W *only in* ∂a *and the intersection is transversal.*
- *(iii) The cut manifold* $W \smallsetminus a := W \smallsetminus a(D^n)$ *is* $(n-1)$*-connected.*

(2) Vertices $a_0,\cdots,a_p$ *span a p-simplex if and only if*

- *(i) The images of the embeddings* a_i *are pairwise disjoint.*
- *(ii) The (jointly) cut manifold* $W \smallsetminus \{a_0,\cdots,a_p\} := W \smallsetminus \bigsqcup_{i=0}^{p} a_i(D^n)$ *is* $(n-1)$*-connected.*

We will need two key properties of the arc complex, which will be shown later in Section 4.4. The first one is an analogue of [GKRW19, Theorem 4.8] and says that the arc complex is highly connected. The second one gives lower bounds on the genus of the (jointly) cut manifold, which is defined to be the genus of its middle homology, see Section 4.3.5 for details. We will state both results here as they will be needed in this section.

Theorem 4.3.3. *If* (W,Δ) *is a valid geometric data then* $\mathcal{A}(W,\Delta)$ *is* $(g(W)-2)$*-connected.*

Proposition 4.3.4. *Let* (W,Δ) *be a valid geometric data and* $\{a_0,\cdots,a_p\} \in \mathcal{A}(W,\Delta)$ *be a p-simplex. Then*

(i) $\mathrm{rk}(H_n(W \smallsetminus \{a_0,\cdots,a_p\})) = \mathrm{rk}(H_n(W)) - (p+1) < \mathrm{rk}(H_n(W))$.

(ii) $g(W \smallsetminus \{a_0,\cdots,a_p\}) \geq g(W)-(p+1)$. *Moreover if* $H_{n-1}(A)=H_{n-1}(\partial W)\neq 0$ *and* $\delta := \Delta_*([S^{n-1}]) \in H_{n-1}(\partial W)$ *generates a direct summand of maximum order then we have* $g(W \smallsetminus \{a_0,\cdots,a_p\}) \geq g(W)-p$.

As we will see in Remark 4.4.10 the proof of the above proposition fundamentally uses n odd. In fact, we will see in Example 5.2.4 that it is false for n even.

4.3.2 Definition of the splitting complexes and posets

Definition 4.3.5. *For* $W \in \mathcal{M}[A]$ *we define the (E_1-)splitting complex* $S^{E_1}_\bullet(W)$ *to be the following semisimplicial space:*

(1) The space of 0*-simplices,* $S^{E_1}_0(W)$*, is given as a set by the collection of triples* (ω,t,ε)*, called "walls", where* $\omega : [t-\varepsilon,t+\varepsilon]\times I^{2n-1} \hookrightarrow W$ *is an embedding,* $0<t-\varepsilon<t<t+\varepsilon<1$*, such that*

- *(i)* ω *agrees pointwise with the inclusion* $[t-\varepsilon,t+\varepsilon]\times I^{2n-1}\hookrightarrow I^{2n}$ *in a neighbourhood of* $[t-\varepsilon,t+\varepsilon]\times \overline{\partial I^{2n-1}\smallsetminus I^{2n-2}\times\{1\}}$ *in* $[t-\varepsilon,t+\varepsilon]\times I^{2n-1}$*. In other words,* ω *looks standard near its boundary except the top face. Moreover,* ω *maps the top face* $[t-\varepsilon,t+\varepsilon]\times I^{2n-2}\times\{1\}$ *inside the interior of the top face* $\mathcal{D}(W)=\partial W\cap(I^{2n-1}\times\{1\}\times\mathbb{R}^\infty)$*, and* ω *maps* $[t-\varepsilon,t+\varepsilon]\times \mathrm{int}(I^{2n-1})$ *inside the interior of* W*.*
- *(ii)* ω *is compatible with the product structure near its top face in the sense of Definition 4.1.1. In particular,* ω *is transversal to* ∂W*.*
- *(iii)* $W\smallsetminus \mathrm{im}(\omega|_{\{t\}\times I^{2n-1}})$ *has precisely two path components. We denote by* $W_{\leq\omega}$ *the left region, i.e. the closure of the path-component containing* $\{0\}\times I^{2n-1}$*, and by* $W_{\geq\omega}$ *right region, i.e. the closure of the other path-component.*
- *(iv) Both* $H_n(W_{\leq\omega})$ *and* $H_n(W_{\geq\omega})$ *are non-zero.*

We then topologise it as a subspace of $\mathrm{Emb}([-1,1]\times I^{2n-1}, I^{2n}\times\mathbb{R}^\infty)\times(0,1)^2$ *by using the parameters* t,ε *to reparametrise the embedding* ω*.*

(2) For $p\geq 0$*,* $S^{E_1}_p(W)\subset (S^{E_1}_0(W))^{p+1}$ *consists of the subspace of* $(p+1)$*-tuples of* 0*-simplices* $((\omega_0,t_0,\varepsilon_0),\cdots,(\omega_p,t_p,\varepsilon_p))$ *such that* $t_i+\varepsilon_i<t_{i+1}-\varepsilon_{i+1}$*,* $\mathrm{im}(\omega_i)\subset \mathrm{int}(W_{\leq\omega_{i+1}})$*, and the region of* W *in between these, denoted* $W_{\omega_i\leq-\leq\omega_{i+1}}$*, has* $H_n(W_{\omega_i\leq-\leq\omega_{i+1}})$ *non-zero for* $0\leq i\leq p-1$*.*

(3) The i-th face map is given by forgetting $(\omega_i,t_i,\varepsilon_i)$*.*

We denote by $S^{E_1}(W) := \|S^{E_1}_\bullet(W)\|$ *its geometric realization.*

The thickness of the walls will be relevant because it gives $W_{\leq\omega}$, $W_{\geq\omega}$ and $W_{\omega\leq-\leq\omega'}$ product structures near their boundaries, but from an intuitive point of view one should think of a wall as a codimension one disc cutting the manifold into two “nice” pieces. In fact, we will often abuse notation and remove t,ε and say “$\omega \in S^{E_1}(W)$ is a wall”.

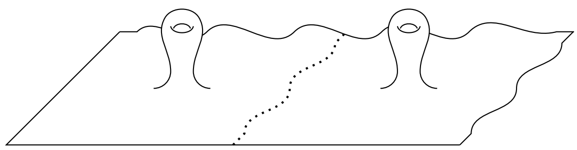


Fig. 4.3 A 0-simplex represented by the dotted line.

In Section 4.3.7 we will study the connection between $S^{E_1}_\bullet(W)$ and E_1-homology, hence justifying the name “E_1-splitting complex”, c.f. [GKRW18, Section 17.2].

In order to understand the above semisimplicial space it will be useful to consider its levelwise discretization too:

Definition 4.3.6. *For $W \in \mathcal{M}[A]$ we define the discretized (E_1-)splitting complex $S^{E_1,\delta}_\bullet(W)$ to be the semisimplicial set obtained by taking the levelwise discretization of $S^{E_1}_\bullet(W)$ (i.e. by using the discrete topology instead of the C^∞-topology on each layer of the semisimplicial object).*

We will view the above semisimplicial objects as nerves of certain posets. To do so, let us firstly define what we mean by topological posets and their nerves.

Definition 4.3.7. *A (non-unital) topological poset $(P,<)$ is a topological space P with a strict partial ordering $<$ on the underlying set of P.*

Any (non-unital) topological poset $(P,<)$ can be viewed as a non-unital topological category in the sense of [ERW19, Definition 3.1] by taking the space of objects to be $P_0 = P$, the space of morphisms to be $P_1 = \{(a,b) \in P^2 : a < b\}$, and the source and target maps given by the projections onto the first and second coordinates respectively. The composition map is then canonically defined by the axioms of a partial ordering. Thus, the nerve $N_\bullet P$ defines a semisimplicial space whose layers can be explicitly described via $P_p = N_pP := \{x_0 < x_1 < \cdots < x_p : x_i \in P\}$, topologized as a subspace of P^{p+1}, where the i-th face map forgets x_i.

Example 4.3.8. *The E_1-splitting complex $S^{E_1}_\bullet(W)$ of Definition 4.3.5 is the nerve of the non-unital topological poset $\mathcal{S}^{E_1}(W)$ whose space of elements is the space of 0-simplices $S^{E_1}_0(W)$, and whose partial ordering $<$ is given by the space of 1-simplices.*

Example 4.3.9. *Given any topological poset $(P,<)$ we can construct its discretization $(P^{\delta},<)$, whose underlying set is P with the discrete topology and where we use the same strict partial ordering. By definition $P^{\delta}_{\bullet}$ is the semisimplicial set given by levelwise discretizing $P_{\bullet}$. In particular, we can form the discretized (E_1)-splitting poset $\mathcal{S}^{E_1,\delta}(W)$ whose nerve is the discretized splitting complex.*

4.3.3 Recollection on simplicial complexes and posets

To state some of the intermediate results to prove Theorem E we need some definitions and constructions about simplicial complexes and posets. All the complexes and posets appearing in this subsection are assumed to be discrete.

Let us begin by recalling the relationship between posets and simplicial complexes: any poset gives a simplicial complex whose vertices are the elements of the poset, and where a set of $p+1$ vertices defines a p-simplex if and only if they are strictly ordered. This simplicial complex has the same geometric realization as the nerve of the poset. Conversely, any simplicial complex has an associated *face poset* whose elements are the simplices and the partial ordering is given by inclusion. The associated simplicial complex to the face poset of any complex is the barycentric subdivision of the original complex, and hence homeomorphic to the original one.

For the rest of the chapter we will refer to topological properties of a poset to mean the properties of its associated simplicial complex. Thus, the topological properties of a simplicial complex agree with the ones of its face poset; in the rest of the chapter we will not make distinctions between topological properties holding for a simplicial complex or its face poset.

If $(P,<)$ is a poset and $x\in P$ we let

$$\dim(x) := \sup\{n : \text{ there is a chain } x_1<\cdots<x_n<x\},$$

and we define the *dimension* of $(P,<)$ via $\dim(P) := \sup_{x\in P}\{\dim(x)\}\in\mathbb{N}\cup\{\infty\}$, which agrees with the dimension of the corresponding simplicial complex.

Definition 4.3.10. *For a given function of sets $f:P\to\mathbb{Z}$ we say that $(P,<)$ is f-weakly Cohen-Macaulay of dimension n if the following holds*

(i) P is $(n-1)$-connected.

(ii) For each $x\in P$ the poset $P_{<x}$ is $(f(x)-2)$-connected.

(iii) For each $x\in P$ the poset $P_{>x}$ is $(n-2-f(x))$-connected.

(iv) For each $x < y$ in P the poset $P_{x<-<y} = P_{>x} \cap P_{<y}$ is $(f(y) - f(x) - 3)$-connected.

When $f = \dim(-)$ we say that P is weakly Cohen-Macaulay of dimension n.

We will say that a simplicial complex is weakly Cohen-Macaulay of dimension n if its associated face poset has the same property. In this case conditions (ii) and (iv) above are automatic as we take $f = \dim(-)$, and condition (iii) is equivalent to saying that for a p-simplex σ, its link $\mathrm{Lk}(\sigma)$ is $(n-p-2)$-connected.

4.3.4 Some technical results

One of the main steps in proving Theorem E will be Corollary 4.3.19, which says that for $W \in \mathcal{M}[I^{2n-1}]$ the discretized splitting poset $\mathcal{S}^{E_1,\delta}(W)$ is f-weakly Cohen-Macaulay of a certain dimension. In this section we will study the posets $\mathcal{S}^{E_1,\delta}(W)_{<\omega}$, $\mathcal{S}^{E_1,\delta}(W)_{>\omega}$ and $\mathcal{S}^{E_1,\delta}(W)_{\omega<-<\omega'}$ and show that they are isomorphic to splitting posets of certain manifolds. Finally we will show a lemma about splitting posets of cut manifolds, which will be used in the proof of Corollary 4.3.19.

Lemma 4.3.11. *Let $W, W' \in \mathcal{M}[A]$ lie in the same path-component. Then $\mathcal{S}_0^{E_1}(W)$ and $\mathcal{S}_0^{E_1}(W')$ are isomorphic topological posets. Thus, their discretizations are also isomorphic.*

Proof. Let $\phi : W \xrightarrow{\cong} W'$ be a diffeomorphism fixing pointwise a neighbourhood of $\partial^- W = \partial^- W'$ and preserving the product structures (the existence of such ϕ follows from Proposition 4.1.2). Then we get an induced map $\phi_* : \mathcal{S}_0^{E_1}(W) \to \mathcal{S}_0^{E_1}(W')$, $(\omega, t, \varepsilon) \mapsto (\phi \circ \omega, t, \varepsilon)$, which is order-preserving and has an inverse given by $(\phi^{-1})_*$. □

The next technical result will allow us to view the manifolds $W_{\leq\omega}$, $W_{\geq\omega}$ as elements in the moduli spaces of Definition 4.1.1.

Lemma 4.3.12. *Let $W \in \mathcal{M}[A]$ and $(\omega, t, \varepsilon) \in S_0^{E_1}(W)$, then*

(a) The inclusion $W_{\leq\omega} \hookrightarrow I^{2n} \times \mathbb{R}^\infty$ is isotopic to an embedding $e : W_{\leq\omega} \hookrightarrow I^{2n} \times \mathbb{R}^\infty$ such that

- *(i)* $\mathrm{im}(e) \in \mathcal{M}[I^{2n-1}]$.
- *(ii) On a neighbourhood of*

$$\partial W_{\leq\omega} \cap (J_{2n-1} \times \mathbb{R}^\infty) = J_{2n-1} \cap \{p = (x_1, \cdots, x_{2n}, y) \in I^{2n} \times \mathbb{R}^\infty : 0 \leq x_1 \leq t\}$$

  it agrees with $(x_1, \cdots, x_{2n}, 0) \mapsto (x_1/t, x_2, \cdots, x_{2n}, 0)$.
- *(iii) $e \circ \omega|_{\{t\} \times I^{2n-1}}$ agrees with the standard inclusion $I^{2n-1} \cong \{1\} \times I^{2n-1} \times \{0\} \subset I^{2n} \times \mathbb{R}^\infty$.*

(iv) e gives a diffeomorphism from $\mathcal{D}(W)\cap\partial W_{\leq\omega}$ to $\mathcal{D}(\mathrm{im}(e))$.

(v) e preserves the product structures (in the sense of Definition 4.1.1).

Moreover, $[\mathrm{im}(e)]\in\pi_0(\mathcal{M}[I^{2n-1}])$ *is independent of the choice of embedding e satisfying all the above conditions.*

(b) The inclusion $W_{\geq\omega}\hookrightarrow I^{2n}\times\mathbb{R}^\infty$ is isotopic to an embedding $e: W_{\geq\omega}\hookrightarrow I^{2n}\times\mathbb{R}^\infty$ such that

(i) $\mathrm{im}(e)\in\mathcal{M}[A]$.

(ii) On a neighbourhood of

$$\partial W_{\geq\omega}\cap(J_{2n-1}\times\mathbb{R}^\infty)=J_{2n-1}\cap\{p=(x_1,\cdots,x_{2n},y)\in I^{2n}\times\mathbb{R}^\infty : t\leq x_1\leq 1\}$$

it agrees with $(x_1,\cdots,x_{2n},0)\mapsto((x_1-t)/(1-t),x_2,\cdots,x_{2n},0)$.

(iii) $e\circ\omega|_{\{t\}\times I^{2n-1}}$ agrees with the standard inclusion $I^{2n-1}\cong\{0\}\times I^{2n-1}\times\{0\}\subset I^{2n}\times\mathbb{R}^\infty$.

(iv) e gives a diffeomorphism from $\mathcal{D}(W)\cap\partial W_{\geq\omega}$ to $\mathcal{D}(\mathrm{im}(e))$.

(v) e preserves the product structures.

Moreover, $[\mathrm{im}(e)]\in\pi_0(\mathcal{M}[A])$ *is independent of the choice of embedding e satisfying all the above conditions.*

Proof. The proofs of (a) and (b) are almost identical, so we will focus on part (a). Let us show the existence of e first and then the uniqueness of the path-component.

We can decompose $\partial W_{\leq\omega}$ as $\partial W_{\leq\omega}\cap(J_{2n-1}\times\mathbb{R}^\infty)\cup\mathrm{im}(\omega|_{\{t\}\times I^{2n-1}})\cup\mathcal{D}(W)\cap\partial W_{\leq\omega}$, and conditions (a)(ii) and (a)(iii) tell us what e should do on the first two pieces of the decomposition. Moreover condition (a)(v) forces the behaviour of e on a closed neighbourhood of the first two pieces of the boundary, and by construction this partial embedding e is isotopic to the inclusion where defined. Now we use isotopy extension to extend it to the remaining piece of the boundary $\mathcal{D}(W)\cap\partial W_{\leq\omega}$ in such a way that e sends it inside $I^{2n-1}\times\{1\}\times\mathbb{R}^\infty$. We then use condition (a)(v) again to extend e to a closed neighbourhood of the whole boundary $\partial W_{\leq\omega}$. Finally we use isotopy extension again to define e on the whole of $W_{\leq\omega}$ such that its interior maps to $\mathrm{int}(I^{2n})\times\mathbb{R}^\infty$.

It remains to check that the manifold $\mathrm{im}(e)\subset I^{2n}\times\mathbb{R}^\infty$ lies in $\mathcal{M}[I^{2n-1}]$: the $(n-1)$-connectivity follows by Seifert-Van Kampen and Mayer-Vietoris applied to the decomposition $W=W_{\leq\omega}\cup_{I^{2n-1}}W_{\geq\omega}$, the s-parallelizability is immediate as W is itself s-parallelizable. Conditions (ii),(iii),(iv)(a),(v) and (vi) of Definition 4.1.1 hold by construction; and hence it remains to verify condition (iv)(b), i.e. that $\mathcal{D}(\mathrm{im}(e))$ is contractible. This follows by another application of Seifert-Van Kampen and Mayer-Vietoris, this time to $\mathcal{D}(W)$ decomposed as a union of two pieces along I^{2n-2}.

For the uniqueness of the path-component suppose that e' is another embedding, then $\partial^-(\mathrm{im}(e)) = \partial^-(\mathrm{im}(e')) = J_{2n-1} \cup \{1\} \times I^{2n-1}$ and $e' \circ e^{-1} \in \mathrm{Emb}^p_{\partial^- \mathrm{im}(e)}(\mathrm{im}(e), I^{2n} \times \mathbb{R}^\infty)$ by condition a(v). Since the embedding space is path-connected by the Whitney embedding theorem then the result follows. □

As a consequence of Lemmas 4.3.11 and 4.3.12 we can meaningfully write $\mathcal{S}^{E_1}(W_{\leq\omega})$, $\mathcal{S}^{E_1}(W_{\geq\omega})$ and their discretized analogues to really mean the corresponding splitting complexes on $\mathrm{im}(e)$ for e as in Lemma 4.3.12. By consecutively applying parts (a) and (b) Lemma 4.3.12 one can also make sense of $\mathcal{S}^{E_1}(W_{\omega\leq-\leq\omega'})$.

Corollary 4.3.13. *For $W \in \mathcal{M}[A]$ and $(\omega, t, \varepsilon) \in S_0^{E_1}(W)$ we have isomorphism of posets*

(i) $\mathcal{S}^{E_1,\delta}(W)_{<\omega} \cong \mathcal{S}^{E_1,\delta}(W_{\leq\omega})$.

(ii) $\mathcal{S}^{E_1,\delta}(W)_{>\omega} \cong \mathcal{S}^{E_1,\delta}(W_{\geq\omega})$.

Moreover for $(\omega', t', \varepsilon') > (\omega, t, \varepsilon)$ we have $\mathcal{S}^{E_1,\delta}(W)_{\omega<-<\omega'} \cong \mathcal{S}^{E_1,\delta}(W_{\omega\leq-\leq\omega'})$.

Proof. Part (i): the map is given by sending $(\tilde{\omega} < \omega) \mapsto e \circ \tilde{\omega}$ and reparametrizing its "thickening coordinate" so that its centre is at coordinate $\tilde{t}/t$ and is compatible with the product. Let us check that it indeed defines a wall: conditions (i), (ii) and (iii) of Definition 4.3.5 hold automatically. Condition (iv) holds because $W_{\leq\tilde{\omega}} = (W_{\leq\omega})_{\leq\tilde{\omega}}$ and $W_{\tilde{\omega}\leq-\leq\omega} = (W_{\leq\omega})_{\geq\tilde{\omega}}$, so both have non-zero n-th homology.

The fact that the correspondence is a bijection is because we can write an inverse map by precomposing with e^{-1} and reparametrizing.

Part (ii) is similar to part (i).

For the moreover part we firstly apply part (i) to identify $\mathcal{S}^{E_1,\delta}(W)_{<\omega'} \cong \mathcal{S}^{E_1,\delta}(W_{\leq\omega'})$ and then part (ii) on the right hand side with ω. □

In a similar flavour we have the following lemma which allows us to make sense of the splitting complex of the manifold obtained by cutting a collection of arcs, and to give an interpretation to it. It will be very useful in the next subsection. Before stating it we will need a small variation of the discretized splitting complex which will also be useful in the next subsection.

Definition 4.3.14. *For $W \in \mathcal{M}[A]$ we define the discrete poset $\tilde{\mathcal{S}}(W)$ to be same as $\mathcal{S}^{E_1,\delta}(W)$ but changing condition (1).(iv) on each wall ω to $H_n(W_{\leq\omega}) \neq 0$ only, so there is no condition on $H_n(W_{\geq\omega})$. The ordering relation is identical to the one of Definition 4.3.5.*

Lemma 4.3.15. *Let (W,Δ) be a valid geometric data where $W \in \mathcal{M}[A]$ and let $\alpha = \{a_0, \cdots, a_p\}$ be a p-simplex in $\mathcal{A}(W,\Delta)$. Denote by A' the result of performing surgery on A along some appropriate framings of the boundaries ∂a_i for $0 \le i \le p$; then A' is valid, there is $W' \in \mathcal{M}[A']$ such that $\mathcal{D}(W') = \mathcal{D}(W)$ and*

(i) The interiors of W' and $W \setminus \alpha$ are diffeomorphic via a diffeomorphism ϕ which is the identity in a neighbourhood of $J_{2n-1} \cup \mathcal{D}(W)$.

(ii) Extending ϕ by the identity on $J_{2n-1} \cup \mathcal{D}(W)$ induces a bijection between the poset of walls $\omega \in \mathcal{S}^{E_1,\delta}(W)$ such that $\mathrm{im}(a_i)$ lies strictly to the right of $\mathrm{im}(\omega)$ for $0 \le i \le p$ and the poset $\tilde{\mathcal{S}}(W')$.

The reason why we need $\tilde{\mathcal{S}}(W')$ in the statement is that the wall ω' we get in W' might satisfy that $H_n(W'_{\ge\omega'}) = 0$. However, condition $H_n(W'_{\le\omega'}) \ne 0$ will be guaranteed because by the construction of the bijection in the following proof we have $W'_{\le\omega'} \cong W_{\le\omega}$.

Proof. For each arc a_i pick a trivialization of its normal bundle to get an embedding $\tilde{a}_i : D^n \times D^n \hookrightarrow W$ such that $\tilde{a}_i|_{D^n\times\{0\}} = a_i$, $\tilde{a}_i(\partial D^n \times D^n) \subset \mathrm{int}(A)$, $\tilde{a}_i$ intersects ∂W transversally and $\tilde{a}_i^{-1}(\partial W) = \partial D^n \times D^n$. Shrink the thickenings if needed to ensure that the images of the $\tilde{a}_i$ are pairwise disjoint. Let $\tilde{W} := W \setminus \bigsqcup_{i=0}^{p} \tilde{a}_i(D^n \times \mathrm{int}(D^n))$. Then $\tilde{W}$ is a compact $2n$-manifold with boundary, whose boundary is the result of performing surgeries to ∂W along the embeddings $\tilde{a}_i|_{\partial D^n\times D^n}$; and hence we can decompose it as $J_{2n-1} \cup \mathcal{D}(W) \cup A'$, where A' is the result of performing the surgeries on A. Since A is $(n-2)$-connected then so is A', and since the surgeries take place in the interior of A then A and A' agree on a neighbourhood of their boundary. Moreover, $\mathrm{int}(\tilde{W})$ is diffeomorphic to $\mathrm{int}(W \setminus \alpha)$ via a diffeomorphism $\tilde{\phi}$ which can be chosen to be the identity near $J_{2n-1} \cup \mathcal{D}(W)$.

Using isotopy extension, we pick an isotopy from the inclusion $A' \hookrightarrow I^{2n} \times \mathbb{R}^\infty$ to an embedding with image contained in the rightmost face $\{1\} \times I^{2n-1} \times \mathbb{R}^\infty$ such that the isotopy is constant near the boundary of A'. We will abuse notation and call its image A' too. We then extend the isotopy to one on $\tilde{W}$ such that it is constant on $J_{2n-1} \cup \mathcal{D}(W)$, and we let $W' \subset I^{2n} \times \mathbb{R}^\infty$ be the resulting manifold at the end of the isotopy. One can proceed as in Lemma 4.3.12 to ensure that W' has a product structure near the boundary compatible with the one on $J_{2n-1} \cup \mathcal{D}(W)$. We let ϕ be the composition of $\tilde{\phi}$ with the end map of the isotopy, so that property (i) is satisfied.

For part (ii) the bijection is given as follows: take a wall $\omega \in \mathcal{S}^{E_1}(W)$ such that $\mathrm{im}(a_i)$ lies strictly to the right of $\mathrm{im}(\omega)$ for all i, then $\mathrm{im}(\omega) \subset W \setminus \alpha$, so we can precompose it with ϕ^{-1} so that it lies in W', and then it will define a wall there because ϕ is the identity near $J_{2n-1} \cup \mathcal{D}(W)$, so the standard part of the wall and the product structure are preserved. The inverse map is given analogously but by precomposing with ϕ instead. All that is left to show is that for any $\omega' \in \tilde{\mathcal{S}}(W')$ we have $H_n(W_{\geq \phi\circ\omega'}) \neq 0$, so that $\omega := \phi \circ \omega'$ is a wall in $\mathcal{S}^{E_1}(W)$. To do so, let us pick an arc a_0 in α, and let $\zeta = (a_0)_*([D^n, \partial D^n]) \in H_n(W_{\geq\omega}, \partial W_{\geq\omega})$. By Proposition 4.4.3 (which will appear later in this thesis, but whose proof is elementary and based on Mayer-Vietoris and excision) we have that ζ is unimodular, hence non-zero, in $H_n(W_{\geq\omega}, \partial W_{\geq\omega}) \cong H^n(W_{\geq\omega})$. Thus, by the universal coefficients theorem and the $(n-1)$-connectivity of $W_{\geq\omega}$ we get the result. $\square$

4.3.5 Connectivity of discretized splitting complexes

The goal of this section is to show that for any $W \in \mathcal{M}[I^{2n-1}]$, its discretized splitting poset is f-weakly Cohen-Macaulay of dimension $g(W)-2$ for some appropriate f. In order to do so we will do an inductive argument on W, and we will allow $W \in \mathcal{M}[A]$ at the inductive step for a general A as in Definition 4.1.1; so we will in fact prove a more general result.

In order to state this general result let us define *genus* for a general $W \in \mathcal{M}[A]$, generalizing the case $A = I^{2n-1}$ treated in Section 4.1.4: the pair $(H_n(W), \lambda_W)$ defines a skew-symmetric bilinear form, and the *genus* of W is

$$g(W) := \sup\{g : \exists \text{ morphism } \phi : H^{\oplus g} \to (H_n(W), \lambda_W)\},$$

which recovers the previous definition when $A = I^{2n-1}$. We will study skew-symmetric forms in more detail and show some basic properties of the genus in Section 4.4.1; but for now the property that we need is: $g(W) = g(W_{\leq\omega}) + g(W_{\geq\omega})$, which is a consequence of the additivity of the genus under orthogonal direct sums, see Section 4.4.1, plus the decomposition $(H_n(W), \lambda_W) \cong (H_n(W_{\leq\omega}), \lambda_{W_{\leq\omega}}) \oplus (H_n(W_{\geq\omega}), \lambda_{W_{\geq\omega}})$ from Mayer-Vietoris.

Theorem 4.3.16. *Let $W \in \mathcal{M}[A]$, then*

(i) $\mathcal{S}^{E_1,\delta}(W)$ is $(g(W)-3+C(A))$-connected, where $C(A) = 0$ if $H_{n-1}(A) = 0$ and $C(A) = 1$ otherwise.

(ii) $\tilde{\mathcal{S}}(W)$ is $(g(W)-2)$-connected.

Before proving it let us show a technical lemma.

Lemma 4.3.17. *(i) If $H_{n-1}(A) \neq 0$ then $\tilde{\mathcal{S}}(W) = \mathcal{S}^{E_1,\delta}(W)$.*

(ii) If $H_{n-1}(A) = 0$ then $\mathcal{S}^{E_1,\delta}(W)$ is a subposet of $\tilde{\mathcal{S}}(W)$ and the poset $\tilde{\mathcal{S}}(W) \setminus \mathcal{S}^{E_1,\delta}(W)$ has no relations, and for any $\omega \in \tilde{\mathcal{S}}(W) \setminus \mathcal{S}^{E_1,\delta}(W)$ we have $\tilde{\mathcal{S}}(W)_{>\omega} = \varnothing$ and $\tilde{\mathcal{S}}(W)_{<\omega} \cong \mathcal{S}^{E_1,\delta}(W_{\leq\omega})$.

(iii) If $H_{n-1}(A) = 0$ then the inclusion $\mathcal{S}^{E_1,\delta}(W) \hookrightarrow \tilde{\mathcal{S}}(W)$ induces the zero map in homotopy groups.

Parts (i),(ii) are some basic properties, whereas part (iii) is more technical and it will be used in the proof of Theorem 4.3.16 to deduce that $\tilde{\mathcal{S}}(W)$ has more connectivity than $\mathcal{S}^{E_1,\delta}(W)$ when $H_{n-1}(A) = 0$.

Proof. There is a natural inclusion of posets $\mathcal{S}^{E_1,\delta}(W) \subset \tilde{\mathcal{S}}(W)$.

Part (i). We will show that the inclusion is surjective. Let $\omega \in \tilde{\mathcal{S}}(W)$, we will show that $\omega \in \mathcal{S}^{E_1,\delta}(W)$, i.e. that $H_n(W_{\geq\omega}) \neq 0$.

By Lemma 4.3.12 we can view $W_{\geq\omega}$ as an element in $\mathcal{M}[A]$, so its boundary is the union of A and a contractible $(2n-1)$-manifold along $\partial A = \partial I^{2n-1}$, hence $H_{n-1}(\partial W_{\geq\omega}) \cong H_{n-1}(A) \neq 0$. Since $W_{\leq\omega}$ is $(n-1)$-connected, the homology long exact sequence of $(W_{\leq\omega}, \partial W_{\leq\omega})$ gives that $H_n(W_{\leq\omega}, \partial W_{\leq\omega}) \to H_{n-1}(A)$ is surjective, and so by Poincaré-Lefschetz it follows that $H^n(W_{\geq\omega}) \neq 0$. Thus $H_n(W_{\geq\omega}) \neq 0$ by the universal coefficient theorem and the $(n-1)$-connectivity of $W_{\geq\omega}$.

Part (ii). For any $\omega \in \tilde{\mathcal{S}}(W) \setminus \mathcal{S}^{E_1,\delta}(W)$ we have $H_n(W_{\geq\omega}) = 0$, and since the piece in between two walls has non-zero n-th homology it follows that $\mathcal{S}^{E_1,\delta}(W)_{>\omega} = \varnothing$ and that $\tilde{\mathcal{S}}(W) \setminus \mathcal{S}^{E_1,\delta}(W)$ has no relations. By an analogous argument to Corollary 4.3.13, $\tilde{\mathcal{S}}(W)_{<\omega} \cong \mathcal{S}^{E_1,\delta}(W_{\leq\omega})$.

Part (iii). By simplicial approximation and considering the map of associated simplicial complexes it suffices to show that given any finite collection of elements $\omega_1, \cdots, \omega_u \in \mathcal{S}^{E_1,\delta}(W)$ we can pick $\omega \in \tilde{\mathcal{S}}(W)$ such that $\omega_i < \omega$ for $1 \leq i \leq u$.

By (1).(i) in Definition 4.3.5, each ω_i is disjoint from A, hence we can use the product structure of W to pick $\varepsilon > 0$ such that $[1-\varepsilon, 1] \times A \subset W$ is contained in $W_{\geq\omega_i}$ for all i. Since by assumption A is $(2n-1)$-dimensional, $(n-2)$-connected, has boundary ∂I^{2n-1} and $H_{n-1}(A) = 0$ then Poincaré-Lefschetz duality and Whitehead's theorem imply that A is contractible. By the h-cobordism theorem we find that A is diffeomorphic to I^{2n-1}, but not necessarily diffeomorphic to I^{2n-1} relative to $\partial A = \partial I^{2n-1}$ (because of the existence of exotic spheres in generic dimensions).

Fix a diffeomorphism $\phi : I^{2n-1} \to A$, which by isotopy extension we can assume to be the identity on a neighbourhood of $\overline{\partial I^{2n-1} \setminus I^{2n-2} \times \{1\}}$. Moreover, by geodesic interpolation we can further isotope it so that it preserves the natural product structure of I^{2n-1} near its top face $I^{2n-2} \times \{1\}$, as in Lemma 4.1.3. Then the composition ω given by $[1-\varepsilon/2, 1-\varepsilon/4] \times I^{2n-1} \xrightarrow{\mathrm{id}\times\phi} [1-\varepsilon/2, 1-\varepsilon/4] \times A \hookrightarrow W$ defines a wall in $\tilde{\mathcal{S}}(W)$ lying to the right of all the ω_i.

Finally, to check that $\omega_i < \omega$ we just need to check that $H_n(W_{\omega_i \leq - \leq \omega}) \neq 0$: we have $H_n(W_{\geq \omega_i}) = H_n(W_{\omega_i \leq - \leq \omega}) \oplus H_n(W_{\geq \omega})$, and the left-hand-side is non-zero by assumption whereas the second summand of the right-hand-side vanishes, giving the result. □

Remark 4.3.18. *The proof of part (iii) explains why in Definition 4.1.1 we wanted to allow $\mathcal{D}(W)$ to be a homotopy disc in general: the wall ω constructed above cannot be chosen to be standard near all its boundary if A is not diffeomorphic to I^{2n-1} relative its whole boundary. Thus we need to allow the walls to be non-standard on the top face, which means that the boundaries of the left pieces of the walls are allowed to be homotopy spheres. As we will see in the proof of Proposition 4.3.25 this suggests that the E_k-algebra under study should contain manifolds whose boundary is a homotopy sphere too.*

Now we can finally proof the main result of the section.

Proof of Theorem 4.3.16. We will prove it by induction on $\mathrm{rk}(H_n(W))$: when $\mathrm{rk}(H_n(W)) = 0$ both (i) and (ii) are vacuously true since $g(W) = 0$ too.

Step 1. We will show part (ii) assuming that part (i) holds for any $W' \in \mathcal{M}[A']$ such that $\mathrm{rk}(H_n(W')) \leq \mathrm{rk}(H_n(W))$, where A' is any valid choice. We have two cases:

If $H_{n-1}(A) \neq 0$ then Lemma 4.3.17 (i) gives $\tilde{\mathcal{S}}(W) = \mathcal{S}^{E_1,\delta}(W)$, which by induction is $(g(W) - 2)$-connected, as required.

If $H_{n-1}(A) = 0$ then view $\mathcal{S}^{E_1,\delta}(W)$ and $\tilde{\mathcal{S}}(W)$ as simplicial complexes and consider the inclusion $\mathcal{S}^{E_1,\delta}(W) \hookrightarrow \tilde{\mathcal{S}}(W)$. The smaller complex is a full subcomplex of the other one; and if σ is a p-simplex in $\tilde{\mathcal{S}}(W)$ with no vertex in $\mathcal{S}^{E_1,\delta}(W)$ then we must have $p = 0$ by Lemma 4.3.17(ii) so that $\sigma = \{\omega\}$. Also, $\mathrm{Lk}(\sigma) \cap \mathcal{S}^{E_1,\delta}(W)$ is the simplicial complex corresponding to the poset $\mathcal{S}^{E_1,\delta}(W_{\leq \omega})$ by Lemma 4.3.17(ii), which is $(g(W_{\leq \omega}) - 3)$-connected by induction. Moreover, additivity of the genus and $\omega \notin \mathcal{S}^{E_1,\delta}(W)$ implies that $g(W_{\leq \omega}) = g(W)$. Therefore the inclusion $\mathcal{S}^{E_1,\delta}(W) \hookrightarrow \tilde{\mathcal{S}}(W)$ is $(g(W) - 2)$-connected by [GRW18, Proposition 2.5]. Thus, by Lemma 4.3.17(iii) we get that $\tilde{\mathcal{S}}(W)$ is $(g(W) - 2)$-connected, as required.

Step 2. We will show part (i) assuming both parts (i) and (ii) hold for any $W' \in \mathcal{M}[A']$ such that $\mathrm{rk}(H_n(W')) < \mathrm{rk}(H_n(W))$, where A' is any valid choice. We will apply the "nerve theorem", [GKRW19, Corollary 4.2]. To do so, we will use the arc complex to index an appropriate family of closed subposets of $\mathcal{S}^{E_1,\delta}(W)$:

Pick δ to be a generator of a direct summand of $H_{n-1}(\partial W) \cong H_{n-1}(A)$ of maximal order, so $\delta = 0$ when $H_{n-1}(A) = 0$, and use that A is $(n-2)$-connected and the Hurewicz theorem to represent δ by a continuous map $f : S^{n-1} \to \mathrm{int}(A)$. Then use transversality to homotope f to a smooth embedding and let Δ be its isotopy class. In principle this isotopy class depends on the choice of smooth embedding, but we just pick one.

Consider the functor of posets

$$F : \mathcal{A}(W,\Delta)^{\mathrm{op}} \to \{\text{closed subposets of } \mathcal{S}^{E_1,\delta}(W)\}$$

given by sending $\alpha = \{a_0,\cdots,a_p\}$ to the subposet of $\mathcal{S}^{E_1,\delta}(W)$ consisting of walls ω such that all the a_i's lie strictly to the right of ω.

Now let us verify the hypotheses of [GKRW19, Corollary 4.2] with $n = g(W)-2+C(A)$ (not to be confused with half the dimension of the manifolds in this proof), $t_{\mathcal{A}}(\alpha) = p$ where $\alpha = \{a_0,\cdots,a_p\}$, and $t_{\mathcal{S}}(\omega) = g(W_{\leq\omega})-1$.

(i) By Theorem 4.3.3 the poset $\mathcal{A}(W,\Delta)$ is $(g(W)-2)$-connected, and in particular $(n-1)$-connected.

(ii) Let $\alpha = \{a_0,\cdots,a_p\} \in \mathcal{A}(W,\Delta)$, then $\mathcal{A}(W,\Delta)_{<\alpha}$ is the boundary of a p-simplex, and so $(p-2) = (t_{\mathcal{A}}(\alpha)-2)$-connected.

On the other hand, by Lemma 4.3.15 the poset $F(\alpha)$ is isomorphic to $\tilde{\mathcal{S}}(W\smallsetminus\alpha)$. By induction hypothesis, which applies by Proposition 4.3.4(i), this is $(g(W\smallsetminus\alpha)-2)$-connected. By our choice of Δ and Proposition 4.3.4(ii) it follows that $g(W\smallsetminus\alpha) \geq g(W)-(p+1)+C(A)$. Thus, $F(\alpha)$ is $(n-(t_{\mathcal{A}}(\alpha)+1))$-connected, as required. Let us remark that this is actually the key point in the argument where n odd is important: by Remark 4.4.10 and Example 5.2.4, Proposition 4.3.4(ii) is only true for n odd.

(iii) Let $\omega \in \mathcal{S}^{E_1,\delta}(W)$. By Corollary 4.3.13, $\mathcal{S}^{E_1,\delta}(W)_{<\omega} \cong \mathcal{S}^{E_1,\delta}(W_{\leq\omega})$. Now we claim that $\mathrm{rk}(H_n(W_{\leq\omega})) < \mathrm{rk}(H_n(W))$.

Since $H_n(W) = H_n(W_{\leq\omega}) \oplus H_n(W_{\geq\omega})$ it suffices to show that $\mathrm{rk}(H_n(W_{\geq\omega})) > 0$. Since $\omega \in \mathcal{S}^{E_1,\delta}(W)$ then $H_n(W_{\geq\omega}) \neq 0$, so it suffices to show that this group is torsion-free. By the universal coefficient theorem it is enough to show that $H^{n+1}(W_{\geq\omega}) = 0$, and by Poincaré-Lefschetz duality it suffices to show that $H_{n-1}(W_{\geq\omega},\partial W_{\geq\omega}) = 0$, which follows from the $(n-1)$-connectivity of $W_{\geq\omega}$ and the $(n-2)$-connectivity of $\partial W_{\geq\omega}$ (see Lemma 4.3.12).

Thus, induction hypothesis applies to $W_{\leq\omega}$ giving that $\mathcal{S}^{E_1,\delta}(W)_{<\omega}$ is $(g(W_{\leq\omega})-3)$-connected, i.e. $(t_{\mathcal{S}}(\omega)-2)$-connected.

On the other hand, the poset $\mathcal{A}(W,\Delta)_\omega := \{\alpha \in \mathcal{A}(W,\Delta) : \ \omega \in F(\alpha)\}$ is isomorphic to $\mathcal{A}(W_{\geq\omega},\Delta)$. Thus, by Theorem 4.3.3 it is $(g(W_{\geq\omega})-2)$-connected. By additivity of the genus we find that $\mathcal{A}(W,\Delta)_\omega$ is $(n-C(A)-(t_{\mathcal{S}}(\omega)+1))$-connected, as required since $C(A) \leq 1$.

The nerve theorem applies giving that $\mathcal{S}^{E_1,\delta}(W)$ is $(g(W)-3+C(A))$-connected, completing the induction step. □

Corollary 4.3.19. *For any allowed A and $W \in \mathcal{M}[A]$, $\mathcal{S}^{E_1,\delta}(W)$ is f-weakly Cohen-Macaulay of dimension $g(W)-2+C(A)$, where $f(\omega) = g(W_{\leq\omega})-1$.*

Proof. By Theorem 4.3.16 we already know that $\mathcal{S}^{E_1,\delta}(W)$ is $(g(w)-3+C(A))$-connected. From Theorem 4.3.16 and Corollary 4.3.13 it follows that

(i) For any $\omega \in \mathcal{S}^{E_1,\delta}(W)$, $\mathcal{S}^{E_1,\delta}(W)_{\leq\omega}$ is $(g(W_{\leq\omega})-3)$-connected, i.e. $(f(\omega)-2)$-connected.

(ii) For any $\omega \in \mathcal{S}^{E_1,\delta}(W)$, $\mathcal{S}^{E_1,\delta}(W)_{\geq\omega}$ is $(g(W_{\geq\omega})-3+C(A))$-connected, which by additivity of the genus is $((g(W)-2+C(A))-2-f(\omega))$-connected.

(iii) For any $\omega < \omega' \in \mathcal{S}^{E_1,\delta}(W)$, $\mathcal{S}^{E_1,\delta}(W)_{\omega<-<\omega'}$ is $(g(W_{\omega\leq-\leq\omega'})-3)$-connected. By additivity of the genus this is $(f(\omega')-f(\omega)-3)$-connected. □

Remark 4.3.20. *It can be shown that* $f(\omega)=\dim(\omega)$ *and* $\dim(\mathcal{S}^{E_1,\delta}(W))=g(W)-2+C(A)$*, so actually* $\mathcal{S}^{E_1,\delta}(W)$ *is Cohen-Macaulay of dimension* $g(W)-2+C(A)$*. However this fact will not be used in this thesis.*

4.3.6 The discretization argument and the connectivity of splitting complexes

In this section we give a general tool which allows to show high connectivity of the nerve of a topological poset by proving that its discretization is f-weakly Cohen-Macaulay. The statement and proof of the following result are inspired by the proof of [GRW18, Theorem 5.6].

Theorem 4.3.21. *Let* $(P,<)$ *be a (non-unital) topological poset such that* P *is a Hausdorff space and the relation* $<$ *is an open subset of* P^2*. If* $(P^\delta,<)$ *is* f*-weakly Cohen-Macaulay of dimension* n *for some function* f*, then* $\|N_\bullet P\|$ *is* $(n-1)$*-connected.*

Proof. Let $\iota : P^\delta \to P$ be the identity map, viewed as a continuous map of posets.

Consider the bi-semisimplicial space $A_{\bullet,\bullet}$ whose space of (p,q)-simplices is given by

$$A_{p,q} := \{(x_0<x_1<\cdots<x_p),(y_0<y_1<\cdots<y_q) : \text{the set } \{x_0,\cdots,x_p,y_0,\cdots,y_q\} \text{ is totally ordered}\}$$

topologized as a subspace of $P^{p+1}\times(P^\delta)^{q+1}$, and whose face maps are given by forgetting the x_i's and the y_j's.

This bi-semisimplicial space has two augmentations $\varepsilon_\bullet : A_{\bullet,\bullet} \to P_\bullet = N_\bullet P$ and $\delta_\bullet : A_{\bullet,\bullet} \to P^\delta_\bullet = N_\bullet P^\delta$.

Claim. The diagram

$$\begin{tikzcd} & {||A_{\bullet,\bullet}||} \arrow[dl, "||\delta_\bullet||"'] \arrow[dr, "||\varepsilon_\bullet||"] & \\ {||P^\delta_\bullet||} \arrow[rr, "||\iota_\bullet||"] & & {||P_\bullet||} \end{tikzcd}$$

is homotopy-commutative.

This claim is similar to [ERW19, Lemma 4.2]

Proof of Claim. For each $p,q \geq 0$ we will construct a map $H_{p,q} : I \times A_{p,q} \times \Delta^p \times \Delta^q \to ||P_\bullet||$ in such a way that the $H_{p,q}$'s are compatible with the face maps, and hence glue to produce the required homotopy.

We define $H_{p,q}$ on a given (p,q)-simplex $\sigma = ((x_0 < \cdots < x_p),(y_0 < \cdots < y_q)) \in A_{p,q}$ to be the map $H_{p,q}(\sigma) : I \times \Delta^p \times \Delta^q \to \Delta^{p+q+1}$ given as follows:

$\varepsilon_\bullet$ at simplex σ gives an inclusion $a : \Delta^p \hookrightarrow \Delta^{p+q+1}$ as a p-face corresponding to the positions of the x_i's in the chain defined by $\{x_0,\cdots,x_p\} \sqcup \{y_0,\cdots,y_q\}$, and similarly $\iota_\bullet \circ \delta_\bullet$ at simplex σ gives an inclusion $b : \Delta^q \hookrightarrow \Delta^{p+q+1}$ as a q-face.

The homotopy H is then given by $H(t,u,v) := ta(u) + (1-t)b(v)$ for $u \in \Delta^p$, $v \in \Delta^q$ and $t \in I$. □

Thus, using that $P^\delta_\bullet$ is $(n-1)$-connected, it suffices to show that $||\varepsilon_\bullet||$ is $(n-1)$-connected.

By [GRW18, Proposition 2.7] it suffices to show that for each p the map $\varepsilon_p : |A_{p,\bullet}| \to P_p$ is $(n-1-p)$-connected. This will be shown by applying [GRW18, Corollary 2.9] with $Y_\bullet = P^\delta_\bullet$, which is a semisimplicial set as P^δ is discrete, $Z = P_p$, which is Hausdorff as P is, and $X_\bullet = A_{p,\bullet} \subset Z \times Y_\bullet$, which is an open subset on each degree as $<$ is open. The only condition to check is that for any given $x_0 < \cdots < x_p \in P_p$ the semisimplicial set $X_\bullet(x_0 < \cdots < x_p)$ defined via $\{x_0 < \cdots < x_p\} \times X_\bullet(x_0 < \cdots < x_p) = X_\bullet \cap (\{x_0 < \cdots < x_p\} \times Y_\bullet)$ is $(n-1-p-1)$-connected.

By definition $X_\bullet(x_0 < \cdots < x_p) = (P^\delta_{<x_0} * P^\delta_{x_0<-<x_1} * \cdots * P^\delta_{x_{p-1}<-<x_p} * P^\delta_{>x_p})_\bullet$, and since P^δ is f-weakly Cohen-Macaulay of dimension n then $P^\delta_{<x_0}$ is $(f(x_0)-2)$-connected, each $P^\delta_{x_{i-1}<-<x_i}$ is $(f(x_i)-f(x_{i-1})-3)$-connected and $P^\delta_{>x_p}$ is $(n-2-f(x_p))$-connected. Thus the join is $(n-2-2-3p+2(p+1))$-connected, i.e. $(n-2-p)$-connected, as required. □

From Corollary 4.3.19 and the above result it follows that

Corollary 4.3.22. *For each $W \in \mathcal{M}[I^{2n-1}]$ the E_1-splitting complex $S^{E_1}(W)$ is $(g(W)-3)$-connected. More generally, for any allowed A and $W \in \mathcal{M}[A]$ the E_1-splitting complex $S^{E_1}(W)$ is $(g(W)-3+C(A))$-connected.*

4.3.7 Splitting complexes and E_1-homology

In this section we will relate the required vanishing in the E_1-homology of $\mathbf{R}$ of Theorem E to the connectivity bound on the splitting complex of Corollary 4.3.22. This will be based on results of [GKRW18, Section 13], which express the E_1-indecomposables in terms of bar constructions. We will make use of different bar constructions defined in [GKRW18, Section 9, Section 13].

Recall also [GKRW18, Section 12.2.1]: given an E_1-algebra $\mathbf{X}$ we can define a unital associative replacement $\overline{\mathbf{X}}$ whose underlying object is given by $([0,\infty)\times\mathbb{1})\sqcup((0,\infty)\times\mathbf{X})$, where $\mathbb{1}$ is the monoidal unit of the underlying category. A small modification of this construction gives a non-unital associative replacement that we shall denote $\mathbf{X}'$ in this section, whose underlying object is $(0,\infty)\times\mathbf{X}$, and such that the inclusion $\mathbf{X}'\to\overline{\mathbf{X}}$ preserves the associative product. In particular, $\mathbf{X}'$ becomes an strict $\overline{\mathbf{X}}$-bimodule in the obvious way.

Our first lemma relates the E_1-homology of $\mathbf{R}$ to a certain bar construction

Lemma 4.3.23. *The homotopy cofibre of $B(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')\to B(\overline{\mathbf{R}},\overline{\mathbf{R}},\mathbf{R}')\simeq\mathbf{R}'$ is homologically $(\mathrm{rk}(x)-2)$-connected in grading $x\in\mathsf{G}_n$ if and only if $H^{E_1}_{x,d}(\mathbf{R})=0$ for $d<\mathrm{rk}(x)-1$.*

Proof. By [GKRW18, Corollary 9.17,Theorem 13.7] and using that $\mathbf{R}'_{\mathbb{Z}}\to\mathbf{R}_{\mathbb{Z}}$ is a weak equivalence of $\overline{\mathbf{R}_{\mathbb{Z}}}$-modules we get that $Q^{E_1}_{\mathbb{L}}(\mathbf{R}_{\mathbb{Z}})\simeq B(\mathbb{1},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})$. Since

$$B(\mathbf{R}'_{\mathbb{Z}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\to B(\overline{\mathbf{R}_{\mathbb{Z}}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\simeq\mathbf{R}'_{\mathbb{Z}}\to B(\mathbb{1},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})$$

is a cofibration then $Q^{E_1}_{\mathbb{L}}(\mathbf{R}_{\mathbb{Z}})$ is the homotopy cofibre of $B(\mathbf{R}'_{\mathbb{Z}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\to B(\overline{\mathbf{R}_{\mathbb{Z}}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\simeq\mathbf{R}'_{\mathbb{Z}}$.

Now perform the associative replacements $\mathbf{R}'$ and $\overline{\mathbf{R}}$ in the original category $\mathsf{Top}^{\mathsf{G}_n}$ and consider the map $B(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')\to B(\overline{\mathbf{R}},\overline{\mathbf{R}},\mathbf{R}')\simeq\mathbf{R}'$. By construction, if we apply the functor $(-)_{\mathbb{Z}}$ then we get $B(\mathbf{R}'_{\mathbb{Z}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\to B(\overline{\mathbf{R}_{\mathbb{Z}}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\simeq\mathbf{R}'_{\mathbb{Z}}$.

Thus, $H^{E_1}_{x,d}(\mathbf{R})=0$ for $d<\mathrm{rk}(x)-1$ if and only if $Q^{E_1}_{\mathbb{L}}(\mathbf{R}_{\mathbb{Z}})$ is homologically $(\mathrm{rk}(x)-2)$-connected in grading x, which is equivalent to the map $B(\mathbf{R}'_{\mathbb{Z}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\to B(\overline{\mathbf{R}_{\mathbb{Z}}},\overline{\mathbf{R}_{\mathbb{Z}}},\mathbf{R}'_{\mathbb{Z}})\simeq\mathbf{R}'_{\mathbb{Z}}$ being homologically $(\mathrm{rk}(x)-2)$-connected in grading x, which in turn is equivalent to the map $B(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')\to B(\overline{\mathbf{R}},\overline{\mathbf{R}},\mathbf{R}')\simeq\mathbf{R}'$ being itself homologically $(\mathrm{rk}(x)-2)$-connected in grading x. $\square$

In $\mathsf{Top}^{\mathsf{G}_n}$ we can also make sense of the bar construction $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}')$, which has an augmentation $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}')\to\mathbf{R}'$ which factors trough the augmentation $B_\bullet(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')\to\mathbf{R}'$. Moreover, we also have a weak equivalence $\mathbf{R}'\to\mathbf{R}$ induced by projection.

Lemma 4.3.24. *The map $B(\mathbf{R}',\mathbf{R}',\mathbf{R}') \to B(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')$ induced by the inclusion $\mathbf{R}' \to \overline{\mathbf{R}}$ is a weak equivalence. Thus, the homotopy cofibres of $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}') \to \mathbf{R}$ and $B(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}') \to \mathbf{R}'$ are equivalent.*

Proof. The object $(\mathbf{R}')^+ := \mathbb{1} \sqcup \mathbf{R}'$ has an obvious unital associative algebra structure, and the inclusion $(\mathbf{R}')^+ \to \overline{\mathbf{R}}$ is a weak equivalence and a map of unital associative algebras, so it induces a map of semisimplicial objects $B_\bullet(\mathbf{R}',(\mathbf{R}')^+,\mathbf{R}') \to B_\bullet(\mathbf{R}',\overline{\mathbf{R}},\mathbf{R}')$, which is levelwise a weak equivalence, and hence a weak equivalence on geometric realizations by [ERW19, Theorem 2.2].

Finally, $B_\bullet(\mathbf{R}',(\mathbf{R}')^+,\mathbf{R}')$ admits the structure of a simplicial graded space where degeneracies are given by inserting the unit $\mathbb{1}$, which consists of a point $*$ in grading $0 \in \mathsf{G}_n$. Thus, this simplicial object is given by freely adding degeneracies to the semisimplicial object $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}')$. By [ERW19, Lemma 2.6] the map $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}') \to B_\bullet(\mathbf{R}',(\mathbf{R}')^+,\mathbf{R}')$ is a weak equivalence in geometric realizations, giving the result. □

Denote by $\pi_\bullet$ the map $B_\bullet(\mathbf{R}',\mathbf{R}',\mathbf{R}') \to \mathbf{R}$, and for each $W \in \mathbf{R}$ let

$$F(W) := \mathrm{hofib}_W(B(\mathbf{R}',\mathbf{R}',\mathbf{R}') \xrightarrow{\pi} \mathbf{R})$$

be the corresponding homotopy fibre.

Proposition 4.3.25. *For $W \in \mathbf{R}$, $F(W)$ is weakly equivalent to $S^{E_1}(W)$.*

Proof. For each $p \geq 0$ let $F_p(W) := \mathrm{hofib}_W(\pi_p : B_p(\mathbf{R}',\mathbf{R}',\mathbf{R}') \to \mathbf{R})$, where $\pi_p : (\mathbf{R}')^{\otimes p+2} \to \mathbf{R}$ is given by using the associative product in $\mathbf{R}'$ followed by the weak equivalence $\mathbf{R}' \to \mathbf{R}$. We will find explicit functorial models of the $F_p(W)$'s so that they define a semisimplicial space $F_\bullet(W)$, and then $F(W) \simeq \|F_\bullet(W)\|$.

The space $\mathrm{Emb}^p_{\partial - W}(W, I^{2n} \times \mathbb{R}^\infty)$ considered in Definition 4.1.1 is weakly contractible by the Whitney embedding theorem, and the map $\mathrm{Emb}^p_{\partial - W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathcal{M}[I^{2n-1}]$ is a fibration by the isotopy extension theorem. Thus, we can fix a model for $F_p(W)$ via the pullback square

$$\begin{tikzcd} F_p(W) \arrow[r] \arrow[d] & \pi_p^{-1}(\mathcal{M}[I^{2n-1};W]) \subset (\mathbf{R}')^{\otimes p+2} \arrow[d] \\ \mathrm{Emb}^p_{\partial - W}(W, I^{2n} \times \mathbb{R}^\infty) \arrow[r] & \mathcal{M}[I^{2n-1};W] \end{tikzcd}$$

where the bottom right corner means the path-component of W in $\mathcal{M}[I^{2n-1}]$, which by Proposition 4.1.2 agrees with the image of $\mathrm{Emb}^p_{\partial - W}(W, I^{2n} \times \mathbb{R}^\infty) \to \mathcal{M}[I^{2n-1}]$.

Hence a model for $F_p(W)$ is the set of tuples $((s_0,W_0),\cdots,(s_{p+1},W_{p+1});e)$ where $(s_i,W_i)\in\mathbf{R}'=(0,\infty)\times\mathbf{R}$ for all i, $e\in\mathrm{Emb}^p_{\partial-W}(W,I^{2n}\times\mathbb{R}^\infty)$, and

$$(s_0,W_0)\bullet\cdots\bullet(s_{p+1},W_{p+1})=(s_0+\cdots+s_{p+1},\mathrm{im}(e))\in\mathbf{R}'$$

where $\bullet$ denotes the product in $\mathbf{R}'$. Changing variables $t_i:=\frac{s_0+\cdots+s_i}{s_0+\cdots s_{p+1}}\in(0,1)$ for $0\le i\le p$, we can also view $F_p(W)$ as the collection of tuples

$$(0<t_0<\cdots<t_p<1;W_0,\cdots,W_{p+1};e)\in\Delta^p\times\mathbf{R}^{p+2}\times\mathrm{Emb}^p_{\partial-W}(W,I^{2n}\times\mathbb{R}^\infty)$$

such that the E_1-multiplication of the W_i's using the partition

$$[0,t_0]\sqcup[t_0,t_1]\sqcup\cdots\sqcup[t_p,1]\in\mathcal{C}_1(p+2)$$

is precisely $\mathrm{im}(e)\in\mathcal{M}[I^{2n-1};W]$.

With this explicit description, it is clear that $F_\bullet(W)$ defines a semisimplicial space, where the i-th face map forgets t_i and glues together W_i and W_{i+1} using the E_1-product with partition $[0,\frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}}]\sqcup[\frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}},1]$.

To finish the proof we will construct a zigzag of semisimplicial spaces

$$F_\bullet(W)\xleftarrow{U_\bullet}F'_\bullet(W)\xrightarrow{\Psi_\bullet}S^{E_1}_\bullet(W)$$

which are levelwise weak equivalences, and hence weak equivalences on geometric realizations by [ERW19, Lemma 2.6].

The semisimplicial space $F'_\bullet(W)$ is defined as follows: $F'_p(W)$ is the collection of tuples

$$(0<t_0<\cdots t_p<1;W_0,\cdots,W_{p+1};e;0<\varepsilon_0,\cdots,\varepsilon_p<1)\in F_p(W)\times(0,1)^{p+1}$$

such that $0<t_0-\varepsilon_0<t_0+\varepsilon_0<t_1-\varepsilon_1<\cdots<t_p-\varepsilon_p<t_p+\varepsilon_p<1$ and for $0\le i\le p$, $\mathrm{im}(e)$ agrees pointwise with $I^{2n}\times\{0\}\subset I^{2n}\times\mathbb{R}^\infty$ in an open neighbourhood of $[t_i-\varepsilon_i,t_i+\varepsilon_i]\times I^{2n-1}\times\mathbb{R}^\infty$.

The map $U_p:F'_p(W)\to F_p(W)$ is given by forgetting the extra data $\varepsilon_0,\cdots,\varepsilon_p$. U_p has contractible fibres, and it is a microfibration because if we pick some valid $(\varepsilon_0,\cdots,\varepsilon_1)$ for a point in $F_p(W)$ then the same choice of $(\varepsilon_0,\cdots,\varepsilon_1)$ works in a neighbourhood of that point. Thus, U_p is a weak equivalence by [Wei05, Lemma 2.2].

The map Ψ_p is defined via

$$(0<t_0<\cdots t_p<1;W_0,\cdots,W_{p+1};e;0<\varepsilon_0,\cdots,\varepsilon_p<1)\mapsto((\omega_0,t_0,\varepsilon_0),\cdots,(\omega_p,t_p,\varepsilon_0))$$

where the wall ω_i is given by the composition $[t_i - \varepsilon_i, t_i + \varepsilon_i] \times I^{2n-1} \hookrightarrow \mathrm{im}(e) \xrightarrow{e^{-1}} W$. By the isotopy extension theorem, $\Psi_p : F'_p(W) \to S_p^{E_1}(W)$ is a Serre fibration, so it suffices to check that it has weakly contractible fibres.

The fibre over a given $((\omega_0, t_0, \varepsilon_0), \cdots, (\omega_p, t_p, \varepsilon_0)) \in S_p^{E_1}(W)$ is the subspace of $e \in \mathrm{Emb}^p_{\partial - W}(W, I^{2n} \times \mathbb{R}^\infty)$ such that the compositions $e \circ \omega_i$ agree with the standard inclusions $[t_i - \varepsilon_i, t_i + \varepsilon_i] \times I^{2n-1} \times \{0\} \subset I^{2n} \times \mathbb{R}^\infty$ for all i. This space is then weakly contractible by the Whitney embedding theorem since the images of the ω_i form a closed submanifold. □

Now we can finally prove Theorem E.

Proof. For each $W \in \mathcal{M}[I^{2n-1}]$ the E_1-splitting complex $S^{E_1}(W)$ is $(g(W) - 3)$-connected by Corollary 4.3.22. Hence Proposition 4.3.25 implies that $F(W)$ is $(g(W) - 3)$-connected for any $W \in \mathbf{R}$. Thus, the cofibre of $B(\mathbf{R}', \mathbf{R}', \mathbf{R}') \xrightarrow{\pi} \mathbf{R}$ is homologically $(\mathrm{rk}(x) - 2)$-connected in degree x. Then Lemmas 4.3.23 and 4.3.24 give the result. □

4.4 The arc complex

In order to finish the proof of Theorem E we need to show Theorem 4.3.3 and Proposition 4.3.4, which are the only results used in Section 4.3 that are left to prove. To do so we will firstly construct an algebraic model for the arc complex, and then we will compare it to the geometric one.

4.4.1 The algebraic arc complex

Definition 4.4.1. *A valid algebraic data consists of a triple (M, λ, δ), where*

(i) M is a finitely generated free $\mathbb{Z}$-module.

(ii) $\lambda : M \otimes M \to \mathbb{Z}$ is a skew-symmetric bilinear form on M. We write $\lambda^\vee : M \to M^\vee$ for the corresponding map $m \mapsto \lambda(m, -)$.

(iii) $\delta \in \partial(M, \lambda)$ is an element, where $\partial(M, \lambda) := \mathrm{coker}(\lambda^\vee) = \frac{M^\vee}{\lambda^\vee(M)}$.

We will usually remove λ from the notation in all future expressions to make them easier to read, for example we shall write ∂M instead of $\partial(M, \lambda)$.

Any valid geometric data (W, Δ) as in Definition 4.3.1 gives a valid algebraic data $(H_n(W), \lambda_W, \delta)$ as follows: compactness of W implies that $H_n(W)$ is finitely generated, the long exact sequence of the pair $(W, \partial W)$ and the connectivity assumptions on W and ∂W give an exact sequence

$$0 \to H_n(\partial W) \to H_n(W) \to H_n(W, \partial W) \to H_{n-1}(\partial W) \to 0$$

and that $H_{n-1}(W, \partial W) = 0$. By Poincaré-Lefschetz duality we have isomorphisms $0 = H_{n-1}(W, \partial W) \cong H^{n+1}(W)$ and $H_n(W, \partial W) \cong H^n(W)$. The universal coefficient theorem implies that $H_n(W)$ is torsion-free and $H^n(W) \cong H_n(W)^\vee$. Composing the isomorphisms $H_n(W, \partial W) \cong H^n(W) \cong H_n(W)^\vee$, the inclusion $H_n(W) \to H_n(W, \partial W)$ becomes $\lambda_W^\vee : H_n(W) \to H_n(W)^\vee$. Thus, the above exact sequence implies that $H_n(\partial W) \cong \ker(\lambda_W^\vee)$ and $H_{n-1}(\partial W) \cong \partial H_n(W)$ in such a way that the first map becomes the inclusion of the kernel and the last one becomes the projection to the cokernel of $\lambda_W^\vee$. In particular, the isotopy class Δ gives a well-defined element $\delta \in \partial H_n(W)$.

Definition 4.4.2. *For a valid algebraic data (M, λ, δ) the algebraic arc complex $\mathcal{A}^{alg}(M, \lambda, \delta)$ is the simplicial complex with vertex set*

$$\{\alpha \in M^\vee : \ \alpha \textit{ is unimodular and } \alpha \bmod \lambda^\vee(M) = \delta\}$$

and where a set $\{\alpha_0, \cdots, \alpha_p\}$ of (distinct) vertices spans a p-simplex if and only if it is unimodular in $M^\vee$.

The geometric interpretation of the above definition is explained by the following result.

Proposition 4.4.3. *If (W, Δ) is valid geometric data then there is a simplexwise injective simplicial map*

$$\Phi : \mathcal{A}(W, \Delta) \to \mathcal{A}^{alg}(H_n(W), \lambda_W, \delta)$$

given (on vertices) by $a \mapsto \alpha := a_*([D^n, \partial D^n]) \in H_n(W, \partial W) \cong H_n(W)^\vee$.

Proof. Firstly we claim that a set $\{a_0, \cdots, a_p\}$ of pairwise disjoint arcs satisfies that the cut manifold $W' := W \smallsetminus \{a_0, \cdots, a_p\}$ is $(n-1)$-connected if and only if the corresponding set $\{\alpha_0, \cdots, \alpha_p\} \subset H_n(W)^\vee$ is unimodular and has size precisely $p+1$, which implies the simplexwise injectivity part.

Indeed, the long exact sequence of the pair (W, W') gives an exact sequence

$$0 \to H_n(W') \to H_n(W) \to H_n(W, W') \to H_{n-1}(W') \to 0.$$

The pairwise disjointness of the arcs and excision imply (by taking tubular neighbourhoods) that

$$H_n(W,W') \cong H_n(\bigsqcup_{i=0}^{p} D^n \times D^n, \bigsqcup_{i=0}^{p} D^n \times \partial D^n) \cong \mathbb{Z}^{p+1}$$

in such a way that $H_n(W) \to H_n(W,W')$ agrees with $\oplus_{i=0}^{p} \alpha_i : H_n(W) \to \mathbb{Z}^{p+1}$, hence giving the result.

Secondly, the boundary map $H_n(W,\partial W) \to H_{n-1}(W)$ agrees with the quotient map $H_n(W)^\vee \to \partial H_n(W)$ and hence any vertex a of the arc complex will map to $\alpha \in H_n(W)^\vee$ such that $[\alpha] = \delta \in \partial H_n(W)$. □

The algebraic argument

In this section we will prove that the algebraic arc complex is weakly Cohen-Macaulay of a certain dimension. To state the precise result we need some notation:
let $t(M) := \max\{\mathrm{rk}(U) : U \subset \lambda^\vee(M) \text{ is a direct summand of } M^\vee\}$.

Theorem 4.4.4. *If (M,λ,δ) is valid algebraic data then $\mathcal{A}^{alg}(M,\lambda,\delta)$ is weakly Cohen-Macaulay of dimension $t(M)-2$.*

We will deduce this result from a more general one inspired by [Fri17, Theorem 1.4] We will use the following notation from [Fri17] to state it and prove it: for X a set we let $\mathcal{O}(X)$ denote the poset of non-empty ordered finite sequences of elements of X, with partial order given by refinement. For V a $\mathbb{Z}$-module let $\mathcal{U}(V)$ denote the subposet of $\mathcal{O}(V)$ consisting of the unimodular sequences. If $F \subset \mathcal{O}(V)$ is a subposet and $(v_1,\cdots,v_k) \in \mathcal{U}(V)$, we write $F \cap \mathcal{U}(V)_{(v_1,\cdots,v_k)}$ for the poset of sequences $(w_1,\cdots,w_l) \in F$ such that $(v_1,\cdots,v_k,w_1,\cdots,w_l) \in \mathcal{U}(V)$.

Theorem 4.4.5. *Let $N' := d_1\mathbb{Z} \oplus \cdots \oplus d_r\mathbb{Z} \oplus \mathbb{Z}^t$, where $d_i \geq 2$. Let $N := \mathbb{Z}^{r+t+l}$, with standard basis elements $x_1,\cdots,x_{r+t+l}$, so that $d_1x_1,\cdots,d_rx_r,x_{r+1},\cdots,x_{r+t}$ is a basis for N'. Let $N^\infty := N \oplus \mathbb{Z}^\infty$, and let the standard basis of $\mathbb{Z}^\infty$ be $e_1,e_2,\cdots$. Fix an element $\delta_0 \in N$.*

For any $(v_1,\cdots,v_k) \in \mathcal{U}(N^\infty)$ with $k \geq 1$

(1) $\mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)$ is $(t-3)$-connected.

(2) $\mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)_{(v_1,\cdots,v_k)}$ is $(t-3-k)$-connected.

(a) $\mathcal{O}(\delta_0 + N' \cup \delta_0 + N' + e_1) \cap \mathcal{U}(N^\infty)$ is $(t-2)$-connected.

(b) $\mathcal{O}(\delta_0 + N' \cup \delta_0 + N' + e_1) \cap \mathcal{U}(N^\infty)_{(v_1,\cdots,v_k)}$ is $(t-2-k)$-connected.

In order to state the application that will be relevant for us we need to fix some extra notation: given a finitely generated free $\mathbb{Z}$-module N and a submodule $N' \subset N$ we let

$$t(N,N') := \max\{\mathrm{rk}(U) : U \subset N' \text{ is a submodule such that } U \subset N \text{ is a direct summand}\}$$

For example, if we take $N = M^\vee$, $N' = \lambda^\vee(M)$ then $t(N,N') = t(M)$.

Given the additional data of an element $\delta \in N/N'$ we define $U^{\mathrm{unord}}(N,N',\delta)$ to be the simplicial complex whose vertices are unimodular elements $x \in N$ such that $x \bmod N' = \delta$ and whose p-simplices are sets of (distinct) $p+1$ vertices $\{x_0,\cdots,x_p\}$ which are unimodular in N.

Corollary 4.4.6. *Let N be a finitely generated free $\mathbb{Z}$-module, let $N' \subset N$ be a submodule and let $\delta \in N/N'$ be an element. Then $U^{unord}(N,N',\delta)$ is weakly Cohen-Macaulay of dimension $t(N,N')-2$.*

This implies Theorem 4.4.4 by taking $N = M^\vee$, $N' = \lambda^\vee(M)$ and the given $\delta \in \partial M = N/N'$.

Proof. By Smith Normal Form it suffices to consider the situation $N' = d_1\mathbb{Z} \oplus \cdots \oplus d_r\mathbb{Z} \oplus \mathbb{Z}^t \subset N = \mathbb{Z}^{r+t+l}$, where $d_i \geq 2$ for all i. Then $t = t(N,N')$, and we can pick a representative $\delta_0 \in N$ of δ.

Let $\mathcal{U}(N,N',\delta)$ be the poset of non-empty finite unimodular sequences in N whose elements lie in the coset $\delta = \delta_0 + N'$. Since a finite sequence of elements in N is unimodular in N if and only if it is unimodular in $N^\infty := N \oplus \mathbb{Z}^\infty$ then $\mathcal{U}(N,N',\delta) = \mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)$ is $(t-3)$-connected by Theorem 4.4.5(1).

Now view $U^{\mathrm{unord}}(N,N',\delta)$ as a poset, and pick a total ordering $<$ in the set of one-element sequences in $\mathcal{U}(N,N',\delta)$. There are poset maps

$$U^{\mathrm{unord}}(N,N',\delta) \xrightarrow{f} \mathcal{U}(N,N',\delta) \xrightarrow{g} U^{\mathrm{unord}}(N,N',\delta)$$

whose composition is the identity, where f sends $\{v_0,\cdots,v_p\} \in U^{\mathrm{unord}}(N,N',\delta)$ to the sequence $(v_{i_0},\cdots,v_{i_p})$ with $v_{i_0} < \cdots < v_{i_p}$ and where g sends a sequence to its underlying set. Thus, the poset $U^{\mathrm{unord}}(N,N',\delta)$ is also $(t-3)$-connected, as required.

Moreover, for a $(k-1)$-simplex $\{v_1,\cdots,v_k\}$ of $U^{\mathrm{unord}}(N,N',\delta)$, we have $(v_1,\cdots,v_k) \in \mathcal{U}(N^\infty)$ so $\mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)_{(v_1,\cdots,v_k)}$ is $(t-3-k)$-connected by Theorem 4.4.5(2). Also,

$$f\big(\mathrm{Lk}_{U^{\mathrm{unord}}(N,N',\delta)}(\{v_1,\cdots,v_k\})\big) \subset \mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)_{(v_1,\cdots,v_k)}$$

and

$$g\big(\mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)_{(v_1,\cdots,v_k)}\big) = \mathrm{Lk}_{U^{\mathrm{unord}}(N,N',\delta)}(\{v_1,\cdots,v_k\}).$$

Thus $\mathrm{Lk}_{U^{\mathrm{unord}}(N,N',\delta)}(\{v_1,\cdots,v_k\})$ is $(t-3-k)$-connected. □

The proof of Theorem 4.4.5 is almost identical to the one of [Fri17, Theorem 1.4], so we will refer to it for the details and only indicate the differences: we always work with the ring $R = \mathbb{Z}$, $sr(R) = 2$. The analogue of the variable g in [Fri17] is what we call t. When $\delta_0 = 0$, $r = l = 0$ we recover the exact same proof as in [Fri17].

Proof of Theorem 4.4.5. We will show the result by induction on t. For $t \leq 1$ parts (1),(2) and (b) are vacuously true since $k \geq 1$ and any poset is (-2)-connected.

Statement (a) is also true for $t \leq 1$ since $\mathcal{O}(\delta_0 + N' \cup \delta_0 + N' + e_1) \cap \mathcal{U}(N^\infty)$ contains the one element sequence $(\delta_0 + e_1)$ and hence is non-empty, so (-1)-connected.

Thus we will assume that $t \geq 2$ throughout the proof. Without loss of generality we can assume that $\delta_0 \in \langle x_1, \cdots, x_r, x_{r+t+1}, \cdots, x_{r+t+l} \rangle$, so that its projection to $\mathbb{Z}^t = \langle x_{r+1}, \cdots, x_{r+t} \rangle$ vanishes.

Proof of (b). Let $Y = \delta_0 + N' \cup \delta_0 + N' + e_1$ and $F := \mathcal{O}(Y) \cap \mathcal{U}(N^\infty)_{(v_1, \cdots, v_k)}$. We need to show that F is $(t-2-k)$-connected. Since $GL_t(\mathbb{Z})$ acts transitively on the set of unimodular vectors of $\mathbb{Z}^t$ for any $t \geq 1$ we don't need to consider two separate cases as in [Fri17], and instead we can use the argument for the case "$g > sr(R)$".

We need to find $f \in GL(N^\infty)$ such that $f(Y) = Y$ and the projection of $f(v_1)$ to $\mathbb{Z}^t = \langle x_{r+1}, \cdots, x_{r+t} \rangle$, denoted $f(v_1)|_{\mathbb{Z}^t}$, is unimodular. Once f is found we can proceed as in [Fri17] to inductively prove the result. The construction of f itself is analogous to the one in [Fri17]: the given f preserves both the set Y and the element δ_0.

Proof of (2). We let

$$X = (\delta_0 + d_1\mathbb{Z} \oplus \cdots \oplus d_r\mathbb{Z} \oplus \mathbb{Z}^{t-1} \oplus 0) \cup (\delta_0 + d_1\mathbb{Z} \oplus \cdots \oplus d_r\mathbb{Z} \oplus \mathbb{Z}^{t-1} \oplus 0 + x_{r+t})$$

and

$$F = \mathcal{O}(\delta_0 + N') \cap \mathcal{U}(N^\infty)_{(v_1, \cdots, v_k)}$$

and then mimic the corresponding step in [Fri17].

Proof of (1) and (a). For (1) we take the same X and F as above. For (2) we take instead $X = \delta_0 + N'$, $F = \mathcal{O}(\delta_0 + N' \cup \delta_0 + N' + e_1) \cap \mathcal{U}(N^\infty)_{(v_1, \cdots, v_k)}$ and $y_0 = \delta_0 + e_1$. In both cases we then follow the exact same proof as in [Fri17]. □

Skew symmetric forms and genus

In this section we state a classification result of skew symmetric forms over the integers and use it derive some algebraic results that will be useful.

The following classification theorem can be found in [New72, Theorem IV.I]:

Theorem 4.4.7. *Any pair (M,λ) where M is a finitely generated free $\mathbb{Z}$-module and $\lambda : M\otimes M \to \mathbb{Z}$ is a skew-symmetric form is isomorphic to a unique canonical form*

$$H^{\oplus g} \oplus \bigoplus_{i=1}^{r} \begin{pmatrix} 0 & d_i \\ -d_i & 0 \end{pmatrix} \oplus (\mathbb{Z}^b, 0)$$

where H is the standard hyperbolic form with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g \geq 0$, $d_1|d_2\cdots|d_r$ and $d_i \geq 2$, $r \geq 0$, $(\mathbb{Z}^b, 0)$ denotes the zero form of rank $b \geq 0$, and $\oplus$ denotes orthogonal direct sum.

The number $g = g(M)$ is called the *genus* of the form, and can also be defined as $g(M) := \max\{k : H^{\oplus k} \hookrightarrow (M,\lambda)\}$, hence generalizing the definition given in Section 4.3.5. The integer b agrees with the rank of the radical $\mathrm{rad}(M) := \ker(\lambda^\vee)$, which is also equal to $\mathrm{rk}(\partial M)$ by rank-nullity. The values of d_i are certain invariants of the form; and the number $r = r(M)$ satisfies that $2r$ is the minimum number of generators of $\mathrm{Tors}(\partial M) \cong \bigoplus_{i=1}^r \mathbb{Z}/d_i\mathbb{Z} \oplus \mathbb{Z}/d_i\mathbb{Z}$. In fact, when (M,λ) is written in canonical form then $\partial M = \bigoplus_{i=1}^r (\mathbb{Z}/d_i\mathbb{Z} \oplus \mathbb{Z}/d_i\mathbb{Z}) \oplus \mathbb{Z}^{\mathrm{rk}(M)-2r-2g}$.

Remark 4.4.8. *Canonical form gives the following formulae for $g(M)$ and $t(M)$ which will be useful latter: $g(M) = \frac{1}{2}(\mathrm{rk}(M) - \mathrm{rk}(\partial M) - 2r(M))$ and $t(M) = 2g(M)$.*

The additivity of the genus under orthogonal direct sum follows from Theorem 4.4.7: by taking Smith Normal Form of the canonical from matrix we see that $g(M)$ is twice the number of 1's in the diagonal. The number of 1's in Smith Normal Form is additive under direct sum of matrices.

4.4.2 Properties of the cut manifold

In this subsection we study the cut manifold and prove Proposition 4.3.4.

Lemma 4.4.9. *Suppose (M,∂,δ) is valid algebraic data such that $\partial M \neq 0$ and $\delta \in \partial M$ is a generator of a cyclic direct summand of ∂M of maximum order. Let $\alpha \in M^\vee$ be unimodular such that $[\alpha] = \delta \in \partial M$, and denote $M' = \ker(\alpha)$ with $\lambda' := \lambda|_{M'}$. Then $g(M') = g(M)$.*

Proof. Let $\delta' \in \partial M'$ be defined as follows: $\alpha \in M^\vee$ is unimodular, so there is $x \in M$ such that $\alpha(x) = 1$, then $\lambda(x,-)|_{M'} \in (M')^\vee$ and we let $\delta' := [\lambda(x,-)|_{M'}] \in \partial M'$. This element is well-defined, i.e. independent of the choice of x, because if we are given x' with $\alpha(x') = 1$ then $x - x' \in \ker(\alpha) = M'$ so $\lambda(x-x',-)|_{M'} \in \lambda^\vee(M')$. We claim that there is an isomorphism $\partial M/\langle\delta\rangle \cong \partial M'/\langle\delta'\rangle$.

Indeed, consider the composition of surjections $\phi : M^\vee \to (M')^\vee \to \partial M'/\langle\delta'\rangle$. It suffices to show that $\ker(\phi)$ is the subgroup of $M^\vee$ generated by $\lambda^\vee(M)$ and α. To do so we check both inclusions: if $\beta \in \ker(\phi)$ then $\beta|_{M'} = \lambda(y,-)|_{M'} + k\lambda(x,-)|_{M'}$ for some $y \in M'$ and $k \in \mathbb{Z}$, since $\delta' = [\lambda(x,-)|_{M'}] \in \partial M'$. Thus, $\beta = \lambda(y+kx,-) - \lambda(y,x)\alpha + \beta(x)\alpha$ because both sides of this equation agree on M' and on x by construction, and M is generated by M' and x; so β lies in the subgroup of $M^\vee$ generated by $\lambda^\vee(M)$ and α. Conversely, if $\beta = \lambda(z,-) + l\alpha$ for some $z \in M$ and $l \in \mathbb{Z}$ then we can write $z = z' + \alpha(z)x$ where $z' \in M'$ and then $\beta|_{M'} = \lambda(z',-)|_{M'} + \alpha(z)\lambda(x,-)|_{M'} + 0$, so $\phi(\beta) = 0$.

By rank-nullity we have $\mathrm{rk}(M') = \mathrm{rk}(M) - 1$. Now we have two cases to consider:

(i)δ has infinite order. We claim that the evaluation map $\mathrm{rad}(M) \to \mathrm{Hom}_{\mathbb{Z}}(\partial M, \mathbb{Z})$, $u \mapsto (\beta \mapsto \beta(u))$ is an isomorphism. Indeed, $\mathrm{Hom}_{\mathbb{Z}}(\partial M, \mathbb{Z}) = \mathrm{Ann}(\lambda^\vee(M) \subset M^\vee)$, and under the evaluation isomorphism $M \xrightarrow{\cong} (M^\vee)^\vee$ this annihilator is precisely $\mathrm{rad}(M)$. (This is the algebraic analogue of Poincaré duality in ∂W.) Since $\delta \in \partial M$ generates a free $\mathbb{Z}$-summand then there is $x \in \mathrm{rad}(M)$ such that $\alpha(x) = 1$. But then $\delta' = 0$ since $x \in \mathrm{rad}(M)$. Thus $\partial M' \cong \frac{\partial M}{\langle \delta \rangle}$ and so $\mathrm{rk}(\partial M') = \mathrm{rk}(\partial M) - 1$ and $r(M) = r(M')$, giving $g(M) = g(M')$ by Remark 4.4.8.

(ii) δ has finite order, so ∂M must be torsion, and by taking canonical form we can assume that δ generates the last $\frac{\mathbb{Z}}{d_r\mathbb{Z}}$ summand. We claim that $\delta' \in \partial M'$ must have infinite order: otherwise let $N \in \mathbb{Z}_{>0}$ be such that $N\delta' = 0$, then there is some $x' \in M'$ such that $N\lambda(x,-)|_{M'} = \lambda(x',-)|_{M'}$, where $x \in M$ satisfies $\alpha(x) = 1$. δ has finite order d_r by assumption so $d_r\alpha = \lambda(t,-)$ for some $t \in M$. Then, $0 \neq Nd_r = \alpha(Nd_r x) = \lambda(t, Nx) = -N\lambda(x,t) = -\lambda(x',t) = \lambda(t,x') = d_r\alpha(x') = 0$, where we have used that $x' \in M' = \ker(\alpha)$ and that $t \in M'$ because $d_r\alpha(t) = \lambda(t,t) = 0$. This gives a contradiction and hence shows the claim. Since ∂M is torsion then so is $\partial M/\langle\delta\rangle$, which is isomorphic to $\partial M'/\langle\delta'\rangle$; thus $\mathrm{rk}(\partial M') = 1$, and $\mathrm{Tors}(\partial M') \subset \mathrm{Tors}(\partial M'/\langle\delta'\rangle) = \partial M/\langle\delta\rangle = \left(\oplus_{i=1}^{r-1} \mathbb{Z}/d_i\mathbb{Z} \oplus \mathbb{Z}/d_i\mathbb{Z}\right) \oplus \mathbb{Z}/d_r\mathbb{Z}$. In particular $r(M') < r(M)$ as the right hand side of the previous formula has less than $2r = 2r(M)$ summands, so $r(M) \geq r(M') + 1$ and hence the formula for genus in Remark 4.4.8 gives $g(M') \geq g(M)$, hence the result as the reverse inequality is trivial because $M' \subset M$. □

Remark 4.4.10. *The above proof uses Remark 4.4.8 plus the fact that $\lambda(t,t) = 0 = \lambda(x,x)$ for t,x as in the proof. These facts use that the form is skew-symmetric, and we will see in Example 5.2.4 that the analogue of Lemma 4.4.9 for symmetric forms is false. This is the key step in which we need to take n odd: Lemma 4.4.9 is used in proving Proposition 4.3.4, which in turn is used in the inductive step of the proof of Theorem 4.3.16, which is needed to prove Theorem E.*

Corollary 4.4.11. *Let (M, ∂, δ) be valid algebraic data such that $\delta \in \partial M$ generates a cyclic direct summand of ∂M of maximum order. Let $\{\alpha_0, \cdots, \alpha_p\}$ be a p-simplex of $\mathcal{A}^{alg}(M, \lambda, \delta)$ and let $M' := \bigcap_{i=0}^p \ker(\alpha_i)$ and $\lambda' = \lambda|_{M'}$, then*

1. $\mathrm{rk}(M') = \mathrm{rk}(M) - (p+1)$.

2. $g(M') \geq g(M) - (p+1)$. *Moreover, if* $\partial M \neq 0$ *then* $g(M') \geq g(M) - p$.

Proof. Rank-nullity implies (i) since $\alpha_0, \cdots, \alpha_p$ is unimodular in $M^\vee$. The first part of (ii) is shown in [GRW18, Corollary 4.2] for the case $p = 0$, and the general case follows by iterating. Lemma 4.4.9 gives the second part of (ii) in the special case $p = 0$, and the general case $p \geq 1$ is a consequence of [GRW18, Corollary 4.2]. □

Now we will show the geometric analogue of the above result, which is the content of Proposition 4.3.4. We will state it again for completeness.

Proposition. *Let* (W, Δ) *be valid geometric data and* $\{a_0, \cdots, a_p\} \in \mathcal{A}(W, \Delta)$ *be a p-simplex. Then*

(i) $\mathrm{rk}(H_n(W \setminus \{a_0, \cdots, a_p\})) = \mathrm{rk}(H_n(W)) - (p+1) < \mathrm{rk}(H_n(W))$.

(ii) $g(W \setminus \{a_0, \cdots, a_p\}) \geq g(W) - (p+1)$. *Moreover if* $H_{n-1}(A) = H_{n-1}(\partial W) \neq 0$ *and* δ *generates a direct summand of maximum order then* $g(W \setminus \{a_0, \cdots, a_p\}) \geq g(W) - p$.

Proof. Let $W' = W \setminus \{a_0, \cdots, a_p\}$ denote the cut manifold and consider the exact sequence of the proof of Proposition 4.4.3

$$0 \to H_n(W') \to H_n(W) \xrightarrow{\oplus_{i=0}^{p} \alpha_i} \mathbb{Z}^{p+1} \to H_{n-1}(W') \to 0$$

where $\alpha_i = (a_i)_*[D^n, \partial D^n] \in H_n(W, \partial W) \cong H_n(W)^\vee$. Then $H_n(W') = \bigcap_{i=0}^{p} \ker(\alpha_i)$ and $\lambda_{W'}$ is the restriction of λ_W by naturality of the intersection product.

The triple $(H_n(W), \lambda_W, \delta)$ defines valid algebraic data, and by Proposition 4.4.3 $\{\alpha_0, \cdots, \alpha_p\}$ is a p-simplex of $\mathcal{A}^{\mathrm{alg}}(H_n(W), \lambda_W, \delta)$. Thus, Corollary 4.4.11 gives the result. □

4.4.3 The connectivity of the arc complex

In this section we will prove Theorem 4.3.3 saying that the arc complex is highly connected. The proof is inspired by the one of [GRW18, Lemma 5.5].

Proof of Theorem 4.3.3. We can assume that $g(W) \geq 1$ as otherwise the result is vacuously true.

Let k be fixed, $0 \leq k \leq g(W) - 2$, we will show that any continuous map $f : S^k \to |\mathcal{A}(W, \Delta)|$ is nullhomotopic. Firstly by the simplicial approximation theorem we can suppose that $S^k \cong |L|$ for L a PL triangulation and that $f : L \to \mathcal{A}(W, \Delta)$ is simplicial. Consider the composition $\Phi \circ f : S^k \xrightarrow{f} |\mathcal{A}(W, \Delta)| \to |\mathcal{A}^{\mathrm{alg}}(H_n(W), \lambda_W, \delta)|$.

By Theorem 4.4.4, the algebraic arc complex $\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)$ is weakly Cohen-Macaulay of dimension $t(H_n(W))-2$, and by Remark 4.4.8 $t(H_n(W))=2g(W)$. Since we are assuming that $g(W)\geq 1$ then $2g(W)-2\geq g(W)-1$, so $\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)$ is also weakly Cohen-Macaulay of dimension $g(W)-1$. Since $k\leq g(W)-2$ then by [GRW18, Theorem 2.4] L can be extended to a PL triangulation K on D^{k+1} with the property that the star of each vertex $v\in K\setminus L$ intersects L at a single (possibly empty) simplex; and such that that there is a simplicial map $g:K\to\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)$ extending $\Phi\circ f$ with the additional property that $g(\mathrm{Lk}_K(v))\subset\mathrm{Lk}_{\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)}(g(v))$ for any vertex $v\in K\setminus L$.

We will prove that $g:K\to\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)$ lifts along Φ to a nullhomotopy $G:K\to\mathcal{A}(W,\Delta)$ of f. Observe that $G|_L=f$ is fixed.

Choose an enumeration of the vertices of $K\setminus L$ as $v_1,\cdots,v_N$. For each $1\leq i\leq N$ we shall inductively pick smooth embeddings $a_i:D^n\hookrightarrow W$ satisfying:

(i) Each ∂a_i has image in $\mathrm{int}(A)\subset W$ and lies in the isotopy class Δ.

(ii) a_i is transverse to ∂W and intersects it precisely along ∂a_i.

(iii) $(a_i)_*([D^n,\partial D^n])=(\Phi\circ f)(v_i)\in\mathcal{A}^{\mathrm{alg}}(H_n(W),\lambda_W,\delta)$.

(iv) If $i\neq j$ then $\mathrm{im}(a_i)$ and $\mathrm{im}(a_j)$ are disjoint.

(v) If $w\in L$ then $\mathrm{im}(a_i)$ and $\mathrm{im}(f(w))$ are disjoint.

Suppose that the embeddings $a_1,\cdots,a_{s-1}$ have already been chosen and satisfy the above properties, we will construct the embedding a_s.

Pick an embedding $\iota:S^{n-1}\hookrightarrow A\subset\partial W$ representing the isotopy class Δ and make it transverse, and hence disjoint, to all the $\partial f(w)$ for $w\in L$ and all the ∂a_i for $i<s$. We have $(\Phi\circ f)(v_s)\in H_n(W,\partial W)\cong\pi_n(W,\partial W)$ such that $(\Phi\circ f)(v_s)\mapsto\delta\in H_{n-1}(\partial W)\cong\pi_{n-1}(\partial W)$ and hence there is a continuous map $a_s:D^n\to W$ such that $\partial a_s=\iota$ and a_s represents the homology class $(\Phi\circ f)(v_s)$. By transversality we can make a_s to be smooth, immersed with at worse double point singularities, transverse to all the a_i for $i<s$, to all the $f(w)$ for $w\in L$ and to ∂W, and intersecting ∂W only along ∂a_s. Moreover we can also make it be in general position with respect to all the a_i for $i<s$, to all the $f(w)$ for $w\in L$, so that there are no triple intersection points.

Now, by the half-Whitney trick, see [GRW14, Claim 6.18], we can isotope a_s to cancel any intersection point with a_i, $i<s$, or with $f(w)$ for $w\in L$. This procedure does not create any new intersection points each time we cancel because we can pick each Whitney disc disjoint from the a_i's and $f(w)$'s not in play by transversality, using that $2+n<2n$ and that there are no triple intersections. This isotopy changes ∂a_s but it keeps it in $\mathrm{int}(A)$ and in the isotopy class Δ, and moreover we can always use transversality at the end of the procedure to ensure condition (ii) above still holds.

We then use Haefliger's trick to homotope a_s so that it becomes embedded, without changing the above disjointness conditions and the behaviour of a_s near the boundary: for each self-intersection point the cancelling procedure takes place in a small neighbourhood of that point, and a_s was already embedded near the boundary.

Thus, the new collection $a_1, \cdots, a_s$ will satisfy all the properties (i)-(v) above, as required.

Finally set $G(v_i) := a_i$ for $1 \leq i \leq N$. By conditions (i),(ii) and (iii) on the choice of the a_i's, G gives a lift of g along Φ on the vertices of K, so to finish the proof it suffices to check that each a_i is indeed non-separating so that it defines a vertex in the arc complex, and that such G is simplicial. Let τ be a simplex in K, we will show that $G(\tau)$ is a simplex in $\mathcal{A}(W, \Delta)$. The pairwise disjointness condition holds because f is itself simplicial and the embeddings a_i are pairwise disjoint and disjoint to all the $f(w)$ for $w \in L$. The connectivity of the jointly cut manifold holds because it is equivalent to the unimodularity of the corresponding elements in the algebraic arc complex by the first step in the proof of Proposition 4.4.3, and by assumption $g(\tau)$ is a simplex as g is simplicial. $\square$

Chapter 5

Partial results for n even

5.1 Moduli spaces and E_k-algebras for n even

Let us begin by explicitly saying how to adapt some of the constructions and results of Section 4.1 to the case n even.

The definition of the moduli spaces $\mathcal{M}[A]$, Definition 4.1.1, makes sense for n even too. Section 4.1.2 also applies without any change, giving $\mathcal{M}[I^{2n-1}]$ the structure of an E_{2n-1}-algebra in Top. However, it will be convenient to also consider the E_{2n-1}-subalgebra, denoted $\mathcal{M}[I^{2n-1}]^{\sigma=0}$, consisting of those $W \in \mathcal{M}[I^{2n-1}]$ with vanishing signature: $\sigma(W) = 0$. Section 4.1.3 applies for n even too, so we can interpret the path-components of $\mathcal{M}[I^{2n-1}]$ as classifying spaces of the form $B\mathrm{Diff}_{\frac{1}{2}\partial}(W)$ for $W \in \mathcal{M}[I^{2n-1}]$.

Section 4.1.4 (where we identify the monoid of path-components of the algebra), can be adapted for n even as follows. Now $(H_n(W), \lambda_W, q_W)$ consists of a symmetric, non-degenerate bilinear form λ_W on a free finitely generated $\mathbb{Z}$-module $H_n(W)$ (by Poincaré duality); and a function $q_W : H_n(W) \to \mathbb{Z}$ such that $q_W(x+y) = q_W(x) + q_W(y) + \lambda_W(x,y)$ and $q_W(k \cdot x) = k^2 q_W(x)$ for all $x, y \in H_n(W)$ and $k \in \mathbb{Z}$ by [Wal62, pages 165-167]. The properties of q_W imply that $q_W(x) = \lambda_W(x,x)/2$ for all $x \in H_n(W)$, so its existence just says that λ_W is an even form, but the value of q_W itself is redundant information.

In order to state the result, let us define the *standard (symmetric) hyperbolic form of genus g*, denoted $H^{\oplus g}$ as in the case n odd, to be given by the block diagonal sum of g copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then, for a manifold $W \in \mathcal{M}[A]$ we define its *genus* via

$$g(W) := \sup\{g : \exists \text{ morphism } \phi : H^{\oplus g} \to (H_n(W), \lambda_W)\}.$$

Proposition 5.1.1. *Let $n \geq 4$ be even, then* *(i)* $\pi_0(\mathcal{M}[I^{2n-1}])$ *is isomorphic to the monoid of even, symmetric, non-degenerate bilinear forms on free finitely generated* $\mathbb{Z}$*-modules (under orthogonal direct sum).*

(ii) $\pi_0(\mathcal{M}[I^{2n-1}]^{\sigma=0}) \cong \mathbb{N}$*, where the isomorphism is given by taking the genus.*

Proof. The proof each part is identical to the one of Proposition 4.1.4, and shows that in each case the monoid of path-components is isomorphic to the monoid of corresponding algebraic data (M,λ). To finish part (ii) we quote [MH73, Theorem 5.3, Chapter 4], which implies that any even, symmetric, non-degenerate bilinear form of zero signature on a free finitely generated $\mathbb{Z}$-module is isomorphic to a unique $H^{\oplus g}$. □

A consequence of the previous proposition is that the algebra $\mathcal{M}[I^{2n-1}]$ seems hard to use to prove homological stability via the cellular E_k-algebra approach: by Proposition 5.1.1(i) the monoid $\pi_0(\mathcal{M}[I^{2n-1}])$ is Artinian in the sense of [GKRW18, Definition 11.10], and so there is a canonical rank functor $\mathrm{rk}: \pi_0(\mathcal{M}[I^{2n-1}]) \to \mathbb{N}$ given by

$$\mathrm{rk}(x) := \sup\{n : x = x_1 \oplus \cdots \oplus x_n \text{ such that } x_i \text{ is not } \oplus - \text{ invertible }\}$$

where the Artinian condition precisely says that the above supremum is attained. But then, all the indecomposable even, symmetric, non-degenerate forms are in rank 1, which means that to understand the E_{2n-1}-cells of $\mathcal{M}[I^{2n-1}]$ in low bidegrees we need to access homology group computations of the diffeomorphism groups of infinitely many $2n$-manifolds. In fact, we do not even know all these manifolds, as we do not know a classification of all even non-degenerate symmetric bilinear forms over the integers. Moreover, if we let σ represent the path-component of $W_{1,1} \in \mathcal{M}[I^{2n-1}]$ (our choice of stabilising element) then $H_{*,0}(\mathcal{M}[I^{2n-1}]/\sigma)$ does not have a vanishing line as it contains classes represented by arbitrary sums of the E_8-form.

However, by Proposition 5.1.1(ii) for any even $n \geq 4$ each path-component of $\mathcal{M}[I^{2n-1}]^{\sigma=0}$ is represented by a $W_{g,1}$, and $\mathcal{M}[I^{2n-1}]^{\sigma=0} \simeq \bigsqcup_{g\geq 0} B\mathrm{Diff}_{\frac{1}{2}\partial}(W_{g,1})$, which looks analogous to what happened when $n = 3,7$ (see Lemma 4.1.3 and Remark 4.1.5). Thus, it seems to us that letting

$$\mathbf{R} := \{W \in \mathcal{M}[I^{2n-1}]^{\sigma=0} : g(W) > 0\} \in \mathrm{Alg}_{E_{2n-1}}(\mathsf{Top}^{\mathbb{N}})$$

is the most reasonable choice: then $H_{*,0}(\overline{\mathbf{R}}) = \mathbb{Z}[\sigma]$ for $\sigma \in H_{1,0}(\mathbf{R})$ by Proposition 5.1.1(ii).

As we already said in Remark 4.4.10, our proof of Theorem E does not work for n even, so we will state it as a conjecture.

Conjecture 5.1.2. *For* $n \geq 4$ *even the* E_{2n-1}*-algebra* $\boldsymbol{R}$ *satisfies* $H^{E_1}_{g,d}(\boldsymbol{R}) = 0$ *for* $d < g - 1$.

Assuming the conjecture we get the immediate corollary that $H^{E_{2n-1}}_{g,d}(\mathbf{R}_{\Bbbk}) = 0$ for $d < g-1$. (This is identical to the proof of Corollary 4.1.8).

The plan for the rest of the chapter is as follows: in Section 5.2 we will study splitting complexes and arc complexes for n even, and we will make some partial progress towards proving the above conjecture by reducing it to the high-connectivity of certain (discretized) splitting complexes. In Section 5.3 we will show that the above conjecture implies a rational stability result for $\{B\mathrm{Diff}_\partial(W_{g,1})\}_{g\geq 1}$ when n is even, assuming a new homological stability result for E_k-algebras, Theorem 5.4.1. Finally, Section 5.4 is devoted to the proof of Theorem 5.4.1.

5.2 Splitting complexes and arc complexes for n even

The aim of this section is to make some partial progress towards the proof of Conjecture 5.1.2 by using splitting complexes analogous to the ones of Section 4.3. Finally, we will also explain why the high-connectivity of splitting complexes for n even, Conjecture 5.2.1, cannot be shown by adapting the ideas of Section 4.3.

The definition of splitting complex, Definition 4.3.5, just needs a minor modification in Part (iv): we need to insist that $W_{\leq\omega}$ has zero signature, so that it is abstractly diffeomorphic to a manifold in $\mathbf{R}$, i.e. to a $W_{g,1}$. Similarly we can define discretized splitting complexes, splitting posets and discretized splitting posets.

Let us formulate a new conjecture, which would be the analogue of Theorem 4.3.16 for n even.

Conjecture 5.2.1. *For $n \geq 4$ even and $W \in \mathbf{R}$ the discretized splitting poset, $\mathcal{S}^{E_1,\delta}(W)$ is $(g(W)-3)$-connected.*

Now let us show that this new conjecture implies the previous one about the a-priori vanishing of E_k-cells of $\mathbf{R}$.

Lemma 5.2.2. *Conjecture 5.2.1 implies Conjecture 5.1.2.*

Proof. Assuming Conjecture 5.2.1 we can prove the analogue of Corollary 4.3.19: that for $W \in \mathbf{R}$ the poset $\mathcal{S}^{E_1,\delta}(W)$ is f-weakly Cohen-Macaulay of dimension $g(W)-2$, where $f(\omega) = g(W_{\leq\omega}) - 1$. (This uses that Section 4.3.4 up to, and including, Corollary 4.3.13 applies for n even without change.)

Thus, we can apply the discretization theorem, Theorem 4.3.21, to show the analogue of Corollary 4.3.22 for n even. The whole of Section 4.3.7 applies in this case too, showing that the splitting complexes are homotopy fibres of a certain map whose cofibre is related to the

E_1-homology of $\mathbf{R}$. (Here it is important the change we made in the definition of splitting by insisting that $\sigma(W_{\leq\omega}) = 0$ for any wall ω.)

Therefore the required vanishing on $H^{E_1}_{g,d}(\mathbf{R})$ follows. $\square$

In order to prove Conjecture 5.2.1, one might try to use similar ideas to the ones in Section 4.3, i.e. trying to find an appropriate "arc complex" and applying a nerve theorem type of argument as in the proof of Theorem 4.3.16. However, as we will explain now, we cannot simply use the same arc complex of Chapter 4 (at least without a substantial modification in the argument).

The arc complex definition, Definition 4.3.2, makes sense for n even without any modification. In fact, let us show that it is still highly connected.

Proposition 5.2.3. *For $n \geq 4$ even and (W, Δ) valid geometric data, the arc complex $\mathcal{A}(W, \Delta)$ is $(g(W)-2)$-connected.*

Proof. We can modify the definition of valid algebraic data, Definition 4.4.1, by saying that λ is symmetric instead of skew-symmetric. Then, Section 4.4.1 applies identically to this situation up to (and including) Corollary 4.4.6. In particular, the analogue of Theorem 4.4.4 holds for n even too: If (M, λ, δ) is valid algebraic data then the simplicial complex $\mathcal{A}^{\mathrm{alg}}(M, \lambda, \delta)$ is weakly Cohen-Macaulay of dimension $t(M)-2$, where $t(M) := \max\{\mathrm{rk}(U) : U \subset \lambda^\vee(M) \text{ is a direct summand of } M^\vee\}$. Remark 4.4.8 is false for symmetric forms in general, but we still have that $t(M) \geq 2g(M)$ by definition.

Then we just proceed as in the proof of Theorem 4.3.3 to "lift" the connectivity from the algebraic arc complex to the geometric one by using half-Whitney tricks, Haefliger cancellation and transversality (here is where $n \geq 4$ becomes relevant). $\square$

The part that we cannot prove for n even is the "moreover" part of Proposition 4.3.4(ii), which says that whenever $W \in \mathcal{M}[A]$ does not belong to $\mathbf{R}$ there is some appropriate choice of Δ such that any p-simplex $\{a_0, \cdots, a_p\}$ in $\mathcal{A}(W, \Delta)$ satisfies that $g(W \smallsetminus \{a_0, \cdots, a_p\}) \geq g(W) - p$. (The rest of Proposition 4.3.4 still holds for n even by the same proof.)

In fact, we will now give a concrete counterexample of Proposition 4.3.4(ii) for n even. We will do it "algebraically" first and then indicate how to get the geometric counterexample.

Example 5.2.4. *Consider the even symmetric bilinear form $(M, \lambda) = (\mathbb{Z}^3, \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right))$, and let us denote the standard basis of M by e, f, m and the dual basis by ε, ϕ, μ. Then, $\lambda^\vee(M)$ is generated by $\phi, \varepsilon, 2\mu$, so $\partial M \cong \mathbb{Z}/2\{\mu\}$. Thus, there are two choices of δ, and we will show that the "moreover" part of Proposition 4.3.4(ii) fails for both of them. Observe that $g(M) = 1$.*

(i) *If $\delta = 0$ then $\varepsilon \in M^\vee$ is a 0-simplex in $\mathcal{A}^{alg}(M,\lambda,\delta)$ and the corresponding "cut form" is* $\ker(\varepsilon)$*, which has basis f, m, and so as a form it is* $(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right))$ *which has genus* 0.

(ii) *If $\delta = \mu$ then for any $N \in 1+2\mathbb{Z}$ the element $\varepsilon + N\mu \in M^\vee$ is a 0-simplex in $\mathcal{A}^{alg}(M,\lambda,\delta)$ and the corresponding "cut form" is* $\ker(\varepsilon + N\mu)$*, which has basis $f, Ne - m$, and so as a form it is* $(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & N \\ N & 2 \end{smallmatrix}\right))$ *which has genus* 0 *for $N \neq \pm 1$. In particular, $N = 3$ gives an example of an algebraic arc, $\varepsilon + 3\mu$, for which the genus decreases when we cut it.*

Now let us explain how to realize this example geometrically: firstly observe that if we start with the standard hyperbolic form $H^{\oplus 2}$ and we cut the arc $\lambda^\vee(e_2 - f_2)$ (written in standard symplectic basis), which is a vertex in $\mathcal{A}^{alg}(H^{\oplus 2}, \delta = 0)$, we precisely get the above form (M,λ). Secondly, to do it geometrically we start with $W_{2,1}$ and cut a geometric arc representing $\lambda^\vee(e_2 - f_2)$ (which exists by an argument similar to the proof of Theorem 4.3.3 for n even). We then get a manifold $W \in \mathcal{M}[A]$ for a valid A such that $(H_n(W), \lambda_W) = (M,\lambda)$, as required.

5.3 Rational homological stability for n even

In this section we will show that Conjecture 5.1.2 implies a rational homological stability result for the family $\{B\mathrm{Diff}_\partial(W_{g,1})\}_{g\geq 1}$ when n is even, which is analogous to Theorem C(iv),(v). In doing so we will also motivate the assumptions for a new generic homological stability result similar to Theorem 2.1.6, which will be shown in the next section.

Let us begin by proving the analogue of Theorem 4.2.13 for n even:

Theorem 5.3.1 (Berglund-Madsen, Krannich)**.** *For $n \geq 4$ even and $g \geq 1$ the stabilization map*

$$H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q}) \to H_d(B\mathrm{Diff}_\partial(W_{g+1,1});\mathbb{Q})$$

is surjective for $d \leq \min\{g-1, 3n-7\}$ and an isomorphism for $d \leq \min\{g-2, 3n-7\}$.

Proof. We proceed as in the proof of Theorem 4.2.13 to reduce the statement to showing that that the stabilisation map

$$H_d(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1});\mathbb{Q}) \to H_d(B\widetilde{\mathrm{Diff}}_\partial(W_{g+1,1});\mathbb{Q})$$

is surjective for $d \leq g-1$ and an isomorphism for $d \leq g-2$.

The isomorphism part of this new statement about Block diffeomorphisms is contained in [Kra20b, Theorem A], so we just need to show the surjectivity part in one degree higher. We proceed as we did in the proof of Theorem 4.2.13 and reduce it to showing that [Kra20b, Theorem 1.1] gives surjectivity in one degree higher when the group called $\mathbf{G_g}$ is the indefinite orthogonal group $O_{g,g}$. By [Tsh19, Theorem 1] it suffices to show that $H_d(BO_{g,g}(\mathbb{Z});\mathbb{Q}) \to H_d(BO_{g+1,g+1}(\mathbb{Z});\mathbb{Q})$ is surjective for $d \leq g-1$, which is precisely [GKT21, Proposition 11]. □

Also, observe that all the results in Sections 4.2.1 and 4.2.2 about comparing homological stability of $\mathbf{R}$ to homological stability of $\{B\mathrm{Diff}_\partial(W_{g,1})\}_g$ hold for $n \geq 4$ even. Thus we get the following result.

Theorem 5.3.2. *Assuming that Conjecture 5.1.2 holds for $n \geq 4$ even, the stabilisation map*

$$H_d(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q}) \to H_d(B\mathrm{Diff}_\partial(W_{g+1,1});\mathbb{Q})$$

is surjective for $d < \frac{3n-7}{3n-5}(g-1)$ and an isomorphism for $d < \frac{3n-7}{3n-5}(g-1)-1$.

Before proving the above theorem, let us remark that in this situation we cannot apply Theorem 2.1.6 since the partial stability result we have for $\mathbf{R}$ is of the form $H_{g,d}(\overline{\mathbf{R}}/\sigma) = 0$ for $d < \min\{g-1, 3n-6\}$, instead of being of the form $d < \min\{g, 3n-6\}$. Thus, we will make use of a different result, Theorem 5.4.1, that we will show in the next section.

Proof. By the comparison theorems between the homological stability of $\mathbf{R}$ and $B\mathrm{Diff}_\partial(W_{g,1})$ it suffices to show the corresponding homological stability result for $\mathbf{R}_\mathbb{Q}$, i.e. that $H_{g,d}(\overline{\mathbf{R}_\mathbb{Q}}/\sigma) = 0$ for $d < \frac{3n-7}{3n-5}(g-1)$.

We will do so by applying Theorem 5.4.1 with $\mathbf{X} = \mathbf{R}_\mathbb{Q}$ and $D = 3n-6$. Let us verify the assumptions: the vanishing of $H_{0,0}(\mathbf{X})$ holds by definition, the vanishing line on E_{2n-1}-homology holds by Conjecture 5.1.2, and $H_{*,0}(\overline{\mathbf{X}}) = \mathbb{Q}[\sigma]$ for $\sigma \in H_{1,0}(\mathbf{X})$ holds as a consequence of Proposition 5.1.1(ii) (c.f. page 110 of this thesis). Finally, the partial stability result for $\mathbf{X}$ holds by the comparison theorems again plus Theorem 5.3.1. □

5.4 A new stability result

The goal of this section is to prove the generic homological stability result that we needed in the previous section. As we will see, despite the similarity of its statement with Theorem 2.1.6, its proof is quite different. In particular, we will use some (relative) formality results of the rationalised E_k-operad in its proof.

Theorem 5.4.1. *Let $k \geq 3$ and $\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}_\mathbb{Q}^\mathbb{N})$ such that $H_{0,0}(\mathbf{X}) = 0$, $H_{g,d}^{E_k}(\mathbf{X}) = 0$ for $d < g-1$, and $H_{*,0}(\overline{\mathbf{X}}) = \mathbb{Q}[\sigma]$ as a ring, for some class $\sigma \in H_{1,0}(\mathbf{X})$. Suppose that there is $D \in \mathbb{Z}_{>1} \cup \{\infty\}$ for which the following partial homological stability result holds: $H_{g,d}(\overline{\mathbf{X}}/\sigma) = 0$ for $d < \min\{g-1, D\}$. Then $H_{g,d}(\overline{\mathbf{X}}/\sigma) = 0$ for $d < \frac{D-1}{D+1}(g-1)$.*

The first step will be to show the above theorem for $k=\infty$. This step is similar to the proof of Theorem 2.1.6 although it also uses some other techniques which are specific to E_∞-algebras over $\mathsf{sMod}_{\mathbb{Q}}$. After showing this step we will explain how to get the result for finite k, which is where the formality appears.

Proof for $k=\infty$. The proof of this case will be divided into several steps. As we will always work with $\mathbb{Q}$-coefficients in this section, we will drop all the $\mathbb{Q}$-subindices.

Step 1. Let $\mathbf{Y}$ be the result of attaching E_∞-cells to $\mathbf{X}$ of dimension $\geq D$ such that

(i) The map $\mathbf{X}\to\mathbf{Y}$ induces isomorphisms $H_{g,d}(\mathbf{X})\xrightarrow{\cong}H_{g,d}(\mathbf{Y})$ for $d<D-1$.

(ii) $H_{g,d}(\mathbf{Y})=0$ for $d\geq D-1$.

Such a $\mathbf{Y}$ can indeed be constructed because attaching an E_∞-cell along a homology class has the effect of killing that class in homology, and does not change the homology on smaller degrees. The details of this are identical to those of the proof of [GKRW18, Theorem 11.21]. By construction $H_{g,d}(\overline{\mathbf{Y}}/\sigma)=0$ for $d<g-1$.

Claim. Let $\mathbf{Z}=\mathbf{Y}\cup_\sigma^{E_\infty}D^{1,1}\sigma'$, then the natural map $\overline{\mathbf{Y}}/\sigma\to\overline{\mathbf{Z}}$ is a weak equivalence (in graded simplicial modules).

This claim is taken from [GKRW20, Proof of Theorem 9.5], and its proof fundamentally uses $k=\infty$.

Proof of Claim. By definition of cell-attachments in the category of $\overline{\mathbf{Y}}$-modules (in the sense of [GKRW18, Section 6.1]) we have $\overline{\mathbf{Y}}/\sigma\simeq\overline{\mathbf{Y}}\cup_\sigma^{\overline{\mathbf{Y}}}D^{1,1}\sigma'\in\overline{\mathbf{Y}}-\mathrm{Mod}$. Consider the induced map of cell-attachment spectral sequences (see [GKRW18, Section 10.3.2]) in the categories of $\overline{\mathbf{Y}}$-modules and E_∞-algebras respectively.

In $\overline{\mathbf{Y}}-\mathrm{Mod}$ the E^1-page is given by $H_*(\overline{\mathbf{Y}}\otimes(\mathbb{1}\oplus S^{1,1}\sigma'))\cong H_*(\overline{\mathbf{Y}})\otimes\Lambda(\sigma')$ where $\overline{\mathbf{Y}}$ is in filtration 0 and σ' in filtration 1. Moreover, $d^1(u\otimes\sigma')=u\cdot\sigma$, and the spectral sequence must collapse at E^2 by filtration-degree reasons.

In $\mathrm{Alg}_{E_\infty}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$ the E^1-page is given by $H_*(\overline{\mathbf{Y}}\oplus^{E_\infty^+}\overline{\mathbf{E}_\infty(S^{1,1}\sigma')})$ where $\overline{\mathbf{X}}$ is in filtration 0 and σ' in filtration 1. By [GKRW18, Corollary 16.5] (which fundamentally uses $k=\infty$) and using the identification $H_*(\overline{\mathbf{E}_\infty(S^{1,1}\sigma')})\cong\Lambda(\sigma')$ we can write the first page as $H_*(\overline{\mathbf{Y}})\otimes\Lambda(\sigma')$ with $d^1(u\otimes\sigma')=u\cdot\sigma$. Thus, this spectral sequence also degenerates at E^2.

The result then follows as the map on spectral sequences is the canonical one, which is an isomorphism on E^1 and preserves d^1 by the above descriptions, hence it induces isomorphisms on $E^2=E^\infty$. □

Thus, we find that $H_{g,d}(\overline{\mathbf{Z}})=0$ for $d<g-1$.

Step 2. In this step we will prove a vanishing on the E_∞-homology of $\mathbf{Z}$ using the vanishing of $H_{*,*}(\overline{\mathbf{Z}})$.

Claim. $H^{E_\infty}_{g,d}(\mathbf{Z}) = 0$ for $d < g-1$.

Proof of Claim. It follows from [GKRW18, Lemmas 13.3, 13.5, Theorem 13.7] that $B(\mathbb{1},\overline{\mathbf{Z}},\mathbb{1}) \simeq \mathbb{1} \oplus S^{0,1} \otimes Q^{E_1}_{\mathbb{L}}(\mathbf{Z})$. Now consider the bar spectral sequence (see [GKRW18, Section 10.2.2]):

$$E^1_{*,p,*} = \mathbb{1} \otimes H_{*,*}(\overline{\mathbf{Z}})^{\otimes p} \otimes \mathbb{1} \Rightarrow H_{*,*}(B(\mathbb{1},\overline{\mathbf{Z}},\mathbb{1})).$$

Our assumption on the vanishing of $H_{*,*}(\overline{\mathbf{Z}})$ implies that $E^1_{g,p,q} = 0$ for $q < g-p$, and hence that $H_{g,d}(B(\mathbb{1},\overline{\mathbf{Z}},\mathbb{1})) = 0$ for $d < g$. Thus, $H^{E_1}_{g,d}(\mathbf{Z}) = 0$ for $d < g-1$. Finally, [GKRW18, Theorem 14.4] gives the result by transferring the vanishing line to E_∞-homology. $\square$

Step 3. Now let $\mathbf{Z_0} = \mathbf{X} \cup^{E_\infty}_{\sigma} D^{1,1}\sigma'$. Then, proceeding as in the claim of Step 1, the natural map $\overline{\mathbf{X}}/\sigma \to \overline{\mathbf{Z_0}}$ is a weak equivalence (in graded simplicial modules). Also, the composition $\mathbf{X} \to \mathbf{Y} \to \mathbf{Z}$ extends canonically to an E_∞-algebra map $\mathbf{Z_0} \to \mathbf{Z}$ such that $\sigma' \mapsto \sigma'$. Now we claim the following

Claim. $H^{E_\infty}_{g,d}(\mathbf{Z},\mathbf{Z_0}) = 0$ for $d < \max\{\frac{D}{D+1}g, D\}$.

Proof of Claim. We will show that $H^{E_\infty}_{g,d}(\mathbf{Z},\mathbf{Z_0}) = 0$ for $d < g-1$ and for any $d < D$, which then implies the result.

By construction, both $H^{E_\infty}_{g,d}(\mathbf{X}) \to H^{E_\infty}_{g,d}(\mathbf{Z_0})$ and $H^{E_\infty}_{g,d}(\mathbf{Y}) \to H^{E_\infty}_{g,d}(\mathbf{Z})$ are isomorphisms for $(g,d) \notin \{(1,0),(1,1)\}$.

In particular, since $H^{E_\infty}_{g,d}(\mathbf{X}) = 0$ for $d < g-1$ then $H^{E_\infty}_{g,d}(\mathbf{Z_0}) = 0$ for $d < g-1$. By Step 2 we have $H^{E_\infty}_{g,d}(\mathbf{Z}) = 0$ for $d < g-1$, and thus $H^{E_\infty}_{g,d}(\mathbf{Z},\mathbf{Z_0}) = 0$ for $d < g-1$, hence proving the first bound.

For the second, consider the map of pairs $(\mathbf{X},\mathbf{Y}) \to (\mathbf{Z_0},\mathbf{Z})$ and the corresponding map of long exact sequences in E_∞-homology. By the four lemma, $H^{E_\infty}_{g,d}(\mathbf{Y},\mathbf{X}) \to H^{E_\infty}_{g,d}(\mathbf{Z},\mathbf{Z_0})$ is surjective for $(g,d) \notin \{(1,2),(1,1)\}$. By construction $H^{E_\infty}_{g,d}(\mathbf{Y},\mathbf{X}) = 0$ for $d < D$ so it suffices to check the cases $(g,d) = (2,1)$ and $(g,d) = (1,1)$.

In the case $(g,d) = (2,1)$ the four lemma still applies because

$$H^{E_\infty}_{1,1}(\mathbf{X}) \to H^{E_\infty}_{1,1}(\mathbf{Z_0}) \to H^{E_\infty}_{1,1}(\mathbf{Z_0},\mathbf{X}) = \mathbb{Q}\{\sigma'\} \xrightarrow{\sigma' \mapsto \sigma} H^{E_\infty}_{1,0}(\mathbf{X}) = \mathbb{Q}\{\sigma\}$$

is exact and hence the first map is surjective. (The identification $H^{E_\infty}_{1,0}(\mathbf{X}) = \mathbb{Q}\{\sigma\}$ follows from the Hurewicz theorem [GKRW18, Corollary 11.14] and our assumption about the zero-th homology of $\overline{\mathbf{X}}$.)

Finally, for the case $(g,d) = (1,1)$ we use the above long exact sequence for the pair $(\mathbf{Z_0},\mathbf{X})$ to show that in fact $H^{E_\infty}_{1,1}(\mathbf{X}) \to H^{E_\infty}_{1,1}(\mathbf{Z_0})$ is an isomorphism and $H^{E_\infty}_{1,0}(\mathbf{Z_0}) = 0$. By the analogous argument, $H^{E_\infty}_{1,1}(\mathbf{Y}) \to H^{E_\infty}_{1,1}(\mathbf{Z})$ is also an isomorphism. Since $D > 1$ and $H^{E_\infty}_{g,d}(\mathbf{Y},\mathbf{X}) = 0$ for $d < D$ then $H^{E_\infty}_{1,1}(\mathbf{Y},\mathbf{X}) = 0$. Thus the commutative diagram

$$\begin{tikzcd} H^{E_\infty}_{1,1}(\mathbf{X}) \arrow[r, two heads] \arrow[d, "\cong"] & H^{E_\infty}_{1,1}(\mathbf{Y}) \arrow[d, "\cong"] \\ H^{E_\infty}_{1,1}(\mathbf{Z_0}) \arrow[r] & H^{E_\infty}_{1,1}(\mathbf{Z}) \end{tikzcd}$$

plus $H^{E_\infty}_{1,0}(\mathbf{Z_0}) = 0$ implies that $H^{E_\infty}_{1,1}(\mathbf{Z},\mathbf{Z_0}) = 0$, as required. □

Step 4. Now we will finally prove the result, which by construction is equivalent to showing that $H_{g,d}(\overline{\mathbf{Z_0}}) = 0$ for $d < \frac{D-1}{D+1}(g-1)$. This step is inspired by [KMP21, Lemma 6.1].

By the CW approximation theorem, [GKRW18, Theorem 11.21], there is a relative CW approximation (in the sense of [GKRW18, Definition 11.20]) of $\mathbf{Z_0} \to \mathbf{Z}$. Moreover, by Step 3 we can pick such a CW approximation in which all its (g,d)-cells satisfy $d \geq \frac{D}{D+1}\max\{g, D+1\}$.

Now consider the corresponding CW spectral sequence, see [GKRW18, Corollary 10.19]:

$$E^1_{g,p,q} = H_{g,p+q,q}(0_*(\overline{\mathbf{Z_0}}) \oplus^{E^+_\infty} \overline{\mathbf{E}_\infty(A)}) \Rightarrow H_{g,p+q}(\overline{\mathbf{Z}})$$

where A is a coproduct of spheres of the form $S^{g,d,d}$, where the last grading is the filtration, and $d \geq \frac{D}{D+1}\max\{g, D+1\}$. By [GKRW18, Corollary 16.5, Theorem 16.4] (which again uses $k = \infty$) we can identify $E^1 = H_*(\overline{\mathbf{Z_0}}) \otimes \Lambda(H_*(A))$ where the first factor is in filtration zero.

Now suppose for a contradiction that the result was false, and pick a minimal (g_0, d_0) for which $H_{g_0,d_0}(\overline{\mathbf{Z_0}}) \neq 0$ and $d_0 < \frac{D-1}{D+1}(g_0-1)$. Fix $u \neq 0 \in H_{g_0,d_0}(\overline{\mathbf{Z_0}})$.

We have $u \otimes \mathbb{1} \in E^1_{g_0,0,d_0}$ and hence $d^r(u \otimes \mathbb{1}) = 0 \forall r \geq 1$ (because d^r decreases filtration by r and there are no elements in negative filtration). Since $H_{g,d}(\overline{\mathbf{Z}}) = 0$ for $d < g-1$ and $d_0 < \frac{D-1}{D+1}(g_0-1) < g_0 - 1$ then $u \otimes \mathbb{1} \in \operatorname{im}(d^r)$ for some $r \geq 1$, so $E^1_{g_0,r,d_0-r+1} \neq 0$ for some $r \geq 1$.

But, any element in $E^1_{g_0,r,d_0-r+1}$ must be a linear combination of elements of the form $v \otimes w \in H_*(\overline{\mathbf{Z_0}}) \otimes \Lambda(H_*(A))$ in the appropriate bidegree and filtration. In particular, as we must have positive filtration then w cannot be a multiple of $\mathbb{1}$. Let (g', d') be the total bidegree of v and (g'', d'') be the total bidegree of w (i.e. we are suppressing the filtration degree here). Then, $g' + g'' = g_0$ and $d' + d'' = d_0 + 1$. By assumption on the support of A we have $d'' \geq \frac{D}{D+1}\max\{g'', D+1\}$. Since $d'' \geq D > 1$ and $g'' \geq 0$ then $d' < d_0$ and $g' \leq g_0$ so the minimality of (g_0, d_0) implies that $d' \geq \frac{D-1}{D+1}(g'-1)$.

Thus, adding up the above inequalities we get

$$\frac{D-1}{D+1}(g_0-1)+1 > d_0+1 = d'+d'' \geq \frac{D-1}{D+1}(g'-1)+\frac{D}{D+1}\max\{g'',D+1\} = \\ \frac{D-1}{D+1}(g'-1+\max\{g'',D+1\})+\frac{1}{D+1}\max\{g'',D+1\} \geq \frac{D-1}{D+1}(g_0-1)+1$$

using that $g_0 = g'-g''$. This gives the required contradiction, finishing the proof. □

Now we want to prove the result for finite $k \geq 3$. This will be based on the following result, which is a new type of result as far as we know, and whose proof is based on different ideas to the rest of this thesis.

Theorem 5.4.2. *Let* $\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$ *with* $k \geq 3$. *Then there is* $\mathbf{X}' \in \mathrm{Alg}_{E_\infty}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$ *such that* $\mathbf{X} \simeq \mathbf{X}'$ *as* E_{k-2}*-algebras.*

Assuming this, we can finish the proof.

Proof for finite k. Let $\mathbf{X}'$ be as in the statement of Theorem 5.4.2. There is a weak equivalence $\mathbf{X} \simeq \mathbf{X}'$ as E_{k-2}-algebras, and hence as E_1-algebras, so $\overline{\mathbf{X}} \simeq \overline{\mathbf{X}'}$ as associative algebras. Thus, $\overline{\mathbf{X}}/\sigma \simeq \overline{\mathbf{X}'}/\sigma$ as graded simplicial modules. By transferring the vanishing lines down, [GKRW18, Theorem 14.6], we have that $H^{E_1}_{g,d}(\mathbf{X}) = 0$ for $d < g-1$, and thus $H^{E_1}_{g,d}(\mathbf{X}') = 0$ for $d < g-1$. By transferring the vanishing lines up, [GKRW18, Theorem 14.4], we then find $H^{E_\infty}_{g,d}(\mathbf{X}') = 0$ for $d < g-1$. Therefore $\mathbf{X}'$ satisfies the assumptions of Theorem 5.4.1 with $k = \infty$. Thus, $\overline{\mathbf{X}'}/\sigma$ has the required vanishing line in homology, and hence so does $\overline{\mathbf{X}}/\sigma$. □

Remark 5.4.3. *We can also use Theorem 5.4.2 to deduce that for any* E_k*-algebra* $\mathbf{X}$ *in* $\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}}$ *and* $\sigma \in H_{1,0}(\mathbf{X})$ *we can give the cofibre* $\mathbf{X}/\sigma$ *the structure of an* E_{k-2}*-algebra over* $\mathbf{X}$. *This could potentially be useful in investigating secondary stability phenomenon, and this type of result might in fact hold in more general categories than* $\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}}$.

Now we will prove Theorem 5.4.2. The main key ingredient of the proof will be the formality result [FW15, Theorems C and D], which says that $E_{k-2} \to E_k$ is formal as a map of operads in rational spaces for $k \geq 3$.

Proof. Let us begin the proof by recalling some technicalities on operads which will be relevant. The definition of an E_k*-operad* is given in [GKRW18, Definition 12.2, Remark 12.3] as follows: an operad $\mathcal{E}_k$ (in sSet or $\mathsf{sMod}_{\Bbbk}$) is an E_k*-operad* if it is Σ-cofibrant and there is a zigzag of weak equivalences of operads between $\mathcal{E}_k$ and the little k-cubes operad $\mathcal{C}_k$ defined in Section 1.3. By [GKRW18, Remark 12.3], the categories of algebras over any two such E_k-operads (with the projective model structures) are Quillen equivalent.

In this proof we will use the operad R_m defined in [FW15, Appendix A], which is a cofibrant replacement of $\mathcal{C}_m$ in the category of operads. In particular, R_m is an E_m-operad by the above discussion (because by definition any cofibrant operad is Σ-cofibrant). Moreover, by [FW15, Theorems C and D] there are maps $R_m \to R_l$ for any $l \geq m \geq 1$ which are compatible with the natural maps $E_m \to E_l$ up to homotopy.

We will also make use of the operad P_m, denoted by $|MC_\bullet(\mathfrak{p}_m)|$ in [FW15], which is defined to be an operad in sSet which is formal as rational operad and quasi-isomorphic to the Poisson operad $Pois_m$ in $\mathbb{Q}$-chain complexes.

We can then re-phrase the content of [FW15, Theorems C and D] as follows: for $k \geq 3$ there is a homotopy commutative diagram of operads in sSet:

$$\begin{tikzcd} R_{k-2} \arrow[r] \arrow[d, "\simeq_{\mathbb{Q}}"] & R_k \arrow[d, "\simeq_{\mathbb{Q}}"] \\ P_{k-2} \arrow[r, "*"] & P_k \end{tikzcd}$$

where $\simeq_{\mathbb{Q}}$ denotes rational weak equivalence, and the bottom map factors trough Com, which in sSet is the terminal operad having $*$ as components.

Now we claim the following

Claim. For $k \geq 3$ there is a homotopy commutative diagram of operads in $\mathsf{sMod}_{\mathbb{Q}}$

$$\begin{tikzcd} R_{k-2} \arrow[rr] \arrow[dr] & & R_k \\ & R_\infty \arrow[ur] & \end{tikzcd}$$

where $R_{k-2} \to R_k$ and $R_{k-2} \to R_\infty$ are the canonical operad maps (up to homotopy).

Proof of Claim. The canonical map $R_\infty \to Com$ is a weak equivalence, and hence R_∞ is a cofibrant replacement of Com such that the canonical map $R_{k-2} \to Com$ factors trough R_∞. Thus we get a homotopy commutative diagram of operads in sSet

$$\begin{tikzcd} R_{k-2} \arrow[r] \arrow[d] & R_k \arrow[d, "\simeq_{\mathbb{Q}}"] \\ R_\infty \arrow[r] & P_k \end{tikzcd}$$

Now we apply the symmetric monoidal free module functor $\mathbb{Q}[-] : \mathsf{sSet} \to \mathsf{sMod}_{\mathbb{Q}}$, to get the corresponding diagram of operads in $\mathsf{sMod}_{\mathbb{Q}}$ (we will call the new operads by the same names as they had in sSet). In this new setting, rational equivalences are now quasi-isomorphisms, hence weak equivalences. The result then follows by the cofibrancy of R_∞, which allows us to lift $R_\infty \to P_k$ along $R_k \xrightarrow{\simeq} P_k$ up to homotopy. □

Now we can apply the symmetric monoidal functor $0_* : \mathsf{sMod}_{\mathbb{Q}} \to \mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}}$ (given by concentrating in degree 0) to get the analogous result of the claim but in the category $\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}}$.

Given $\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$ we view it as $\mathbf{X} \in \mathrm{Alg}_{R_k}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$ and use the map of operads $R_\infty \to R_k$ of the previous claim to produce $\mathbf{X}' \in \mathrm{Alg}_{R_\infty}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$, which then can be viewed as $\mathbf{X}' \in \mathrm{Alg}_{E_\infty}(\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}})$. Then, we just need to show that $\mathbf{X} \simeq \mathbf{X}'$ as E_{k-2}-algebras (more precisely as R_{k-2}-algebras), which is a consequence of the previous claim plus the following general result

Claim. Let $\mathcal{O}$ be an operad in $\mathsf{sMod}_{\mathbb{Q}}^{\mathbb{N}}$, and X be an object. Denote by $End(X)$ the endomorphism operad of X, so that an operad map $f : \mathcal{O} \to End(X)$ is the same as an $\mathcal{O}$-algebra structure on X, which we denote by $\mathbf{X_f}$. If $f, g : \mathcal{O} \to End(X)$ are homotopic maps of operads, then $\mathbf{X_f} \simeq \mathbf{X_g}$ are weakly equivalent $\mathcal{O}$-algebras.

Proof of Claim. The proof actually works in any simplicial model category C, as the proof only requires a cotensor $- \times - : \mathsf{sSet} \times \mathsf{C} \to \mathsf{C}$ and a copower $(-)^{-} : \mathsf{sSet} \times \mathsf{C} \to \mathsf{C}$.

By assumption we have a map of symmetric sequences $H : I \times \mathcal{O} \to End(X)$, where $I = [0,1]$, such that for each $t \in I$ $H(t,-)$ is a map of operads.

The object X^I admits an $\mathcal{O}$-algebra structure, which we denote by $\mathbf{X^I}$, via

$$(\phi, \gamma_1, \cdots, \gamma_n) \in \mathcal{O} \otimes (X^I)^{\otimes n} \mapsto (t \mapsto H(t,\phi)(\gamma_1(t), \cdots, \gamma_n(t))) \in X^I.$$

Then, $\mathbf{X_f} \xleftarrow{ev_0} \mathbf{X^I} \xrightarrow{ev_1} \mathbf{X_g}$ is a zigzag of $\mathcal{O}$-algebra maps which are weak equivalences in the underlying category, as required. □

□

References

[BKK23] Mauricio Bustamante, Manuel Krannich, and Alexander Kupers, *Finiteness properties of automorphism spaces of manifolds with finite fundamental group*, Mathematische Annalen (2023), 1–51.

[BM20] Alexander Berglund and Ib Madsen, *Rational homotopy theory of automorphisms of manifolds*, Acta Math. **224** (2020), no. 1, 67–185. MR 4086715

[Bor74] Armand Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975). MR 387496

[ERW19] Johannes Ebert and Oscar Randal-Williams, *Semisimplicial spaces*, Algebr. Geom. Topol. **19** (2019), no. 4, 2099–2150. MR 3995026

[Fri17] Nina Friedrich, *Homological stability of automorphism groups of quadratic modules and manifolds*, Doc. Math. **22** (2017), 1729–1774. MR 3760514

[FW15] Benoit Fresse and Thomas Willwacher, *The intrinsic formality of E_n-operads*, arXiv e-prints (2015), arXiv:1503.08699.

[Gal06] Søren Galatius, *Mod 2 homology of the stable spin mapping class group*, Math. Ann. **334** (2006), no. 2, 439–455. MR 2207707

[GKRW18] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *Cellular E_k-algebras*, arXiv e-prints (2018), arXiv:1805.07184.

[GKRW19] ———, *E_2-cells and mapping class groups*, Publ. Math. Inst. Hautes Études Sci. **130** (2019), 1–61. MR 4028513

[GKRW20] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *E_∞-cells and general linear groups of infinite fields*, arXiv e-prints (2020), arXiv:2005.05620.

[GKT21] Jeffrey Giansiracusa, Alexander Kupers, and Bena Tshishiku, *Characteristic classes of bundles of K3 manifolds and the Nielsen realization problem*, Tunis. J. Math. **3** (2021), no. 1, 75–92. MR 4103767

[GRW14] Søren Galatius and Oscar Randal-Williams, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377. MR 3207759

[GRW18] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. I*, J. Amer. Math. Soc. **31** (2018), no. 1, 215–264. MR 3718454

[Har85] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), no. 2, 215–249. MR 786348

[Har90] ______, *Stability of the homology of the moduli spaces of Riemann surfaces with spin structure*, Math. Ann. **287** (1990), no. 2, 323–334. MR 1054572

[JM90] Dennis Johnson and John J. Millson, *Modular Lagrangians and the theta multiplier*, Invent. Math. **100** (1990), no. 1, 143–165. MR 1037145

[Joh80] Dennis Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. (2) **22** (1980), no. 2, 365–373. MR 588283

[KMP21] Alexander Kupers, Jeremy Miller, and Peter Patzt, *Improved homological stability for certain general linear groups*, arXiv e-prints (2021), arXiv:2106.12363.

[Kor02] Mustafa Korkmaz, *Low-dimensional homology groups of mapping class groups: a survey*, Turkish J. Math. **26** (2002), no. 1, 101–114. MR 1892804

[Kos67] A. Kosiński, *On the inertia group of π-manifolds*, Amer. J. Math. **89** (1967), 227–248. MR 214085

[Kra20a] Manuel Krannich, *Mapping class groups of highly connected $(4k+2)$-manifolds*, Selecta Math. (N.S.) **26** (2020), no. 5, Paper No. 81, 49. MR 4182837

[Kra20b] ______, *A note on rational homological stability of automorphisms of manifolds*, Q. J. Math. **71** (2020), no. 3, 1069–1079. MR 4142724

[Kre79] M. Kreck, *Isotopy classes of diffeomorphisms of $(k-1)$-connected almost-parallelizable $2k$-manifolds*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 643–663. MR 561244

[KRW20] Alexander Kupers and Oscar Randal-Williams, *On diffeomorphisms of even-dimensional discs*, arXiv e-prints (2020), arXiv:2007.13884.

[KRW21a] Manuel Krannich and Oscar Randal-Williams, *Diffeomorphisms of discs and the second Weiss derivative of BTop(-)*, arXiv e-prints (2021), arXiv:2109.03500.

[KRW21b] Alexander Kupers and Oscar Randal-Williams, *Framings of $W_{g,1}$*, Q. J. Math. **72** (2021), no. 3, 1029–1053. MR 4310307

[Lu92] Ning Lu, *A simple presentation of the Siegel modular groups*, Linear Algebra Appl. **166** (1992), 185–194. MR 1152494

[Maa79] Hendrik Maazen, *Stabilité de l'homologie de* GL_n, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 15, 707–708. MR 532394

[MH73] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer-Verlag, New York-Heidelberg, 1973. MR 0506372

[Mic80] Peter W. Michor, *Manifolds of differentiable mappings*, Shiva Mathematics Series, vol. 3, Shiva Publishing Ltd., Nantwich, 1980. MR 583436

[New72] Morris Newman, *Integral matrices*, Pure and Applied Mathematics, Vol. 45, Academic Press, New York-London, 1972. MR 0340283

[Qui72] Daniel Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586. MR 315016

[Ram] Maxime Ramzi, *f2=0*, Personal note https://drive.google.com/file/d/1_ctuCFe5ficctsfkcqjNP9jwYg2ssgwZ/view.

[RW12] Oscar Randal-Williams, *The Picard group of the moduli space of r-spin Riemann surfaces*, Adv. Math. **231** (2012), no. 1, 482–515. MR 2935396

[RW14] ———, *Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces*, J. Topol. **7** (2014), no. 1, 155–186. MR 3180616

[Sie22a] Ismael Sierra, *Homological stability of diffeomorphism groups of high dimensional manifolds via E_k-algebras*, arXiv e-prints (2022), arXiv:2209.10243.

[Sie22b] ———, *Homological stability of spin mapping class groups and quadratic symplectic groups*, arXiv e-prints (2022), arXiv:2211.06219.

[SW20] David Sprehn and Nathalie Wahl, *Homological stability for classical groups*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 4807–4861. MR 4127863

[Tsh19] Bena Tshishiku, *Borel's stable range for the cohomology of arithmetic groups*, J. Lie Theory **29** (2019), no. 4, 1093–1102. MR 4022145

[vdK80] Wilberd van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), no. 3, 269–295. MR 586429

[vdKL11] Wilberd van der Kallen and Eduard Looijenga, *Spherical complexes attached to symplectic lattices*, Geom. Dedicata **152** (2011), 197–211. MR 2795243

[Vog81] Karen Vogtmann, *Spherical posets and homology stability for* $\mathrm{O}_{n,n}$, Topology **20** (1981), no. 2, 119–132. MR 605652

[Wah08] Nathalie Wahl, *Homological stability for the mapping class groups of non-orientable surfaces*, Invent. Math. **171** (2008), no. 2, 389–424. MR 2367024

[Waj99] Bronislaw Wajnryb, *An elementary approach to the mapping class group of a surface*, Geom. Topol. **3** (1999), 405–466. MR 1726532

[Wal62] C. T. C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann. of Math. (2) **75** (1962), 163–189. MR 145540

[Wei05] Michael Weiss, *What does the classifying space of a category classify?*, Homology Homotopy Appl. **7** (2005), no. 1, 185–195. MR 2175298