Herding and Contrarian Behavior
in Financial Markets[*]

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September 30, 2009

Abstract

Rational herd behavior and informationally efficient security prices have long been considered to be mutually exclusive but for exceptional cases. In this paper we describe the conditions on the underlying information structure that are necessary and sufficient for informational herding and contrarianism. In a standard sequential security trading model, subject to sufficient noise trading, people herd if and only if, loosely, their information is sufficiently dispersed so that they consider extreme outcomes more likely than moderate ones. Likewise, people act as contrarians if and only if their information leads them to concentrate on middle values. Both herding and contrarianism generate more volatile prices, and they lower liquidity. They are also resilient phenomena, although by themselves herding trades are self enforcing whereas contrarian trades are self-defeating. We complete the characterization by providing conditions for the absence of herding and contrarianism.

JEL Classification: D80, D82, G10, G14.

Keywords: Social learning, herding, contrarians, asset price volatility, market transparency.

1 Introduction

In times of great economic uncertainty, financial markets often appear to behave frantically, displaying substantial spikes as well as drops. The recent financial crisis is a striking example. Such extreme price fluctuations are possible only if there are dramatic changes in behaviour with investors switching from buying to selling or the reverse. This pattern of behavior and the resulting price volatility is often claimed to be inconsistent with rational traders and informationally efficient asset prices and is attributed to investors' animal instincts. We argue in this paper, however, that such behavior need not be triggered by 'animal spirits' but that it can be the result of fully rational social learning where agents change their beliefs and behavior as a result of observing the action of others.

One example of social learning is herd behavior in which agents switch behavior (from buying to selling or the reverse) *following the crowd*. So-called 'rational herding' can occur in situations with information externalities, when agents' private information is swamped by the information derived from observing others' actions. Such 'herders' rationally act against their private information and follow the crowd.[1]

At first sight, rational herding seems tailor-made to explain financial market frenzies, crashes and panics. However, when prices are assumed to be informationally efficient, reflecting all public information, it is not clear that herd behavior can occur at all. For example, suppose a crowd of people buys a stock frantically and consider the case of an investor with unfavorable private information about this security. Such an investor will update his information, and, indeed many buys will increase his expectation. At the same time, prices also adjust upward. Then it is not clear that at this stage the investor with an unfavorable signal buys — to him the security may still be overvalued. So, for herding to take place, private expectations and prices must diverge substantially — once unfavorable expectations must rise faster or favorable expectations must drop faster than prices.

In models with only two states of the world, it turns out that such a divergence is impossible as prices always adjust so that there is no herding (more generally no social learning).[2] Two states models however are rather special and rational herding can emerge in richer models. In this paper we identify economically appealing necessary and sufficient conditions on traders' private information that allow for herding in the context of a simple informationally efficient financial market. Moreover, we show that (i) during herding prices can move substantially and (ii) herding can induce lower liquidity and higher price volatility than if there were no herding. In other words, the kind of herd behavior that we identify can have exactly the features that have long been suspected to be present in financial markets.

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[1] See Banerjee (1992) or Bikhchandani, Hirshleifer, and Welch (1992) for early work on herding.

[2] With two states traders with favourable information always buy and those with unfavourable information always sell irrespective of what the traders observe.

Herd behavior in our set-up is defined as any history-switching behavior in the direction of the crowd (a kind of momentum trading).[3] Social learning can also arise as a result of traders switching behavior by acting *against the crowd*. Such contrarian behaviour is the natural counterpart of herding, and in this paper we also characterize conditions under which contrarian behavior arises. Contrary to received wisdom that contrarianism is stabilizing, we also show that contrarian behavior leads to higher volatility and lower liquidity, just as herd behavior. Finally, by contraposition we obtain the conditions under which neither herding nor contrarian behavior is possible — the case of no social learning. We thus provide a complete characterization of trading behavior.

The key insight of our characterization result is that social learning in financial markets with informationally efficient prices occurs if and only if the investors in question receive information that satisfies some compelling and intuitive property. Loosely (see below), herding happens if and only if private information satisfies a property that we call "U shaped". Namely, an investor who receives such information believes that extreme states are more likely to have generated the information than more moderate ones. Therefore, when forming his posterior belief, the recipient of such a signal will shift weight away from the center to the extremes so that the posterior distribution of the trader is "fat-tailed". Thus, the recipient of a U shaped signal discounts the possibility of the intermediate value and as a consequence will update the probabilities of extreme values faster than an agent who receives only the public information. So, even if this investor's prior belief is pessimistic, after observing a large number of buys (favourable news), he updates his belief and puts more weight on the best outcome than the market, and hence starts buying. Similarly, such an investor will sell after a large number of sales because his updated belief puts more weight on the worse outcome than the market's. Therefore, the behavior of an investor with a U shaped signal can be volatile.

On the other hand, contrarianism occurs if and only if the investor's signal indicates that moderate states are more likely to have generated the signal than extreme states. We describe such signals as being Hill shaped. The recipient of a Hill shaped signal updates extreme outcomes slower and always puts more weight on the middle outcome relative to the market so that this trader's posterior distribution becomes "thin-tailed". This causes him to take actions that move prices towards this middle outcome: if prices rise too much he sells and if prices fall too much he buys. Thus, he may act against the crowd.

Finally, an informed investor trades the same way irrespective of the history (no social learning) if and only if his signal is neither U shaped nor Hill shaped.

We follow the microstructure literature and establish our results in the context of a

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[3]Herding in our set-up does not signify that everyone acts alike or that there is an informational cascade. In fact, in an efficient financial market all traders acting alike would not be such an interesting phenomenon as actions would not reveal any private information and therefore prices would not react.

stylized trading model in the tradition of Glosten and Milgrom (1985). In such models the bid and ask prices are set by a competitive market maker. Investors trade with the market maker either because they receive private information about the asset's fundamental value or because they are "noise traders" and trade for reasons outside of the model, e.g. liquidity.

The simplest possible Glosten and Milgrom (1985) type trading model that would allow herding or contrarianism is one with at least three states or liquidation values for the asset (as we mentioned before, with two states social learning is not possible). For this case, we show that (i) a U shaped (Hill shaped) signal is necessary for herding (contrarianism) and (ii) herding (contrarianism) occurs with a positive probability if there exists at least one U shaped (Hill shaped) signal and there is a sufficient amount of noise traders. The latter assumption on the minimum level of noise trading is not necessary in all cases and is made because otherwise the bid and ask spread may be too large to induce appropriate trading. In Section 8 we will further show that the intuition for our three states characterization carries over to a setup with an arbitrary number of states.

We obtain our characterization results without restrictions on the signal structure. In the literature on asymmetric information (for instance, in rational expectations models or auctions) it is often assumed that information structures satisfy the monotone likelihood ratio property (MLRP). Such information structures are "well-behaved" because, for example, investors' expectations are ordered. At first, it may appear that such a very strong monotonicity requirement would prohibit herding or contrarianism. Yet MLRP does not only admit the possibility of U shaped signals (and thus herding) or Hill shaped signals (and thus contrarianism), but it also turns out that the proofs for our sufficiency results are significantly simpler with MLRP signals. When the well-behaved MLRP is violated, on the other hand, the proofs are substantially more complicated.

Having characterized both herding and contrarianism, we next show that the range of price movements can be very large during both contrarianism and herding. In fact we show that with MLRP signals both herding and contrarianism can occur for almost the entire range of feasible prices.[4]

Our second main result concerns the impact of social learning on liquidity (measured by the bid-ask-spread) and price volatility. We show that the former declines when the herding or contrarian candidate switches the trading direction (compared to a situation where he does not switch). To understand the impact of social learning on the latter, we compare price movements in our set-up, which we call the transparent economy, with those in a hypothetical economy, which we call opaque, that is otherwise identical to our set-up except that the informed traders do not switch behavior as a result of observing

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[4]In the model asymptotically the true state is revealed and prices converge to the true value. In this paper, however, we shall not be concerned with such long-run results.

the behavior of others. In contrast to the transparent economy, in the opaque economy there is no social learning either because traders do not have access to the information regarding the behavior others or because they cannot (e.g. for cognitive reasons) extract any information about the true state from the behavior of others. We also restrict attention to MLRP signals. We then show that once herding or contrarianism begins, prices respond more to individual trades relative to the situation without social learning so that price rises *and* price drops are greater in the transparent set-up than in the opaque one.[5]

Some of the above results on the impact of social learning on price paths (in particular the impact on volatility in the transparent case relative to the opaque one) are surprising for both the herding and the contrarianism case. For casual intuition suggests: (i) during herding there is little informational content in herd trades and thus price movements and spreads are small and (ii) contrarianism is often stabilizing. These intuitions are, however, incorrect because volatility is higher when social learning occurs (for *both* herding and contrarianism) compared to the opaque case where there is no social learning. These results are even more surprising as they arise with the 'well-behaved' MLRP restriction.

The liquidity and price volatility results also have direct and important implications for the discussion on the merits of 'market transparency'. The price path in the opaque economy without social learning can be interpreted as the outcome of a trading mechanism in which people submit orders without knowing the behavior of others and without knowing the market price. Our results thus indicate that in the less transparent setup, price movements are less pronounced.

While the results on price ranges, volatility and liquidity indicate similarities between herding and contrarianism, there is also a stark difference. Contrarian trades are self-defeating because a large number of such trades will cause prices to move 'against the crowd' thus ending contrarianism. During herding, on the other hand, investors continue to herd when trades are 'in the direction of the crowd', so herding is self-enforcing.

We complete this introduction by outlining several real-world situations where signals can have the structure necessary for herding or contrarian behavior. The current financial crisis provides a good set of examples for such signals, and our characterization results for herding and contrarianism may cast some light on the idea that extreme uncertainty may foster increases in trading activities and volatility. For example, the collapse of Lehman Brothers might have changed investors' beliefs of the possibility of extreme events, such as unconditional bailouts, nationalizations or further collapses of financial institutions. With all the rumours, and extreme and divergent commentaries, many investors might have concluded that in the aftermath of the Lehman collapse the authorities either knew what

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[5]The increase in price-volatility associated with herding is only relative to a hypothetical scenario. Even when herding is possible, in the long-run volatility settles down and prices react less to individual trades. It is well known that the variance of Martingale price-processes such as ours is bounded by model primitives.

they were doing and that the remaining financial institutions must be sound and would eventually come out well or that the authorities have no idea what they are doing and that the contagion effect will be so large that a complete collapse of the financial system will be inevitable. According to our theory, such assessments that either extreme outcomes are likely could have resulted in herding in the aftermath of the Lehman collapse.

As another example consider September 29, 2008, the day the bailout bill (TARP) was first rejected. Loosely, after the rejection there were three possible outcomes: (i) the bill is re-introduced and passed, (ii) a new bill with a foul compromise is passed, or (iii) no bill at all is passed and wide-spread banking panics follow. Thus, (i) and (iii) correspond to the extreme outcomes and (ii) is the middle outcome.

In this environment, investors' information might have implied that the most likely outcome is either that Congress will ultimately pass the original bill and a good outcome will occur or that Congress will block any attempted bailout and the doomsday outcome will happen. Such information is an example of U shaped signal. Alternatively, some investors' assessment might have implied that the compromise bill is the most likely outcome, for policy makers would neither allow the initial bill to be passed nor would they conceivably allow the doomsday outcome by passing no bill at all. Such information is an example of a Hill shaped signal.

It is conceivable that in Autumn 2008 many believed that the two extreme states (the bill passes or there is no bill at all) were the most likely outcomes. Then our theory predicts the potential for herd behavior, with investors changing behavior in the direction of the crowd, causing strong short-term price fluctuations. Hill shaped private signals, signifying that the compromise is considered the most likely outcome, may also be part of the explanation as contrarians display rapid changes of behaviour that cause volatility.

Finally, U shaped and Hill shaped signals may also be good descriptions of situations with a potential upcoming event that has an uncertain impact. For example, consider the case of a company or institution that contemplates appointing a new leader who is an uncompromising "reformer". If this person takes power, then either the necessary reforms take place or there will be strife with calamitous outcomes. Thus the new leader will be either very good or disastrous, and the institution will certainly not be the same. In this situation, private information signifying that the person is likely to be appointed exemplifies a U shaped signal and any information revealing that this person is unlikely to be appointed (and thus the institution will carry on as before) represents a Hill shaped signal.[6]

Overview. The next section discusses some of the related literature. Section 3 outlines the setup. Section 4 defines herding and contrarian behavior, outlines properties of

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[6]Other examples are an upcoming merger or takeover with uncertain merits, the possibility of a government stepping down, announcements of FDA drug approvals, outcomes of lawsuits etc. Degenerate examples for such uncertain events were first mentioned in Easley and O'Hara (1987).

signal distributions and derives some basic properties. Section 5 discusses the necessary and sufficient conditions that ensure herding and contrarianism. Section 6 considers the resiliency, and fragility of herding and contrarianism and describes the range of prices for which there may be herding and contrarianism. Section 7 discusses the impact of social learning on prices with respect to liquidity and volatility. Section 8 extends the result to a setting with an arbitrary number of states. Section 9 concludes. Proofs that are not in the text are either in the appendix or in the supplementary material.

2   Related Literature

Extensive literature surveys on herding in financial markets are in Brunnermeier (2001), Chamley (2004) and Vives (2008). The work closest to ours is Avery and Zemsky (1998). Probably best known for its no-herding result with informationally efficient prices, it also presents an intuitive appealing example in which herding is possible. Avery and Zemsky argue that herd behavior with informationally efficient asset prices is not possible unless signals are "non-monotonic" and attribute the herding result in their example to "multidimensional uncertainty/risk" (investors have a finer information structure than the market). In their example, however, there are hardly any price movements under herding.[7]

The profession, for instance Brunnermeier (2001), Bikhchandani and Sunil (2000), Chamley (2004) have derived three messages from Avery and Zemsky (1998)'s paper. First, with "monotonic" signals, herding is impossible. Second, for herding one needs "multidimensionality" of risk. Third, herding does not involve violent price movements except in the most unlikely environments. Therefore, since in Avery and Zemsky's example the information structure needed to induce herding is very special and large price movements cannot easily be attributed to herd-type behavior, it has been concluded that rational herding models are not so relevant to understanding the functioning of efficient financial markets. The results of our paper demonstrate that the conclusions derived from Avery and Zemsky and the profession's perception need to be corrected. First, we show that it is U shaped signals, and not multidimensionality (or non-monotonicity) of the information structure, that generate herding. Therefore there may be a great deal more rational informational herding than is currently expected in the literature. Second, extreme price movements with herding are possible under not so unlikely situations. And third, price volatility may even be exacerbated by herding and contrarian behavior.

There are several related contributions in the literature that highlight how certain facets

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[7]To generate extreme price movements (bubbles) with herding Avery and Zemsky have an example with further information asymmetries and thus more 'risk dimensions'. However, even with these further informational asymmetries, the likelihood of large price movements in their set-up during a herd phase is extremely small; see Chamley (2004). Section 9 outlines Avery and Zemsky's notions of monotonicity and dimensions of risk and discusses further relations of their results to ours.

of market organization or incentives lead to conformism and informational cascades. In Lee (1998), there are fixed transaction costs which temporarily prevent traders from revealing their information. This hidden information gets revealed suddenly when a large number of traders enters the market simultaneously. The market maker absorbs these trades at a fixed price, which leads to large price jumps after the avalanche. In Cipriani and Guarino (2008), traders have private benefits from trading in addition to the fundamental value payoff. As the private and public expectations converge, private benefits gain importance to the point when they overwhelm the informational rents. Then learning breaks down and an informational cascade arises. In Dasgupta and Prat (2005) an informational cascade is triggered by traders' reputation concerns, which eventually outweigh the possible benefit from trading on information. Chari and Kehoe (2004) also study a financial market with efficient prices; herding in their model arises with respect to a capital investment that is made outside of the financial market.

All of the above contributions highlight important aspects, facets, and mechanisms that can trigger conformism in financial markets. Our findings complement and complete the analysis by providing a complete description of trading behavior when prices do account correctly for all the information that is revealed by agents' trading at any point in time.

3   The Model

We model financial market sequential trading in the tradition of Glosten and Milgrom (1985).

Security: There is a single risky asset with a liquidation value V from a set of three potential values $\mathbb{V} = \{V_1, V_2, V_3\}$ with $V_1 < V_2 < V_3$. The prior distribution over $\mathbb{V}$ is denoted by $\mathsf{Pr}(\cdot)$. To simplify the computation we assume that $\{V_1, V_2, V_3\} = \{0, \mathcal{V}, 2\mathcal{V}\}$, $\mathcal{V} > 0$ and that the prior distribution is symmetric around V_2; thus $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$.[8]

Traders: There is a pool of traders consisting of two kinds of agents: *Noise Traders* and *Informed Traders*. At each discrete date t one trader arrives at the market in an exogenous and random sequence. Each trader can only trade once at the point in time at which he arrives. We assume that at each date the entering trader is an informed agent with probability $\mu > 0$ and a noise trader with probability $1 - \mu > 0$.

The informed agents are risk neutral and rational. Each receives a private, conditionally i.i.d. signal about the true value of the asset V. The set of possible signals is denoted by $\mathbb{S}$ and consists of three elements S_1, S_2 and S_3. The signal structure of the informed agent can therefore be described by a 3-by-3 matrix $\mathcal{I} = \{\mathsf{Pr}(S_i|V_j)\}_{i,j=1,2,3}$ where $\mathsf{Pr}(S_i|V_j)$ is the probability of signal S_i if the true value of the asset is V_j.

Noise traders have no information and trade randomly. These traders are not necessarily

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[8]The ideas of this paper remain valid without these symmetry assumptions.

irrational, but they trade for reasons not included in this model, such as liquidity.[9]

Market Maker: Trade in the market is organised by a market maker who has no private information. He is subject to competition and thus makes zero-expected profits.[10] In every period t, prior to the arrival of a trader, he posts a bid-price bid^t at which he is willing to buy the security and an ask-price ask^t at which he is willing to sell the security. Consequently he sets prices in the interval $[V_1, V_3]$.

Traders' Actions: Each trader can buy or sell *one* unit of the security at prices posted by the market maker, or he can be inactive. So the set of possible actions for any trader is $\{\text{buy}, \text{hold}, \text{sell}\}$. We denote the action taken in period t by the trader that arrives at that date by a^t. We assume that noise traders trade with equal probability. Therefore, in any period, a noise-trader buy, hold or sale occurs with probability $\gamma = (1 - \mu)/3$ each.

Public History: The structure of the model is common knowledge among all market participants. The identity of a trader and his signal are private information, but everyone can observe past trades and transaction prices. The history (public information) at any date $t > 1$, the sequence of the traders' past actions together with the realised past transaction prices, is denoted by $H^t = ((a^1, p^1), \ldots, (a^{t-1}, p^{t-1}))$ for $t > 1$, where a^τ and p^τ are traders' actions and realised transaction prices at any date $\tau < t$ respectively. Also, H^1 refers to the initial history before any trade takes place.

At any date t and any history H^t the public belief of the probability that the true liquidation value of the asset is V_i is denoted by $q_i^t = \Pr(V_i | H^t)$, for each $i = 1, 2, 3$. The public expectation of the value is given by $\mathsf{E}[V | H^t] = \Sigma q_i^t V_i$.

The Informed Trader's Optimal Choice: The game played by the informed agents is one of incomplete information; therefore the optimal strategies correspond to a Perfect Bayesian equilibrium. Here, the equilibrium strategy for each trader simply involves comparing the quoted prices with his expected value taking into account both the public history and his own private information. For simplicity, we restrict ourselves to equilibria in which each agent trades *only if* he is strictly better off (in the case of indifference the agents do not trade). Therefore, the equilibrium strategy of an informed trader that enters the market in period t, receives signal S^t and observes history H^t is (i) to *buy* if $\mathsf{E}[V | H^t, S^t] > \mathsf{ask}^t$, (ii) to *sell* if $\mathsf{bid}^t > \mathsf{E}[V | H^t, S^t]$, and (iii) to *hold* in all other cases.

The Market Maker's Price-Setting: To ensure that the market maker receives zero expected profits, bid and ask prices must satisfy the following at any date t and any history H^t: $\mathsf{ask}^t = \mathsf{E}[V | a^t = \text{buy at } \mathsf{ask}^t, H^t]$ and $\mathsf{bid}^t = \mathsf{E}[V | a^t = \text{sell at } \mathsf{bid}^t, H^t]$.

If the market maker always sets prices equal to the public expectation, $\mathsf{E}[V | H^t]$, he makes an expected loss on trades with an informed agent (note that each informed agent

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[9]As is common in the microstructure literature with asymmetric information, we assume that noise traders have positive weight ($\mu < 1$) to prevent "no-trade" outcomes a la Milgrom and Stokey (1982).

[10]Alternatively, we could assume a Bertrand model with many identical market makers setting prices.

trades only if he makes a strict gain). However, if the market maker sets an ask-price and a bid-price respectively above and below the public expectation, he gains on noise traders, as their trades have no information value. Thus, in equilibrium the market maker may have to make a profit on trades with noise traders to compensate for any losses against informed types. This implies that if at any history H^t, there is a possibility that the market maker trades with an informed trader, then there is a spread between the bid and ask prices at H^t and the public expectation $\mathsf{E}[V|H^t]$, henceforth also referred to as the "average price", satisfies $\mathsf{ask}^t > \mathsf{E}[V|H^t] > \mathsf{bid}^t$.

Trading by the Informed Types and No Cascade Condition: At any history H^t either informed types do not trade and every trade is by a noise trader or there is an informed type that would trade at the quoted prices. The game played by the informed types in the former case is trivial as there will be no trade by the informed from H^t onwards and an informational cascade occurs.[11] In this paper, we thus consider the latter case in which at every history there is an informed type that would trade at the quoted prices.

Informative Private Signals: The private signals of the informed traders are informative at history H^t if

$$\text{there exists } S \in \mathbb{S} \text{ such that } \mathsf{E}[V|H^t, S] \neq \mathsf{E}[V|H^t]. \tag{1}$$

First note that (1) implies that at H^t there is an informed type that buys and an informed type that sells.[12] Second, if there is informed type that trades at H^t then (1) must hold. Otherwise, for every signal $S \in \mathbb{S}$, $\mathsf{E}[V|H^t, S] = \mathsf{E}[V|H^t] = \mathsf{ask}^t = \mathsf{bid}^t$, in which case the informed types would not trade at H^t. Therefore, it follows from the above that (1) is both necessary and sufficient for trading by an informed type at H^t. Since we are interested in the case when the informed types trade, we therefore assume throughout this paper that (1) holds at every history H^t.[13]

Long-run behavior of the model. Since price formation in our model is standard, (1) also ensures that standard asymptotic results on efficient prices hold. More specifically, by standard arguments as in Glosten and Milgrom (1985) we have that transaction prices form a martingale process. Since by (1) buys and sells have some information content (at every date there is an informed type that buys and one that sells), it also follows that beliefs and prices converge to the truth in the *long run*. However, as we mentioned before (see Footnote 4), here we are solely interested in *short-run* behavior and fluctuations.

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[11]If there are no trades by the informed at H^t then no information will be revealed and the expectations and prices remain unchanged irrespective of the outcome at H^t. Hence, by induction, we would have no trading by the informed and no information revelation at any date after H^t.

[12]By (1) there must exist two signals S' and S'' such that $\mathsf{E}[V|H^t, S'] < \mathsf{E}[V|H^t] < \mathsf{E}[V|H^t, S'']$. If no informed type buys at H^t then there is no informational content in a buy and $\mathsf{ask}^t = \mathsf{E}[V|H^t]$. Then, by $\mathsf{E}[V|H^t] < \mathsf{E}[V|H^t, S'']$, type S'' must be buying at H^t; a contradiction. Similarly, if no informed type sells at H^t then $\mathsf{bid}^t = \mathsf{E}[V|H^t]$. Then, by $\mathsf{E}[V|H^t] > \mathsf{E}[V|H^t, S']$, type S' must be selling at H^t; a contradiction.

[13]A sufficient condition for (1) to hold at every H_t is that all minors of order two of the information matrix $\mathcal{I}$ are non-zero.

4 Some Definitions and Basic Properties

4.1 Definitions of Herding and Contrarian Behavior

The definitions of herding and contrarianism that we adopt here refer to the behavior of a particular signal type and they capture the history-dependent (or social learning) element of behavior in an informationally efficient financial market. We differentiate between herding and contrarianism by describing the former as a history-induced switch of opinion *in the direction of the crowd* and the latter as a history-induced switch *against the direction of the crowd*. Thus in our setup there is a symmetry in our definitions, making herding the intuitive counterpart to contrarianism — which itself is not a mass phenomenon.

Definition (Herding and Contrarianism)
HERDING. *A trader with signal S buy herds in period t at history H^t if and only if* (*i*) $\mathsf{E}[V|S] < \mathsf{bid}^1$, (*ii*) $\mathsf{E}[V|S, H^t] > \mathsf{ask}^t$, (*iii-h*) $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. *Sell herding at history H^t is defined analogously with the required conditions* $\mathsf{E}[V|S] > \mathsf{ask}^1$, $\mathsf{E}[V|S, H^t] < \mathsf{bid}^t$, *and* $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. *Type S herds if he either buy herds or sell herds.*
CONTRARIANISM. *A trader with signal S is engages in buy contrarianism in period t at history H^t if and only if* (*i*) $\mathsf{E}[V|S] < \mathsf{bid}^1$, (*ii*) $\mathsf{E}[V|S, H^t] > \mathsf{ask}^t$, (*iii-c*) $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. *Sell contrarianism at history H^t is defined analogously with the required conditions* $\mathsf{E}[V|S] > \mathsf{ask}^1$, $\mathsf{E}[V|S, H^t] < \mathsf{bid}^t$, *and* $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. *Type S engages in contrarianism if he engages either in buy contrarianism or sell contrarianism.*

Both with buy herding and buy contrarianism, type S prefers to sell at the initial history, before observing other traders' actions (condition (*i*)), but prefers to buy after observing the history H^t (condition (*ii*)). The key differences between a herd-buy and a contrarian-buy are conditions (*iii-h*) and (*iii-c*). The former requires the average price to rise at history H^t so that a change of action from selling to buying at H^t is *with* the general movement of the prices (crowd), whereas the latter condition requires the average price to have dropped so that a trader who buys at H^t acts *against* the movement of prices.[14]

Our definition of herding is the same as that in Avery and Zemsky (1998).[15] In the literature, there are other definitions of herding (and informational cascades). For instance,

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[14]There are several points to note about our definition. First, the benchmark for a switch is the decision that a trader would take at the initial history. Thus someone acts as a buy herder (contrarian) if he would have sold at the initial history, but buys after observing a history with rising (falling) prices. Second, a 'history with buy herding or buy contrarianism' only implies that there could be types that buy-herd or act as buy contrarians — it does not mean that the actual trades are by these types. Third, herding and contrarianism here refer to extreme switches of behavior from selling to buying or the reverse. One could expand the definition to switches from holding to buying or to selling (or the reverse). To ensure consistency with the earlier literature, we focus on the extreme cases where switches do not include holding.

[15]Avery and Zemsky (1998)'s definition of contrarianism is stronger than ours (they also impose an additional bound on price movements that reflects the expectations that would obtain if the traders receives an infinite number of draws of the same signal). We adopt the definition of contrarianism above because, as we explained before, it is a natural and simple counterpart to the definition of herd behavior.

some papers require 'action convergence' or even complete informational cascades where all types take the same action, irrespective of their information (see Chamley (2004)). However, as we discussed above (see footnote 3), in a standard sequential security trading model with informationally efficient prices it is impossible for all informed agents to trade on the same side of the market, if there are informative private signals (see (1)). Furthermore, if all traders act alike, actions would be uninformative, and prices would not move; and, therefore, one cannot explain excess volatility or booms and busts.[16]

4.2   Information Structure

Conditional signal distributions. As we outlined in the introduction, the possibility of herding or a contrarian behaviour for any informed agent with signal $S \in \mathbb{S}$ depends critically on the shape of the conditional signal distribution of S. Henceforth, we refer to the conditional signal distribution as the *csd*. Furthermore, we will also employ the following terminology to describe six different types of csds:

increasing:	$\Pr(S|V_1) \leq \Pr(S|V_2) \leq \Pr(S|V_3);$	*decreasing:*	$\Pr(S|V_1) \geq \Pr(S|V_2) \geq \Pr(S|V_3);$
U shaped:	$\Pr(S|V_i) > \Pr(S|V_2)$ for $i = 1, 3;$	*Hill shaped:*	$\Pr(S|V_i) < \Pr(S|V_2)$ for $i = 1, 3;$
Negative bias:	$\Pr(S|V_1) > \Pr(S|V_3);$	*Positive bias:*	$\Pr(S|V_1) < \Pr(S|V_3);$
zero bias:	$\Pr(S|V_1) = \Pr(S|V_3);$	*non-zero bias:*	$\Pr(S|V_1) \neq \Pr(S|V_3).$

We shall call a signal's csd *monotonic* if its csd is either increasing or decreasing (monotonic signals thus include the case of an uninformative signal). Also, a monotonic csd is said to be strictly monotonic if all three conditional probabilities for the signal are distinct. Furthermore, we use the term nU (pU) shaped csd to describe a U shaped csd with negative (positive) bias. Similarly, we use nHill and (pHill) to describe a Hill shape with a negative (positive) bias. In describing the above properties of a type of csd for a signal we shall henceforth drop the reference to the csd and attribute the property to the signal itself, when the meaning is clear. Similarly, when describing the behavior of a signal recipient we attribute the behavior to the signal itself.

An immediate property implied by a monotonic signal is that the recipient of such a signal cannot buy at some history and sell at another.

Proposition 1 (No Herding or Contrarianism with Monotonic Signals) *If S is increasing then type S does not sell at any history. If S is decreasing then type S does not buy at any history. Thus recipients of monotonic signals cannot herd or behave as contrarians.*

To see the intuition suppose that S is decreasing. Then for any $V_l < V_h$ we have $\Pr(S|V_i) \leq \Pr(S|V_j)$. Thus at any history H^t the likelihood that trader S attaches to V_ℓ relative to V_h

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[16]For instance, models with informational cascades such Cipriani and Guarino (2008) and Dasgupta and Prat (2005) have the feature that prices no longer move once the informational cascade starts.

is no more than that for the market maker: $\frac{\mathsf{Pr}(V_l|S,H^t)}{\mathsf{Pr}(V_h|S,H^t)} = \frac{\mathsf{Pr}(V_l|H^t)\mathsf{Pr}(S|V_l)}{\mathsf{Pr}(V_h|H^t)\mathsf{Pr}(S|V_h)} \leq \frac{\mathsf{Pr}(V_l|H^t)}{\mathsf{Pr}(V_h|H^t)}$. Then S's expectation does not exceed that of the market maker, and hence S cannot be buying at any history. By a similar reasoning an increasing signal type cannot be selling at any history. The following lemma formally yields the result.

Lemma 1 *For any S, time t and history H^t, $\mathsf{E}[V|S,H^t] - \mathsf{E}[V|H^t]$ has the same sign as*

$$q_1^t q_2^t \left[\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1)\right] + q_2^t q_3^t \left[\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2)\right] + 2q_1^t q_3^t \left[\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_1)\right]. \quad (2)$$

To show Proposition 1, fix any history H^t. If signal S is decreasing then every term in (2) is non-positive and $\mathsf{E}[V|S,H^t] \leq \mathsf{E}[V|H^t]$. Since $\mathsf{E}[V|H^t] < \mathsf{ask}^t$, S does not buy at H^t. Similarly, if the signal type S is increasing then every term in (2) is non-negative and $\mathsf{E}[V|S,H^t] \geq \mathsf{E}[V|H^t]$. Since $\mathsf{E}[V|H^t] > \mathsf{bid}^t$, S does not sell at H^t.

Monotone likelihood ratio property (MLRP). As we mentioned before, the literature on asymmetric information often assumes that the information structure satisfies MLRP. Here this means that for any signals $S_l, S_h \in \mathbb{S}$ and values $V_l, V_h \in \mathbb{V}$ such that $S_l < S_h$ and $V_l < V_h$, $\mathsf{Pr}(S_h|V_l)\mathsf{Pr}(S_l|V_h) < \mathsf{Pr}(S_h|V_h)\mathsf{Pr}(S_l|V_l)$ holds. Thus, MLRP holds if and only if all minors of order two of the information matrix $\mathcal{I}$ are positive.

The MLRP imposes a natural order on the signals in terms of their conditional expectations after any history. In particular, this implies that the lowest signal is always selling and the highest signal is always buying. Also, with MLRP signals, the extreme signals have monotonic csds. Formally, we have the following.

Lemma 2 *Assume $S_1 < S_2 < S_3$ and the information structure satisfies MLRP. Then*
(i) $\mathsf{E}[V|S_1,H^t] < \mathsf{E}[V|S_2,H^t] < \mathsf{E}[V|S_3,H^t]$ *at any t and any H^t.*
(ii) *In any equilibrium informed traders with signal S_1 (S_3) always sell (buy).*
(iii) *The csd for S_1 is strictly decreasing and the csd for S_3 is strictly increasing.*

We derive our main results on herding and contrarianism with no assumptions on the information structure (other than the informativeness of the private signals as described by (1)). As it turns out the properties described in Lemma 2 enable us to derive a sharper (and easier to establish) sets of results when the information structure satisfies MLRP.

Finally, note that MLRP is a set of restrictions on the conditional probabilities for the entire signal structure, whereas the properties associated with a csd are a restriction on the conditional probabilities of a given signal. In particular, note that although MLRP implies that the csd of the lowest and highest signals are strictly monotonic (Lemma 2 (c)), the csd for the middle signal S_2 can be decreasing, increasing, hill shaped or U shaped with a negative or a positive bias — Table 1 displays examples of all these possibilities.[17]

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[17]Avery and Zemsky (1998) define a notion of monotonicity that is different from both MLRP and csd monotonicity (see the concluding Section 9 below).

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{2}{9}$
S_2	$\frac{6}{18}$	$\frac{4}{18}$	$\frac{3}{18}$
S_3	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{11}{18}$

decreasing for S_2

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{9}$
S_2	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{9}{27}$
S_3	$\frac{7}{27}$	$\frac{1}{3}$	$\frac{5}{9}$

increasing for S_2

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{5}{6}$	$\frac{1}{3}$	0
S_2	$\frac{11}{120}$	$\frac{40}{120}$	$\frac{20}{120}$
S_3	$\frac{3}{40}$	$\frac{1}{3}$	$\frac{5}{6}$

pHill shape for S_2

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{3}{40}$
S_2	$\frac{20}{120}$	$\frac{40}{120}$	$\frac{11}{120}$
S_3	0	$\frac{1}{3}$	$\frac{5}{6}$

nHill shape for S_2

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{31}{100}$	$\frac{1}{5}$	$\frac{1}{100}$
S_2	$\frac{59}{100}$	$\frac{50}{100}$	$\frac{60}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

pU shape for S_2

$\Pr(S\mid V)$	V_1	V_2	V_3
S_1	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{50}$
S_2	$\frac{60}{100}$	$\frac{50}{100}$	$\frac{59}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

nU shape for S_2

Table 1: **Examples of MLRP Signal distributions** Each entry represents the probability of the row-signal given the true liquidation value in the column. In all the above examples, the signal distributions for S_1 and S_3 are csd-monotonic. MLRP is satisfied as each minor of order 2 is positive.

5 Characterizing Herding and Contrarian Behavior

5.1 Necessary Conditions

Before stating the necessary conditions on csds that make herding and contrarianism possible, we state two useful lemmas. First, since the prior on the liquidation values is symmetric it follows that at the initial history the expectation of the informed is less (greater) than the average price if and only if the signal type is negatively (positively) biased.

Lemma 3 *For any signal S, $\mathsf{E}[V|S]$ is less than $\mathsf{E}[V]$ if and only if S has a negative bias, and $\mathsf{E}[V|S]$ is greater than $\mathsf{E}[V]$ if and only if S has a positive bias.*

Second, note that when the average price rises (falls), as is the case when buy herding (buy contrarian) occurs, the public belief must attach a lower (higher) probability to the lowest value, V_1, than to the highest value, V_3.

Lemma 4 *If $\mathsf{E}[V|H^t] > \mathsf{E}[V]$ then $q_3^t > q_1^t$ and if $\mathsf{E}[V] > \mathsf{E}[V|H^t]$ then $q_1^t > q_3^t$.*

We are now in the position to describe our main necessary conditions:

Proposition 2 (Necessary Conditions for Herding and Contrarianism)
(a) *Signal type S buy herds only if S is nU shaped, sell herds only if S is pU shaped and herds only if S is U shaped.*
(b) *Signal type S acts as a buy contrarian only if S is nHill shaped, acts as a sell contrarian only if S is pHill shaped and acts as a contrarian only if S is Hill shaped.*

A sketch of the proof of Proposition 2 is as follows. Suppose that S buy herds or acts as a buy contrarian (the cases of sell herding or sell contrarian are analogous). Then it must be

that at the initial history H^1 type S sells and therefore that his expectation is below the market price. By Lemma 3, this implies that S is negatively biased.[18] By Proposition 1, S cannot be monotonic. Thus, it follows that S is either nU shaped or nHill shaped.

The proof is completed by showing that buy herding is inconsistent with an nHill shaped csd and that buy contrarianism is inconsistent with an nU shaped csd. To see the intuition, for example, for the case of buy herding, note that in forming his belief a trader with an nHill shaped csd puts less weight on the tails of his belief (and thus more on the center) relative to the market maker; furthermore, the shift from the tails towards the center is more for value V_3 than for V_1 because of the negative bias. Since when buy herding occurs prices must have risen and therefore, by Lemma 4, the public belief attaches more weight to V_3 relative to V_1 (i.e. $q_1^t < q_3^t$), such a redistribution of probability mass ensures that S's expectation is less than that of the market maker. Hence S cannot be buying.

5.2   Sufficient Conditions: Informal Discussion

The above necessary conditions —U shape for herding and Hill shape for contrarianism— also turn out to be almost sufficient. We present the sufficiency results in the next subsection; here we provide some intuition by discussing three underlying insights for the results.

(I) With a U shaped or a Hill shaped signal an informed trader has an expectation that is below that of the market maker (the average price) at some history and above it at another.

To see the intuition for this, consider first any history H^t at which the probability of state V_1 is sufficiently small relative to the probability of the other states (both q_1^t/q_2^t and q_1^t/q_3^t are close to zero). Then at such a history there are effectively two states V_2 and V_3. This means that at such a history the expectation of a trader S exceeds (is less than) that of the market maker (who has no private information) if S induces a higher (lower) weight to the higher state V_3 than to the lower state V_2. Formally, at any H^t at which both q_1^t/q_2^t and q_1^t/q_3^t are close to zero, the first and the third terms in (2) are arbitrarily small; moreover, the second term in (2) has the same sign as $\Pr(S|V_3) - \Pr(S|V_2)$; therefore, it follows from from Lemma 1 that $\mathsf{E}[V|S, H_t] - \mathsf{E}[V|H_t]$ has the same sign as $\Pr(S|V_3) - \Pr(S|V_2)$. But the latter is positive for a U shaped S and negative for a Hill shaped S. Thus when q_1^t/q_2^t and q_1^t/q_3^t are sufficiently small, the expectation of a U shaped type exceeds that of the market maker and the expectation of a Hill shaped type is below that of the market maker.

By a similar reasoning the opposite happens at any history H^t at which at the probability of state V_3 is sufficiently small relative to the probability of the other states (both q_3^t/q_1^t and q_3^t/q_2^t are close to zero). At such a history there are effectively two states, V_1 and V_2; therefore, by Lemma 1, $\mathsf{E}[V|S, H^t] - \mathsf{E}[V|H^t]$ has the same sign as $\Pr(S|V_2) - \Pr(S|V_1)$.

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[18]Note that the bias is only required because we assume that priors are symmetric. If, for instance, the prior would favor the highest state, then herding or contrarian behavior can also arise when the signal distributions have no bias.

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Since the latter is negative for a U shaped S and positive for a Hill shaped S, it follows that at such a history the expectation of a U shaped type is less than that of the market maker and the expectation of a Hill shaped type exceeds that of the market maker.

The above arguments assume that there are histories at which the probability of each extreme state (V_1 or V_3) is sufficiently small relative to the probabilities of the other states. Demonstrating the existence of such histories is simple in some cases: for example, when the signal structure satisfies MLRP, the probability of V_1 (V_3) relative to the remaining states can be made arbitrarily small if there is a sufficient number of buys (sales). For arbitrary signal structures, however, it can be quite complex to demonstrate these possibilities, as can be seen from the proof of Proposition 3 (see the discussion in Subsection 5.3 below).

(II) If the informed type has an nU (pU) shaped signal then his expectation is less than (above) the market maker's initially and rises above (falls below) it in the direction of the crowd at some history. The same holds for any Hill shaped signal (with the bias determining the relative expectation at the initial history) except that the switch is against the crowd.

For instance, consider the cases of nU shape and nHill shape. If the informed trader has an nU shaped signal, then it follows from Lemma 3 and from the insights in (I) respectively that his expectation is below the market maker's expectation initially and subsequently it rises above it at any history at which the relative likelihood of the lowest state V_1 is arbitrarily small. Since by Lemma 4 the average price rises when the probability of V_1 is small relative to that of V_3, it follows that such a history induces the private expectation of the informed to rise above that of the market maker in the direction of the crowd.

Similarly, if the informed has an nHill shape then his expectation is below the market maker's expectation initially and subsequently it rises above it at any history at which the likelihood of the highest state V_3 is arbitrarily small. Since the average price falls when the probability of V_3 is small, with Hill shape such a history induces the private expectation of the informed to rise above that of the market maker against the movement of the crowd.

(III) The probability of noise trading may have to be sufficiently large to ensure that the bid-ask spread is not too large both initially and later at the point of the switch.

In (I) and (II) we have compared the private expectation of the informed trader with that of the market maker. To establish the existence of herding or contrarian behavior, however, we must compare the private expectations with the bid- and ask-*prices*. The difference is that bid- and ask-prices form a spread around the market maker's expectation. To ensure that this spread has no adverse effect on the possibility of herding or contrarianism, we must ensure that the spread is sufficiently 'tight'. Tightness of the spread, in turn, depends on the extent of noise trading: the more noise there is (the smaller the likelihood of the informed types, μ), the tighter the spread.

More specifically, consider the case of buy herding. Here, the spreads may need to

be tight so that the informed type with signal S sells initially and then buys after some history H^t in the direction of the crowd. This means that one may need a minimal amount of noise trading so that (a) the expectation of the informed is less than the bid price at the initial history and (b) the expectation of the informed is greater than the ask price at H^t. Analogous restrictions apply to sell herding, buy contrarianism and sell contrarianism.

For each of the cases of herding and contrarian behaviour, we formalize these minimal noise trading restrictions by introducing two bounds μ^{in} and $\mu^{ch} \in (0, 1]$ and require the likelihood of informed trading μ to be less than both μ^{in} (to ensure the initial trade by the informed trader) and μ^{ch} (to ensure the subsequent change of behaviour).

There are two further points to note concerning the minimal noise trading restrictions. First, to demonstrate herding or contrarian behaviour by type S, a minimal amount of noise trading is needed only if type S's expectation is in between those of the other informed types. For example, if at the initial history the buy herding candidate type S has the lowest expectation amongst all the informed then his expectation must be less than the bid price at the initial history. In this case, a minimal noise trading condition is not needed to ensure the initial sale (μ^{in} can be set to equal 1). Similarly, at any potential history H^t at which buy herding occurs if type S has the highest expectation, then his expectation must be greater than the ask price at H^t for all values of μ (μ^{ch} can be set to equal 1).

Second, the minimal amount of noise trading needed to ensure that a particular type S trades at some history H^t depends on the actions of the other informed types at H^t. If the actions of the others are always the same at every history (as is the case with MLRP), then there is a unique upper bound on the size of the informed μ^{ch} that is independent of the trading history; otherwise, μ^{ch} will depend on the history considered.

5.3   Sufficient Conditions: Formal results

We now show formally that a U shape and a Hill shape, combined with some minimal amount of noise trading, are respectively sufficient for herding and contrarian behaviour.

We fix an informed type and denote it by S and consider first the decision problem of type S at the initial history. If S has a negative bias then, by Lemma 3, $\mathsf{E}[V|S] < \mathsf{E}[V]$. Also, note that $\mathsf{E}[V] - \mathsf{bid}^1 > 0$ and $\lim_{\mu \to 0} \mathsf{E}[V] - \mathsf{bid}^1 = 0$. Therefore, there must exist a critical level of noise μ_s^{in} such that $\mathsf{E}[V|S] - \mathsf{bid}^1 < 0$ if and only if $\mu < \mu_s^{in}$. By a similar argument, if S has a positive bias there must exist a critical level $\mu_b^{in} \in (0, 1]$ such that $\mathsf{E}[V|S] - \mathsf{ask}^1 > 0$ if and only if $\mu < \mu_b^{in}$. Therefore, we have the following result.

Lemma 5 (Minimal Noise Levels at the Initial History)

(i) If $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_1)$ *then there exists* $\mu_b^{in} \in (0, 1]$ *such that S buys at the initial history if and only if* $\mu < \mu_b^{in}$.

(*ii*) *If* $\Pr(S|V_3) < \Pr(S|V_1)$ *then there exists* $\mu_s^{in} \in (0,1]$ *such that* S *sells at the initial history if and only if* $\mu < \mu_s^{in}$.

Next, at date t, after history H^t and in state V_i denote the probability of a buy by $\beta_i^t = \Pr(\text{buy}|H^t, V_i)$ and the probability of a sale by $\sigma_i^t = \Pr(\text{sell}|H^t, V_i)$. Then the following is a useful characterization of the decision problem of the informed with signal S at any H^t.

Lemma 6 (Expectation Minus Price)
(*i*) $\mathsf{E}[V|S, H^t] - \text{ask}^t$ *has the same sign as*
$$q_1^t q_2^t [\beta_1^t \Pr(S|V_2) - \beta_2^t \Pr(S|V_1)] + q_2^t q_3^t [\beta_2^t \Pr(S|V_3) - \beta_3^t \Pr(S|V_2)] + 2q_1^t q_3^t [\beta_1^t \Pr(S|V_3) - \beta_3^t \Pr(S|V_1)].$$
$$(3)$$

(*ii*) $\mathsf{E}[V|S, H^t] - \text{bid}^t$ *has the same sign as*
$$q_1^t q_2^t [\sigma_1^t \Pr(S|V_2) - \sigma_2^t \Pr(S|V_1)] + q_2^t q_3^t [\sigma_2^t \Pr(S|V_3) - \sigma_3^t \Pr(S|V_2)] + 2q_1^t q_3^t [\sigma_1^t \Pr(S|V_3) - \sigma_3^t \Pr(S|V_1)].$$
$$(4)$$

To establish buy herding or buy contrarianism we need to show that (3) is positive at some history H^t. Similarly, for sell herding or sell contrarianism we need to show that (4) is negative at some history H^t. To analyze the sign of the expressions in (3) and (4) we shall first consider the first two terms in both of these expressions and show that the sign of these two terms are determined by either $\Pr(S|V_2) - \Pr(S|V_1)$ or by $\Pr(S|V_3) - \Pr(S|V_2)$ if and only if μ is sufficiently small. To establish this, let, for any $i = 1,2$ and any signal type S', $m^i \equiv \Pr(S|V_{i+1}) - \Pr(S|V_i)$, $M^i(S') \equiv \Pr(S'|V_i)\Pr(S|V_{i+1}) - \Pr(S'|V_{i+1})\Pr(S|V_i)$ and

$$\mu_i^{ch}(S') \equiv \begin{cases} \frac{m^i}{m^i - 3M^i(S')} & \text{if } m^i \text{ and } M^i(S') \text{ are non-zero and have opposite signs,} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, for each i, $\mu_i^{ch}(S') \in (0,1]$. The next lemma shows that for some S', $1 - \mu_1^{ch}(S')$ is the critical minimum amount of noise trading necessary to characterize the signs of the first term in both (3) and (4), and $1 - \mu_2^{ch}(S')$ is the critical minimum amount of noise trading necessary to characterize the signs of the second term in both (3) and (4).

Lemma 7 (Critical Noise Levels) *In any equilibrium the following holds:*

(*i*) *Suppose that* $\Pr(S|V_3) > \Pr(S|V_2)$. *Then at any* H^t *at which* S' *buys and* $S'' \neq S, S'$ *does not, the second term in (3) is positive if and only if* $\mu < \mu_2^{ch}(S')$.

(*ii*) *Suppose that* $\Pr(S|V_1) > \Pr(S|V_2)$. *Then at any* H^t *at which* S' *sells and* $S'' \neq S, S'$ *does not, the first term in (4) is negative if and only if* $\mu < \mu_1^{ch}(S')$.

(*iii*) *Suppose that* $\Pr(S|V_2) > \Pr(S|V_1)$. *Then at any* H^t *at which* S' *buys and* $S'' \neq S, S'$ *does not, the first term in (3) is positive if and only if* $\mu < \mu_1^{ch}(S')$.

(*iv*) *Suppose that* $\Pr(S|V_2) > \Pr(S|V_3)$. *Then at any* H^t *at which* S' *sells and* $S'' \neq S, S'$ *does not, the second term in (4) is negative if and only if* $\mu < \mu_2^{ch}(S')$.

Remark *Note that the critical value $\mu_i^{ch}(S')$ for the size of the informed could be 1; this happens if either both m^i and $M^i(S')$ have the same sign or if one of them is 0. Then the assumption $\mu < \mu_i^{ch}(S')$ in Lemma 7 is trivially satisfied. For example, suppose that $\Pr(S|V_3) > \Pr(S|V_2)$ and S' is Hill shaped so that $\Pr(S'|V_2) > \Pr(S'|V_3)$. Then both m^2 and $M^2(S')$ are positive and thus $\mu_2^{ch}(S') = 1$; therefore by part (i) of Lemma 7 the second term in (3) is positive irrespective of the value of μ.*

Each of the four cases in Lemma 7 provides a set of conditions that determine the sign of one of the terms in either (3) or (4). These conditions also determine the signs of (3) and (4) as a whole if the other terms in (3) and (4) are sufficiently small. This can happen at the following two types of extreme histories. First, suppose that there is a history H^t such that q_1^t is arbitrarily small relative to q_2^t and q_3^t. Since in both (3) and (4) the first and the last terms are multiplied by q_1^t (and the second term is not), it follows that these terms are close to zero and can be ignored. Thus, at such H^t type S buys if the second term in (3) is positive and sells if the second term in (4) is negative.

Second, suppose that there is a history H^t such that q_3^t is arbitrarily small relative to q_1^t and q_2^t. Since in both (3) and (4) the last two terms are multiplied by q_3^t (and the first term is not) it follows that these terms in both (3) and (4) are close to zero. Then type S buys at H^t if the first term in (3) is positive and sells if the first term in (4) is negative.

Appealing to Lemma 7, we can then show that with a sufficient amount of noise traders, at one of the extreme histories described above, type S buy (sell) herds if S is nU (pU) shaped and acts as a buy (sell) contrarian if S is pHill (nHill) shaped.

The argument for buy herding and buy contrarian is as follows (the cases of sell herding and sell contrarian are analogous). U shaped S implies that $\Pr(S|V_3) > \Pr(S|V_2)$. Then at any H^t at which q_1^t is arbitrarily small relative to q_2^t and q_3^t, by part (i) of Lemma 7, U shaped S must be buying if μ is sufficiently small. Also, at such H^t since q_1^t is small relative to q_3^t, by Lemma 4, the prices must have risen and, therefore, S would be buying in the direction of the crowd.

Hill shape S, on the other hand, implies that $\Pr(S|V_2) > \Pr(S|V_1)$. Then, by part (iii) of Lemma 7, at any H^t at which q_3^t is arbitrarily small relative to q_1^t and q_2^t, Hill shaped S must be buying if μ is sufficiently small. Furthermore, the buying in this case is against the crowd, as the price has fallen.

The argument is completed by noting that if S has also a negative bias and there is a sufficient level of noise trading, by part (i) of Lemma 5, he sells at the initial history. For our formal result we define the following two properties:

For any $\epsilon > 0$ there exists a history H^t such that $q_1^t/q_l^t < \epsilon$ for all $l = 2, 3$.   (5)

For any $\epsilon > 0$ there exists a history H^t such that $q_3^t/q_l^t < \epsilon$ for all $l = 1, 2$.   (6)

Then we can establish the following:

Lemma 8 (Possibility of Herding and Contrarian Behavior)
(i) *Suppose S is nU and (5) holds. Then S buy herds if $\mu < \min\{\mu_s^{in}, \mu_2^{ch}\}$.*
(ii) *Suppose S is pU and (6) holds. Then S sell herds if $\mu < \min\{\mu_b^{in}, \mu_1^{ch}\}$.*
(iii) *Suppose S is $nHill$ and (6) holds. Then S is a buy contrarian if $\mu < \min\{\mu_s^{in}, \mu_1^{ch}\}$.*
(iv) *Suppose S is $pHill$ and (5) holds. Then S is a sell contrarian if $\mu < \min\{\mu_b^{in}, \mu_2^{ch}\}$.*

To complete the analysis of the sufficiency results we need to show that properties (5) and (6) *can* hold. For this, we will first turn to the case where the information structure is MLRP, where the analysis is simple and the result is sharpest.

The Special Case of Monotone Likelihood Ratio Signals. Assume that the signals are ordered so that $S_1 < S_2 < S_3$ and that the signal structure satisfies MLRP. Then by Lemma 2 we can make the following two observations. First, S_1 and S_3 are strictly monotonic; therefore S_2 is the only type of informed agent that could be U shaped or hill shaped. Second, type S_1 will always be selling and type S_3 will always be buying; thus the only possible herding or contrarian candidate is an informed agent with middle signal S_2. These observations allow us to state the following result for the case of MLRP signals.

Theorem 1 (Herding and Contrarianism with MLRP) *Assume signals are ordered, $S_1 < S_2 < S_3$, and that the signal structure satisfies MLRP. Then the following holds:*
(a) *Suppose S_2 is nU. Then S_2 buy herds if and only if $\mu < \min\{\mu_s^{in}, \mu_2^{ch}\}$.*
(b) *Suppose S_2 is pU. Then S_2 sell herds if and only if $\mu < \min\{\mu_b^{in}, \mu_1^{ch}\}$.*
(c) *Suppose S_2 is $nHill$. Then S_2 is a buy contrarian if and only if $\mu < \min\{\mu_s^{in}, \mu_1^{ch}\}$.*
(d) *Suppose S_2 is $pHill$. Then S_2 is a sell contrarian if and only if $\mu < \min\{\mu_b^{in}, \mu_2^{ch}\}$.*

The "only if" part of the above results follows immediately from Lemmas 5 and 7 (for details see the Appendix). To establish the "if" parts, by Lemma 8, it suffices to show that (5) and (6) hold. To show this note that the probability of a buy (sale) increases (decreases) in the value of V by a positive amount (independent of the past history) if S_3 and S_1 are monotonic, as is the case with MLRP signals. Formally, we have the following.[19]

Lemma 9 *Let signals satisfy MLRP. Then there exists $\delta \in (0,1)$ such that for every H^t and for any V_h and V_ℓ with $V_h > V_\ell$, the following holds: $\beta_\ell^t/\beta_h^t < \delta$ and $\sigma_h^t/\sigma_\ell^t < \delta$.*

Next consider any history H^t that consists of only buys and note that $q_1^{\tau+1}/q_\ell^{\tau+1} = \beta_1^\tau q_1^\tau / \beta_\ell^\tau q_\ell^\tau$ for all $\tau < t$ and $\ell = 2,3$. Then with MLRP signals it follows from Lemma 9 that q_1^τ/q_ℓ^τ is decreasing by a factor that is independent of the past for all $\ell = 2,3$; hence, for sufficiently large t, q_1^t/q_ℓ^t must be arbitrarily close to zero and (5) holds. Similarly, with MLRP, for histories with sufficiently many sales Lemma 9 implies that q_3^t/q_ℓ^t can be made arbitrarily close to zero for all $\ell = 1,2$, and hence (6) holds. This proves Theorem 1.

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[19]The result actually follows more generally: all that is required is that there are two monotonic signals.

The General Case. Our main result for the general information structure is as follows.

Theorem 2 *There exists a $\overline{\mu} \in (0, 1]$ such that the following holds:*

(a) *If there is a non-zero biased U shaped signal and $\mu < \overline{\mu}$, then there exist an informed type that herds.*

(b) *If there is a non-zero biased Hill shaped signal and $\mu < \overline{\mu}$, then there exists an informed type that acts as a contrarian.*

Theorem 2 provides a sufficient condition for herding and contrarian behaviour; yet, in contrast to Theorem 1 for the case of MLRP signals, it does not differentiate between buy and sell herding or between buy and sell contrarians. We will now highlight the issues involved in making such differentiations for the general case.

First, the constructions for MLRP signal structures that ensure both (5) and (6) do not extend to non-MLRP signal structures. The reason is that without MLRP a buy does not necessarily reduce *both* q_1^t/q_2^t and q_1^t/q_3^t and therefore, histories consisting of only buys do not always ensure (5). Similarly, a sale does not necessarily reduce *both* q_3^t/q_1^t and q_3^t/q_2^t and therefore, histories consisting of only sales do not always ensure (6).

Without MLRP we need a different construction to ensure that (5) and (6) hold. Such a construction can be particularly difficult because in some cases there are no paths that result in both q_1^t/q_2^t and q_1^t/q_3^t decreasing at *every* t or in both q_3^t/q_1^t and q_3^t/q_2^t decreasing at *every* t. For these difficult cases, we construct outcome paths, each consisting of two different stages, to ensure (5) and (6). For example, to ensure (5), the path is constructed so that in the first stage q_1^t/q_2^t becomes small while ensuring that q_1^t/q_3^t does not increase by too much. Then in the second stage, once q_1^t/q_2^t is sufficiently small, the continuation path makes q_1^t/q_3^t small while ensuring that q_1^t/q_2^t does not increase by too much. A similar construction is used to ensure (6). Such constructions work for most signal distributions so that we obtain a similar set of conclusions with respect to herding and contrarian behaviour as in Theorem 1. The exception are cases with two U shaped signals or two Hill shaped signals. In these cases we can show, depending on the bias of the third signal, that either (5) or (6) holds, but not both; thus in these cases we can only establish (i) either buy or sell herding and (ii) either buy or sell contrarianism. Thus our result for herding and contrarian behaviour in these cases is weaker than with MLRP signals. Formally,

Proposition 3 (Taxonomy of Herding and Contrarianism: The General Case)

(a) *Suppose S is nU shaped. Assume also that if one other signal is pU shaped then the third signal has a non negative bias. Then S buy-herds if $\mu < \min\{\mu_s^{in}, \mu_2^{ch}\}$.*

(b) *Suppose S is pU shaped. Assume also that if one other signal is nU shaped then the third signal has a non positive bias. Then S sell-herds if $\mu < \min\{\mu_b^{in}, \mu_1^{ch}\}$.*

(c) *Suppose S is nHill shaped. Assume also that if one other signal is pHill shaped then the third signal has non positive bias. Then S is a buy contrarian if $\mu < \min\{\mu_s^{in}, \mu_1^{ch}\}$.*

(d) *Suppose S is pHill shaped. Assume also that if one other signal is nHill shaped then the third signal has a non negative bias. Then S is a sell contrarian if $\mu < \min\{\mu_b^{in}, \mu_2^{ch}\}$.*

Note first, that by setting $\overline{\mu} = \min\{\mu_s^{in}, \mu_b^{in}, \mu_1^{ch}, \mu_2^{ch}\}$, Theorem 2 follows immediately from Proposition 3. To see this suppose that there exist a U shaped signal as in part (i) of Theorem 2. Then there are two possibilities: either there is another U shaped signal with an opposite bias or there is not. If there is no other U shaped signal with an opposite bias, then by parts (a) and (b) of Proposition 3, the U shaped type buy herds if the signal has a negative bias and sell herds if it has a positive bias. If there is another U shaped signal with the opposite bias, then by parts (a) and (b) of Proposition 3, one of the U shaped signals must herd: if the third signal is weakly positive then the U shaped signal with a negative bias buy herds, and if the third signal is weakly negative then the U shaped signal with a positive bias sell herds. The reasoning for part (b) of Theorem 2 is analogous.

Second, the sufficiency result in Theorem 1 for the case of MLRP signal structures is a special case of Proposition 3 (with MLRP there is at most one U shaped type).

Third, as we explained before, the noise restrictions in Proposition 3 (and in Theorem 2) may be trivially satisfied if the upper bounds on the value of μ are equal to 1.

Fourth, consider the minimal noise trading restrictions that require μ to be less than some critical level $\min\{\mu_\ell^{in}, \mu_j^{ch}\}$ for some $\ell = b, s$ and $j = 1, 2$. In Theorem 1 with MLRP signal structures, a U shaped type herds or a Hill shaped type acts as a contrarian if and *only if* such a minimal noise trading restriction holds. By contrast, in Proposition 3 (and Theorem 2), we can only show a similar herding or contrarian behaviour if μ is less than some critical level $\min\{\mu_\ell^{in}, \mu_j^{ch}\}$. The reason is that there may be levels of μ *above* these critical levels that are also compatible with herding or contrarian behaviour.

To see this consider the case of buy herding for an nU shaped signal S. As Lemma 7 (i) shows, the upper bound on the value of μ that allows S to change behaviour and buy at some history H^t depends uniquely on the identity of the other type (if any) that might also buy at H^t. With MLRP the expectations of the different types are ordered so that the type with the lowest and the highest signals always trade the same way. This implies that when S changes behaviour the identity of any other type that buys together with S is always the same and therefore, the upper bound for μ is uniquely defined by μ_2^{ch} for any history. With a non-MLRP signal structure, on the other hand, the expectations of the signal types may cross. It is thus possible that for different histories, different types are paired to buy or sell so that the upper bound on the value of μ may be different for different histories.

It turns out however that the above difficulty with respect to the necessity of the minimal noise trading conditions for herding (contrarian) behaviour only arises if there are two U shaped (Hill Shaped) signals. In the supplementary material (Proposition 3a) we show that, even without MLRP, the minimal noise trading condition remains necessary for

herding (contrarian) behaviour as long as there is at most one U (Hill) shaped signal.

So in summary, the results on herding and contrarianism in the general case are almost as strong as in the case with MLRP signals.

6   Resilience, Fragility and Large Price Movements

We now consider the robustness of herding and contrarianism and describe the range of prices for which herding and contrarianism can occur. Throughout this section we assume that signals satisfy the well-behaved case of MLRP (we will return to this later) and perform the analysis for buy herding and buy contrarianism; the other cases are analogous.

We first show that buy herding persists if and only if the number of sales during an episode of buy herding is not too large. This implies in particular that buy herding behavior persists if the buy herding episode consists of only buys. We also show that during a buy herding episode as the number of buys increases, it takes more sales to break the herd. For buy contrarianism the impact of buys and sales work in reverse: in particular, buy contrarianism persists if and only if the number of buys during an episode of buy contrarianism is not too large. This means that buy contrarianism does not end if the buy contrarianism episode consists of only sales. We also show that during a buy contrarianism episode as the number of sales increases, it takes more buys to break the contrarianism.

Proposition 4 (Persistent Herding and Self-Defeating Contrarianism)
Assume MLRP. Consider any history $H^r = (a^1, \ldots, a^{r-1})$ and suppose that H^r is followed by $b \geq 0$ buys and $s \geq 0$ sales in some order; denote this history by $H^t = (a^1, \ldots, a^{r+b+s-1})$.[20]

(a) If there is buy herding by S at H^r then there exists an increasing function $\bar{s}(\cdot) > 1$ such that S continues to buy herd at H^t if and only if $s < \bar{s}(b)$.

(b) If there is buy contrarianism by S at H^r then there exists an increasing function $\bar{b}(\cdot) > 1$ such that S continues to act as a buy contrarianism at H^t if and only if $b < \bar{b}(s)$.

One implication of the above result is that herding is resilient and contrarianism is self defeating. The reason is that when buy herding or buy contrarianism begins, buys become more likely relative to a situation where the herding or contrarian type does not switch. Thus, in both buy herding and buy contrarianism there is a general bias towards buying (relative to the case of no social learning). By Proposition 4, buy herding behavior persists if there are not too many sales and buy contrarian ends if there is a sufficiently large number of buys. Thus herding is more likely to persist whereas contrarianism is more likely to end.

To see the intuition for Proposition 4 consider first the case of buy herding in part (a). At any history the difference between the expectation of the herding type S and that of

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[20] We will henceforth omit past prices from the history H^t to simplify the exposition.

the market maker is determined by the relative likelihood that they attach to each of the three states. Since the herding type S must have an nU shaped csd it follows that in comparing the expectation of the herding type S with that of the market maker there are two effects: first, S attaches more weight to V_3 relative to V_2 than the market maker and, second, S attaches more weight to V_1 relative to both V_2 and V_3 than the market maker. Since (i) $V_1 < V_2 < V_3$ and (ii) at any history H^r with buy herding the expectation of the herding type S exceeds that of the market maker, it then follows that at H^r the first effect must dominate the second one, i.e. q_1^r/q_2^r and q_1^r/q_3^r are sufficiently small so that the first effect dominates. Also, by Lemma 9, when the MLRP holds, buys reduce the probability of V_1 relative to the other states. Therefore, further buys after H^r make the second effect more insignificant. Further buys thereby ensure that the expectation of the herding type S remains above the ask price.

On the other hand, by Lemma 9 when MLRP holds, sales reduce the probability of V_3 relative to the other states; thus sales after H^r make the first effect less significant. Therefore, with sufficiently many sales, the expectation of the herding type S will move below the ask price so that type S will no longer buy. This ends herding.

The intuition for the buy contrarian case is analogous except that the effect of further buys and further sales work in the opposite direction.

Next, we consider the range of prices for which herding and contrarianism is possible. Casual intuition may suggest that prices may not move significantly during any herding episode. Yet with MLRP signals this is not true and large price movements are consistent with both herding and contrarianism. In fact, in both cases the range of price movements can (almost) include the entire set of feasible prices.

More specifically, for buy herding the range of feasible prices is $[V_2, V_3]$ and for buy contrarianism the range is $[V_1, V_2]$.[21] To show that price movements during herding and contrarianism can span almost the entire feasible range of prices note that, as argued above, with MLRP buys increase prices, and sales decrease prices. Furthermore, by Proposition 4, buy herding persists when there are only buys and buy contrarianism persists when there are only sales. Then we can conclude that (i) once buy herding starts, a large number of buys can induce prices to rise to levels arbitrarily close to V_3 without ending buy herding and (ii) once buy contrarianism starts, large numbers of sales can induce prices to fall to levels arbitrarily close to V_1 without ending buy contrarianism.

Finally, we complete the analysis by showing that there exists a set of priors on $\mathbb{V}$ such that herding and contrarianism can start when prices are close to the middle value, V_2. Together with the arguments in the last paragraph, we have that herding and contrarian prices can span almost the entire range of feasible prices. Formally, we have the following.

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[21]By Lemma 4, at any date t buy herding implies $q_3^t > q_1^t$ and buy contrarian implies $q_1^t > q_3^t$.

Proposition 5 (Social Learning and the Price Range) *Let signals obey MLRP.*

(a) Consider any history $H^r = (a^1, \ldots, a^{r-1})$ at which there is buy herding (contrarianism). Then for any $\epsilon > 0$, there exists history $H^t = (a^1, \ldots, a^{t-1})$ following H^r such that there is buy herding (contrarianism) at every $H^\tau = (a^1, \ldots, a^{\tau-1})$, $r \leq \tau \leq t$, and the average price $\mathsf{E}[V|H^{r+\tau}]$ exceeds $V_3 - \epsilon$ (is less than $V_1 + \epsilon$).

(b) Let μ admit buy herding (contrarianism) as in Theorem 1. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mathsf{Pr}(V_2) > 1 - \delta$ there is a history $H^t = (a^1, \ldots, a^{t-1})$ and a date $r < t$ such that (i) there is buy herding (contrarianism) at every $H^\tau = (a^1, \ldots, a^{\tau-1})$, $r \leq \tau \leq t$, (ii) $\mathsf{E}[V|H^r] < V_2 + \epsilon$ ($\mathsf{E}[V|H^r] > V_2 - \epsilon$) and (iii) $\mathsf{E}[V|H^t] > V_3 - \epsilon$ ($\mathsf{E}[V|H^t] > V_1 + \epsilon$).

The results of this section (and the ones in the next section on volatility) assume that the information structure satisfies the well behaved case of the MLRP. This ensures that the probabilities of buys and the sales are monotonic in V (see Lemma 9). As a result, we have that the relative probability q_1^t / q_ℓ^t falls with buys and rises with sales for all $\ell = 2, 3$, and the opposite holds for q_3^t / q_ℓ^t for all $\ell = 1, 2$. This monotonicity in the relative probabilities of the extreme states is the feature allow us to establish our persistence and fragility results. If MLRP were not to hold, then the probability of buys and sales may not be monotonic in V, and the results of this section may not hold.[22] An example of such possibility is the herding example in Avery and Zemsky (1998); see Section 9 for a discussion.

7 The Impact of Social Learning on Liquidity and Volatility

In this section we consider the consequences of both herding and contrarianism for liquidity and price volatility. The key feature of such social learning is that traders change their behavior. To assess the impact of herding and contrarianism we now compare the outcomes with social learning to a scenario when informed types do not change their actions as prescribed by the theory.

This exercise is important because it highlights the impact of the history induced switches of behavior. Furthermore, there are many realistic instances in which traders may not change their behavior. For instance, traders may be unable to observe past prices and actions due to a lack of transparency in the market. Or there may be some constraint on rationality resulting in traders not remembering past observations or not being able to make the right inference from observing past histories. We will discuss these instances in more detail towards the end of this section. The exercise below provides a comparison between the outcomes with history dependent social learning with those in other situations in which traders follow the simple strategy of always taking the same actions irrespective

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[22]Since the monotonicity of the probability of buys and sales in V (Lemma 9) also holds if two signals have monotonic csds; all the results of this paper that assume MLRP hold under this weaker assumption.

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of the past. Our conclusion is that transparency, in terms of observing the past actions of other players, is less attractive (as measured by liquidity and volatility) than no transparency. For the former can generate herding and contrarian behavior that would be absent if such details were not known.

7.1 Liquidity

In sequential trading models in the tradition of Glosten and Milgrom (1985) liquidity is measured by the size of the bid-ask-spread because a larger spread implies higher adverse selection costs and thus lower liquidity. We will compare this measure of liquidity when a (rational) informed trader herds or acts as a contrarian by switching his behavior with that when he chooses not to switch according to the theory (prices in both cases accurately reflect behavior). We show that the spread is larger in the former case than in the latter.

To do this consider the case of buy herding by a signal type S. Casual intuition suggests that buy herding hampers the information transmission and thus the ask-price is lower when the herding candidate switches than when he does not switch. (The idea is that when S type switches to buy herding there are more types that are buying compared to when he does not switch and therefore a buy conveys less information in the former case than in the latter one). This intuition is, however, misleading, and the ask-price is *higher* with herding than when there is no switch in behavior. The reason lies in the herding candidate's U shaped csd: the difference in the ask prices in the two cases reflects the fact that the herding candidate S buys in one case and not in the other. When buy herding starts the likelihood of V_1 is small relative to both V_2 and V_3. As type S puts larger weight on signal V_3 relative to V_2, it then follows that in the case when S buys the ask-price must be higher.

By a similar reasoning, the bid price is lower when S (rationally) buy herds than when S sells: When buy herding happens the relative likelihood of V_1 is small; since type S puts larger weight on signal V_3 relative to V_2 it then follows that a sale in the case when S is selling must involve a higher price than in the case when S is buy herding.

This increase in the spread that we obtain for herding compared to the case when the informed types do not switch also extends to the case of buy contrarianism. The reason here lies in the hill shaped nature of a contrarian candidate's signal. First, when a hill shaped S type engages in buy contrarianism, prices have dropped and state V_3 is considered to be unlikely relative to V_1 and V_2. Second, the contrarian type S puts more weight on V_2 than V_1 and thus when S buys, the price must be larger than when S does not buy. Thus when S switches and acts as a buy contrarian the ask-price must be higher and the bid-price lower.

Proposition 6 (The Impact of Herding and Contrarian Behavior on Liquidity)
Consider any history H^t at which type S engages in buy herding or buy contrarianism.
(a) The ask price when the buy herding or buy contrarian candidate S rationally buys

exceeds the ask price when he chooses not to buy.

(b) *The bid price when the buy herding or buy contrarian candidate S rationally buys is lower than the bid price when he chooses to sell.*

The above result is only formulated for the cases of buy herding and buy contrarianism; an identical set of results holds for sell herding and sell contrarianism.

7.2 Volatility

In this subsection we address the impact of switches in behavior due to herding and contrarian on price volatility. Specifically we address the following questions. Will buys move prices less with than without herding? Will sales move prices more with than without herding? We also ask the same questions about contrarian behavior. To answer these questions we compare prices when herding or contrarian behavior happens in the model described thus far with prices in a hypothetical world that is otherwise identical except that each informed type always takes the same action as the one that he would take at the initial history. The market maker in the hypothetical economy correctly accounts for such behavior and the whole history when setting prices.

In the hypothetical economy informed traders act as if they do not observe prices and past actions of others; we thus refer to this world as the *opaque* market. In contrast, in the standard setting traders observe and learn from the actions of their predecessors. To highlight the difference, in this section we refer to the standard case as the *transparent* market.

Figure 1 presents particular sequences of simulated transaction prices for the two markets. In these simulated figures, it appears that volatility is higher in the transparent market than in the opaque one and this conclusion holds both for when herding is possible (left panel) and for when contrarian behaviour is possible (right panel).

Clearly, the conclusions that can be drawn from these figures are only suggestive. In Proposition 7 below, we formally confirm these conclusions concerning volatility at the point when herding or contrarianism begins. In fact, as the average price after a buy is the last period's ask price, and the average price after a sale is the last bid price, the liquidity result, Proposition 6, implies that when herding or contrarianism starts, the first trade has a larger impact on the average price in the transparent market than in the opaque one (the average price is higher after a buy and lower after a sale). We show below that at histories at which herding or contrarian behavior occurs, this effect applies more generally. Namely, we show that further trades move prices more in the transparent market than in the opaque one where traders, contrary to herding or contrarian behavior, always do the same thing.

We will derive this strong result on volatility for the case of MLRP signals. This is surprising because, taken at face value, MLRP corresponds to a "well-behaved" information structure. Also, we focus on the case of buy herding and buy contrarianism; the results for
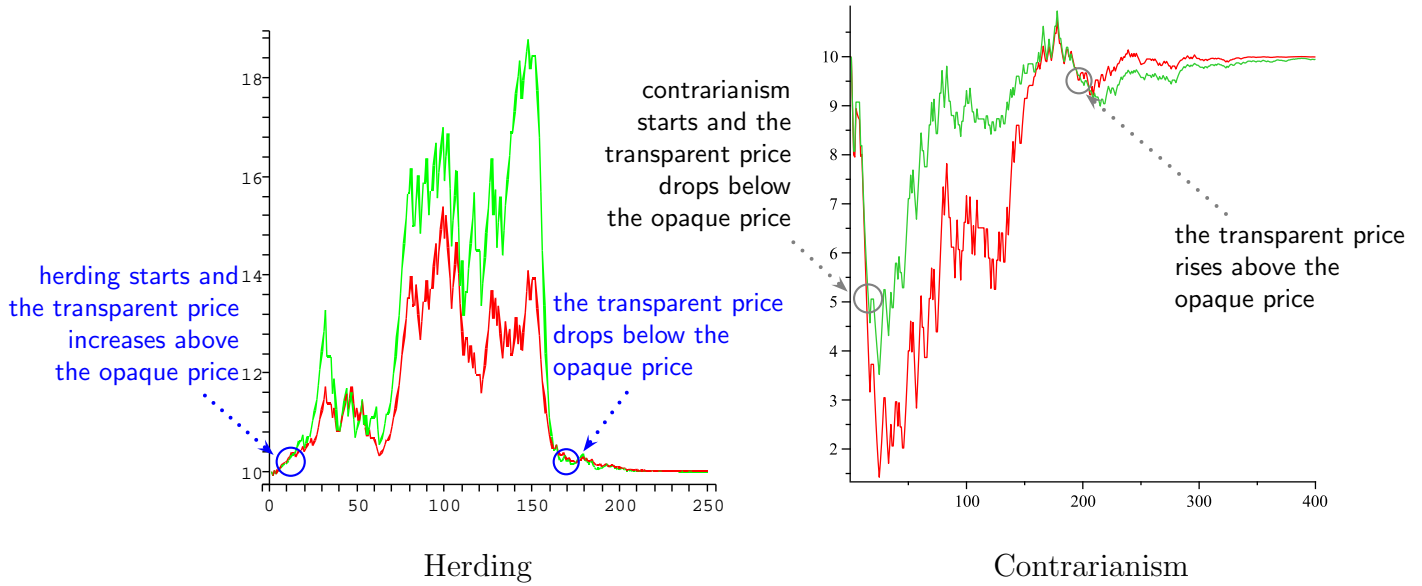
Figure 1: **Simulated Transaction Prices.** The left panel is an example for a history with herding, the right panel is an example for a history with contrarianism ($V_1 = 0$, $V_2 = 10$, $V_3 = 20$). In the LEFT PANEL the gray line plots the outcome of the simulated prices for the transparent market (where there may be herding). Herding starts for prices close to V_2, and prices during herding can move up substantially. The dark line plots transaction prices for the same sequence of traders, but for an opaque market. In the RIGHT PANEL, the dark line plots the outcome of the simulated prices for the transparent market with contrarianism. Here, prices with contrarianism drop below those that would transpire in the opaque market. Moreover, the transparent prices move more than the opaque prices. The underlying signal distributions are listed in the Supplementary Appendix; we use the smallest possible amount of noise trading. The underlying trader sequence is random but for five 'artificial' buys in the herding example and six 'artificial' sales in the contrarian example in the early rounds of trading.

sell herding and sell contrarianism are identical and will thus be omitted.

Specifically, fix any history H^r at which buy herding starts and consider the difference between the average price in the transparent market with that in the opaque market at any buy herding history that follows H^r. Assuming MLRP signals, we show (a) that the difference between the two prices is positive if the history since H^r consists of only buys, (b) that the difference is negative if the history since H^r consists of only sales and the number of sales is not too large,[23] and (c) that the difference is positive if the history following H^r is such that the number of buys is arbitrarily large relative to the number of sales. We also show an analogous result for buy contrarianism.

Formally, for any history H^t let $\mathsf{E}_o[V|H^t]$, $q_{i,o}^t$, $\beta_{i,o}^t$ and $\sigma_{i,o}^t$ be respectively the average price, the probability of V_i, the probability of a buy in state V_i and the probability of a sale in state V_i in the opaque market at H^t. Then we can show the following.

Proposition 7 (Relative Volatility) *Assume MLRP. Consider any finite history $H^r = (a^1, \ldots, a^{r-1})$ at which the priors in the two markets coincide: $q_i^r = q_{i,o}^r$ for $i = 1, 2, 3$.*

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[23]Note that buy herding cannot persist with an arbitrarily large number of sales; therefore, here by assumption, we are considering the differences between the prices in the two worlds only at histories at which the number of sales is not too large after buy herding has started.

Suppose that H^r is followed by $b \geq 0$ buys and $s \geq 0$ sales in some order; denote this history by $H^t = (a^1 \ldots, a^{r+b+s-1})$.

(1) *Assume that there is buy herding at H^τ, for every $\tau = r, \ldots, r + b + s$.*

    (a) *Suppose $s = 0$. Then $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b > 0$.*

    (b) *Suppose $b = 0$. Then there exists $\bar{s} \geq 1$ such that $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s \leq \bar{s}$.*

    (c) *For any s there exists $\bar{b}$ such that $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b > \bar{b}$.*

(2) *Assume that there is buy contrarianism at H^τ, for every $\tau = r, \ldots, r + b + s$.*

    (a) *Suppose $b = 0$. Then $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s > 0$.*

    (b) *Suppose $s = 0$. Then there exists $\bar{b} \geq 1$ such that $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$ for any $b \leq \bar{b}$.*

    (c) *For any b there exists $\bar{s}$ such that $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$ for any $s > \bar{s}$.*

The intuition for the above proposition is similar to the insights from the liquidity result before. The critical element in demonstrating the result is the U shaped nature of the herding candidate's signal and the Hill shaped nature of the contrarian candidate's signal. To see this consider any buy herding history $H^t = (a^1, \ldots, a^{r+b+s-1})$ satisfying the above proposition for the case described in part (1) of the proposition — the arguments for a buy contrarian history described in part (2) of the proposition are analogous. Then the prices in the transparent and opaque markets differ because at any buy herding history in the transparent market the market maker assumes that the buy herding candidate S buys whereas in the opaque market the market maker assumes that S sells.[24] Since the buy herding type must have a U shaped signal we also have $\Pr(S|V_3) > \Pr(S|V_2)$. Then the following must hold: (i) the market maker upon observing a buy increases his belief about the likelihood of V_3 relative to that of V_2 faster in the transparent market (where S is a buyer) than in the opaque market (where S is a seller) and (ii) the market maker upon observing a sale decreases his belief about the likelihood of V_3 relative to V_2 faster in the transparent market than in the opaque market. Now if it is also the case that the likelihood of V_1 is small relative to that of V_3 in both worlds then it follows from (i) and (ii), respectively, that the average price in the transparent market exceeds that in the opaque market after a buy and it is less after a sale.

At H^r in both markets the likelihoods of each state coincide ($q_i^r = q_{i,o}^r$); moreover the likelihood of V_1 in both markets is small relative to that V_3 (to ensure buy herding). Then the following two conclusions follow from the discussion in the previous paragraph: First, if H^t involves only a single buy after H^r (i.e. if $s = 0$ and $b = 1$) then $\mathsf{E}[V|H^t] > \mathsf{E}_o[V|H^t]$. Second, if H^t involves only a single sale after H^r (i.e. if $b = 0$ and $s = 1$) then $\mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t]$. Thus Proposition 7 part 1(b) follows immediately. To complete the intuition for 1(a) and 1(c) in Proposition 7, note that further buys after H^r reduce the probabilities of V_1 relative to V_3 in both markets (see Lemma 9); therefore if either the history after H^r

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[24]If we assume $S_1 < S_2 < S_3$, then with MLRP the buy herding candidate must be S_2.

involves no sale (as in part $1(a)$ of Proposition 7) or the number of buys is large relative to the number of sales (as in part $1(c)$ of Proposition 7) then the first conclusion is simply reinforced, and $\mathsf{E}[V|H^t]$ remains above $\mathsf{E}_o[V|H^t]$ after any such histories.

Notice that with MLRP signals, any sale beyond H^r increases the probability of V_1 relative to V_3 (and relative to V_2) both in the transparent and in the opaque market. Furthermore, the increase may be larger in the latter than in the former. As a result, for the herding case we cannot show that in general average prices in the transparent market fall more than in the opaque market after *any arbitrary* number of sales. However, if the relative likelihood of a sale in state V_1 to V_3 in the transparent market is no less than that in the opaque market, then we can extend the conclusion in part $1(b)$ of the proposition to show that the price in the transparent market falls more than in the opaque market after *any arbitrary* number of sales (the proof is in the supplementary material):[25]

$$\text{if } (\sigma_1/\sigma_3) \geq (\sigma_{1,o}/\sigma_{3,o}) \text{ and } b = 0 \text{ then } \mathsf{E}[V|H^t] < \mathsf{E}_o[V|H^t] \text{ for any } s. \tag{7}$$

Proposition 7 of course does not address the likelihood of a buy or a sale after herding or contrarianism begins. However, it is important to note that once buy herding or buy contrarianism starts there will also be more buys in the transparent market compared to the opaque market because the herding type buys at such histories. Thus, given the conclusions of Proposition 7, price paths must have a stronger upward bias in the transparent market than in the opaque market (this is consistent with the simulations in Figure 1).

Finally, it is often claimed that herding generates excess volatility whereas contrarianism tends to stabilize markets because the contrarian types act against the crowd. The conclusions of this section are consistent with the former claim but contradict the latter. Both herding and contrarianism increase price movements (compared to the opaque market) and they do so for similar reasons — namely because of the U shaped nature of the herding type's csd and the Hill shaped nature of the contrarian type's csd.

Interpretation of the Opaque Market. To illustrate the importance of social learning, we have compared the outcomes with learning and switches of behaviour (herding and contrarianism) to a stark situation in which traders always take the same action (the optimal one from the initial history) independently of the public history of actions and prices.

One can think of the traders in the opaque market as automata that always buy or sell depending on their signals. One justification for such naive behavior is that traders do not observe or remember the public history of actions and prices (including current prices).

Alternatively, the non changing behavior may represent actions of rational traders in a trading mechanism where traders submit their orders, possibly through an intermediary, some time *before* the orders get executed. The market maker would receive these orders in some sequence and he would execute them sequentially at prices which reflect all the

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[25]The condition $(\sigma_1/\sigma_3) \geq (\sigma_{1,o}/\sigma_{3,o})$ is satisfied if, e.g., the herding candidate has an almost zero bias.

information contained in the orders received so far. The actions of other traders and the prices are unknown at the time of the order submission and thus, as in the opaque market, the order submissions of each trader are independent of these variables.

In this set-up the traders effectively commit to a particular trade before any information is revealed and the market maker receives orders sequentially and sets efficient prices. Therefore, the price sequence in this alternative model would coincide with the price sequence in the opaque market if each informed trader chooses the same action in this alternative set-up as he would choose at the initial history in our original model and if the order of arrivals of the informed is the same in the two models. There is thus an exact mapping between the price in this alternative set-up and the price in the opaque market. Our result in Proposition 7 demonstrates the excess volatility resulting from social learning in the standard sequential transparent market compared to this alternative set-up in which all orders are submitted before any execution and in which there is no social learning.

In the opaque market the informed traders always take the same action because the traders either ignore prices and the public history or because they do not have access to them at the time they have to make a decision. A slightly more transparent market than the opaque one is one where each trader with signal S compares his prior expectations, $\mathsf{E}[V|S]$, with the current price and buys if $\mathsf{E}[V|S]$ exceed the ask price, sells if $\mathsf{E}[V|S]$ is less than the bid price and does not trade otherwise. In this "almost opaque" market there is a different kind of non-transparency in that at each period the traders do not observe or recall past actions and prices but they know the bid and ask prices at that period; furthermore they act semi-rationally by comparing their private expectation with current prices without learning about the fundamental value from the current price (e.g., for cognitive reasons).

For the case of herding, the same excess volatility result as in part (1) of Proposition 7 can also be demonstrated if we compare the transparent market with the above almost opaque market. To see this note that by assumption at the initial history every buy herding type S sells. Also, at every buy herding history the prices are higher than at the initial history; therefore it must be that in an almost opaque market the herding type must also sell at every buy herding history.[26] Since Proposition 7 compares price volatility *only* at histories at which buy herding (or buy contrarianism) occurs, it follows that the same excess volatility result holds if we compare the transparent with the almost opaque market.

Finally, in comparing the opaque market with our standard model, we have focussed solely on price volatility (in the sense of Proposition 7). There are other aspects of transparency that one could consider. Although transparency has many benefits, our results point to a possible adverse effect caused by social learning.

[26]Since at a buy contrarian history prices are lower than at the initial history, the same claim cannot be made for the contrarian case (a buy contrarian type in an almost opaque market may change behaviour and buy at a buy contrarian history).

8   Herding and Contrarianism with Many States

Our results intuitively extend to cases with more signals and more values. In fact, with three states and arbitrary number of signals our characterization results (in terms of U shape signals for herding and Hill shaped signals for contrarianism)[27] and all our conclusions in the previous two sections with respect to fragility, persistence, large price movements, liquidity and price volatility remain unchanged.[28]

With more than 3 states, U shape and Hill shape are no longer the only possible signal structures that can lead to herding and contrarianism. The intuition for our results with many states does, however, remain the same: the herding type must distribute probability weight to the tails, the contrarian types must distribute weight to the middle.

To illustrate this, assume that there are $n > 2$ states and n signals. Then an analogous result to Proposition 1 holds regarding csd monotonic signals.[29]

Lemma 10 (No switching with csd monotonic signals) *If signal type S has a monotonic csd, then type S will never switch from buying to selling or vice versa.*

To see this point, it can be shown, analogously to Lemma 6, that $\mathsf{E}[V|S, H^t] - \mathsf{E}[V|H^t]$ has the same sign as

$$\sum_{j=1}^{n-1} \sum_{i=1}^{n-j} j \cdot q_i q_{i+j} [\mathsf{Pr}(S|V_{i+j}) - \mathsf{Pr}(S|V_i)]. \tag{8}$$

For increasing csds, $\mathsf{Pr}(S|V_{i+j}) - \mathsf{Pr}(S|V_i)$ will be non-negative for all i, j, and for decreasing csds non-positive. Thus, (8) is either always non-negative or always non-positive for increasing and decreasing csds respectively, so that the necessary condition for switching cannot be satisfied. To observe switches, the underlying signal must thus have non-monotonic csd. In what follows we will outline two sufficient conditions that yield herding and contrarian behavior respectively, and that have a similar flavour as our sufficiency results in Section 5. We will focus only on buy herding and buy contrarianism; sell herding and sell contrarianism is analogous.

In line with the previous analysis we assume that values are on an equal grid and that the prior probability distribution is symmetric. This means $\{V_1, V_2, \ldots, V_n\} = \{0, \mathcal{V}, 2\mathcal{V}, \ldots, (n-1)\mathcal{V}\}$ and $\mathsf{Pr}(V_i) = \mathsf{Pr}(V_{n+1-j})$, respectively.

Next, recall that a buy herding and buy contrarian candidate must sell at the initial history. A necessary condition for this is that $\mathsf{E}[V|S] < \mathsf{E}[V]$. In the supplementary material

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[27]With 3 states, hill- and U shape are still well-defined, irrespective of the number of signals; even with a continuum of signals these concepts can be defined in terms of conditional densities.

[28]With MLRP signals and three states there is at most one type with a U shaped signal, and thus, depending on the bias of this type, there is either buy or sell herding but not both. With MLRP and more than three signals, there may be more than one U shaped type. If these have different biases, then both buy and sell herding may be feasible.

[29]Of course, with n states, signal S has an increasing csd if $\mathsf{Pr}(S|V_i) \leq \mathsf{Pr}(S|V_{i+1})$ for all $i = 1, \ldots, n-1$ and it has a decreasing csd if $\mathsf{Pr}(S|V_i) \geq \mathsf{Pr}(S|V_{i+1})$ for all $i = 1, \ldots, n-1$.

we show this arises if signal S is *negatively biased* in the sense that $\Pr(S|V_i) > \Pr(S|V_{n+1-i})$ for all $i < (n+1)/2$, i.e. the signal happens more frequently in low states compared to high states that are equally far from the middle value. Moreover, to be selling at the initial history, type S's expectation must also be lower than the bid price. As in Lemma 5, if $\mathsf{E}[V|S] < \mathsf{E}[V]$ this is indeed the case if μ is sufficiently small.

Next we consider sufficient conditions for switching behavior. As with three states, we need to consider histories at which the probabilities of extreme states are small. When the probabilities of the lowest states are small (and therefore these states can be effectively ignored), the expectation of the informed will be larger than that of the market maker if the informed puts more weight on high than middle states. Moreover at such histories the price must have risen. Analogously, when the probabilities of the highest states are small enough (so they can be ignored), the trader may act as a buy contrarian if he puts more weight on middle relative to low states. The sufficient conditions that we describe here for the switches are very simple and impose restrictions only on the most extreme states. Specifically, for buy herding we assume $\Pr(S|V_{n-1}) < \Pr(S|V_n)$, and for buy contrarianism we assume $\Pr(S|V_1) < \Pr(S|V_2)$.

As in the three state model the simplest way of ensuring the existence of histories at which the probabilities of the extreme states are small is the case of MLRP. Then, as in Lemma 2 for the three states case, the probability of a buy is increasing and the probability of a sale is decreasing in V. As a result, at any history H^t at which the number of buys is sufficiently large, we can ignore all but the highest two states V_{n-1} and V_n. Then at any such H^t (i) the price must have risen and (ii) the expectation of type S must exceed that of the public expectation if $\Pr(S|V_{n-1}) < \Pr(S|V_n)$. Furthermore, if in addition the bid-ask spread is not too large (enough noise trading), the expectation of S will also exceed the ask price at H^t and S switches from selling to buying after a price rise.

Similarly, with MLRP, at any history H^t at which the number of sales is large relative to the number of buys, we can ignore all but the first two states V_1 and V_2. At any such H^t (i) the price must have fallen and (ii) the expectation of type S must exceed that of the market expectation if $\Pr(S|V_1) < \Pr(S|V_2)$. Further, if in addition the bid-ask spread is not too large (there is enough noise trading), the expectation of type S will also exceed the ask price at H^t and S switches from selling to buying after a price fall. Formally we have:[30]

Theorem 3 (Herding and Contrarianism with n States)
Assume that signals satisfy MLRP and let signal S be negatively biased.
(a) If $\Pr(S|V_{n-1}) < \Pr(S|V_n)$ then there exist noise levels μ^{in} and $\mu^{ch} \in (0,1]$ such that

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[30]Conditions that ensure sell herding and sell contrarian can be defined analogously. In particular, to ensure the initial buy we need to assume a positive bias defined $\Pr(S|V_i) < \Pr(S|V_{n+1-i})$ for all $i < (n+1)/2$. For the switches we reverse the two conditions that ensure switching for buy herding and buy contrarian: for sell herding we need $\Pr(S|V_1) > \Pr(S|V_2)$ and for sell contrarian we need $\Pr(S|V_{n-1}) > \Pr(S|V_n)$.

type S buy herds with positive probability if $\mu < \min\{\mu^{in}, \mu^{ch}\}$.

(b) If $\Pr(S|V_1) < \Pr(S|V_2)$ there exists noise levels μ^{in} and $\mu^{ch} \in (0, 1]$ such that type S acts as a buy contrarian with positive probability if $\mu < \min\{\mu^{in}, \mu^{ch}\}$.

9   Extensions, Discussion and Conclusion

In the first part of this paper we showed under which specific circumstances herding and contrarian behavior can and cannot occur in markets with efficient prices. Our second main set of findings has been that both herding and contrarianism can reduce liquidity and increase volatility relative to situations where these kinds of social learning are absent. This is a surprising result with a potential conclusion that transparency of trading activities may hamper liquidity and increase volatility. Our characterisation result also reveals which structure of information can prevent herding or contrarian behavior; for example, it shows that mixed messages predicting extreme outcomes (U shaped signal) should be avoided, as herding is a result of such information.

In the paper we have presented the results for which we were able to obtain clear-cut analytical results. In the supplementary material, we also explore other implications with numerical simulation. First, as some types of traders change their trading modes during herding or contrarianism, prices become history-dependent. Thus as the entry order of traders is permutated, prices with the same population of traders can be strikingly different. Second, herding results in price paths that are very sensitive to changes in some key parameters. Specifically, in the case with MLRP, comparing the situation where the proportion of informed agents is just below the critical levels described in Theorem 1 with that where the proportion is just above that threshold (so there is no herding), prices deviate substantially in the two cases. Third, herding slows down the convergence to the true value if the herd moves away from that true value, but it accelerates convergence if the herd moves into the right direction. The differences in speeds of convergence speak to the prevalence of herding.

As mentioned in the introduction, Avery and Zemsky (1998), AZ, argue that herd behavior with informationally efficient asset prices is not possible unless signals are "non-monotonic" and risk is "multi-dimensional". In the rest of this concluding section, we explain why our conclusions differ from theirs.

AZ reach their conclusions by (i) showing that herding is not possible when the information structure satisfies their definition of monotonicity and (ii) demonstrating the possibility of herding by providing a specific example that that has a special "multi-dimensional" information structure. AZ argue that it is this information structure's inherent non-monotonicity that triggers herding. Our herding characterization (in terms of U shaped signals) however holds even with (and also without) the standard MLRP

monotonicity assumption. AZs' conclusion differs from ours because their definition is non-standard. Specifically, they define monotonicity by:

$$\forall S, \ \exists w(S) \text{ s.t. } \forall H^t, \ \mathsf{E}(V|H^t, S) \text{ is weakly between } w(S) \text{ and } \mathsf{E}(V|H^t) \qquad (9)$$

This definition does not imply nor is implied by the standard MLRP definition of monotonicity. Also, it is not a condition on the primitives, i.e. on the signal distribution, but it is a requirement on endogenous variables that must hold for *all trading histories*.[31] Furthermore, it precludes herding almost by definition; for example, if $\mathsf{E}[V|S] < \mathsf{E}[V]$ and the price rises, which must hold for buy herding, (9) implies immediately that $w(S) \leq \mathsf{E}[V|H^t, S] \leq \mathsf{E}[V|H^t]$ for any H^t and hence buying is not possible.

AZ's example of herding also has the same three-states–three-signals structure as in our set-up, and uses *Event Uncertainty*, a concept first employed by Easley and O'Hara (1992). Event Uncertainty is an interesting example and a special case of the information structures that we identify as the causes for herding and contrarian behavior. Many of the real-world examples that we list in the introduction were inspired by the general intuition behind event uncertainty. The idea behind Event Uncertainty as used by AZ is that first, informed agents know *if* something has happened. Second, they receive noisy information about how this event has influenced the asset's liquidation value. Formally, the information matrix in AZ's example has $\mathsf{Pr}(S_i|V_i) = q > 1/2$ and $\mathsf{Pr}(S_i|V_2) = 0$ for $i = 1, 3$, and $\mathsf{Pr}(S_2|V_2) = 1$.

AZ attribute herding in their example to the signals of the informed investors inducing a finer partition of the set of states than the market maker: an informed trader excludes V_2 if their signal is S_1 or S_3 and excludes V_1 and V_3 if he receives signal S_2. As we have shown, it is not the finer partition (multidimensionality of risk) per se that generates herding but it is the U shaped nature of a signal that is both necessary and (almost) sufficient for herding. In fact, the event uncertainty in AZ is a good example of U shaped signals (with degenerate csds): The two types S_1 and S_3 who know that the event has happened are the herding candidates. These two types have U shaped signals as these signals do not happen in the middle state, V_2.[32] Also, notice that our sufficiency results in Section 5 demonstrate that there would also be herding if the AZ example is perturbed in such a way that all signals occur with positive probability in all states, while maintaining the U shaped nature

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[31]Condition (9) does not imply that each signal has a monotonic csd; however, one can to show that if S has a monotonic csd then S satisfies (9).

[32]It is important to note that in AZ's example herding does not constitute an informational cascade either, as not all types take the same action. To see this observe that in AZ's example the informed's private information is either $\{S_1, S_3\}$ or S_2. Thus, when an informed trader receives information S_1 or S_3 and herds, there is no informed trader who receives signal S_2. In this case, at the point at which herding takes place one can say that all potential informed traders (receiving either S_1 or S_3) act alike. However, at any herding history (at which an informed trader with either S_1 or S_3 herds) if there is an informed trader who receives the middle signal S_2 then he will trade in the opposite direction to the S_1 and S_3 types. Further, at any herding history the market maker must take into account that there may be an S_2 type.

of signals S_1 and S_3. Since such a perturbed information structure would no longer have multidimensional risk (the partition of the informed trader would be the same as that of the market maker) it follows that multidimensionality is relevant to herding only to the extent that it generates a U shaped signal.

Finally, AZ's herding outcome has limited capacity to explain price volatility as price movements during herding are strictly limited. For informed agents, herding trades do not convey information, thus traders' expectations do not move. To break buy herding (sell herding), it suffices that prices rise above (fall below) the (constant) expectation of S_1 (S_3) types, and this is generally a very small movement.[33] This lack of price movements and the fragility of buy herding (sell herding) after small price rises (falls) is in sharp contrast to our results where herding can persist and prices may move significantly during herding.

A Appendix: Omitted Proofs

A.1 Proof of Proposition 2

To save space we shall prove the result for the case of buy herding and buy contrarian; the proof for the sell cases are analogous. Therefore, suppose that S buy herds or acts as a buy contrarian at some H^t. Then the proof is in several steps.

Step 1: *S must have a negative bias:* It follows from the definition of buy herding and buy contrarian that $\mathsf{E}[V|S] < \mathsf{bid}_1$. Since $\mathsf{bid}_1 < \mathsf{E}[V]$ we must have $\mathsf{E}[V|S] < \mathsf{E}[V]$. Then by Lemma 3, S must have a negative bias.

Step 2: $(\mathsf{Pr}(S|V_1) - \mathsf{Pr}(S|V_2))(q_3^t - q_1^t) > 0$: It follows from the definition of buy herding and buy contrarian that $\mathsf{E}[V|S, H^t] > \mathsf{ask}^t$. Since $\mathsf{E}[V|H^t] < \mathsf{ask}^t$ we must have $\mathsf{E}[V|S, H^t] > \mathsf{E}[V|H^t]$. By Lemma 1, this implies that the RHS of (2) is positive at H^t. Also, by negative bias (Step 1), the third term in the RHS of (2) is negative. Therefore, the sum of the first two terms in the RHS of (2) is positive: $q_3^t(\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2)) + q_1^t(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1)) > 0$. But this means, by negative bias, that $(\mathsf{Pr}(S|V_1) - \mathsf{Pr}(S|V_2))(q_3^t - q_1^t) > 0$.

Step 3a: *If S buy herds at H^t then S is nU shaped*: It follows from the definition of buy herding that $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. By Lemma 4, this implies that $q_3^t > q_1^t$. Then it follows from Step 2 that $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2)$. Also, since S buy-herds, by Lemma 1, S cannot have a decreasing csd. Therefore, we must have $\mathsf{Pr}(S|V_2) < \mathsf{Pr}(S|V_3)$. Thus, S is U shaped.

Step 3b: *If S acts as a buy contrarian at H^t then S is nHill shaped.* It follows from the definition of buy contrarian that $\mathsf{E}[V|H^t] < \mathsf{E}[V]$. By Lemma 4, this implies that $q_3^t < q_1^t$. But then it follows from Step 2 that $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$. Since by Step 1 S has a negative bias, we have $\mathsf{Pr}(S|V_2) > \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_3)$. Thus S is nHill shaped.

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[33]In fact, in AZ the required price movement during herding vanishes in the limit as $\mu \to 0$ and $q \to 1/2$ (as the informativeness of the signals of the informed agents disappears); see Proposition 8 in AZ.

A.2 Proof of Lemma 7

First we establish (i) and (iv). Consider the second terms in (3) and (4) given by $\beta_2^t \mathsf{Pr}(S|V_3) - \beta_3^t \mathsf{Pr}(S|V_2)$ and $\sigma_2^t \mathsf{Pr}(S|V_3) - \sigma_3^t \mathsf{Pr}(S|V_2)$, respectively. In case (i) the former and in case (iv) the latter equal $\gamma m^2 + \mu M^2(S')$. Also, note that $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$ in (i) and $\mathsf{Pr}(S|V_3) < \mathsf{Pr}(S|V_2)$ in (iv). Therefore, it follows from the definition of m^2 and $M^2(S')$ that in case (i) we have $\beta_2^t \mathsf{Pr}(S|V_3) - \beta_3^t \mathsf{Pr}(S|V_2) > 0$ if and only if $\mu < \mu_2^{ch}(S')$ and in case (iv) we have $\sigma_2^t \mathsf{Pr}(S|V_3) - \sigma_3^t \mathsf{Pr}(S|V_2) < 0$ if and only if $\mu < \mu_2^{ch}(S')$.

The proofs of (ii) and (iii) are analogous. Consider the first term in (3) and (4) given by $\sigma_1^t \mathsf{Pr}(S|V_2) - \sigma_2^t \mathsf{Pr}(S|V_1)$ and $\beta_1^t \mathsf{Pr}(S|V_2) - \beta_2^t \mathsf{Pr}(S|V_1)$, respectively. In case (ii) the former and in case (iii) the latter equal $\gamma m^1 + \mu M^1(S')$. Also, note that $\mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2)$ in (ii) and $\mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2)$ in (iii). Therefore, it follows from the definition of m^1 and $M^1(S')$ that in case (ii) we have $\sigma_1^t \mathsf{Pr}(S|V_2) - \sigma_2^t \mathsf{Pr}(S|V_1) < 0$ if and only if $\mu < \mu_1^{ch}(S')$ and in case (iii) we have $\beta_1^t \mathsf{Pr}(S|V_2) - \beta_2^t \mathsf{Pr}(S|V_1) > 0$ if and only if $\mu < \mu_1^{ch}(S')$.

A.3 Proof of Lemma 8

Consider case (i). Since by assumption S has a negative bias and $\mu < \mu_s^{in}$, it follows from Lemma 5 that S sells at the initial history. Also, since S has a U shape we have $\mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Therefore, by $\mu < \mu_2^{ch}$ and Lemma 7 (i), there exists some $\eta > 0$ such that the second term in (3) always exceeds η.

By condition (5) there exists a history H^t such that $q_1^t / q_3^t < 1$ and $\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t} < \eta$. Then by the former inequality and Lemma 4 we have $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. Also, since the sum of the second and the third term in (3) is greater than $-q_2 q_3(\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t})$, it follows from $\frac{q_1^t}{q_3^t} + \frac{2q_1^t}{q_2^t} < \eta$ that the sum must also be greater than $-\eta$. This, together with the first term in (3) exceeding η, implies that (3) is greater than zero, and hence S must be buying at H^t.

The proofs of $(ii) - (iv)$ are analogous and will be omitted to save space.

A.4 Proof of the Necessity Part of Theorem 1

Consider case (i) of the theorem in which S_2 buy herds (the proof of the other cases are similar and therefore omitted). Since at the initial history S_2 sells, by Lemma 5, we must have $\mu < \mu_s^{in}$. Now by Lemma 2, type S_3 always buys and S_1 always sells; therefore since S_2 buy herds at some history, it follows from Lemma 7 that $\mu < \mu_2^{ch}(S_3)$. Also by Lemma 2, type S_1 is strictly decreasing and S_3 is strictly increasing; thus it follows from the definition of $\mu_2^{ch}(.)$ that $\mu_2^{ch}(S_1) = 1$ and therefore $\mu_2^{ch}(S_3) = \mu_2^{ch}$.

A.5 Proof of Lemma 9

Consider any arbitrary history H^t and any two values $V_l < V_h$. Then $\beta_l^t / \beta_h^t < 1$. To see this note, first, as S_1 is strictly decreasing, by Proposition 1 S_1 never buys. Then if there

is a buy at H^t it must be that either S_3 types buy and S_1 and S_2 types sell or S_2 and S_3 types buy and S_1 types sell. In the former case, since S_3 is strictly increasing, it follows immediately that $\beta_l^t/\beta_h^t < 1$. In the latter case,

$$
\begin{aligned}
\beta_h^t - \beta_l^t &= \mu \left(\mathsf{Pr}(S_3|V_h) + \mathsf{Pr}(S_2|V_h) - \mathsf{Pr}(S_3|V_l) - \mathsf{Pr}(S_2|V_l) \right) \\
&= \mu \left(1 - \mathsf{Pr}(S_1|V_h) - (1 - \mathsf{Pr}(S_1|V_l)) \right) = \mu \left(\mathsf{Pr}(S_1|V_l) - \mathsf{Pr}(S_1|V_h) \right).
\end{aligned}
$$

Since S_1 is strictly decreasing, we have $\beta_l^t/\beta_h^t < 1$ in this case as well. But then, by the finiteness of the set of states, there must exist $\delta \in (0, 1)$ such that for every H^t and for any V_l and V_h with $V_l < V_h$, we have $\beta_l^t/\beta_h^t < \delta$.

By a similar reasoning it can be shown that there must exist $\delta \in (0, 1)$ so that $\sigma_h^t/\sigma_l^t < \delta$.

A.6 Proof of Proposition 3

Below we provide a proof for part (i) of the proposition; the arguments for the other parts are analogous and therefore omitted.

The proof of part (i) is by contradiction. Suppose that S is nU shaped and that all the other assumptions in part (i) of the proposition hold. Also assume, contrary to the claim in part (i), that S does not buy herd. Then, by Lemma 8 (i), we have a contradiction if it can be shown that (5) holds. This is indeed what we establish in the rest of the proof.

First note that the no buy herding supposition implies that S does not buy at any history H^t. Otherwise, since S has a negative bias, by Step 2 in the proof of Proposition 2, $(\mathsf{Pr}(S|V_1) - \mathsf{Pr}(S|V_2))(q_3^t - q_1^t) > 0$. Since S is U shaped this implies that $q_3^t > q_1^t$; but then since by assumption $\mu < \mu_s^{in}$, it follows from Lemma 5 that S buy herds; a contradiction.

Next, we describe conditions that ensure that q_1^t/q_l^t are decreasing $(q_1^{t+1}/q_l^{t+1} < q_1^t/q_l^t)$ for any $l = 2, 3$. Denote an infinite path of actions by $H^\infty = \{a^1, a^2, \ldots\}$. For any date t and any finite history $H^t = \{a^1, \ldots, a^{t-1}\}$, let a_k^t be the action that would be taken by type $S_k \in \mathbb{S}\backslash S$ at H^t; thus if the informed trader at date t receives a signal $S_k \in \mathbb{S}\backslash S$ then a^t, the actual action taken at H^t, equals a_k^t. Also denote the action taken by S at H^t by $a^t(S)$. Then we have the following.

Lemma I *Fix any infinite path $H^\infty = \{a^1, a^2, \ldots\}$ and any signal $S_k \in \mathbb{S}\backslash S$. Let $S_{k'} \in \mathbb{S}\backslash S$ be such that $S_{k'} \neq S_k$. Suppose that $a^t = a_k^t$. Then for any date t and $l = 2, 3$ we have:*
A. If $a_k^t = a_{k'}^t$ then q_1^t/q_l^t is strictly decreasing.
B. If $a_k^t = a^t(S)$ and the inequality $\mathsf{Pr}(S_{k'}|V_l) \leq \mathsf{Pr}(S_{k'}|V_1)$ holds then q_1^t/q_l^t is decreasing; furthermore, if the inequality is strict then q_1^t/q_l^t is strictly decreasing .
C. If $a_k^t \neq a_{k'}^t$ and $a_k^t \neq a^t(S)$ and the inequality $\mathsf{Pr}(S_k|V_l) \geq \mathsf{Pr}(S_k|V_1)$ holds then q_1^t/q_l^t is decreasing; furthermore, if the inequality is strict then q_1^t/q_l^t is strictly decreasing.

Proof of Lemma I: Fix any $l = 2, 3$. Since $\frac{q_1^{t+1}}{q_l^{t+1}} = \frac{q_1^t \mathsf{Pr}(a^t|H^t, V_1)}{q_l^t \mathsf{Pr}(a^t|H^t, V_l)}$, to establish that

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q_1^t/q_l^t is (strictly) decreasing it suffices to show that $\Pr(a^t|H^t, V_l)$ is (greater) no less than $\Pr(a^t|H^t, V_1)$. Now consider each of the three cases $A. - C.$

A. Since signal S is nU shaped, the combination of S_k and $S_{k'}$ is pHill shaped. This together with $a^t = a_k^t = a_{k'}^t$ imply that $\Pr(a^t|H^t, V_l)$ exceeds $\Pr(a^t|H^t, V_1)$.

B. If $\Pr(S_{k'}|V_l) \leq \Pr(S_{k'}|V_1)$ we have $\Pr(S_k|V_l) + \Pr(S|V_l) \geq \Pr(S_k|V_1) + \Pr(S|V_1)$. This, together with $a^t = a_k^t = a^t(S)$ imply that $\Pr(a^t|H^t, V_l) \geq \Pr(a^t|H^t, V_1)$. Furthermore, the latter inequality must be strict if $\Pr(S_{k'}|V_l)$ were less than $\Pr(S_{k'}|V_1)$.

C. If $\Pr(S_k|V_l) \geq \Pr(S_k|V_1)$ and $a_k^t \neq a_{k'}^t$ and $a_k^t \neq a^t(S)$ we have immediately that $\Pr(a^t|H^t, V_l) \geq \Pr(a^t|H^t, V_1)$. Furthermore, the latter inequality is strict if $\Pr(S_k|V_l)$ were less than $\Pr(S_k|V_1)$. This concludes the proof of Lemma I.

Now we show that (5) holds and thereby obtain the required contradiction. This will be done for each feasible csd combination of signals.

Case A: Either there exists a signal that is decreasing or there are two Hill shaped signals each with a non-negative bias.

Consider an infinite path of actions consisting of an infinite number of buys. We demonstrate (5) by showing that along this infinite history at any date t both q_1^t/q_2^t and q_1^t/q_3^t are decreasing, and hence converge to zero (note that there are a finite number of states and signals). We show this in several steps.

Step 1: If more than one informed type buy at t then q_1^t/q_2^t and q_1^t/q_3^t are both decreasing at any t: Since S does not buy at any t, this follows immediately from Lemma I.A.

Step 2: If exactly one informed type buys at period t then (i) q_1^t/q_2^t is decreasing and (ii) q_1^t/q_3^t is decreasing if the informed type that buys has a non-zero bias, and is non-increasing otherwise: Let S_i be the only type that buys at t. This implies that S_i cannot be decreasing; therefore, by assumption, S_i must be pHill shaped and the step follows from Lemma I.C.

Step 3: If a type has a zero bias he cannot be a buyer at any date t: Suppose not. Then there exist a type S_i with a zero bias such that $\mathsf{E}[V|H^t, S_i] - \mathsf{E}[V|H^t] > 0$. By Lemma 1 we then have

$$[\Pr(S_i|V_3) - \Pr(S_i|V_2)](q_3^t - q_1^t) > 0. \tag{10}$$

Also, by Steps 1 and 2, q_1^t/q_3^t is non-increasing at every t. Moreover by assumption $q_1^1/q_3^1 = 1$. Therefore, $q_1^t/q_3^t \leq 1$. Since S_i buys at t, S_i must be Hill shaped, contradicting (10).

Step 4: q_1^t/q_2^t and q_1^t/q_3^t are both decreasing at any t. This follows Steps 1-3.

Case B: There exists an increasing S_i s.t. $\Pr(S_i|V_k) \neq \Pr(S_i|V_{k'})$ for some k and k'.

Let S_j be the third signal other than S and S_i. Now we obtain (5) in two steps.

Step 1: If $\Pr(S_i|V_1) = \Pr(S_i|V_2)$ then for any $\epsilon > 0$ there exists a finite history $H^\tau = \{a^1, \ldots, a^{\tau-1}\}$ such that $q_1^\tau/q_2^\tau < \epsilon$. Consider an infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t = a_j^t$ (recall that a_j^t is the action taken by S_j at history $H^t = (a^1, \ldots, a^{t-1})$). Note that S is nU shaped and $\Pr(S_i|V_1) = \Pr(S_i|V_2) < \Pr(S_i|V_3)$. Therefore, $\Pr(S_j|V_2) >$

$\max\{\Pr(S_j|V_1), \Pr(S_j|V_3)\}$.

Then it follows from Lemma I that q_1^t/q_2^t is decreasing if $a^t \neq a^t(S)$ and it is constant if $a^t = a^t(S)$. To establish the claim it suffices to show that $a^t \neq a^t(S)$ infinitely often. Suppose not. Then there exists T such that for all $t > T$, $a_j^t = a^t(S)$. Since type S does not buy at any date and there cannot be more than one informed type holding at any date (there is always a buyer or a seller), we must have S_j (and S) selling at at every $t > T$. Then, by Lemma 1 we have

$$\frac{q_3^t}{q_1^t}[\Pr(S_j|V_3) - \Pr(S_j|V_2)] + [\Pr(S_j|V_2) - \Pr(S_j|V_1)] + 2\frac{q_3^t}{q_2^t}[\Pr(S_j|V_3) - \Pr(S_j|V_1)] < 0. \quad (11)$$

for all $t > T$. Also, by $\Pr(S_i|V_1) = \Pr(S_i|V_2) < \Pr(S_i|V_3)$ we have $\Pr(S_j|V_l) + \Pr(S|V_l) > \Pr(S_j|V_3) + \Pr(S|V_3)$ for $l = 1, 2$. Therefore, $\frac{q_3^t}{q_l^t} \to 0$ as $t \to \infty$ for any $l = 1, 2$. This, together with $\Pr(S_j|V_2) > \Pr(S_j|V_1)$, contradict (11).

Step 2: For any $\epsilon > 0$ there exists a history H^t s.t. $q_1^t/q_l^t < \epsilon$ for any $l = 2, 3$: Fix any $\epsilon > 0$. Let H^τ be such that $q_1^\tau/q_2^\tau < \epsilon$ if $\Pr(S_i|V_1) = \Pr(S_i|V_2)$ (by the previous step such a history exists) and be the empty history H^1, otherwise. Consider any infinite path $H^\infty = \{H^\tau, a^\tau, a^{\tau+1}, \dots\}$, where for any $t \geq \tau$, a^t is the action that type S_i takes at history $H^t = \{H^\tau, a^\tau, \dots, a^{t-1}\}$; i.e. we first have the history H^τ and then we look at a subsequent history that consists only of the actions that type S_i takes.

Since S_i is increasing it follows from Proposition 1 that at any history S_i does not sell. Also, by the supposition S does not buy at any history. Therefore, S_i and S always differ at every history H^t with $t \geq \tau$ (there cannot be more than one type holding). But since a^t is the action that type S_i takes at history H^t, S_i is increasing and $\Pr(S_i|V_k) \neq \Pr(S_i|V_{k'})$ for some k and k', it then follows from part A and C of Lemma I that for every $t \geq \tau$ (i) $\frac{q_1^t}{q_3^t}$ is decreasing, (ii) $\frac{q_1^t}{q_2^t}$ is non-increasing. This, together with $q_1^\tau/q_2^\tau < \epsilon$ when $\Pr(S_i|V_1) = \Pr(S_i|V_2)$, establishes that there exists t such that $q_1^t/q_l^t < \epsilon$ for any $l = 2, 3$.

Case C: There are two hill shaped signals and one has a negative bias.

Let S_i be the Hill shaped signal with the negative bias. Also, let S_j be the other Hill shaped signal. Since both S and S_i have negative biases, S_j must have a positive bias.

Next fix any $\epsilon > 0$ and define y and φ_{lm}, for any $l, m = 1, 2, 3$, as follows:

$$y := \frac{[\Pr(S_i|V_2) - \Pr(S_i|V_1)]}{2[\Pr(S_i|V_1) - \Pr(S_i|V_3)]} > 0$$

$$\varphi_{lm} := \max\left\{\frac{\gamma + \mu\Pr(S_i|V_l)}{\gamma + \mu\Pr(S_i|V_m)}, \frac{\gamma + \mu(1 - \Pr(S|V_l))}{\gamma + \mu(1 - \Pr(S|V_m))}, \frac{\gamma + \mu\Pr(S_j|V_l)}{\gamma + \mu\Pr(S_j|V_m)}\right\}. \quad (12)$$

Since both S_i and S_j are hill shaped we have $\varphi_{12} < 1$. This implies that there exists an integer $M > 0$ and $\delta \in (0, \epsilon)$ such that $y(\varphi_{12})^M < \epsilon$ and $\delta(\varphi_{13})^M < \epsilon$.

Consider the infinite path $H^\infty = \{a^1, a^2, \dots\}$ where $a^t = a_j^t$ at every t. Then we have:

Claim 1: q_1^t/q_3^t is decreasing at every t: As S_i and S_j have a negative and a positive bias respectively, by Lemma I, q_1^t/q_3^t is decreasing at every t.

Claim 2: q_1^t/q_2^t converge to zero if there exists T such that $a_i^t \neq a_j^t$ for all $t > T$: Since S_j is Hill shaped this follows immediately from parts A and C of Lemma I.

Claim 3: There exists a history H^τ s.t. $q_1^\tau/q_3^\tau < \delta$ and $q_1^\tau/q_2^\tau < y$: Suppose not; then by Claims 1 and 2 there exists a date τ such that $q_1^\tau/q_3^\tau < \delta$ and $a_i^\tau = a_j^\tau$. Since S does not buy at any history, it follows that S_i and S_j must be buying at τ (there is always at least one buyer and seller; thus S_i and S_j cannot both be holding at τ). Then, $\mathsf{E}[V|S_i, H^t] - \mathsf{E}[V|H^t] > 0$. By Proposition 2, this implies

$$[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\Pr(S_i|V_2) - \Pr(S_i|V_1)] + \frac{2q_1^\tau}{q_2^\tau}[\Pr(S_i|V_3) - \Pr(S_i|V_1)] > 0.$$

Since S_i is nHill shaped, it follows from the last inequality that

$$\frac{q_1^\tau}{q_2^\tau} < \frac{[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\Pr(S_i|V_2) - \Pr(S_i|V_1)]}{2[\Pr(S_i|V_1) - \Pr(S_i|V_3)]} < \frac{\frac{q_1^\tau}{q_3^\tau}[\Pr(S_i|V_2) - \Pr(S_i|V_1)]}{2[\Pr(S_i|V_1) - \Pr(S_i|V_3)]}. \tag{13}$$

As $q_1^\tau/q_3^\tau < \delta$ and $\delta < 1$, we have $q_1^\tau/q_2^\tau < y$. This contradicts the supposition.

To complete the proof for this case, fix any τ and H^τ such that $q_1^\tau/q_3^\tau < \delta$ and $q_1^\tau/q_2^\tau < y$ (by Claim 3 such a history exists). Consider a history $\overline{H}^t$ that consists of path $H^\tau = (a^1, \ldots, a^{\tau-1})$ followed by M periods of buys. Thus $t = \tau + M$ and $H^t = \{H^\tau, \overline{a}^1, \ldots, \overline{a}^M\}$, where for any $m \leq M$, $\overline{a}^m = $ buy. Since a buy must be either from S_j or S_i or both, it then follows from the definitions of φ_{13}, M and δ, and from $q_1^\tau/q_3^\tau < \delta$ that

$$q_1^t/q_3^t \leq (\varphi_{13})^M(q_1^\tau/q_3^\tau) < (\varphi_{13})^M\delta < \epsilon. \tag{14}$$

Also, since $q_1^\tau/q_2^\tau < y$ we have
$$q_1^t/q_2^t < (\varphi_{12})^M(q_1^\tau/q_2^\tau) < (\varphi_{12})^M y < \epsilon. \tag{15}$$

Since the initial choice of ϵ was arbitrary, (5) follows immediately from (14) and (15).

Case D: There exists a U shaped signal $S_i \in \mathbb{S}\backslash S$.

Since both S and S_i are U shaped it follows that the third signal S_j is Hill shaped. Moreover, by assumption S_j must have a non-negative bias.

To establish (5) fix any $\epsilon > 0$ and consider the two possible subcases that may arise.

Subcase D1: Either S_i or S_j has a zero bias.

Consider the infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t = $ buy. Since a buy must be either from S_j or S_i or both, and either S_i or S_j has a zero bias, it follows from parts A and C of Lemma I that q_1^t/q_3^t is non-increasing at every t. Furthermore, q_1^t/q_3^t is decreasing if $a^t = a_l^t$ and S_l has a positive bias. Next we establish the following claims.

Claim 1: There exists a history H^τ such that $q_1^\tau/q_3^\tau < \epsilon$: By assumption either S_i or S_j has a zero bias. First assume that S_i has a zero bias; thus S_j must have a positive bias.

Since q_1^t/q_3^t is decreasing if $a^t = a_j^t$, the claim follows if S_j buys infinitely often along the path H^∞. To show the latter suppose not; then there exists T such that for all $t \geq T$, $a_j^t \neq a_i^t =$buy. But then for all $t > T$, by Lemma 6,

$$\frac{q_2^t}{q_1^t}[\beta_2^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_2)] + \frac{q_2^t}{q_3^t}[\beta_1^t\mathsf{Pr}(S_j|V_2) - \beta_2^t\mathsf{Pr}(S_j|V_1)] + [\beta_1^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_1)] < 0, \quad (16)$$

Also, since S_i is U shaped both $\frac{q_2^t}{q_1^t}$ and $\frac{q_2^t}{q_3^t}$ must be decreasing at every $t > T$. But this is a contradiction because at every $t > T$, the last term in (16) is positive: $\beta_1^t\mathsf{Pr}(S_j|V_3) - \beta_3^t\mathsf{Pr}(S_j|V_1) = \gamma(\mathsf{Pr}(S_j|V_3) - \mathsf{Pr}(S_j|V_1)) > 0$ (the equality follows from S_i's zero bias).

Second, assume that S_j has a zero bias; thus S_i has a positive bias. Since S_j is also Hill shaped, by exactly the same reasoning as in Step 3 in Case A, S_j cannot be a buyer. Therefore, $a_j^t \neq a_i^t =$ buy at every t. But this, together with the positive bias of S_i, implies that q_1^t/q_3^t is decreasing at all t and therefore the claim must hold.

Claim 2: There exists a history H^t such that $q_1^t/q_l^t < \epsilon$ for all $l = 2, 3$: By the previous claim there exists a history H^τ such that $q_1^\tau/q_3^\tau < \epsilon$. Next, consider a history $H^\infty = \{H^\tau, a^\tau, a^{\tau+1}, \ldots\}$ that consists of path H^τ followed by a sequence of actions $\{a^\tau, a^{\tau+1}, \ldots\}$ such that $a^t = a_j^t$ at every history $H^t = \{H^\tau, a^\tau, \ldots, a^{t-1}\}$. Since either S_i or S_j has a zero bias, it follows from Lemma I that at every $t > \tau$, q_1^t/q_3^t is non-increasing. Also, we have $q_1^\tau/q_3^\tau < \epsilon$; therefore we have that at every $t > \tau$, $q_1^t/q_3^t < \epsilon$. Furthermore, since S and S_i are U shaped, and S_j is Hill shaped, by Lemma I, q_1^t/q_2^t is decreasing at every $t > \tau$; hence there must exists $t > \tau$ such that $q_1^t/q_2^t < \epsilon$.

Since the initial choice of ϵ was arbitrary, (5) follows from Claim 2.

Subcase D2: Both S_i and S_j have non-zero bias.

Consider first the infinite path $H^\infty = \{a^1, a^2, \ldots\}$ such that $a^t = a_j^t$ at every history $H^t = \{a^1, \ldots, a^{t-1}\}$. Then the following claims must hold.

Claim 1: q_1^t/q_2^t is decreasing at every t : Since S_j and S_i are respectively Hill shaped and U shaped, it follows from Lemma I that q_1^t/q_2^t is decreasing.

Claim 2: If S_i has a negative bias then q_1^t/q_3^t is decreasing at every t: Since S_j has a positive bias and S_i has a negative bias, by Lemma I, q_1^t/q_3^t is decreasing at every t.

Claim 3: If there exists a period T such that for all $t > T$, $a_j^t = buy$ then q_1^t/q_3^t is decreasing at every $t > T$: Since S_j has a positive bias and S does not buy at any date, by Lemma I, q_1^t/q_3^t must be decreasing at every $t > T$.

Before stating the next claim, consider φ_{ml} defined in (12). Since by assumption S has a negative bias and S_j have a positive bias, it follows that $\varphi_{13} < 1$ if S_i has positive bias. Thus, in this case there exist an integer M and a positive real number $\delta < \epsilon$ such that

$$(\varphi_{13})^M < \epsilon \text{ and } \delta(\varphi_{12})^M < \epsilon. \quad (17)$$

Claim 4: If S_i has positive bias, then there exists a history H^τ s.t. $q_1^\tau/q_2^\tau < \delta$ and

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$q_1^\tau/q_3^\tau < x$, *where satisfies δ satisfies (17) and*

$$x \equiv \frac{[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + 2\epsilon[\Pr(S_i|V_3) - \Pr(S_i|V_1)]}{[\Pr(S_i|V_1) - \Pr(S_i|V_2)]} :$$

Suppose not. Then by Claims 1 and 3 there exists date τ such that $q_1^\tau/q_2^\tau < \delta$ and $a_j^\tau \neq$ buy. Since S also does not buy at H^τ, it follows that only S_i buys at τ. Then $\mathsf{E}[V|S_i, H^\tau] > \mathsf{E}[V|H^\tau]$. By Proposition 2, this implies

$$[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + \frac{q_1^\tau}{q_3^\tau}[\Pr(S_i|V_2) - \Pr(S_i|V_1)] + \frac{2q_1^\tau}{q_2^\tau}[\Pr(S_i|V_3) - \Pr(S_i|V_1)] > 0.$$

Since $q_1^\tau/q_2^\tau < \delta < \epsilon$ and S_i is pU shaped, we can rearrange the above to show that

$$\frac{q_1^\tau}{q_3^\tau} \leq \frac{[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + \frac{2q_1^\tau}{q_2^\tau}[\Pr(S_i|V_3) - \Pr(S_i|V_1)]}{\Pr(S_i|V_1) - \Pr(S_i|V_2)} < \frac{[\Pr(S_i|V_3) - \Pr(S_i|V_2)] + 2\epsilon[\Pr(S_i|V_3) - \Pr(S_i|V_1)]}{[\Pr(S_i|V_1) - \Pr(S_i|V_2)]} = x.$$

Claim 5: If S_i has a positive bias, then there exists a history $\overline{H}^t$ s.t. $q_1^t/q_l^t < \epsilon$ for any $l = 2, 3$: Fix any history $H^\tau = (a^1, \ldots, a^{\tau-1})$ s.t. $q_1^\tau/q_2^\tau < \delta$ and $q_1^\tau/q_3^\tau < x$ (by the previous claim such a history exists). Next, consider a history $\overline{H}^t$ that consists of path H^τ followed by M periods of buys. Thus $t = \tau + M$ and $H^t = \{h^\tau, \overline{a}^1, \ldots, \overline{a}^M\}$, where for any $m \leq M$, $\overline{a}^m =$ buy. Since a buy must be either from S_j or S_i or both, it then follows from the definitions of φ_{12} in (12), from (17) and from $q_1^\tau/q_2^\tau < \delta$ that $\frac{q_1^t}{q_2^t} \leq \frac{q_1^\tau}{q_2^\tau}(\varphi_{12})^M < \delta(\varphi_{12})^M < \epsilon$. Also, since $q_1^\tau/q_3^\tau < x$, we have $\frac{q_1^t}{q_3^t} < \frac{q_1^\tau}{q_3^\tau}(\varphi_{13})^M < x(\varphi_{13})^M < \epsilon$.

Since the initial choice of ϵ was arbitrary, (5) follows from Claims 1,2 and 5.

A.7 Resilience and Fragility: Proof of Proposition 4

(a) First we demonstrate the existence of the function $\overline{s}$. Since at H^t buy herding occurs if and only if $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t > 0$ and $\mathsf{E}[V|H^t] - \mathsf{E}[V] > 0$, the existence of $\overline{s}$ is obtained by fixing b and showing (i) these two inequalities hold when $s = 0$ and (ii) neither hold for large enough s.

Let $\beta_i = \Pr(\text{buy}|V_i)$ and $\sigma_i = \Pr(\text{sale}|V_3)$ at every buy herding history (these probabilities are are always the same at every history at which S buy herds). Note that by Lemma 6(i), $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ has the same sign as

$$\left(\frac{\beta_1}{\beta_3}\right)^b \left(\frac{\sigma_1}{\sigma_3}\right)^s q_2^r q_1^r[\beta_1\Pr(S|V_2) - \beta_2\Pr(S|V_1)] + q_3^r q_2^r[\beta_2\Pr(S|V_3) - \beta_3\Pr(S|V_2)]$$
$$+ 2\left(\frac{\beta_1}{\beta_2}\right)^b \left(\frac{\sigma_1}{\sigma_2}\right)^s q_3^r q_1^r[\beta_1\Pr(S|V_3) - \beta_3\Pr(S|V_1)]. \tag{18}$$

Also, by MLRP and Lemma 9 we have

$$\beta_1 < \beta_2 < \beta_3 \text{ and } \sigma_1 > \sigma_2 > \sigma_3 \tag{19}$$

Since by Proposition 2, S must have an nU shaped csd it then follows that

$$\beta_1\Pr(S|V_2) - \beta_2\Pr(S|V_1) < 0, \quad \beta_1\Pr(S|V_3) - \beta_3\Pr(S|V_1) < 0, \quad \beta_2\Pr(S|V_3) - \beta_3\Pr(S|V_2) > 0. \tag{20}$$

(the last inequality in (20) follows from the first two and from (3) being positive at H^r). Thus, the first and the third terms in (18) are negative, the second is positive. Hence, it follows from (19) that the expression in (18) increases in b and decreases in s. Also, by (19), we must have that $\mathsf{E}(V|H^t)$ increases in b and decreases in s (note that q_3^t/q_1^t is increasing in b and decreasing in s).

By assumption there is buy herding at H^r. Therefore, both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive when both b and s are equal to zero. Since both (18) and $\mathsf{E}(V|H^t)$ are increasing in b and decreasing in s, it then follows respectively that (i) for any b both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive when $s = 0$ and (ii) for any b both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are negative for large enough values of s. These two conclusions, together with (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ being decreasing in s, imply that there exists an integer $\bar{s} > 1$ such that both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive for any integer $s < \bar{s}$, and either (18) or $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are non-positive for any integer $s \geq \bar{s}$.

To complete the proof of this part we need to show that $\bar{s}$ is increasing in b. To show this suppose otherwise; then there exists b' and b'' such that $b' < b''$ and $\bar{s}' > \bar{s}''$ where $\bar{s}'$ and $\bar{s}''$ are respectively the critical values of sales corresponding to b' and b'' described in the previous paragraph. Now since $\bar{s}' > \bar{s}''$ it follows that both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive if $b = b'$ and $s = \bar{s}''$. But since both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are increasing in b, we must then have that both (18) and $\mathsf{E}(V|H^t) - \mathsf{E}(V)$ are positive if $b = b''$ and $s = \bar{s}''$. By the definition of $\bar{s}''$ this is a contradiction.

(b) By Lemma 6(i), $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ has the same sign as

$$
q_2^r q_1^r [\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)] + \left(\frac{\beta_3}{\beta_1}\right)^b \left(\frac{\sigma_3}{\sigma_1}\right)^s q_3^r q_2^r [\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)] \\
+ 2 \left(\frac{\beta_3}{\beta_2}\right)^b \left(\frac{\sigma_3}{\sigma_2}\right)^s q_3^r q_1^r [\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)]. \tag{21}
$$

Also, with buy contrarianism S must have an nHill shaped csd and therefore $\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1) > 0$, $\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1) < 0$, and $\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2) < 0$. Thus, the second and the third terms in (21) are negative, and the first is positive. Hence, it follows from (19) that the expression in (21) increases in s and decreases in b.

The remainder of the argument is then analogous to that for part (a), with reversed roles for buys and sales (i.e. one needs to show first that for any s (i) (21) is positive and $\mathsf{E}(V|H^t) - \mathsf{E}(V) < 0$ when $b = 0$ and (ii) (21) is negative and $\mathsf{E}(V|H^t) - \mathsf{E}(V) > 0$ when b is sufficiently large; then use (i) and (ii) to demonstrate the existence of $\bar{b}$).

A.8 The Range of Herding Prices: Proof of Proposition 5

(a) In the proof of Proposition 4 we have shown for the case of buy herding that if the history following H^r consists only of buys, then type S herds at any point during that history. What remains to be shown is that for an arbitrary number of buys after herding

has started, the price will approach V_3. Observe that $\mathsf{E}[V|H^t] = \sum_i V_i q_i^t = q_3^t \left(\frac{q_2^t}{q_3^t} V_2 + V_3 \right)$. Also, q_2^t/q_3^t is arbitrarily small at any history H^t that includes a sufficiently large number of buys (see the discussion that follows Lemma 9). Consequently, for every $\epsilon > 0$, there exists a finite history of length $t = r + b$, consisting of H^r followed by sufficiently many buys b, such that $\mathsf{E}[V|H^t] > V_3 - \epsilon$.

Similarly, for the case of buy contrarianism, if the history following H^r consists only of sales then type S acts as a contrarian at that history. Also, $\mathsf{E}[V|H^t] = \sum_i V_i q_i^t = q_1^t \left(V_2 \frac{q_2^t}{q_1^t} + V_3 \frac{q_3^t}{q_1^t} \right)$. Furthermore, q_i^t/q_1^t, $i = 2, 3$, is arbitrarily small at any history H^t, $t = r + s$, that includes a sufficiently large number of sales s. Hence, for any ϵ there is exists a history of length $t = r + s$, consisting of H^r followed by sufficiently many sales s during buy contrarianism, such that $\mathsf{E}[V|H^t] < V_1 + \epsilon$.

(b) First note that since $\mu < \mu_2^{ch}$ there exists $\eta > 0$ such that

$$[\beta_2^t \mathsf{Pr}(S|V_3) - \beta_3^t \mathsf{Pr}(S|V_2)] > \eta, \text{ for every } t. \tag{22}$$

As signals satisfy MLRP, without loss of generality signal S_3 is strictly increasing so that type S_3 always buys. Now fix any $r > 1$ such that $\frac{q_i^r}{q_3^r} = \left(\frac{\gamma + \mu \mathsf{Pr}(S_3|V_i)}{\gamma + \mu \mathsf{Pr}(S_3|V_3)} \right)^{r-1} < \eta/2$, for $i = 1, 2$. (Since $\mathsf{Pr}(S_3|V_i) < \mathsf{Pr}(S_3|V_3)$ such a r exists.) Let H^r be the history consisting only of $r - 1$ buys. Then it follows from (22) that

$$[\beta_2^r \mathsf{Pr}(S|V_3) - \beta_3^r \mathsf{Pr}(S|V_2)] + \frac{q_1^r}{q_3^r}[\beta_1^r \mathsf{Pr}(S|V_2) - \beta_2^r \mathsf{Pr}(S|V_1)] > \eta/2. \tag{23}$$

Next, fix any $\epsilon > 0$ and note that there exists $\delta > 0$ such that if $q_2^1 > 1 - \delta$ then $\mathsf{ask}^r = \mathsf{E}[V|H^r, \text{buy}] = q_2^r V_2 + q_3^r V_3 \in (V_2, V_2 + \epsilon)$ and

$$2 \frac{q_1^r}{q_2^r}[\beta_3^r \mathsf{Pr}(S|V_1) - \beta_1^r \mathsf{Pr}(S|V_3)] < \eta/2. \tag{24}$$

Hence, if $q_2^1 > 1 - \delta$ it follows from (23) and (24) that

$$q_2^r q_3^r [\beta_2^r \mathsf{Pr}(S|V_3) - \beta_3^r \mathsf{Pr}(S|V_2)] + q_1^r q_2^r [\beta_1^r \mathsf{Pr}(S|V_2) - \beta_2^r \mathsf{Pr}(S|V_1)] + 2\, q_1^r q_3^r [\beta_1^r \mathsf{Pr}(S|V_3) - \beta_3^r \mathsf{Pr}(S|V_1)] > 0.$$

This, together with $\mathsf{ask}^r \in (V_2, V_2 + \epsilon)$, establish that if $q_2^1 > 1 - \delta$ then at H^r there is buy herding and the ask price belongs to the interval $(V_2, V_2 + \epsilon)$.

Next, as shown in part (a), there must also exist a history H^t with $t = r + b$ following H^r such that there is buy herding at any history H^τ, $r \le \tau \le t$, and $\mathsf{E}[V|H^t] > V_3 - \epsilon$.

The arguments for buy contrarianism are analogous except that one uses $s - 1$ sales.

A.9   Proof of Propositions 6 and 7

We shall prove the two results for the case of buy herding; the proof for the buy contrarian case is analogous and will be omitted.

Proof of part (a) of Propositions 6 and part 1(a) of 7. Let β_i and σ_i be respectively the probability of a buy and the probability of a sale in the transparent world

at any date $\tau = r, .., r + b + s$. Also, let $\beta_{i,o}$ and $\sigma_{i,o}$ be the analogous probabilities in the opaque world. Then $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] = \mathcal{V}\{(q_2^t - q_{2,o}^t) + 2(q_3^t - q_{3,o}^t)\}$

$$= \mathcal{V}\left\{ q_2^r \left(\frac{\beta_2^b \sigma_2^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{2,o}^b \sigma_{2,o}^s}{\sum_i q_{i,o}^r \beta_{i,o}^b \sigma_{i,o}^s} \right) + 2q_3^r \left(\frac{\beta_3^b \sigma_3^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{3,o}^b \sigma_{3,o}^s}{\sum_i q_i^r \beta_{i,o}^b \sigma_{i,o}^s} \right) \right\}.$$

Therefore, $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r [(\beta_2 \beta_{1,o})^b (\sigma_2 \sigma_{1,o})^s - (\beta_{2,o} \beta_1)^b (\sigma_{2,o} \sigma_1)^s] + q_3^r q_2^r [(\beta_3 \beta_{2,o})^b (\sigma_3 \sigma_{2n})^s - (\beta_{3,o} \beta_2)^b (\sigma_{3,o} \sigma_2)^s]$$
$$+ 2q_3^r q_1^r [(\beta_3 \beta_{1,o})^b (\sigma_3 \sigma_{1,o})^s - (\beta_{3,o} \beta_1)^b (\sigma_{3,o} \sigma_1)^s]. \tag{25}$$

Suppose that S buy herds at H^r. Then, by Step 1 of Lemma 6, we have

$$q_2^r q_1^r [\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)] + q_3^r q_2^r [\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)]$$
$$+ 2\, q_3^r q_1^r [\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)] > 0. \tag{26}$$

By simple computation we also have

$$\begin{aligned}
\beta_2 \beta_{1,o} - \beta_{2,o} \beta_1 &= \mu[\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)], \\
\beta_3 \beta_{1,o} - \beta_{3,o} \beta_1 &= \mu[\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1)], \\
\beta_3 \beta_{2,o} - \beta_{3,o} \beta_2 &= \mu[\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2)].
\end{aligned} \tag{27}$$

Therefore, it follows from (26) that

$$q_2^r q_1^r [\beta_2 \beta_{1,o} - \beta_{2,o} \beta_1] + q_3^r q_2^r [\beta_3 \beta_{2,o} - \beta_{3,o} \beta_2] + 2q_3^r q_1^r [\beta_3 \beta_{1,o} - \beta_{3,o} \beta_1] > 0. \tag{28}$$

Next suppose that $b = 1$ and $s = 0$ (thus $t = r + 1$). Then, by from (25) and (28), we have $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] > 0$. Since in this case H^t is simply H^r followed by a buy, it follows that $\mathsf{E}[V|H^t]$ and $\mathsf{E}_o[V|H^t]$ are respectively the ask price at H^r when S buys and the ask price when S does not. This, together with $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] > 0$, completes the proof of (a) in Proposition 6.

To prove $1(a)$ in Proposition 7 suppose that $s = 0$ (thus $t = b$). Then by expanding (25) it must be that $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \left\{ (\beta_2 \beta_{1,o} - \beta_{2,o} \beta_1) \sum_{\tau=0}^{b-1} (\beta_2 \beta_{1,o})^{b-1-\tau} (\beta_{2,o} \beta_1)^\tau \right\} \tag{29}$$
$$+ q_3^r q_2^r [(\beta_3 \beta_{2,o}) - (\beta_{3,o} \beta_2)] \sum_{\tau=0}^{b-1} (\beta_3 \beta_{2,o})^{b-1-\tau} (\beta_{3,o} \beta_2)^\tau$$
$$+ 2q_3^r q_1^r \left\{ (\beta_3 \beta_{1,o} - \beta_{3,o} \beta_1) \sum_{\tau=0}^{b-1} (\beta_3 \beta_{1,o})^{b-1-\tau} (\beta_{3,o} \beta_1)^\tau \right\}.$$

Also, by MLRP $\beta_3 > \beta_2 > \beta_1$ and $\beta_{3,o} > \beta_{2,o} > \beta_{1,o}$. Therefore,

$$\sum_{\tau=0}^{b-1} (\beta_3 \beta_{2,o})^{b-1-\tau} (\beta_{3,o} \beta_2)^\tau > \sum_{\tau=0}^{b-1} (\beta_2 \beta_{1,o})^{b-1-\tau} (\beta_{2,o} \beta_1)^\tau, \tag{30}$$

$$\sum_{\tau=0}^{b-1} (\beta_3 \beta_{2,o})^{b-1-\tau} (\beta_{3,o} \beta_2)^\tau > \sum_{\tau=0}^{b-1} (\beta_3 \beta_{1,o})^{b-1-\tau} (\beta_{3,o} \beta_1)^\tau. \tag{31}$$

Also, by (20) and (27) the first and the third terms in (29) are negative and the second is positive. Therefore, by (28), (30), and (31), $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] > 0$ for $s = 0$. This completes the proof of part $1(a)$ of Proposition 7.

Proof of part (b) of Proposition 6 and part 1(b) of Proposition 7. Suppose that $b = 0$ and $s = 1$ $(t = r + 1)$. Since S buys in the transparent world, $\mathsf{bid}^r - \mathsf{E}[V|S, H^r] < 0$. Thus, by Lemma 6, we have

$$q_2^r q_1^r[\sigma_1\mathsf{Pr}(S|V_2) - \sigma_2\mathsf{Pr}(S|V_1)] + q_3^r q_2^r[\sigma_2\mathsf{Pr}(S|V_3) - \sigma_3\mathsf{Pr}(S|V_2)] + 2q_3^r q_1^r[\sigma_1\mathsf{Pr}(S|V_3) - \sigma_3\mathsf{Pr}(S|V_1)] < 0 \tag{32}$$

Also, by the definition of σ_i and σ_i we have

$$\begin{aligned}
\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2 &= -\mu[\sigma_2\mathsf{Pr}(S|V_3) - \sigma_3\mathsf{Pr}(S|V_2)], \\
\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1 &= -\mu[\sigma_1\mathsf{Pr}(S|V_3) - \sigma_3\mathsf{Pr}(S|V_1)], \\
\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1 &= -\mu[\sigma_1\mathsf{Pr}(S|V_2) - \sigma_2\mathsf{Pr}(S|V_1)].
\end{aligned} \tag{33}$$

Therefore, (32) is equivalent to

$$q_2^r q_1^r[\sigma_2\sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^r q_2^r[\sigma_3\sigma_{2,o} - \sigma_{3,o}\sigma_2] + 2q_3^r q_1^r[\sigma_3\sigma_{1,o} - \sigma_{3,o}\sigma_1] > 0. \tag{34}$$

Since the LHS of (34) is the same as the RHS of (25) when $b = 0$ and $s = 1$, it follows that in this case $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] < 0$. This completes both the proof of part (b) of Propositions 6[34] and part 1(b) of 7.

Proof of $1(c)$ **in Proposition 7.** First, note that (25) can be written as:

$$q_2^r q_1^r \frac{(\beta_{2,o}\beta_1)^b}{(\beta_3\beta_{2,o})^b} \left[\frac{(\beta_2\beta_{1,o})^b}{(\beta_{2,o}\beta_1)^b}(\sigma_2\sigma_{1,o})^s - (\sigma_{2,o}\sigma_1)^s \right] + q_3^r q_2^r \left[(\sigma_3\sigma_{2n})^s - \frac{(\beta_{3,o}\beta_2)^b}{(\beta_3\beta_{2,o})^b}(\sigma_3\sigma_2)^s \right] $$
$$+ 2q_3^r q_1^r \frac{(\beta_{3,o}\beta_1)^b}{(\beta_3\beta_{2,o})^b} \left[\frac{(\beta_3\beta_{1,o})^b}{(\beta_{3,o}\beta_1)^b}(\sigma_3\sigma_{1,o})^s - (\sigma_{3,o}\sigma_1)^s \right]. \tag{35}$$

Fix s and let $b \to \infty$. Then since by (20) and (27) $\beta_3\beta_{2,o} > \beta_{3,o}\beta_2$ we have that the second term in (35) converges to $q_3^r q_2^r(\sigma_3\sigma_{2n})^s$ as $b \to \infty$. Also, since $\beta_3 > \beta_2 > \beta_1$ it follows that $\beta_{2,o}\beta_1 < \beta_{2,o}\beta_3$ and $\beta_{3,o}\beta_2 > \beta_{3,0}\beta_1$. The former, together with (20) and (27), imply that the first term in (35) vanishes as $b \to \infty$. The latter, together with (20) and (27), imply that $\beta_3\beta_{2,o} > \beta_{3,o}\beta_1$; therefore, using (20) and (27) again, the last term in (35) also vanishes. Consequently, as $b \to \infty$ the expression in (35) converges to $q_3^r q_2^r(\sigma_3\sigma_{2n})^s$. Since $(\sigma_3\sigma_{2n})^s > 0$ and $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as the expression in (35), the claim in $1(c)$ of the proposition is established.

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[34]Since in this case H^t correspond to H^r followed by a single sale, $\mathsf{E}_o[V|H^t]$ and $\mathsf{E}[V|H^t]$ correspond respectively to bid prices at H^r when S sells and when S does not.

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Supplementary Material

B Proofs Omitted from the Paper

B.1 Proof of Lemma 1

Observe first that

$$\mathsf{E}[V|S, H^t] - \mathsf{E}[V|H^t] = \mathcal{V}q_2^t \left(\frac{\mathsf{Pr}(S|V_2)}{\mathsf{Pr}(S)} - 1 \right) + 2\mathcal{V}q_3^t \left(\frac{\mathsf{Pr}(S|V_3)}{\mathsf{Pr}(S)} - 1 \right).$$

The RHS of the the above equality has the same sign as

$$
\begin{aligned}
& q_2^t \left(\mathsf{Pr}(S|V_2) \sum_j q_j^t - \sum_j \mathsf{Pr}(S|V_j)q_j^t \right) + 2\, q_3^t \left(\mathsf{Pr}(S|V_3) \sum_j q_j^t - \sum_j \mathsf{Pr}(S|V_j)q_j^t \right) \\
=\ & q_1^t q_2^t \left(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_1) \right) + q_2^t q_3^t \left(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_3) \right) \\
& + 2\, q_3^t \left(q_1^t (\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_1) + q_2^t \left(\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_2) \right) \right) = \text{expression (2)}.
\end{aligned}
$$

B.2 Proof of Lemma 2

(i) By standard results on MLRP and stochastic dominance it must be that $\mathsf{E}[V|S_l] < \mathsf{E}[V|S_h]$. By a similar reasoning, at any history H^t, $\mathsf{E}[V|S_l, H^t] < \mathsf{E}[V|S_h, H^t]$ if the following MLRP condition holds at H^t: for any $S_l < S_h$ and any $V_l < V_h$

$$\frac{\mathsf{Pr}(S_h|V_h, H^t)}{\mathsf{Pr}(S_l|V_h, H^t)} > \frac{\mathsf{Pr}(S_h|V_l, H^t)}{\mathsf{Pr}(S_l|V_l, H^t)}. \tag{B-1}$$

To show this note first that $\mathsf{Pr}(V|H^t, S) = \mathsf{Pr}(V|S)\mathsf{Pr}(H^t|V)/\sum_{V' \in \mathbb{V}} \mathsf{Pr}(V'|S)\mathsf{Pr}(H^t|V')$. Then we have by the following manipulations that the MLRP condition $\frac{\mathsf{Pr}(S_h|V_h)}{\mathsf{Pr}(S_l|V_h)} > \frac{\mathsf{Pr}(S_h|V_l)}{\mathsf{Pr}(S_l|V_l)}$ implies the MLRP condition (B-1) at any H^t:

$$
\begin{aligned}
\mathsf{Pr}(S_l|V_l)\mathsf{Pr}(S_h|V_h) \ &>\ \mathsf{Pr}(S_l|V_h)\mathsf{Pr}(S_h|V_l) \\
\Leftrightarrow\ \mathsf{Pr}(V_l|S_l)\mathsf{Pr}(V_h|S_h) \ &>\ \mathsf{Pr}(V_h|S_l)\mathsf{Pr}(V_l|S_h) \\
\Leftrightarrow\ \frac{\mathsf{Pr}(V_l|S_l)\mathsf{Pr}(H^t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H^t|V)} \frac{\mathsf{Pr}(V_h|S_h)\mathsf{Pr}(H^t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H^t|V)} \ &>\ \frac{\mathsf{Pr}(V_h|S_l)\mathsf{Pr}(H^t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H^t|V)} \frac{\mathsf{Pr}(V_l|S_h)\mathsf{Pr}(H^t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H^t|V)} \\
\Leftrightarrow\ \mathsf{Pr}(V_l|H^t, S_l)\mathsf{Pr}(V_h|H^t, S_h) \ &>\ \mathsf{Pr}(V_h|H^t, S_l)\mathsf{Pr}(V_l|H^t, S_h).
\end{aligned}
$$

(ii) Suppose contrary to the claim, that an informed trader with signal S_1 does not sell at some history H^t. Then by part (i) no informed trader sells at H^t. This implies that at history H^t, $\mathsf{bid}^t = \mathsf{E}[V|H^t]$. But since, by part (i), $\mathsf{E}[V|H^t]$ exceeds $\mathsf{E}[V|S_1, H^t]$, we

have $\mathsf{bid}^t > \mathsf{E}[V|S_1, H^t]$. Hence, an informed trader with signal S_1 sells at H^t. This is a contradiction.

The proof that informed traders with signal S_3 always buy is analogous.

(iii) First we show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_3)$. Suppose otherwise; thus $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$. Then the two MLRP conditions $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1)\mathsf{Pr}(S_1|V_3)$ and $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_3|V_3) > \mathsf{Pr}(S_3|V_1)\mathsf{Pr}(S_1|V_3)$ imply respectively that $\mathsf{Pr}(S_2|V_1) < \mathsf{Pr}(S_2|V_3)$ and $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_3)$. Hence, since $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$ we have $\sum_{i=1}^{3} \mathsf{Pr}(S_i|V_3) > \sum_{i=1}^{3} \mathsf{Pr}(S_i|V_1)$. But this contradicts $\sum_{i=1}^{3} \mathsf{Pr}(S_i|V_j) = 1$ for every j.

The same argument can be applied to show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_2)$ and $\mathsf{Pr}(S_1|V_2) > \mathsf{Pr}(S_1|V_3)$, and also in the reverse direction for $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_2) < \mathsf{Pr}(S_3|V_3)$.

B.3 Proof of Lemma 3

This follows from Lemma 1: By the symmetry assumption on the priors ($q_1^1 = q_3^1$), the RHS of (2) is negative (positive) at $t = 1$ if and only if $(\mathsf{Pr}(S|V_3) - \mathsf{Pr}(S|V_1))(q_2^1 + 2q_1^1)q_3^1$ is less (greater) than 0; the latter is equivalent to S having a negative (positive) bias.

B.4 Proof of Lemma 4

The claim follows from $\mathsf{E}[V|H^t] - \mathsf{E}[V] = \mathcal{V}[(1 - q_1^t - q_3^t) + 2q_3^t] - \mathcal{V} = \mathcal{V}(q_3^t - q_1^t)$.

B.5 Proof of Lemma 6

The proof is analogous to the derivation in the proof of Lemma 1. To show (i) note that

$$\mathsf{E}[V|S, H^t] - \mathsf{ask}^t = \mathcal{V}q_2 \left(\frac{\mathsf{Pr}(S|V_2)}{\mathsf{Pr}(S)} - \frac{\beta_2}{\mathsf{Pr}(\mathsf{buy}|H^t)} \right) + 2\mathcal{V}q_3 \left(\frac{\mathsf{Pr}(S|V_3)}{\mathsf{Pr}(S)} - \frac{\beta_3}{\mathsf{Pr}(\mathsf{buy}|H^t)} \right).$$

The RHS of the above has the same sign as

$$
\begin{aligned}
& q_2 \left(\mathsf{Pr}(S|V_2) \sum_j \beta_j q_j - \beta_2 \sum_j \mathsf{Pr}(S|V_j)q_j \right) + 2 \, q_3 \left(\mathsf{Pr}(S|V_3) \sum_j \beta_j q_j - \beta_3 \sum_j \mathsf{Pr}(S|V_j)q_j \right) \\
= \; & q_1 q_2 \left(\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1) \right) + q_2 q_3 \left(\beta_3 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_3) \right) \\
& + 2 \, q_3 \left(q_1 \left(\beta_1 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_1) \right) + q_2 \left(\beta_2 \mathsf{Pr}(S|V_3) - \beta_3 \mathsf{Pr}(S|V_2) \right) \right) = \text{expression (3)}.
\end{aligned}
$$

The proof of (ii) is analogous to that of (i).

B.6 Proof of Theorem 3

We first prove that any informed type buys initially if it has a negative bias and if there are enough noise traders.

Lemma II *Let S be negatively biased. Then $\mathsf{E}[V|S] < \mathsf{E}[V]$. Hence, there exists $\mu^{in} \in (0,1]$ such that S sells at the initial history if $\mu < \mu^{in}$.*

Proof of Lemma II: Without loss of generality, we present the proof only for the case when the number of states $n > 2$ is even so that $n = 2k$ for some integer k. Then by the symmetry of the prior $\mathsf{E}[V] = (2k-1)/2$. Also, $\mathsf{E}[V|S] = \sum_{i=1}^{2k}(i-1)\mathsf{Pr}(V_i|S)$. Thus, we need to show

$$\sum_{i=1}^{2k}(i-1)\mathsf{Pr}(V_i|S) < \frac{2k-1}{2}. \tag{B-2}$$

Next note that by $\mathsf{Pr}(S|V_i) > \mathsf{Pr}(S|V_{n+1-i})$ together with the symmetry of the initial prior, we have $\mathsf{Pr}(V_i|S) > \mathsf{Pr}(V_{n+1-i}|S)$ for all $i < (2k+1)/2$. Using this and $\sum_{i=1}^{n}\mathsf{Pr}(V_i|S) = 1$, we have $\sum_{i=1}^{k}\mathsf{Pr}(V_i|S) > \frac{1}{2} > \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S)$. Therefore

$$(k-1) + \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) < (k-1) + \frac{1}{2} = \frac{2k-1}{2}. \tag{B-3}$$

Then by (B-2) it is sufficient to show that

$$\sum_{i=1}^{k}(i-1)\mathsf{Pr}(V_i|S) + \sum_{i=k+1}^{2k}(i-1)\mathsf{Pr}(V_i|S) < (k-1) + \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S). \tag{B-4}$$

But the second term on the left hand side of (B-4) is

$$\sum_{i=k+1}^{2k}(i-1)\mathsf{Pr}(V_i|S) = \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + (k-1)\mathsf{Pr}(V_{k+1}|S) + \ldots + (2k-2)\mathsf{Pr}(V_{2k}|S)$$

$$< \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + (k-1)\mathsf{Pr}(V_{k+1}|S) + [(k-1)\mathsf{Pr}(V_{k+2}|S) + \mathsf{Pr}(V_{k-1}|S)]$$

$$+ [(k-1)\mathsf{Pr}(V_{k+3}|S) + 2\mathsf{Pr}(V_{k-2}|S)] + \ldots + [(k-1)\mathsf{Pr}(V_{2k}|S) + (k-1)\mathsf{Pr}(V_1|S)]$$

$$= \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + (k-1)\sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + \sum_{j=1}^{k}(k-j)\mathsf{Pr}(V_j|S).$$

Therefore,

$$\text{LHS of (B-4)} < \sum_{i=1}^{k}(i-1)\mathsf{Pr}(V_i|S) + \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + (k-1)\sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) + \sum_{i=1}^{k}(k-i)\mathsf{Pr}(V_i|S)$$

$$= (k-1) + \sum_{i=k+1}^{2k}\mathsf{Pr}(V_i|S) = \text{RHS of (B-4)}.$$

This demonstrates that $\mathsf{E}[V|S] < \mathsf{E}[V]$. To complete the proof of the lemma we also need to show that there exists $\mu^{in} \in (0,1]$ such that $\mathsf{E}[V|S] < \mathsf{bid}^1$ if $\mu < \mu^{in}$. As in Lemma 5 this follows immediately from $\mathsf{E}[V|S] < \mathsf{E}[V]$ and from $\lim_{\mu \to 0} \mathsf{E}[V] - \mathsf{bid}^1 = 0$. This completes the proof of Lemma II.

Next, we turn to the switching of behavior. Before doing that note that with MLRP the probability of a buy is increasing in the liquidation values and probability of a sale is decreasing in the liquidation values. To show this assume without any loss of generality that $S_1 < S_2 < \ldots < S_n$. Then we have the following.

Lemma III *When signals satisfy the MLRP, $\beta_1^t < \beta_2^t < \ldots < \beta_n^t$ and $\sigma_1^t > \sigma_2^t > \ldots > \sigma_n^t$.*

Proof of Lemma III: We will show only $\beta_1^t < \beta_2^t < \ldots < \beta_n^t$, the result on sales σ_i follows analogously. To show the former, observe that with MLRP signals, expectations are ordered in signals: for $i > j$, $\mathsf{E}[V|H^t, S_i] > \mathsf{E}[V|H^t, S_j]$. Thus, if signal type S_k buys, so will all $S_l > S_k$. Thus for $i > j$, $\beta_i - \beta_j$ has the same sign as

$$
\begin{aligned}
\sum_{l=m}^{n} \mathsf{Pr}(S_l|V_i) - \sum_{l=m}^{n} \mathsf{Pr}(S_l|V_j) &= 1 - \sum_{l=1}^{m-1} \mathsf{Pr}(S_l|V_i) - \left(1 - \sum_{l=1}^{m-1} \mathsf{Pr}(S_l|V_j) \right) \\
&= \sum_{l=1}^{m-1} \mathsf{Pr}(S_l|V_j) - \sum_{l=1}^{m-1} \mathsf{Pr}(S_l|V_i), \text{ for some } m \leq n.
\end{aligned}
$$

This latter expression is positive since MLRP implies First Order Stochastic dominance. This completes the proof of Lemma III.

Proof of part (a) of Theorem 3: Fix S. Analogously to Lemma 6, by simple calculations, it can be shown that $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t$ has the same sign as

$$
q_n^t q_{n-1}^t \cdot \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} j \cdot \frac{q_i^t q_{i+j}^t}{q_{n-1}^t q_n^t} (\beta_i^t \mathsf{Pr}(S|V_{i+j}) - \beta_{i+j}^t \mathsf{Pr}(S|V_i)). \tag{B-5}
$$

Next consider the infinite path consisting of only buys at every date. By MLRP and Lemma III, $\beta_1^t < \beta_2^t < \ldots < \beta_n^t$. Then we have that q_i^t/q_n^t converges to zero for all $i < n$. But this implies that along this path

$$
\lim_t \mathsf{E}[V|H^t] = V_n > \mathsf{E}[V] \tag{B-6}
$$

Also, since for any $i < n-1$ and $j \geq 1$ and for any t,

$$
\frac{q_i^{t+1} q_{i+j}^{t+1}}{q_{n-1}^{t+1} q_n^{t+1}} = \frac{\beta_i^t \beta_{i+j}^t}{\beta_{n-1}^t \beta_n^t} \frac{q_i^t q_{i+j}^t}{q_{n-1}^t q_n^t} < \frac{q_i^t q_{i+j}^t}{q_{n-1}^t q_n^t},
$$

we have that $(q_i^t q_{i+j}^t)/(q_{n-1}^t q_n^t)$ converges to zero along this infinite path of buys. Thus, as $t \to \infty$

$$\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} j \cdot \frac{q_i^t q_{i+j}^t}{q_{n-1}^t q_n^t}(\beta_i^t \mathsf{Pr}(S|V_{i+j}) - \beta_{i+j}^t \mathsf{Pr}(S|V_i)) \to \lim_t[\beta_{n-1}^t \mathsf{Pr}(S|V_n) - \beta_n^t \mathsf{Pr}(S|V_{n-1})]. \quad \text{(B-7)}$$

Let S_i be the lowest type that buys at t. Then with MLRP all S_j with $j \in \{i, \ldots, n\}$ will buy. This implies that

$$
\begin{aligned}
\beta_{n-1}^t \mathsf{Pr}(S|V_n) - \beta_n^t \mathsf{Pr}(S|V_{n-1}) &> 0 \\
\Leftrightarrow (\gamma + \mu \sum_{j=i}^n \mathsf{Pr}(S_j|V_{n-1}))\mathsf{Pr}(S|V_n) &> (\gamma + \mu \sum_{j=i}^n \mathsf{Pr}(S_j|V_n))\mathsf{Pr}(S|V_{n-1}) \\
\Leftrightarrow (\mathsf{Pr}(S|V_n) - \mathsf{Pr}(S|V_{n-1})) &> \tfrac{\mu}{\gamma}\sum_{j=i}^n\{\mathsf{Pr}(S_j|V_n)\mathsf{Pr}(S|V_{n-1}) - \mathsf{Pr}(S_j|V_{n-1})\mathsf{Pr}(S|V_n)\}.
\end{aligned}
$$
(B-8)

Since $\mathsf{Pr}(S|V_n) > \mathsf{Pr}(S|V_{n-1})$, the left hand side of the last inequality in (B-8) is positive. Therefore, by (B-8), for μ sufficiently small, $\beta_{n-1}^t\mathsf{Pr}(S|V_n) - \beta_n^t\mathsf{Pr}(S|V_{n-1}) > 0$. Since there is a finite number of types, it then follows from (B-5), (B-6), and (B-7) that there exist a critical level of informed trading $\mu^{ch} > 0$ and a history H^t along the infinite path of buys such that $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t > 0$ and $\mathsf{E}[V|H^t] > \mathsf{E}[V]$. This together with Lemma II completes the proof.

Proof of part (b) of Theorem 3: Analogously to (a), we can rewrite $\mathsf{E}[V|S, H^t] - \mathsf{ask}^t =$

$$q_1 q_2 \cdot \sum_{j=1}^{n-1}\sum_{i=1}^{n-j} j \cdot \frac{q_i q_{i+j}}{q_1 q_2}(\beta_i^t \mathsf{Pr}(S|V_{i+j}) - \beta_{i+j}^t \mathsf{Pr}(S|V_i)). \quad \text{(B-9)}$$

Now consider the infinite path consisting of only sales at every date. By MLRP and Lemma III, $\sigma_1^t > \sigma_2^t > \ldots > \sigma_n^t$. Then we have that q_i^t/q_1^t converges to zero for all $i > 1$. But this implies that

$$\lim_t \mathsf{E}[V|H^t] = V_1 < \mathsf{E}[V] \quad \text{(B-10)}$$

Also, since for any $i, j \geq 1$ such that either i or $j > 1$, and any t,

$$\frac{q_i^{t+1} q_{i+j}^{t+1}}{q_1^{t+1} q_2^{t+1}} = \frac{\sigma_i^t \sigma_{i+j}^t}{\sigma_1^t \sigma_2^t}\frac{q_i^t q_{i+j}^t}{q_1^t q_2^t} < \frac{q_i^t q_{i+j}^t}{q_1^t q_2^t},$$

we have that $(q_i^t q_{i+j}^t)/(q_1^t q_2^t)$ converges to zero along this infinite path of buys. Thus, as $t \to \infty$

$$\sum_{j=1}^{n-1}\sum_{i=1}^{n-j} j \cdot \frac{q_i q_{i+j}}{q_1 q_2}(\beta_i \mathsf{Pr}(S|V_{i+j}) - \beta_{i+j} \mathsf{Pr}(S|V_i)) \to \lim_t[\beta_1 \mathsf{Pr}(S|V_2) - \beta_2 \mathsf{Pr}(S|V_1)]. \quad \text{(B-11)}$$

Let S_i be the highest type that buys at t. Then with MLRP all S_j with $j \in \{i, \dots, n\}$ will buy at t. This implies that $\beta_1^t \text{Pr}(S|V_2) - \beta_2^t \text{Pr}(S|V_1) > 0$ if and only if

$$\Leftrightarrow (\text{Pr}(S|V_2) - \text{Pr}(S|V_1)) > \frac{\mu}{\gamma} \sum_{j=i}^{n} \{\text{Pr}(S_j|V_2)\text{Pr}(S|V_1) - \text{Pr}(S_j|V_1)\text{Pr}(S|V_2)\}. \tag{B-12}$$

Since $\text{Pr}(S|V_2) > \text{Pr}(S|V_1)$, the left hand side of (B-12) is positive. Thus, for μ sufficiently small, $\beta_1^t \text{Pr}(S|V_2) - \beta_2^t \text{Pr}(S|V_1) > 0$. As there are finite number of types, it then follows from (B-9), (B-10), and (B-11) that there exist a critical level of informed trading $\mu^{ch} > 0$ and a history H^t along the infinite path of sales such that $\text{E}[V|S, H^t] - \text{ask}^t > 0$ and $\text{E}[V|H^t] < \text{E}[V]$. This together with Lemma II completes the proof.

C   Additional Results

C.1   Proposition 3a

(i)   *Suppose that S buy herds and there is at most one U shaped signal.*
   Then $\mu < \min\{\mu_s^{in}, \mu_2^{ch}\}$.

(ii)   *Suppose that S sell herds and there is at most one U shaped signal.*
   Then $\mu < \min\{\mu_b^{in}, \mu_1^{ch}\}$.

(iii) *Suppose that S acts as a buy contrarian and there is at most one hill shaped signal.*
   Then $\mu < \min\{\mu_s^{in}, \mu_1^{ch}\}$.

(iv) *Suppose that S acts as a sell contrarian and there is at most one hill shaped signal.*
   Then $\mu < \min\{\mu_b^{in}, \mu_2^{ch}\}$.

We shall prove (i); the proof of $(ii) - (iv)$ are analogous.

Since S sells initially it follows from Lemma 5 that $\mu < \mu_s^{in}$. To show that $\mu < \mu_2^{ch}$ first note that by Proposition 1, S must be nU shaped. Next consider the different possibilities separately.

Case A. There is no signal $S' \neq S$ such that $\text{Pr}(S'|V_3) > \text{Pr}(S'|V_2)$. Then it must be that $\mu_2^{ch}(S') = 1$ for all S' and therefore it must be that $\mu < \mu_2^{ch} = 1$.

Case B. There is a signal $S' \neq S$ such that $\text{Pr}(S'|V_3) > \text{Pr}(S'|V_2)$. Since S is U shaped it must be that $\text{Pr}(S|V_3) > \text{Pr}(S|V_2)$ and $\text{Pr}(S''|V_3) \leq \text{Pr}(S''|V_2)$ for $S'' \neq S, S'$. This implies that $\mu_2^{ch}(S'') = 1$ and hence, $\mu_2^{ch}(S') = \mu_2^{ch}$.

Now there are two cases. First, if $\mu_2^{ch}(S')$ also equals 1 then clearly $\mu_2^{ch} = 1$ and the claim is trivially true.

Second, assume that $\mu_2^{ch}(S') = \mu_2^{ch} < 1$. Since S buy herds at H^t, to show that $\mu < \min\{\mu_s^{in}, \mu_2^{ch}\}$ it suffices to show that S' also buys whenever S buys (the alternative

is that S' does not buy so that $\mu^{ch} = 1 > \mu_2^{ch}$). When S' buys, $\mathsf{E}[V|S', H^t] - \mathsf{ask}^t > 0$. Suppose S' does not buy. As the sign of $\mathsf{E}[V|S', H^t] - \mathsf{ask}^t$ is given by equation (3), it must then hold that

$$
\begin{aligned}
q_1 q_2 \left[\beta_1 \mathsf{Pr}(S'|V_2) - \beta_2 \mathsf{Pr}(S'|V_1) \right] + q_2 q_3 [\beta_2 \mathsf{Pr}(S'|V_3) - \beta_3 \mathsf{Pr}(S'|V_2)] \\
+ 2 \, q_1 q_3 \left[\beta_1 \mathsf{Pr}(S'|V_3) - \beta_3 \mathsf{Pr}(S'|V_1) \right] \leq 0.
\end{aligned}
\tag{C-13}
$$

Also, since there is at most one U shaped signal it must be that

$$
\mathsf{Pr}(S'|V_3) > \mathsf{Pr}(S'|V_2) \geq \mathsf{Pr}(S'|V_1).
\tag{C-14}
$$

By Proposition 1 this implies that S' does not sell. By supposition S' does not buy and therefore S is the only buyer at H^t (S'' is selling). Since S is nU shaped we must also have $\beta_1^t > \beta_3^t \geq \beta_2^t$. This, together with (C-14) imply that the first and the third term in (C-13) are positive. Furthermore, the second term equals

$$
\gamma(\mathsf{Pr}(S'|V_3) - \mathsf{Pr}(S'|V_2)) + \mu(\mathsf{Pr}(S|V_2)\mathsf{Pr}(S'|V_3) - \mathsf{Pr}(S|V_3)\mathsf{Pr}(S'|V_2)).
\tag{C-15}
$$

By (C-14) the first term in the last expression is positive; furthermore, since S is nU, we have $m^2 = \mathsf{Pr}(S|V_3) > \mathsf{Pr}(S|V_2)$. Since $\mu_2^{ch}(S') < 1$ we must have that $M^2(S') < 1$ is negative. But $-\mu M^2(S')$ is the second term in the last expression and it is thus positive. Consequently, (C-15) is positive. Therefore, the second term in (C-13) must also be positive. Therefore, S' must be buying at any H^t at which S buys and thus $\mu_2^{ch} < 1$ is unique.

C.2 Proof of the statement in condition (7) in Section 7 of the main text

First, note that, by (33) in the proof of Proposition 7, we have

$$
\begin{aligned}
\sigma_3 \sigma_{2,o} - \sigma_{3,o} \sigma_2 &= -\mu^2 \rho_{12}^{23} + \mu\gamma(\mathsf{Pr}(S|V_2) - \mathsf{Pr}(S|V_3)) < 0 \\
\sigma_2 \sigma_{1,o} - \sigma_{2,o} \sigma_1 &> \sigma_3 \sigma_{1,o} - \sigma_{3,o} \sigma_1
\end{aligned}
\tag{C-16}
$$

Also, since for herding we require $\mathsf{E}[V|S, H^1] < \mathsf{bid}^1$, it follows from Lemma 6 and (32) that

$$
q_2^1 q_1^1 [\sigma_2 \sigma_{1,o} - \sigma_{2,o} \sigma_1] + q_3^1 q_2^1 [\sigma_3 \sigma_{2,o} - \sigma_{3,o} \sigma_2] + 2 q_3^1 q_1^1 [\sigma_3 \sigma_{1,o} - \sigma_{3,o} \sigma_1] > 0.
$$

But then by (C-16) we have

$$
\sigma_2 \sigma_{1,o} - \sigma_{2,o} \sigma_1 > 0.
\tag{C-17}
$$

Since $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as the expression in (29), by simple expansion of this expression we have that if $b = 0$ then $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \left\{ (\sigma_2 \sigma_{1,o} - \sigma_{2,o}\sigma_1) \sum_{\tau=0}^{s-1} (\sigma_2 \sigma_{1,o})^{s-1-\tau} (\sigma_{2,o}\sigma_1)^\tau \right\}$$

$$+ q_3^r q_2^r \left\{ [(\sigma_3 \sigma_{2,o}) - (\sigma_{3,o}\sigma_2)] \sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau \right\}$$

$$+ 2q_3^r q_1^r \left\{ (\sigma_3 \sigma_{1,o} - \sigma_{3,o}\sigma_1) \sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{1,o})^{s-1-\tau} (\sigma_3 \sigma_1)^\tau \right\}.$$

Rearranging, we have that for $b = 0$, $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t]$ has the same sign as

$$q_2^r q_1^r \frac{\sum_{\tau=0}^{s-1} (\sigma_2 \sigma_{1,o})^{s-1-\tau} (\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau} [\sigma_2 \sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^r q_1^r [\sigma_3 \sigma_{2,o} - \sigma_{3,o}\sigma_2]$$

$$+ \; 2q_3^r q_1^r \frac{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{1,o})^{s-1-\tau} (\sigma_3 \sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau} [\sigma_3 \sigma_{1,o} - \sigma_{3,o}\sigma_1]. \tag{C-18}$$

Further manipulations show that

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1} (\sigma_2 \sigma_{1,o})^{s-1-\tau} (\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1} \sigma_2^{s-1-\tau} \sigma_{2,o}^\tau (\sigma_1 \sigma_3)^\tau \left((\sigma_1 \sigma_{3,o})^{s-1-\tau} - (\sigma_3 \sigma_{1,o})^{s-1-\tau} \right) > 0.$$

Also, by assumption we have $\frac{\sigma_1}{\sigma_3} > \frac{\sigma_{1,o}}{\sigma_{3,o}}$. Therefore, we must have

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1} (\sigma_2 \sigma_{1,o})^{s-1-\tau} (\sigma_{2,o}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau}. \tag{C-19}$$

Similar manipulations show that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{1,o})^{s-1-\tau} (\sigma_3 \sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1} \sigma_3^{s-1-\tau} \sigma_{3,o}^\tau (\sigma_1 \sigma_2)^\tau \left((\sigma_2 \sigma_{1,o})^{s-1-\tau} - (\sigma_1 \sigma_{2,o})^{s-1-\tau} \right) > 0.$$

This together with (C-17), implies that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{1,o})^{s-1-\tau} (\sigma_3 \sigma_1)^\tau}{\sum_{\tau=0}^{s-1} (\sigma_3 \sigma_{2,o})^{s-1-\tau} (\sigma_{3,o}\sigma_2)^\tau}. \tag{C-20}$$

Also, since $\mathsf{E}[V|S, H^t] - \mathsf{bid}^t > 0$, by Lemma 6, we have that if $b = 0$ then

$$q_2^r q_1^r \left(\frac{\sigma_1}{\sigma_3}\right)^s [\sigma_2 \sigma_{1,o} - \sigma_{2,o}\sigma_1] + q_3^r q_2^r [\sigma_3 \sigma_{2,o} - \sigma_{3,o}\sigma_2]$$

$$+ 2q_3^r q_1^r \left(\frac{\sigma_1}{\sigma_2}\right)^s [\sigma_3 \sigma_{1,o} - \sigma_{3,o}\sigma_1] \quad < \quad 0. \tag{C-21}$$

Then it follows from (C-21), together with $\frac{\sigma_1}{\sigma_3} > \frac{\sigma_{1,o}}{\sigma_{3,o}}$, (C-17), (C-19) and (C-20), that the expression in (C-18) is negative. Thus $\mathsf{E}[V|H^t] - \mathsf{E}_o[V|H^t] < 0$ and (7) follows.

D   The Parameters used for Figure 1

<table>
<tr><td colspan="4" align="center">Herding Example</td></tr>
<tr><td>$\Pr(S|V)$</td><td>V_1</td><td>V_2</td><td>V_3</td></tr>
<tr><td>S_1</td><td>$\frac{601}{1000}$</td><td>$\frac{270}{1000}$</td><td>0</td></tr>
<tr><td>S_2</td><td>$\frac{399}{1000}$</td><td>$\frac{180}{1000}$</td><td>$\frac{245}{1000}$</td></tr>
<tr><td>S_3</td><td>0</td><td>$\frac{550}{1000}$</td><td>$\frac{755}{1000}$</td></tr>
</table>

$$\mathbb{V} = (0, 10, 20)$$
$$\Pr(V) = (1/100, 98/100, 1/100)$$
$$\mu_b^{ch} = 0.9496, \ \mu_b^{in} = 0.4294$$
$$\Rightarrow \mu = 0.4294 - 0.0001$$

<table>
<tr><td colspan="4" align="center">Contrarian Example</td></tr>
<tr><td>$\Pr(S|V)$</td><td>V_1</td><td>V_2</td><td>V_3</td></tr>
<tr><td>S_1</td><td>$\frac{7}{10}$</td><td>$\frac{1}{4}$</td><td>0</td></tr>
<tr><td>S_2</td><td>$\frac{3}{10}$</td><td>$\frac{1}{2}$</td><td>$\frac{3}{20}$</td></tr>
<tr><td>S_3</td><td>0</td><td>$\frac{1}{4}$</td><td>$\frac{17}{20}$</td></tr>
</table>

$$\mathbb{V} = (0, 10, 20)$$
$$\Pr(V) = (1/4, 1/2, 1/4)$$
$$\mu_b^{ch} = 0.4706, \ \mu_b^{in} = 0.1922$$
$$\Rightarrow \mu = 0.1922 - 0.001$$

E   The Parameters used for Figure 2

$$\mu_b^{ch} = \kappa_b/(3 + \kappa_b) = 0.7656 \equiv \mu_b$$
$$\mu_b^{in} = \theta_b/(3 + \theta_b) = 0.9215$$
$$\mathbb{V} = (0, 10, 20),$$
$$\Pr(V) = (1/10, 4/5, 1/10), \text{ and}$$

<table>
<tr><td>$\Pr(S|V)$</td><td>V_1</td><td>V_2</td><td>V_3</td></tr>
<tr><td>S_1</td><td>$\frac{40049}{49000}$</td><td>$\frac{4}{49}$</td><td>0</td></tr>
<tr><td>S_2</td><td>$\frac{8951}{49000}$</td><td>$\frac{9}{490}$</td><td>$\frac{243}{12250}$</td></tr>
<tr><td>S_3</td><td>0</td><td>$\frac{9}{10}$</td><td>$\frac{12007}{12250}$</td></tr>
</table>

(E-22)

F   Further Features of Herding

Simple History Dependence.   The order of trades and traders does not affect the price path as long as the model primitives do not allow any type of trader to change behavior. Clearly, herding or contrarian behavior involve such a change of behavior; changes from buying to holding or selling to holding also qualify as a change of behavior.

Without changes in behavior, it suffices to study the order imbalance (number of buys minus number of sales) to determine prices, but with changes, the order of arrival matters a great deal. Consider the following numerical example[35] of an MLRP signal structure with U-shaped and negatively biased csd for S_2

<table>
<tr><td>$\Pr(S|V)$</td><td>V_1</td><td>V_2</td><td>V_3</td></tr>
<tr><td>S_1</td><td>$\frac{40}{49}$</td><td>$\frac{4}{49}$</td><td>0</td></tr>
<tr><td>S_2</td><td>$\frac{9}{49}$</td><td>$\frac{9}{490}$</td><td>$\frac{243}{12250}$</td></tr>
<tr><td>S_3</td><td>0</td><td>$\frac{9}{10}$</td><td>$\frac{12007}{12250}$</td></tr>
</table>

$$\mu = \frac{1209}{1600},$$
$$\mathbb{V} = (0, 10, 20), \text{ and}$$
$$\Pr(V) = (1/6, 2/3, 1/6).$$

For illustrative purposes, assume that the first fifteen traders are all informed and each signal S_i, $i = 1, 2, 3$, is received by five of the first fifteen traders. Next, we compare the

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[35]We chose the numbers so that there can be herding after a small number of trades.

price paths for different arrival orders of these traders.

SERIES 1:   The arrival order is $5\times S_1$–$5 \times S_2$–$5\times S_3$ (meaning the first five receive S_1, the next five S_2 and the last five S_3). The S_1 types, who move first, all sell and thus the price drops. The S_2 types also sell and the S_3 types buy. Computations show that after these 15 trades the public expectation will drop from 10 to .15.

SERIES 2: $5\times S_1$–$5\times S_3$–$5\times S_2$. Here the outcome is the same as in the previous series with S_1 traders selling, S_3 types' buying and finally the S_2 types selling. The public expectation also drops from 10 to .15.

SERIES 3: $5\times S_3$–$5\times S_2$–$5\times S_1$. The S_3 traders move first and buy. The S_2 types will now behave differently from the previous two series and will be buy-herding. The public expectation now rises to about 13.5. Finally, the five S_1 type sell, and then the public expectation drops to 10.31.

The difference between the outcome for Series 3 with those of Series 1 and 2 illustrates how the arrival order of traders matters: since there are S_2 types who trade, this type's change in trading-mode (from selling to buying) strongly affects the price-path.

Note, however, that even if there are no S_2-types directly involved in trading, the market maker has to consider the possibility that this type trades and thus has to account for this type's change of trading mode. To illustrate this, we next compare the outcome when the same number of buys and sales occurs, but in different orders.

SERIES 4: 20 BUYS FOLLOWED BY 20 SALES. After 20 buys, the public expectation is 15.36, after 20 subsequent sales it is 3.12.

SERIES 4: 20 SALES FOLLOWED BY 20 BUYS. After 20 sales, the public expectation is 1.16×10^{-13}, after 20 subsequent buys it is 10.0064.

In summary, the S_2-type can change trading modes in response to observing the order flow; thus the order flow affects prices and the frequency of different types of future trades. Although there is convergence in the long run, in the short run the fluctuations may be influenced by the precise order of trades.

Price Sensitivity.   To further elaborate on the price sensitivity induced by herding, consider the following simulations of our model which uses the specification outlined in Appendix D, expression (E-22). The simulated prices paths are plotted in Figure 2.

In the left panel, there are two relevant price paths: the first (in gray) is for a setting with $\mu = \mu_b - \epsilon$, $\epsilon = 1/10,000$; in other words, there is just enough noise so that herding is possible. The second price path (in red) is for $\mu = \mu_b$ so that there cannot be herding.[36] The entry series for the graph is as follows: first, there is a long series of S_3 types, who all

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[36]The third price path (in blue) is for the case of naïve agents as described in the preceding section. For the naïve case the differences in prices for the two levels of μ are negligible.
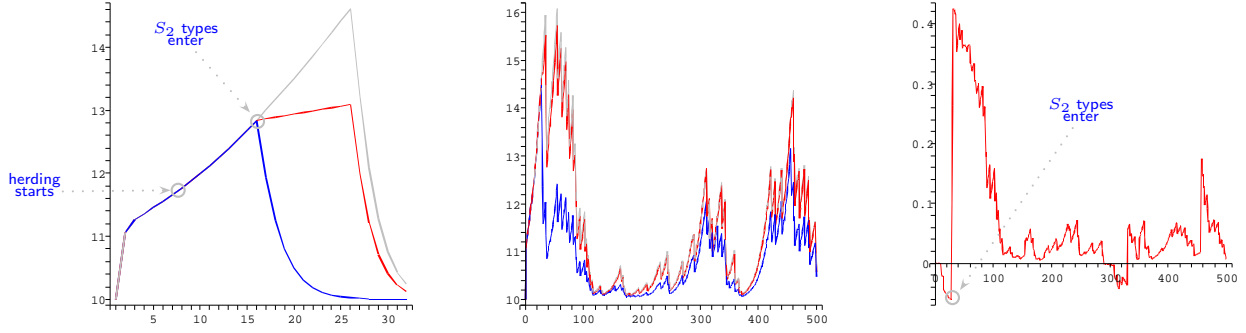
Figure 2: **Illustrations of the Sensitivity in Prices Paths with and without Herding.**

buy; this is followed by a group of S_2 types and eventually by some S_1 types. The point when S_2 types start entering is clearly marked; the S_1 types enter at the point when both curves peak. The point at which herding starts is marked too.

The series is constructed so that there are S_3 types who enter during herding. When the S_2 types enter, in the herding case, they buy, in the no-herding case, they hold. Even with holds, however, prices increase (this is due to the U-shaped csd).[37]

In the middle panel we plot prices for the same specifications, this time for a random sequence of traders; both series have the same sequence of traders but due to herding their actions may differ.[38] In the right panel we plot the difference of the two rational price-series from the middle panel. The series with herding-prices has more noise (because $\mu < \mu_b$). Thus initially, the price for the no-herding series is above the price of the herd series. Once herding starts (here after 8 trades), and once an S_2 type enters, this relation flips; this illustrates that due to herding prices move stronger in the direction of the herd than in the no-herding case.

Does Herding Hamper Learning? The common perception of herding is that it slows down learning. With rational agents and informationally efficient prices, this is not so obvious. With U-shaped signal distributions, the S_2-herding-type occurs with high probability in both the highest and the lowest state — so their herding may speed up learning.

To explore this more generally, we use Monte Carlo simulations and compare the two scenarios outlined when discussing price sensitivity. That is, for the first series, there is just enough noise so that buy-herding can be triggered, $\mu = \mu_b - \epsilon$, $\epsilon \approx 1/10,000$. In the second series, herding cannot occur, because there is too much informed trading, $\mu = \mu_b^{ch} \equiv \mu_b$.

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[37]The same simulation for the naïve case of the proceeding section results in S_2 types selling and prices falling for both levels of μ.

[38]There is also a series for the naïve case which, not surprisingly, is entirely below both rational series. Again, the naïve price series for $\mu = \mu_b$ and $\mu = \mu_b - \epsilon$ are almost identical.
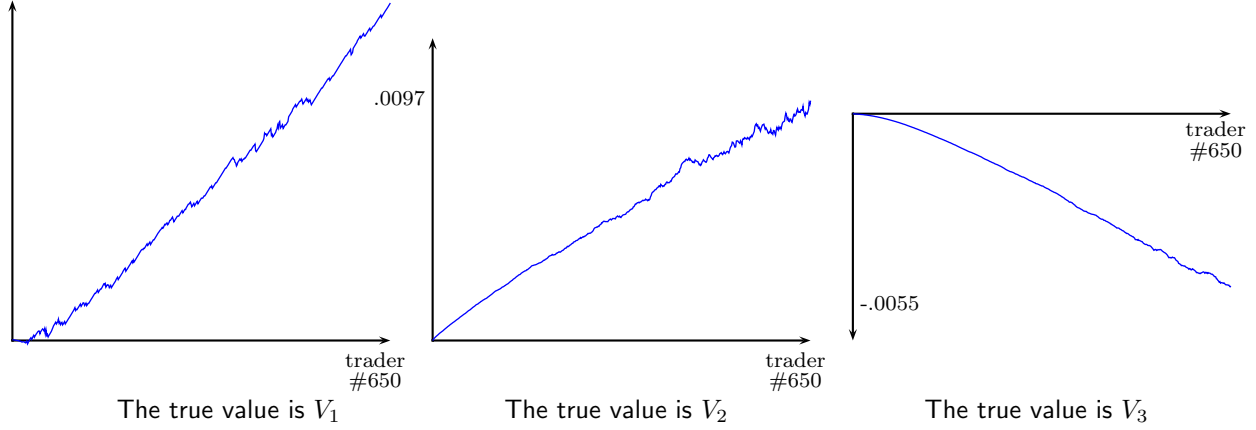
Figure 3: **The Difference in Speeds of Convergence.** Each graph plots the difference of the negative of the average log-distance of the transaction prices of herding and no-herding case. An up-sloping line thus indicates that for any t herding-prices are further from the true value than no-herding prices. All graphs are scaled to fit the page. The underlying signal distribution is listed in Appendix D.

We will refer to prices in the first setting as herding-prices, irrespective of whether or not herding actually occurred; we refer to prices in the second setting as no-herding prices. Comparing the speeds of convergence for our two sets of simulations we note the following two observations:

1. if the true value is V_1 or V_2, then herding-prices converge slower;

2. if the true value is V_3, then convergence with herding is faster.

These observations are based on the following: For the simulations we again used the specification of the parameters given by (E-22) in Appendix D. Fixing the true liquidation values, we then drew 650 traders at random (noise and informed) assuming that $\mu_b \approx .766$. Since the proportion of the informed agents μ is large — approximately three quarters for both simulations — the 650 trades are almost always sufficient to obtain convergence to the true value. Next, we computed the time series of the transaction prices for both the herding and the no-herding case, and then recorded for each t and for both cases the absolute distance of the transaction price from the true value (which we know). We then repeated this procedure a large number of times, and calculated for each t and for each case the average distance from the true value. Since prices converge to the true value, these average distances decline in t. In the simulations, this distance declines approximately exponentially to zero. Thus the slope of the logarithm of the average distance measures the speed of convergence.

As the final step, we subtract at each t the log-averages for the no-herding from the herding series. A positive number indicates that the herding series is slower, i.e. that the average herding price is further away from the true value. Figure 3 plots these differences

and the graphs are striking; they confirm our two observations mentioned above.[39]

To see the intuition for the these observations compare the effects of buy-herding on the herding and no-herding prices. First, when buy-herding occurs, S_2 types buy in the herding case and thus there are more buys with herding than in the no-herding case. Second, in the case of a buy, prices in the herding case tend to be higher than in the no-herding case. Since the no-herding prices here are the same as the ones that arise in the 'naïve' economy of the previous section (only S_3 types buy in both cases), this second effect follows from the same reasoning used in the previous section to explain why, in the case of a buy, prices in the rational world, when herding starts, exceed those in the naïve hypothetical economy (see Proposition 7(a)). Third, when there is a sale, prices in the herding and no-herding cases are almost identical and unaffected by buy-herding. This is because in both cases only S_1 types sell: in the herding case this is so by definition and in the no-herding case, the S_2 type's expectation is almost equal to the ask-price (expression (3) is almost zero) and thus larger than the bid-price.[40]

Now it follows from the above that if the true value is V_1 or V_2, herding prices converge slower: during herding, herd-buys move prices *away* from the true value by a larger magnitude and there are more such buys than in the no-herding case (sales have a similar effect in both cases). If, however, the true value is V_3 then once herding starts, prices in the herding-case move up more strongly because of the first two effects and thus they move faster *towards* the true value. This leads to a higher speed of convergence in the herding case. Figure 3 documents these three cases.

The Probability of the Fastest Herd. The shortest sequence of trades that leads to buy-herding is one with only buys; this is the 'fastest' herd. We now want get a sense of how likely this sequence is. Keeping the csd and the prior distribution fixed but varying the proportion of informed trading, we compute first how many buys are needed for buy-herding to begin, and then we determine how likely this sequence of buys is. The same type of analysis clearly applies to sell-herding.

As was explained before, S_2 types buy at any history H_t if the expression in (3) is positive. As the amount of informed trading increases from 0 to μ_b, there are then two opposing effects. First, as noise decreases, the positive term in expression (3) (the first

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[39]We have also made a formal analysis by regressing the log-distance on time and, using the Chow test, checking if one slope is steeper than the other. The results were highly significant.

[40]The herding and no-herding price paths may also differ even if no buy-herding occurs (if S_2 types behave the same way in the two cases) because the proportions of informed trading μ are different for the two cases. In particular, when S_2 types do not buy-herd, since μ is smaller in the herding case, each price-movement in the herding-price series is smaller than than in the no-herding case, and as a result speed of convergence is slower in the former series. However, since for the simulations the difference between the values of μ is small ($\epsilon = 1/10,000$), the consequence of this effect is small relative to the first two effects mentioned above.

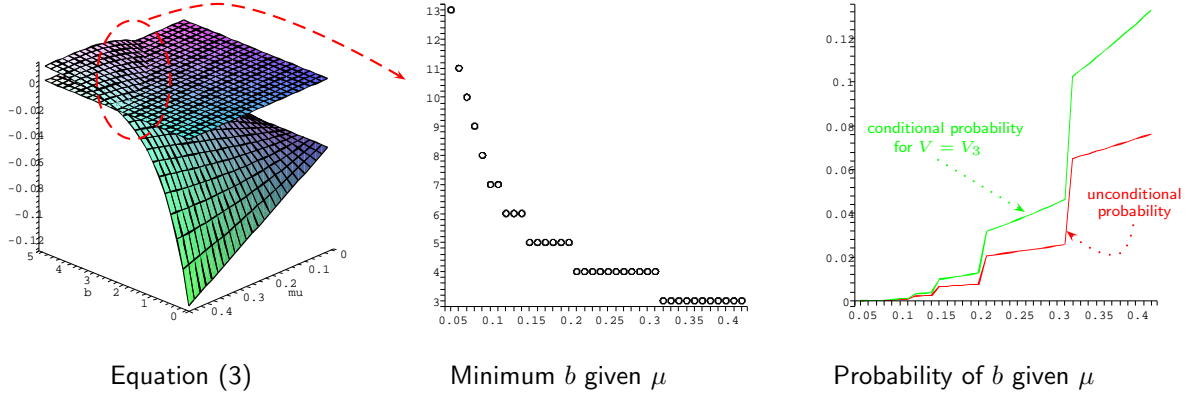| Equation (3) | Minimum b given μ | Probability of b given μ |

Figure 4: **Trades needed for Herding the Probabilities for these trades.** The left panel plots the value of expression (3) as a function of μ, with $\mu \in (0, \mu_b)$, and of no-herd buys b. Whenever the bend curve crosses the 0-surface from below, herding is triggered. The middle panel computes the minimum integer number of no-herd buys that would trigger herding as a function of noise level μ. The right panel computes two probabilities: the first is the probability of having exactly the threshold number of buys at the beginning of trade (the thresholds are taken from the middle panel) conditional on the true state being V_3. The second probability is the unconditional likelihood of this threshold number. The plots in the right panel are functions of the μ. The signal distribution that underlies these plots is listed in Appendix D.

term) becomes smaller. This implies that for any history, the difference between the market maker's and the S_2 type's expectation becomes smaller; thus to get buy-herding one needs more buys. Second, as noise decreases, the informational content of past behavior (public information) improves and this makes herding more likely. Formally, the second and third terms in (3), the negative terms, decline as μ increases. This is because for any $i = 2, 3$, $\frac{\beta_1}{\beta_i} = \frac{\mu \Pr(S_3|V_1) + \gamma}{\mu \Pr(S_3|V_i) + \gamma}$, $\frac{\partial(\beta_1/\beta_i)}{\partial \mu} = (\Pr(S_3|V_1) - \Pr(S_3|V_i))/\beta_i^2$ and thus, since S_1's csd is decreasing, $\frac{\partial(\beta_1/\beta_i)}{\partial \mu} < 0$.

While we do not have an analytical result on the net effect of increasing μ from 0 to μ_b, in all numerical examples that we computed the second effect dominates. Thus as noise trading declines (μ increases to μ_b) it takes *fewer* buys to trigger buy-herding. Figure 4 plots the minimum number of such consecutive time-zero buys needed to trigger buy-herding for our simulations. As the amount of noise decreases, ex ante it gets more likely that these consecutive buy-trades occur. (Figure 4's right panel illustrates these probabilities.)