Hitchin Functionals, h-Principles and Spectral Invariants

Laurence Hamilton Mayther

Department of Pure Mathematics and Mathematical Statistics
&
St John's College
University of Cambridge

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Dedicated to my Mother.

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration. It is not substantially the same as any work that has already been submitted before for any degree, or other qualification.

Laurence Hamilton Mayther

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Abstract

This thesis investigates Hitchin functionals and h-principles for stable forms on oriented manifolds, with a special focus on G_2 and $\widetilde{\mathrm{G}}_2$ 3- and 4-forms. Additionally, it introduces two new spectral invariants of torsion-free G_2-structures.

Part I begins by investigating an open problem posed by Bryant, *viz.* whether the Hitchin functional $\mathcal{H}_3$ on closed G_2 3-forms is unbounded above. Chapter 3 uses a scaling argument to obtain sufficient conditions for the functional $\mathcal{H}_3$ to be unbounded above and applies this result to prove the unboundedness above of $\mathcal{H}_3$ on two explicit examples of closed 7-manifolds with closed G_2 3-forms. Chapter 3 then proceeds to interpret this unboundedness geometrically, demonstrating an unexpected link between the functional $\mathcal{H}_3$ and fibrations, proving that the 'large volume limit' of $\mathcal{H}_3$ in each case corresponds to the adiabatic limit of a suitable fibration. The proof utilises a new, general collapsing result for singular fibrations between orbifolds, without assumptions on curvature, which is proved in Chapter 4. Chapter 5 broadens the focus of Part I to include the Hitchin functionals $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ on closed G_2 4-forms, $\widetilde{\mathrm{G}}_2$ 3-forms and $\widetilde{\mathrm{G}}_2$ 4-forms respectively. In its main result, Chapter 5 proves that $\mathcal{H}_4, \widetilde{\mathcal{H}}_3, \widetilde{\mathcal{H}}_4$ are always unbounded above and below (whenever defined), and also that $\mathcal{H}_3$ is always unbounded below (whenever defined). As scholia, the critical points of the functionals $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ are shown to be saddle points, and initial conditions of the Laplacian coflow which cannot lead to convergent solutions are shown to be dense. Part I ends with a short discussion of open questions, in Chapter 6.

Part II investigates relative h-principles for closed, stable forms. After establishing some prerequisite algebraic results, Chapter 7 begins by proving that if a class of closed, stable forms satisfies the relative h-principle, then its corresponding Hitchin functional is automatically unbounded above. By utilising the technique of convex integration, Chapter 7 then obtains sufficient conditions for a class of closed, stable forms to satisfy the relative h-principle, a result which subsumes all previously established h-principles for closed stable forms. Until now, 12 of the 16 possible classes of closed stable forms have remained open questions with regard to the relative h-principle. In the main result of Part II, Chapters 7 and 8 prove the relative h-principle in 5 of these open cases. The remaining 7 cases are addressed in the final chapter of Part II, where it is conjectured that the relative h-principle holds in each case. Chapter 9 applies the h-principles established in this thesis to prove various results on the topological properties of closed $\widetilde{\mathrm{G}}_2$, $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ forms. Firstly, it characterises which oriented 7-manifolds admit closed $\widetilde{\mathrm{G}}_2$ forms, in the process introducing a new technique for proving the vanishing of natural cohomology classes on non-closed manifolds. Next, it introduces $\widetilde{\mathrm{G}}_2$-cobordisms of closed $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms and proves that homotopic forms are $\widetilde{\mathrm{G}}_2$-cobordant. Additionally, Chapter 9 classifies $\mathrm{SL}(3;\mathbb{C})$ 3-forms up to homotopy and provides a partial classification result on homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms. Part II ends with a short discussion of open questions, in Chapter 10.

Part III introduces and examines two new spectral invariants of torsion-free G_2-structures. Although the notion of an invariant is a central theme in geometry and topology, currently, there is only one known invariant of torsion-free G_2-structures: the $\overline{\nu}$-invariant of Crowley–Goette–Nordström. Part III defines two new invariants of torsion-free G_2-structures, termed μ_3- and μ_4-invariants, by regularising the classical notion of Morse index for the Hitchin functionals $\mathcal{H}_3$ and $\mathcal{H}_4$ at their critical points. In general, there is no known way to compute $\overline{\nu}$ for G_2-manifolds constructed via Joyce's 'generalised Kummer construction'. Chapter 11 obtains closed formulae for μ_3 and μ_4 on the orbifolds used in Joyce's construction, leading to a conjectural discussion in Chapter 12 of how to compute μ_3 and μ_4 on Joyce's manifolds.

Contents

Notation

- Unless stated otherwise, $(e_1, ..., e_n)$ shall denote the standard basis of $\mathbb{R}^n$, so that e_i is given by the transpose of
$$e_i = (0, ..., 0, \underbrace{1}_{i^{\text{th}}\text{ position}}, 0, ..., 0),$$
and $\left(\theta^1, ..., \theta^7\right)$ shall denote the corresponding dual basis of $(\mathbb{R}^n)^*$. Multi-index notation:
$$\theta^{ij...k} = \theta^i \wedge \theta^j \wedge ... \wedge \theta^k \quad \text{and} \quad e_{ij...k} = e_i \wedge e_j \wedge ... \wedge e_k$$
is used throughout this thesis. The canonical orientation on $\mathbb{R}^n$ is then fixed by declaring $\theta^{1...n} > 0$.

- Use the symbol $\bigodot^n$ to denote the n^{th} exterior tensor power. Thus given a real vector space $\mathbb{A}$, $\bigodot^2 \mathbb{A}^*$ is the space of symmetric bilinear forms on $\mathbb{A}$. Write $\bigodot^2_+ \mathbb{A}^*$ for the space of positive-definite symmetric bilinear forms (i.e. inner-products) on $\mathbb{A}$ and write $\bigodot^2_{\geqslant 0} \mathbb{A}^*$ for the space of non-negative definite symmetic bilinear forms on $\mathbb{A}$. Define a partial order on $\bigodot^2 \mathbb{A}^*$ by declaring $g \geqslant g'$ if and only if $g - g' \in \bigodot^2_{\geqslant 0} \mathbb{A}^*$. Given $g \in \bigodot^2 \mathbb{A}^*$ and $a \in \mathbb{A}$, write $g(a)$ as an abbreviation for $g(a,a)$. Finally, given $g \in \bigodot^2 \mathbb{A}^*$, define the kernel of g to be the kernel of the linear map $()^\flat : a \in \mathbb{A} \mapsto g(a,-) \in \mathbb{A}^*$.

- Following [62], given a topological space X and a subset $A \subset X$, write $\mathcal{O}p(A)$ for an arbitrarily small but unspecified open neighbourhood of A in M, which may be shrunk whenever necessary, and given $x \in X$, write $\mathcal{O}p(x)$ as a short-hand for $\mathcal{O}p(\{x\})$.

- Given a manifold M and a bundle $\pi : E \to \mathrm{M}$, unless alternative notation is defined, use $\Gamma(E,-)$ to denote the sheaf of smooth sections of E.

- Define symbols $\gtrless$ and $\gtreqless$ be analogy with $\pm$ to 'mean greater than or less than, respectively' and 'greater than or equal to, or less than or equal to, respectively', where, by convention, the upper-most symbol should be read first. Thus the equation $a \pm b \gtrless c$ should be interpreted as the pair of equations:
$$a + b > c \quad \text{and} \quad a - b < c$$
while the equation $a \mp b \gtreqless c$ denotes:
$$a - b \geqslant c \quad \text{and} \quad a - b \leqslant c.$$

Chapter 1

Introduction

In 1955, Berger [16] published the first list of possible pseudo-Riemannian holonomy groups of simply connected, non-locally symmetric manifolds whose holonomy group acts irreducibly. Since then, Berger's list has been perfected through the work of many authors, including [3, 4, 20, 26, 21, 54, 55] and others, and is now known to contain 6 exceptional holonomy groups, *viz.* G_2, $\widetilde{\mathrm{G}}_2$ $\mathrm{G}_2^{\mathbb{C}}$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(3,4)$ and $\mathrm{Spin}(7;\mathbb{C})$, which can occur only in dimensions 7, 7, 14, 8, 8 and 16 respectively (see [53, §2.2] for the full classification). The initial motivation for this thesis stems from one of these exceptional groups, *viz.* G_2.

The existence of (incomplete) manifolds with holonomy G_2 was first established by Bryant [21], with complete examples later independently constructed by Bryant–Salamon [23] and Gibbons–Page–Pope [56]. The first compact manifolds with holonomy G_2 were constructed by Joyce [76, 77, 78] with further examples constructed by Kovalev [87], Kovalev–Lee [88], Corti–Haskins–Nordström–Pacini [29], Joyce–Kariagiannis [80], Nordström [110] and others.

One further candidate for constructing compact manifolds with holonomy G_2 was proposed by Bryant (jointly with Altschuler) in [22]. Their proposal is most appropriately phrased in terms of Hitchin functionals, introduced in [71, 72]. Given an oriented 7-manifold M, a G_2-structure on M may be characterised by a 3-form ϕ of a certain algebraic type, called a G_2 3-form (see §2.2 for a precise defintion). Since $\mathrm{G}_2 \subset \mathrm{SO}(7)$, ϕ induces a metric g_ϕ and orientation on M, and hence a volume form vol_ϕ and Hodge star $\bigstar_\phi$. When $\mathrm{d}\phi = 0$ and M is closed, the Hitchin functional $\mathcal{H}_3$ on (M,ϕ) is defined by:

$$\begin{aligned} \mathcal{H}_3 : [\phi]_+ = \left\{\phi' \in [\phi] \in H^3_{\mathrm{dR}}(\mathrm{M}) \;\middle|\; \phi' \text{ is of } \mathrm{G}_2\text{-type}\right\} &\longrightarrow (0,\infty) \\ \phi' &\longmapsto \int_{\mathrm{M}} vol_{\phi'} \end{aligned}$$

Since G_2 3-forms are stable in the sense of Hitchin [72], $[\phi]_+ \subset [\phi]$ is open in the C^0-topology. The critical points ϕ' of $\mathcal{H}_3$ are then characterised by the condition $\mathrm{d}\bigstar_{\phi'}\phi' = 0$, which is equivalent to $\mathrm{Hol}(g_{\phi'}) \subseteq \mathrm{G}_2$ by a well-known result of Fernández–Gray [49]; such ϕ' are termed torsion-free. In view of this, Altschuler–Bryant proposed a construction whereby closed manifolds with holonomy G_2 would be constructed as the limit of the gradient flow of the $\mathcal{H}_3$. As remarked in [22, Remark 17], for this construction to be feasible, it would be desirable for the functional $\mathcal{H}_3$ to be bounded above. This motivated the initial question for this thesis:

Question 1.0.1. *Is the functional $\mathcal{H}_3$ unbounded above?*

Part I of this thesis examines Question 1.0.1 in depth. Chapter 3 introduces a new scaling argument (Proposition 3.1.9) for proving the unboundedness above of the functional $\mathcal{H}_3$ and applies the argument to prove the unboundedness above of the Hitchin functional $\mathcal{H}_3$ on two examples of closed 7-manifolds with closed G_2-structures. The first is a 4-dimensional family $(N, \phi(\alpha,\beta,\lambda))_{(\alpha,\beta,\lambda)\in(\mathbb{R}\backslash\{0\})^2\times(\mathbb{C}\backslash\{0\})}$ of closed 7-manifolds equipped with closed G_2 3-forms inspired by Fernández' short paper [46], where N is the product of S^1 with the Nakamura manifold constructed by de Bartolomeis–Tomassini [14]). The second is the manifold $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$ constructed by Fernández–Fino–Kovalev–Muñoz in [48].

Theorem 1.0.2 (Theorems 3.2.9 and 3.4.12)**.**

1. *The map:*

$$\begin{aligned}(\mathbb{R}\backslash\{0\})^2\times(\mathbb{C}\backslash\{0\}) &\longrightarrow H^3_{\mathrm{dR}}(N)\\ (\alpha,\beta,\lambda) &\longmapsto [\phi(\alpha,\beta,\lambda)]\end{aligned}$$

   is injective, and for all $(\alpha,\beta,\lambda)\in(\mathbb{R}\backslash\{0\})^2\times(\mathbb{C}\backslash\{0\})$, there exists a family $\phi(\alpha,\beta,\lambda;\mu)\in[\phi(\alpha,\beta,\lambda)]_+$, $\mu\in[1,\infty)$ such that:

$$\lim_{\mu\to\infty}\mathcal{H}_3(\phi(\alpha,\beta,\lambda;\mu)) = \infty;$$

2. *There exists a family $\breve{\phi}^\mu\in\left[\breve{\phi}\right]$, $\mu\in[1,\infty)$ such that:*

$$\lim_{\mu\to\infty}\mathcal{H}_3\left(\breve{\phi}^\mu\right) = \infty.$$

In proving (2), careful treatment of the resolution of singularities in the construction of the manifold $\breve{\mathrm{M}}$ is required, in order to ensure that the rescaled forms are cohomologically constant.

The explicit families $(\phi(\alpha,\beta,\lambda;\mu))_{\mu\in[0,\infty)}$ and $\left(\breve{\phi}^\mu\right)_{\mu\in[0,\infty)}$ constructed in Theorem 1.0.2 are each of further geometric interest. Recall from [48] that the manifold $\breve{\mathrm{M}}$ can be regarded as the total space of a singular fibration π over $S^1\times\mathbb{T}^2\big\backslash\{\pm1\}$ with generic fibre $\mathbb{T}^4$. Chapter 3 shows that N can also be regarded as the total space of a (twisted) $\mathbb{T}^6$-fibration $\mathfrak{p}$ over S^1. The families $(\phi(\alpha,\beta,\lambda;\mu))_{\mu\in[0,\infty)}$ and $\left(\breve{\phi}^\mu\right)_{\mu\in[0,\infty)}$ are closely related to the fibrations $\mathfrak{p}$ and π; this relation is made precise by the following theorem:

Theorem 1.0.3 (Theorem 3.3.1 and 3.5.8)**.**

- *Let $(\alpha,\beta,\lambda)\in(\mathbb{R}\backslash\{0\})^2\times(\mathbb{C}\backslash\{0\})$ and let $(N,\phi(\alpha,\beta,\lambda;\mu))_{\mu\in[1,\infty)}$ be as in Theorem 1.0.2. Then the large volume limit of $(N,\phi(\alpha,\beta,\lambda;\mu))$ corresponds to an adiabatic limit of $\mathfrak{p}$. Specifically, writing $\ell = \log\frac{3+\sqrt{5}}{2}$ for the constant arising in the construction of the Nakamura manifold X:*

$$\left(N,\mu^{-12}\phi(\alpha,\beta,\lambda;\mu)\right)\to\left(\mathbb{R}\big/\ell\mathbb{Z}, \alpha^2\left(\lambda\bar{\lambda}\right)^{-\frac{2}{3}}g_{\mathrm{Eucl}}\right)\quad \textit{as}\quad \mu\to\infty,$$

  where the convergence is in the Gromov–Hausdorff sense.

- *Let* $\left(\breve{\mathrm{M}}, \breve{\phi}^{\mu}\right)_{\mu\in[1,\infty)}$ *be as in Theorem 1.0.2. Then the large volume limit of* $\left(\breve{\mathrm{M}}, \breve{\phi}^{\mu}\right)$ *corresponds to an adiabatic limit of the fibration* π*. Specifically, let* B *denote the orbifold* $\{\pm1\}\backslash\mathbb{T}^2 \times S^1$*. Then:*
$$\left(\breve{\mathrm{M}}, \mu^{-6}\breve{\phi}^{\mu}\right) \to (B, d) \quad \text{as} \quad \mu \to \infty$$
in the Gromov–Hausdorff sense, where d *is some suitable metric (i.e. distance function) on* B*.*

I remark that, since neither N nor $\breve{\mathrm{M}}$ admit torsion-free G_2-structures, Theorem 1.0.3 demonstrates the potential geometric relevance of the functional $\mathcal{H}_3$ even to manifolds which do not admit torsion-free G_2-structures.

The limiting metric d on the base space B cited in Theorem 1.0.3 is locally Euclidean outside a neighbourhood of the singular locus of B, but globally is not induced by a Riemannian metric; rather it is induced by a certain 'stratified' geometric structure, which I term a stratified quasi-Finslerian structure. This new class of structures on orbifolds is defined formally in §3.5.1. The proof of Theorem 1.0.3 combines suitable geometric estimates on the Riemannian metrics induced by $\mu^{-12}\phi(\alpha, \beta, \lambda; \mu)$ with $\mu^{-6}\breve{\phi}^{\mu}$ with a technical collapsing result for singular fibrations between orbifolds, stated in Theorem 4.2.5. This theorem is distinct from similar theorems in the literature since it does not require bounds on curvature or injectivity radius of $(E, \widehat{g}^{\mu})$ and thus allows for Gromov–Hausdorff limits which have strictly lower dimension than the family whose limit is under consideration. The proof of Theorem 4.2.5 occupies Chapter 4, and requires the introduction and investigation of a new class of stratified fibrations between orbifolds, termed weak submersions.

Part I ends by considering two possible generalisations of Question 1.0.1. Firstly, attention is broadened from the functional $\mathcal{H}_3$ on G_2 3-forms, to include the analogous Hitchin functionals $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ on G_2 4-forms, $\widetilde{\mathrm{G}}_2$ 3-forms and $\widetilde{\mathrm{G}}_2$ 4-forms respectively (see §2.2.3 for a precise definition). Secondly, it is asked whether the functionals $\mathcal{H}_3$, $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ are bounded bounded below, in a logarithmic sense (i.e. bounded away from 0). As its main result, Chapter 5 proves:

Theorem 1.0.4. *The functionals* $\mathcal{H}_4$*,* $\widetilde{\mathcal{H}}_3$ *and* $\widetilde{\mathcal{H}}_4$ *are always unbounded above and below, and* $\mathcal{H}_3$ *is always unbounded below. Specifically, let* M *be a closed 7-manifold (or, more generally, 7-orbifold) and let* ϕ*,* ψ*,* $\widetilde{\phi}$ *and* $\widetilde{\psi}$ *be closed* G_2 *3-forms,* G_2 *4-forms,* $\widetilde{\mathrm{G}}_2$ *3-forms and* $\widetilde{\mathrm{G}}_2$ *4-forms on* M *respectively and let* $\left[\widetilde{\phi}\right]_\sim$ *and* $\left[\widetilde{\psi}\right]_\sim$ *be as defined in §2.2.1. Then:*

- $\inf_{\psi' \in [\psi]_+} \mathcal{H}_4(\psi') = 0$ *and* $\sup_{\psi' \in [\psi]_+} \mathcal{H}_4(\psi') = \infty$*;*
- $\inf_{\widetilde{\phi}' \in [\widetilde{\phi}]_\sim} \widetilde{\mathcal{H}}_3\left(\widetilde{\phi}'\right) = 0$ *and* $\sup_{\widetilde{\phi}' \in [\widetilde{\phi}]_\sim} \widetilde{\mathcal{H}}_3\left(\widetilde{\phi}'\right) = \infty$*;*
- $\inf_{\widetilde{\psi}' \in [\widetilde{\psi}]_\sim} \widetilde{\mathcal{H}}_4\left(\widetilde{\psi}'\right) = 0$ *and* $\sup_{\widetilde{\psi}' \in [\widetilde{\psi}]_\sim} \widetilde{\mathcal{H}}_4\left(\widetilde{\psi}'\right) = \infty$*;*
- $\inf_{\phi' \in [\phi]_+} \mathcal{H}_3(\phi') = 0$*.*

Recall that it was shown by Hitchin [71] that the critical points of $\mathcal{H}_3$ are local maxima. Previously, however, the nature of the critical points of $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ has not been established. As a scholium of Theorem 1.0.4, Chapter 5 obtains:

Theorem 1.0.5 (Theorems 5.1.11 and 5.5.5)**.** *The critical points of $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ are always saddles. Specifically, let* M *be a closed, oriented 7-manifold (or, more generally, 7-orbifold) and let ψ be a torsion-free* G_2 *4-form on* M*. Then there exist infinite-dimensional subspaces $\mathcal{S}_4^{\pm}(\psi) \subset \mathrm{T}_\psi[\psi]_+$ along which $\mathcal{D}^2\mathcal{H}_4|_\psi$ is positive definite and negative definite respectively. The analogous statement holds for the functionals $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$.*

As a second scholium, let M be an oriented 7-manifold and recall that, given a closed G_2 4-form ψ on M, the Laplacian coflow of ψ is defined to be the solution of the evolution PDE:

$$\frac{\partial\psi(t)}{\partial t} = \Delta_{\psi(t)}\psi(t) = \mathrm{dd}^*_{\psi(t)}\psi(t) \quad \text{and} \quad \psi(0) = \psi. \tag{1.0.6}$$

(Note that I adopt the sign convention for Laplacian coflow used in [59], rather than that used in the the original paper [83].) Whilst the existence and uniqueness of the Laplacian coflow have yet to be proven, Laplacian coflow can be regarded as the gradient flow of the Hitchin functional $\mathcal{H}_4$ [59]. Consequently, Theorems 1.0.4 and 1.0.5 intuitively suggest that most solutions of the Laplacian coflow on a given manifold M (when they exist) will not converge to a torsion-free G_2 4-form as $t \to \infty$. Chapter 5 confirms this expectation, by proving the following result:

Theorem 1.0.7. *Let* M *be an oriented 7-manifold (not necessarily closed) and let $\psi \in \Omega^4_+(\mathrm{M})$ be a closed* G_2 *4-form. Consider the space:*

$$\mathcal{O}_{[\psi]_+} = \left\{\psi' \in [\psi]_+ \;\middle|\; \begin{matrix}\textit{no solution to the Laplacian coflow started at}\\ \psi' \textit{ converges to a torsion-free } \mathrm{G}_2 \textit{ 4-form}\end{matrix}\right\}.$$

Then $\mathcal{O}_{[\psi]_+} \subset [\psi]_+$ is dense in the C^0 topology.

Part II broadens the scope of investigation from G_2 and $\widetilde{\mathrm{G}}_2$ forms to more general classes of geometric structures. Recall that, in the terminology of Hitchin [72], a p-form $\sigma_0 \in \Lambda^p(\mathbb{R}^n)^*$ is stable if its $\mathrm{GL}_+(n;\mathbb{R})$-orbit in $\Lambda^p(\mathbb{R}^n)^*$ is open; examples include the G_2 and $\widetilde{\mathrm{G}}_2$ forms considered in Part I. Given an oriented n-manifold M, say that $\sigma \in \Omega^p(\mathrm{M})$ is a σ_0-form if, for each $p \in \mathrm{M}$, $(\mathrm{T}_p\mathrm{M}, \sigma|_p)$ is oriented-isomorphic to $(\mathbb{R}^n, \sigma_0)$. If M is closed and $\mathrm{d}\sigma = 0$, then provided $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0) \subseteq \mathrm{SL}(n;\mathbb{R})$, there is a natural Hitchin functional $\mathcal{H}$ on the set $\mathcal{C}l^p_{\sigma_0}([\sigma])$ of σ_0-forms in the de Rham class $[\sigma]$, defined by analogy with $\mathcal{H}_3$. The initial motivation for Part II was to generalise the questions posed in Part I to these more general Hitchin functionals, and study their unboundedness above via the notion of relative h-principles, which I now briefly define.

Given $\sigma_0 \in \Lambda^p(\mathbb{R}^n)^*$, an oriented n-manifold M and a fixed cohomology class $\alpha \in H^p_{\mathrm{dR}}(\mathrm{M})$, write $\mathcal{C}l^p_{\sigma_0}(\mathrm{M})$ for the set of closed σ_0-forms on M and recall the set $\mathcal{C}l^p_{\sigma_0}(\alpha)$. More generally, given a possibly empty submanifold $A \subset \mathrm{M}$, let σ_r be a closed σ_0-form on $\mathcal{O}p(A)$ such that $[\sigma_r] = \alpha|_{\mathcal{O}p(A)} \in$

$H^p_{\mathrm{dR}}(\mathcal{O}p(A))$ and write:

$$\Omega^p_{\sigma_0}(\mathrm{M};\sigma_r) = \left\{\sigma \in \Omega^p_{\sigma_0}(\mathrm{M}) \;\middle|\; \sigma|_{\mathcal{O}p(A)} = \sigma_r\right\}$$
$$\mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) = \left\{\sigma \in \Omega^p_{\sigma_0}(\mathrm{M};\sigma_r) \;\middle|\; \mathrm{d}\sigma = 0\right\}$$
$$\mathcal{C}l^p_{\sigma_0}(\alpha;\sigma_r) = \left\{\sigma \in \mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) \;\middle|\; [\sigma] = \alpha \in H^p_{\mathrm{dR}}(\mathrm{M})\right\}.$$

Say that σ_0-forms satisfy the relative h-principle if for every M, A, α and σ_r, the inclusions:

$$\mathcal{C}l^p_{\sigma_0}(\alpha;\sigma_r) \hookrightarrow \mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M};\sigma_r)$$

are homotopy equivalences. (In fact, this thesis uses a slightly stronger notion of h-principle; see §7.2.) Such an h-principle is of significant independent geometric interest: indeed, taking $A = \varnothing$, the inclusions:

$$\mathcal{C}l^p_{\sigma_0}(\alpha) \hookrightarrow \mathcal{C}l^p_{\sigma_0}(\mathrm{M}) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M})$$

are also homotopy equivalences and thus, if M admits any σ_0-form (a question which can be answered using purely topological methods), then every degree p de Rham class on M can be represented by a σ_0-form. In addition to this, the relative h-principle is relevant to the study of Hitchin functionals, as the following result demonstrates:

Theorem 1.0.8 (Theorem 7.2.3)**.** *Let σ_0 be a stable form such that $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0) \subseteq \mathrm{SL}(n;\mathbb{R})$ and suppose that σ_0-forms satisfy the relative h-principle. For any closed, oriented n-manifold M admitting σ_0-forms and any $\alpha \in H^p_{\mathrm{dR}}(\mathrm{M})$, the Hitchin functional:*

$$\mathcal{H} : \mathcal{C}l^p_{\sigma_0}(\alpha) \to (0,\infty)$$

is unbounded above. More generally, if M is a closed, oriented n-orbifold and $\mathcal{C}l^p_{\sigma_0}(\alpha) \neq \varnothing$, then the same conclusion applies.

If $A = \varnothing$ and M is open (i.e. not closed), the inclusions $\mathcal{C}l^p_{\sigma_0}(\alpha) \hookrightarrow \mathcal{C}l^p_{\sigma_0}(\mathrm{M}) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M})$ are known to be homotopy equivalences for any σ_0, by the techniques introduced [60] (see also [42, 32]). However if M is closed, or $A \neq \varnothing$, the question of which σ_0 satisfy the relative h-principle remains an open problem. More specifically, there are essentially 16 classes of closed stable forms (see §7.1.3 and Remark 7.2.4). Of these 16 classes, only 3 are known to satisfy the relative h-principle, *viz.* stable 2-forms on odd-dimensional manifolds (McDuff [104]), G_2 4-forms (Crowley–Nordström [32]) and $\mathrm{SL}(3;\mathbb{C})$ 3-forms (Donaldson [37]). Conversely, symplectic forms are widely known not to satisfy the relative h-principle (see, e.g. [42]). The answer in all remaining 12 classes has remained open. The main result of Part II, proven in Chapters 7 and 8, resolves 5 of these open cases:

Theorem 1.0.9 ((Theorems 7.6.4, 7.6.5, 7.7.5, 7.7.44, 8.2.1))**.** *The relative h-principle holds for each of the following classes of closed, stable forms:*

- *Co-symplectic forms (i.e. stable $(2k-2)$-forms in dimension $2k$, $k \geqslant 3$);*
- *Co-pseudoplectic forms (i.e. stable $(2k-1)$-forms in dimension $2k+1$, $k \geqslant 2$);*

- $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms;*

- $\widetilde{\mathrm{G}}_2$ *3-forms;*

- $\widetilde{\mathrm{G}}_2$ *4-forms.*

As an immediate consequence of Theorem 1.0.9 and Theorem 1.0.8, one obtains:

Theorem 1.0.10. *If* M *admits any* $\widetilde{\mathrm{G}}_2$ *3-form, then every degree 3 de Rham class on* M *can be represented by a* $\widetilde{\mathrm{G}}_2$ *3-form and likewise for* $\widetilde{\mathrm{G}}_2$ *4-forms,* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms, co-symplectic forms and co-pseudoplectic forms. Moreover, the Hitchin functionals on* $\widetilde{\mathrm{G}}_2$ *3-forms,* $\widetilde{\mathrm{G}}_2$ *4-forms,* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms,* $\mathrm{SL}(3;\mathbb{C})$ *3-forms and co-symplectic forms are always unbounded above, whenever defined (note that (co-)pseudoplectic forms do not have a corresponding Hitchin functional): e.g. in the case of* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms, let* M *be any closed, oriented 6-manifold admitting* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms. Then for each* $\alpha \in H^3_{\mathrm{dR}}(\mathrm{M})$*, the functional:*

$$\mathcal{H} : \mathcal{C}l^3_{\rho_+}(\alpha) \to (0,\infty)$$

is unbounded above. More generally, if M *is a closed, oriented* 6*-orbifold and* $\mathcal{C}l^3_{\rho_+}(\alpha) \neq \varnothing$*, then the same conclusion applies.*

In particular, note that Theorem 1.0.10 provides an alternative proof of the unboundedness above of $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ shown in Part I.

Essential to the proof of Theorem 1.0.9 is the technique of convex integration, introduced by Gromov in [61] and developed in [62, 116, 42, 98]. In particular, by using convex integration, the following result, which plays a key role in the proof of Theorem 1.0.9, is established:

Theorem 1.0.11 (Theorem 7.3.1)**.** *Let* $\sigma_0 \in \bigwedge^p(\mathbb{R}^n)^*$ *be stable. Given an arbitrary* p *form* τ *on* $\mathbb{R}^{n-1}$*, define:*

$$\mathcal{N}_{\sigma_0}(\tau) = \left\{\nu \in \bigwedge^{p-1}\left(\mathbb{R}^{n-1}\right)^* \;\middle|\; \theta\wedge\nu + \tau \in \bigwedge^p_{\sigma_0}\left(\mathbb{R}\oplus\mathbb{R}^{n-1}\right)^*\right\} \subset \bigwedge^{p-1}\left(\mathbb{R}^{n-1}\right)^*$$

where θ *is the standard annihilator of* $\mathbb{R}^{n-1} \subset \mathbb{R}\oplus\mathbb{R}^{n-1}$*. Suppose that, for every* τ*, the set* $\mathcal{N}_{\sigma_0}(\tau)$ *is ample in the sense of affine geometry, i.e.* $\mathcal{N}_{\sigma_0}(\tau)$ *is either empty, or the convex hull of every path component of* $\mathcal{N}_{\sigma_0}(\tau)$ *equals* $\bigwedge^{p-1}\left(\mathbb{R}^{n-1}\right)^*$ *(in such cases, say that* σ_0 *itself is ample). Then* σ_0*-forms satisfy the relative* h*-principle.*

I remark that Theorem 1.0.11 subsumes all three previously known h-principles for stable forms, *viz.* the relative h-principles for stable 2-forms in $(2k+1)$-dimensions ($k \geqslant 2$), G_2 4-forms and $\mathrm{SL}(3;\mathbb{C})$ 3-forms; see §7.5.

Theorem 1.0.9, together with the relative h-principle for $\mathrm{SL}(3;\mathbb{C})$ 3-forms, shows that the topological properties of the spaces of $\widetilde{\mathrm{G}}_2$ 3- and 4-forms, $\mathrm{SL}(3;\mathbb{C})$ 3-forms and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms which are closed, or which lie in any given cohomology class, can be understood by studying the spaces of all $\widetilde{\mathrm{G}}_2$ 3- and 4-forms, $\mathrm{SL}(3;\mathbb{C})$ 3-forms and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, respectively. These spaces can be investigated using the standard bundle-theoretic techniques of characteristic classes and obstruction theory; Part II ends in Chapter 9 by carrying out such an investigation. Chapter 9 begins by proving the following conjecture of Lê in [92]:

Theorem 1.0.12. *Let* M *be an oriented 7-manifold (not necessarily closed). Then* M *admits* $\widetilde{\mathrm{G}}_2$*-structures if and only if it is spin.*

In the process, the following result is established, which the author hopes will have many applications beyond those used in this thesis:

Theorem 1.0.13. *Suppose there is an assignment to each n-manifold* M *(with – possibly empty – boundary) of a degree p cohomology class* $\nu(\mathrm{M}) \in \mathrm{H}^p(\mathrm{M}; G)$*, where G is either a field or a finite Abelian group, and suppose moreover that the assignment is natural, in the sense that for each embedding* $f : \mathrm{M} \hookrightarrow \mathrm{M}'$ *of n-manifolds with boundary, the identity:*

$$\nu(\mathrm{M}) = f^*\nu(\mathrm{M}')$$

holds. Finally, suppose that ν *vanishes on all closed (resp. closed, oriented) n-manifolds. Then* ν *vanishes on all (resp. all oriented) n-manifolds with boundary.*

Combining Theorem 1.0.12 with Theorem 1.0.10 yields the following corollary:

Theorem 1.0.14. *Let* M *be an oriented 7-manifold. If* M *is spin, then every degree 3 de Rham class can be represented by a* $\widetilde{\mathrm{G}}_2$ *3-form and every degree 4 de Rham class can be represented by a* $\widetilde{\mathrm{G}}_2$ *4-form.*

Next, Chapter 9 investigates the link between closed $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms in 6-dimensions and closed $\widetilde{\mathrm{G}}_2$ 3-forms in 7-dimensions. Say that an $\mathrm{SL}(3;\mathbb{C})$ or $\mathrm{SL}(3;\mathbb{R})^2$ 3-form ρ on an oriented 6-manifold N is extendible if there exists an oriented 7-manifold with boundary M such that $\partial\mathrm{M}$ contains N as a connected component, and a closed $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi}$ on M such that $\widetilde{\phi}|_{\mathrm{N}} = \rho$. Motivated by Donaldson's notion of G_2-cobordism introduced in [37], say that two oriented 6-manifolds (N_1, ρ_1) and (N_2, ρ_2) equipped with closed, extendible, $\mathrm{SL}(3;\mathbb{C})$ (resp. $\mathrm{SL}(3;\mathbb{R})^2$) 3-forms are $\widetilde{\mathrm{G}}_2$-cobordant if there exists an oriented 7-manifold M with boundary $\partial\mathrm{M} = \mathrm{N}_1 \coprod \overline{\mathrm{N}}_2$ and a closed $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi}$ on M such that:

$$\widetilde{\phi}|_{\mathrm{N}_1} = \rho_1 \quad \text{and} \quad \widetilde{\phi}|_{\mathrm{N}_2} = \rho_2$$

(where overline denotes orientation-reversal).

Theorem 1.0.15. *Let* N *be a 6-manifold and let* ρ, ρ' *be closed, extendible* $\mathrm{SL}(3;\mathbb{C})$ *(resp.* $\mathrm{SL}(3;\mathbb{R})^2$*) 3-forms on* N*. Suppose that* ρ *and* ρ' *are homotopic and lie in the same cohomology class. Then* (N, ρ) *and* (N, ρ') *are* $\widetilde{\mathrm{G}}_2$*-cobordant.*

I remark that, in contrast, the analogous result for G_2-cobordisms is not known; see [37], particularly the discussion on p. 116.

Motivated by Theorem 9.3.3, the remainder of Chapter 9 investigates when two closed $\mathrm{SL}(3;\mathbb{C})$ (resp. $\mathrm{SL}(3;\mathbb{R})^2$) 3-forms are homotopic, and when a given $\mathrm{SL}(3;\mathbb{C})$ (resp. $\mathrm{SL}(3;\mathbb{R})^2$) 3-form is extendible. Let N be an oriented 6-manifold and let $\mathcal{SL}_{\mathbb{C}}(\mathrm{N})$ denote the set of homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on N. Since $\mathrm{SL}(3;\mathbb{C})$ deformation retracts onto the simply-connected subgroup $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, each $\mathrm{SL}(3;\mathbb{C})$ 3-form ρ defines a choice of spin structure on N, which depends only on the homotopy class of ρ. Thus there is a map:

$$\sigma : \mathcal{SL}_{\mathbb{C}}(\mathrm{N}) \to Spin(\mathrm{N})$$

Theorem 1.0.16. *The map σ is bijective. In particular, there is a 1-1 correspondence between homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on N (equivalently closed $\mathrm{SL}(3;\mathbb{C})$ 3-forms, or $\mathrm{SL}(3;\mathbb{C})$ 3-forms in any fixed degree 3 de Rham class) and spin structures on N, which in turn correspond non-canonically with elements of $H^1\left(\mathrm{N}, \mathbb{Z}/2\mathbb{Z}\right)$.*

I remark that Theorem 1.0.16 corrects an error in Donaldson's paper [37, p. 116], where it is stated that any two $\mathrm{SL}(3;\mathbb{C})$ 3-forms on a given oriented 6-manifold are homotopic.

Theorem 1.0.17. *Let N be an oriented 6-manifold. If the Euler class $e(\mathrm{N}) = 0$, then any $\mathrm{SL}(3;\mathbb{C})$ 3-form on N is extendible. In particular:*

- *If N is open, then any $\mathrm{SL}(3;\mathbb{C})$ 3-form on N is extendible.*
- *If N is closed and the Euler characteristic $\chi(\mathrm{N}) = 0$, then any $\mathrm{SL}(3;\mathbb{C})$ 3-form on N is extendible.*

Conversely, if $e(\mathrm{N}) \neq 0$ and in addition $b^2 = 0$ (i.e. $\mathrm{H}^2(\mathrm{N};\mathbb{Z})$ and $\mathrm{H}^4(\mathrm{N};\mathbb{Z})$ are pure torsion), then no $\mathrm{SL}(3;\mathbb{C})$ 3-form on N is extendible.

Turning to the case of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, by using the results of Thomas [119, Cor. 1.7], a lower bound on the number of homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms is obtained:

Theorem 1.0.18. *Let N be a closed, oriented, 6-manifold with $e(\mathrm{N}) = 0$ and suppose $w_2(\mathrm{N})^2 = 0$. Write $\rho_2 : \mathrm{H}^4(\mathrm{N};\mathbb{Z}) \to \mathrm{H}^4\left(\mathrm{N};\mathbb{Z}/2\mathbb{Z}\right)$ for reduction modulo 2 and define:*

$$\mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2} = \left\{u \in \mathrm{H}^4(\mathrm{N};\mathbb{Z}) \;\middle|\; \rho_2 u \cup w_2(\mathrm{N}) = 0\right\}.$$

Then there is an injection from $\mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2} \big/ \text{2-torsion}$ into the set of homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on N (equivalently closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, or $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms in any fixed degree 3 de Rham class). In particular, if N is spin and $b^4(\mathrm{N}) > 0$, then each of these sets is infinite.

As an immediate corollary of Theorem 9.5.2, one obtains:

Corollary 1.0.19. *Let N be a closed, oriented, spin 6-manifold. Then N admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms if and only if $e(\mathrm{N}) = 0$.*

Finally, Chapter 9 ends by investigating the extendibility of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms. Firstly, the manifold $\mathbb{T}^2 \times \mathcal{E}$, where $\mathcal{E}$ denotes the Enriques surface, is shown to admit infinitely many distinct homotopy classes of (closed) $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, none of which are extendible. Secondly 652 distinct homotopy classes of (closed) extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on $\mathbb{T}^6$ are constructed.

In **Part III**, the thesis adopts a new focus. Recall that there is currently only one known invariant of 'G_2-manifolds' (i.e. oriented 7-manifolds equipped with torsion-free G_2-structures) *viz.* the $\overline{\nu}$-invariant defined by Crowley–Goette–Nordström [31]. Whilst the value of $\overline{\nu}$ can be effectively computed for the 'twisted connected sum' G_2-manifolds constructed in [87, 29, 110] (see [31, 57]) there is no known general method of computing the $\overline{\nu}$-invariant for the G_2-manifolds constructed by Joyce [76, 77, 78].

Part III introduces two new invariants of closed G_2-manifolds (and, more generally, of G_2-orbifolds), which I denote μ_3 and μ_4. It was proven in [71] that the critical points of the functional $\mathcal{H}_3$ on a closed, oriented 7-manifold are non-degenerate local maxima, modulo the actions of diffeomorphisms. The same argument also proves the corresponding result for orbifolds. Likewise, Proposition 11.4.1 proves that the critical points of $\mathcal{H}_4$ on a closed, oriented 7-orbifold are non-degenerate saddles, modulo the action of diffeomorphisms.[1] Motivated by classical Morse theory, Part III addresses the question of whether a torsion-free G_2-structure on an oriented 7-orbifold has a well-defined notion of Morse index, when viewed as a critical point of the functionals $\mathcal{H}_3$ and $\mathcal{H}_4$. Whilst the classical Morse indices of the critical points are not well-defined, by using the theory of spectral invariants developed by Seeley [115] and Atiyah–Patodi–Singer [8, 9, 10], and later elaborated by Kawasaki [85] and Farsi [44], I show that the critical points of $\mathcal{H}_3$ and $\mathcal{H}_4$ both have a well-defined regularised notion of Morse index, denoted μ_3 and μ_4 respectively. Explicitly, given a closed, oriented orbifold M and a torsion-free G_2 3-form ϕ on M, $\mu_3(\phi)$ is the value at 0 of the meromorphic extension to $\mathbb{C}$ of the holomorphic function:

$$\begin{aligned} \mu_\phi : \left\{ s \in \mathbb{C} \;\middle|\; \mathfrak{Re}\, s > \tfrac{7}{2} \right\} &\longrightarrow \mathbb{C} \\ s &\longmapsto \sum_{\substack{\lambda \in \mathrm{Spec}(\mathcal{D}^2\mathcal{H}_3) \\ \lambda < 0}} |\lambda|^{-s} \end{aligned}$$

where $\mathcal{D}^2\mathcal{H}_3$ is viewed as a linear operator (rather than a bilinear form) via a suitable L^2-inner product. μ_4 is defined analogously. The main result of Part III is the following theorem:

Theorem 1.0.20. *Given $A \in \mathrm{End}(\mathbb{R}^7)$, define:*

$$\mathrm{Tr}_8^{\mathrm{SU}(3)}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2\,\mathrm{Tr}_{\mathbb{R}^7}(A) + 1$$

[1] After proving Proposition 11.4.1, the author discovered that a related result was obtained in [59]. Note, however, that Proposition 11.4.1 differs from [59, Prop. 3.4] firstly, since it proves not only that the critical points of $\mathcal{H}_4$ are non-degenerate, but also that they are saddles; and secondly, since it considers not only manifolds, but also orbifolds.

and:

$$\mathrm{Tr}^{\mathrm{SU}(3)}_{12}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^3 + 2\,\mathrm{Tr}_{\mathbb{R}^7}(A^3) - 3\,\mathrm{Tr}_{\mathbb{R}^7}(A^2)\,\mathrm{Tr}_{\mathbb{R}^7}(A)}{6} - \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2.$$

Let $\mathrm{M}_\Gamma = {}_\Gamma\backslash^{T}$ *be a Joyce orbifold and let* $\mathscr{G}_2^{TF}(\mathrm{M}_\Gamma)$ *denote the moduli space of torsion-free* G_2-*structures on* M_Γ. *Then the invariants:*

$$\mu_3 : \mathscr{G}_2^{TF}(\mathrm{M}_\Gamma) \to \mathbb{R}^2 \quad \text{and} \quad \mu_4 : \mathscr{G}_2^{TF}(\mathrm{M}_\Gamma) \to \mathbb{R}^2$$

are constant, given by the formulae:

$$\mu_3(\mathrm{M}_\Gamma) = \frac{-1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \mathrm{Tr}^{\mathrm{SU}(3)}_{8}(A) \quad \text{and} \quad \mu_4(\mathrm{M}_\Gamma) = \frac{-1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \mathrm{Tr}^{\mathrm{SU}(3)}_{12}(A).$$

The proof of Theorem 1.0.20 reveals an interesting link between the μ-invariants and twisted Epstein ζ-functions, as introduced in [43]; see §11.6 for details. Theorem 1.0.20 serves as the introduction to what will of necessity be a much larger project, which seeks to compute the μ-invariants on the G_2-manifolds constructed by Joyce in [76, 77, 78]. The conjectural shape of this project is briefly discussed at the end of Part III.

Chapter 2

Preliminaries

This chapter recounts the prerequisite theory regarding orbifolds, G_2- and $\widetilde{\mathrm{G}}_2$-forms, stable forms and Hitchin functionals, metric geometry, and h-principles which will be assumed in this thesis.

2.1 Differential topology of orbifolds

The main references for this section are [1, §§1.1–1.3] and [38, §14.1].

2.1.1 Basic definitions

Let E be a topological space.

Definition 2.1.1. An n-dimensional orbifold chart Ξ is the data of a connected, open neighbourhood U in E, a finite subgroup $\Gamma \subset \mathrm{GL}(n;\mathbb{R})$, a connected, Γ-invariant open neighbourhood $\widetilde{U}$ of $0 \in \mathbb{R}^n$ and a homeomorphism $\chi : \Gamma\backslash\widetilde{U} \to U$. Write $\widetilde{\chi}$ for the composite $\widetilde{U} \xrightarrow{quot} \Gamma\backslash\widetilde{U} \xrightarrow{\chi} U$. Say that Ξ is centred at $e \in E$ if $e = \widetilde{\chi}(0)$. In this case, Γ is called the orbifold group of e, denoted Γ_e. e is called a smooth point if $\Gamma_e = 0$, and a singular point if $\Gamma_e \neq 0$.

Now consider two orbifold charts $\Xi_1 = \left(U_1, \Gamma_1, \widetilde{U}_1, \chi_1\right)$ and $\Xi_2 = \left(U_2, \Gamma_2, \widetilde{U}_2, \chi_2\right)$ with $U_1 \subseteq U_2$. An embedding of Ξ_1 into Ξ_2 is the data of a smooth, open embedding $\iota_{12} : \widetilde{U}_1 \hookrightarrow \widetilde{U}_2$ and a group isomorphism $\lambda_{12} : \Gamma_1 \to \mathrm{Stab}_{\Gamma_2}(\iota_{12}(0))$ such that for all $x \in \widetilde{U}_1$ and all $\sigma \in \Gamma_1$: $\iota_{12}(\sigma \cdot x) = \lambda_{12}(\sigma) \cdot \iota_{12}(x)$, and such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{U}_1 \arrow[r, "\iota_{12}"] \arrow[d, "\widetilde{\chi}_1"] & \widetilde{U}_2 \arrow[d, "\widetilde{\chi}_2"] \\
U_1 \arrow[r, hook, "incl"] & U_2
\end{tikzcd}$$

Now let Ξ_1 and Ξ_2 be arbitrary. Ξ_1 and Ξ_2 are compatible if for every $e \in U_1 \cap U_2$, there exists a chart $\Xi_e = \left(U_e, \Gamma_e, \widetilde{U}_e, \chi_e\right)$ centred at e together with embeddings $(\iota_{e1}, \lambda_{e1}) : \Xi_e \hookrightarrow \Xi_1$ and $(\iota_{e2}, \lambda_{e2}) : \Xi_e \hookrightarrow \Xi_2$. If $U_1 \cap U_2 = \varnothing$, then Ξ_1 and Ξ_2 are automatically compatible, however if $U_1 \cap U_2 \neq \varnothing$ and Ξ_1 and Ξ_2 are compatible, then Ξ_1 and Ξ_2 have the same dimension; moreover, if

Ξ_1 and Ξ_2 are centred at the same point $e \in E$, then $\Gamma_1 \cong \Gamma_2$ and therefore the orbifold group Γ_e is well-defined up to isomorphism.

An orbifold atlas for E is a collection of compatible orbifold charts $\mathfrak{A}$ which is maximal in the sense that if a chart Ξ is compatible with every chart in $\mathfrak{A}$, then $\Xi \in \mathfrak{A}$. An orbifold is a connected, Hausdorff, second-countable topological space E equipped with an orbifold atlas $\mathfrak{A}$. Every chart of E has the same dimension n; call this the dimension of the orbifold.

Definition 2.1.2. Let E_1, E_2 be orbifolds. A continuous map $f : E_1 \to E_2$ is termed smooth if for any point $e \in E_1$, there exists a chart $\Xi_e = \left(U_e, \Gamma_e, \widetilde{U}_e, \chi_e\right)$ for E_1 centred at e, a chart $\Xi_{f(e)} = \left(U_{f(e)}, \Gamma_{f(e)}, \widetilde{U}_{f(e)}, \chi_{f(e)}\right)$ for E_2 centred at $f(e)$, a group homomorphism $\kappa_f : \Gamma_e \to \Gamma_{f(e)}$ and a smooth map $\widetilde{f} : \widetilde{U}_e \to \widetilde{U}_{f(e)}$ satisfying $\widetilde{f}(\sigma \cdot x) = \kappa_f(\sigma) \cdot \widetilde{f}(x)$ for all $x \in \widetilde{U}_e$ and $\sigma \in \Gamma_e$, such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{U}_e \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\chi}_e"] & \widetilde{U}_{f(e)} \arrow[d, "\widetilde{\chi}_{f(e)}"] \\ U_e \arrow[r, "f"] & U_{f(e)} \end{tikzcd} \tag{2.1.3}$$

The lift $\widetilde{f}$ need not be unique, even modulo the action of the groups Γ_e and $\Gamma_{f(e)}$; see, e.g. [27, Example 1.4.3]. Nevertheless, Definition 2.1.2 is independent of the choice of charts Ξ_e and $\Xi_{f(e)}$ and the map f has a well-defined differential in the following sense: the bottom arrow in the diagram:

$$\begin{tikzcd} \mathbb{R}^{n_e} \arrow[r, "D\widetilde{f}|_0"] \arrow[d, "proj"] & \mathbb{R}^{n_{f(e)}} \arrow[d, "proj"] \\ \Gamma_e \backslash \mathbb{R}^{n_e} \arrow[r] & \Gamma_{f(e)} \backslash \mathbb{R}^{n_{f(e)}} \end{tikzcd}$$

is independent of the choice of $\widetilde{f}$.

2.1.2 Suborbifolds and stratifications

Definition 2.1.4 (See [121, Defn. 13.2.7])**.** Let E be an orbifold. A subset $S \subseteq E$ is termed a suborbifold if for each $e \in S$, there exists a chart $\Xi_e = \left(U_e, \Gamma_e, \widetilde{U}_e, \chi_e\right)$ for E centred at e and a Γ-invariant subspace $\mathbb{I}_e \subset \mathbb{R}^n$ such that:

$$\widetilde{\chi}_e^{-1}(S \cap U_e) = \widetilde{U}_e \cap \mathbb{I}_e. \tag{2.1.5}$$

Call such a chart regular for S and call $\mathbb{I}_e$ the regular subspace. If the action of Γ on $\mathbb{I}_e$ is trivial for all $e \in S$, then call S a submanifold.

A subset S can have at most one suborbifold structure, as the following (readily verified) proposition demonstrates:

Proposition 2.1.6. *Let E be an orbifold, $S \subseteq E$ be a subset, $e, f \in S$ and let $\Xi_e = (U_e, \Gamma_e, \widetilde{U}_e, \chi_e)$, $\Xi_f = (U_f, \Gamma_f, \widetilde{U}_f, \chi_f)$ be regular charts for S centred at e and f respectively. Then Ξ_e and Ξ_f are* *compatible via regular charts. Specifically, let $g \in U_e \cap U_f \cap S$. Then there exists a regular chart Ξ_g centred at g together with embeddings $(\iota_{ge}, \lambda_{ge})$ and $(\iota_{gf}, \lambda_{gf})$ into Ξ_e and Ξ_f respectively. In particular, suborbifolds inherit a natural orbifold structure.*

Using this terminology, one can make the following generalisation of Mather's terminology [100, §5] to orbifolds:

Definition 2.1.7. Let E be an orbifold. A stratification Σ of E is a partition of E into disjoint submanifolds $E = \bigcup_{i=0}^{n} E_i$. Say that Σ satisfies the condition of the frontier if, in addition, for each $i \in \{0, ..., n\}$, there exists $I(i) \subseteq \{0, ..., n\}$ such that:

$$\overline{E_i} = \bigcup_{j \in I(i)} E_j, \tag{2.1.8}$$

where $\overline{E_i}$ denotes the topological closure of E_i in E.

Now let $\Sigma = \{E_i\}$, $\Sigma' = \left\{E'_j\right\}$ be two stratifications of E. Say that Σ' is a refinement of Σ if for every j, there exists and i such that $E'_j \subseteq E_i$. Finally, given stratified orbifolds $\left(E_1, \Sigma_1 = \{E_{1,i}\}_{i=1}^{n}\right)$ and $\left(E_2, \Sigma_2 = \{E_{2,i}\}_{i=1}^{n})\right)$, a smooth map $f : E_1 \to E_2$ is a stratified diffeomorphism if it is an orbifold diffeomorphism (i.e. it has a smooth inverse) and $f(E_{1,i}) = E_{2,i}$ for each i (in particular, Σ_1 and Σ_2 have the same number of strata).

Remark 2.1.9. A stratification as defined above is not the same as a Whitney stratification, the latter being a strictly stronger notion; see [100, §5] for further details. Indeed, the strong notion of a Whitney stratification will not be required for the purposes of this thesis.

A key source of stratifications is provided by the following definition:

Definition 2.1.10. Let E be an orbifold. For each isomorphism class of finite groups $[\Gamma]$, the set:

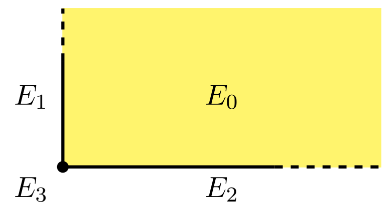
$$E([\Gamma]) = \{e \in E \mid \Gamma_e \in [\Gamma]\}$$

is either empty or its connected components are submanifolds of E, with $E(\mathbf{1})$ being always an open and dense subset of E. Take E_i to be an enumeration of the connected components of the $E([\Gamma])$ as $[\Gamma]$ varies. Then one can show that (at least when E is compact) there are only a finite number of strata E_i and that $\Sigma_{can} = \{E_i\}$ defines a stratification of E known as the canonical stratification of E. One may verify that this stratification satisfies the condition of the frontier.

Example 2.1.11. Consider:

$$\Gamma = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\} \subset \mathrm{GL}(2; \mathbb{R}). \tag{2.1.12}$$

The quotient $E = {}^{\Gamma}\backslash^{\mathbb{R}^2}$ is a 2-dimensional orbifold. The canonical stratification of E is given by:



Note that whilst each E_i is a submanifold of E, the singular locus $S = E_1 \cup E_2 \cup E_3$ is not even a suborbifold of E: indeed, in the natural (global) orbifold chart for E, S corresponds to the subset $\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2$, which is not a linear subspace.

2.1.3 Vector bundles over orbifolds

Definition 2.1.13 (Cf. [38, §14.1])**.** Let $\pi : E \to B$ be a smooth map of orbifolds. An (orbifold) vector bundle chart Θ_b for π about $b \in B$ is the data of:

- A chart Ξ_b for B centred at b;
- A chart Ξ_e for E centred at a suitable $e \in \pi^{-1}(b)$;
- A local lift $\widetilde{\pi} : \widetilde{U}_e \to \widetilde{U}_b$ for π and a homomorphism $\kappa_\pi : \Gamma_e \to \Gamma_b$ as in Definition 2.1.2,

such that:

1. $\widetilde{\pi} : \widetilde{U}_e \to \widetilde{U}_b$ is a vector bundle of some rank k and $0 \in \widetilde{U}_e$ is the zero of the fibre over $0 \in \widetilde{U}_b$;
2. $\kappa_\pi : \Gamma_e \to \Gamma_b$ is an isomorphism and $\Gamma_e \cong \Gamma_b$ acts on $\widetilde{U}_e$ via vector bundle automorphisms.

An embedding of a chart $\Theta_{b_1} = (\Xi_{b_1}, \Xi_{e_1}, \widetilde{\pi}_1, (\kappa_\pi)_1)$ into $\Theta_{b_2} = (\Xi'_{b_2}, \Xi'_{e_2}, \widetilde{\pi}_2, (\kappa_\pi)_2)$ is the data of embeddings of orbifold charts $(\iota_{b_1 b_2}, \lambda_{b_1 b_2}) : \Xi_{b_1} \hookrightarrow \Xi_{b_2}$ and $(\iota_{e_1 e_2}, \lambda_{e_1 e_2}) : \Xi_{e_1} \hookrightarrow \Xi'_{e_2}$ such that the induced equivariant commutative square:

$$\begin{tikzcd} \widetilde{U}_{e_1} \arrow[r, "\iota_{e_1 e_2}"] \arrow[d, "\widetilde{\pi}_2"] & \widetilde{U}_{e_2} \arrow[d, "\widetilde{\pi}_2"] \\ \widetilde{U}_{b_1} \arrow[r, "\iota_{b_1 b_2}"] & \widetilde{U}_{b_2} \end{tikzcd}$$

is a bundle isomorphism (and thus the ranks of the two bundles are equal). In particular, given a point $b \in B$, the distinguished point $e \in \pi^{-1}(b)$ is unique and does not depend on the choice of chart.

As for orbifolds, two vector bundle charts Θ_{b_1} and Θ_{b_2} are called compatible if for all $b \in U_{b_1} \cap U_{b_2}$, there exists a chart Θ_b centred at b which embeds into both Θ_{b_1} and Θ_{b_2}. An (orbifold) vector bundle is then a smooth map $\pi : E \to B$ of orbifolds together with a maximal atlas of compatible vector bundle charts. Note that π has a well defined rank k. The fibre E_b over a point $b \in B$ is naturally identified with the space $\Gamma_b \backslash \mathbb{R}^k$. A section of E is then simply a continuous map $X : B \to E$ such that for each chart $\Theta_b = (\Xi_b, \Xi_e, \widetilde{\pi}, \kappa_\pi)$ for E, there is a smooth, $\Gamma_b \cong \Gamma_e$-equivariant, section $\widetilde{X} : \widetilde{U}_b \to \widetilde{U}_e$

such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{U}_b \arrow[r, "\widetilde{X}"] \arrow[d, "\widetilde{\chi}_b"] & \widetilde{U}_e \arrow[d, "\widetilde{\chi}_e"] \\ U_b \arrow[r, "X"] & U_e \end{tikzcd} \tag{2.1.14}$$

Note in particular, that $\widetilde{X}|_0$ is invariant under the action of $\Gamma_e \cong \Gamma_b$ on the fibre $\widetilde{U}_e|_0$. Thus $X|_b$ can be regarded as an element of the subspace of $\mathbb{R}^k$ upon which Γ_b acts trivially, denoted $Fix_{\Gamma_b}(\mathbb{R}^k)$, which in turn, can be regarded as a subspace of $\Gamma_b \backslash \mathbb{R}^k$. Finally, let $\pi : E \to B$ be a vector bundle and let $F \subseteq E$ be a suborbifold. The smooth map $\pi|_F : F \to B$ is called a sub-vector bundle of $\pi : E \to B$ if for any chart $\Theta_b = (\Xi_b, \Xi_e, \widetilde{\pi}, \kappa_\pi)$ for π, the subset $\widetilde{\chi}_e^{-1}(U_e \cap F) \subseteq \widetilde{U}_e$ is a $\Gamma_b \cong \Gamma_e$-invariant sub-vector bundle of $\widetilde{\pi} : \widetilde{U}_e \to \widetilde{U}_b$.

Let E be an n-orbifold. Given any chart $\Xi = \left(U, \Gamma, \widetilde{U}, \chi\right)$ for E, the action of Γ on $\widetilde{U}$ naturally lifts to an action of Γ on $\mathrm{T}\widetilde{U}$ by bundle automorphisms. Given a second chart $\Xi' = (U', \Gamma', \widetilde{U}', \chi')$ embedding into Ξ, the map $\widetilde{U}' \hookrightarrow \widetilde{U}$ induces an equivariant embedding $\mathrm{T}\widetilde{U}' \hookrightarrow \mathrm{T}\widetilde{U}$. Define:

$$\mathrm{T}E = \left[\amalg_\Xi \left(\Gamma \backslash \mathrm{T}\widetilde{U} \right) \right] \Big/ \sim \tag{2.1.15}$$

where the quotient by $\sim$ denotes that one should glue along the embeddings $\mathrm{T}\widetilde{U}' \hookrightarrow \mathrm{T}\widetilde{U}$. The resulting space $\mathrm{T}E$ is an orbifold vector bundle over E and is termed the tangent bundle of E. Given $e \in E$, the tangent space at e, denoted $\mathrm{T}_e E$, is the preimage of e under the map $\mathrm{T}E \to E$ and may be identified with the quotient space $\Gamma_e \backslash \mathbb{R}^n$, where Γ_e is the orbifold group at e. In a similar way, one may define the cotangent bundle of an orbifold, tensor bundles, bundles of exterior forms, etc., denoted in the usual way. Say that an n-orbifold E is orientable if $\bigwedge^n \mathrm{T}^* E$ and $E \times \mathbb{R}$ are isomorphic as orbifold vector bundles; in particular, all of the orbifold groups of E are necessarily orientation preserving. (When this latter condition holds, I say that E is pre-orientable, although this terminology is non-standard.)

Now let $\pi : F \to E$ be such a vector bundle over E of rank k and let $\mathcal{A} \subseteq \mathbb{R}^k$ be a $\mathrm{GL}(k;\mathbb{R})$-invariant subset. For each $e \in E$, the subset $\Gamma_e \backslash \mathcal{A} \subseteq \Gamma_e \backslash \mathbb{R}^k$ is well-defined and gives rise to a subset of $\pi^{-1}(e)$ under the identification $\pi^{-1}(e) \cong \Gamma_e \backslash \mathbb{R}^k$. As $e \in E$ varies, this defines a subbundle of F.

Definition 2.1.16. Let E be an n-orbifold and consider the bundle $\bigodot^2 \mathrm{T}^* E$. The subspace $\bigodot^2_+ (\mathbb{R}^n)^* \subset \bigodot^2 (\mathbb{R}^n)^*$ is $\mathrm{GL}(n;\mathbb{R})$-invariant and hence defines a corresponding subbundle of $\bigodot^2 \mathrm{T}^* E$, denoted $\bigodot^2_+ \mathrm{T}^* E$. An (orbifold) Riemannian metric on E is then simply a section g of $\bigodot^2_+ \mathrm{T}^* E$. Likewise, the set $\bigodot^2_{\geqslant 0} (\mathbb{R}^n)^* \subset \bigodot^2 (\mathbb{R}^n)^*$ is also $\mathrm{GL}(n;\mathbb{R})$-invariant and thus gives rise to subbundle $\bigodot^2_{\geqslant 0} \mathrm{T}^* E \subset \bigodot^2 \mathrm{T}^* E$; sections h of this bundle are termed (orbifold) Riemannian semi-metrics.

Given a Riemannian (semi)-metric g on E, recall that for each $e \in E$, $g|_e$ can be regarded as an element of the space $Fix_{\Gamma_e}\left(\bigodot^2 (\mathbb{R}^n)^*\right) \subseteq \Gamma_e \backslash \bigodot^2 (\mathbb{R}^n)^*$. Thus, given $u, u' \in \mathrm{T}_e E \cong \Gamma_e \backslash \mathbb{R}^n$, the quantity $g(u,u')$ is well-defined. In particular, given a Riemannian metric g on E and a C^1 curve

$\gamma : [a,b] \to X$, one can define:

$$\ell^g(\gamma) = \int_{[a,b]} g(\dot{\gamma})\, \mathrm{d}\mathscr{L} \tag{2.1.17}$$

where $\mathscr{L}$ denotes the Lebesgue measure on $[a,b]$. If γ is merely piecewise-C^1, i.e. $\gamma = \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_k$ is the concatenation of C^1-curves, then define:

$$\ell^g(\gamma) = \sum_{i=1}^{k} \ell^g(\gamma_i). \tag{2.1.18}$$

Analogous definitions apply if g is only assumed to be a Riemannian semi-metric.

2.1.4 Stratified distributions

A distribution on an orbifold E is simply a sub-vector bundle $\mathcal{D}$ of $\mathrm{T}E$. Given a Riemannian metric g on E, define the orthocomplement $\mathcal{D}^\perp$ to $\mathcal{D}$ via the formula:

$$\mathcal{D}^\perp|_e = \{u \in \mathrm{T}_e E \mid g(u,u') = 0 \text{ for all } u' \in \mathcal{D}|_e\}.$$

Then $\mathcal{D}^\perp$ is also a distribution over E. Indeed recall that, locally, $\mathcal{D}$ is given by $\Gamma_e \backslash \widetilde{\mathcal{D}}$ for some Γ_e-invariant distribution $\widetilde{\mathcal{D}} \subseteq \mathrm{T}\widetilde{U}$, where $\widetilde{U}$ is a local chart for E, and g is induced by a Γ_e-invariant Riemannian metric $\widetilde{g}$ over $\widetilde{U}$; from this the result is clear.

Now let Σ be a stratification on E. Even in the case where E is a manifold, a general distribution $\mathcal{D}$ can be 'incompatible' with Σ in the following sense:

Example 2.1.19. Consider the distribution $\mathcal{D}$ over $E = \mathbb{R}^2$ given by $\mathcal{D} = \langle \partial_1 + x^1 \partial_2 \rangle$ and the stratification Σ of E given by:

$$E_0 = \mathbb{R}^2 \backslash (\mathbb{R} \times \{0\}) \quad \text{and} \quad E_1 = \mathbb{R} \times \{0\} \cong \mathbb{R}.$$

Then $\mathcal{D} \cap \mathrm{T}E_1$ is not a distribution over E_1, since over the non-zero points of $E_1 \cong \mathbb{R}$, $\mathcal{D}$ only intersects $\mathrm{T}E_1$ along its zero-section, however over the point 0 the fibres of $\mathcal{D}$ and $\mathrm{T}E_1$ coincide.

This potential for incompatibility motivates the following definition, which cannot (to the author's knowledge) be found in the literature:

Definition 2.1.20. Let $(E, \Sigma = \{E_i\})$ be a stratified orbifold. A distribution $\mathcal{D}$ on E is termed stratified if $\mathcal{D}_i = \mathcal{D} \cap \mathrm{T}E_i \subseteq \mathrm{T}E_i$ is a distribution over E_i, for all i.

I remark if $\mathcal{D}$ is stratified, then for every Riemannian metric g, the orthocomplement $\mathcal{C} = \mathcal{D}^\perp$ is also stratified. Indeed, for each i:

$$\mathcal{C} \cap \mathrm{T}E_i = (\mathcal{D}_i)^\perp,$$

where the orthocomplement is defined using Riemannian metric $g|_{E_i}$ on the stratum E_i.

2.2 3- and 4-forms of G_2- and $\widetilde{\mathrm{G}}_2$-type

The main references for this section are [78, 71, 68, 21].

2.2.1 Basic definitions

Consider the 3-form:

$$\varphi_0 = \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356} \in \bigwedge^3(\mathbb{R}^7)^*. \tag{2.2.1}$$

The stabiliser of φ_0 in $\mathrm{GL}_+(7;\mathbb{R})$ is 14-dimensional, compact, connected and simply-connected, with Lie algebra a compact real form of the exceptional simple Lie algebra $\mathfrak{g}_{2,\mathbb{C}}$; thus the stabiliser may be identified with the exceptional Lie group G_2 [21, §2, Thm. 1]. Since G_2 is 14-dimensional, writing $\bigwedge^3_+\left(\mathbb{R}^7\right)^*$ for the $\mathrm{GL}_+(7;\mathbb{R})$-orbit of φ_0, one finds that:

$$\dim \bigwedge^3_+\left(\mathbb{R}^7\right)^* = \dim \mathrm{GL}_+(7;\mathbb{R}) - \dim \mathrm{G}_2 = 49 - 14 = 35 = \dim \bigwedge^3\left(\mathbb{R}^7\right)^*,$$

and hence $\bigwedge^3_+\left(\mathbb{R}^7\right)^* \subset \bigwedge^3\left(\mathbb{R}^7\right)^*$ is open (in particular, φ_0 is stable; see §2.3). The geometric interest in φ_0 lies in its link with G_2-structures on 7-manifolds. Let M be an oriented 7-manifold, define a subbundle $\bigwedge^3_+\mathrm{T}^*\mathrm{M} \subset \bigwedge^3\mathrm{T}^*\mathrm{M}$ via, for $x \in \mathrm{M}$:

$$\begin{aligned}\bigwedge^3_+\mathrm{T}^*_x\mathrm{M} = \Big\{\phi \in \bigwedge^3\mathrm{T}^*_x\mathrm{M} \;\Big|\; \exists\, &\text{orientation preserving isomorphism} \\ &\alpha : \mathrm{T}_x\mathrm{M} \xrightarrow{\sim} \mathbb{R}^7 \text{ with } \phi|_x = \alpha^*(\varphi_0)\Big\}\end{aligned} \tag{2.2.2}$$

and write Ω^3_+ for the corresponding sheaf of sections. Since $\mathrm{Stab}_{\mathrm{GL}_+(7;\mathbb{R})}(\varphi_0) = \mathrm{G}_2$, given a section $\phi \in \Omega^3_+(\mathrm{M})$, the collection of all orientation-preserving isomorphisms $\alpha : \mathrm{T}_x\mathrm{M} \to \mathbb{R}^7$ identifying $\phi|_x$ with φ_0 defines a G_2-structure on M, i.e. a principal G_2-subbundle of the frame bundle of M. Accordingly, sections of $\bigwedge^3_+\mathrm{T}^*\mathrm{M}$ are termed G_2 3-forms.

G_2-structures on oriented 7-manifolds can equivalently be defined via suitable 4-forms, as I now describe. Recall from [21, §2, Thm. 1] that $\mathrm{G}_2 \subset \mathrm{SO}(7)$. Thus any G_2 3-form $\phi \in \Omega^3_+(\mathrm{M})$ induces a Riemannian metric g_ϕ on M, defined at each $x \in \mathrm{M}$ by pulling back the Euclidean inner product on $\mathbb{R}^7$ along any isomorphism α identifying $\phi|_x$ with φ_0. Hence ϕ also defines a volume form vol_ϕ, Hodge star operator $\bigstar_\phi$ and Levi-Civita connection ∇^ϕ. The 4-form $\bigstar_\phi\phi$ is pointwise oriented-isomorphic to the 4-form:

$$\psi_0 = \bigstar_0\varphi_0 = \theta^{4567} + \theta^{2367} + \theta^{2345} + \theta^{1357} - \theta^{1346} - \theta^{1256} - \theta^{1247} \in \bigwedge^4(\mathbb{R}^7)^*, \tag{2.2.3}$$

where $\bigstar_0$ denotes the Euclidean Hodge star on $\mathbb{R}^7$; 4-forms with this property are termed G_2 4-forms. The stabiliser of ψ_0 in $\mathrm{GL}_+(7;\mathbb{R})$ is also G_2; consequently the $\mathrm{GL}_+(7;\mathbb{R})$-orbit of ψ_0 in $\bigwedge^4\left(\mathbb{R}^7\right)^*$ is once again open and G_2-structures on oriented 7-manifolds can be equivalently defined using G_2 4-forms. Write $\bigwedge^4_+\mathrm{T}^*\mathrm{M}$ for the bundle of G_2 4-forms on M and write $\Omega^4_+(\mathrm{M})$ for the corresponding sheaf of sections.

Remark 2.2.4. Although φ_0 and ψ_0 have the same stabiliser in $\mathrm{GL}_+(7;\mathbb{R})$, if one also considers orientation-reversing endomorphisms, one finds:

$$\mathrm{Stab}_{\mathrm{GL}(7;\mathbb{R})}(\varphi_0) = \mathrm{Stab}_{\mathrm{GL}_+(7;\mathbb{R})}(\varphi_0) = \mathrm{G}_2$$

whilst:

$$\mathrm{Stab}_{\mathrm{GL}(7;\mathbb{R})}(\psi_0) = \mathrm{Stab}_{\mathrm{GL}_+(7;\mathbb{R})}(\psi_0) \times \{\pm 1\} \cong \mathrm{G}_2 \times \{\pm 1\}.$$

Thus, there is a subtle difference between G_2 3- and 4-forms on manifolds M which are orientable but unoriented: if ϕ is a 3-form on M such that, for all $x \in \mathrm{M}$, there exists an isomorphism $\alpha : \mathrm{T}_x\mathrm{M} \to \mathbb{R}^7$ satisfying $\alpha^*\varphi_0 = \phi|_x$, then ϕ still defines a G_2-structure (and hence an orientation) on M, whilst a 4-form which is pointwise isomorphic to ψ_0 only induces a $\mathrm{G}_2 \times \{\pm 1\}$-structure on M and does not induce a preferred choice of orientation. However, this thesis takes the perspective described in [78], that the orientation on M should be considered primary and that one should restrict attention to those G_2-structures compatible with the chosen orientation. Thus, the above subtlety will not arise.

Now consider the 3-form:

$$\widetilde{\varphi}_0 = \theta^{123} - \theta^{145} - \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356} \in \bigwedge^3(\mathbb{R}^7)^*. \tag{2.2.5}$$

The stabiliser of $\widetilde{\varphi}_0$ in $\mathrm{GL}_+(7;\mathbb{R})$ is 14-dimensional, connected, centreless and doubly connected (i.e. has first fundamental group $\mathbb{Z}/2$), with Lie algebra a split real form of the exceptional Lie algebra $\mathfrak{g}_{2,\mathbb{C}}$; thus the stabiliser may be identified with the exceptional Lie group $\widetilde{\mathrm{G}}_2$ [21, §2, Thm. 2] (cf. also [68]). Write $\bigwedge^3_\sim\left(\mathbb{R}^7\right)^*$ for the $\mathrm{GL}_+(7;\mathbb{R})$-orbit of $\widetilde{\varphi}_0$ in $\bigwedge^3\left(\mathbb{R}^7\right)^*$; as for G_2 3-forms, since $\widetilde{\mathrm{G}}_2$ is 14-dimensional, $\bigwedge^3_\sim\left(\mathbb{R}^7\right)^* \subset \bigwedge^3_\sim\left(\mathbb{R}^7\right)^*$ is open. Given an oriented 7-manifold M, write $\bigwedge^3_\sim\mathrm{T}^*\mathrm{M}$ for the bundle of 3-forms which are pointwise oriented-isomorphic to $\widetilde{\varphi}_0$, and write $\Omega^3_\sim$ for the corresponding sheaf of section. Then sections of $\bigwedge^3_\sim\left(\mathbb{R}^7\right)^*$ are equivalent to $\widetilde{\mathrm{G}}_2$-structures on M; accordingly, such 3-forms are termed $\widetilde{\mathrm{G}}_2$ 3-forms. As for G_2-forms, it can be shown [21, §2, Thm. 2] that $\widetilde{\mathrm{G}}_2 \subset \mathrm{SO}(3,4)$ (again, cf. [68]). Thus any $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi} \in \Omega^3_\sim(\mathrm{M})$ induces a pseudo-Riemannian metric $g_{\widetilde{\phi}}$ on M, defined at each $x \in \mathrm{M}$ by pulling back the indefinite inner-product:

$$\widetilde{g}_0 = \sum_{i=1}^{3}\left(\theta^i\right)^{\otimes 2} - \sum_{i=4}^{7}\left(\theta^i\right)^{\otimes 2}$$

on $\mathbb{R}^7$ along any isomorphism identifying $\widetilde{\phi}|_x$ with $\widetilde{\varphi}_0$. (See [18, Ch. 1.C] for an exposition of the elementary properties of pseudo-Riemannian metrics.) Hence $\widetilde{\phi}$ also defines a volume form $vol_{\widetilde{\phi}}$, Hodge star operator $\bigstar_{\widetilde{\phi}}$ and Levi-Civita connection $\nabla^{\widetilde{\phi}}$. The 4-form $\bigstar_{\widetilde{\phi}}\widetilde{\phi}$ is pointwise oriented-isomorphic to the 4-form:

$$\widetilde{\psi}_0 = \widetilde{\bigstar}_0\widetilde{\varphi}_0 = \theta^{4567} - \theta^{2367} - \theta^{2345} + \theta^{1357} - \theta^{1346} - \theta^{1256} - \theta^{1247} \in \bigwedge^4(\mathbb{R}^7)^*, \tag{2.2.6}$$

where $\widetilde{\bigstar}_0$ denotes the Hodge star defined by the metric $\widetilde{g}_0$. The stabiliser of $\widetilde{\psi}_0$ in $\mathrm{GL}_+(7;\mathbb{R})$ is also $\widetilde{\mathrm{G}}_2$; consequently the $\mathrm{GL}_+(7;\mathbb{R})$-orbit of $\widetilde{\psi}_0$ in $\bigwedge^4\left(\mathbb{R}^7\right)^*$ is open and $\widetilde{\mathrm{G}}_2$-structures on oriented 7-

manifolds can be equivalently defined using G_2 4-forms. Write $\bigwedge^4_\sim T^*M$ for the bundle of G_2 4-forms on M and write $\Omega^4_\sim(M)$ for the corresponding sheaf of sections.

Since $\bigwedge^3_+ T^*M$, $\bigwedge^4_+ T^*M$, $\bigwedge^3_\sim T^*M$ and $\bigwedge^4_\sim T^*M$ are fibre bundles over M with non-trivial fibres, the bundles need not admit any global sections over a general oriented manifold M. It is a well-known theorem of Gray [58] that $\bigwedge^3_+ T^*M$ (equivalently $\bigwedge^4_+ T^*M$) admits a global section if and only if M is spin. The corresponding result for the bundle $\bigwedge^3_\sim T^*M$ (and thus equivalently $\bigwedge^4_\sim T^*M$ was conjectured by Lê in [92], who proved the result in the special case of closed manifolds. A full proof of the conjecture is provided in Chapter 9.

Remark 2.2.7. Since the subsets $\bigwedge^3_+ \left(\mathbb{R}^7\right)^*, \bigwedge^3_\sim \left(\mathbb{R}^7\right)^* \subset \bigwedge^3 \left(\mathbb{R}^7\right)^*$ and $\bigwedge^4_+ \left(\mathbb{R}^7\right)^*, \bigwedge^4_\sim \left(\mathbb{R}^7\right)^* \subset \bigwedge^4 \left(\mathbb{R}^7\right)^*$ are $GL_+(7;\mathbb{R})$-invariant, the bundles $\bigwedge^3_+ T^*M$, $\bigwedge^3_\sim T^*M$, $\bigwedge^4_+ T^*M$ and $\bigwedge^4_\sim T^*M$ can be defined over any pre-orientable orbifold (see §2.1.3). In this way, the discussion regarding G_2- and $\widetilde{G}_2$-structures in this section can be generalised to orbifolds. For simplicity I shall state the results for manifolds; from these statements, the corresponding results for orbifolds can be written down without extra work.

2.2.2 Type decomposition induced by G_2- and $\widetilde{G}_2$-structures

Recall that the groups G_2 and $\widetilde{G}_2$ have identical real representation theories, each coinciding with the complex representation theory of the simple Lie algebra $\mathfrak{g}_{2,\mathbb{C}}$. Given a G_2- (resp. $\widetilde{G}_2$-) structure on a 7-manifold M, the induced fibrewise action of G_2 (resp. $\widetilde{G}_2$) on the exterior bundles of M is, in general, reducible. The corresponding decomposition of the exterior bundles into subbundles of fibrewise simple modules was first computed by Fernández–Gray in 1982 [49] in the G_2 case and Kath [84] in the $\widetilde{G}_2$ case, leading to the following result:

Proposition 2.2.8 (Cf. [78, Prop. 10.1.4])**.** *Let* M *be an oriented 7-manifold with* G_2*- (resp.* $\widetilde{G}_2$*-) structure, with corresponding metric* g *and Hodge star* $\bigstar$*. Then the fibres of the bundles* $\bigwedge^0 T^*M$, $\bigwedge^1 T^*M$, $\bigwedge^6 T^*M$ *and* $\bigwedge^7 T^*M$ *are simple* G_2 *(resp.* $\widetilde{G}_2$*) modules. For the remaining exterior bundles, there are natural decompositions:*

$$\begin{aligned}
\bigwedge^2 T^*M &= \bigwedge^2_7 T^*M \oplus \bigwedge^2_{14} T^*M;\\
\bigwedge^3 T^*M &= \bigwedge^3_1 T^*M \oplus \bigwedge^3_7 T^*M \oplus \bigwedge^3_{27} T^*M;\\
\bigwedge^4 T^*M &= \bigwedge^4_1 T^*M \oplus \bigwedge^4_7 T^*M \oplus \bigwedge^4_{27} T^*M;\\
\bigwedge^5 T^*M &= \bigwedge^5_7 T^*M \oplus \bigwedge^5_{14} T^*M,
\end{aligned} \tag{2.2.9}$$

where the subscript in each case denotes the rank of the bundle, the fibres of each $\bigwedge^p_q T^*M$ *are simple* G_2 *(resp.* $\widetilde{G}_2$*) modules and any two bundles of a given rank are isomorphic; in particular* $\bigstar : \bigwedge^p_q T^*M \xrightarrow{\sim} \bigwedge^{7-p}_q T^*M$ *is a* G_2*- (resp.* $\widetilde{G}_2$*-) equivariant isomorphism for each* p, q*. Write* $\pi_q : \bigwedge^\bullet T^*M \to \bigwedge^\bullet_q T^*M$ *for the* g*-orthogonal projection; since for each exterior power no subscript occurs more than once, no ambiguity should arise from this notation.*

The subbundles in Proposition 2.2.8 admit very explicit descriptions. Indeed let ϕ and ψ be the

3- and 4-forms corresponding to the G_2- (resp. $\widetilde{\mathrm{G}}_2$-) structure on M. The for any $x \in \mathrm{M}$:

$$\begin{aligned}
\textstyle\bigwedge^2_7 \mathrm{T}^*_x\mathrm{M} &= \{v \lrcorner \phi_x \mid v \in \mathrm{T}_x\mathrm{M}\} \cong \mathrm{T}_x\mathrm{M};\\
\textstyle\bigwedge^2_{14} \mathrm{T}^*_x\mathrm{M} &= \left\{\alpha \in \textstyle\bigwedge^2 \mathrm{T}^*_x\mathrm{M} \;\middle|\; \alpha \wedge \psi_x = 0\right\} \cong \mathfrak{g}_2 \text{ (resp. } \widetilde{\mathfrak{g}}_2);\\
\textstyle\bigwedge^3_1 \mathrm{T}^*_x\mathrm{M} &= \mathbb{R}\cdot\phi_x \cong \mathbb{R};\\
\textstyle\bigwedge^3_7 \mathrm{T}^*_x\mathrm{M} &= \{v \lrcorner \psi_x \mid v \in \mathrm{T}_x\mathrm{M}\} \cong \mathrm{T}_x\mathrm{M};\\
\textstyle\bigwedge^3_{27} \mathrm{T}^*\mathrm{M} &= \left\{a \in \textstyle\bigwedge^3 \mathrm{T}^*_x\mathrm{M} \;\middle|\; a \wedge \phi_x = 0 \,\text{and}\, a \wedge \psi_x = 0\right\} \cong \textstyle\bigodot^2_0 \mathrm{T}^*_x\mathrm{M},
\end{aligned} \tag{2.2.10}$$

(where $\bigodot^2_0 \mathrm{T}^*_x\mathrm{M}$ denotes the space symmetric bilinear forms on $\mathrm{T}_x\mathrm{M}$ which are trace-free with respect to g_ϕ) with descriptions of all the other simple modules following from the Hodge star. For an arbitrary p-form σ, the decomposition $\sigma = \sum_q \pi_q(\sigma)$ is called the type decomposition of σ.

Write $\Theta : \wedge^3_\bullet(\mathbb{R}^*) \to \wedge^4_\bullet\left(\mathbb{R}^7\right)^*$ for the map given by $\Theta(\varphi) = \bigstar_\varphi$ ($\bullet = +, \sim$) and write $\Sigma(\psi) = \bigstar_\psi\psi$ for the inverse map. The derivatives of Θ and Σ can be explicitly computed using type-decomposition. For the map Θ in the G_2 case, this was first stated by Joyce [76], and later re-proved by Hitchin in [71] using representation theoretic arguments. Since G_2 and $\widetilde{\mathrm{G}}_2$ have identical representation theories, the same formula for $\mathrm{D}\Theta$ holds in the $\widetilde{\mathrm{G}}_2$ case. The formula for $\mathrm{D}\Sigma$ then follows at once:

Proposition 2.2.11. *Let $\bullet = +, \sim$ as appropriate. Then the differentials of Θ and Σ at $\varphi \in \wedge^3_\bullet\left(\mathbb{R}^7\right)^*$ and $\psi \in \wedge^4_\bullet\left(\mathbb{R}^7\right)^*$ respectively are given by:*

$$\begin{aligned}
\mathrm{D}\Theta_\varphi : \wedge^3\left(\mathbb{R}^7\right)^* &\longrightarrow \wedge^4\left(\mathbb{R}^7\right)^* & \qquad \mathrm{D}\Sigma_\psi : \wedge^4\left(\mathbb{R}^7\right)^* &\longrightarrow \wedge^3\left(\mathbb{R}^7\right)^*\\
\sigma &\longmapsto \bigstar_\varphi I_\varphi(\sigma) & \sigma &\longmapsto \bigstar_\psi J_\psi(\sigma),
\end{aligned} \tag{2.2.12}$$

where:

$$I_\varphi(\sigma) = \frac{4}{3}\pi_1(\sigma) + \pi_7(\sigma) - \pi_{27}(\sigma) \quad \text{and} \quad J_\psi(\sigma) = \frac{3}{4}\pi_1(\sigma) + \pi_7(\sigma) - \pi_{27}(\sigma).$$

Here, the projections $\pi_\bullet$ are defined with respect to φ and ψ respectively.

2.2.3 Torsion-free structures and Hitchin functionals for G_2- and $\widetilde{\mathrm{G}}_2$-forms

A G_2- or $\widetilde{\mathrm{G}}_2$-structure is called torsion-free if the corresponding 3-form ϕ satisfies the non-linear PDE $\nabla^\phi\phi = 0$ (or, equivalently, the corresponding 4-form satisfies $\nabla^\psi\psi = 0$). The name derives from the fact that $\tau_\phi = \nabla^\phi\phi$ can be identified with the intrinsic torsion of the G_2- or $\widetilde{\mathrm{G}}_2$-structure induced by ϕ [114, Cor. 2.2] (see [79, §2.6] for the definition of intrinsic torsion). A G_2-manifold is simply an oriented 7-manifold equipped with a torsion-free G_2 3- (equivalently, 4-) form; the term $\widetilde{\mathrm{G}}_2$-manifold is defined analogously.

Further significance of the torsion-free condition is provided by the following result:

Proposition 2.2.13 ([21, §1])**.** *Let (M, g) be a Riemannian manifold with holonomy contained in G_2. Then there is a G_2 3-form $\phi \in \Omega^3_+(\mathrm{M})$ such that:*

$$g = g_\phi \quad \text{and} \quad \nabla^\phi\phi = 0.$$

The analogous result holds for pseudo-Riemannian metrics with holonomy contained in $\widetilde{\mathrm{G}}_2$.

The torsion-free condition can alternatively be characterised via the following result:

Theorem 2.2.14 ([21, §3]; see also [49])**.** *Let* M *be an oriented 7-manifold with* G_2 *(resp.* $\widetilde{\mathrm{G}}_2$*) 3- and 4-forms* ϕ *and* ψ*. Then* (ϕ, ψ) *is torsion-free if and only if:*

$$\mathrm{d}\phi = 0 \quad \textit{and} \quad \mathrm{d}\psi = 0.$$

It should be noted that whilst each of $\mathrm{d}\phi = 0$ and $\mathrm{d}\psi = 0$ are individually linear, the relationship between ϕ and ψ is non-linear. Thus, the combination of these two equations defines a non-linear PDE, as expected. It is common practice to refer to the underlying oriented G_2- or $\widetilde{\mathrm{G}}_2$-structure as closed if $\mathrm{d}\phi = 0$ and coclosed if $\mathrm{d}\psi = 0$.

On closed manifolds, an alternative perspective on the torsion-free condition is provided by the notion of a Hitchin functional, introduced in [71]. Given a closed G_2 3-form ϕ on a closed, oriented manifold M, define a functional $\mathcal{H}_3$ by:

$$\begin{aligned} \mathcal{H}_3 : [\phi]_+ = \left\{\phi' \in [\phi] \in H^3_{\mathrm{dR}}(\mathrm{M}) \;\middle|\; \phi' \text{ is of } \mathrm{G}_2\text{-type}\right\} &\longrightarrow (0, \infty) \\ \phi' &\longmapsto \int_{\mathrm{M}} vol_{\phi'}. \end{aligned}$$

Likewise, given a closed G_2 4-form ψ on M, define a functional $\mathcal{H}_4$ by:

$$\begin{aligned} \mathcal{H}_4 : [\psi]_+ = \{\psi' \in [\psi] \mid \psi' \text{ is of } \mathrm{G}_2\text{-type}\} &\longrightarrow (0, \infty) \\ \psi' &\longmapsto \int_{\mathrm{M}} vol_{\psi'}. \end{aligned} \tag{2.2.15}$$

The definitions can naturally be generalised to $\widetilde{\mathrm{G}}_2$-structures, yielding the following defintion:

Definition 2.2.16. Let M be a closed, oriented 7-manifold and let $\widetilde{\phi}$ be a closed $\widetilde{\mathrm{G}}_2$ 3-form on M. Define the functional $\widetilde{\mathcal{H}}_3$ on (M, ϕ) by:

$$\begin{aligned} \widetilde{\mathcal{H}}_3 : \left[\widetilde{\phi}\right]_\sim = \left\{\widetilde{\phi}' \in \left[\widetilde{\phi}\right] \in H^3_{\mathrm{dR}}(\mathrm{M}) \;\middle|\; \widetilde{\phi}' \text{ is of } \widetilde{\mathrm{G}}_2\text{-type}\right\} &\longrightarrow (0, \infty) \\ \widetilde{\phi}' &\longmapsto \int_{\mathrm{M}} vol_{\widetilde{\phi}'}. \end{aligned}$$

Likewise, given a closed $\widetilde{\mathrm{G}}_2$ 4-form $\widetilde{\psi}$ on M, define the functional $\widetilde{\mathcal{H}}_4$ on (M, ψ) by:

$$\begin{aligned} \widetilde{\mathcal{H}}_4 : \left[\widetilde{\psi}\right]_\sim = \left\{\widetilde{\psi}' \in \left[\widetilde{\psi}\right] \;\middle|\; \widetilde{\psi}' \text{ is of } \widetilde{\mathrm{G}}_2\text{-type}\right\} &\longrightarrow (0, \infty) \\ \widetilde{\psi}' &\longmapsto \int_{\mathrm{M}} vol_{\widetilde{\psi}'}. \end{aligned}$$

Since $\bigwedge^3_+ \mathrm{T}^*\mathrm{M} \subset \bigwedge^3 \mathrm{T}^*\mathrm{M}$ is an open subbundle and M is closed, the subset $[\phi]_+ \subset [\phi]$ is open in the C^0-topology and thus one can identify $\mathrm{T}_{\phi'}[\phi]_+ \cong \mathrm{d}\Omega^2(\mathrm{M})$ for all $\phi' \in [\phi]_+$. A similar argument applies to $[\psi]_+$, $\left[\widetilde{\phi}\right]_\sim$ and $\left[\widetilde{\psi}\right]_\sim$. Using type decomposition, one can explicitly compute the functional derivatives of $\mathcal{H}_3$, $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$. For $\mathcal{H}_3$, this was accomplished by Hitchin in [71, Thm. 19 & Lem. 20].[1]. Analogous arguments can be used to compute the first and second derivatives of $\mathcal{H}_4$; see [72, Thm. 1] and also [59, Prop. 3.3 & 3.4]. Since the computation of these derivatives is completely representation-theoretic, and since G_2 and $\widetilde{\mathrm{G}}_2$ have identical representation theories, the formulae for $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ are identical. Thus one obtains:

Proposition 2.2.17. *The first and second derivatives of $\mathcal{H}_3$ (equivalently $\widetilde{\mathcal{H}}_3$) are given by:*

$$\begin{aligned} \overset{(\sim)}{\mathcal{D}\mathcal{H}}_3|_\phi : \mathrm{d}\Omega^2(\mathrm{M}) &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto \tfrac{1}{3}\int_{\mathrm{M}} \sigma \wedge \bigstar_\phi \phi \end{aligned} \qquad \begin{aligned} \mathcal{D}^2\overset{(\sim)}{\mathcal{H}}_3|_\phi : \mathrm{d}\Omega^2(\mathrm{M}) \times \mathrm{d}\Omega^2(\mathrm{M}) &\longrightarrow \mathbb{R} \\ (\sigma_1, \sigma_2) &\longmapsto \tfrac{1}{3}\int_{\mathrm{M}} \sigma_1 \wedge \bigstar_\phi I_\phi(\sigma_2) \end{aligned}$$

and the first and second derivatives of $\mathcal{H}_4$ (equivalently $\widetilde{\mathcal{H}}_4$) are given by:

$$\begin{aligned} \overset{(\sim)}{\mathcal{D}\mathcal{H}}_4|_\psi : \mathrm{d}\Omega^3(\mathrm{M}) &\longrightarrow \mathbb{R} \\ \varpi &\longmapsto \tfrac{1}{4}\int_{\mathrm{M}} \varpi \wedge \bigstar_\psi \psi \end{aligned} \qquad \begin{aligned} \mathcal{D}^2\overset{(\sim)}{\mathcal{H}}_4|_\psi : \mathrm{d}\Omega^3(\mathrm{M}) \times \mathrm{d}\Omega^3(\mathrm{M}) &\longrightarrow \mathbb{R} \\ (\varpi_1 \varpi_2) &\longmapsto \tfrac{1}{4}\int_{\mathrm{M}} \varpi_1 \wedge \bigstar_\psi J_\psi(\varpi_2) \end{aligned}$$

where:

$$I_\phi(\sigma) = \frac{4}{3}\pi_1(\sigma) + \pi_7(\sigma) - \pi_{27}(\sigma) \quad \textit{and} \quad J_\psi(\sigma) = \frac{3}{4}\pi_1(\sigma) + \pi_7(\sigma) - \pi_{27}(\sigma). \tag{2.2.18}$$

(Here, the projections $\pi_\bullet$ are defined with respect to ϕ and ψ respectively.)

In particular, ϕ' is a critical point of $\mathcal{H}_3$ if and only if it satisfies $\mathrm{d}\bigstar_{\phi'}\phi' = 0$, i.e. it is torsion-free, and similarly for the other three functionals. Moreover, in [71], Hitchin proved that the Hessian $\mathcal{D}^2\mathcal{H}_3$ was non-positive definite, and negative definite transverse to the action of diffeomorphisms; in particular, the critical points of $\mathcal{H}_3$ are all local maxima. However before this thesis, the corresponding results for the functionals $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ have remained unknown. I resolve this issue in Chapter 5 (see also Proposition 11.4.1 in Chapter 11).

2.2.4 Results specific to G_2-structures

Algebraically, G_2- and $\widetilde{\mathrm{G}}_2$-structures are very similar, largely due to the equality of their representation theories. However, since G_2-structures induce positive definite metrics, whereas $\widetilde{\mathrm{G}}_2$-structures induce indefinite metrics, there are notable differences between the analytic properties of G_2- and $\widetilde{\mathrm{G}}_2$-structures. It is for this reason that Part I of this thesis considers both G_2- and $\widetilde{\mathrm{G}}_2$-structures, whilst Part III only considers G_2-structures.

Firstly, since torsion-free G_2-structures induce Ricci-flat Riemannian metrics, Bochner's technique can be applied to G_2-manifolds, yielding:

[1] Note that the formulae for $\mathcal{D}\mathcal{H}_3|_\phi$ and $\mathcal{D}^2\mathcal{H}_3|_\phi$ differ from those in [71], as the author of this thesis has discovered an error in the numerical factor of $\frac{7}{18}$ used *op. cit.*, which has been corrected to $\frac{1}{3}$ in the formulae here presented.

Theorem 2.2.19 ([78, Thm. 3.5.4 and 3.5.5]). *Let* (M, ϕ) *be a closed* G_2*-manifold. Then* $b^1(\mathrm{M}) \leqslant \dim \mathrm{M}$ *and the universal cover* $\left(\widetilde{\mathrm{M}}, \widetilde{g}_\phi\right)$ *of* (M, g_ϕ) *is isometric to a Riemannian product* $\mathbb{R}^{b^1(\mathrm{M})} \times \mathrm{N}$, *where* $\mathbb{R}^{b^1(\mathrm{M})}$ *has the (flat) Euclidean metric and* N *is Ricci-flat, closed and simply-connected.*

Significantly, this theorem places topological restrictions on which manifolds can admit torsion-free G_2-structures. In particular, using Theorem 2.2.19, it can be proven that every manifold with holonomy G_2 has finite fundamental group. More specifically:

Theorem 2.2.20 ([78, Prop. 10.2.2]). *Let* (M, ϕ) *be a closed* G_2*-manifold. Then* $\mathrm{Hol}(g_\phi) = \mathrm{G}_2$ *if and only if* $|\pi_1(\mathrm{M})| < \infty$.

A further topological condition on G_2-manifolds arises from considering its first Pontryagin class $p_1(\mathrm{M})$. Recall that, on a general Riemannian manifold (M, g), the Riemann tensor R takes values in the bundle $\odot^2\left(\wedge^2 \mathrm{T}^*\mathrm{M}\right)$ and that, according to Chern–Weil Theory, the real first Pontryagin class can be represented by the closed 4-form $\frac{1}{8\pi^2}\mathrm{Tr}(R \wedge R)$, where $\wedge$ acts (say) on the first factor of $\wedge^2 \mathrm{T}^*\mathrm{M}$ and Tr acts on the second. In the case of a G_2-manifold (M, ϕ), it can be shown that the Riemann tensor R takes values in the bundle $\odot^2\left(\wedge^2_{14} \mathrm{T}^*\mathrm{M}\right)$. Thus, by using the identity $\alpha \wedge \alpha \wedge \phi = -\|\alpha\|^2_\phi vol_\phi$ for $\alpha \in \wedge^2_{14}\mathrm{T}^*\mathrm{M}$ together with the fact that $\mathrm{d}\phi = 0$, it follows that:

$$\langle p_1(\mathrm{M}) \cup [\phi], [\mathrm{M}] \rangle = \int_\mathrm{M} \frac{1}{8\pi^2}\mathrm{Tr}(R \wedge R) \wedge \phi = -\int_\mathrm{M} \frac{1}{8\pi^2}\|R\|^2_\phi vol_\phi,$$

where $p_1(\mathrm{M})$ denotes the real first Pontryagin class of M, $\langle , \rangle$ denotes the usual pairing between cohomology and homology and M denotes the fundamental class of M. In particular, one obtains:

Theorem 2.2.21 (cf. [78, Prop. 10.2.7]). *Let* M *be a closed, oriented 7-manifold with vanishing real first Pontryagin class. Then, any torsion-free* G_2*-structure on* M *induces a flat metric, and thus the corresponding holonomy group is discrete. In particular, if a closed oriented 7-manifold admits a torsion-free* G_2*-structure with holonomy* G_2, *then* $p_1(\mathrm{M}) \neq 0$.

Secondly, since G_2-structures induce positive definite metrics, the Hodge Laplacian induced by a G_2-structure is an elliptic (rather than hyperbolic) operator. Thus G_2-structures have a well-defined notion of Hodge Theory, which is compatible with the G_2-structure when the structure is torsion-free:

Theorem 2.2.22 (See [78, Thm. 3.5.3]). *Let* (M, ϕ) *be a* G_2*-manifold and let* Δ *denote the Hodge Laplacian determined by* ϕ. *Then:*

$$\Delta \circ \pi_q = \pi_q \circ \Delta \tag{2.2.23}$$

for any q. *In particular, if* M *is closed, then the Hodge decomposition on* M *may be refined to give:*

$$\Omega^k(\mathrm{M}) = \bigoplus_q \mathscr{H}^k_q(\mathrm{M}) \oplus \Delta\Omega^k_q(\mathrm{M}) \tag{2.2.24}$$

where $\mathscr{H}^k_q(\mathrm{M}) = \mathscr{H}^k(\mathrm{M}) \cap \Omega^k_q(\mathrm{M})$ *is the space of harmonic k-forms of type* q *and* q *runs over the appropriate irreducible representations of* G_2. *In particular, there is a decomposition of the de Rham cohomology of M which is analogous to the type decomposition described in eqn. (2.2.8); once more, any two groups* $H^k_q(\mathrm{M})$ *and* $H^{k'}_q(\mathrm{M})$ *with the same* q *are isomorphic.*

The relationship between Hodge Theory and type-decomposition can be made yet more explicit: as in Kähler geometry, for G_2-manifolds one may decompose the exterior derivative according to type. Indeed, define the following 'refined' exterior differential operators:

$$\begin{aligned} \mathrm{d}^1_7 : \Omega^0(\mathrm{M}) &\to \Omega^1(\mathrm{M}) & \mathrm{d}^7_7 : \Omega^1(\mathrm{M}) &\to \Omega^1(\mathrm{M}) & \mathrm{d}^7_{14} : \Omega^1(\mathrm{M}) &\to \Omega^2_{14}(\mathrm{M}) \\ f &\mapsto \mathrm{d}f & \alpha &\mapsto \star_\phi \mathrm{d}(\alpha \wedge \psi) & \alpha &\mapsto \pi_{14}(\mathrm{d}\alpha) \end{aligned}$$

$$\begin{aligned} \mathrm{d}^7_{27} : \Omega^1(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) & \mathrm{d}^{14}_{27} : \Omega^2_{14}(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) & \mathrm{d}^{27}_{27} : \Omega^3_{27}(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) \\ \alpha &\mapsto \pi_{27}\mathrm{d}\star_\phi(\alpha \wedge \psi) & \beta &\mapsto \pi_{27}(\mathrm{d}\beta) & \gamma &\mapsto \star_\phi \pi_{27}(\mathrm{d}\beta). \end{aligned}$$

Note that d^7_7 and d^{27}_{27} are both formally L^2-self-adjoint. Analogously, define $\mathrm{d}^7_1 = (\mathrm{d}^1_7)^*$, $\mathrm{d}^{14}_7 = (\mathrm{d}^7_{14})^*$, $\mathrm{d}^{27}_7 = (\mathrm{d}^7_{27})^*$ and $\mathrm{d}^{27}_{14} = (\mathrm{d}^{14}_{27})^*$. Then one has:

Theorem 2.2.25 (Bryant-Harvey, [22, §5])**.** *All exterior and co-exterior derivatives on the* G_2*-manifold* (M, ϕ, ψ) *can be expressed purely in terms of the operators* d^1_7, d^7_1, d^7_7, d^7_{14}, d^{14}_7, d^7_{27}, d^{27}_7, d^{14}_{27}, d^{27}_{14} *and* d^{27}_{27}. *In particular, the Hodge Laplacian operator can be expressed in terms of the same operators.*

The explicit formulae are presented in Appendix A. They will be needed for calculations in Part III of this thesis.

2.3 Stable forms and Hitchin functionals in 6- and 7-dimensions

The main references for this section are [71, 72].

Definition 2.3.1. A p-form $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)^*$ is termed stable if its $\mathrm{GL}_+(n;\mathbb{R})$-orbit in $\bigwedge^p (\mathbb{R}^n)^*$ is open (equivalently, if its $\mathrm{GL}(n;\mathbb{R})$-orbit is open). Thus if σ_0 is stable, all sufficiently small perturbations of σ_0 have the same algebraic properties as σ_0. I shall further term σ_0 a Hitchin form if it additionally satisfies $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0) \subseteq \mathrm{SL}(n;\mathbb{R})$. In particular, note that φ_0, ψ_0, $\widetilde{\varphi}_0$ and $\widetilde{\psi}_0$ are all Hitchin forms.

Given an orbit $\mathcal{O} \subset \bigwedge^p (\mathbb{R}^n)^*$ of Hitchin forms, fix σ_0 in $\mathcal{O}$ and define a map $vol : \mathcal{O} \to \bigwedge^n (\mathbb{R}^n)^*$ by $vol_\sigma = \alpha^* vol_0$ for any $\alpha \in \mathrm{GL}_+(n;\mathbb{R})$ such that $\sigma = \alpha^*\sigma_0$. Then vol is well-defined up to an overall positive constant multiple. Since $\mathcal{O} \subseteq \bigwedge^p (\mathbb{R}^n)^*$ is open, the derivative of vol at σ is a linear map $\bigwedge^p (\mathbb{R}^n)^* \to \bigwedge^n (\mathbb{R}^n)^*$, i.e. an element of the space $\bigwedge^p \mathbb{R}^n \otimes \bigwedge^n (\mathbb{R}^n)^* \cong \bigwedge^{n-p} (\mathbb{R}^n)^*$. Thus there is an element of $\bigwedge^{n-p} (\mathbb{R}^n)^*$, denoted $\Xi(\sigma)$, such that:

$$\mathcal{D}vol|_\sigma(\alpha) = \alpha \wedge \Xi(\sigma).$$

I term Ξ the Hitchin duality map. It defines a $\mathrm{GL}_+(n;\mathbb{R})$-equivariant map from $\mathcal{O}$ to an open orbit in $\bigwedge^{n-p} (\mathbb{R}^n)^*$ and is unique up to a constant positive multiple. Since $vol(\sigma)$ is homogeneous of degree $\frac{n}{p}$ in σ, Euler's Theorem for homogeneous functions gives:

$$\sigma \wedge \Xi(\sigma) = \frac{n}{p} vol(\sigma). \tag{2.3.2}$$

In the case $\sigma = \varphi_0, \psi_0, \widetilde{\varphi}_0$ or $\widetilde{\psi}_0$, note that Ξ is simply the map Θ or Σ, as appropriate (up to a constant positive multiple).

Hitchin forms are of particular interest when considering stable forms on manifolds. Let M be an oriented n-manifold and let $\sigma_0 \in \wedge^p (\mathbb{R}^n)^*$ be a stable p-form on $\mathbb{R}^n$. By analogy with the notions of G_2 and $\widetilde{\mathrm{G}}_2$-forms, say that $\sigma \in \Omega^p(\mathrm{M})$ is a σ_0-form if, for all $x \in \mathrm{M}$, there exists an orientation-preserving isomorphism $\alpha : \mathrm{T}_x\mathrm{M} \to \mathbb{R}^n$ such that $\sigma|_x = \alpha^*(\sigma_0)$. Allowing $x \in \mathrm{M}$ to vary, the set of all such α defines a $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0)$-structure on M. Write $\wedge^p_{\sigma_0}\mathrm{T}^*\mathrm{M}$ for the bundle of σ_0-forms and $\Omega^p_{\sigma_0}$ for the corresponding sheaf of sections.

Now suppose that σ_0 is a Hitchin form. Then for each $x \in \mathrm{M}$, $\sigma|_x$ induces a volume form $vol_\sigma|_x$ which may be integrated over all of M. In the special case where $\mathrm{d}\sigma = 0$ one defines the Hitchin functional:

$$\begin{aligned} \mathcal{H} : \mathcal{C}l^p_{\sigma_0}([\sigma]) = \left\{\sigma' \in [\sigma] \in H^p_{\mathrm{dR}}(\mathrm{M}) \;\middle|\; \sigma' \text{ is a } \sigma_0\text{-form}\right\} &\longrightarrow (0,\infty) \\ \sigma' &\longmapsto \int_{\mathrm{M}} vol_{\sigma'}. \end{aligned}$$

The functional derivative of $\mathcal{H}$ is then given by:

$$\begin{aligned} \mathcal{D}\mathcal{H}|_{\sigma'} : \mathrm{d}\Omega^{p-1}(\mathrm{M}) &\longrightarrow \mathbb{R} \\ \mathrm{d}\gamma &\longmapsto \int_{\mathrm{M}} \mathrm{d}\gamma \wedge \Xi(\sigma') \end{aligned}$$

In particular, σ' is a critical point of the functional $\mathcal{H}$ if and only if $\mathrm{d}\Xi(\sigma') = 0$. This motivates the following definition:

Definition 2.3.3. Say that a Hitchin form σ on M is biclosed if $\mathrm{d}\sigma = 0$ and $\mathrm{d}\Xi(\sigma) = 0$.

Biclosed stable forms are often of significant geometric interest. E.g. a G_2- or $\widetilde{\mathrm{G}}_2$-form is biclosed if and only if it is torsion-free. As a second example, it was proven in [71, Thm. 12] that an $\mathrm{SL}(3;\mathbb{C})$ 3-form (defined in the next subsection) is biclosed if and only if it defines an (integrable) complex structure with trivial canonical bundle.

2.3.1 Stable 3-forms in 6-dimensions

The classification of stable 3-forms in 6-dimensions was accomplished by Hitchin in [71]. I recount his construction below.

Given a 3-form $\rho \in \wedge^3 \left(\mathbb{R}^6\right)^*$, consider the linear map $K_\rho : \mathbb{R}^6 \to \mathbb{R}^6 \otimes \wedge^6 \left(\mathbb{R}^6\right)^*$ defined by composing the map:

$$\begin{aligned} \mathbb{R}^6 &\to \bigwedge^5 \left(\mathbb{R}^6\right)^* \\ v &\mapsto (v \lrcorner \rho) \wedge \rho \end{aligned}$$

with the canonical isomorphism $\wedge^5 \left(\mathbb{R}^6\right)^* \cong \mathbb{R}^6 \otimes \wedge^6 \left(\mathbb{R}^6\right)^*$. Using K_ρ, one can define a $\mathrm{GL}_+(6;\mathbb{R})$-equivariant map $\Lambda : \wedge^3 \left(\mathbb{R}^6\right)^* \to \left(\wedge^6 \left(\mathbb{R}^6\right)^*\right)^2$ by:

$$\Lambda(\rho) = \frac{1}{6} \mathrm{Tr}\left(K_\rho^2\right).$$

(Recall that the space $\left(\bigwedge^6 \left(\mathbb{R}^6\right)^*\right)^2$ is canonically oriented by declaring $s \otimes s > 0$ for any $s \neq 0 \in \bigwedge^6 \left(\mathbb{R}^6\right)^*$.) The following result is essentially proved in [71, §2] (although see [28, Prop. 1.5] for the expression for I_ρ and the explicit formulae for vol, I_ρ and J_ρ when $\rho = \rho_\pm$):

Proposition 2.3.4. *The action of* $\mathrm{GL}_+(6;\mathbb{R})$ *on* $\bigwedge^3 \left(\mathbb{R}^6\right)^*$ *has precisely two open orbits, namely:*

$$\bigwedge^3_+ \left(\mathbb{R}^6\right)^* = \left\{\rho \in \bigwedge^3 \left(\mathbb{R}^6\right)^* \;\middle|\; \Lambda(\rho) > 0\right\} \quad \textit{and} \quad \bigwedge^3_- \left(\mathbb{R}^6\right)^* = \left\{\rho \in \bigwedge^3 \left(\mathbb{R}^6\right)^* \;\middle|\; \Lambda(\rho) < 0\right\},$$

both of which are invariant under $\mathrm{GL}(6;\mathbb{R})$*. Representatives of* $\bigwedge^3_+ \left(\mathbb{R}^6\right)^*$ *and* $\bigwedge^3_- \left(\mathbb{R}^6\right)^*$ *may be taken to be:*

$$\rho_+ = \theta^{123} + \theta^{456} \quad \textit{and} \quad \rho_- = \theta^{135} - \theta^{146} - \theta^{236} - \theta^{245} = \mathfrak{Re}\Big(\left(\theta^1 + i\theta^2\right) \wedge \left(\theta^3 + i\theta^4\right) \wedge \left(\theta^5 + i\theta^6\right) \Big) \tag{2.3.5}$$

respectively. Each $\rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^*$ *induces a volume form* $vol_\rho = (\Lambda(\rho))^{\frac{1}{2}}$ *and para-complex structure* $I_\rho = vol_\rho^{-1} K_\rho$ *on* $\mathbb{R}^6$ *(i.e.* I_ρ *is an automorphism of* $\mathbb{R}^6$ *such that* $I_\rho^2 = \mathrm{Id}$*, with* $+1$ *and* -1 *eigenspaces* $E_{\pm,\rho}$ *each having dimension 3), and* $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho) \cong \mathrm{SL}(3;\mathbb{R})^2$ *acting diagonally on* $\mathbb{R}^6 \cong E_{+,\rho} \oplus E_{-,\rho}$*. Explicitly for* $\rho = \rho_+$*:*

$$vol_{\rho_+} = \theta^{123456}; \quad I_{\rho_+} = (e_1, e_2, e_3, e_4, e_5, e_6) \mapsto (e_1, e_2, e_3, -e_4, -e_5, -e_6);$$
$$E_{+,\rho_+} = \langle e_1, e_2, e_3\rangle \quad \textit{and} \quad E_{-,\rho_+} = \langle e_4, e_5, e_6\rangle.$$

By contrast, each $\rho \in \bigwedge^3_- \left(\mathbb{R}^6\right)^*$ *induces a volume form* $vol_\rho = \frac{1}{2}\left(-\Lambda(\rho)\right)^{\frac{1}{2}}$ *and a complex structure* $J_\rho = -\frac{1}{2} vol_\rho^{-1} K_\rho$ *on* $\mathbb{R}^6$ *, and* $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho) \cong \mathrm{SL}(3;\mathbb{C})$*. Explicitly for* $\rho = \rho_-$*:*

$$vol_{\rho_-} = \theta^{123456} \quad \textit{and} \quad J_{\rho_-} = (e_1, e_2, e_3, e_4, e_5, e_6) \mapsto (e_2, -e_1, e_4, -e_3, e_6, -e_5).$$

Note in particular that both $\mathrm{SL}(3;\mathbb{C})$ 3-forms and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms are Hitchin forms.

2.3.2 Stable 3- and 4-forms in 7-dimensions

As described in §2.2, 3- and 4-forms of G_2- and $\widetilde{\mathrm{G}}_2$-type are all stable (and indeed, Hitchin) forms. These are, in fact, essentially the only stable 3- and 4-forms in 7-dimensions. Given $\phi \in \bigwedge^3 \left(\mathbb{R}^7\right)^*$, define a quadratic form Q_ϕ on $\mathbb{R}^7$ valued in $\bigwedge^7 \left(\mathbb{R}^7\right)^*$ by $Q_\phi(v) = \frac{1}{6}\left(v \lrcorner \phi\right)^2 \wedge \phi \in \bigwedge^7 \left(\mathbb{R}^7\right)^*$. The determinant of Q_ϕ is a polynomial in ϕ and thus $\left\{\phi \;\middle|\; Q_\phi \text{ is degenerate}\right\}$ is an affine subvariety of $\bigwedge^3 \left(\mathbb{R}^7\right)^*$ of positive codimension; hence Q_ϕ must be non-degenerate whenever ϕ is stable. The following proposition is easily deduced from the results of [68]:

Proposition 2.3.6. *The action of* $\mathrm{GL}_+(7;\mathbb{R})$ *has precisely 4 open orbits, corresponding to* Q *having signature* $(7,0)$*,* $(3,4)$*,* $(4,3)$ *and* $(0,7)$*. Explicitly:*

$$\left\{\phi \;\middle|\; Q_\phi \textit{ has signature } (7,0)\right\} = \bigwedge^3_+ \left(\mathbb{R}^7\right)^* \qquad \left\{\phi \;\middle|\; Q_\phi \textit{ has signature } (3,4)\right\} = \bigwedge^3_\sim \left(\mathbb{R}^7\right)^*$$
$$\left\{\phi \;\middle|\; Q_\phi \textit{ has signature } (4,3)\right\} = -\bigwedge^3_\sim \left(\mathbb{R}^7\right)^* \qquad \left\{\phi \;\middle|\; Q_\phi \textit{ has signature } (0,7)\right\} = -\bigwedge^3_+ \left(\mathbb{R}^7\right)^*.$$

For $\phi \in \Lambda^3_+ \left(\mathbb{R}^7\right)^*$: $Q_\phi = g_\phi \otimes vol_\phi$, where g_ϕ and vol_ϕ are as defined in §2.2 and can be characterised by the condition $\|\phi\|^2_{g_\phi} = 7$. Likewise, for $\widetilde{\phi} \in \Lambda^3_\sim \left(\mathbb{R}^7\right)^*$: $Q_{\widetilde{\phi}} = g_{\widetilde{\phi}} \otimes vol_{\widetilde{\phi}}$ where $g_{\widetilde{\phi}}$ and $vol_{\widetilde{\phi}}$ are as in §2.2 and can be characterised by the condition $\|\phi\|^2_{g_\phi} = 7$.

Using Proposition 2.3.6, the following result is easily deduced (see also [21, p. 541]):

Proposition 2.3.7. *The action of* $\mathrm{GL}_+(7;\mathbb{R})$ *on* $\Lambda^4_\sim \left(\mathbb{R}^7\right)^*$ *has precisely 4 open orbits, given by* $\Lambda^4_+ \left(\mathbb{R}^7\right)^*$, $\Lambda^4_\sim \left(\mathbb{R}^7\right)^*$, $-\Lambda^4_\sim \left(\mathbb{R}^7\right)^*$ *and* $-\Lambda^4_+ \left(\mathbb{R}^7\right)^*$, *each of which are also orbits of* $\mathrm{GL}(7;\mathbb{R})$.

2.4 Metric spaces and Gromov–Hausdorff distance

2.4.1 Gromov–Hausdorff distance and forwards discrepancy

The main reference for the material on Gromov–Hausdorff distance in this subsection is [25, §7.3].

Let (X, d) be a metric space. Given $Y \subseteq X$, let $\mathcal{N}_\eta(Y) = \{x \in X \mid d(x, Y) < \eta\}$ be the open η-neighbourhood of Y, where $d(x, Y) = \inf\{d(x, y) \mid y \in Y\}$. Given $A, B \subseteq X$ non-empty, closed and bounded, define the Hausdorff distance between A and B to be:

$$d_{\mathcal{H}}(A, B) = \inf\{\eta > 0 \mid A \subseteq \mathcal{N}_\eta(B) \text{ and } B \subseteq \mathcal{N}_\eta(A)\}.$$

Now let (Y, d_Y) and $(Y', d_{Y'})$ be compact metric spaces. The Gromov–Hausdorff distance between (Y, d_Y) and $(Y', d_{Y'})$ is defined to be:

$$d_{\mathcal{GH}}\left[(Y, d_Y), (Y', d_{Y'})\right] = \inf\left\{\eta > 0 \;\middle|\; \begin{array}{l}\text{there exists } (X, d) \text{ together with isometric embeddings}\\ \iota : Y \hookrightarrow X, \iota' : Y' \hookrightarrow X \text{ such that } d_{\mathcal{H}}\left[\iota(Y), \iota'(Y')\right] \leqslant \eta\end{array}\right\}.$$

It can be shown that $d_{\mathcal{GH}}\left[(Y, d_Y), (Y', d_{Y'})\right] = 0$ if and only if (Y, d_Y) and $(Y', \mathrm{d}_{Y'})$ are isometric, and thus $d_{\mathcal{GH}}$ defines a metric on the collection of isometry classes of compact metric spaces. In light of this, a family $(Y_i, d_i)_{i\in\mathbb{N}}$ of compact metric spaces is said to converge to a compact metric space (Y, d) in the Gromov–Hausdorff sense as $i \to \infty$ if $d_{\mathcal{GH}}\left[(Y_i, d_i), (Y, d)\right] \to 0$.

Gromov–Hausdorff distance is closely related to the notion of ε-isometry. Recall that for metric spaces (X, d), (X', d') and $\varepsilon > 0$, an ε-isometry is a set-theoretic function $f : X \to X'$ (which need not be continuous) satisfying:

- For all $x' \in X'$, there exists $x \in X$ such that $d'(f(x), x') \leqslant \varepsilon$ ($f(X)$ is an 'ε-net' in X');
- For all $x, y \in X$: $|d'(f(x), f(y)) - d(x, y)| \leqslant \varepsilon$.

The relation between Gromov–Hausdorff distance and ε-isometries can be quantified as follows:

Proposition 2.4.1 ([25, Cor. 7.3.28])**.** *Let (X, d) and (X', d') be compact metric spaces. Then:*

$$d_{\mathcal{GH}}\left[(X, d), (X', d')\right] \leqslant 2 \inf\{\varepsilon > 0 \mid \textit{There exists an } \varepsilon\textit{-isometry } f : (X, d) \to (X', d')\}.$$

Motivated by this result, I make the following definition:

Definition 2.4.2. Let (X,d) and (X',d') be compact metric spaces. The forwards discrepancy between (X,d) and (X',d'), denoted $\mathfrak{D}[(X,d)\to(X',d')]$, is defined to be:

$$\mathfrak{D}[(X,d)\to(X',d')] = \inf\{\varepsilon>0 \mid \text{There exists an } \varepsilon\text{-isometry } f:(X,d)\to(X',d')\}.$$

(Note that the infimum is finite since, choosing $\varepsilon = \max[\mathrm{diam}(X,d),(X',d')]$, any map $f: X\to X'$ is an ε-isometry.)

The definition of forwards discrepancy naturally extends to semi-metric spaces. Recall from [25, §1.1] that a semi-metric d satisfies all the usual conditions of a metric, except that distinct points x and y are permitted to satisfy $d(x,y)=0$. Given a semi-metric space (X,d), define an equivalence relation $\sim_d$ on X via $x\sim_d y$ if and only if $d(x,y)=0$. Then d descends to define a metric $d_\sim$ on the quotient $X/\sim_d$. In this thesis, I term $\left(X/\sim_d, d_\sim\right)$ the free metric space on (X,d) (the name deriving from the fact that the assignment $(X,d)\mapsto\left(X/\sim_d, d_\sim\right)$ is left-adjoint to the natural inclusion functor of the category **Met** of metric spaces and non-expansive maps into the category **SMet** of semi-metric spaces and non-expansive maps). I say that a semi-metric space is compact if its corresponding free metric space is compact in the usual sense. Then it is clear that the notion of forwards discrepancy is well-defined not just on the class of compact metric spaces, but also on the class of compact semi-metric spaces.

The key result concerning Gromov–Hausdorff distance and forwards discrepancy which I shall require is the following:

Proposition 2.4.3. *Let (X,d) be a compact metric space and let (X',d') be a compact semi-metric space. Then:*

$$d_{\mathcal{GH}}\left[(X,d),\left(X'/\sim_{d'}, d'_\sim\right)\right] \leqslant 2\mathfrak{D}[(X,d)\to(X',d')].$$

In particular, given a family of compact metric spaces $(X^\mu,d^\mu)_{\mu\in[1,\infty)}$ such that $2\mathfrak{D}[(X^\mu,d^\mu)\to(X',d')]\to 0$ as $\mu\to\infty$, the spaces (X^μ,d^μ) converge in the Gromov–Hausdorff sense to $\left(X'/\sim_{d'}, d'_\sim\right)$ as $\mu\to\infty$.

Proof. Firstly note that whilst forwards discrepancy does not satisfy the triangle inequality, it does satisfy the weaker inequality:

$$\mathfrak{D}[(X,d)\to(X'',d'')] \leqslant \mathfrak{D}[(X,d)\to(X',d')] + 2\mathfrak{D}[(X',d')\to(X'',d'')].$$

for any compact semi-metric spaces (X,d), (X',d') and (X'',d''): indeed, given ε-isometries $f: X\to X'$ and $f': X'\to X''$ for some $\varepsilon,\varepsilon'>0$, one can verify directly that $f'\circ f: X\to X''$ is an $(\varepsilon+2\varepsilon')$-isometry. Secondly, note that given a compact semi-metric space (X,d), the quotient map $f:(X,d)\to\left(X/\sim_d, d_\sim\right)$ is an ε-isometry for any $\varepsilon>0$ and thus $\mathfrak{D}\left[(X,d),\left(X/\sim_d, d_\sim\right)\right]=0$. The result follows by combining these two observations with Proposition 2.4.1.

□

Proposition 2.4.3 decouples the task of computing Gromov–Hausdorff limits into two distinct stages: firstly, given a family $(X^\mu,d^\mu)_{\mu\in[1,\infty)}$ of metric spaces, one finds a compact semi-metric space

(X^∞, d^∞) such that $\mathfrak{D}[(X^\mu, d^\mu) \to (X^\infty, d^\infty)] \to 0$ as $\mu \to \infty$. By applying Proposition 2.4.3, it follows that $(X^\mu, d^\mu) \to \left(X^\infty /_{\sim_{d^\infty}}, d^\infty_\sim \right)$ in the Gromov–Hausdorff sense, as $\mu \to \infty$; thus, the task of computing the Gromov–Hausdorff limit of the family (X^μ, d^μ) is reduced to describing the free metric space $\left(X^\infty /_{\sim_{d^\infty}}, d^\infty_\sim \right)$. This two stage process will underpin the treatment of Gromov–Hausdorff convergence in Chapter 4.

2.4.2 Length structures

The main reference for this subsection is [25, Ch. 2]. Let X be a topological space. A length structure on X is a class $\mathcal{A}$ of continuous paths in X together with an assignment $\ell : \mathcal{A} \to (0, \infty]$ satisfying the following four conditions:

1. If $(\gamma : [a,b] \to X) \in \mathcal{A}$ and $c \in [a,b]$, then $\left(\gamma|_{[a,c]} : [a,c] \to X\right), \left(\gamma|_{[c,b]} : [c,b] \to X\right) \in \mathcal{A}$ and:

$$\ell(\gamma) = \ell\left(\gamma|_{[a,c]}\right) + \ell\left(\gamma|_{[c,b]}\right).$$

   Moreover, $\ell\left(\gamma|_{[a,c]}\right)$ is continuous, when viewed as a function of c.

2. If $\gamma : [a,b] \to X$ is continuous and $c \in [a,b]$ is such that $\left(\gamma|_{[a,c]} : [a,c] \to X\right), \left(\gamma|_{[c,b]} : [c,b] \to X\right) \in \mathcal{A}$, then $(\gamma : [a,b] \to X) \in \mathcal{A}$;

3. If $(\gamma : [a,b] \to X) \in \mathcal{A}$ and $\phi : [c,d] \to [a,b]$ is a homeomorphism of the form $t \mapsto \alpha t + \beta$ $(\alpha \neq 0)$, then $(\gamma \circ \phi : [c,d] \to X) \in \mathcal{A}$ and $\ell(\gamma \circ \phi) = \ell(\gamma)$.

4. For all $x \in X$ and all open neighbourhoods U of x:

$$\inf\{\ell(\gamma) \mid (\gamma : [a,b] \to X) \in \mathcal{A} \text{ satisfies } \gamma(a) = x \text{ and } \gamma(b) \in X \backslash U\} > 0.$$

Every length structure $(\mathcal{A}, \ell)$ on X defines a metric $d_{(\mathcal{A},\ell)}$ via:

$$d_{(\mathcal{A},\ell)}(x,y) = \inf\{\ell(\gamma) \mid (\gamma : [a,b] \to X) \in \mathcal{A} \text{ satisfies } \gamma(a) = x \text{ and } \gamma(b) = y\}.$$

Such metrics are termed intrinsic. The following example of length structures will be of particular significance in this thesis:

Example 2.4.4. Let E be an orbifold, let $\mathcal{A}$ denote the set of piecewise-C^1 curves in E and let g be a Riemannian metric on E. Then the map $\ell^g : \mathcal{A} \to (0, \infty]$ defined in eqns. (2.1.17) and (2.1.18) defines a length structure on E; write d^g for the induced intrinsic metric.

I remark that, whilst in general the topology on X induced by an intrinsic metric need only be no coarser than the original topology (in the sense that if $U \subseteq X$ is open, then it is open with respect to $d_{(\mathcal{A},\ell)}$ for any length-structure $(\mathcal{A}, \ell)$), for intrinsic metrics of the form d^g, and for all other intrinsic metrics considered in this thesis, the two topologies in fact coincide.

Finally, I say that $(\mathcal{A}, \ell)$ is a weak length structure if it satisfies conditions 1–3 above. In this case, $(\mathcal{A}, \ell)$ naturally induces a semi-metric $d(\mathcal{A}, \ell)$ on X. As a key example of this notion, given a

Riemannian semi-metric on an orbifold E, $(\mathcal{A}, \ell^g)$ defines a weak length structure on E and hence induces a semi-metric d^g.

2.5 Differential relations and h-principles

The main reference for this section is [42, Chs. 1, 5, 6].

Definition 2.5.1. Let M^n be a manifold, let $F \to \mathrm{M}$ be a fibre bundle of dimension $n+q$ over M and fix $r \in \mathbb{N}$. Given a point $p \in \mathrm{M}$, two sections $f \in \Gamma(\mathcal{O}p(p), F)$ and $g \in \Gamma(\mathcal{O}p(p), F)$ are said to be r-tangent at p if:

- $f(p) = g(p)$;
- For any system of coordinates $(x^1, ..., x^n)$ on $\mathcal{O}p(p)$, any local trivialisation $F|_{\mathcal{O}p(p)} \cong \mathcal{O}p(p) \times F_p$ and any coordinate neighbourhood $U \subseteq \mathbb{R}^q$ in F_p such that $f(p) = g(p) \in \mathcal{O}p(p) \times U$, regarding f and g as maps $\mathcal{O}p(p) \to U \subseteq \mathbb{R}^q$ via this trivialisation, one has:

$$\left.\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\right|_p = \left.\frac{\partial^{|\alpha|} g}{\partial x^\alpha}\right|_p \quad \text{for all multi-indices } \alpha \text{ such that } |\alpha| \leqslant r.$$

The notion of r-tangency is independent of any choices and defines an equivalence relation $\sim_r$ on the set of pairs $\left\{\left(\mathcal{O}p(p), f \in \Gamma(\mathcal{O}p(p), F)\right)\right\}$. The r^{th} jet space $F_p^{(r)}$ of F at p is the set of equivalence classes under this relation, i.e.:

$$F_p^{(r)} = \left\{\left(\mathcal{O}p(p), f \in \Gamma(\mathcal{O}p(p), E)\right)\right\} / \sim_r.$$

The r^{th} jet bundle of F is then defined to be $F^{(r)} = \coprod_{p \in \mathrm{M}} F_p^{(r)}$, together with its natural (smooth) bundle structure. Since $\sim_R$ is a finer equivalence relation than $\sim_r$ when $R > r$, there are natural maps $p_{R,r} : F^{(R)} \to F^{(r)}$ for $R > r$. In particular, there is a map $p_r : F^{(r)} \to F$ which assigns to each r-jet its underlying value in F. Conversely, given a section $f : \mathrm{M} \to F$, define the r^{th} jet extension of f by:

$$\begin{aligned} j_r(f) : \mathrm{M} &\to F^{(r)} \\ p &\mapsto [(\mathcal{O}p(p), f)]_{\sim_r}. \end{aligned}$$

Clearly $p_r \circ j_r(f) = f$.

This thesis restricts attention to the case where $F = E$ is a vector bundle and $r = 1$. In this case, given a connection ∇ on E, by [111, §9, Cor. to Thm. 7] there is a bundle isomorphism $E^{(1)} \cong E \oplus (\mathrm{T}^*\mathrm{M} \otimes E)$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma\left(\mathrm{M}, E^{(1)}\right) \arrow[rr, "\cong"] & & \Gamma\left(\mathrm{M}, E \oplus (\mathrm{T}^*\mathrm{M} \otimes E)\right) \\ & \Gamma(\mathrm{M}, E) \arrow[ul, "j_1"] \arrow[ur, "s \mapsto s \oplus \nabla s"'] & \end{tikzcd} \tag{2.5.2}$$

In particular, note that $E^{(1)}$ naturally has the structure of a vector bundle over M. More generally, given $q \geqslant 0$, let D^q denote the q-dimensional disc and write E_{D^q} for the pullback of the vector bundle E along the projection $D^q \times \mathrm{M} \to \mathrm{M}$; explicitly, E_{D^q} is the vector bundle $D^q \times E \xrightarrow{\mathrm{Id}\times\pi} D^q \times \mathrm{M}$. By a section of E_{D^q}, I shall mean a continuous map $\varsigma : D^q \times \mathrm{M} \to D^q \times E$ satisfying $\pi_{E_{D^q}} \circ s = \mathrm{Id}_{D^q\times\mathrm{M}}$, and depending smoothly on $x \in \mathrm{M}$; in particular, sections of E_{D^q} over $D^q \times \mathrm{M}$ correspond to continuous maps $D^q \to \Gamma(E, \mathrm{M})$. Write $E^{(1)}_{D^q}$ for the vector bundle $\left(E^{(1)}\right)_{D^q}$ and note that $E^{(1)}_{D^q} \neq (E_{D^q})^{(1)}$, since only derivatives in the 'M-direction' are considered in the bundle $E^{(1)}_{D^q}$. A section of $E^{(1)}_{D^q}$ is termed holonomic if it is the 1-jet of a section of E_{D^q}, i.e., using the identification in eqn. (2.5.2), if it can be written as $s \oplus \nabla s$ for some section s of E_{D^q}. A fibred differential relation (of order 1) on D^q-indexed families of sections of E is simply a subset $\mathscr{R} \subseteq E^{(1)}_{D^q}$. $\mathscr{R}$ is termed an open relation if it is open as a subset of $E^{(1)}_{D^q}$.

Definition 2.5.3. Let M be an n-manifold. A subset $A \subseteq \mathrm{M}$ is termed a polyhedron if there exists a smooth triangulation $\mathscr{K}$ of M identifying A with a subcomplex of $\mathscr{K}$ (in particular, A is a closed subset of M). I define the boundary of A to be $\partial A = A \backslash \mathring{A}$, where $\mathring{A}$ denotes the topological interior of A in M. Then ∂A is a subpolyhedron of A and $A = \partial A$ if and only if A has positive codimension in M.

Note that every sufficiently small open neighbourhood of a polyhedron $A \subseteq \mathrm{M}$ deformation retracts onto A. I will always implicitly assume that $\mathcal{O}p(A)$ has been chosen small enough to ensure that $\mathcal{O}p(A)$ deformation retracts onto A; in particular, the cohomology rings of A and $\mathcal{O}p(A)$ are identical.

Definition 2.5.4. Let $\mathscr{R}$ be a fibred differential relation over a manifold M. Say that $\mathscr{R}$ satisfies the relative h-principle if for every polyhedron $A \subseteq \mathrm{M}$, every $q \geqslant 0$ and every section F_0 of $\mathscr{R}$ over $D^q \times \mathrm{M}$ which is holonomic over $(\partial D^q \times \mathrm{M}) \cup (D^q \times \mathcal{O}p(A))$, there exists a homotopy $(F_t)_{t\in[0,1]}$ of sections of $\mathscr{R}$, constant over $(\partial D^q \times \mathrm{M}) \cup (D^q \times \mathcal{O}p(A))$, such that F_1 is a holonomic section of $\mathscr{R}$. Say that $\mathscr{R}$ satisfies the C^0-dense relative h-principle if, in addition, the induced homotopy $p_1(F_t)$ of sections of E can be taken to be arbitrarily small in the C^0-topology.

Remark 2.5.5. Note that the case $A = \mathrm{M}$ is vacuous in the above defintion; thus without loss of generality one can always assume $A \neq \mathrm{M}$. Moreover, one can also assume without loss of generality that A has positive codimension in M: indeed, the relative h-principle for the pair (M, A) is equivalent to the relative h-principle for the pair $\left(\mathrm{M}\backslash\mathring{A}, A\backslash\mathring{A}\right)$, with $A\backslash\mathring{A}$ having positive codimension in $\mathrm{M}\backslash\mathring{A}$ for $A \neq \mathrm{M}$. (Note that, although $\mathrm{M}\backslash\mathring{A}$ need not be a manifold, it is triangulable and thus the techniques for proving h-principles used in [42] apply.) Consequently, although [42] only considers the case where A has positive codimension, its results are equally valid for the codimension-0 case.

2.5.1 Convex integration

The main reference for this subsection is [42, Chs. 17, 18].

Definition 2.5.6. Let $\mathbb{A}$ be a real affine space. Say that a subset $S \subseteq \mathbb{A}$ is ample if the convex hull of each path-component of S is equal to $\mathbb{A}$. In particular, note that the empty set $\varnothing$ is formally ample.

Now fix a point $p \in \mathrm{M}$. Identifying $E^{(1)} \cong E \oplus \mathrm{T}^*\mathrm{M} \otimes E$, the fibre of the map $p_1 : E^{(1)} \to E$ over $e \in E_p$ is isomorphic to the space $p_1^{-1}(e) = \{e\} \times \mathrm{T}_p^*\mathrm{M} \otimes E_p = \{e\} \times \mathrm{Hom}(\mathrm{T}_p\mathrm{M}, E_p)$. Each codimension-1 hyperplane $\mathbb{B} \subset \mathrm{T}_p\mathrm{M}$ and linear map $\lambda : \mathbb{B} \to E_p$ thus define a so-called principal subspace of $p_1^{-1}(e)$ by:

$$\Pi_e(\mathbb{B}, \lambda) = \{e\} \times \underbrace{\{L \in \mathrm{Hom}(\mathrm{T}_p\mathrm{M}, E_p) \mid L|_{\mathbb{B}} = \lambda\}}_{\Pi(\mathbb{B},\lambda)}.$$

$\Pi_e(\mathbb{B}, \lambda)$ is an affine subspace of $p_1^{-1}(e)$ modelled on E_p, though not, in general, a linear subspace. (Note also that changing the choice of connection ∇ on E changes the identification $p_1^{-1}(e) = \{e\} \times \mathrm{T}_p^*\mathrm{M} \otimes E_p$ by an affine linear map and so the collection of principal subspaces of $p_1^{-1}(e)$ is independent of the choice of connection.)

Definition 2.5.7. Let $E \to \mathrm{M}$ be a vector bundle, let $q \geqslant 0$ and let $\mathscr{R} \subseteq E^{(1)}_{D^q}$ be an open fibred differential relation. For each $s \in D^q$, define $\mathscr{R}_s \subseteq E^{(1)}$ by the formula:

$$\{s\} \times \mathscr{R}_s = \mathscr{R} \cap \left(\{s\} \times E^{(1)}\right).$$

Say that $\mathscr{R}$ is ample if, for every $s \in D^q$, $e \in E$ and principal subspace $\Pi_e \subset p_1^{-1}(e)$, the subset $\mathscr{R}_s \cap \Pi_e \subseteq \Pi_e$ is ample in the sense of Definition 2.5.6.

The following special case will be of particular interest in this thesis. Suppose $\mathscr{R}$ has the form:

$$\mathscr{R} = E_{D^q} \times_{(D^q \times \mathrm{M})} \mathscr{R}' \subseteq E_{D^q} \oplus (\mathrm{T}^*\mathrm{M} \otimes E)_{D^q}$$

for some subbundle $\mathscr{R}' \subseteq (\mathrm{T}^*\mathrm{M} \otimes E)_{D^q}$, where $\times_{(D^q \times \mathrm{M})}$ denotes the fibrewise Cartesian product of bundles over $D^q \times \mathrm{M}$. Define $\mathscr{R}'_s$ by the equation:

$$\{s\} \times \mathscr{R}'_s = \mathscr{R}' \cap [\{s\} \times (\mathrm{T}^*\mathrm{M} \otimes E)] \subset (\mathrm{T}^*\mathrm{M} \otimes E)_{D^q}.$$

Then for all $s \in D^q$, $e \in E$ and principal subspaces $\Pi_e(\mathbb{B}, \lambda) \subset \pi^{-1}(e)$:

$$\mathscr{R}_s \cap \Pi_e(\mathbb{B}, \lambda) = \{s\} \times \{e\} \times (\mathscr{R}'_s \cap \Pi(\mathbb{B}, \lambda)),$$

and thus $\mathscr{R}$ is ample if and only if $\mathscr{R}'_s \cap \Pi(\mathbb{B}, \lambda) \subseteq \Pi(\mathbb{B}, \lambda)$ is ample for all $\mathbb{B}$ and λ. In particular, the underlying point e is irrelevant for relations of this form.

The significance of ample differential relations lies in the following result:

Theorem 2.5.8 ([42, §§17–18])**.** *Let $E \to \mathrm{M}$ be a vector bundle, let $q \geqslant 0$ and let $\mathscr{R} \subseteq E^{(1)}_{D^q}$ be a fibred differential relation which is open and ample. Then $\mathscr{R}$ satisfies the C^0-dense, relative h-principle.*

2.5.2 Convex integration with avoidance

The main reference for this subsection is [98] (although note that the presentation and notation used below differs from that *op. cit.*).

Configuration spaces for hyperplanes

Let $\mathbb{A}$ be a n-dimensional vector space and write $\mathrm{Gr}_{n-1}^{(\infty)}(\mathbb{A})$ for the collection of all finite subsets of $\mathrm{Gr}_{n-1}(\mathbb{A})$. $\mathrm{Gr}_{n-1}^{(\infty)}(\mathbb{A})$ shall be termed the configuration space for hyperplanes in $\mathbb{A}$ and can be given a natural 'smooth structure' as follows. For any $k \geqslant 1$, consider the manifold $\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})$ parameterising ordered k-tuples of hyperplanes in $\mathbb{A}$. The symmetric group Sym_k acts on $\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})$ by permuting the factors, however this action is not free and thus the resulting quotient is not a smooth manifold, but rather an orbifold. Now define the subset:

$$\left(\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})\right)_{sing} = \left\{(\mathbb{B}_1, ..., \mathbb{B}_k) \in \prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A}) \;\middle|\; \text{for some } i \neq j\text{: } \mathbb{B}_i = \mathbb{B}_j\right\}$$

of tuples whose elements are not distinct. This set consists precisely of those elements of $\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})$ with a non-trivial stabiliser in Sym_k and may naturally be regarded as a stratified submanifold of $\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})$ of codimension $n-1 = \dim \mathrm{Gr}_{n-1}(\mathbb{A})$. The complement of this set:

$$\prod_1^k \widetilde{\mathrm{Gr}_{n-1}}(\mathbb{A}) = \prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A}) \Big\backslash \left(\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})\right)_{sing}$$

is thus an open and dense subset of $\prod_1^k \mathrm{Gr}_{n-1}(\mathbb{A})$ on which the group Sym_k acts freely. In particular, the space $\prod_1^k \widetilde{\mathrm{Gr}_{n-1}}(\mathbb{A}) \Big/ Sym_k$ is naturally a smooth manifold. Denote this manifold by $\mathrm{Gr}_{n-1}^{(k)}(\mathbb{A})$ and denote the natural quotient map by $\sigma : \prod_1^k \widetilde{\mathrm{Gr}_{n-1}}(\mathbb{A}) \to \mathrm{Gr}_{n-1}^{(k)}(\mathbb{A})$. Since $\mathrm{Gr}_{n-1}^{(\infty)}(\mathbb{A}) = \coprod_{k=1}^{\infty} \mathrm{Gr}_{n-1}^{(k)}(\mathbb{A})$ as sets, $\mathrm{Gr}_{n-1}^{(\infty)}(\mathbb{A})$ inherits a natural topology such that each connected component is a smooth manifold.

Avoidance templates

Consider the vector bundles TM over M and TM_{D^q} over $D^q \times \mathrm{M}$. Applying the construction of §2.5.2 to each fibre of these vector bundles yields bundles $\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM})$ and $\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})$ over M and $D^q \times \mathrm{M}$ respectively, termed the bundle of configurations of hyperplanes over M, resp. $D^q \times \mathrm{M}$. (Note that $\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})$ is simply the bundle $D^q \times \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}) \to D^q \times \mathrm{M}$.) Write $\mathscr{R} \times_{(D^q \times \mathrm{M})} \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})$ for the bundle over $D^q \times \mathrm{M}$ given by taking the fibrewise product of $\mathscr{R}$ and $\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})$; explicitly:

$$\mathscr{R} \times_{(D^q \times \mathrm{M})} \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q}) = \left\{[(s,T),(s,\Xi)] \in \underbrace{\mathscr{R} \times \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})}_{\subseteq (D^q \times E^{(1)}) \times \left(D^q \times \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM})\right)} \;\middle|\; \pi_{E^{(1)}}(T) = \pi_{\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM})}(\Xi)\right\},$$

where $\pi_{E^{(1)}}$ and $\pi_{\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM})}$ denote the bundle projections $E^{(1)} \to \mathrm{M}$ and $\mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}) \to \mathrm{M}$ respectively. Let $\mathscr{A} \subseteq \mathscr{R} \times_{(D^q \times \mathrm{M})} \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})$. Given $s \in D^q$, $x \in \mathrm{M}$ and a configuration of hyperplanes $(s,\Xi) \in \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{TM}_{D^q})_{(s,x)} = \{s\} \times \mathrm{Gr}_{n-1}^{(\infty)}(\mathrm{T}_x\mathrm{M})$, there is a natural subset $\mathscr{A}(s,\Xi) \subseteq E_x^{(1)}$ given by:

$$\mathscr{A}(s,\Xi) = \{T \in E_x^{(1)} \mid [(s,T),(s,\Xi)] \in \mathscr{A}_{(s,x)}\}.$$

Similarly, given a 1-jet $(s,T)\in\mathscr{R}_{(s,x)}$, there is a natural subset $\mathscr{A}(s,T)\subseteq\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})$ given by:

$$\mathscr{A}(s,T)=\{\Xi\in\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})\mid[(s,T),(s,\Xi)]\in\mathscr{A}_{(s,x)}\}.$$

Definition 2.5.9 ([98, Defn. 4.1])**.** Let M, q and $\mathscr{R}$ be as above. Say $\mathscr{A}\subseteq\mathscr{R}\times_{D^q\times\mathrm{M}}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})$ is a fibred avoidance pre-template for $\mathscr{R}$ if:

1. $\mathscr{A}\subseteq\mathscr{R}\times_{(D^q\times\mathrm{M})}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})$ is an open subset;
2. For all $s\in D^q$, $x\in\mathrm{M}$ and all pairs $\Xi'\subseteq\Xi\in\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})$, there is an inclusion $\mathscr{A}(s,\Xi)\subseteq\mathscr{A}(s,\Xi')$.

Say that $\mathscr{A}$ is a fibred avoidance template for $\mathscr{R}$ if, in addition, it satisfies the following two conditions:

3. For all $s\in D^q$, $x\in\mathrm{M}$ and $(s,T)\in\mathscr{R}_{(s,x)}$, the subset $\mathscr{A}(s,T)\subseteq\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})$ is dense (and open);
4. For all $s\in D^q$, $x\in\mathrm{M}$, $\Xi\in\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})$, $\mathbb{B}\in\Xi$, $\lambda\in\mathrm{Hom}(\mathbb{B},E_x)$ and $e\in E_x$, the subset $\mathscr{A}(s,\Xi)\cap\Pi_e(\mathbb{B},\lambda)\subseteq\Pi_e(\mathbb{B},\lambda)$ is ample.

Again, the following special case will be of particular interest in this thesis. Suppose an avoidance pre-template $\mathscr{A}$ has the form:

$$\mathscr{A}=E_{D^q}\times_{(D^q\times\mathrm{M})}\mathscr{A}'\subseteq E_{D^q}\times_{(D^q\times\mathrm{M})}\left[(\mathrm{T}^*\mathrm{M}\otimes E)_{D^q}\times_{(D^q\times\mathrm{M})}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})\right]$$

for some subbundle $\mathscr{A}'\subseteq(\mathrm{T}^*\mathrm{M}\otimes E)_{D^q}\times_{(D^q\times\mathrm{M})}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})$. In this case, given $s\in D^q$, $x\in\mathrm{M}$ and $\Xi\in\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{T}_x\mathrm{M})$, define

$$\mathscr{A}'(s,\Xi)=\left\{T\in\mathrm{T}^*_x\mathrm{M}\otimes E_x\;\middle|\;[(s,T),(s,\Xi)]\in\mathscr{A}'_{(s,x)}\right\}.$$

Then for all $\mathbb{B}\in\Xi$, $\lambda\in\mathrm{Hom}(\mathbb{B},E_x)$ and $e\in E_x$:

$$\mathscr{A}(s,\Xi)\cap\Pi_e(\mathbb{B},\lambda)=\{e\}\times\left[\mathscr{A}(s,\Xi)'\cap\Pi(\mathbb{B},\lambda)\right]$$

and thus $\mathscr{A}(s,\Xi)\cap\Pi_e(\mathbb{B},\lambda)\subseteq\Pi_e(\mathbb{B},\lambda)$ is ample if and only if $\mathscr{A}'(s,\Xi)\cap\Pi(\mathbb{B},\lambda)\subseteq\Pi(\mathbb{B},\lambda)$ is ample for all $\mathbb{B}$ and λ.

Theorem 2.5.10 ([98, Thm. 5.1])**.** *Let* M *be an n-manifold, let $E\to\mathrm{M}$ be a vector bundle, let $q\geqslant 0$ and let $\mathscr{R}\subseteq E^{(1)}_{D^q}$ be an open fibred differential relation on sections of E. Suppose that $\mathscr{R}$ admits an avoidance template $\mathscr{A}\subseteq\mathscr{R}\times_{(D^q\times\mathrm{M})}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})$. Then $\mathscr{R}$ satisfies the relative h-principle.*

As remarked in [98, Cor. 5.5], Theorem 2.5.10 is a special case of Gromov's general theory of convex integration via convex hull extensions introduced in [62] and developed in [116]. Note also that $\mathscr{A}=\mathscr{R}\times_{(D^q\times\mathrm{M})}\mathrm{Gr}^{(\infty)}_{n-1}(\mathrm{TM}_{D^q})$ is an avoidance template for $\mathscr{R}$ if and only if $\mathscr{R}$ is an ample fibred relation in the classical sense and thus, in this case, Theorem 2.5.10 recovers the classical convex integration theorem as proved in [42, Chs. 17–18].

Part I

The unboundedness of Hitchin volume functionals on G_2- and $\widetilde{\mathrm{G}}_2$-structures

Chapter 3

Large volume limits of the Hitchin functional on G_2 3-forms and associated collapsing results

This chapter uses scaling arguments to prove the unboundedness above of the Hitchin functional $\mathcal{H}_3$ on two examples of closed 7-manifolds with closed G_2-structures. The first is a 4-dimensional family of closed G_2 3-forms on the product $S^1 \times X$ (where X is the Nakamura manifold constructed by de Bartolomeis–Tomassini [14]) inspired by Fernández' short paper [46]. The second is the manifold constructed by Fernández–Fino–Kovalev–Muñoz in [48]. In the latter example, careful resolution of singularities is required, in order to ensure that the rescaled forms are cohomologically constant. By combining suitable geometric estimates with a general collapsing theorem for orbifolds (proved in Chapter 4) explicit descriptions of the large volume limits in both examples are also obtained.

3.1 A general unboundedness result for $\mathcal{H}_3$

I begin with an algebraic lemma:

Lemma 3.1.1. *1. Recall the standard* G_2 *3-form* φ_0 *and write:*

$$\begin{aligned}\varphi_0 &= \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356} \\ &= \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7.\end{aligned}$$

Then for all $\lambda_1, ..., \lambda_7 \in (0, \infty)$*:*

$$\phi_{(\lambda_1,...,\lambda_7)} = \sum_{i=1}^{7} \lambda_i \phi_i$$

is of G_2*-type and:*

$$vol_{\phi_{(\lambda_1,...,\lambda_7)}} = \left(\prod_{i=0}^{7} \lambda_i\right)^{\frac{1}{3}} vol_\phi.$$

2. Let $\mathbb{F}$ *be a 4-dimensional real vector space equipped with a complex structure* J*. Let* ω *be a real, positive* $(1,1)$*-form on* $\mathbb{F}$ *and let* Ω *be a complex* $(2,0)$*-form on* $\mathbb{F}$*. Define a constant* $\nu > 0$ *by the equation:*

$$2\omega^2 = \nu^2 \Omega \wedge \overline{\Omega}. \tag{3.1.2}$$

Then given a 3-dimensional real vector space $\mathbb{G}$ with basis $\left(g^1, g^2, g^3\right)$ of $\mathbb{G}^*$, the 3-form on $\mathbb{F} \oplus \mathbb{G}$ defined by:

$$\phi' = g^{123} + g^1 \wedge \omega - g^2 \wedge \mathfrak{Re}\Omega + g^3 \wedge \mathfrak{Im}\Omega \tag{3.1.3}$$

is of G_2*-type. Moreover:*

$$\begin{aligned} g_{\phi'} &= \nu^{\frac{4}{3}} \left(g^1\right)^{\otimes 2} + \nu^{-\frac{2}{3}} \left[\left(g^2\right)^{\otimes 2} + \left(g^3\right)^{\otimes 2}\right] + \nu^{-\frac{2}{3}} g_\omega \\ vol_{\phi'} &= \frac{\nu^{\frac{2}{3}}}{4} g^{123} \wedge \Omega \wedge \overline{\Omega} \\ \bigstar_{\phi'}\phi' &= \frac{\nu^{\frac{2}{3}}}{4} \Omega \wedge \overline{\Omega} + \nu^{-\frac{4}{3}} g^{23} \wedge \omega + \nu^{\frac{2}{3}} g^{13} \wedge \mathfrak{Re}\Omega + \nu^{\frac{2}{3}} g^{12} \wedge \mathfrak{Im}\Omega \end{aligned} \tag{3.1.4}$$

where g_ω is the metric on $\mathbb{F}$ induced by J and the real, positive $(1,1)$-form ω.

Proof. Begin with (1). Let $\mu_1, ..., \mu_7 \in (0, \infty)$ be chosen later, define $\vartheta^i = \mu_i \theta^i$ for all i and consider the G_2 3-form:

$$\begin{aligned} \varphi(\mu_1, ..., \mu_7) &= \vartheta^{123} + \vartheta^{145} + \vartheta^{167} + \vartheta^{246} - \vartheta^{257} - \vartheta^{347} - \vartheta^{356} \\ &= \mu_{123}\phi_1 + \mu_{145}\phi_2 + \mu_{167}\phi_3 + \mu_{246}\phi_4 + \mu_{257}\phi_5 + \mu_{347}\phi_6 + \mu_{356}\phi_7, \end{aligned}$$

where $\mu_{ijk} = \mu_i \mu_j \mu_k$. Clearly:

$$vol_{\varphi(\mu_1,...,\mu_7)} = \mu_{1234567}\theta^{1234567} = \mu_{1234567} vol_\phi. \tag{3.1.5}$$

I claim that $\varphi(\mu_1, ..., \mu_7) = \phi_{(\lambda_1,...,\lambda_7)}$ for suitable μ_i. Indeed, this equation is equivalent to the system of equations:

$$\begin{gathered} \mu_{123} = \lambda_1 \qquad \mu_{145} = \lambda_2 \qquad \mu_{167} = \lambda_3 \\ \mu_{246} = \lambda_4 \qquad \mu_{257} = \lambda_5 \qquad \mu_{347} = \lambda_6 \\ \mu_{356} = \lambda_7, \end{gathered} \tag{3.1.6}$$

and taking log (which is possible since all μ_i and λ_i are positive) yields the invertible linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \log \mu_1 \\ \log \mu_2 \\ \log \mu_3 \\ \log \mu_4 \\ \log \mu_5 \\ \log \mu_6 \\ \log \mu_7 \end{pmatrix} = \begin{pmatrix} \log \lambda_1 \\ \log \lambda_2 \\ \log \lambda_3 \\ \log \lambda_4 \\ \log \lambda_5 \\ \log \lambda_6 \\ \log \lambda_7 \end{pmatrix}. \tag{3.1.7}$$

Taking the product of the equations in eqn. (3.1.6) yields:

$$\prod_{i=1}^{7} \lambda_i = \left(\prod_{i=1}^{7} \mu_i\right)^3.$$

The formula for $vol_{\phi_{(\lambda_1,\ldots,\lambda_7)}}$ now follows from eqn. (3.1.5).

Now let $\mathbb{F}$, $\mathbb{G}$, J, ω, Ω and ϕ' be as in (2). Since ω is a positive $(1,1)$-form and Ω is a $(2,0)$-form with respect to J, one can choose a basis $\left(f^1, f^2, f^3, f^4\right)$ of $\mathbb{F}^*$ such that $f^1 + if^2$ and $f^3 + if^4$ are $(1,0)$-forms with respect to J and:

$$\omega = f^{12} + f^{34} \quad \text{and} \quad \Omega = \nu^{-1}\left(f^1 + if^2\right) \wedge \left(f^3 + if^4\right), \tag{3.1.8}$$

where ν is defined in eqn. (3.1.2). Consider the (correctly oriented) basis:

$$\left(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^7\right) = \left(f^1, -f^2, -g^1, -g^2, -f^3, -f^4, -g^3\right)$$

of $(\mathbb{F} \oplus \mathbb{G})^*$; then with respect to this basis:

$$\phi' = \theta^{123} + \nu^{-1}\theta^{145} + \nu^{-1}\theta^{167} + \nu^{-1}\theta^{246} - \nu^{-1}\theta^{257} - \theta^{347} - \theta^{356}.$$

This is of G_2-type by (1). The explicit formulae for g_ϕ, vol_ϕ and $\bigstar_\phi \phi$ follow by solving the linear system in eqn. (3.1.7) explicitly to obtain the 'G_2 basis':

$$\left(\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4, \vartheta^5, \vartheta^6, \vartheta^7\right) = \left(\nu^{-\frac{1}{3}}\theta^1, \nu^{-\frac{1}{3}}\theta^2, \nu^{\frac{2}{3}}\theta^3, \nu^{-\frac{1}{3}}\theta^4, \nu^{-\frac{1}{3}}\theta^5, \nu^{-\frac{1}{3}}\theta^6, \nu^{-\frac{1}{3}}\theta^7\right).$$

□

Applying Lemma 3.1.1 to manifolds yields the following unboundedness result for the functional $\mathcal{H}_3$:

Proposition 3.1.9. *1. Let* M *be a closed, parallelisable 7-manifold, let* $\theta^1, \ldots, \theta^7$ *be a basis of 1-forms and let* ϕ *be the* G_2 *3-form:*

$$\begin{aligned} \phi &= \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356} \\ &= \phi_1 \;\; + \phi_2 \;\; + \phi_3 \;\; + \phi_4 \;\; + \phi_5 \;\; + \phi_6 \;\; + \phi_7. \end{aligned} \tag{3.1.10}$$

Suppose that $\mathrm{d}\phi = 0$ *and that there exists* $I \subseteq \{1, \ldots, 7\}$ *such that* $\phi_I = \sum_{i \in I} \phi_i$ *is exact. Then for all* $\lambda \geqslant 0$, $\phi(\lambda) = \phi + \lambda\phi_I$ *is a closed* G_2 *3-form in the same cohomology class as* ϕ *satisfying* $\mathcal{H}_3(\phi(\lambda)) = (1+\lambda)^{\frac{|I|}{3}}\mathcal{H}_3(\phi)$*. In particular:*

$$\sup_{\phi' \in [\phi]_+} \mathcal{H}_3(\phi') = \infty.$$

2. Let M *be a closed, oriented 7-manifold, let* $\mathrm{TM} \cong \mathbb{R}^3 \oplus \mathcal{F}$ *for some rank 4 distribution* $\mathcal{F}$ *on* M *(such a splitting always exists by [39, Table 1]), let* (g^1, g^2, g^3) *be a basis of 1-forms for the trivial bundle* $\left(\mathbb{R}^3\right)^* \subset \left(\mathbb{R}^3 \oplus \mathcal{F}\right)^* \cong \mathrm{T^*M}$*, let* J *be a section of* $\mathrm{End}(\mathcal{F})$ *satisfying* $J^2 = -\mathrm{Id}$*, let* (ω, Ω) *be* $(1,1)$ *and* $(2,0)$*-forms on* $\mathcal{F}$ *with respect to* J *and let* ϕ' *be the* G_2 *3-form:*

$$\phi' = g^{123} + g^1 \wedge \omega - g^2 \wedge \mathfrak{Re}\Omega + g^3 \wedge \mathfrak{Im}\Omega.$$

Suppose that $\mathrm{d}\phi' = 0$ *and that* $g^1 \wedge \omega$ *is exact. Then for all* $\lambda \geqslant 0$, $\phi'(\lambda) = \phi' + \lambda g^1 \wedge \omega$ *is a closed* G_2 *3-form in the same cohomology class as* ϕ' *satisfying* $\mathcal{H}_3(\phi'(\lambda)) = (1+\lambda)^{\frac{2}{3}}\mathcal{H}_3(\phi')$. *In particular:*

$$\sup_{\phi'' \in [\phi']_+} \mathcal{H}_3(\phi'') = \infty.$$

Likewise, if $g^2 \wedge \mathfrak{Re}\Omega - g^3 \wedge \mathfrak{Im}\Omega$ *is exact, then for all* $\lambda > 0$, $\phi''(\lambda) = \phi' - \lambda\left(g^2 \wedge \mathfrak{Re}\Omega - g^3 \wedge \mathfrak{Im}\Omega\right)$ *is a closed* G_2 *3-form in the same cohomology class as* ϕ' *satisfying* $\mathcal{H}_3(\phi''(\lambda)) = (1+\lambda)^{\frac{4}{3}}\mathcal{H}_3(\phi')$, *so that once again:*

$$\sup_{\phi'' \in [\phi']_+} \mathcal{H}_3(\phi'') = \infty.$$

3.2 The unboundedness above of $\mathcal{H}_3$ on $(N, \phi(\alpha, \beta, \lambda))$

I begin by recalling the construction of the Nakamura manifold X from [14]. Define a product $*$ on $\mathbb{C}^3$ via the formula:

$$\left(u^1, u^2, u^3\right) * \left(w^1, w^2, w^3\right) = \left(u^1 + w^1, e^{-w^1}u^2 + w^2, e^{w^1}u^3 + w^3\right).$$

Then $\left(\mathbb{C}^3, *\right)$ is a complex, soluble, non-nilpotent Lie group, which I denote H. Equivalently, one may identify:

$$H = \left\{ \begin{pmatrix} e^{w^1} & 0 & 0 & 0 \\ 0 & e^{-w^1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ w^3 & w^2 & w^1 & 1 \end{pmatrix} \in \mathrm{SL}(4;\mathbb{C}) \;\middle|\; (w^1, w^2, w^3) \in \mathbb{C}^3 \right\}. \tag{3.2.1}$$

The basis (complex) right-invariant 1-forms on H are given by:

$$\Theta^1 = \mathrm{d}w^1, \quad \Theta^2 = e^{w^1}\mathrm{d}w^2 \quad \text{and} \quad \Theta^3 = e^{-w^1}\mathrm{d}w^3. \tag{3.2.2}$$

Let $\ell = \log\frac{3+\sqrt{5}}{2}$, $m = \frac{\sqrt{5}-1}{2}$ and define $\Delta \subset H$ to be the uniform (i.e. discrete and co-compact) subgroup generated by the six elements[1]:

$$\begin{aligned} &h_1 = (\ell, 0, 0), && h_2 = (2\pi i, 0, 0), && h_3 = (0, -m, 1) \\ &h_4 = (0, 1, m), && h_5 = (0, -2\pi i m, 2\pi i), && h_6 = (0, 2\pi i, 2\pi i m). \end{aligned}$$

[1]These formulae differ from those in [14], as the author of this paper has discovered an error *op. cit.*, which has been corrected in the formulae here presented.

The quotient $X = {}^{H}\!/_{\Delta}$ is a compact soluble manifold called a Nakamura manifold (the first such examples being constructed by Nakamura in [109]). Explicitly, write $P = \begin{pmatrix} -m & 1 \\ 1 & m \end{pmatrix}$; then clearly:

$$^{H}\!\Big/\langle h_2, h_3, h_4, h_5, h_6 \rangle \cong \underbrace{^{\mathbb{C}}\!\big/_{2\pi i\mathbb{Z}}}_{\cong S^1\times\mathbb{R}} \times \underbrace{^{\mathbb{C}^2}\!\Big/_{P\left(\mathbb{Z}^2 + 2\pi i\mathbb{Z}^2\right)}}_{=\mathfrak{T}}. \tag{3.2.3}$$

Moreover, from the equation:

$$P\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}P^{-1} = \begin{pmatrix} e^{-\ell} & 0 \\ 0 & e^{\ell} \end{pmatrix}$$

it follows that the linear map $\begin{pmatrix} e^{-\ell} & 0 \\ 0 & e^{\ell} \end{pmatrix}$ on $\mathbb{C}^2$ descends to define a map Λ on the complex torus $\mathfrak{T}$. One can then write:

$$X = S^1 \times {}^{(\mathbb{R}\times\mathfrak{T})}\!\Big/_{\langle T\rangle}$$

where T is the automorphism given by $(w,p) \in \mathbb{R}\times\mathfrak{T} \mapsto (w+\ell, \Lambda(p)) \in \mathbb{R}\times\mathfrak{T}$. The right-invariant 1-forms Θ^i descend to a basis of (complex) 1-forms on X, again denoted Θ^i, which satisfy:

$$\mathrm{d}\Theta^1 = 0, \qquad \mathrm{d}\Theta^2 = \Theta^1\wedge\Theta^2, \qquad \mathrm{d}\Theta^3 = -\Theta^1\wedge\Theta^3. \tag{3.2.4}$$

Write $g^1 = \mathfrak{Re}\Theta^1$ and $g^2 = \mathfrak{Im}\Theta^1$, so that in particular $\mathrm{d}g^1 = \mathrm{d}g^2 = 0$.

Now consider the manifold $N = X \times S^1$. I begin by recording the following result, which is not proved in the literature but which nevertheless appears known to some authors (cf. [22, §6, Example 2]):

Proposition 3.2.5. *The manifold N admits no torsion-free* G_2*-structures.*

Proof. The argument is largely topological in nature. It follows from [14, Thm. 4.1] that $b^1(N) = 3$. Thus by Theorem 2.2.19, if N admitted a torsion-free G_2-structure, then the universal cover of N would be homeomorphic to $\mathbb{R}^3\times\widetilde{N}$ for some simply-connected, closed 4-manifold $\widetilde{N}$. However the universal cover of X is $H \overset{homeo}{\cong} \mathbb{C}^3$ (a fact which holds more generally for any complex soluble manifold [109, p. 86]) and thus the universal cover of N is $\mathbb{R}^7$, not $\mathbb{R}^3\times\widetilde{N}$. Thus no torsion-free G_2-structures on N can exist.

□

N does, however, admit closed G_2 3-forms. Consider the (complex) rank 2 distribution on X given by $\mathcal{F} = \mathrm{Ker}\,\Theta^1$ and define $(1,1)$ and $(2,0)$-forms on $\mathcal{F}$ by:

$$\omega = \frac{i}{2}\left[\Theta^2\wedge\overline{\Theta}^2 + \Theta^3\wedge\overline{\Theta}^3\right], \qquad \rho = \frac{i}{2}\left[\Theta^2\wedge\overline{\Theta}^2 - \Theta^3\wedge\overline{\Theta}^3\right] \quad \text{and} \quad \Omega = \Theta^2\wedge\Theta^3.$$

Then (ω,Ω) satisfies:

$$\mathrm{d}\omega = 2g^1\wedge\rho \quad \text{and} \quad \mathrm{d}\Omega = 0. \tag{3.2.6}$$

Write g^3 for the canonical 1-form on S^1. For each $\alpha \in \mathbb{R}\backslash\{0\}$, $\beta \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}\backslash\{0\}$ define a 3-form on N by:

$$\phi(\alpha,\beta,\lambda) = \alpha\beta g^{123} + \alpha g^1 \wedge \omega - \beta g^2 \wedge \mathfrak{Re}\lambda\Omega + g^3 \wedge \mathfrak{Im}\lambda\Omega. \tag{3.2.7}$$

$\phi(\alpha,\beta,\lambda)$ defines a G_2-structure on N, by Lemma 3.1.1 (applied to the forms $\alpha g^1, \beta g^2, g^3, \omega, \lambda\Omega$). Moreover $\mathrm{d}\phi(\alpha,\beta,\lambda) = 0$, by eqn. (3.2.6).

Remark 3.2.8. The construction of $\phi(\alpha,\beta,\lambda)$ above was inspired by the G_2 3-forms defined by Fernández' [46]. Indeed, in the special case that $\lambda = 1$ and $\alpha = \beta$ lies in the discrete set $\left\{\kappa \in \mathbb{R} \;\middle|\; e^{\frac{1}{\kappa}} + e^{-\frac{1}{\kappa}} \in \mathbb{Z}\backslash\{2\}\right\}$, the forms $\phi(\alpha,\beta,\lambda)$ are closely related to Fernandez' definition. The key differences are firstly that, for H as defined above, Fernández considers a left-quotient of H and thus constructs left-invariant G_2 3-forms rather than right-invariant G_2 3-forms, and secondly that Fernández reverses the roles of g^2 and the canonical 1-form on S^1; this arises since [46] uses the opposite convention for the orientation of G_2-structures to the one used in this paper; see [82, §2.1] for a discussion of the two conventions. Also, Fernández' treatment in [46] focuses almost exclusively on the manifold N from the perspective of real differential geometry, and thus does not notice the natural $\mathrm{SU}(2)$-structure (with torsion) underlying the construction *op. cit.*.

Theorem 3.2.9. *The map:*

$$\begin{aligned} (\mathbb{R}\backslash\{0\})^2 \times (\mathbb{C}\backslash\{0\}) &\longrightarrow H^3_{\mathrm{dR}}(N) \\ (\alpha,\beta,\lambda) &\longmapsto [\phi(\alpha,\beta,\lambda)] \end{aligned}$$

is injective, and for all $(\alpha,\beta,\lambda) \in (\mathbb{R}\backslash\{0\})^2 \times (\mathbb{C}\backslash\{0\})$, the functional:

$$\mathcal{H}_3 : [\phi(\alpha,\beta,\lambda)]_+ \to (0,\infty)$$

is unbounded above.

Proof. Since each of $(\mathfrak{Re}\Omega)^2$, $g^{13} \wedge \mathfrak{Re}\Omega$, $-g^{13} \wedge \mathfrak{Im}\Omega$, $g^{12} \wedge \mathfrak{Re}\Omega$ and $g^{12} \wedge \mathfrak{Im}\Omega$ are closed, there is a map:

$$\begin{aligned} H^3_{\mathrm{dR}}(N) &\xrightarrow{\quad\chi\quad} \mathbb{R}^5 \\ [\xi] &\longmapsto \begin{pmatrix} \int_N \xi \wedge (\mathfrak{Re}\Omega)^2 \\ \int_N \xi \wedge g^{13} \wedge \mathfrak{Re}\Omega \\ \int_N \xi \wedge -g^{13} \wedge \mathfrak{Im}\Omega \\ \int_N \xi \wedge g^{12} \wedge \mathfrak{Im}\Omega \\ \int_N \xi \wedge g^{12} \wedge \mathfrak{Re}\Omega \end{pmatrix} \end{aligned}$$

A direct calculation shows that, writing $A = \int_N g^{123} \wedge (\mathfrak{Re}\Omega)^2 = \int_N g^{123} \wedge (\mathfrak{Im}\Omega)^2 > 0$, one has:

$$\chi\left([\phi(\alpha,\beta,\lambda)]\right) = A \cdot \begin{pmatrix} \alpha\beta \\ \mathfrak{Re}\lambda \\ \mathfrak{Im}\lambda \\ \beta\mathfrak{Re}\lambda \\ \beta\mathfrak{Im}\lambda \end{pmatrix}.$$

Thus the composite $(\mathbb{R}\backslash\{0\})^2 \times (\mathbb{C}\backslash\{0\}) \xrightarrow{\phi(-,-,-)} H^3_{\mathrm{dR}}(N) \xrightarrow{\chi} \mathbb{R}^5$ is injective, and hence so too is $(\mathbb{R}\backslash\{0\})^2 \times (\mathbb{C}\backslash\{0\}) \xrightarrow{\phi(-,-,-)} H^3_{\mathrm{dR}}(N)$.

Finally, note that $g^1 \wedge \omega = \frac{1}{2}\mathrm{d}\rho$ is exact. Thus the unboundedness of $\mathcal{H}_3$ on the classes $[\phi(\alpha,\beta,\lambda)]_+$ follows immediately from Proposition 3.1.9(2); in particular, writing:

$$\phi(\alpha,\beta,\lambda;\mu) = \alpha\beta g^{123} + \alpha\mu^6 g^1 \wedge \omega - \beta g^2 \wedge \mathfrak{Re}\lambda\Omega + g^3 \wedge \mathfrak{Im}\lambda\Omega$$

for $\mu \geqslant 1$, Proposition 3.1.9(2) shows that:

$$\mathcal{H}_3\left(\phi(\alpha,\beta,\lambda;\mu)\right) = \mu^4 \mathcal{H}_3\left(\phi(\alpha,\beta,\lambda)\right) \to \infty \quad \text{as} \quad \mu \to \infty,$$

completing the proof.

□

3.3 The large volume limit of $(N, \phi(\alpha,\beta,\lambda;\mu))$

The aim of this section is to describe the geometry of $(N, \phi(\alpha,\beta,\lambda;\mu))$ as $\mu \to \infty$. Recall the group H defined in eqn. (3.2.1) and the uniform subgroup Δ. The subgroup $K \subset H$ corresponding to $w^1 = 0$ is a connected, normal, Abelian Lie subgroup of H, which is maximal nilpotent since H is non-nilpotent and $\mathrm{codim}(K,H) = 1$. By Mostow's Theorem [109, p. 87], there is a fibration:

$$\mathfrak{f} : X = {}^{H}\!/\!_{\Delta} \to \left({}^{H}\!/\!_{K}\right)\Big/\left(\Delta \cdot {}^{K}\!/\!_{K}\right)$$

with fibre ${}^{K}\!/\!_{\Delta \cap K}$. Explicitly, recall that $X \cong S^1 \times {}^{(\mathbb{R} \times \mathfrak{T})}\!/\!_{\langle T \rangle}$, where $\mathfrak{T}$ is the 4-torus defined in eqn. (3.2.3) and T is the automorphism given by $(w,p) \in \mathbb{R} \times \mathfrak{T} \mapsto (w + \ell, \Lambda(p)) \in \mathbb{R} \times \mathfrak{T}$. Then $\mathfrak{f}$ is simply the natural projection:

$$S^1 \times {}^{(\mathbb{R} \times \mathfrak{T})}\!/\!_{\langle T \rangle} \xrightarrow{proj} S^1 \times {}^{\mathbb{R}}\!/\!_{\ell\mathbb{Z}},$$

with fibre $\mathfrak{T}$. Using $\mathfrak{f}$, define a fibration $\mathfrak{p} : N \to {}^{\mathbb{R}}\!/\!_{\ell\mathbb{Z}}$ via:

$$\mathfrak{p} : N = S^1 \times X \xrightarrow{proj_2} X \xrightarrow{\mathfrak{f}} S^1 \times {}^{\mathbb{R}}\!/\!_{\ell\mathbb{Z}} \xrightarrow{proj_2} {}^{\mathbb{R}}\!/\!_{\ell\mathbb{Z}}.$$

Theorem 3.3.1. *Let $(\alpha, \beta, \lambda) \in (\mathbb{R}\backslash\{0\})^2 \times (\mathbb{C}\backslash\{0\})$ and let $(N, \phi(\alpha, \beta, \lambda; \mu))_{\mu \in [1,\infty)}$ be the family constructed in the proof of Theorem 3.2.9. Then the large volume limit of $(N, \phi(\alpha, \beta, \lambda; \mu))$ corresponds to an adiabatic limit of the fibration $\mathfrak{p}$. Specifically:*

$$(N, \mu^{-12}\phi(\alpha, \beta, \lambda; \mu)) \to \left(\mathbb{R}/_{\ell\mathbb{Z}}, \alpha^2 \left(\lambda\overline{\lambda}\right)^{-\frac{2}{3}} g_{\mathrm{Eucl}}\right) \quad \textit{as} \quad \mu \to \infty,$$

where the convergence is in the Gromov–Hausdorff sense.

The proof uses the following convergence result:

Theorem 3.3.2. *Let E and B be closed manifolds, let $\pi : E \to B$ be a submersion, let g^μ be a family of Riemannian metrics on E and let g be a Riemannian metric on B. If $g^\mu \to \pi^* g$ uniformly and there exist constants $\Lambda_\mu \geqslant 0$ such that:*

$$\lim_{\mu \to \infty} \Lambda_\mu = 1 \quad \textit{and} \quad g^\mu \geqslant \Lambda_\mu^2 \pi^* g \textit{ for all } \mu \in [1, \infty), \tag{3.3.3}$$

then (E, g^μ) converges to (B, g) in the Gromov–Hausdorff sense as $\mu \to \infty$.

Since Theorem 3.3.2 is a result in metric geometry, rather than G_2 geometry, the proof is postponed to Chapter 4 of this thesis (see Theorem 4.2.5), so as not to detract from the main thrust of the current chapter. Using Theorem 3.3.2, I now prove Theorem 3.3.1:

Proof. By Proposition 3.1.1 applied to the forms $\alpha g^1, \beta g^2, g^3, \mu^6\omega, \lambda\Omega$ (so that $\nu = \frac{\mu^6}{\sqrt{\lambda\overline{\lambda}}}$) one may compute that:

$$g_{\phi(\alpha,\beta,\lambda;\mu)} = \frac{\mu^8\alpha^2}{\left(\lambda\overline{\lambda}\right)^{\frac{2}{3}}} \left(g^1\right)^{\otimes 2} + \mu^2 \left(\lambda\overline{\lambda}\right)^{\frac{1}{3}} g_\omega + \frac{\left(\lambda\overline{\lambda}\right)^{\frac{1}{3}}}{\mu^4} \left[\beta^2 \left(g^2\right)^{\otimes 2} + \left(g^3\right)^{\otimes 2}\right].$$

Rescaling the G_2 3-forms $\phi(\alpha, \beta, \lambda; \mu) \mapsto \mu^{-12}\phi(\alpha, \beta, \lambda; \mu)$, one finds that:

$$\begin{aligned} g_{\mu^{-12}\phi(\alpha,\beta,\lambda;\mu)} &= \frac{\alpha^2}{\left(\lambda\overline{\lambda}\right)^{\frac{2}{3}}} \left(g^1\right)^{\otimes 2} + \frac{\left(\lambda\overline{\lambda}\right)^{\frac{1}{3}}}{\mu^6} g_\omega + \frac{\left(\lambda\overline{\lambda}\right)^{\frac{1}{3}}}{\mu^{12}} \left[\beta^2 \left(g^2\right)^{\otimes 2} + \left(g^3\right)^{\otimes 2}\right] \\ &\to \alpha^2 \left(\lambda\overline{\lambda}\right)^{-\frac{2}{3}} \left(g^1\right)^{\otimes 2} = \mathfrak{p}^* \left[\alpha^2 \left(\lambda\overline{\lambda}\right)^{-\frac{2}{3}} g_{\mathrm{Eucl}}\right] \text{ uniformly as } \mu \to \infty, \end{aligned} \tag{3.3.4}$$

where g_{Eucl} denotes the Euclidean metric on $\mathbb{R}/_{\ell\mathbb{Z}}$. Moreover:

$$g_{\mu^{-12}\phi(\alpha,\beta,\lambda;\mu)} \geqslant \mathfrak{p}^* \left[\alpha^2 \left(\lambda\overline{\lambda}\right)^{-\frac{2}{3}} g_{\mathrm{Eucl}}\right] \quad \text{for all } \mu. \tag{3.3.5}$$

The result now follows from Theorem 3.3.2.

$\square$

3.4 The unboundedness above of $\mathcal{H}_3$ on $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$

3.4.1 The construction of $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$

For full details of the construction, see [48]. Let $G = \left\{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathrm{SL}(12;\mathbb{R})\right\}$, where, for $(x^1,..,x^7) \in \mathbb{R}^7$:

$$A_1 = \begin{pmatrix} 1 & -x^2 & x^1 & x^4 & -x^1x^2 & x^6 \\ 0 & 1 & 0 & -x^1 & x^1 & \frac{1}{2}(x^1)^2 \\ 0 & 0 & 1 & 0 & -x^2 & -x^4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & -x^3 & x^1 & x^5 & -x^1x^3 & x^7 \\ 0 & 1 & 0 & -x^1 & x^1 & \frac{1}{2}(x^1)^2 \\ 0 & 0 & 1 & 0 & -x^3 & -x^5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.4.1}$$

Write $\Gamma \subset G$ for the discrete subgroup corresponding to $(x^1,...,x^7) \in 2\mathbb{Z} \times \mathbb{Z}^6$ and define $\mathrm{M} = {}_{\Gamma}\backslash^{G}$, a closed nilmanifold. G admits a basis of left-invariant 1-forms given by:

$$\begin{gathered} \theta^1 = \mathrm{d}x^1, \quad \theta^2 = \mathrm{d}x^2, \quad \theta^3 = \mathrm{d}x^3, \quad \theta^4 = \mathrm{d}x^4 - x^2\mathrm{d}x^1 \\ \theta^5 = dx^5 - x^3\mathrm{d}x^1, \quad \theta^6 = \mathrm{d}x^6 + x^1\mathrm{d}x^4, \quad \theta^7 = \mathrm{d}x^7 + x^1\mathrm{d}x^5 \end{gathered} \tag{3.4.2}$$

which descend to define a basis of 1-forms on M (also denoted θ^i) satisfying:

$$\mathrm{d}\theta^i = 0 \ (i = 1,2,3), \quad \mathrm{d}\theta^4 = \theta^{12}, \quad \mathrm{d}\theta^5 = \theta^{13}, \quad \mathrm{d}\theta^6 = \theta^{14} \quad \text{and} \quad \mathrm{d}\theta^7 = \theta^{15}. \tag{3.4.3}$$

Define a closed G_2 3-form on M by:

$$\varphi = \theta^{123} + \theta^{145} + \theta^{167} - \theta^{246} + \theta^{257} + \theta^{347} + \theta^{356}. \tag{3.4.4}$$

Then M admits a (non-free) involution $\mathcal{I}$ given by:

$$\mathcal{I} : \Gamma \cdot (x^1, x^2, x^3, x^4, x^5, x^6, x^7) \mapsto \Gamma \cdot (-x^1, -x^2, x^3, x^4, -x^5, -x^6, x^7) \tag{3.4.5}$$

which preserves φ and hence φ descends to define a closed (orbifold) G_2 3-form $\widehat{\varphi}$ on $\widehat{\mathrm{M}} = {}_{\mathcal{I}}\backslash^{\mathrm{M}}$.

Let $\widehat{S}$ denote the singular locus of $\widehat{\mathrm{M}}$ and write S for the preimage of $\widehat{S}$ under the natural projection $\mathrm{M} \to \widehat{\mathrm{M}}$. By eqn. (3.4.5) (see §5 of the arXiv version of [48])[2] $S = \coprod_{\mathbf{a}\in\mathfrak{A}} S_{\mathbf{a}}$ where $\mathbf{a} = \left(a^1, a^2, a^5, a^6\right) \in \mathfrak{A} = \{0,1\} \times \left\{0, \frac{1}{2}\right\}^3$ and:

$$S_{\mathbf{a}} = \begin{cases} \left\{\Gamma \cdot \left(0, a^2, x^3, x^4, a^5, a^6, x^7\right) \;\middle|\; x^3, x^4, x^7 \in \mathbb{R}\right\} & \text{if } a^1 = 0 \\ \left\{\Gamma \cdot \left(1, a^2, x^3, x^4, a^5, \frac{3}{2}a^2 + a^6 - x^4, x^7\right) \;\middle|\; x^3, x^4, x^7 \in \mathbb{R}\right\} & \text{if } a^1 = 1. \end{cases}$$

[2] My investigation of the manifold $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$ revealed some errors in the journal version of [48], which have been communicated to the authors of [48] and since corrected in the arXiv version of the article.

Similarly, write $\widehat{S} = \coprod_{\mathbf{a}\in\mathfrak{A}} \widehat{S}_{\mathbf{a}}$ where each $\widehat{S}_{\mathbf{a}}$ is the image of $S_{\mathbf{a}}$ under the projection $\mathrm{M} \to \widehat{\mathrm{M}}$. The map:

$$\begin{gathered}\Phi_{\mathbf{0}} : \mathbb{T}^3 \times \mathrm{B}^4_\varepsilon \to \mathrm{M} \\ \left[\left(y^3, y^4, y^7\right) + \mathbb{Z}^3, \left(y^1, y^2, y^5, y^6\right)\right] \mapsto \Gamma \cdot \left(y^1, y^2, y^3, y^4, y^5 + y^1 y^3, y^6, y^7 - \tfrac{1}{2}\left(y^1\right)^2 y^3\right)\end{gathered} \tag{3.4.6}$$

defines an embedding onto an open neighbourhood of $S_{\mathbf{0}}$ identifying $\mathbb{T}^3$ with $S_{\mathbf{0}}$ and $\mathcal{I}$ with $\mathrm{Id}_{\mathbb{T}^3} \times -\mathrm{Id}_{\mathrm{B}^4_\varepsilon}$ for $\varepsilon > 0$ sufficiently small. Similarly, for each $\mathbf{a} = \left(0, a^2, a^5, a^6\right)$, there is an embedding $\Phi_{\mathbf{a}} : \mathbb{T}^3 \times \mathrm{B}^4_\varepsilon \to \mathrm{M}$ given by $\Phi_{\mathbf{a}} = f_{\mathbf{a}} \circ \Phi_{\mathbf{0}}$, where $f_{\mathbf{a}}$ is a diffeomorphism of M induced by a left-translation of G, commuting with $\mathcal{I}$ and mapping $S_{\mathbf{0}}$ to $S_{\mathbf{a}}$. For the other components of the singular locus, define a lattice $\Lambda = \mathbb{Z} \cdot \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \mathbb{Z} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{Z} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \subset \mathbb{R}^3$ and write $\widetilde{\mathbb{T}^3} = \mathbb{R}^3 \big/ \Lambda$. Writing $\mathbf{1} = (1, 0, 0, 0)$, the map:

$$\begin{gathered}\Phi_{\mathbf{1}} : \widetilde{\mathbb{T}^3} \times \mathrm{B}^4_\varepsilon \to \mathrm{M} \\ \left[\left(y^3, y^4, y^7\right) + \Lambda, \left(y^1, y^2, y^5, y^6\right)\right] \mapsto \Gamma \cdot \left(y^1 + 1, y^2, y^3, y^4, y^5 + y^1 y^3, y^6 - y^4, y^7 - y^5 - y^1 y^3 - \tfrac{1}{2}\left(y^1\right)^2 y^3\right).\end{gathered} \tag{3.4.7}$$

is an embedding onto an open neighbourhood of $S_{\mathbf{1}}$ for $\varepsilon > 0$ sufficiently small identifying $\widetilde{\mathbb{T}^3}$ with $S_{\mathbf{1}}$ and $\mathcal{I}$ with $\mathrm{Id}_{\widetilde{\mathbb{T}^3}} \times -\mathrm{Id}_{\mathrm{B}^4_\varepsilon}$. Similarly, for each $\mathbf{a} = \left(1, a^2, a^5, a^6\right)$, there is an embedding $\Phi_{\mathbf{a}} : \widetilde{\mathbb{T}^3} \times \mathrm{B}^4_\varepsilon \to \mathrm{M}$ given by $\Phi_{\mathbf{a}} = g_{\mathbf{a}} \circ \Phi_{\mathbf{1}}$, where $g_{\mathbf{a}}$ is a diffeomorphism of M induced by a left-translation of G, commuting with $\mathcal{I}$ and mapping $S_{\mathbf{1}}$ to $S_{\mathbf{a}}$.

Let T denote either $\mathbb{T}^3$ or $\widetilde{\mathbb{T}^3}$ as appropriate and write $\widehat{\Phi}_{\mathbf{a}}$ for the composite $T \times \mathrm{B}^4_\varepsilon \xrightarrow{\Phi_{\mathbf{a}}} \mathrm{M} \to \widehat{\mathrm{M}}$. For each $\mathbf{a} = \left(a^1, a^2, a^5, a^6\right) \in \mathfrak{A}$, define $U_{\mathbf{a}} = \Phi_{\mathbf{a}}\left(T \times \mathrm{B}^4_\varepsilon\right)$ and $\widehat{U}_{\mathbf{a}} = \widehat{\Phi}_{\mathbf{a}}\left(T \times {\{\pm1\}}\backslash{\mathrm{B}^4_\varepsilon}\right)$; for sufficiently small $\varepsilon > 0$ the $U_{\mathbf{a}}$ are disjoint. Then shrinking $\varepsilon > 0$ still further if necessary, there exists a closed, orbifold G_2 3-form $\widehat{\phi}$ on $\widehat{\mathrm{M}}$ such that $\widehat{\phi} = \widehat{\varphi}$ on $\widehat{\mathrm{M}} \backslash \coprod_{\mathbf{a}\in\mathfrak{A}} \widehat{U}_{\mathbf{a}}$ and on each $\widehat{W}_{\mathbf{a}} = \widehat{\Phi}_{\mathbf{a}}\left(T \times {\{\pm1\}}\backslash{\mathrm{B}^4_{\varepsilon/2}}\right)$ one has:

$$\widehat{\phi} = \mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356},$$

where the y^i are defined in eqns. (3.4.6) and (3.4.7). Identify ${\{\pm1\}}\backslash{\mathrm{B}^4_{\varepsilon/2}} \subset {\{\pm1\}}\backslash{\mathbb{C}^2}$ by writing $w^1 = y^1 + iy^2$ and $w^2 = y^5 + iy^6$ and define:

$$\widehat{\omega} = \frac{i}{2}\left(\mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^1 + \mathrm{d}w^2 \wedge \mathrm{d}\overline{w}^2\right) \quad \text{and} \quad \widehat{\Omega} = \mathrm{d}w^1 \wedge \mathrm{d}w^2.$$

Then on $\widehat{W}_{\mathbf{a}}$, one has:

$$\widehat{\phi} = \mathrm{d}y^{347} + \mathrm{d}y^3 \wedge \omega - \mathrm{d}y^4 \mathfrak{Re}\Omega + \mathrm{d}y^7 \mathfrak{Im}\Omega.$$

Now recall the space [97, §2]:

$$\widetilde{X} = \mathcal{O}_{\mathbb{CP}^1}(-2) = \left\{\left(\left(U^1, U^2\right), \left[W^1 : W^2\right]\right) \in \mathbb{C}^2 \times \mathbb{CP}^1 \;\middle|\; U^1\left(W^2\right)^2 = U^2\left(W^1\right)^2\right\} \cong \mathrm{T}^*\mathbb{CP}^1.$$

together with the continuous (non-smooth) blow-up map $\rho : \mathcal{O}_{\mathbb{CP}^1}(-2) \to {}^{\mathbb{C}^2}\big/_{\{\pm1\}}$ given by:

$$\left(\left(U^1, U^2\right), \left[W^1 : W^2\right]\right) \mapsto \pm\left(\sqrt{U^1}, \sqrt{U^2}\right), \tag{3.4.8}$$

where the square-roots on the right-hand side are constrained by the condition $\sqrt{U^1}W^2 = \sqrt{U^2}W^1$, and write $\mathfrak{E} = \rho^{-1}(\{0\})$ for the exceptional divisor. Using the map ρ, identify the spaces $\widetilde{X}\backslash\mathfrak{E}$ and $\left({}^{\mathbb{C}^2}\big/_{\{\pm1\}}\right)\backslash\{0\}$. It can be shown that the form $\widehat{\Omega}$ on $\widetilde{X}\backslash\mathfrak{E}$ extends over all of $\widetilde{X}$ to define a smooth, closed, $(2,0)$-form $\widetilde{\Omega}$, however the form $\widehat{\omega}$ on $\widetilde{X}\backslash\mathfrak{E}$ cannot be extended over $\mathfrak{E}$. Instead, one considers the so-called Eguchi–Hanson metrics $\widetilde{\omega}_t$ on $\widetilde{X}$ defined as follows: let $r^2 = \left|w^1\right|^2 + \left|w^2\right|^2$ denote the distance squared from the origin in ${}^{\mathbb{C}^2}\big/_{\{\pm1\}}$ and define:

$$\widetilde{\omega}_t = \frac{1}{4}\mathrm{dd}^c\left[\sqrt{r^4+t^4} + t^2\log\left(\frac{r^2}{\sqrt{r^4+t^4}+t^2}\right)\right] \text{ on } \left({}^{\mathbb{C}^2}\big/_{\{\pm1\}}\right)\backslash\{0\}. \tag{3.4.9}$$

Then $\widetilde{\omega}_t$ can be extended smoothly over $\mathfrak{E}$ to define a Ricci-flat Kähler form on $\widetilde{X}$ [78, p. 60]. The forms $\widetilde{\omega}_t$ can be used to 'extend' $\widehat{\omega}$ over the exceptional divisor in the following sense: for $\varepsilon > 0$, write $\widetilde{X}(\varepsilon)$ for the pre-image of ${}^{\mathrm{B}^4_\varepsilon}\big/_{\{\pm1\}}$ under the map $\rho : \widetilde{X} \to {}^{\mathbb{C}^2}\big/_{\{\pm1\}}$. Then for every $\varepsilon > 0$, there exists t sufficiently small (depending on ε) and a Kähler form $\breve{\omega}_t$ on $\widetilde{X}$ such that:

$$\breve{\omega}_t = \widehat{\omega} \text{ on } \widetilde{X}\Big\backslash\widetilde{X}\left(\frac{1}{2}\varepsilon\right) \quad \text{ and } \quad \breve{\omega}_t = \widetilde{\omega}_t \text{ on } \widetilde{X}\left(\frac{1}{4}\varepsilon\right).$$

Now define a new manifold $\breve{\mathrm{M}}$ by:

$$\breve{\mathrm{M}} = \left(\widehat{\mathrm{M}}\Big\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\widehat{W}_{\mathbf{a}}\right)\underset{\sim}{\bigcup}\left(\coprod_{\mathbf{a}\in\mathfrak{A}} T\times\widetilde{X}(\varepsilon)\right)$$

where $\underset{\sim}{\bigcup}$ denotes that for each $\mathbf{a}\in\mathfrak{A}$, the region $\widehat{U}_{\mathbf{a}}\backslash\widehat{W}_{\mathbf{a}}$ should be identified with the region $T\times\widetilde{X}(\varepsilon)\backslash\widetilde{X}\left(\frac{1}{2}\varepsilon\right) \cong T\times\left({}^{\mathrm{B}^4_\varepsilon}\big/_{\{\pm1\}}\right)\Big\backslash\left({}^{\mathrm{B}^4_{\frac{1}{2}\varepsilon}}\big/_{\{\pm1\}}\right)$ using $\widehat{\Phi}_{\mathbf{a}}$. Denote the image of $T\times\widetilde{X}(\varepsilon)$ in $\breve{\mathrm{M}}$ corresponding to $\mathbf{a}\in\mathfrak{A}$ by $\breve{U}_{\mathbf{a}}$ and the image of $T\times\widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ by $\breve{W}_{\mathbf{a}}$. Define a three-form $\breve{\phi}$ on $\breve{\mathrm{M}}$ by setting $\breve{\phi} = \widehat{\phi}$ on $\left(\widehat{\mathrm{M}}\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\widehat{W}_{\mathbf{a}}\right)$ and setting:

$$\breve{\phi} = \mathrm{d}y^{347} + \mathrm{d}y^3\wedge\breve{\omega}_t - \mathrm{d}y^4\wedge\mathfrak{Re}\widetilde{\Omega} + \mathrm{d}y^7\wedge\mathfrak{Im}\widetilde{\Omega}$$

on the region $\breve{W}_{\mathbf{a}}$ for each $\mathbf{a}$. This yields:

Theorem 3.4.10 ([48, Thm. 21])**.** *Let $\rho : \breve{\mathrm{M}} \to \widehat{\mathrm{M}}$ denote the 'blow-down' map. Then there exists a smooth, closed* G_2 *3-form $\breve{\phi}$ on $\breve{\mathrm{M}}$ such that $\rho_*\breve{\phi} = \widehat{\phi}$ outside of a neighbourhood of the singular locus S.*

Remark 3.4.11. It is well-known that $\widetilde{\omega}_t$ defines a non-zero cohomology class on $\widetilde{X}$ which depends on t. Using [48, Prop. 22], it follows that the cohomology class of $\breve{\phi}$ depends on the choice of t and hence ε. To prove the unboundedness of $\mathcal{H}_3$ on $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$, I construct a family of closed G_2 3-forms $\breve{\phi}^\mu$

with unbounded volume in the fixed cohomology class $\left[\breve{\phi}\right]$; thus they must all have the same 'choice of ε'. This is an important technical subtlety in the construction of the forms $\breve{\phi}^\mu$.

3.4.2 The unboundedness of $\mathcal{H}_3$

Theorem 3.4.12. *The Hitchin functional:*

$$\mathcal{H}_3 : \left[\breve{\phi}\right]_+ \to (0, \infty)$$

is unbounded above.

Whilst the manifold $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$ does not satisfy the hypotheses of Proposition 3.1.9, the manifold (M, φ) does satisfies the hypotheses of Proposition 3.1.9(1), since, by eqn. (3.4.3), the 3-form $\theta^{123} = \mathrm{d}\left(\theta^{25}\right)$ is exact. Thus, by Proposition 3.1.9(1), for each $\mu \geqslant 1$, the 3-form:

$$\varphi^\mu = \mu^6\theta^{123} + \theta^{145} + \theta^{167} - \theta^{246} + \theta^{257} + \theta^{347} + \theta^{356}$$

is of G_2-type and satisfies $vol_{\varphi^\mu} = \mu^2 vol_\varphi$. By eqn. (3.4.5), both the 3-form θ^{123} and the 2-form θ^{25} are $\mathcal{I}$-invariant and thus descend to the orbifold $\widehat{\mathrm{M}}$. Hence the forms φ^μ descend to define closed G_2 3-forms $\widehat{\varphi}^\mu$ on $\widehat{\mathrm{M}}$ with unbounded volume, which lie in the fixed cohomology class $[\widehat{\varphi}]$.

To complete the proof of Theorem 3.4.12 therefore, it suffices to 'resolve the singularities' of $\left(\widehat{\mathrm{M}}, \widehat{\varphi}^\mu\right)$. The obvious approach is to mimic the construction of $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$, by first deforming $\widehat{\varphi}^\mu$ into the form:

$$\widehat{\xi}^\mu = \mu^6\mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356} \tag{3.4.13}$$

in a neighbourhood of the singular locus, and then resolving the singularity in $\widehat{\xi}$ using $\breve{\omega}_t$ as above. However this approach fails: in order to deform $\widehat{\varphi}^\mu$ into $\widehat{\xi}^\mu$ on the region $\widehat{U}_{\mathbf{a}} \cong T \times {}_{\{\pm 1\}}\backslash \mathrm{B}^4_\varepsilon$, it is necessary for ε to depend on μ. This implies that the cohomology class of the resolved 3-form $\breve{\phi}^\mu$ also depends on μ (see Remark 3.4.11) and thus this construction fails to demonstrate the unboundedness of the Hitchin functional $\mathcal{H}_3$ on the fixed cohomology class $\left[\breve{\phi}\right]$. Thus instead, I deform $\widehat{\varphi}^\mu$ into the form:

$$\widehat{\xi}^\mu + y^1\mathrm{d}y^{147} = \mu^6\mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356} + y^1\mathrm{d}y^{147} \tag{3.4.14}$$

near the singular locus. This deformation can be performed on $\widehat{U}_{\mathbf{a}} \cong T \times {}_{\{\pm 1\}}\backslash \mathrm{B}^4_\varepsilon$ with ε chosen independently of μ. The additional term $y^1\mathrm{d}y^{147}$ persists during the resolution of singularities, before being cut-off near the exceptional divisor, at some distance from the exceptional divisor depending on μ. This enables the resolved 3-forms $\breve{\phi}^\mu$ to lie in a fixed cohomology class, completing the proof of Theorem 3.4.12.

Remark 3.4.15. The reader will recall that Joyce [76, 77, 78] constructed numerous G_2-manifolds by resolving the singularities in finite quotients of the torus $\left(\mathbb{T}^7, \phi_0\right)$. Despite the similarities between Joyce's construction and the construction of $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$, the results of this chapter do not apply to Joyce's manifolds since, unlike (M, φ), the torus $\left(\mathbb{T}^7, \phi_0\right)$ itself does not satisfy the hypotheses of Proposition

3.1.9. Thus, the question of whether $\mathcal{H}_3$ is unbounded above on manifolds admitting torsion-free G_2 3-forms appears to remain beyond our current understanding.

I begin with the following lemma:

Lemma 3.4.16. *Let $\mathbf{a} \in \mathfrak{A}$, let $r \geqslant 0$ denote the radial distance from the singular locus in $\widehat{U}_{\mathbf{a}}$, i.e.:*

$$r^2 = \left(y^1\right)^2 + \left(y^2\right)^2 + \left(y^5\right)^2 + \left(y^6\right)^2,$$

where the y^i are defined in eqns. (3.4.6) and (3.4.7), and define:

$$\widehat{\xi}^\mu = \mu^6 \mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356}.$$

Then there exist a constant $C > 0$ and a 2-form $\widehat{\alpha}_{\mathbf{a}}$ on $\widehat{U}_{\mathbf{a}}$, both independent of μ, satisfying:

$$\left|\widehat{\alpha}_{\mathbf{a}}\right|_{\widehat{\xi}^\mu} \leqslant C\mu^{-1}r^2 \quad \text{and} \quad \left|\mathrm{d}\widehat{\alpha}_{\mathbf{a}}\right|_{\widehat{\xi}^\mu} \leqslant Cr \tag{3.4.17}$$

such that:

$$\widehat{\varphi}^\mu - \widehat{\xi}^\mu = y^1 \mathrm{d}y^{147} + \mathrm{d}\widehat{\alpha}_{\mathbf{a}}$$

(Here $|\cdot|_{\widehat{\xi}^\mu}$ denotes the pointwise norm induced by the G_2 3-form $\widehat{\xi}^\mu$. E.g. in the case $\mu = 1$, this is just the Euclidean norm in the y^i coordinates, denoted $|\cdot|_{\mathrm{Eucl}}$.)

Proof. Begin by working on $U_{\mathbf{a}}$. Using the equation:

$$\Phi_{\mathbf{a}} = \begin{cases} f_{\mathbf{a}} \circ \Phi_{\mathbf{0}} & \text{if } a^1 = 0; \\ g_{\mathbf{a}} \circ \Phi_{\mathbf{1}} & \text{if } a^1 = 1, \end{cases}$$

together with the fact that both $f_{\mathbf{a}}$ and $g_{\mathbf{a}}$ are induced by left-translations, and hence preserve each θ^i, one sees that:

$$\Phi_{\mathbf{a}}^* \theta^i = \begin{cases} \Phi_{\mathbf{0}}^* \theta^i & \text{if } a^1 = 0; \\ \Phi_{\mathbf{1}}^* \theta^i & \text{if } a^1 = 1. \end{cases}$$

Using the explicit expressions for $\Phi_{\mathbf{0}}$ and $\Phi_{\mathbf{1}}$ given in eqns. (3.4.6) and (3.4.7), together with eqn. (3.4.2), it follows that:

$$\Phi_{\mathbf{0}}^* \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \\ \theta^7 \end{pmatrix} = \Phi_{\mathbf{1}}^* \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \\ \theta^7 \end{pmatrix} = \begin{pmatrix} \mathrm{d}y^1 \\ \mathrm{d}y^2 \\ \mathrm{d}y^3 \\ \mathrm{d}y^4 - y^2 \mathrm{d}y^1 \\ \mathrm{d}y^5 + y^1 \mathrm{d}y^3 \\ \mathrm{d}y^6 + y^1 \mathrm{d}y^4 \\ \mathrm{d}y^7 + y^1 \mathrm{d}y^5 + \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^3 \end{pmatrix}. \tag{3.4.18}$$

Therefore:

$$\begin{aligned}\varphi^\mu - \xi^\mu &= y^1\left(\mathrm{d}y^{147} - \mathrm{d}y^{156} - \mathrm{d}y^{134} + \mathrm{d}y^{237}\right) + y^2\left(\mathrm{d}y^{137} - \mathrm{d}y^{126}\right) \\ &\qquad + \tfrac{1}{2}\left(y^1\right)^2\left(2\mathrm{d}y^{145} - \mathrm{d}y^{136} + \mathrm{d}y^{235}\right) + y^1y^2\left(\mathrm{d}y^{135} - \mathrm{d}y^{124}\right) - \tfrac{1}{2}\left(y^1\right)^3\mathrm{d}y^{134} \\ &= y^1\mathrm{d}y^{147} + \mathrm{d}\alpha_{\mathbf{a}},\end{aligned}$$

where:

$$\begin{aligned}\alpha_{\mathbf{a}} = \mathrm{d}y^1 \wedge &\underbrace{\left[\left(y^1y^5 + \tfrac{1}{2}\left(y^2\right)^2\right)\mathrm{d}y^6 + \left(\left(y^1\right)^2y^5 + \tfrac{1}{2}y^1\left(y^2\right)^2\right)\mathrm{d}y^4 - \tfrac{1}{2}\left(y^1\right)^2y^6\mathrm{d}y^3\right]}_{=\beta_{\mathbf{a}}} \\ &+ \mathrm{d}y^3 \wedge \underbrace{\left[\left(-\tfrac{1}{2}\left(y^1\right)^2 - \tfrac{1}{8}\left(y^1\right)^4\right)\mathrm{d}y^4 + y^1y^2\mathrm{d}y^7 + \tfrac{1}{2}\left(y^1\right)^2y^2\mathrm{d}y^5\right]}_{=\gamma_{\mathbf{a}}}.\end{aligned}$$

Observe that there exists $C > 0$ independent of μ such that:

$$|\beta_{\mathbf{a}}|_{\mathrm{Eucl}}, |\gamma_{\mathbf{a}}|_{\mathrm{Eucl}} \leqslant \frac{C}{2}r^2 \quad \text{and} \quad |\mathrm{d}\beta_{\mathbf{a}}|_{\mathrm{Eucl}}, |\mathrm{d}\gamma_{\mathbf{a}}|_{\mathrm{Eucl}} \leqslant \frac{C}{2}r.$$

Also, by solving the linear system in eqn. (3.1.7), one may verify that:

$$g_{\xi^\mu} = \mu^4\left(\left(\mathrm{d}y^1\right)^{\otimes 2} + \left(\mathrm{d}y^2\right)^{\otimes 2} + \left(\mathrm{d}y^3\right)^{\otimes 2}\right) + \mu^{-2}\left(\left(\mathrm{d}y^4\right)^{\otimes 2} + \left(\mathrm{d}y^5\right)^{\otimes 2} + \left(\mathrm{d}y^6\right)^{\otimes 2} + \left(\mathrm{d}y^7\right)^{\otimes 2}\right). \quad (3.4.19)$$

In particular $g_{\xi^\mu} \geqslant \mu^{-2}g_{\mathrm{Eucl}}$ when acting on vectors. It follows that $|\cdot|_{\xi^\mu} \leqslant \mu|\cdot|_{\mathrm{Eucl}}$ when acting on 1-forms, and $|\cdot|_{\xi^\mu} \leqslant \mu^2|\cdot|_{\mathrm{Eucl}}$ when acting on 2-forms. Hence:

$$|\beta_{\mathbf{a}}|_{\xi^\mu}, |\gamma_{\mathbf{a}}|_{\xi^\mu} \leqslant \frac{C}{2}\mu r^2 \quad \text{and} \quad |\mathrm{d}\beta_{\mathbf{a}}|_{\xi^\mu}, |\mathrm{d}\gamma_{\mathbf{a}}|_{\xi^\mu} \leqslant \frac{C}{2}\mu^2 r.$$

One may also compute that $\left|\mathrm{d}y^1\right|_{\xi^\mu} = \left|\mathrm{d}y^3\right|_{\xi^\mu} = \mu^{-2}$. Therefore:

$$\begin{aligned}|\alpha_{\mathbf{a}}|_{\xi^\mu} &\leqslant \left|\mathrm{d}y^1\right|_{\xi^\mu}|\beta_{\mathbf{a}}|_{\xi^\mu} + \left|\mathrm{d}y^3\right|_{\xi^\mu}|\gamma_{\mathbf{a}}|_{\xi^\mu} \\ &\leqslant C\mu^{-1}r^2,\end{aligned}$$

as required. Likewise $\mathrm{d}\alpha_{\mathbf{a}} = \mathrm{d}y^1 \wedge \mathrm{d}\beta_{\mathbf{a}} + \mathrm{d}y^3 \wedge \mathrm{d}\gamma_{\mathbf{a}}$ and hence $|\mathrm{d}\alpha_{\mathbf{a}}|_{\xi^\mu} \leqslant Cr$. Since $\mathcal{I}^*\alpha_{\mathbf{a}} = \alpha_{\mathbf{a}}$, $\alpha_{\mathbf{a}}$ descends to define the required 2-form $\widehat{\alpha}_{\mathbf{a}}$ on $\widehat{U}_{\mathbf{a}}$.

□

Remark 3.4.20. The term $y^1\mathrm{d}y^{147}$ is also exact with primitive $\frac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47}$, however one may calculate that:

$$\left|\frac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47}\right|_{\widehat{\xi}^\mu} = \frac{\mu^2}{2}\left|y^1\right|^2;$$

thus this primitive does not satisfy the bounds in eqn. (3.4.17). It is for this reason that the term $y^1\mathrm{d}y^{147}$ is dealt with separately to the other terms in the expression for $\widehat{\varphi}^\mu - \widehat{\xi}^\mu$.

Using Lemma 3.4.16, I now prove:

Proposition 3.4.21. *There exists $\varepsilon_0 > 0$, independent of μ, such that for all $\varepsilon \in (0, \varepsilon_0]$, the following is true:*

For all $\mu \geqslant 1$, there exists a closed, orbifold G_2 3-form $\widehat{\phi}^\mu$ on $\widehat{\mathrm{M}}$ such that:

$$\widehat{\phi}^\mu = \widehat{\varphi}^\mu \ \text{on} \ \widehat{\mathrm{M}} \Big\backslash \coprod_{\mathbf{a} \in \mathfrak{A}} \widehat{U}_{\mathbf{a}}$$

and on each $\widehat{W}_{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{A}$, one has:

$$\widehat{\phi}^\mu = \mu^6 \mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356} + y^1 \mathrm{d}y^{147}.$$

Proof. Again, begin by working at the level of M. Let $f : [0, \infty) \to [0, 1]$ be a smooth function such that:

$$\begin{aligned}
&\bullet \quad f \equiv 0 \text{ on an open neighbourhood of } \left[0, \tfrac{1}{2}\right]; \\
&\bullet \quad f \equiv 1 \text{ on an open neighbourhood of } [1, \infty); \\
&\bullet \quad \|f'\|_\infty \leqslant 3.
\end{aligned} \tag{3.4.22}$$

Consider the 3-form ϕ on $U_{\mathbf{a}}$ defined by:

$$\phi^\mu = \xi^\mu + y^1 \mathrm{d}y^{147} + \mathrm{d}\left[f\left(\frac{r}{\varepsilon}\right)\alpha_{\mathbf{a}}\right]. \tag{3.4.23}$$

Clearly ϕ^μ is closed and satisfies:

$$\phi^\mu = \begin{cases} \xi^\mu + y^1 \mathrm{d}y^{147} & \text{on } W_{\mathbf{a}}; \\ \varphi^\mu & \text{near the boundary of } U_{\mathbf{a}}. \end{cases}$$

On $U_{\mathbf{a}}$, using eqns. (3.4.19), (3.4.17) and (3.4.22), one may compute that:

$$\begin{aligned}
|\phi^\mu - \xi^\mu|_{\xi^\mu} &\leqslant \left|y^1 \mathrm{d}y^{147}\right|_{\xi^\mu} + |\mathrm{d}\alpha_{\mathbf{a}}|_{\xi^\mu} + \frac{\|f'\|_\infty}{\varepsilon} |\mathrm{d}r|_{\xi^\mu} |\alpha_{\mathbf{a}}|_{\xi^\mu} \\
&\leqslant (4C + 1)\varepsilon
\end{aligned} \tag{3.4.24}$$

where $C > 0$ is as in Lemma 3.4.16 (recall that $|\mathrm{d}r|_{\xi^\mu} \leqslant \mu$, as in the proof of Lemma 3.4.16). Thus ϕ^μ is of G_2-type for all $\varepsilon > 0$ sufficiently small, independent of μ, by the stability of G_2 3-forms. Since ξ^μ, $y^1 \mathrm{d}y^{147}$ and $\alpha_{\mathbf{a}}$ are all $\mathcal{I}$-invariant, the form ϕ^μ descends to define an orbifold G_2 3-form $\widehat{\phi}^\mu$ on $\widehat{\mathrm{M}}$, completing the proof.

□

One can also use Lemma 3.4.16 to give an explicit formula for $\widehat{\phi}$ on the region $\widehat{U}_{\mathbf{a}}$. Explicitly, one takes:

$$\widehat{\phi} = \widehat{\xi} + \mathrm{d}\left[f\left(\frac{r}{\varepsilon}\right)\left(\tfrac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47} + \alpha_{\mathbf{a}}\right)\right]. \tag{3.4.25}$$

In particular, note that whilst $\widehat{\varphi}^1 = \widehat{\varphi}$, it is not true that $\widehat{\phi}^1 = \widehat{\phi}$.

The task now is to resolve the singularities in $\widehat{\phi}^\mu$. I begin by introducing some notation. Firstly, for $k \in (0,\infty)$, define:

$$\mathrm{B}^4\left(\tfrac{1}{2}\varepsilon, k\right) = \left\{ \left(w^1, w^2\right) \in \mathbb{C}^2 \;\middle|\; k^6 \left|w^1\right|^2 + \left|w^2\right|^2 < \frac{1}{2}\varepsilon \right\}.$$

Thus $\mathrm{B}^4\left(\frac{1}{2}\varepsilon, k\right)$ is a complex ellipse with radius $\frac{1}{2}k^{-3}\varepsilon$ in the w^1-direction and radius $\frac{1}{2}\varepsilon$ in the w^2-direction. Also define $\widetilde{X}\left(\frac{1}{2}\varepsilon, k\right)$ to be the pre-image of ${}^{\mathrm{B}^4\left(\frac{1}{2}\varepsilon,k\right)}\big\backslash_{\{\pm 1\}}$ under the blow-down map ρ and, for $k \in \left[2^{-\frac{1}{3}}, \infty\right)$, define $\breve{W}_{\mathbf{a},k}$ to be the subset of $\breve{U}_{\mathbf{a}}$ corresponding to $T \times \widetilde{X}\left(\frac{1}{2}\varepsilon, k\right)$. ($k \geqslant 2^{-\frac{1}{3}}$ is needed to ensure that $\breve{W}_{\mathbf{a},k} \subset \breve{U}_{\mathbf{a}}$.) Secondly, define:

$$\mathbb{T}^3_\mu = {}^{\mathbb{R}^3}\big/_{\mu^3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}.$$

Analogously, let Λ_μ denote the image of Λ under the map $(y^3, y^4, y^7) \in \mathbb{R}^3 \mapsto (\mu^3 y^3, y^4, y^7) \in \mathbb{R}^3$ and define:

$$\widetilde{\mathbb{T}^3_\mu} = {}^{\mathbb{R}^3}\big/_{\Lambda_\mu}.$$

As above, use T_μ to denote either $\mathbb{T}^3_\mu$ or $\widetilde{\mathbb{T}^3_\mu}$ as appropriate.

Begin by considering the space $R_\mu = (T_\mu)_{y^3,y^4,y^7} \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right)_{y^1,y^2,y^5,y^6} \supseteq T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$. Define a 3-form σ on R_μ via:

$$\sigma = \mathrm{d}\left[f\left(\frac{2r}{\varepsilon}\right) \cdot \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47} \right], \tag{3.4.26}$$

where f is as defined in eqn. (3.4.22). Clearly, σ vanishes near the exceptional locus and thus σ defines a smooth 3-form over all of R_μ via extension by zero. Moreover, outside the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ (i.e. on the region $\left\{r \geqslant \frac{1}{2}\varepsilon\right\}$) σ is simply given by $y^1 \mathrm{d}y^{147}$.

Next, define a 3-form ζ on R_μ via:

$$\zeta = \mathrm{d}y^{347} + \mathrm{d}y^3 \wedge \breve{\omega}_t - \mathrm{d}y^4 \wedge \mathfrak{Re}\widetilde{\Omega} + \mathrm{d}y^7 \wedge \mathfrak{Im}\widetilde{\Omega} \tag{3.4.27}$$

for $\breve{\omega}_t$ as above. ζ defines a G_2 3-form on R_μ by Lemma 3.1.1. Finally, define a 3-form ζ^μ on R_μ as follows:

$$\zeta^\mu = \zeta + \frac{1}{\mu^3}\sigma.$$

Lemma 3.4.28. *For $\varepsilon > 0$ sufficiently small, independent of μ, and for μ sufficiently large, ζ^μ is of G_2-type on R_μ.*

Proof. The proof is, again, an application of the stability of G_2 3-forms. Firstly, consider the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right) \backslash \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ (i.e. the region $\left\{r \geqslant \frac{1}{2}\varepsilon\right\}$). Here, $\zeta = \widehat{\xi}$ and $\sigma = y^1 \mathrm{d}y^{147}$, so:

$$|\sigma|_\zeta = \left|y^1\right| \leqslant \frac{\varepsilon\mu^3}{2}$$

and thus:

$$|\zeta^\mu - \zeta|_\zeta \leqslant \frac{1}{2}\varepsilon.$$

Hence ζ_μ is of G_2-type on $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right) \Big\backslash \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ for all μ if ε is sufficiently small, independent of μ.

Now fix ε and consider the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$. On this region $|\sigma|_\zeta \leqslant C$ for some fixed $C > 0$ independent of μ. Thus:

$$|\zeta^\mu - \zeta|_\zeta \leqslant \frac{C}{\mu^3}.$$

Thus for μ sufficiently large, ζ^μ is also of G_2-type on the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$. Thus ζ^μ is of G_2-type on all of R_μ and the result is proven.

□

Using this lemma, the G_2 3-forms required for the resolution can be constructed. Firstly, consider the map ${}_{\{\pm 1\}}\Big\backslash^{\mathbb{C}^2} \to {}_{\{\pm 1\}}\Big\backslash^{\mathbb{C}^2}$ given by:

$$\left(w^1, w^2\right) \mapsto \left(\mu^3 w^1, w^2\right).$$

Restricting this map to the region $\left({}_{\{\pm 1\}}\Big\backslash^{\mathbb{C}^2}\right) \backslash \{0\}$ and using the blow-up map ρ, this gives rise to a map $\mathfrak{h}^\mu : \widetilde{X} \Big\backslash \mathfrak{E} \to \widetilde{X} \Big\backslash \mathfrak{E}$ which extends to all of $\widetilde{X}$. Now define:

$$\mathfrak{H}^\mu : \breve{W}_{\mathbf{a}} \cong (T)_{y^3,y^4,y^7} \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon\right)_{y^1,y^2,y^5,y^6} \to (T_\mu)_{y^3,y^4,y^7} \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon, \mu^{-1}\right)_{y^1,y^2,y^5,y^6}, \tag{3.4.29}$$

where the action of $\mathfrak{H}^\mu$ on T is induced by the map $(y^3, y^4, y^7) \in \mathbb{R}^3 \mapsto (\mu^3 y^3, y^4, y^7) \in \mathbb{R}^3$ and $\mathfrak{H}^\mu$ acts on $\widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ by $\mathfrak{h}^\mu$, and write:

$$\breve{\zeta}^\mu = \mu^{-3} \left(\mathfrak{H}^\mu\right)^* \zeta^\mu. \tag{3.4.30}$$

By Lemma 3.4.28, this is a smooth, closed, G_2 3-form on $\breve{W}_{\mathbf{a}}$. An explicit computation shows that near the boundary of $\breve{W}_{\mathbf{a}}$ (and, more generally, on an open neighbourhood of the region $\breve{W}_{\mathbf{a}} \Big\backslash \breve{W}_{\mathbf{a},\mu}$):

$$\breve{\zeta}^\mu = \mu^6 \mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356} + y^1 \mathrm{d}y^{147}, \tag{3.4.31}$$

which is exactly the 'boundary-conditions' required for the resolution. Thus for each $\mu \in [1, \infty)$, one obtains a smooth, closed G_2 3-form $\breve{\phi}^\mu$ on $\breve{\mathrm{M}}$ by setting $\breve{\phi}^\mu = \widehat{\phi}^\mu$ outside $\breve{W}_{\mathbf{a}}$ for each $\mathbf{a} \in \mathfrak{A}$ and $\breve{\phi}^\mu = \breve{\zeta}^\mu$ on each $\breve{W}_{\mathbf{a}}$.

Now let $\mathring{\mathrm{M}} = \breve{\mathrm{M}} \Big\backslash \coprod_{\mathbf{a} \in \mathfrak{A}} \breve{U}_{\mathbf{a}}$. Then on $\mathring{\mathrm{M}}$ one has $\breve{\phi}^\mu = \widehat{\varphi}^\mu$ and hence $vol_{\breve{\phi}^\mu} = \mu^2 \theta^{1\dots7}$ by Proposition 3.1.9. Hence, one may compute that:

$$\mathcal{H}_3\left(\breve{\phi}^\mu\right) \geqslant \int_{\mathring{\mathrm{M}}} vol_{\breve{\phi}^\mu} = \mu^2 \int_{\mathring{\mathrm{M}}} \theta^{1\dots7} \to \infty \text{ as } \mu \to \infty. \tag{3.4.32}$$

Thus, the proof of Theorem 3.4.12 is completed by the following result:

Proposition 3.4.33. *Let $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$ be as defined in Theorem 3.4.10 and let $\breve{\phi}^\mu$ be as defined above. Then:*

$$\left[\breve{\phi}^\mu\right] = \left[\breve{\phi}\right] \in H^3_{\mathrm{dR}}\left(\breve{\mathrm{M}}\right) \textit{ for all } \mu \geqslant 1.$$

Proof. It suffices to prove that the difference $\breve{\phi}^\mu - \breve{\phi}$ is exact for each $\mu \geqslant 1$. The strategy is to prove that $\breve{\phi}^\mu - \breve{\phi}$ is exact on each of the regions:

- $\breve{W}_{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{A}$;
- $\breve{\mathrm{M}} \Big\backslash \coprod_{\mathbf{a} \in \mathfrak{A}} \breve{U}_{\mathbf{a}}$;
- $\breve{U}_{\mathbf{a}} \Big\backslash \breve{W}_{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{A}$,

and then to verify that the primitives may be combined to define a global primitive on all of $\breve{\mathrm{M}}$.

$\underline{\breve{W}_{\mathbf{a}} \text{ for } \mathbf{a} \in \mathfrak{A}}$: Recall the map $\mathfrak{h}^\mu$ defined above. One can verify that:

$$\breve{\zeta}^\mu = \mathrm{d}y^{347} + \mathrm{d}y^3 \wedge \left(\mathfrak{h}^\mu\right)^* \breve{\omega}_t - \mathrm{d}y^4 \wedge \mathfrak{Re}\widetilde{\Omega} + \mathrm{d}y^7 \wedge \mathfrak{Im}\widetilde{\Omega} + \mathrm{d}\left[f\left(\frac{2\left(\mathfrak{h}^\mu\right)^* r}{\varepsilon}\right) \cdot \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47}\right]$$

and thus:

$$\breve{\phi}^\mu - \breve{\phi} = \mathrm{d}y^3 \wedge \left[\left(\mathfrak{h}^\mu\right)^* \breve{\omega}_t - \breve{\omega}_t\right] + \mathrm{d}\left[f\left(\frac{2\left(\mathfrak{h}^\mu\right)^* r}{\varepsilon}\right) \cdot \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47}\right] \text{ on } \breve{W}_{\mathbf{a}}.$$

The second term is manifestly exact. For the first term, recall the Generalised Poincaré Lemma [94, Prop. 17.10]:

Theorem 3.4.34. *Let X, Y be smooth manifolds, let $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y$ be smooth maps and let $F : f_1 \Rightarrow f_2$ be a smooth homotopy. Then the maps $H^\bullet_{\mathrm{dR}}(Y) \underset{f_2^*}{\overset{f_1^*}{\rightrightarrows}} H^\bullet_{\mathrm{dR}}(X)$ are equal.*

Since $\mathfrak{h}^\mu$ is homotopic to the identity on $\widetilde{X}$, it follows that $\left(\mathfrak{h}^\mu\right)^* \breve{\omega}_t - \breve{\omega}_t = \mathrm{d}\tau$ for some suitable τ. Thus, on the region $\breve{W}_{\mathbf{a}}$, one finds that:

$$\begin{aligned}\breve{\phi}^\mu - \breve{\phi} &= \mathrm{d}\left[\tau \wedge \mathrm{d}y^3 + f\left(\frac{2\left(\mathfrak{h}^\mu\right)^* r}{\varepsilon}\right) \cdot \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47}\right] \\ &= \mathrm{d}\varpi.\end{aligned}$$

In order to extend ϖ to all of $\breve{\mathrm{M}}$ below, it is necessary to compute ϖ explicitly near the boundary of $\breve{W}_{\mathbf{a}}$. For the second term in ϖ, since $\left(\mathfrak{h}^\mu\right)^*(r) \geqslant r$, one finds that:

$$f\left(\frac{2\left(\mathfrak{h}^\mu\right)^* r}{\varepsilon}\right) \cdot \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47} = \frac{1}{2}\left(y^1\right)^2 \mathrm{d}y^{47} \text{ near the boundary of } \breve{W}_{\mathbf{a}}.$$

For the first term in ϖ, recall that the Generalised Poincaré Lemma stated above may be proved by constructing an explicit chain homotopy $\Omega^\bullet(Y) \underset{f_2^*}{\overset{f_1^*}{\rightrightarrows}} \Omega^\bullet(X)$ (with $\Downarrow\mathfrak{F}$) defined by:

$$\begin{aligned}\mathfrak{F}:\Omega^\bullet(Y) &\to \Omega^{\bullet-1}(X)\\ \omega &\mapsto \int_{[0,1]} \iota_s^*\left(\partial_s \lrcorner \left(F^*\omega\right)\right)\mathrm{d}s\end{aligned} \tag{3.4.35}$$

and calculating that:

$$\mathrm{d}\mathfrak{F} + \mathfrak{F}\mathrm{d} = f_2^* - f_1^*, \tag{3.4.36}$$

where the homotopy F is viewed as a map $F : [0,1] \times X \to Y$ and ι_s denotes the embedding $X \cong \{s\} \times X \hookrightarrow [0,1] \times X$. Using the specific homotopy $\mathfrak{h}^{\sqrt[3]{1+s(\mu^3-1)}} = F_s$ of $\widetilde{X}$ connecting Id to $\mathfrak{h}^\mu$, one may calculate that:

$$\tau = \frac{\mu^6-1}{2}\left(y^1\mathrm{d}y^2 - y^2\mathrm{d}y^1\right) \text{ near the boundary of } \breve{W}_{\mathbf{a}}.$$

Thus:

$$\varpi = \frac{\mu^6-1}{2}\left(y^1\mathrm{d}y^{23} - y^2\mathrm{d}y^{13}\right) + \frac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47} \text{ near the boundary of } \breve{W}_{\mathbf{a}}. \tag{3.4.37}$$

$\underline{\breve{\mathrm{M}}\setminus\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{U}_{\mathbf{a}}}$: As discussed at the start of §3.4.2, by eqn. (3.4.3) one has:

$$\breve{\phi}^\mu - \breve{\phi} = \left(\mu^6-1\right)\theta^{123} = \left(\mu^6-1\right)\mathrm{d}\left(\theta^{25}\right) = \mathrm{d}\varpi.$$

Using eqns. (3.4.2), (3.4.6) and (3.4.7), one finds that:

$$\varpi = \left(\mu^6-1\right)\left(\mathrm{d}y^{25} + y^1\mathrm{d}y^{23}\right) \text{ near the boundary of } \breve{U}_{\mathbf{a}}. \tag{3.4.38}$$

$\underline{\breve{U}_{\mathbf{a}}\setminus\breve{W}_{\mathbf{a}} \text{ for } \mathbf{a}\in\mathfrak{A}}$: Finally, using eqns. (3.4.23) and (3.4.25), one finds that on $\breve{U}_{\mathbf{a}}\setminus\breve{W}_{\mathbf{a}}$ for $\mathbf{a}\in\mathfrak{A}$:

$$\breve{\phi}^\mu - \breve{\phi} = \left(\mu^6-1\right)\mathrm{d}y^{123} + \mathrm{d}\left\{\left[1 - f\left(\frac{r}{\varepsilon}\right)\right]\left(\tfrac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47}\right)\right\}.$$

Thus $\breve{\phi}^\mu - \breve{\phi} = \mathrm{d}\varpi$, where:

$$\begin{aligned}\varpi = \frac{\mu^6-1}{2}\left(y^1\mathrm{d}y^{23} - y^2\mathrm{d}y^{13}\right) + \left(\mu^6-1\right)\mathrm{d}\left(f\left(\frac{r}{\varepsilon}\right)\left(y^2\mathrm{d}y^5 + \tfrac{1}{2}y^1y^2\mathrm{d}y^3\right)\right)&\\ + \left[1 - f\left(\frac{r}{\varepsilon}\right)\right]\left(\tfrac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47}\right).&\end{aligned}$$

This satisfies:

$$\begin{aligned}\varpi &= \frac{\mu^6-1}{2}\left(y^1\mathrm{d}y^{23} - y^2\mathrm{d}y^{13}\right) + \left(\tfrac{1}{2}\left(y^1\right)^2\mathrm{d}y^{47}\right) \text{ near the boundary of } \breve{W}_{\mathbf{a}}\\ \varpi &= \left(\mu^6-1\right)\left(\mathrm{d}y^{25} + y^1\mathrm{d}y^{23}\right) \text{ near the boundary of } \breve{U}_{\mathbf{a}}.\end{aligned} \tag{3.4.39}$$

Combining eqns. (3.4.37), (3.4.38), (3.4.39), one sees that ϖ defines a smooth 2-form on all of $\breve{\mathrm{M}}$ such that:

$$\breve{\phi}^{\mu} - \breve{\phi} = \mathrm{d}\varpi,$$

as required.

$\square$

This completes the proof of Theorem 3.4.12.

Remark 3.4.40. Recall that, for a closed G_2 3-form ϕ, the Laplacian flow of ϕ is the solution of the evolution PDE [22, §6]:

$$\frac{\partial \phi(t)}{\partial t} = \Delta_{\phi(t)}\phi(t) = -\mathrm{d}\bigstar_{\phi(t)}\mathrm{d}\Theta(\phi(t)) \quad \text{and} \quad \phi(0) = \phi.$$

Laplacian flow can be regarded as the gradient flow of $\mathcal{H}_3$ [24, §1.5]; in particular, $\mathcal{H}_3$ increases strictly along the flow. Accordingly, Laplacian flow has been used in the literature to provide examples of 7-manifolds on which $\mathcal{H}_3$ is unbounded above; see, e.g., [22, §6] and [50, §5].

The family $\phi(\alpha,\beta,\lambda;\mu)$ constructed in §3.2 can also be interpreted via Laplacian flow. Using Lemma 3.1.1, eqn. (3.2.4) and eqn. (3.2.6), one may compute that:

$$-\mathrm{d}\bigstar_{\phi(\alpha,\beta,\lambda;\mu)}\mathrm{d}\Theta(\phi(\alpha,\beta,\lambda;\mu)) = \frac{4\left(\lambda\overline{\lambda}\right)^{\frac{2}{3}}}{\alpha\mu^2} g^1 \wedge \omega.$$

On the other hand, allowing $\mu = \mu(t)$ gives:

$$\frac{\partial \phi(\alpha,\beta,\lambda;\mu)}{\partial t} = 6\mu^5 \frac{\mathrm{d}\mu}{\mathrm{d}t}\alpha g^1 \wedge \omega.$$

Thus $\phi(\alpha,\beta,\lambda;\mu(t))$ is a flow line of Laplacian flow starting from $\phi(\alpha,\beta,\lambda)$ if μ satisfies the ODE:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{2\left(\lambda\overline{\lambda}\right)^{\frac{2}{3}}}{3\alpha^2\mu^7} \quad \text{and} \quad \mu(0) = 1.$$

It follows that the Laplacian flow starting from $\phi(\alpha,\beta,\lambda)$ exists for all $t > 0$ and is given by:

$$\phi\left(\alpha,\beta,\lambda; \sqrt[8]{\frac{16\left(\lambda\overline{\lambda}\right)^{\frac{2}{3}} t}{3\alpha^2} + 1}\right).$$

In general, however, Laplacian flow can only be explicitly solved on manifolds with a high degree of symmetry, and thus cannot be used to investigate the unboundedness above of $\mathcal{H}_3$ on more complicated manifolds. As a illustration, note that, even at the level of the manifold (M,φ):

$$\Delta_\varphi\varphi = -\mathrm{d}\bigstar_\varphi\mathrm{d}\Theta(\varphi) = 2\theta^{123} + 2\theta^{145} - \theta^{136} + \theta^{127},$$

and consequently the equation $\frac{\partial\varphi_t}{\partial t} = \Delta_{\varphi_t}\varphi_t$ cannot (to the author's knowledge) explicitly be solved starting from $\varphi_0 = \varphi$. However, §3.4 has shown that the scaling arguments described in §3.1 can be applied to successfully prove the unboundedness above of $\mathcal{H}_3$ on $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$. This suggests that Proposition 3.1.9 is a more widely applicable technique for proving the unboundedness above of $\mathcal{H}_3$ than Laplacian flow.

3.5 The large volume limit of $\left(\breve{\mathrm{M}}, \breve{\phi}\right)$

The aim of this section is to describe the geometry of $\left(\breve{\mathrm{M}}, \breve{\phi}^\mu\right)$ as $\mu \to \infty$. The arguments presented require new notions of geometric structures on orbifolds known as stratified Riemannian metrics (and other stratified geometric structures) to be introduced, so I begin by defining these concepts.

3.5.1 Stratified (semi-)Riemannian and quasi-Finslerian structures on orbifolds

I begin by recalling the following definition [81, §15.10]:

Definition 3.5.1. Let $\mathbb{A}$ be a real vector space. A quasinorm on $\mathbb{A}$ is a map $\mathcal{L} : \mathbb{A} \to \mathbb{R}$ satisfying the following three properties:

1. For all $a \in \mathbb{A}$: $\mathcal{L}(a) \geqslant 0$, with equality if and only if $a = 0$ ($\mathcal{L}$ is 'positive definite');
2. For all $\lambda \in \mathbb{R}$, $a \in \mathbb{A}$:
$$\mathcal{L}(\lambda \cdot a) = |\lambda| \cdot \mathcal{L}(a);$$
3. There exists some $k = k(\mathcal{L}) > 0$ such that for all $a, a' \in \mathbb{A}$:
$$\mathcal{L}(a + a') \leqslant k(\mathcal{L}(a) + \mathcal{L}(a')).$$

Note that in the case $k = 1$, this reduces to the definition of a norm.

In this thesis, I restrict attention to continuous quasinorms. In this case, condition (3) above becomes automatic:

Proposition 3.5.2. *Let $\mathbb{A}$ be a finite-dimensional real vector space and let $\mathcal{L} : \mathbb{A} \to \mathbb{R}$ be a continuous map satisfying conditions (1) and (2) from Definition 3.5.1. Then $\mathcal{L}$ is a quasinorm.*

Proof. Consider the continuous map:

$$\begin{aligned} f : (\mathbb{A} \times \mathbb{A})\backslash\{0\} &\to [0, \infty) \\ (a, a') &\mapsto \frac{\mathcal{L}(a + a')}{\mathcal{L}(a) + \mathcal{L}(a')} \end{aligned}$$

(Note that f is well-defined by condition (1) in Definition 3.5.1.) For a contradiction, suppose f is unbounded and pick a sequence $(a_i, a_i') \in (\mathbb{A} \times \mathbb{A})\backslash\{0\}$ such that $f(a_i, a_i') \to \infty$ as $i \to \infty$. Choose some norm $\|-\|$ on $\mathbb{A}$ and consider the new sequence:

$$(\mathfrak{a}_i, \mathfrak{a}'_i) = \left(\frac{a_i}{\|a_i\| + \|a'_i\|}, \frac{a'_i}{\|a_i\| + \|a'_i\|} \right) \in \mathbb{A} \times \mathbb{A}.$$

Clearly $(\mathfrak{a}_i, \mathfrak{a}'_i)$ is bounded in the norm $\|-\|$ and hence converges subsequentially to some $(\mathfrak{a}, \mathfrak{a}') \in \mathbb{A}\times\mathbb{A}$ (since $\mathbb{A}$ is finite-dimensional). Moreover, by construction, the sequence $(\mathfrak{a}_i, \mathfrak{a}'_i)$ satisfies $\|\mathfrak{a}_i\| + \|\mathfrak{a}'_i\| = 1$ and thus $\|\mathfrak{a}\| + \|\mathfrak{a}'\| = 1$. Hence $(\mathfrak{a}, \mathfrak{a}') \in (\mathbb{A} \times \mathbb{A})\backslash\{0\}$ and thus $f(\mathfrak{a}, \mathfrak{a}')$ is well-defined and finite. By condition (2) in Definition 3.5.1, f satisfies $f(\lambda -, \lambda -) = f(-,-)$ for any $\lambda \neq 0$. Therefore:

$$f(a_i, a'_i) = f\left(\frac{a_i}{\|a_i\| + \|a'_i\|}, \frac{a'_i}{\|a_i\| + \|a'_i\|} \right) = f(\mathfrak{a}_i, \mathfrak{a}'_i) \xrightarrow{\text{subsequentially}} f(\mathfrak{a}, \mathfrak{a}') < \infty,$$

contradicting the fact that $f(a_i, a'_i) \to \infty$ as $i \to \infty$. Thus f is bounded and $\mathcal{L}$ is a quasinorm.

□

Let E be a manifold. A quasi-Finslerian structure on E shall mean a continuous map $\mathcal{L} : \mathrm{T}E \to \mathbb{R}$ such that the restriction of $\mathcal{L}$ to any fixed tangent space is a (continuous) quasinorm. (Note that in the case where $\mathcal{L}$ is smooth and a fibrewise norm, this recovers the usual definition of a Finslerian structure.) Using this terminology, I now define the required generalisations of Riemannian metrics to stratified orbifolds:

Definition 3.5.3. Let $(E, \Sigma = \{E_i\}_i)$ be a stratified orbifold.

- A stratified Riemannian metric $\widehat{g} = \{g_i\}_i$ on E is the data of a Riemannian metric g_i on each stratum E_i satisfying the extendibility condition that for each i, there exists a continuous orbifold Riemannian metric $\overline{g}_i$ on E whose tangential component along E_i is g_i.

- A stratified Riemannian semi-metric $\widehat{g} = \{g_i\}_i$ on E is the data of a Riemannian semi-metric g_i on each stratum E_i satisfying the analogous condition that for each i, there exists a continuous orbifold Riemannian semi-metric $\overline{g}_i$ on $\mathrm{T}E$ whose tangential component along E_i is g_i. If, in addition, $\mathcal{D}$ is a stratified distribution on E, then $\widehat{g}$ is regular with respect to $\mathcal{D}$ if for each $i = 0, ..., n$, the kernel of the Riemannian semi-metric $\overline{g}_i$ is precisely the distribution $\mathcal{D}$. In particular, this implies that the kernel of the Riemannian semi-metric g_i on E_i is precisely $\mathcal{D}_i = \mathcal{D} \cap \mathrm{T}E_i$.

- A stratified quasi-Finslerian structure on E is the data of a quasi-Finslerian structure $\mathcal{L}_i$ on each E_i satisfying the property that for every continuous orbifold Riemannian metric h on $\mathrm{T}E$ and each index i, there exists a continuous function $C : \overline{E_i} \to (0, \infty)$ such that

$$\frac{1}{C}\| - \|_h \leqslant \mathcal{L}_i \leqslant C\| - \|_h \quad \text{on} \quad E_i. \tag{3.5.4}$$

Remarks 3.5.5.

- Any two continuous quasinorms $\mathcal{L}$ and $\mathcal{L}'$ on a finite-dimensional vector space $\mathbb{A}$ are Lipschitz equivalent. Indeed, let $\mathcal{S} \subset \mathbb{A}$ be the unit sphere with respect to some norm on $\mathbb{A}$; then

$\frac{\mathcal{L}}{\mathcal{L}'} : \mathcal{S} \to (0, \infty)$ is well-defined and continuous, and hence has compact image (as $\mathcal{S}$ is compact). In light of this, the function C in eqn. (3.5.4) automatically exists on E_i. The significance of eqn. (3.5.4) is that C can be extended continuously over the boundary of E_i, i.e. over the set $\overline{E_i} \backslash E_i$.

- The extendibility condition for stratified Riemannian metrics can alternatively be stated as follows: for every subset $K \subseteq E_i$ which is relatively compact in E:

  1. $g_i|_K$ is uniformly continuous;
  2. g_i is uniformly Lipschitz equivalent to $g|_{E_i}$ for any continuous Riemannian metric g on E.

  The reader will note, by contrast, that condition (1) is not imposed on stratified quasi-Finlserian structures. This extra condition is required for stratified Riemannian metrics to facilitate some technical steps in Chapter 4.

The stratified structures defined in Definition 3.5.3 naturally induce (semi-)metrics on the underlying orbifold E, in the following way:

Definition 3.5.6. Let $(E, \Sigma = \{E_i\}_i)$ be a stratified orbifold and recall the set $\mathcal{A}$ of piecewise-C^1 curves in E. Let $\widehat{g} = (g_i)_i$ be a stratified Riemannian (semi-)metric on E and let $(\gamma : [a, b] \to E) \in \mathcal{A}$. Since each stratum $E_i \subseteq E$ is locally-closed, $I_i = \gamma^{-1}(E_i) \subseteq [a, b]$ is also locally closed and hence measurable. Moreover, since γ is piecewise-C^1 on the compact interval $[a, b]$, it is Lipschitz continuous on $[a, b]$ and hence on each I_i (with respect to any Riemannian metric on E). It follows from [45, Lem. 3.1.7] that $\dot{\gamma}$ lies in the subspace $\mathrm{T}E_i \subseteq \mathrm{T}E$, and hence $g_i(\dot{\gamma})$ is well-defined, almost everywhere on I_i. Now define $\widehat{g}(\dot{\gamma}) : I \to [0, \infty)$ by:

$$\widehat{g}(\dot{\gamma}) = g_i(\dot{\gamma}) \text{ on } I_i.$$

Since each g_i can be extended to a continuous (semi-)metric $\overline{g}_i$ on all of E, one has:

$$|\widehat{g}(\dot{\gamma})| \leqslant \max_i \sup_{t \in I} |\overline{g}_i(\dot{\gamma}(t))| \text{ almost everywhere}$$

and thus $\widehat{g}(\dot{\gamma})$ defines a non-negative element of $L^\infty(I)$. One then defines:

$$\ell^{\widehat{g}}(\gamma) = \int_I \widehat{g}(\dot{\gamma})^{\frac{1}{2}} \, \mathrm{d}\mathscr{L}$$

where $\mathscr{L}$ denotes the Lebesgue measure on I. $(\mathcal{A}, \ell^{\widehat{g}})$ defines a (weak) length structure on E; denote the corresponding (semi-)metric by $d^{\widehat{g}}$. A similar construction applies to stratified quasi-Finslerian structures $\widehat{\mathcal{L}}$, resulting in a metric $d^{\widehat{\mathcal{L}}}$.

Note that any two stratified Riemannian metrics $\widehat{g}$ and $\widehat{h}$ on E are locally uniformly Lipschitz equivalent on E, in the sense that for all $K \Subset E$ compact, there exists a constant $C(K) > 0$ such that for all E_i:

$$\frac{1}{C(K)}g_i \leqslant h_i \leqslant C(K)g_i \text{ on } E_i \cap K.$$

(Indeed, by compactness of K, for each E_i, there exists $C_i(K) > 0$ such that $\frac{1}{C_i(K)}\overline{g}_i \leqslant \overline{h}_i \leqslant C_i(K)\overline{g}_i$ on K; hence the result follows by setting $C(K) = \max_i C_i(K)$.) In particular, the metric $d^{\widehat{g}}$ induces the usual topology on E. The analogous result holds for stratified quasi-Finslerian structures.

By contrast, two stratified Riemannian semi-metrics are not, in general, even pointwise Lipschitz equivalent. However if one fixes a stratified distribution $\mathcal{D}$ on E, then any two stratified Riemannian semi-metrics $\widehat{g}$ and $\widehat{h}$ on E which are regular with respect to $\mathcal{D}$ are locally uniformly Lipschitz equivalent.

Remark 3.5.7 (Refinement of stratification). Every orbifold Riemannian (semi-) metric g on a stratified orbifold $(E, \Sigma = \{E_i\}_i)$ defines a stratified Riemannian (semi-) metric $\widehat{g}$ on E in the obvious way, by setting $g_i = g|_{E_i}$ (where $g|_{E_i}$ denotes the tangential component of g along E_i). Then $\ell^{\widehat{g}}(\gamma) = \ell^g(\gamma)$ for all piecewise-C^1 paths in E and hence $d^g = d^{\widehat{g}}$, i.e. the (semi-) metrics induced by g and $\widehat{g}$ are the same.

More generally, given a stratified orbifold $(E, \Sigma = \{E_i\}_i)$ and a refinement Σ' of Σ (see Definition 2.1.7), every stratified Riemannian (semi-) metric $\widehat{g} = \{g_i\}_i$ on E with respect to the stratification Σ also defines a stratified Riemannian (semi-) metric $\widehat{g}' = \{g'_j\}_j$ with respect to Σ' via $g'_j = g_{i(j)}|_{E'_j}$. It is again clear that $d^{\widehat{g}} = d^{\widehat{g}'}$. The corresponding results for stratified quasi-Finslerian structures are also valid.

Aside. Let E be a stratified orbifold and let $\widehat{\mathcal{L}}$ be a stratified quasi-Finslerian structure on E. Then the length-structure $\ell^{\widehat{\mathcal{L}}}$ has the surprising property that the quantity $\ell^{\widehat{\mathcal{L}}}(\gamma)$ does not depend continuously on the piecewise-C^1 curve γ. This phenomenon can be observed even on an unstratified manifold; see [25, Example 2.4.4]. This observation will not, however, be significant to this thesis.

3.5.2 A collapsing result for $\left(\check{\mathrm{M}}, \check{\phi}\right)$

Recall that there is a natural fibration [48, §9]:

$$\begin{aligned} q : \mathrm{M} &\longrightarrow \mathbb{T}^3 \\ \Gamma \cdot (x^1, ..., x^7) &\longmapsto \left(\tfrac{x^1}{2} + \mathbb{Z}, x^2 + \mathbb{Z}, x^3 + \mathbb{Z}\right) \end{aligned}$$

with (non-calibrated coassociative) fibres diffeomorphic to $\mathbb{T}^4$. Let $\mathcal{I}$ denote the involution of M defined in eqn. (3.4.5) and define a non-free involution $\mathfrak{I}$ of $\mathbb{T}^3$ by acting on the first two factors of $\mathbb{T}^3$ by $-\mathrm{Id}$ and on the final factor by Id. Then $q \circ \mathcal{I} = \mathfrak{I} \circ q$ and so q descends to define a singular fibration:

$$\widehat{q} : \widehat{\mathrm{M}} \to {}_{\mathfrak{I}}\backslash^{\mathbb{T}^3} = {}_{\{\pm 1\}}\backslash^{\mathbb{T}^2} \times S^1 = B,$$

with ${}_{\{\pm 1\}}\backslash^{\mathbb{T}^2}$ being homeomorphic (although obviously not diffeomorphic) to $\mathbb{CP}^1$. The fibres of $\widehat{q}$ are all path-connected, the generic fibres being 4-tori and the fibres over the singular locus of B being diffeomorphic to ${}_{\{\pm 1\}}\backslash^{\mathbb{T}^2} \times \mathbb{T}^2$. Combining $\widehat{q}$ with the natural 'blow-down' map $\rho : \check{\mathrm{M}} \to \widehat{\mathrm{M}}$ similarly yields a fibration π of $\breve{\mathrm{M}}$ over B:

$$\begin{tikzcd} \breve{\mathrm{M}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \widehat{\mathrm{M}} \arrow[d, "\widehat{q}"] \\ & \left.\{\pm1\}\middle\backslash\mathbb{T}^2\right. \times S^1. \end{tikzcd}$$

Away from the exceptional locus of $\breve{\mathrm{M}}$ the map π is a smooth surjection, with fibre $\mathbb{T}^4$. Near the exceptional locus, the map π is modelled on:

$$\widetilde{X} \times \mathbb{T}^3 \to \left(\{\pm1\}\backslash\mathbb{C}^2 \right) \times \mathbb{T}^3 \xrightarrow{proj_1 \times proj_1} \left(\{\pm1\}\backslash\mathbb{C} \right) \times S^1,$$

where $\widetilde{X}$ is the blow-up of $\{\pm1\}\backslash\mathbb{C}^2$ at the origin as in §3.4.1. The fibre of $\widetilde{X} \to \{\pm1\}\backslash\mathbb{C}^2 \to \{\pm1\}\backslash\mathbb{C}$ over 0 is the union of the proper transform of $\{\pm1\}\backslash(\{0\} \times \mathbb{C})$ – denoted $\{\pm1\}\backslash(\{0\} \times \mathbb{C})_{PT}$ – and the exceptional divisor $\mathbb{CP}^1$ intersecting transversally at a single point; hence for each $y^3 \in S^1$, the fibre of π over $\{0\}\times\{y^3\}$ is the union of $\{\pm1\}\backslash(\{0\} \times \mathbb{C})_{PT} \times \{y^3\} \times \mathbb{T}^2$ and $\mathbb{CP}^1 \times \{y^3\} \times \mathbb{T}^2$, intersecting transversally along a single $\mathbb{T}^2$. It follows that the singular fibres of π are homeomorphic to four copies of $\mathbb{CP}^1 \times \mathbb{T}^2$ intersecting a fifth copy of $\mathbb{CP}^1 \times \mathbb{T}^2$ transversally along four distinct copies of $\mathbb{T}^2$.[3]

Since $\widehat{q}$ is induced by the submersive map $q : \mathrm{M} \to \mathbb{T}^3$, $\widehat{q}$ induces a natural stratification Σ on $\widehat{\mathrm{M}}$ by 'pulling back' the canonical stratification on B (see Corollary 4.1.6 for a proof of this fact): explicitly, the strata of Σ consist firstly of the pre-image under $\widehat{q}$ of the smooth locus of B, i.e. the collection of all smooth fibres of the map $\widehat{q}$. Secondly, they consist of the smooth locus of the four singular fibres of $\widehat{q}$. Finally, they consist of the 16 components of the singular locus of $\widehat{\mathrm{M}}$. By pulling Σ back along the blow-down map $\rho : \breve{\mathrm{M}} \to \widehat{\mathrm{M}}$, one obtains a stratification of $\breve{\mathrm{M}}$, say Σ'. Explicitly, the stratification Σ' consists of firstly the collection of all smooth fibres of $\breve{q}$, secondly the four singular fibres of $\breve{q}$ with their exceptional loci removed, and thirdly the 16 exceptional loci of $\breve{\mathrm{M}}$.

Theorem 3.5.8. *Let $\left(\breve{\mathrm{M}}, \breve{\phi}^\mu\right)_{\mu \in [1,\infty)}$ be the family constructed in the proof of Theorem 3.4.12. Then the large volume limit of $\left(\breve{\mathrm{M}}, \breve{\phi}^\mu\right)$ corresponds to an adiabatic limit of the fibration π. Specifically:*

$$\left(\breve{\mathrm{M}}, \mu^{-6}\breve{\phi}^\mu\right) \to \left(B, \widehat{\mathcal{L}}\right) \quad \text{as} \quad \mu \to \infty$$

in the Gromov–Hausdorff sense, where $\widehat{\mathcal{L}}$ is a stratified quasi-Finslerian structure on B (with respect to the canonical stratification of B) defined explicitly as follows: fix a stratum B_i in the canonical stratification of B and write $\pi^{-1}(B_i) = \bigcup_{j=0}^k S_j$, where each S_j is a stratum in the stratification Σ of $\widehat{\mathrm{M}}$. Then given $p \in B_i$ and $u \in \mathrm{T}_p B_i$, define

$$\mathcal{L}_i(u) = \min_{j=0}^{k} \inf\left\{ \|u'\|_{g_j^\infty} \;\middle|\; u' \in \mathrm{T}_x S_j \text{ such that } \mathrm{d}\widehat{q}(u') = u \text{ (and in particular } \widehat{q}(x) = p) \right\}. \quad (3.5.9)$$

[3]Note that [48, p. 35] contains an error in its description of the singular fibres of π, of which the authors of [48] have been informed.

Moreover, outside a neighbourhood of the singular locus of B, $\widehat{\mathcal{L}}$ is simply given by the Euclidean norm.

Consider the map $\mathfrak{f} : \mathrm{M} \to S^1$ given by:

$$\Gamma \cdot (x^1, ..., x^7) \mapsto x^3 + \mathbb{Z}.$$

Then $\mathfrak{f}$ descends to a map $\widehat{\mathfrak{f}} : \widehat{\mathrm{M}} \to S^1$. Define $\breve{\mathfrak{f}} = \widehat{\mathfrak{f}} \circ \rho : \breve{\mathrm{M}} \to S^1$ and, for each $\mathbf{a} \in \mathfrak{A}$, define:

$$\breve{\mathfrak{f}}_{\mathbf{a}} = \breve{\mathfrak{f}}|_{\breve{W}_{\mathbf{a},1}} \quad \text{and} \quad \widehat{\mathfrak{f}}_{\mathbf{a}} = \widehat{\mathfrak{f}}|_{\widehat{W}_{\mathbf{a},1}}. \tag{3.5.10}$$

Explicitly, the maps $\breve{\mathfrak{f}}_{\mathbf{a}}$ and $\widehat{\mathfrak{f}}_{\mathbf{a}}$ may be described as follows: writing $\breve{W}_{\mathbf{a},1} \cong T_{y^3,y^4,y^7} \times {\{\pm 1\}}\backslash^{\mathrm{B}^4(\varepsilon)}$, one finds:

$$\begin{aligned} \breve{\mathfrak{f}}_{\mathbf{a}} : \breve{W}_{\mathbf{a},1} &\to S^1 \\ \left(y^i\right) &\mapsto y^3 \end{aligned}$$

and similarly for $\widehat{\mathfrak{f}}_{\mathbf{a}}$.

Now let g^μ be the Riemannian metric on $\breve{\mathrm{M}}$ induced by the G_2 3-form $\mu^{-6}\breve{\phi}$ and write $\widehat{g}^\mu$ for the induced stratified Riemannian metric on $\breve{\mathrm{M}}$ as in Example 3.5.7. Write d^μ for the metric on $\breve{\mathrm{M}}$ induced by $\widehat{g}^\mu$. For all $k \in [1, \infty)$, consider the space:

$$\breve{\mathrm{M}}^{(k)} = \breve{\mathrm{M}} \Big\backslash \coprod_{\mathbf{a} \in \mathfrak{A}} \breve{W}_{\mathbf{a},k}$$

where $\breve{W}_{\mathbf{a},k}$ was defined in §3.4.2 and write $\widehat{W}_{\mathbf{a},k} = \rho\left(\breve{W}_{\mathbf{a},k}\right)$. The proof of Theorem 3.5.8 starts from the following result, which should be regarded as a stratified generalisation of Theorem 3.3.2:

Theorem 3.5.11. *Suppose that there exists a stratified Riemannian semi-metric $\widehat{g}^\infty$ on $\widehat{\mathrm{M}}$ such that the following five conditions hold:*

Conditions 3.5.12.

1. Write $\mathcal{D}$ for the distribution over $\widehat{\mathrm{M}}$ given by $\ker \mathrm{d}\widehat{q}$ and note that $\mathcal{D}$ is stratified with respect to Σ. Then $\widehat{g}^\infty$ is regular with respect to $\mathcal{D}$;

2. On each $\breve{\mathrm{M}}^{(k)}$, $k \in [1, \infty)$:

$$\widehat{g}^\mu \to \widehat{g}^\infty \text{ uniformly as } \mu \to \infty,$$

   and there exist constants $\Lambda_\mu(k) \geqslant 0$ such that:

$$\lim_{\mu \to \infty} \Lambda_\mu(k) = 1 \quad \text{and} \quad \widehat{g}^\mu \geqslant \Lambda_\mu(k)^2 \widehat{g}^\infty \text{ for all } \mu \in [1, \infty), \tag{3.5.13}$$

   where $\widehat{g}^\infty$ is regarded as a stratified Riemannian semi-metric on $\breve{\mathrm{M}}^{(k)}$ using the blow-up map ρ;

3. $$\lim_{k \to \infty} \limsup_{\mu \to \infty} \max_{\mathbf{a} \in \mathfrak{A}} \sup_{p \in S^1} \operatorname{diam}_{d^\mu} \left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \breve{W}_{\mathbf{a},k}\right] = 0;$$

4.

$$\lim_{k\to\infty} \max_{\mathbf{a}\in\mathfrak{A}} \sup_{p\in S^1} \operatorname{diam}_{d^\infty}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap \widehat{W}_{\mathbf{a},k}\right] = 0;$$

5.

$$\lim_{k\to\infty} \limsup_{\mu\to\infty} \max_{\mathbf{a}\in\mathfrak{A}} \sup_{\partial_{\mathbf{a},k}} |d^\mu - d^\infty| = 0,$$

where d^∞ is the semi-metric on $\widehat{\mathrm{M}}$ induced by $\widehat{g}^\infty$ (regarded as a semi-metric on $\partial_{\mathbf{a},k}$ using the identification ρ) and, for simplicity of notation, I write $\partial_{\mathbf{a},k}$ for the subset $\partial\breve{W}_{\mathbf{a},k} \overset{\rho}{\cong} \partial\widehat{W}_{\mathbf{a},k}$.

Then the manifolds $\left(\breve{\mathrm{M}}, \mu^{-6}\breve{\phi}^\mu\right)$ *converge to* $(B,\mathcal{L})$*, as claimed in Theorem 3.5.8.*

(As for Theorem 3.3.2, since Theorem 3.5.11 is a result in metric geometry, rather than G_2 geometry, the proof is postponed until Chapter 4 of this thesis (see Theorem 4.2.5), so as not to detract from the main thrust of the current chapter. For comparative purposes, the reader may wish to note that $\breve{M}$, $\widehat{\mathrm{M}}$ and ρ above correspond to E_1, E_2 and Φ in Chapter 4 respectively, and that $\breve{W}_{\mathbf{a},k}$, $\widehat{W}_{\mathbf{a},k}$ above correspond respectively to $U_1^{(k^{-1})}(j)$, $U_2^{(k^{-1})}(j)$ in Chapter 4, the rest of the notation being obviously equivalent.)

Thus, to prove Theorem 3.5.8, it suffices to establishing the five conditions in Conditions 3.5.12. The remainder of this chapter will be devoted to this task.

3.5.3 Bounding the volume form induced by $\breve{\omega}_t$

Recall the Kähler forms $\breve{\omega}_t$ interpolating between $\widehat{\omega}$ and the Eguchi–Hanson metric $\widetilde{\omega}_t$ on $\widetilde{X}$ used in the construction of $\breve{\phi}$. For the purpose of proving Conditions 3.5.12, I require a lower bound on the volume forms induced by $\breve{\omega}_t$ which is both sharp and t-independent. The purpose of this subsection is to derive this bound.

I begin by providing an alternative perspective on the Eguchi–Hanson forms $\widetilde{\omega}_t$. Consider the problem of trying to construct Ricci-flat Kähler metrics on $\left(\{\pm1\}\backslash\mathbb{C}^2\right)\backslash\{0\}$. One possible approach is as follows: suppose one is given a closed, positive, real $(1,1)$-form $\widetilde{\omega}$ on $\left(\{\pm1\}\backslash\mathbb{C}^2\right)\backslash\{0\}$ with the following property:

$$\widetilde{\omega}^2 = \left(\mathfrak{Re}\widehat{\Omega}\right)^2 = \left(\mathfrak{Im}\widehat{\Omega}\right)^2 = 2vol_0. \tag{3.5.14}$$

(Here $\widehat{\Omega} = \mathrm{d}w^1 \wedge \mathrm{d}w^2$ as usual and vol_0 denotes the Euclidean volume form on $\{\pm1\}\backslash\mathbb{C}^2$ given by $\mathrm{d}x^1\wedge\mathrm{d}y^1\wedge\mathrm{d}x^2\wedge\mathrm{d}y^2$, where $w^1 = x^1 + iy^1$ and $w^2 = x^2 + iy^2$.) Then the triple $\left(\widetilde{\omega}, \mathfrak{Re}\widehat{\Omega}, \mathfrak{Im}\widehat{\Omega}\right)$ defines an $\mathrm{Sp}(1)$-structure on $\left(\{\pm1\}\backslash\mathbb{C}^2\right)\backslash\{0\}$ and the condition $\mathrm{d}\widetilde{\omega} = \mathrm{d}\mathfrak{Re}\widehat{\Omega} = \mathrm{d}\mathfrak{Im}\widehat{\Omega} = 0$ implies the vanishing of the torsion of this $\mathrm{Sp}(1)$-structure [73, Lem. 6.8, p. 91], i.e. the triple defines a hyper-Kähler structure. This implies that the holonomy of the Kähler metric induced by $\widetilde{\omega}$ is contained in $\mathrm{Sp}(1) = \mathrm{SU}(2)$ which is a Ricci-flat holonomy group [79, p. 55].

To ensure that $\widetilde{\omega}$ is closed and a $(1,1)$-form, apply the ansatz:

$$\widetilde{\omega} = \frac{1}{4}\mathrm{dd}^c\left[a(\lambda)\right] = a'(\lambda)\widehat{\omega} + \frac{1}{4}a''(\lambda)\mathrm{d}(\lambda)\wedge\mathrm{d}^c(\lambda), \tag{3.5.15}$$

where $\lambda = r^2 = \left|w^1\right|^2 + \left|w^2\right|^2$ is the radial distance squared from $0 \in {\{\pm 1\}}\backslash{\mathbb{C}^2}$, $a : (0,\infty) \to \mathbb{R}$ is a smooth, real-valued function and $\widehat{\omega} = \frac{i}{2}\left(\mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^1 + \mathrm{d}w^2 \wedge \mathrm{d}\overline{w}^2\right)$ is the standard Euclidean Kähler form on ${\{\pm 1\}}\backslash{\mathbb{C}^2}$ (in particular, note that $\widetilde{\omega}$ only depends on a'). A long but elementary calculation yields:

$$\widetilde{\omega}^2 = \frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^2 a'(\lambda)^2\right]}{\lambda} vol_0. \tag{3.5.16}$$

Thus eqn. (3.5.14) is reduced to the second-order ODE:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^2 a'(\lambda)^2\right] = 2\lambda.$$

Integrating this equation gives:

$$a_t'(\lambda) = \sqrt{1 + \frac{t^4}{\lambda^2}} \tag{3.5.17}$$

for some $t \geqslant 0$ (the positive square root is needed to ensure that $\widetilde{\omega}$ is a positive $(1,1)$-form). In the case where $t = 0$, $\widetilde{\omega}_0 = \frac{1}{4}\mathrm{d}\mathrm{d}^c a_0(\lambda)$ is simply the Euclidean form $\widehat{\omega}$, however in the case $t > 0$, one recovers the Eguchi–Hanson metrics $\widetilde{\omega}_t$ defined in §3.4.1, eqn. (3.4.9).

Remark 3.5.18. The fact that the 1-parameter family $\widetilde{\omega}_t$ of Eguchi–Hanson metrics can naturally be extended to include the metric $\widehat{\omega}$ (corresponding to the case $t = 0$) may appear initially surprising, since the metrics $\widetilde{\omega}_t$ for $t > 0$ are defined on the manifold $\widetilde{X}$ whereas the metric $\widehat{\omega}$ is defined on the orbifold ${\{\pm 1\}}\backslash{\mathbb{C}^2}$. However as $t \to 0$, the diameter of the exceptional divisor $\mathfrak{E}$ tends to zero and so the manifolds $\left(\widetilde{X}, \widetilde{\omega}_t\right)$ converge in the Gromov–Hausdorff sense to the orbifold $\left({\{\pm 1\}}\backslash{\mathbb{C}^2}, \widehat{\omega}\right)$, and hence the result is not as surprising as it first appears. See also [5, p. 21].

Using this perspective, I now prove the required bound on the volume form of $\breve{\omega}_t$:

Proposition 3.5.19. *There exist $R > 0$, $\upsilon \in (0,1)$, independent of $t > 0$, such that the following is true:*

For every $t > 0$, there exists a closed, real, positive $(1,1)$-form $\breve{\omega}_t$ on $\widetilde{X}$ satisfying the following three properties:

1. *$\breve{\omega}_t = \widetilde{\omega}_t$ on the region $\left\{p \in \widetilde{X} \;\middle|\; r(p) \leqslant \frac{tR}{2}\right\}$;*
2. *$\breve{\omega}_t = \hat{\omega}$ on a neighbourhood of the region $\left\{p \in \widetilde{X} \;\middle|\; r(p) \geqslant tR\right\}$;*
3. *$\breve{\omega}_t^2 \geqslant 2\upsilon^2 vol_0$ on all of $\widetilde{X}$, with equality holding at least on $\left\{p \in \widetilde{X} \;\middle|\; r(p) = \mathfrak{r}_t\right\}$, where $\mathfrak{r}_t \in \left(\frac{tR}{2}, tR\right)$.*

Remark 3.5.20. It is not difficult to show that any $\breve{\omega}_t$ of the form considered in eqn. (3.5.15) which satisfies points (1) and (2) cannot also satisfy $\breve{\omega}_t^2 \geqslant 2 vol_0$ on all of $\widetilde{X}$, and thus there is some $\upsilon = \upsilon(t) \in (0,1)$ such that $\breve{\omega}_t^2 \geqslant 2\upsilon(t)^2 vol_0$, with equality realised at some point. The significance of the above result is that υ can be taken to be independent of t.

Proof. Let $\upsilon \in (0,1)$ be a chosen later and write $c = 2 - 2\upsilon^2$. I begin with the following auxiliary claim:

Claim 3.5.21. *There exists $R_0 = R_0(c) > 0$ (i.e. depending on c but independent of t) such that the following is true:*

For all $R \geqslant R_0$ and all $t > 0$, there exists a smooth function $k_t : [0, t^2R^2] \to (-\infty, 0]$ satisfying the following three properties:

(i) $k_t \equiv 0$ on $\left[0, \frac{t^2R^2}{4}\right]$ and on a neighbourhood of t^2R^2 in $[0, t^2R^2]$;

(ii) $\int_0^{t^2R^2} k_t(\lambda)\mathrm{d}\lambda = -t^4$;

(iii) $k_t(\lambda) \geqslant -c\lambda$ for all $\lambda \in [0, t^2R^2]$, with equality holding at some point $\mathfrak{r}_t^2 \in [0, t^2R^2]$.

Proof of Claim. For such a function k_t to exist, it is necessary and sufficient that:

$$\int_{\frac{t^2R^2}{4}}^{t^2R^2} c\lambda \, \mathrm{d}\lambda > t^4.$$

(Given this, one constructs k_t by smoothing out the piecewise constant function:

$$\hat{k}_t(\lambda) = \begin{cases} 0 & \text{on } \left[0, \frac{t^2R^2}{4}\right) \\ -c\lambda & \text{on } \left[\frac{t^2R^2}{4}, t^2R^2\right) \\ 0 & \text{at } \lambda = t^2R^2 \end{cases}$$

whilst ensuring that $\int_0^{t^2R^2} k_t(\lambda)\mathrm{d}\lambda = -t^4$ and that $k_t(\mathfrak{r}_t^2) = -c\mathfrak{r}_t^2$ still holds at some point $\mathfrak{r}_t^2 \in [0, t^2R^2]$. The converse is clear.) However:

$$\int_{\frac{t^2R^2}{4}}^{t^2R^2} c\lambda \, \mathrm{d}\lambda = \frac{15ct^4R^4}{32} > t^4$$

whenever $R_0(c) > \sqrt[4]{\frac{32}{15c}}$, completing the proof.

$\square$

The proof of Proposition 3.5.19 now proceeds as follows. Define:

$$h_t(\lambda) = \int_0^\lambda k_t(s)\mathrm{d}s \geqslant -t^4$$

and define:

$$\breve{\omega}_t = \frac{1}{4}\mathrm{dd}^c\left[\alpha_t(r^2)\right],$$

where $\alpha_t : (0, t^2R^2] \to \mathbb{R}$ satisfies:

$$\alpha_t'(\lambda) = \sqrt{1 + \frac{t^4}{\lambda^2} + \frac{h_t(\lambda)}{\lambda^2}}. \tag{3.5.22}$$

For $\lambda \in \left(0, \frac{t^2R^2}{4}\right]$, $h_t \equiv 0$ by Claim 3.5.21(i), hence $\alpha_t' = a_t'$ (see eqn. (3.5.17)) and whence $\breve{\omega}_t = \widetilde{\omega}_t$ for $r \in \left(0, \frac{tR}{2}\right)$. It follows that $\breve{\omega}_t$ extends over the exceptional divisor in $\widetilde{X}$ and satisfies property (1) in Proposition 3.5.19. Similarly, for λ in a neighbourhood of t^2R^2 in $\left(0, t^2R^2\right]$, $h_t \equiv -t^4$ by Claim 3.5.21(i)–(ii), hence $\alpha_t' = a_0$ and whence $\breve{\omega}_t = \widehat{\omega}$ for r in a neighbourhood of tR in $(0, tR]$. Thus $\breve{\omega}_t$ extends over the whole of $\widetilde{X}$ and satisfies property (2) in Proposition 3.5.19. Thus to complete the proof of Proposition 3.5.19, it suffices to prove that $\breve{\omega}_t$ is positive and satisfies property (3).

To prove positivity, I use the following well-known fact: if ω is a real, positive $(1,1)$-form on an almost complex manifold M and ω' is a real $(1,1)$-form on M with $|\omega' - \omega|_\omega < 1$, then ω' is also positive (this can easily be verified by working in local coordinates). To apply this to $\breve{\omega}_t$, firstly note that since $\widetilde{\omega}_t$ and $\widehat{\omega}$ are both positive, one can restrict attention to the region $r \in \left(\frac{tR}{2}, tR\right)$. Thus it suffices to prove that:

$$\left|\widehat{\omega} - \left(\alpha_t'(r^2)\widehat{\omega} + \frac{1}{4}\alpha_t''(r^2)\mathrm{d}(r^2) \wedge \mathrm{d}^c(r^2)\right)\right|_{\widehat{\omega}} < 1 \quad \text{for } r \in \left(\frac{tR}{2}, tR\right).$$

Using the triangle inequality:

$$\left|\widehat{\omega} - \left(\alpha_t'(r^2)\widehat{\omega} + \frac{1}{4}\alpha_t''(r^2)\mathrm{d}(r^2) \wedge \mathrm{d}^c(r^2)\right)\right|_{\widehat{\omega}} \leqslant |\alpha_t'(r^2) - 1| \cdot |\widehat{\omega}|_{\widehat{\omega}} + \frac{1}{4}\left|\alpha_t''(r^2)\right| \cdot \left|\mathrm{d}(r^2) \wedge \mathrm{d}^c(r^2)\right|_{\widehat{\omega}}. \tag{3.5.23}$$

Using eqn. (3.5.22), it follows that:

$$\left|\alpha_t'(r^2) - 1\right| \leqslant \frac{\sqrt{t^4 + h_t(r^2)}}{r^2} \leqslant \frac{4}{R^2} \quad \text{for } r \in \left(\frac{tR}{2}, tR\right), \tag{3.5.24}$$

since $h_t \leqslant 0$. Using eqn. (3.5.22) once more, one sees:

$$\alpha_t''(r^2) = \frac{-\frac{t^4 + h_t(r^2)}{r^6} + \frac{h_t'(r^2)}{2r^4}}{\sqrt{1 + \frac{t^4}{r^4} + \frac{h_t(r^2)}{r^4}}}$$

and hence:

$$r^2\left|\alpha_t''(r^2)\right| \leqslant \frac{\left|t^4 + h_t(r^2)\right|}{r^4} + \frac{\left|h_t'(r^2)\right|}{2r^2}.$$

Since $-t^4 \leqslant h_t(r^2) \leqslant 0$, $-cr^2 \leqslant h_t'(r^2) \leqslant 0$ and $\frac{tR}{2} \leqslant r \leqslant tR$, it follows that:

$$r^2\left|\alpha_t''(r^2)\right| \leqslant \frac{16}{R^4} + \frac{c}{2}. \tag{3.5.25}$$

A simple calculation shows that $|\widehat{\omega}|_{\widehat{\omega}} = \sqrt{2}$ and $\left|\mathrm{d}(r^2) \wedge \mathrm{d}^c(r^2)\right|_{\widehat{\omega}} \leqslant 4Cr^2$ for some $C > 0$ independent of r^2, R and t. Thus, by combining eqns. (3.5.23), (3.5.24) and (3.5.25):

$$|\widehat{\omega} - \breve{\omega}_t|_{\widehat{\omega}} \leqslant \frac{4\sqrt{2}}{R^2} + \frac{16C}{R^4} + \frac{Cc}{2}.$$

Define $c = C^{-1}$ (note that C is independent of R and t) and choose $R > R_0(c)$ such that:

$$\frac{4\sqrt{2}}{R^2} + \frac{16C}{R^4} < \frac{1}{2}.$$

Then $|\widehat{\omega}-\breve{\omega}_t|_{\widehat{\omega}}<1$ for $r\in\left(\frac{tR}{2},tR\right)$ and thus the positivity of $\breve{\omega}_t$ has been verified.

Finally, let me establish property (3) of Proposition 3.5.19. Clearly for $r\notin\left(\frac{tR}{2},tR\right)$, one has $\breve{\omega}_t^2=2vol_0$, since both $\widehat{\omega}$ and $\widetilde{\omega}_t$ have this property. For $r\in\left(\frac{tR}{2},tR\right)$, by combining eqn. (3.5.16) and (3.5.22), one computes:

$$\widetilde{\omega}_t^2=\left(2+\frac{h_t'(r^2)}{r^2}\right)vol_0.$$

Now by property (iii) in Claim 3.5.21, $\frac{h_t'(r^2)}{r^2}\geqslant -c=2v^2-2$ with equality holding at $r=\mathfrak{r}_t\in[0,tR]$, as required. This completes the proof.

□

3.5.4 Defining a suitable $\widehat{g}^{\infty}$

For simplicity of notation, write $\breve{\varphi}^{\mu}=\mu^{-6}\breve{\phi}^{\mu}$ so that $g^{\mu}=g_{\breve{\varphi}^{\mu}}$. The next task is to understand the limit of the Riemannian metrics g^{μ} away from the exceptional locus $\breve{S}$.

Define:

$$\breve{\mathfrak{W}}_{\mathbf{a}}=\bigcap_{k\geqslant 1}\breve{W}_{\mathbf{a},k}\supset\breve{S},$$

so that in local coordinates:

$$\breve{\mathfrak{W}}_{\mathbf{a}}=\left\{y^1,y^2=0,\left(y^5\right)^2+\left(y^6\right)^2\leqslant\frac{1}{2}\varepsilon\right\},$$

and write:

$$\breve{\mathrm{M}}\Big\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{\mathfrak{W}}_{\mathbf{a}}=\underbrace{\left(\breve{\mathrm{M}}\Big\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{U}_{\mathbf{a}}\right)}_{\mathring{\mathrm{M}}}\bigcup\underbrace{\left(\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{U}_{\mathbf{a}}\backslash\breve{W}_{\mathbf{a}}\right)}_{\breve{\mathrm{M}}_{int}}\bigcup\left(\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{W}_{\mathbf{a}}\backslash\breve{\mathfrak{W}}_{\mathbf{a}}\right).$$

I shall consider the behaviour of g^{μ} on each of these three regions in turn.

The region $\mathring{\mathrm{M}}$

Recall that on $\mathring{\mathrm{M}}$:

$$\breve{\varphi}^{\mu}=\theta^{123}+\mu^{-6}\left(\theta^{145}+\theta^{167}-\theta^{246}+\theta^{257}+\theta^{347}+\theta^{356}\right).$$

Using the G_2-basis $\left(\theta^1,\theta^2,\theta^3,\mu^{-3}\theta^4,\mu^{-3}\theta^5,\mu^{-3}\theta^6,\mu^{-3}\theta^7\right)$, it follows that

$$\begin{aligned}g^{\mu}&=\left(\left(\theta^1\right)^{\otimes 2}+\left(\theta^2\right)^{\otimes 2}+\left(\theta^3\right)^{\otimes 2}\right)+\mu^{-6}\left(\left(\theta^4\right)^{\otimes 2}+\left(\theta^5\right)^{\otimes 2}+\left(\theta^6\right)^{\otimes 2}+\left(\theta^7\right)^{\otimes 2}\right)\\&\to\left(\theta^1\right)^{\otimes 2}+\left(\theta^2\right)^{\otimes 2}+\left(\theta^3\right)^{\otimes 2}\text{ uniformly on }\mathring{\mathrm{M}}\text{ as }\mu\to\infty.\end{aligned}$$

Thus define:

$$g^{\infty}=\left(\theta^1\right)^{\otimes 2}+\left(\theta^2\right)^{\otimes 2}+\left(\theta^3\right)^{\otimes 2}\text{ on }\mathring{\mathrm{M}}.\tag{3.5.26}$$

Then $g^\mu - g^\infty$ is non-negative definite for all $\mu \in [1,\infty)$ and thus eqn. (4.2.7) holds with $\Lambda_\mu = 1$ for all μ. Moreover (see [48, §9]) it can be shown that $\ker \mathrm{d}\widehat{q} = \langle e_4, e_5, e_6, e_7\rangle$, where (e_i) denotes the basis of left-invariant vector fields on M dual to the basis of left-invariant 1-forms (θ^i). (Note that whilst the forms e_i do not themselves descend to the orbifold $\widehat{\mathrm{M}}$, the distribution $\langle e_4, ..., e_7\rangle$ is invariant under the involution $\mathcal{I}$ defined in eqn. (3.4.5) and thus does descend to $\widehat{\mathrm{M}}$.) Thus from eqn. (3.5.26), one sees that g^∞ is positive definite transverse to $\ker \mathrm{d}\widehat{q}$ on $\mathring{\mathrm{M}}$.

The region $\coprod_{\mathbf{a}\in\mathfrak{A}} \breve{W}_{\mathbf{a}} \backslash \breve{\mathfrak{W}}_{\mathbf{a}}$

For simplicity, fix some choice of $\mathbf{a} \in \mathfrak{A}$. Recall from eqn. (3.4.31) that on the region $\breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},\mu}$:

$$\breve{\phi}^\mu = \mathrm{d}y^{123} + \mu^{-6}\left\{\mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356} + y^1\mathrm{d}y^{147}\right\}.$$

In particular, for any given $k \in [1,\infty)$ and all $\mu \geqslant k$, since $\breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},k} \subseteq \breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},\mu}$ one may calculate that on $\breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},k}$:

$$\begin{aligned} g_{\breve{\phi}^\mu} = \left(1 - \frac{(y^1)^2}{4}\right)^{\frac{-1}{3}} \Big\{&\left[(\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2 + (\mathrm{d}y^3)^2 + y^1\mathrm{d}y^1 \odot \mathrm{d}y^3\right] \\ &+ \mu^{-6}\left[(\mathrm{d}y^4)^2 + (\mathrm{d}y^5)^2 + (\mathrm{d}y^6)^2 + (\mathrm{d}y^7)^2 + y^1\mathrm{d}y^4 \odot \mathrm{d}y^6 + y^1\mathrm{d}y^5 \odot \mathrm{d}y^7\right]\Big\}. \end{aligned}$$

Define:

$$g^\infty = \left(1 - \frac{(y^1)^2}{4}\right)^{\frac{-1}{3}} \left\{(\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2 + (\mathrm{d}y^3)^2 + y^1\mathrm{d}y^1 \odot \mathrm{d}y^3\right\} \text{ on } \breve{W}_{\mathbf{a}} \backslash \breve{\mathfrak{W}}_{\mathbf{a}}. \tag{3.5.27}$$

Then $g^\mu \to g^\infty$ uniformly on $\breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},k}$ as $\mu \to \infty$. Moreover when $\mu \geqslant k$, $g^\mu - g^\infty$ is non-negative definite for $\varepsilon > 0$ sufficiently small, independent of μ (where ε is the size of the surgery region used in the construction of $\breve{\mathrm{M}}$). Thus eqn. (4.2.7) holds once again by setting $\Lambda_\mu(k) = 1$ for all $\mu \in [k,\infty)$. Moreover, using eqns. (3.4.6) and (3.4.7), one can show that on the region $\widehat{U}_{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{A}$:

$$\ker \mathrm{d}\widehat{q} = \langle \frac{\partial}{\partial y^4}, \frac{\partial}{\partial y^5}, \frac{\partial}{\partial y^6}, \frac{\partial}{\partial y^7}\rangle. \tag{3.5.28}$$

Thus by eqn. (3.5.27), g^∞ is positive definite transverse to $\ker \mathrm{d}\widehat{q}$ on $\breve{W}_{\mathbf{a}} \backslash \breve{W}_{\mathbf{a},k}$ for all $\varepsilon > 0$ sufficiently small, independent of μ.

The region $\breve{\mathrm{M}}_{int}$

By analogy with the notation $\breve{\phi}^\mu = \mu^{-6}\breve{\varphi}^\mu$, define $\widehat{\xi}^\mu$ on $\breve{\mathrm{M}}_{int}$ by:

$$\widehat{\xi}^\mu = \mu^{-6}\widehat{\xi}^\mu = \mathrm{d}y^{123} + \mu^{-6}\left\{\mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356}\right\}.$$

Recall also that, by Proposition 3.4.21, on the region $\breve{\mathrm{M}}_{int}$:

$$\breve{\phi}^{\mu} = \widehat{\xi}^{\mu} + \mu^{-6}\left\{y^1 \mathrm{d}y^{147} + \mathrm{d}\left[f\left(\frac{r}{\varepsilon}\right)\alpha_{\mathbf{a}}\right]\right\} \tag{3.5.29}$$

where $\alpha_{\mathbf{a}}$ (defined in Lemma 3.4.16) is independent of μ and at least quadratic in (y^1, y^2, y^5, y^6).

To analyse the behaviour of g^{μ} on $\breve{\mathrm{M}}_{int}$ as $\mu \to \infty$, it is useful to introduce a third G_2 3-form Ξ^{μ} on $\breve{\mathrm{M}}_{int}$. To define Ξ^{μ}, firstly write $\breve{\phi}^{\mu} = \widehat{\xi}^{\mu} + \sum_{1 \leqslant i<j<k \leqslant 7} \mu^{-6}\sigma_{ijk}\mathrm{d}y^{ijk}$, where each coefficient σ_{ijk} is a smooth function on $\breve{\mathrm{M}}_{int}$ independent of μ and satisfying:

$$|\sigma_{ijk}| \leqslant Cr \tag{3.5.30}$$

for some fixed $C > 0$, independent of μ, ε and r. Then define:

$$\Xi^{\mu} = \widehat{\xi}^{\mu} + \mu^{-6} \sum_{\substack{1 \leqslant i \leqslant 3 \\ 4 \leqslant j<k \leqslant 7}} \sigma_{ijk}\mathrm{d}y^{ijk}. \tag{3.5.31}$$

Lemma 3.5.32. *There exist constants C_1, C_2 independent of μ and r such that:*

1. $\left\|\widehat{\xi}^{\mu} - \Xi^{\mu}\right\|_{\widehat{\xi}^{\mu}} \leqslant C_1 r$;

2. $\left\|\breve{\phi}^{\mu} - \Xi^{\mu}\right\|_{\widehat{\xi}^{\mu}} \leqslant C_2\mu^{-3} r$.

Proof. Observe that $\widehat{\xi}^{\mu}$ has the G_2-basis $\left(\vartheta^1, ..., \vartheta^7\right) = \left(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3, \mu^{-3}\mathrm{d}y^4, \mu^{-3}\mathrm{d}y^5, \mu^{-3}\mathrm{d}y^6, \mu^{-3}\mathrm{d}y^7\right)$. With respect to this basis one can write:

$$\Xi^{\mu} = \underbrace{\vartheta^{123} + \vartheta^{145} + \vartheta^{167} - \vartheta^{246} + \vartheta^{257} + \vartheta^{347} + \vartheta^{356}}_{\widehat{\xi}^{\mu}} + \sum_{\substack{1 \leqslant i \leqslant 3 \\ 4 \leqslant j<k \leqslant 7}} \sigma_{ijk}\vartheta^{ijk}. \tag{3.5.33}$$

(1) then immediately follows from eqn. (3.5.30).

For (2), note that from eqn. (3.5.31):

$$\breve{\phi}^{\mu} - \Xi^{\mu} = \mu^{-6}\left(\sigma_{123}\mathrm{d}y^{123} + \sum_{\substack{1 \leqslant i<j \leqslant 3 \\ 4 \leqslant k \leqslant 7}} \sigma_{ijk}\mathrm{d}y^{ijk}\right).$$

(The fact that there are no terms of the form $\mathrm{d}y^{ijk}$ with $4 \leqslant i < j < k \leqslant 7$ follows from eqn. (3.5.29) and the precise expression for $\alpha_{\mathbf{a}}$ given in Lemma 3.4.16.) Writing this in terms of the G_2-basis (ϑ^i) gives:

$$\breve{\phi}^{\mu} - \Xi^{\mu} = \mu^{-3}\left(\mu^{-3}\sigma_{123}\vartheta^{123} + \sum_{\substack{1 \leqslant i<j \leqslant 3 \\ 4 \leqslant k \leqslant 7}} \sigma_{ijk}\vartheta^{ijk}\right),$$

from which, together with eqn. (3.5.30), (2) is immediately clear.

$\square$

Informally, this result says that (a) Ξ^μ is of G_2-type and 'close' to $\widehat{\xi}^\mu$ if $\varepsilon > 0$ is sufficiently small uniformly in μ and (b) the difference between Ξ^μ and $\breve{\phi}^\mu$ is negligible as $\mu \to \infty$. Thus to understand the behaviour of $\breve{\phi}^\mu$ as $\mu \to \infty$, I begin by studying Ξ^μ as $\mu \to \infty$.

Lemma 3.5.34. *One can write:*

$$g_{\Xi^\mu} = \sum_{1\leqslant i,j\leqslant 3} (\delta_{ij} + g_{ij})\mathrm{d}y^i \odot \mathrm{d}y^j + \mu^{-6} \sum_{4\leqslant i,j\leqslant 7} (\delta_{ij} + g_{ij})\mathrm{d}y^i \odot \mathrm{d}y^j$$

for some smooth functions g_{ij} on $\breve{\mathrm{M}}_{int}$ independent of μ and satisfying:

$$|g_{ij}| \leqslant C_3 r \tag{3.5.35}$$

for some constant $C_3 > 0$ independent of μ, ε and r.

Proof. Recall eqn. (3.5.33):

$$\Xi^\mu = \underbrace{\vartheta^{123} + \vartheta^{145} + \vartheta^{167} - \vartheta^{246} + \vartheta^{257} + \vartheta^{347} + \vartheta^{356}}_{\widehat{\xi}^\mu} + \sum_{\substack{1\leqslant i\leqslant 3\\ 4\leqslant j<k\leqslant 7}} \sigma_{ijk}\vartheta^{ijk}.$$

Then one can automatically write:

$$g_{\Xi^\mu} = \sum_{1\leqslant i,j\leqslant 7} (\delta_{ij} + g_{ij})\vartheta^i \odot \vartheta^j$$

for some g_{ij} independent of μ and satisfying eqn. (3.5.35). Recalling the definition of the ϑ^i, to complete the proof, it suffices to prove that $g_{ij} = 0$ if $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 7$.

To this end, recall from [21, §2, Thm. 1] that:

$$g_{\Xi^\mu} vol_{\Xi^\mu} = [(-) \lrcorner \Xi^\mu] \wedge [(-) \lrcorner \Xi^\mu] \wedge \Xi^\mu.$$

Thus, it suffices to prove that for all $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 7$:

$$[\vartheta_i \lrcorner \Xi^\mu] \wedge [\vartheta_j \lrcorner \Xi^\mu] \wedge \Xi^\mu = 0,$$

where $(\vartheta_1, ..., \vartheta_7)$ is the basis of vectors dual to $(\vartheta^1, ..., \vartheta^7)$. Define:

$$\mathcal{D} = \langle \vartheta_1, \vartheta_2, \vartheta_3 \rangle \quad \text{and} \quad \mathcal{T} = \langle \vartheta_4, ..., \vartheta_7 \rangle$$

so that $\mathrm{T}\breve{\mathrm{M}}_{int} = \mathcal{D} \oplus \mathcal{T}$. By examining eqn. (3.5.33), one can verify that:

$$\Xi^\mu \in \bigwedge^3 \mathcal{D}^* + \mathcal{D}^* \otimes \bigwedge^2 \mathcal{T}^*.$$

It follows that for $1 \leqslant i \leqslant 3$:

$$\vartheta_i \lrcorner \Xi^\mu \in \bigwedge^2 \mathcal{D}^* + \bigwedge^2 \mathcal{T}^* \subset \bigwedge^2 \mathrm{T}^*\breve{\mathrm{M}}_{int}$$

and that for $4 \leqslant j \leqslant 7$:

$$\vartheta_j \lrcorner \Xi^\mu \in \mathcal{D}^* \otimes \mathcal{T}^* \subset \bigwedge^2 \mathrm{T}^* \breve{\mathrm{M}}_{int}.$$

Thus:

$$[\vartheta_i \lrcorner \Xi^\mu] \wedge [\vartheta_j \lrcorner \Xi^\mu] \wedge \Xi^\mu \in \bigwedge^6 \mathcal{D}^* \otimes \mathcal{T}^* + \bigwedge^4 \mathcal{D}^* \otimes \bigwedge^3 \mathcal{T}^* + \bigwedge^2 \mathcal{D}^* \otimes \bigwedge^5 \mathcal{T}^*,$$

with all three summands vanishing since $\operatorname{rank}\mathcal{D} = 3$ and $\operatorname{rank}\mathcal{T} = 4$. This completes the proof. □

It follows at once that g_{Ξ^μ} converges uniformly to:

$$g^\infty = \sum_{1 \leqslant i,j \leqslant 3} (\delta_{ij} + g_{ij}) \mathrm{d}y^i \odot \mathrm{d}y^j \tag{3.5.36}$$

on $\breve{\mathrm{M}}_{int}$ as $\mu \to \infty$, and moreover that $g_{\Xi^\mu} - g^\infty$ is non-negative definite for all μ. Once again, recalling that:

$$\ker \mathrm{d}\widehat{q} = \langle \frac{\partial}{\partial y^4}, \frac{\partial}{\partial y^5}, \frac{\partial}{\partial y^6}, \frac{\partial}{\partial y^7} \rangle.$$

(see eqn. (3.5.28)) one sees that g^∞ is positive definite transverse to $\ker \mathrm{d}\widehat{q}$ on $\breve{\mathrm{M}}_{int}$.

I now return to the metrics $g^\mu = g_{\breve{\phi}^\mu}$:

Proposition 3.5.37. *The metrics g^μ converge uniformly on $\breve{\mathrm{M}}_{int}$ to g^∞ (as defined above), and moreover there exist constants $\Lambda'_\mu \to 1$ as $\mu \to \infty$ such that $g^\mu \geqslant \left(\Lambda'_\mu\right)^2 g^\infty$ for all $\mu \in [1, \infty)$.*

Proof. Recall the 'standard' G_2 3-form φ_0 on $\mathbb{R}^7$ defined in §2.2. Since the assignment $\varphi \in \bigwedge^3_+ \left(\mathbb{R}^7\right)^* \mapsto g_\varphi \in \bigodot^2_+ \left(\mathbb{R}^7\right)^*$ is smooth, there exist constants $\delta_1, \Delta_0 > 0$ such that if $|\varphi - \varphi_0|_{\mathrm{Eucl}} < \delta_1$, then:

$$|g_\varphi - g_{\mathrm{Eucl}}|_{\mathrm{Eucl}} \leqslant \Delta_0 \, |\varphi - \varphi_0|_{\mathrm{Eucl}} \,. \tag{3.5.38}$$

Since every G_2 3-form on a 7-manifold is pointwise isomorphic to φ_0, it follows that eqn. (3.5.38) holds for general G_2 3-forms on manifolds, with the same values of δ_1 and Δ_0.

The proof now proceeds via repeated application of eqn. (3.5.38), together with the following result:

Lemma 3.5.39. *Let $(\mathbb{A}, g)$ be a finite-dimensional inner product space and write $\|-\|_g$ for the norm on $\bigodot^2 \mathbb{A}^*$ induced by g. Then for any symmetric bilinear form h on $\mathbb{A}$:*

$$h \leqslant \|h\|_g \cdot g. \tag{3.5.40}$$

Proof. Firstly, recall the definition of $\|h\|_g$. Pick any g-orthonormal basis $(a_1, ..., a_n)$ of $\mathbb{A}$. Then:

$$\|h\|_g = \sqrt{\sum_{i,j=1}^n h(a_i, a_j)^2}.$$

Now fix any vector $a \in \mathbb{A}$. By scale invariance of eqn. (3.5.40), one may assume without loss of generality that $g(a) = 1$. One can then extend a to a g-orthonormal basis $(a_1 = a, ..., a_n)$ of $\mathbb{A}$ and compute:

$$\begin{aligned} h(a) &\leqslant |h(a)| \\ &= \sqrt{h(a)^2} \\ &\leqslant \sqrt{\sum_{i,j=1}^{n} h(a_i, a_j)^2} = \|h\|_g, \end{aligned}$$

as required.

$\square$

By combining point 1 from Lemma 3.5.32 with eqn. (3.5.38), for all $\varepsilon \leqslant \frac{\delta_1}{C_1}$ (a condition which is independent of μ), one obtains:

$$\left\| g_{\Xi^\mu} - g_{\widehat{\xi}^\mu} \right\|_{g_{\widehat{\xi}^\mu}} \leqslant \Delta_0 \left\| \Xi^\mu - \widehat{\xi}^\mu \right\|_{g_{\widehat{\xi}^\mu}} \leqslant \Delta_0 C_1 \varepsilon.$$

Applying Lemma 3.5.39, it follows that:

$$g_{\Xi^\mu} - g_{\widehat{\xi}^\mu} \leqslant \Delta_0 C_1 \varepsilon g_{\widehat{\xi}^\mu} \quad \text{and} \quad g_{\widehat{\xi}^\mu} - g_{\Xi^\mu} \leqslant \Delta_0 C_1 \varepsilon g_{\widehat{\xi}^\mu}$$

and hence:

$$(1 - \Delta_0 C_1 \varepsilon) g_{\widehat{\xi}^\mu} \leqslant g_{\Xi^\mu} \leqslant (1 + \Delta_0 C_1 \varepsilon) g_{\widehat{\xi}^\mu}.$$

In particular, for $\varepsilon < \frac{1}{\Delta_0 C_1}$ (a condition which is independent of μ) $g_{\widehat{\xi}^\mu}$ and g_{Ξ^μ} are Lipschitz equivalent on $\breve{\mathrm{M}}_{int}$, uniformly in μ. Now by Lemma 3.5.32, point 2: $\left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\widehat{\xi}^\mu}} \to 0$ as $\mu \to \infty$ and hence by the Lipschitz equivalence just established $\left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\Xi^\mu}} \to 0$ as $\mu \to \infty$. Using eqn. (3.5.38) again, one sees that for all μ sufficiently large:

$$\| g^\mu - g_{\Xi^\mu} \|_{g_{\Xi^\mu}} \leqslant \Delta_0 \left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\Xi^\mu}}$$

and hence by using Lemma 3.5.39 again:

$$g^\mu \geqslant \left(1 - \Delta_0 \left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\Xi^\mu}} \right) g_{\Xi^\mu} \geqslant \left(1 - \Delta_0 \left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\Xi^\mu}} \right) g^\infty,$$

where in the final line I have used that $g_{\Xi^\mu} \geqslant g^\infty$ for all μ as above. Thus, setting $\Lambda'_\mu = \sqrt{1 - \Delta_0 \left\| \breve{\phi}^\mu - \Xi^\mu \right\|_{g_{\Xi^\mu}}}$ for all μ sufficiently large, one has $\Lambda'_\mu \to 1$ as required. Therefore to conclude the proof, it suffices to prove that $g^\mu \to g^\infty$ uniformly on $\breve{\mathrm{M}}_{int}$ as $\mu \to \infty$.

To this end, fix a reference metric g on $\breve{\mathrm{M}}_{int}$. Since $g_{\Xi^\mu} \to g^\infty$ uniformly, it follows that $\|g_{\Xi^\mu}\|_g \leqslant 2 \|g^\infty\|_g = D$ for all μ sufficiently large. Thus by applying Lemma 3.5.39 one final time it follows that:

$$g_{\Xi^\mu} \leqslant Dg$$

for all μ sufficiently large. Thus $g_{\Xi^\mu} \geqslant D^{-3} g$ when acting on 3-forms. In particular:

$$\begin{aligned}\left\|g^\mu - g_{\Xi^\mu}\right\|_g &\leqslant D^3 \left\|g^\mu - g_{\Xi^\mu}\right\|_{g_{\Xi^\mu}}\\ &\leqslant D^3 \Delta_0 \left\|\breve{\phi}^\mu - \Xi^\mu\right\|_{g_{\Xi^\mu}} \to 0 \text{ as } \mu \to \infty.\end{aligned}$$

Thus since g_{Ξ^μ} tends to g^∞ uniformly, it follows that g^μ also tends to g^∞ uniformly. This completes the proof.

□

In summary, it has been shown that one may define a Riemannian semi-metric g^∞ on $\breve{\mathrm{M}}\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{\mathfrak{W}}_{\mathbf{a}}$ with kernel precisely $\ker \mathrm{d}\widehat{q}$ such that on each $\breve{\mathrm{M}}^{(k)}$, $g^\mu \to g^\infty$ uniformly and there exist constants $\Lambda_\mu(k) \geqslant 0$ such that:

$$\lim_{\mu\to\infty} \Lambda_\mu(k) = 1 \quad \text{ and } \quad g^\mu \geqslant \Lambda_\mu(k)^2 g^\infty \text{ for all } \mu \in [1, \infty).$$

I now explain how to define the limiting stratified Riemannian semi-metric $\widehat{g}^\infty$ on all of $\widehat{\mathrm{M}}$. Using ρ, one may identify $\breve{\mathrm{M}}\backslash\coprod_{\mathbf{a}\in\mathfrak{A}}\breve{\mathfrak{W}}_{\mathbf{a}}$ with a subset of $\widehat{\mathrm{M}}$. By examining eqn. (3.5.27), one may verify that g^∞ can be smoothly extended to a semi-metric on all of $\widehat{\mathrm{M}}$.

Recall that the strata of Σ (the stratification of $\widehat{\mathrm{M}}$ induced by $\widehat{q}$) consist firstly of the preimage under $\widehat{q}$ of the smooth locus of B, secondly of the smooth loci of the singular fibres of $\widehat{q}$ and thirdly of the components of the singular locus $\widehat{S} \subset \widehat{\mathrm{M}}$. On the first two types of strata of Σ, simply define $\widehat{g}^\infty$ to be the restriction of the semi-metric g^∞ to the stratum. On the strata $\widehat{S}_{\mathbf{a}}$, define $\widehat{g}^\infty$ to be the semi-metric:

$$g^\infty_{\mathbf{a}} = \upsilon^{\frac{4}{3}} \left(\mathrm{d}y^3\right)^{\otimes 2}, \tag{3.5.41}$$

where υ is defined in Proposition 3.5.19. (The motivation for this definition will become apparent in the next section.) Again, this can be extended to a semi-metric on all of $\widehat{\mathrm{M}}$; indeed, it is easy to verify that:

$$\upsilon^{\frac{4}{3}} g^\infty|_{\widehat{S}_{\mathbf{a}}} = g^\infty_{\mathbf{a}}. \tag{3.5.42}$$

Thus $\widehat{g}^\infty$ defines a stratified Riemannian semi-metric on $\widehat{\mathrm{M}}$. Moreover, since:

$$\ker \mathrm{d}\widehat{q} \cap \mathrm{T}\widehat{S}_{\mathbf{a}} = \langle \frac{\mathrm{d}}{\mathrm{d}y^4}, \frac{\mathrm{d}}{\mathrm{d}y^7} \rangle \subset \langle \frac{\mathrm{d}}{\mathrm{d}y^3}, \frac{\mathrm{d}}{\mathrm{d}y^4}, \frac{\mathrm{d}}{\mathrm{d}y^7} \rangle = \mathrm{T}\widehat{S}_{\mathbf{a}}$$

(cf. eqn. (3.5.28)) it follows that $g^\infty_{\mathbf{a}}$ is positive definite transverse to $\ker \mathrm{d}\widehat{q} \cap \mathbb{T}\widehat{S}_{\mathbf{a}}$. Thus $\widehat{g}^\infty$ is regular with respect to the stratified distribution $\mathcal{D}$.

Thus in summary, I have defined the stratified Riemannian semi-metric on all of $\widehat{\mathrm{M}}$, and have shown that with this definition, points 1 and 2 of Conditions 3.5.12 hold.

3.5.5 Estimates on $\widehat{g}^\infty$

The purpose of this subsection is to verify points 3, 4 and 5 of Conditions 3.5.12. Specifically:

Proposition 3.5.43.

3.
$$\lim_{k\to\infty}\ \limsup_{\mu\to\infty}\ \max_{\mathbf{a}\in\mathfrak{A}}\ \sup_{p\in S^1}\ \mathrm{diam}_{d^\mu}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k}\right]=0;$$

4.
$$\lim_{k\to\infty}\ \max_{\mathbf{a}\in\mathfrak{A}}\ \sup_{p\in S^1}\ \mathrm{diam}_{d^\infty}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\right]=0;$$

5.
$$\lim_{k\to\infty}\ \limsup_{\mu\to\infty}\ \max_{\mathbf{a}\in\mathfrak{A}}\ \sup_{\partial_{\mathbf{a},k}}|d^\mu-d^\infty|=0,$$

where d^∞ is the metric on $\widehat{\mathrm{M}}$ induced by $\widehat{g}^\infty$ (regarded as a metric on $\partial_{\mathbf{a},k}$ using the identification ρ) and, for simplicity of notation, I write $\partial_{\mathbf{a},k}$ for the subset $\partial\breve{W}_{\mathbf{a},k}\overset{\rho}{\cong}\partial\widehat{W}_{\mathbf{a},k}$.

Proof.
(3) For each $p\in S^1$, write:
$$\mathrm{diam}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k},g^\mu\right]$$
for the diameter of the space $\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k}$ with respect to the intrinsic metric induced by the Riemannian metric g^μ, i.e. the metric defined using paths contained entirely within $\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k}$. Then clearly:
$$\mathrm{diam}_{d^\mu}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k}\right]\leqslant\mathrm{diam}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k},g^\mu\right]$$
and so it suffices to prove that:
$$\lim_{k\to\infty}\ \limsup_{\mu\to\infty}\ \max_{\mathbf{a}\in\mathfrak{A}}\ \sup_{p\in S^1}\ \mathrm{diam}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k},g^\mu\right]=0.$$

Initially, fix $k\in[1,\infty)$, $\mathbf{a}\in\mathfrak{A}$ and consider $\mu\geqslant k$. Recall from eqn. (3.4.29) that there is a homothety:
$$\mathfrak{H}^\mu:\breve{W}_{\mathbf{a}}\cong(T)_{y^3,y^4,y^7}\times\widetilde{X}\left(\tfrac{1}{2}\varepsilon\right)_{y^1,y^2,y^5,y^6}\to(T_\mu)_{y^3,y^4,y^7}\times\widetilde{X}\left(\tfrac{1}{2}\varepsilon,\mu^{-1}\right)_{y^1,y^2,y^5,y^6}$$
(given by rescaling the y^1, y^2 and y^3 directions by μ^3) which identifies the G_2 3-form $\breve{\phi}^\mu$ on the left-hand side with the G_2 3-form $\mu^{-3}\zeta^\mu=\mu^{-3}\left(\zeta+\mu^{-3}\sigma\right)$ on the right-hand side, where ζ and σ are defined in eqns. (3.4.27) and (3.4.26) respectively. In particular, the homothety identifies $\breve{\phi}^\mu=\mu^{-6}\breve{\phi}^\mu$ on the left-hand side with the G_2 3-form $\mu^{-9}\zeta^\mu$ on the right-hand side.

Note also that there is a natural map $f:T_\mu\to\mu^3S^1$ given by projecting onto the first coordinate (i.e. y^3). Given $p\in S^1$, write $T_{\mu,p}$ for the fibre of this map over the point $\mu^3p\in\mu^3S^1$, which can naturally be identified with the torus $\mathbb{T}^2$, irrespective of whether $T_\mu=\mathbb{T}^3_\mu$ or $\widetilde{\mathbb{T}^3_\mu}$. Then, using the diffeomorphism invariance of intrinsic diameter, one may compute that:
$$\begin{aligned}\mathrm{diam}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k},g^\mu\right]&=\mathrm{diam}\left[\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p},\mu^{-9}\zeta^\mu\right]\\&=\mu^{-3}\,\mathrm{diam}\left[\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p},\zeta^\mu\right].\end{aligned}\qquad(3.5.44)$$

Now as seen in Lemma 3.4.28, $\left\|\mu^{-3}\sigma\right\|_\zeta$ is bounded on $\widetilde{X}\left(\frac{1}{2}\varepsilon,\mu^{-1}\right)\times T_\mu$ by some absolute constant $C'>0$ independent of μ, which may be made arbitrarily small by choosing ε sufficiently small (independent of μ) and μ sufficiently large. So without loss of generality one may assume that $C'<\delta_1$ for δ_1 as in eqn. (3.5.38) and thus:

$$\|g_{\zeta^\mu}-g_\zeta\|_\zeta \leqslant \Delta_0 C' \text{ on } \widetilde{X}\left(\tfrac{1}{2}\varepsilon,\mu^{-1}\right)\times T_\mu.$$

Since $\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p}\subset \widetilde{X}\left(\frac{1}{2}\varepsilon,\mu^{-1}\right)\times T_\mu$ it follows by Lemma 3.5.39 that:

$$g_{\zeta^\mu}\leqslant (1+\Delta_0C')g_\zeta \text{ on } \widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p}.$$

Therefore:

$$\operatorname{diam}\left[\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p},\zeta^\mu\right]\leqslant\sqrt{1+\Delta_0C'}\operatorname{diam}\left[\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p},\zeta\right]. \tag{3.5.45}$$

Now outside the region $\widetilde{X}\left(\frac{1}{2}\varepsilon\right)=\widetilde{X}\left(\frac{1}{2}\varepsilon,1\right)$, the metric induced by ζ is just Euclidean. It follows that:

$$\operatorname{diam}\left[\widetilde{X}\left(\frac{1}{2}\varepsilon,\frac{k}{\mu}\right)\times T_{\mu,p},\zeta\right]\leqslant C''\left(\frac{\mu}{k}\right)^3 \tag{3.5.46}$$

for some $C''>0$ independent of k, μ and p. Combining eqns. (3.5.44), (3.5.45) and (3.5.46) gives:

$$\operatorname{diam}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\breve{W}_{\mathbf{a},k},g^\mu\right]\leqslant\mu^{-3}\sqrt{1+C'\Delta_0}C''\left(\frac{\mu}{k}\right)^3=C''\sqrt{1+C'\Delta_0}k^{-3}.$$

Taking supremum over $p\in S^1$, maximum over $\mathbf{a}\in\mathfrak{A}$, limit superior over $\mu\to\infty$ and then the limit over $k\to\infty$ then gives the required result.

(4) Since every point of $\widehat{S}_{\mathbf{a}}$ is a limit point of $\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}$, one clearly has:

$$\lim_{k\to\infty}\max_{\mathbf{a}\in\mathfrak{A}}\sup_{p\in S^1}\operatorname{diam}_{d^\infty}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\right]=\lim_{k\to\infty}\max_{\mathbf{a}\in\mathfrak{A}}\sup_{p\in S^1}\operatorname{diam}_{d^\infty}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\right].$$

Now $\widehat{g}^\infty$ is simply given by g^∞ on $\widehat{\mathrm{M}}\backslash\widehat{S}$. Thus, one has:

$$\lim_{k\to\infty}\max_{\mathbf{a}\in\mathfrak{A}}\sup_{p\in S^1}\operatorname{diam}_{d^\infty}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\right]\leqslant\lim_{k\to\infty}\max_{\mathbf{a}\in\mathfrak{A}}\sup_{p\in S^1}\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}},g^\infty\right]$$

where again $\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}},g^\infty\right]$ denotes the diameter of $\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}$ with respect to the intrinsic semi-metric defined by g^∞, i.e. the semi-metric defined using paths contained entirely within $\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}$.

Now fix $\mathbf{a}\in\mathfrak{A}$ and recall that:

$$\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\})\cap\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}=\left[\left({}_{\{\pm1\}}\backslash\mathrm{B}^4\left(\tfrac{1}{2}\varepsilon,k\right)\right)\backslash\{0\}\right]\times T_p,$$

where T_p is the fibre over $p \in S^1$ of the projection $T \to S^1$ onto the first coordinate, and:

$$g^\infty = \left(1 - \frac{\left(y^1\right)^2}{4}\right)^{\frac{-1}{3}} \left\{\left(\mathrm{d}y^1\right)^2 + \left(\mathrm{d}y^2\right)^2 + \left(\mathrm{d}y^3\right)^2 + y^1 \mathrm{d}y^1 \odot \mathrm{d}y^3\right\}.$$

Write $g^\infty = g_{s\mathrm{Eucl}} + \varpi$ where:

$$g_{s\mathrm{Eucl}} = \left(\mathrm{d}y^1\right)^2 + \left(\mathrm{d}y^2\right)^2 + \left(\mathrm{d}y^3\right)^2 \quad \text{and} \quad \varpi = g^\infty - g_{s\mathrm{Eucl}}.$$

Note that g^∞ and $g_{s\mathrm{Eucl}}$ are not Riemannian metrics on $\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}$ and so *a priori* it is not clear that Lemma 3.5.39 applies. However, if one restricts attention to the distribution $\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\rangle$, then both g^∞ and $g_{s\mathrm{Eucl}}$ are non-degenerate (i.e. inner-products) and hence Lemma 3.5.39 applies. One may compute that on this distribution, over the region $\widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}$:

$$\|\varpi\|_{g_{s\mathrm{Eucl}}} \leqslant Dk^{-3}$$

for some $D > 0$ independent of k. Thus by Lemma 3.5.39:

$$g^\infty \leqslant \left(1 + Dk^{-3}\right) g_{s\mathrm{Eucl}}. \tag{3.5.47}$$

Hence:

$$\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\ , g^\infty\right] \leqslant \sqrt{1 + Dk^{-3}}\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\ , g_{s\mathrm{Eucl}}\right]$$

Clearly $\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\ , g_{s\mathrm{Eucl}}\right]$ is bounded by $D'k^{-3}$ for some $D' > 0$ independent of k and p. Thus one has:

$$\operatorname{diam}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \widehat{W}_{\mathbf{a},k}\backslash\widehat{S}_{\mathbf{a}}\ , g^\infty\right] \leqslant D'k^{-3}\sqrt{1 + Dk^{-3}}.$$

Taking supremum over $p \in S^1_{y^3}$, maximum over $\mathbf{a} \in \mathfrak{A}$ and the limit as $k \to \infty$ gives the required result.

(5) To prove this result, it is useful to introduce a third semi-metric on the region $\partial_{\mathbf{a},k}$ as follows. Equip S^1 with the metric $\upsilon^{\frac{2}{3}} d_{\mathrm{Eucl}}$, where υ is as defined in Proposition 3.5.19. Pulling this metric back along the restriction $\breve{\mathfrak{f}}_{\mathbf{a}} \cong \widehat{\mathfrak{f}}_{\mathbf{a}} : \partial_{\mathbf{a},k} \to S^1$ defines a (μ-independent) semi-metric on $\partial_{\mathbf{a},k}$, which I shall denote d_k. Explicitly:

$$d_k(x,y) = \upsilon^{\frac{2}{3}} d_{\mathrm{Eucl}}\left(\breve{\mathfrak{f}}_{\mathbf{a}}(x), \breve{\mathfrak{f}}_{\mathbf{a}}(y)\right).$$

Then to prove (5) in Proposition 3.5.43, I shall prove the following two statements:

5i.

$$\lim_{k\to\infty} \limsup_{\mu\to\infty} \max_{\mathbf{a}\in\mathfrak{A}} \sup_{\partial_{\mathbf{a},k}} |d^\mu - d_k| = 0;$$

5ii.

$$\lim_{k\to\infty} \limsup_{\mu\to\infty} \max_{\mathbf{a}\in\mathfrak{A}} \sup_{\partial_{\mathbf{a},k}} |d_k - d^\infty| = 0.$$

Clearly, these collectively imply (5).

To prove ($5i$), I employ the following strategy: it is necessary to bound both $d_k - d^\mu$ and $d^\mu - d_k$ from above. For the first of these two quantities, I show that on all of $\breve{\mathrm{M}}$: '$g^\mu \geqslant \upsilon^{\frac{4}{3}}\breve{\mathfrak{f}}^* g_{\mathrm{Eucl}}$' in the limit as $\mu \to \infty$. Thus '$d^\mu \geqslant d^k$' (again, in a limiting sense) and hence $d_k - d^\mu$ can be bounded above. For the second, for any two points $x, y \in \partial_k$ I write down an explicit path $\gamma : x \to y$ whose length with respect to g^μ is approximately $d_k(x,y)$, with this approximation becoming exact in the limit as first $\mu \to \infty$ and then $k \to \infty$. The strategy for ($5ii$) is similar.

($5i$) Recall the following decomposition of $\breve{\mathrm{M}}$:

$$\breve{\mathrm{M}} = \mathring{\mathrm{M}} \bigcup \breve{\mathrm{M}}_{int} \bigcup \left(\coprod_{\mathbf{a} \in \mathfrak{A}} \breve{W}_{\mathbf{a}} \right).$$

As in §3.5.4, I shall consider each region in turn.

$\underline{\mathring{\mathrm{M}}}$: Here:

$$g^\mu = \left(\left(\theta^1\right)^{\otimes 2} + \left(\theta^2\right)^{\otimes 2} + \left(\theta^3\right)^{\otimes 2} \right) + \mu^{-6} \left(\left(\theta^4\right)^{\otimes 2} + \left(\theta^5\right)^{\otimes 2} + \left(\theta^6\right)^{\otimes 2} + \left(\theta^7\right)^{\otimes 2} \right)$$

and thus evidently:

$$g^\mu \geqslant \left(\theta^3\right)^{\otimes 2} = \left(\mathrm{d}x^3\right)^{\otimes 2} = \breve{\mathfrak{f}}^* g_{\mathrm{Eucl}} \geqslant \upsilon^{\frac{4}{3}} \breve{\mathfrak{f}}^* g_{\mathrm{Eucl}} \tag{3.5.48}$$

on $\mathring{\mathrm{M}}$ (recall that $\upsilon < 1$).

$\underline{\breve{\mathrm{M}}_{int}}$: On this region, recall that $\breve{\phi}^\mu = \mu^{-6}\widehat{\phi}^\mu$, where $\widehat{\phi}^\mu$ was defined in Proposition 3.4.21. Moreover, recall from the proof of the same proposition that:

$$\left|\widehat{\phi}^\mu - \widehat{\xi}^\mu\right|_{\widehat{\xi}^\mu} \leqslant (4C+1)\varepsilon$$

for some constant $C > 0$ independent of μ, where $\widehat{\xi}^\mu$ is given by:

$$\widehat{\xi}^\mu = \mu^6 \mathrm{d}y^{123} + \mathrm{d}y^{145} + \mathrm{d}y^{167} - \mathrm{d}y^{246} + \mathrm{d}y^{257} + \mathrm{d}y^{347} + \mathrm{d}y^{356}.$$

Hence, by applying a simple rescaling:

$$\left|\breve{\phi}^\mu - \widehat{\xi}^\mu\right|_{\widehat{\xi}^\mu} \leqslant (4C+1)\varepsilon,$$

where $\widehat{\xi}^\mu = \mu^{-6}\widehat{\xi}^\mu$ as in §3.5.4. It follows from eqn. (3.5.38) that for all $\varepsilon > 0$ sufficiently small (independent of μ) the metric g^μ induced by $\breve{\phi}^\mu$ satisfies:

$$\left|g^\mu - g_{\widehat{\xi}^\mu}\right|_{g_{\widehat{\xi}^\mu}} \leqslant C'\varepsilon$$

for some constant $C' > 0$ independent of μ. By applying Lemma 3.5.39, it follows that on $\breve{\mathrm{M}}_{int}$:

$$g^{\mu} \geqslant (1 - C'\varepsilon)\, g_{\widehat{\xi}^{\mu}}.$$

However an explicit calculation shows that:

$$g_{\widehat{\xi}^{\mu}} = \left(\left(\mathrm{d}y^1\right)^{\otimes 2} + \left(\mathrm{d}y^2\right)^{\otimes 2} + \left(\mathrm{d}y^3\right)^{\otimes 2}\right) + \mu^{-6}\left(\left(\mathrm{d}y^4\right)^{\otimes 2} + \left(\mathrm{d}y^5\right)^{\otimes 2} + \left(\mathrm{d}y^6\right)^{\otimes 2} + \left(\mathrm{d}y^7\right)^{\otimes 2}\right)$$

and thus:

$$g_{\widehat{\xi}^{\mu}} \geqslant \left(\mathrm{d}y^3\right)^{\otimes 2} = \left(\mathrm{d}x^3\right)^{\otimes 2} = \breve{\mathfrak{f}}^* g_{\mathrm{Eucl}}.$$

Thus on $\breve{\mathrm{M}}_{int}$:

$$g^{\mu} \geqslant (1 - C'\varepsilon)\, \breve{\mathfrak{f}}^* g_{\mathrm{Eucl}}.$$

Now recall that $\upsilon < 1$ is independent of μ and t and hence also independent of ε. Thus for ε sufficiently small independent of μ, one has that:

$$1 - C'\varepsilon \geqslant \upsilon^{\frac{4}{3}}.$$

Thus, reducing ε equally for all μ if necessary, one has that on $\breve{\mathrm{M}}_{int}$:

$$g^{\mu} \geqslant \upsilon^{\frac{4}{3}} \breve{\mathfrak{f}}^* g_{\mathrm{Eucl}}. \tag{3.5.49}$$

$\underline{\coprod_{\mathbf{a}\in\mathfrak{A}} \breve{W}_{\mathbf{a}}}$: Fix some $\breve{W}_{\mathbf{a}}$, $\mathbf{a} \in \mathfrak{A}$, and as for point 3 begin by considering the homothetic region $T_{\mu} \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right)$ equipped with the G_2 3-form $\zeta^{\mu} = \zeta + \mu^{-3}\sigma$ (cf. Lemma 3.4.28).

Initially, consider the region outside $T_{\mu} \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$, i.e. the region $\left\{r \geqslant \frac{1}{2}\varepsilon\right\}$. On this region $\zeta = \hat{\xi}$ is just the standard (Euclidean) G_2 3-form in the coordinates $\left(y^i\right)$ and so:

$$g_{\zeta} \geqslant \left(\mathrm{d}y^3\right)^{\otimes 2} = f^* g_{\mathrm{Eucl}},$$

where f denotes the composite:

$$T_{\mu} \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon, \mu^{-1}\right) \xrightarrow{proj_1} T_{\mu} \xrightarrow{\left(y^3, y^4, y^7\right) \mapsto y^3} \mu^3 S^1,$$

as above. Moreover, recall from Lemma 3.4.28 that outside $T_{\mu} \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$:

$$\left\|\zeta^{\mu} - \zeta\right\|_{\zeta} \leqslant \frac{1}{2}\varepsilon.$$

and thus by eqn. (3.5.38), for all $\varepsilon > 0$ sufficiently small (independent of μ):

$$\left\|g_{\zeta^{\mu}} - g_{\zeta}\right\|_{\zeta} \leqslant \frac{\Delta_0 \varepsilon}{2}.$$

Hence by applying Lemma 3.5.39:

$$g_{\zeta^\mu} \geqslant \left(1 - \frac{\Delta_0 \varepsilon}{2}\right) g_\zeta \geqslant \left(1 - \frac{\Delta_0 \varepsilon}{2}\right) f^* g_{\mathrm{Eucl}}.$$

As before, reducing ε equally for all μ if necessary, of may assume that $1 - \frac{\Delta_0 \varepsilon}{2} \geqslant \upsilon^{\frac{4}{3}}$ and thus that:

$$g_{\zeta^\mu} \geqslant \upsilon^{\frac{4}{3}} f^* g_{\mathrm{Eucl}} \tag{3.5.50}$$

outside $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$.

Now consider the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$. By Lemma 3.1.1, on this region one can write:

$$g_\zeta = \nu^{\frac{4}{3}} \left(\mathrm{d}y^3\right)^{\otimes 2} + \nu^{-\frac{2}{3}} \left[\left(\mathrm{d}y^4\right)^{\otimes 2} + \left(\mathrm{d}y^7\right)^{\otimes 2}\right] + \nu^{-\frac{2}{3}} g_{\breve{\omega}_t}$$

where ν is defined by the equation:

$$\nu^2 \breve{\omega}_t^2 = \mathfrak{Re}\widetilde{\Omega}^2 = \mathfrak{Im}\widetilde{\Omega}^2.$$

Then clearly:

$$g_\zeta \geqslant \nu^{\frac{4}{3}} \left(\mathrm{d}y^3\right)^{\otimes 2} \geqslant \upsilon^{\frac{4}{3}} \left(\mathrm{d}y^3\right)^{\otimes 2}, \tag{3.5.51}$$

where the final equality follows from the fact that υ was defined precisely as the minimum value of ν. Now recall that on the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$ one has:

$$\left\| \frac{1}{\mu^3} \sigma \right\|_\zeta \leqslant \frac{C}{\mu^3}$$

for some $C > 0$ independent of μ. Thus, by eqn. (3.5.38), for all μ sufficiently large one can write:

$$\left\| g_{\zeta^\mu} - g_\zeta \right\|_{g_\zeta} \leqslant \frac{C \Delta_0}{\mu^3}. \tag{3.5.52}$$

Applying Lemma 3.5.39 yields:

$$g_{\zeta^\mu} \geqslant \left(1 - \frac{C\Delta_0}{\mu^3}\right) g_\zeta \geqslant \left(1 - \frac{C\Delta_0}{\mu^3}\right) \upsilon^{\frac{4}{3}} f^* g_{\mathrm{Eucl}} \tag{3.5.53}$$

on the region $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right)$. Combining eqns. (3.5.50) and (3.5.53) then yields the estimate:

$$g_{\zeta^\mu} \geqslant \left(1 - \frac{C\Delta_0}{\mu^3}\right) \upsilon^{\frac{4}{3}} f^* g_{\mathrm{Eucl}} \tag{3.5.54}$$

on all of $T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right)$.

Now recall the homothety $\breve{W}_{\mathbf{a}} \cong T \times \widetilde{X}\left(\frac{1}{2}\varepsilon\right) \xrightarrow{\mathfrak{H}^\mu} T_\mu \times \widetilde{X}\left(\frac{1}{2}\varepsilon, \mu^{-1}\right)$ and recall from eqn. (3.4.30) that the 3-form $\breve{\phi}^\mu$ on $\breve{W}_{\mathbf{a}}$ is defined by:

$$\breve{\phi}^\mu = \mu^{-9} \left(\mathfrak{H}^\mu\right)^* \zeta^\mu.$$

Note also that the diagram:

$$\begin{array}{ccc} \breve{W}_{\mathbf{a}} \cong T \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon\right) & \xrightarrow{\mathfrak{H}^{\mu}} & T_{\mu} \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon, \mu^{-1}\right) \\ \downarrow{\scriptstyle \breve{\mathfrak{f}}} & & \downarrow{\scriptstyle f} \\ S^1 & \xrightarrow{\times \mu^3} & \mu^3 S^1 \end{array}$$

commutes. Applying the homothety $\mathfrak{H}^{\mu}$ to eqn. (3.5.54) yields:

$$g^{\mu} = g_{\breve{\Phi}^{\mu}} \geqslant \left(1 - \frac{C\Delta_0}{\mu^3}\right) \upsilon^{\frac{4}{3}} \breve{\mathfrak{f}}^* g_{\text{Eucl}}$$

on the region $\breve{W}_{\mathbf{a}}$.

Combining this with eqn. (3.5.48) and (3.5.49) yields:

$$g^{\mu} \geqslant \left(1 - \frac{C\Delta_0}{\mu^3}\right) \upsilon^{\frac{4}{3}} \breve{\mathfrak{f}}^* g_{\text{Eucl}} \tag{3.5.55}$$

on all of $\breve{\mathrm{M}}$, hence:

$$d^{\mu} \geqslant \sqrt{1 - \frac{C\Delta_0}{\mu^3}} d_k$$

and whence:

$$d_k - d^{\mu} \leqslant \frac{C\Delta_0}{\mu^3}, \tag{3.5.56}$$

since manifestly $d_k(x,y) \leqslant \upsilon^{\frac{4}{3}} < 1$ for all $x, y \in \breve{\mathrm{M}}$. Thus, the quantity $d_k - d^{\mu}$ has been bounded above uniformly on all of $\breve{\mathrm{M}}$.

Now fix $\mathbf{a} \in \mathfrak{A}$ and turn attention to bounding the quantity $d^{\mu} - d_k$ from above on the subset $\partial_{\mathbf{a},k}$. Recall the distance $\mathfrak{r}$ defined in Proposition 3.5.19. For each $p \in S^1$, define a point $\breve{p} \in \breve{W}_{\mathbf{a},k}$ using the local coordinates $(y^1, ..., y^7)$ on $\breve{W}_{\mathbf{a},k}$ as:

$$\breve{p} = \left(\frac{\mathfrak{r}}{\mu^3}, 0, p, 0, 0, 0, 0\right) \in \breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \breve{W}_{\mathbf{a},\mu} \subseteq \breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \breve{W}_{\mathbf{a},k},$$

where the final membership holds when $\mu \geqslant k$. The first task is to understand the distance between points of the form $\breve{p}$ with respect to the metric d^{μ}.

Given $p, q \in S^1$, pick the shorter segment γ connecting $p \to q$ in S^1 and use it to define a path $\breve{\gamma}$ in $\breve{W}_{\mathbf{a},\mu}$ connecting $\breve{p} \to \breve{q}$ via:

$$\breve{\gamma} = \left(\frac{\mathfrak{r}}{\mu^3}, 0, \gamma, 0, 0, 0, 0\right).$$

Then clearly:

$$d^{\mu}(\breve{p}, \breve{q}) \leqslant \ell_{g^{\mu}}(\breve{\gamma}).$$

To compute $\ell_{g^{\mu}}(\breve{\gamma})$, consider the homothety $\breve{W}_{\mathbf{a},\mu} \cong T \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon, \mu\right) \xrightarrow{\mathfrak{H}^{\mu}} T_{\mu} \times \widetilde{X}\left(\tfrac{1}{2}\varepsilon\right)$ and recall that

$\breve{\phi}^{\mu} = \mu^{-9}\left(\mathfrak{H}^{\mu}\right)^{*}\zeta^{\mu}$. Under $\mathfrak{H}^{\mu}$, $\breve{\gamma}$ becomes the path:

$$\breve{\gamma}' = \left(\mathfrak{r}, 0, \mu^{3}\gamma, 0, 0, 0, 0\right).$$

Begin by considering the G_2 3-form ζ. This induces the metric:

$$g_{\zeta} = \nu^{\frac{4}{3}}\left(\mathrm{d}y^{3}\right)^{\otimes 2} + \nu^{-\frac{2}{3}}\left[\left(\mathrm{d}y^{4}\right)^{\otimes 2} + \left(\mathrm{d}y^{7}\right)^{\otimes 2}\right] + \nu^{-\frac{2}{3}}g_{\widetilde{\omega}_t}$$

where $\nu|_{r = \mathfrak{r}} = \upsilon$. The length of $\breve{\gamma}'$ with respect to the metric induced by ζ is then clearly $\upsilon^{\frac{2}{3}}\mu^{3}d_{\mathrm{Eucl}}(p,q)$. Applying Lemma 3.5.39 to eqn. (3.5.52) once again yields:

$$g_{\zeta^{\mu}} \leqslant \left(1 + \frac{C\Delta_0}{\mu^{3}}\right)g_{\zeta}. \tag{3.5.57}$$

and so:

$$\ell_{g_{\zeta^{\mu}}}\left(\breve{\gamma}'\right) \leqslant \sqrt{1 + \frac{C\Delta_0}{\mu^{3}}}\,\upsilon^{\frac{2}{3}}\mu^{3}d_{\mathrm{Eucl}}(p,q).$$

Pulling this equation back along the homothety $\mathfrak{H}^{\mu}$ (and rescaling by μ^{-3}) gives:

$$\begin{aligned} d^{\mu}\left(\breve{p},\breve{q}\right) &\leqslant \ell_{g^{\mu}}\left(\breve{\gamma}\right) \\ &\leqslant \sqrt{1 + \frac{C\Delta_0}{\mu^{3}}}\,\upsilon^{\frac{2}{3}}d_{\mathrm{Eucl}}(p,q) \\ &\leqslant d_k\left(\breve{p},\breve{q}\right) + \frac{C\Delta_0}{\mu^{3}}, \end{aligned} \tag{3.5.58}$$

where again, I have used $\upsilon^{\frac{2}{3}}d_{\mathrm{Eucl}}(p,q) \leqslant 1$ for all $p,q \in S^1$.

Now pick two arbitrary points $x,y \in \partial_k$. Define $p = \breve{\mathfrak{f}}(x)$ and $q = \breve{\mathfrak{f}}(y)$. Then clearly:

$$\begin{aligned} d^{\mu}(x,y) &\leqslant \underbrace{d^{\mu}\left(x,\breve{p}\right)}_{\leqslant \operatorname{diam}_{d^{\mu}}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \breve{W}_{\mathbf{a},k}\right]} + d^{\mu}\left(\breve{p},\breve{q}\right) + \underbrace{d^{\mu}\left(\breve{q},y\right)}_{\leqslant \operatorname{diam}_{d^{\mu}}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{q\}) \cap \breve{W}_{\mathbf{a},k}\right]} \\ &\leqslant d^{\mu}\left(\breve{p},\breve{q}\right) + 2\sup_{s \in S^1}\operatorname{diam}_{d^{\mu}}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \breve{W}_{\mathbf{a},k}\right] \\ &\leqslant d_k(x,y) + \frac{C\Delta_0}{\mu^{3}} + 2\sup_{s \in S^1}\operatorname{diam}_{d^{\mu}}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \breve{W}_{\mathbf{a},k}\right], \end{aligned} \tag{3.5.59}$$

where eqn. (3.5.58) was used in passing to the final line. Combining eqns. (3.5.56) and (3.5.59) gives:

$$\left|d^{\mu}(x,y) - d_k(x,y)\right| \leqslant \frac{C\Delta_0}{\mu^{3}} + 2\sup_{s \in S^1}\operatorname{diam}_{d^{\mu}}\left[\breve{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \breve{W}_{\mathbf{a},k}\right]$$

uniformly in x and y. Taking max over $\mathbf{a} \in \mathfrak{A}$, $\limsup$ over $\mu \to \infty$ and subsequently the limit over $k \to \infty$, together with point 2 from Proposition 3.5.43 gives the required result.

$(5ii)$ The argument in this case is similar but easier. Firstly, by taking the (pointwise) limit of eqn.

(3.5.55) as $\mu \to \infty$, one obtains that on $\widehat{\mathrm{M}} \backslash \coprod_{\mathbf{a} \in \mathfrak{A}} \widehat{\mathfrak{W}}_{\mathbf{a}}$:

$$g^{\infty} \geqslant \upsilon^{\frac{4}{3}} \widehat{\mathfrak{f}}^{*} g_{\mathrm{Eucl}}$$

and hence by continuity this inequality holds on all of $\widehat{\mathrm{M}} \backslash \widehat{S}$. Moreover, for each $\mathbf{a} \in \mathfrak{A}$:

$$g_{\mathbf{a}}^{\infty} = \upsilon^{\frac{4}{3}} \widehat{\mathfrak{f}}^{*} g_{\mathrm{Eucl}}.$$

It follows that:

$$d^{\infty} \geqslant d_{k} \tag{3.5.60}$$

on all of $\widehat{\mathrm{M}}$.

For the converse bound, given $x, y \in \partial_{k}$, define $p = \widehat{\mathfrak{f}}(x)$, $q = \widehat{\mathfrak{f}}(y)$ and consider the points:

$$\widehat{p} = (0,0,p,0,0,0,0) \quad \text{and} \quad \widehat{q} = (0,0,q,0,0,0,0) \in \widehat{W}_{\mathbf{a},k}.$$

Then clearly:

$$\begin{aligned} d^{\infty}(x,y) \leqslant & \underbrace{d^{\infty}(x,\widehat{p})}_{\leqslant \operatorname{diam}_{d^{\infty}}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{p\}) \cap \widehat{W}_{\mathbf{a},k}\right]} + d^{\infty}(\widehat{p},\widehat{q}) + \underbrace{d^{\infty}(\widehat{q},y)}_{\leqslant \operatorname{diam}_{d^{\infty}}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{q\}) \cap \widehat{W}_{\mathbf{a},k}\right]} \\ \leqslant & \, d^{\infty}(\widehat{p},\widehat{q}) + 2 \sup_{s \in S^{1}} \operatorname{diam}_{d^{\infty}}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \widehat{W}_{\mathbf{a},k}\right]. \end{aligned}$$

However $d^{\infty}(\widehat{p},\widehat{q})$ can easily be bounded as follows: choose the shorter segment $\gamma : p \to q$ in S^{1}. This defines a path $\widehat{\gamma}$ from $\widehat{p}$ to $\widehat{q}$ in $\widehat{W}_{\mathbf{a},k}$ via $(0,0,\gamma,0,0,0,0)$ which has length $\upsilon^{\frac{2}{3}} \mathrm{d}_{\mathrm{Eucl}}(p,q) = d_{k}(x,y)$ with respect to the $g_{\mathbf{a}}^{\infty}$. Thus:

$$d^{\infty}(x,y) \leqslant d_{k}(x,y) + 2 \sup_{s \in S^{1}} \operatorname{diam}_{d^{\infty}}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \widehat{W}_{\mathbf{a},k}\right]. \tag{3.5.61}$$

Finally, combining eqns. (3.5.60) and (3.5.61) gives:

$$\left|d^{\infty}(x,y) - d_{k}(x,y)\right| \leqslant 2 \sup_{s \in S^{1}} \operatorname{diam}_{d^{\infty}}\left[\widehat{\mathfrak{f}}_{\mathbf{a}}^{-1}(\{s\}) \cap \widehat{W}_{\mathbf{a},k}\right]$$

uniformly in $x, y \in \partial_{\mathbf{a},k}$. Taking max over $\mathbf{a} \in \mathfrak{A}$, $\limsup$ over $\mu \to \infty$ and subsequently the limit over $k \to \infty$, together with point 4 of Proposition 3.5.43 gives the required result.

$\square$

Chapter 4

Convergence and collapsing result for orbifolds

This chapter proves a general collapsing result (Theorem 4.2.5) for families of stratified Riemannian metrics $\widehat{g}^{\mu}$ on a compact orbifold E, subject to suitable limiting conditions on the metrics $\widehat{g}^{\mu}$ as $\mu \to \infty$, which subsumes Theorems 3.3.2 and Theorem 3.5.11. The result is distinct from similar theorems in the literature since it does not require bounds on curvature or injectivity radius of $(E, \widehat{g}^{\mu})$ and thus allows for Gromov–Hausdorff limits of $(E, \widehat{g}^{\mu})$ which have strictly lower dimension than E. The chapter also introduces and studies a new class of stratified fibrations between orbifolds, termed weak submersions, which play a key role in the proof of the main theorem.

4.1 Stratified fibrations between orbifolds

Following [27, §3.2], I call a smooth map $f : E \to B$ a submersion if for all $e \in E$, there exists a chart Ξ_e centred at e, a chart $\Xi_{f(e)}$ centred at $f(e)$, and a local representation $(\widetilde{f}, \kappa_f)$ in these charts such that $\widetilde{f}$ is submersive and κ_f is surjective. The following result may not, to the author's knowledge, be found in the literature. It is the analogue of Ehresmann's Theorem for orbifolds (see e.g. [122, Thm. 9.3] for the classical statement):

Proposition 4.1.1. *Let E, B be orbifolds and let $\pi : E \to B$ a proper, surjective submersion. Let* $\ker \mathrm{d}\pi$ *be the vertical distribution of π, pick a Riemannian metric g on E and let $\mathcal{C} = \ker \mathrm{d}\pi^{\perp}$ be the corresponding horizontal distribution.*

1. *Let $\gamma : (-1, 1) \to B$ be an embedded curve. Then $\pi^{-1}(\gamma(-1, 1)) \subseteq E$ is a suborbifold, denoted E_γ. Write $E_0 = \pi^{-1}(\gamma(0))$. Then there is an orbifold diffeomorphism:*

$$E_\gamma \cong E_0 \times \gamma(-1, 1)$$

identifying π with projection onto the second factor and $\mathcal{C}$ with the product connection;

2. *Let $U \subseteq B$ be an open ball and write $E_U = \pi^{-1}(U) \subseteq E$. Write b for the centre of the ball U and write $E_b = \pi^{-1}(b)$. Then E_U is an a suborbifold of E and there is an orbifold diffeomorphism:*

$$E_U \cong E_b \times U$$

identifying π with projection onto the second factor (although the identification does not identify $\mathcal{C}$ with the product distribution, in general).

To prove Proposition 4.1.1, I begin by recording the following equivariant version of the Implicit Function Theorem:

Proposition 4.1.2. *Let $\Gamma_i \subset \mathrm{GL}(n_i;\mathbb{R})$ be finite subgroups, $i = 1,2$, and let $\widetilde{U} \subseteq \mathbb{R}^{n_1}$ be a Γ_1-invariant open neighbourhood of 0. Suppose one is given a group homomorphism $\iota : \Gamma_1 \to \Gamma_2$ and a smooth map $f : \widetilde{U} \to \mathbb{R}^{n_2}$ which is ι-equivariant, which maps $0 \in \mathbb{R}^{n_1} \mapsto 0 \in \mathbb{R}^{n_2}$ and has surjective derivative at 0. Write $\mathbb{K} = \ker(\mathrm{d}f|_0)$, a Γ_1-invariant subspace of $\mathbb{R}^{n_1}$ of dimension $n_1 - n_2$ and let $\mathbb{T}$ be any Γ_1-invariant complementary subspace to $\mathbb{K}$ in $\mathbb{R}^{n_1}$.*

Then (shrinking $\widetilde{U}$ if necessary) there is a Γ_1-equivariant diffeomorphism:

$$F : \widetilde{U} \subseteq \mathbb{K} \times \mathbb{T} \to F\left(\widetilde{U}\right) \subseteq \mathbb{K} \times \mathbb{R}^{n_2}$$

(where Γ_1 acts on $\mathbb{R}^{n_2}$ via the map $\iota : \Gamma_1 \to \Gamma_2$) such that $\mathrm{d}F|_0$ identifies $\mathbb{T}$ with $\mathbb{R}^{n_2}$ and such that the diagram:

$$\begin{tikzcd}
\widetilde{U} \subseteq \mathbb{R}^{n_1} \arrow[rr, "F"] \arrow[dr, "f"'] & & F\left(\widetilde{U}\right) \subseteq \mathbb{K} \times \mathbb{R}^{n_2} \arrow[dl, "\pi_2"] \\
& \mathbb{R}^{n_2} &
\end{tikzcd}$$

commutes and is equivariant. In particular, if ι is surjective, then the bottom arrow in the following diagram is an isomorphism:

$$\begin{tikzcd}
0 \times \mathbb{T} \arrow[r, "\mathrm{d}f|_0"] \arrow[d, "proj"] & \mathbb{R}^{n_2} \arrow[d, "proj"] \\
\Gamma_1 \backslash (0 \times \mathbb{T}) \arrow[r] & \Gamma_2 \backslash \mathbb{R}^{n_2}
\end{tikzcd}$$

The proof is a simple application of the Inverse Function Theorem, and so it is omitted here.

Proof of Proposition 4.1.1. That E_γ and E_U are suborbifolds of E follows at once from the local description of π afforded by Proposition 4.1.2. Moreover, Proposition 4.1.2 shows that for each $e \in E$, $\mathrm{d}\pi|_e$ defines an isomorphism $\mathcal{C}_e \to \mathrm{T}_{\pi(e)}B$.

Now consider (1). Choose a point $e \in E_0$. The derivative of γ defines a natural map $\dot{\gamma} : (-1,1) \to \mathrm{T}B|_\gamma$. Using $\mathrm{d}\pi$, one may lift this uniquely to a map $(-1,1) \to \mathcal{C}$; integrating this vector field along $(-1,1)$ defines the horizontal lift of γ starting from e, denoted γ_e. Now define a map:

$$\begin{aligned} E_0 \times (-1,1) &\to E_\gamma \\ (e,t) &\mapsto \gamma_e(t). \end{aligned}$$

One may verify that this is the required diffeomorphism. Given (1), (2) follows as for manifolds, by trivialising π along radial paths emanating from b.

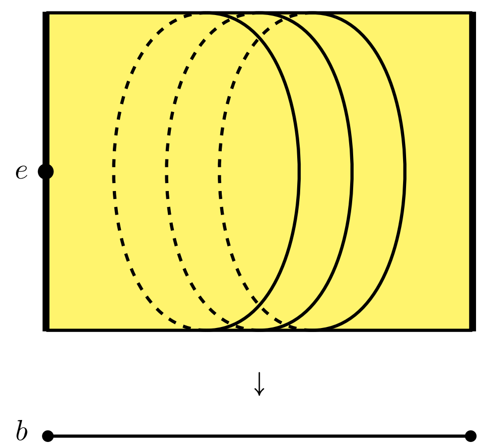
∎

In general, the requirement that the homomorphisms κ_f be surjective can be rather strict. The following definition relaxes this condition:

Definition 4.1.3. A smooth map of orbifolds $f : E \to B$ is a weak submersion if for all $e \in E$, there exists a chart Ξ_e centred at e, a chart $\Xi_{f(e)}$ centred at $f(e)$, and a local representation $(\widetilde{f}, \kappa_f)$ in these charts such that $\widetilde{f}$ is submersive. (In particular, κ_f is not assumed to be surjective.)

Clearly Proposition 4.1.1 does not apply to weak submersions in general, as the following example illustrates:

Example 4.1.4. Let $E = {\{\pm 1\}}\backslash{\mathbb{T}^2}$, $B = {\{\pm 1\}}\backslash{[\mathbb{T}^1 \times \{0\}]}$ and let $\pi : E \to B$ denote the canonical projection. Then E is the 'pillowcase', homeomorphic to a 2-sphere, with singular points precisely the four corners of the 'pillowcase' and B is a closed interval, with singular points precisely the endpoints of the interval.



(4.1.5)

π is a surjective, proper, weak submersion, however whilst the preimage of a smooth point in B is topologically a circle, the preimage of a singular point is topologically a closed interval. Thus π is not a locally-trivially fibration (or even a Serre fibration, since the homotopy groups of its fibres are not constant over the connected base space). Note also that the points e and b satisfy $\Gamma_e = \mathbf{1}$ and $\Gamma_b = {\mathbb{Z}}/{2}$. Thus $\mathrm{d}\pi_e : \mathcal{C}_e \to \mathrm{T}_b B$ is not an isomorphism (in fact, it is a 2:1 quotient).

Nevertheless, a stratified version of Proposition 4.1.1 still holds for weak submersions:

Corollary 4.1.6. *Let E and B be orbifolds and let $\pi : E \to B$ be a proper, surjective, weak submersion and let $\Sigma(B) = \{B_i\}$ be a stratification of B. For each B_i, the subset $\pi^{-1}(B_i) \subseteq E$ is a suborbifold and the restriction of π to $\pi^{-1}(B_i) \subseteq E$ defines a submersion onto the submanifold $B_i \subseteq B$ in the usual orbifold sense. Let $\left\{E_i^j\right\}_j$ denote the strata in the canonical stratification of $\pi^{-1}(B_i)$. Then $\pi : E_i^j \to B_i$ is a surjective submersion for all i, j. Moreover, the collection $\left\{E_i^j\right\}_{i,j}$ define a stratification of E, denoted $\Sigma(\pi, E, B)$ with respect to which the distribution $\mathcal{D} = \ker(\mathrm{d}\pi)$ is a stratified distribution. In particular, the orthocomplement of the vertical distribution (with respect to any Riemannian metric) is also stratified.*

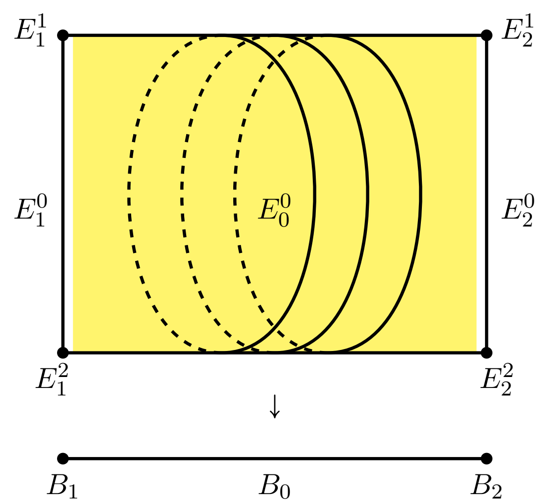
Proof. Firstly, note that any weak submersion from an orbifold to a manifold is (trivially) a submersion, since any group homomorphism to the trivial group is surjective. The rest of the corollary follow simply by working in appropriate local charts as in the proof of Proposition 4.1.1. For example, to see that the distribution $\mathcal{D}$ is stratified with respect to the induced stratification $\Sigma(\pi, E, B)$, one notes that by Proposition 4.1.1, if one picks a stratum B_i in B and a point $b \in B_i$, then each canonical stratum of the orbifold $\pi^{-1}(B_i)$ is locally given by the product of a union of canonical strata in $\pi^{-1}(b)$ with open discs in the base space B_i; from this, the result is clear.

□

Remark 4.1.7. Note that in the proof above, each canonical stratum of $\pi^{-1}(B_i)$ need not be given locally as the product of a single canonical stratum of $\pi^{-1}(b)$ with an open disc in the base: indeed, consider the Möbius band M (viewed as an orbifold with singular set precisely its boundary) and let $\pi : \mathrm{M} \to S^1$ be the usual projection. Then the stratification of M induced by π is simply its canonical stratification (which has two strata), whereas the canonical stratification of the preimage of any point in the base has 3 strata, the two endpoints being different strata of the preimage, but belonging to the same stratum of M.

More generally, in Proposition 4.1.1, if one writes $\{F_j\}_j$ for the canonical stratification of $F = E_b$ and $\{E_i\}_i$ for the stratification of E induced by π, then the stratification of E_U given by $\{F_j \times U_b\}_j$ is a refinement of the stratification $\{E_i \cap E_U\}_i$. Phrased differently, the stratification of E induced by π is stable under 'horizontal transport maps' used in the proof of Proposition 4.1.1.

Example 4.1.8. Return to eqn. (4.1.5). The stratification $\Sigma(\pi, E, B)$ is depicted below:



4.2 Statement of main result

The purpose of this section is to present a precise statement of the main theorem of this chapter. I begin by introducing the necessary notation:

Notation 4.2.1.

1. Let E_2 be a compact orbifold, let $(B, \Sigma_B = \{B_i\}_i)$ be a stratified orbifold, let $\pi : E_2 \to B$ be a surjective, weak submersion with path-connected fibres and let $\Sigma_2 = \{(E_2)_j\}_j$ be the induced stratification on E_2 (see Corollary 4.1.6). Let $\widehat{g}^\infty$ be a stratified Riemannian semi-metric on E_2 which is regular with respect to the stratified distribution $\ker \mathrm{d}\pi$ and write d^∞ for the semi-metric on E_2 induced by $\widehat{g}^\infty$.
2. Let (E_1, Σ_1) be a second compact, stratified orbifold and, for $i = 1, 2$, let $S_i(j) \subseteq E_i$ $(j = 1, \dots, N)$ be disjoint, closed, subsets. Write:
$$S_i = \coprod_{j=1}^{N} S_i(j)$$
and suppose there is a stratified orbifold diffeomorphism $\Phi : E_1 \backslash S_1 \to E_2 \backslash S_2$.
3. For each $j = 1, \dots, N$, let $\left(U_1^{(r)}(j)\right)_{r \in (0,1]}$ be a family of open neighbourhoods of $S_1(j) \subseteq E_1$ such that $U_1^{(r)} \subseteq U_1^{(s)}$ for $r < s$. Suppose moreover that for all $j \neq j' \in \{1, \dots, N\}$:
$$\overline{U_1^{(1)}(j)} \cap \overline{U_1^{(1)}(j')} = \varnothing. \tag{4.2.2}$$
Write $U_2^{(r)}(j) = E_2 \Big\backslash \Phi\left(E_1 \Big\backslash U_1^{(r)}(j)\right)$ for the corresponding nested open neighbourhoods of $S_2(j) \subseteq E_2$, where $\overline{U_2^{(1)}(j)}$ and $\overline{U_2^{(1)}(j')}$ are, again, disjoint for distinct j and j'. Write:
$$U_i^{(r)} = \coprod_{j=1}^{N} U_i^{(r)}(j), \quad i = 1, 2.$$
4. For each $j = 1, \dots, N$, let $S(j)$ be a set and let $\mathfrak{f}_{i,j} : U_i^{(1)}(j) \to S(j)$ be surjective maps such that the following diagram commutes:
$$\begin{tikzcd} U_1^{(1)}(j)\Big\backslash S_1(j) \arrow[rr, "\Phi"] \arrow[dr, "\mathfrak{f}_{1,j}"'] & & U_2^{(1)}(j)\Big\backslash S_2(j) \arrow[dl, "\mathfrak{f}_{2,j}"] \\ & S(j) & \end{tikzcd} \tag{4.2.3}$$

Intuitively, one should think of S_1 and S_2 as representing 'singular' regions in E_1 and E_2 respectively. The existence of Φ then asserts that the orbifolds E_1 and E_2 are diffeomorphic 'away from their singular regions' and condition 4 states that, for each j, the singular regions $S_1(j)$ and $S_2(j)$ are 'fibred' over a common base space $S(j)$. Using Φ, I shall henceforth identify $E_1 \backslash S_1$ with $E_2 \backslash S_2$ and write $E^{(r)} = E_1 \backslash U_1^{(r)} \cong E_2 \backslash U_2^{(r)}$. Similarly, I shall write $\partial^{(r)}(j) = \partial U_1^{(r)}(j) \cong \partial U_2^{(r)}(j)$.

Remark 4.2.4. In the case of the manifold $\check{\mathrm{M}}$ considered in Chapter 3, the above definition takes the following concrete form:

1. $E_1 = \check{\mathrm{M}}$ (a manifold), $E_2 = \widehat{\mathrm{M}}$, Σ_2 is the stratification $\Sigma = \Sigma\left(\widehat{q}, \widehat{\mathrm{M}}, B\right)$ induced by the submersion $\widehat{q} \colon \widehat{\mathrm{M}} \to B$, and Σ_1 is Σ', the pullback of Σ along ρ.
2. Φ is the restriction of $\rho : \check{\mathrm{M}} \to \widehat{\mathrm{M}}$ to $\check{M} \backslash \check{S}$.

3. The index $j \in \{1, ..., N\}$ is simply $\mathbf{a} \in \mathfrak{A}$ and for $r \in (0,1]$:

$$U_1^{(r)}(\mathbf{a}) = \breve{W}_{\mathbf{a},\frac{1}{r}}.$$

Likewise:

$$U_2^{(r)}(\mathbf{a}) = \widehat{W}_{\mathbf{a},\frac{1}{r}}.$$

4. $S(\mathbf{a}) = S^1$ for all $\mathbf{a}$ and:

$$\mathfrak{f}_{1,\mathbf{a}} = \breve{\mathfrak{f}}_{\mathbf{a}} \quad \text{and} \quad \mathfrak{f}_{2,\mathbf{a}} = \widehat{\mathfrak{f}}_{\mathbf{a}},$$

where $\breve{\mathfrak{f}}_{\mathbf{a}}$ and $\widehat{\mathfrak{f}}_{\mathbf{a}}$ are defined in eqn. (3.5.10).

5. For each $\mu \in [1, \infty)$, take $\widehat{g}^{\mu}$ to be the stratified Riemannian metric obtained from $g_{\mu^{-6}\breve{\phi}^{\mu}}$ by the refinement procedure detailed in Remark 3.5.7.

Theorem 4.2.5.

1. Fix a stratum B_i in B, write $\pi^{-1}(B_i) = \bigcup_{l=0}^{k} (E_2)_{j(l)}$ and write $g_{j(l)}$ for the component of $\widehat{g}^{\infty}$ on the stratum $(E_2)_{j(l)}$. Define a map $\mathcal{L}_i : \mathrm{T}B_i \to \mathbb{R}$ as follows: given $p \in B_i$ and $u \in \mathrm{T}_p B_i$, define

$$\mathcal{L}_i(u) = \min_{l=0}^{k} \inf_{x \in (E_2)_{j(l)} \cap \pi^{-1}(p)} \left\{ \|u'\|_{g_{j(l)}} \;\middle|\; u' \in \mathrm{T}_x (E_2)_{j(l)} \text{ such that } \mathrm{d}\pi(u') = u \right\}. \tag{4.2.6}$$

Then $\widehat{\mathcal{L}} = \{\mathcal{L}_i\}_i$ defines a stratified quasi-Finslerian structure on B (see Definition 3.5.3) and $\left(B, \widehat{\mathcal{L}}\right)$ is the free metric space on $(E_2, \widehat{g}^{\infty})$.

2. Now suppose further that $(\widehat{g}^{\mu})_{\mu \in [1,\infty)}$ are stratified Riemannian metrics on E_1 inducing metrics d^{μ}, such that the following 4 conditions are satisfied:

(i) For all $r \in (0,1]$:

$$\widehat{g}^{\mu} \to \widehat{g}^{\infty} \text{ uniformly as } \mu \to \infty \text{ on the space } E^{(r)}$$

and there exist constants $\Lambda_{\mu}(r) \geqslant 0$ such that:

$$\lim_{\mu \to \infty} \Lambda_{\mu}(r) = 1 \quad \text{and} \quad \widehat{g}^{\mu} \geqslant \Lambda_{\mu}(r)^2 \widehat{g}^{\infty} \text{ on } E^{(r)} \text{ for all } \mu \in [1, \infty); \tag{4.2.7}$$

(ii)

$$\limsup_{r \to 0} \limsup_{\mu \to \infty} \max_{j \in \{1,\ldots,N\}} \sup_{p \in S(j)} \mathrm{diam}_{d^{\mu}} \left[\mathfrak{f}_{1,j}^{-1}(\{p\}) \cap U_1^{(r)}(j) \right] = 0;$$

(iii)

$$\limsup_{r \to 0} \max_{j \in \{1,\ldots,N\}} \sup_{p \in S} \mathrm{diam}_{d^{\infty}} \left[\mathfrak{f}_{2,j}^{-1}(\{p\}) \cap U_2^{(r)}(j) \right] = 0;$$

(iv)
$$\limsup_{r\to 0}\ \limsup_{\mu\to\infty}\ \max_{j\in\{1,\dots,N\}}\ \sup_{\partial^{(r)}(j)}\left|d^\mu - d^\infty\right| = 0.$$

Then:
$$\left(E_1, d^\mu\right) \to \left(B, \widehat{\mathcal{L}}\right) \quad \text{as} \quad \mu \to \infty,$$

in the Gromov–Hausdorff sense.

(By '$\widehat{g}^\mu \to \widehat{g}^\infty$ uniformly on $E^{(r)}$', I mean that for each fixed reference stratified Riemannian metric $\widehat{h}$ on E_1, one has:
$$\left\|g_i^\mu - g_i^\infty\right\|_{h_i} \to 0 \quad \text{as} \quad \mu \to \infty \text{ on } \left(E_1\right)_i \cap E^{(r)}.$$

Since any two stratified Riemannian metrics on a compact orbifold are uniformly Lipschitz equivalent, this definition is independent of the choice of $\widehat{h}$.)

Remark 4.2.8. Note that if the bilinear form $\widehat{g}^\infty$ is given on each stratum $\left(E_2\right)_j$ lying over B_i as $\pi^* h_i$, where $\widehat{h} = \{h_i\}_i$ is a stratified Riemannian metric on B (write $\widehat{g}^\infty = \pi^*\widehat{h}$), then $\widehat{\mathcal{L}} = \widehat{h}$ and $\left(B, d^{\widehat{h}}\right)$ is clearly the free metric space on $\left(E_2, d^\infty\right)$. More generally, if $\widehat{g}^\infty = \pi^*\widehat{h}$ on some proper subset U of E, then once again it is clear that $\widehat{\mathcal{L}} = \widehat{h}$ over U, since the value of $\widehat{\mathcal{L}}$ at a point $b \in B$ only depends on the values of $\widehat{g}^\infty$ on the fibre over b. However due to the global definition of the metrics d_B and $d^{\widehat{\mathcal{L}}}$, the assumption $\widehat{g}^\infty = \pi^*\widehat{h}$ on U provides no simplification and the proof that $d_B = d^{\widehat{\mathcal{L}}}$ – far from being trivial – assumes its general form in this case.

Note also that Theorem 4.2.5 clearly subsumes Theorems 3.3.2 and 3.5.11, as stated in Chapter 3.

Intuitively, condition (i) states that the metrics $\widehat{g}^\mu$ on E_1 converge locally uniformly away from the singular region S_1 to a stratified Riemannian semi-metric $\widehat{g}^\infty$, which extends to some given compactification E_2 of $E_1\backslash S_1$; conditions (ii) and (iii) state that the fibres of the maps $f_{i,j} : U_i^{(r)}(j) \to S(j)$ are 'small' with respect to $\widehat{g}^\mu$ and $\widehat{g}^\infty$ respectively, provided that μ is sufficiently large and r is sufficiently small and, finally, condition (iv) states that the two metrics d^μ and d^∞ approximately agree near the singular regions S_1 and S_2. Further intuition can be gained by considering the case where the map $\Phi : E_1\backslash S_1 \to E_2\backslash S_2$ admits a non-smooth extension to a map $\overline{\Phi} : E_1 \to E_2$. In this case, by considering the composite $\pi \circ \overline{\Phi} : E_1 \to B$, one can regard E_1 as fibred over the space B and Theorem 4.2.5 can then be regarded as a 'collapsing' theorem for this fibration which states, informally, that if the diameter of the fibres of $E_1 \to B$ (with respect to the metrics $\widehat{g}^\mu$) tends to zero away from some singular set S_1 and if the limiting size of the region S_1 is 'not too large', then the orbifold E_1 collapses to the orbifold B in the limit as $\mu \to \infty$. The use of Theorem 4.2.5 in Chapter 3 is an example of such an application.

The proof of Theorem 4.2.5 occupies the rest of this chapter. I begin by explaining why the hypotheses of Theorem 4.2.5 are necessary.

Firstly, let me explain why it is necessary to use quasi-Finslerian structures, rather than the more usual Finslerian or Riemannian structures, to describe the free metric space on $\left(E_2, \widehat{g}^\infty\right)$. Consider the following simple (unstratified) example, where $\lambda > 2$ is a constant:

$$\pi : E = \mathbb{T}^2_{\theta^1,\theta^2} \times \mathbb{T}^1_\alpha \xrightarrow{proj} \mathbb{T}^2_{\theta^1,\theta^2} = B$$
$$g = \left(1 + \lambda \cos^2 \alpha\right) \left(\mathrm{d}\theta^1\right)^{\otimes 2} + \left(1 + \lambda \sin^2 \alpha\right) \left(\mathrm{d}\theta^2\right)^{\otimes 2}$$

Write $\partial_i = \frac{\partial}{\partial \theta^i}$ $(i = 1, 2)$, let $p = \left(\theta^1, \theta^2, \alpha\right) \in E$ and for each $a, b \in \mathbb{R}$, let $u(a, b)$ be any vector in $\mathrm{T}_p\mathbb{T}^3$ such that $\mathrm{d}\pi|_p(u(a, b)) = a\partial_1 + b\partial_2$. Then:

$$\|u(a, b)\|_g = \sqrt{a^2 + (\lambda + 1)b^2 + \lambda\left(a^2 - b^2\right)\cos^2 \alpha} = \sqrt{(\lambda + 1)a^2 + b^2 + \lambda\left(b^2 - a^2\right)\sin^2 \alpha}$$

and thus:

$$\mathcal{L}(a\partial_1 + b\partial_2) = \begin{cases} \sqrt{a^2 + (\lambda + 1)b^2} & \text{if } |a| \geqslant |b| \\ \sqrt{(\lambda + 1)a^2 + b^2} & \text{if } |a| \leqslant |b|. \end{cases}$$

Whilst this function is continuous, it is not differentiable along $a = b$. Moreover:

$$\mathcal{L}(\partial_1 + \partial_2) = \sqrt{\lambda + 2} > 2 = \mathcal{L}(\partial_1) + \mathcal{L}(\partial_2)$$

and thus $\mathcal{L}$ does not satisfy the triangle inequality. This is the motivation behind the definition of stratified quasi-Finslerian structures in §3.5.1.

Secondly, let me explain why the existence of $\Lambda_\mu(r) \to 1$ in condition (i) is necessary for the second conclusion of Theorem 4.2.5 to be valid. Take $E_1 = E_2 = \mathbb{T}^2_{\theta^1,\theta^2}$ with the trivial (1-stratum) stratifications, let $U_i^{(r)} = S_i = \varnothing$ for $i = 1, 2$, let $\widehat{g}^\infty = \left(\mathrm{d}\theta^1\right)^{\otimes 2}$ and let:

$$\widehat{g}^\mu = \left(1 + \mu^{-1}\right) \left(\mathrm{d}\theta^1\right)^{\otimes 2} - 2\mu^{-1}\mathrm{d}\theta^1 \odot \mathrm{d}\theta^2 + \mu^{-2} \left(\mathrm{d}\theta^2\right)^{\otimes 2}.$$

Since $U_i^{(r)} = S_i = \varnothing$, conditions (ii)–(iv) in Theorem 4.2.5 are automatically satisfied. Moreover, since $\widehat{g}^\mu \to \widehat{g}^\infty$ uniformly as $\mu \to \infty$, condition (i) is also satisfied, expect for the existence of suitable Λ_μ. However $d^\mu \to 0$ uniformly as $\mu \to \infty$. Indeed, for each $a \in [0, 1]$, consider the path:

$$\begin{aligned} \gamma : [0, a] &\to \mathbb{T}^2 \\ s &\mapsto (s, \mu \cdot s). \end{aligned}$$

Then one may calculate that $g^\mu(\dot{\gamma}) = \mu^{-1}$ and thus:

$$d^\mu\left[(0, 0), (a, \mu \cdot a)\right] \leqslant a\mu^{-\frac{1}{2}} \leqslant \mu^{-\frac{1}{2}}.$$

Likewise, by considering a vertical path between (a, b) and $(a, \mu{\cdot}b)$ one sees that $d^\mu\left[(a, b), (a, \mu \cdot a)\right] \leqslant \mu^{-1}$ for any $b \in [0, 1]$ and thus for all $(a, b) \in \mathbb{T}^2$:

$$d^\mu\left[(0, 0), (a, b)\right] \leqslant \mu^{-1} + \mu^{-\frac{1}{2}} \to 0 \quad \text{as} \quad \mu \to \infty.$$

By [25, Example 7.4.4] it follows that $\left(\mathbb{T}^2, \widehat{g}^\mu\right)$ converges to the one-point space as $\mu \to \infty$. However, the free metric space on $\left(\mathbb{T}^2, \widehat{g}^\infty\right)$ is $\left(\mathbb{T}^1_{\theta^1}, \left(\mathrm{d}\theta^1\right)^{\otimes 2}\right)$. This shows the necessity of the existence of the $\Lambda_\mu \to 1$. An analogous phenomenon was observed (albeit from a very different perspective) in [89].

(To see why no such $\Lambda_\mu \to 1$ can exist, suppose that the inequalities $g^\mu \geqslant \Lambda_\mu^2 g^\infty$ held for all $\mu \in [1, \infty)$. Then the bilinear form $g^\mu - \Lambda_\mu^2 g^\infty$ would be non-negative definite and hence would have non-negative determinant, i.e.:

$$\mu^{-2}\left(1 + \mu^{-1} - \Lambda_\mu^2\right) - \mu^{-2} \geqslant 0.$$

This rearranges to $\Lambda_\mu^2 \leqslant \mu^{-1}$ and hence would force $\Lambda_\mu \to 0$ as $\mu \to \infty$.)

4.3 Proof of Theorem 4.2.5: Part 1

The purpose of this section is to prove the first part of Theorem 4.2.5. The reader should note that the orbifold E_1 (and also the sets $S_i(j)$, $U_i^{(r)}(j)$ and their associated data) plays no role in this section. Thus in this section, for simplicity of notation, I denote E_2 by E, Σ_2 by Σ and $\widehat{g}^\infty$ by $\widehat{g}$. The reader should also note that the results of this section remain valid when E is non-compact, provided that the map π is proper.

The first task is to verify that $\widehat{\mathcal{L}}$ is a well-defined stratified quasi-Finslerian structure on B:

Proposition 4.3.1. *For each stratum B_i in B, recall the definition:*

$$\mathcal{L}_i(u) = \min_{l=0}^{k} \inf_{x \in E_{j(l)} \cap \pi^{-1}(p)} \left\{ \|u'\|_{g_{j(l)}} \;\middle|\; u' \in \mathrm{T}_x E_{j(l)} \text{ such that } \mathrm{d}\pi(u') = u \right\}, \tag{4.3.2}$$

for $u \in \mathrm{T}_p B_i$, where $\pi^{-1}(B_i) = \bigcup_{l=0}^k E_{j(l)}$ and $g_{j(l)}$ is the component of $\widehat{g}$ on the stratum $E_{j(l)}$. Then $\widehat{\mathcal{L}} = \{\mathcal{L}_i\}_i$ defines a stratified quasi-Finslerian structure on B.

Proof. Firstly, note that $\mathcal{L}_i$ is well-defined. Indeed $\pi|_{E_{j(l)}} : E_{j(l)} \to B_i$ is a surjective submersion for each $l \in \{0, ..., k\}$ and so for each $x \in E_{j(l)} \cap \pi^{-1}(p)$ there exist vectors $u' \in \mathrm{T}_x E_{j(l)}$ satisfying $\mathrm{d}\pi(u') = u$. Moreover $\|u'\|_{g_{j(l)}}$ is independent of the choice of u', since any two such choices of u' differ by an element of $\ker(\mathrm{d}\pi)|_x \cap \mathrm{T}_x E_{j(l)}$ which is in turn precisely the kernel of $g_{j(l)}$, since $\widehat{g}$ is regular with respect to $\ker \mathrm{d}\pi$.

The proof now breaks into two cases of increasing generality:

Case 1: B is a manifold. Let $p \in B$ and $u \in \mathrm{T}_p B$. It is beneficial to have a preferred choice of preimage u' of u under $\mathrm{d}\pi : \mathrm{T}_x E \to \mathrm{T}_p B$ for each $x \in \pi^{-1}(p)$. To this end, choose a stratified 'horizontal distribution' $\mathcal{C}$, complementary to $\mathcal{D}$. Then (cf. Proposition 4.1.1):

$$\mathrm{d}\pi_x : \mathcal{C}_x \to \mathrm{T}_p B$$

is an isomorphism for all $x \in E_i$; let u_x denote the preimage of u under this isomorphism. Then since

$\mathcal{C} \subseteq \mathrm{T}E_i$ over the stratum E_i for all i, one has

$$\mathcal{L}(u) = \min_i \underbrace{\inf\left\{\|u_x\|_{g_i} \;\middle|\; x \in \pi^{-1}(p) \cap E_i\right\}}_{=\mathcal{L}_i}. \tag{4.3.3}$$

Next, note that given quasi-Finlserian structures $\mathcal{L}_1$, $\mathcal{L}_2$ on B, their pointwise minimum $\mathcal{L} = \min(\mathcal{L}_1, \mathcal{L}_2)$ is also a quasi-Finslerian structure. Thus, it suffices to prove that each $\mathcal{L}_i$ defines a quasi-Finlserian structure on B.

It is clear from eqn. (4.3.3) that $\mathcal{L}_i$ is non-negative and satisfies:

$$\mathcal{L}_i(\lambda \cdot u) = |\lambda| \mathcal{L}_i(u)$$

for all $u \in \mathrm{T}B$ and $\lambda \in \mathbb{R}$. To see that $\mathcal{L}_i$ is positive definite, let $u \in \mathrm{T}_p B$ satisfy $\mathcal{L}_i(u) = 0$. Choose a sequence $x_n \in \pi^{-1}(p) \cap E_i$ such that $\|u_{x_n}\|_{g_i} \to 0$ as $n \to \infty$. Since π is proper, $\pi^{-1}(p)$ is compact and thus x_n converges subsequentially to some $x \in \pi^{-1}(p)$; passing to a subsequence, one may assume without loss of generality that $x_n \to x$. Since g_i may be extended to a continuous Riemannian metric $\overline{g}_i$ on all of E, one has:

$$\|u_{x_n}\|_{g_i} = \|u_{x_n}\|_{\overline{g}_i} \to \|u_x\|_{\overline{g}_i} \quad \text{as} \quad n \to \infty.$$

Thus $\|u_x\|_{\overline{g}_i} = 0$ and hence $u_x = 0$, since $\overline{g}_i$ is non-degenerate on $\mathcal{C}$, which in turn follows from the regularity of $\widehat{g}$ with respect to $\mathcal{D}$. Thus:

$$u = \mathrm{d}\pi(u_x) = 0,$$

as required. Thus to prove that $\mathcal{L}_i$ is a quasi-Finlserian structure, it suffices to prove continuity.

To this end, choose $b \in B$, $u \in \mathrm{T}_b B$ and pick a sequence $u_n \in \mathrm{T}_{b_n} B$ tending to u as $n \to \infty$ (in particular, $b_n \to b$ as $n \to \infty$). Pick a sequence $x_m \in \pi^{-1}(b) \cap E_i$ such that:

$$\mathcal{L}_i(u) = \lim_{m \to \infty} \|u_{x_m}\|_{g_i}.$$

Then, by properness of π, $\pi^{-1}(b)$ is compact and so x_m converges subsequentially to some $x \in \pi^{-1}(b)$; by passing to a subsequence, without loss of generality $x_m \to x$ as $m \to \infty$.

By Proposition 4.1.1, one may choose a neighbourhood U_b of b satisfying $\pi^{-1}(U_b) \cong U_b \times \pi^{-1}(b)$. Without loss of generality assume that all b_n lie in U_b. Note that both $(u_n)_{(b_n, x_n)}$ and $u_{(b, x_n)}$ converge to $u_{(b,x)} \in \mathcal{C}$ as $n \to \infty$. Thus:

$$\|(u_n)_{(b_n,x_n)}\|_{g_i}, \|u_{(b,x_n)}\|_{g_i} \to \|u_{(b,x)}\|_{\overline{g}_i} \quad \text{as} \quad n \to \infty \tag{4.3.4}$$

(note that $(b, x_n) \in E_i$ implies that $(b_n, x_n) \in E_i$; see Remark 4.1.7). Now clearly:

$$\|(u_n)_{(b_n,x_n)}\|_{g_i} \geqslant \mathcal{L}_i(u_n)$$

for each n. Thus eqn. (4.3.4) implies that:

$$\limsup_{n\to\infty} \mathcal{L}_i(u_n) \leqslant \lim_{n\to\infty} \|u_{(b,x_n)}\|_{g_i} = \mathcal{L}_i(u).$$

Thus suppose, for the sake of contradiction, that $\liminf_{n\to\infty} \mathcal{L}_i(u_n) \leqslant \mathcal{L}_i(u) - \eta$ for some $\eta > 0$. Then $\mathcal{L}_i(u_n) \leqslant \mathcal{L}_i(u) - \eta$ for infinitely many n, and without loss of generality for all n by passing to a subsequence if necessary. For each n, choose $y_n \in \pi^{-1}(b) \cap E_i$ such that:

$$\left\|(u_n)_{b_n,y_n}\right\|_{g_i} \leqslant \mathcal{L}_i(u_n) + \frac{\eta}{3}. \tag{4.3.5}$$

Using compactness of $\pi^{-1}(b)$ again, by passing to a subsequence if necessary $y_n \to y \in \pi^{-1}(b)$. Thus as above:

$$\|(u_n)_{(b_n,y_n)}\|_{g_i}, \|u_{(b,y_n)}\|_{g_i} \to \|u_{(b,y)}\|_{\overline{g}_i} \quad \text{as} \quad n \to \infty. \tag{4.3.6}$$

Choose n sufficiently large so that $\left|\|(u_n)_{(b_n,y_n)}\|_{g_i} - \|u_{(b,y_n)}\|_{g_i}\right| \leqslant \frac{\eta}{3}$. Then:

$$\begin{aligned}
\mathcal{L}_i(u) &\leqslant \|u_{(b,y_n)}\|_{g_i} \\
&\leqslant \|(u_n)_{(b_n,y_n)}\|_{g_i} + \frac{\eta}{3} \\
&\leqslant \mathcal{L}_i(u_n) + \frac{2\eta}{3} \qquad \text{(by eqn. (4.3.5))}.
\end{aligned}$$

However by assumption $\mathcal{L}_i(u_n) \leqslant \mathcal{L}_i(u) - \eta$ and thus $\mathcal{L}_i(u) \leqslant \mathcal{L}_i(u) - \frac{\eta}{3}$, a contradiction. Thus $\liminf_{n\to\infty} \mathcal{L}_i(u_n) \geqslant \mathcal{L}_i(u)$ and so $\mathcal{L}_i(u_n) \to \mathcal{L}_i(u)$ as $n \to \infty$, proving that $\mathcal{L}_i$ is continuous, as required.

Case 2: General Case. Using case 1, for each stratum B_i of B, the function $\mathcal{L}_i$ is a quasi-Finslerian structure on B_i. Thus, to prove that $\widehat{\mathcal{L}} = \{\mathcal{L}_i\}_i$ is a stratified quasi-Finslerian structure on B – and hence to complete the proof of Proposition 4.3.1 – it suffices to prove that given a Riemannian metric h on B, each $\mathcal{L}_i$ is Lipschitz equivalent to h up to the boundary of B_i. However this is clear: fix a stratum E_j of E and recall the extension $\overline{g}_j$ of g_j to all of E. Since both the Riemannian semi-metrics π^*h and $\overline{g}_j$ vanish on $\mathcal{D} = \ker \mathrm{d}\pi$ and are positive definite on $\mathcal{C}$, it follows that there is a continuous map $D : E \to (0,\infty)$ such that:

$$\frac{1}{D}\pi^*h \leqslant \overline{g}_j \leqslant D\pi^*h \text{ on all of } E.$$

Since π is proper, one may define a continuous map $C : B \to (0,\infty)$ such that for all $e \in E$:

$$D(e) \leqslant C(\pi(e)).$$

Then it follows immediately from eqn. (4.3.3) that:

$$\frac{1}{C}\| - \|_h \leqslant \mathcal{L}_i \leqslant C\| - \|_h \text{ on } B_i.$$

This completes the proof.

$\square$

I now prove the first part of Theorem 4.2.5:

Proposition 4.3.7. *Let $\widehat{\mathcal{L}}$ be as in Proposition 4.3.1. Then $\left(B, d^{\widehat{\mathcal{L}}}\right)$ is the free metric space on $\left(E, d^{\widehat{g}}\right)$.*

The proof proceeds by a series of lemmas.

Lemma 4.3.8 ('*A Priori* Bound')**.** *Let $\pi : E \to B$ be as above and let γ be a piecewise-C^1 path in B. There exists a constant $C > 0$ depending only on γ such that for all piecewise-C^1 lifts $\widetilde{\gamma}$ of γ along π:*

$$\widehat{g}\left(\dot{\widetilde{\gamma}}\right) \leqslant C \text{ almost everywhere.}$$

Proof. The argument is similar to the proof of the general case in Proposition 4.3.1. By considering each C^1 portion of γ separately, without loss of generality assume that γ is C^1. Let h be a Riemannian metric on B and recall that there is a continuous map $c : B \to (0, \infty)$ such that, for each stratum E_j of E, writing $\overline{g}_j$ for the extension of g_j to all of E:

$$\overline{g}_j|_e \leqslant c(\pi(e))\pi^* h|_e \text{ for all } e \in E.$$

Thus for all t in the domain of definition of γ:

$$\overline{g}_j\left(\dot{\widetilde{\gamma}}\right)|_{\widetilde{\gamma}(t)} \leqslant c(\gamma(t))h\left(\dot{\gamma}\right)|_{\gamma(t)}.$$

Since the domain of definition of γ is compact, the right-hand side may be bounded uniformly above by some $C > 0$. The result follows.

$\square$

Lemma 4.3.9 ('Piecewise-C^1 Lifts with Specified Endpoints')**.** *Let $\gamma : [0,1] \to B$ be a C^1 path in B, let $b_i = \gamma(i)$ and let $e_i \in \pi^{-1}(b_i)$ for $i = 0,1$ respectively. Then there exists a piecewise-C^1 $\widetilde{\gamma} : [0,1] \to E$ such that $\pi\left(\widetilde{\gamma}\right) = \gamma$ and $\widetilde{\gamma}(i) = e_i$ for $i = 0,1$ respectively.*

Remark 4.3.10. Note that the ability to lift paths along π is a non-trivial result since, as explained in Example 4.1.5, π need not even be a Serre fibration. In particular, Lemma 4.3.9 does not appear to follow from any known result in the literature.

Proof. Firstly, I claim that there exists a piecewise-C^1 lift $\widehat{\gamma}$ of γ along π satisfying $\widehat{\gamma}(0) = e_0$. Indeed, by applying Proposition 4.1.2, one can choose a chart $\Xi_{e_0} = (U_{e_0}, \Gamma_{e_0}, \mathbb{A}_1 \times \mathbb{A}_2, \chi_{e_0})$ about e_0 in E, a chart $\Xi_{b_0} = (U_{b_0}, \Gamma_{b_0}, \mathbb{A}_2, \chi_{b_0})$ about b_0 in B and a homomorphism $\kappa_\pi : \Gamma_{e_0} \to \Gamma_{b_0}$ such that π may be locally lifted as:

$$\begin{array}{ccc} \underbrace{\mathbb{A}_1 \times \mathbb{A}_2}_{\circlearrowleft} & \xrightarrow{\widetilde{\pi}=proj} & \underbrace{\mathbb{A}_2}_{\circlearrowleft} \\ \Gamma_{e_0} & \xrightarrow[\kappa_\pi]{} & \Gamma_{b_0} \end{array}$$

and where Γ_{e_0} acts on $\mathbb{A}_2$ via κ_π. Let $\overline{\gamma} : [0,\varepsilon) \to \mathbb{A}_2$ for some $\varepsilon > 0$ be a local representation of γ in the chart Ξ_{b_0} (note that the equivariance of this local representation is trivial, since $[0,\varepsilon)$ is a manifold and so its orbifold group about each point vanishes). Lift $\overline{\gamma}$ to a map $[0,\varepsilon) \to \mathbb{A}_1 \times \mathbb{A}_2$ as:

$$0 \times \overline{\gamma} : [0,\varepsilon) \to \mathbb{A}_1 \times \mathbb{A}_2.$$

Under the projection $\mathbb{A}_1 \times \mathbb{A}_2 \xrightarrow{proj} \Gamma_{e_0}\backslash\mathbb{A}_1 \times \mathbb{A}_2$, the map $0 \times \overline{\gamma}$ defines a C^1 lift $\widehat{\gamma}$ of γ along π on the interval $[0,\varepsilon)$. To obtain a piecewise-C^1 lift of the path γ over the whole interval $[0,1]$, one then proceeds inductively, repeating the above process starting from some point $\varepsilon' \in [0,\varepsilon)$; the inductive process can be made to terminate in finite-time by compactness of $[0,1]$ and properness of the map π.

Next, I claim that one may 'improve' the lift from $\widehat{\gamma}$ to a lift $\widetilde{\gamma}$ such that $\widetilde{\gamma}(b_i) = e_i$ for both $i = 0$ and $i = 1$. To this end, define $e_1' = \widehat{\gamma}(1)$. Since π has path-connected fibres, one may choose a piecewise-C^1 path $\sigma : [0,1] \to \pi^{-1}(b_1)$ such that $\sigma(0) = e_1'$ and $\sigma(1) = e_1$. The task is to deform the endpoint of $\widehat{\gamma}$ along the path σ.

The argument is very similar to the construction of $\widehat{\gamma}$. Initially, one chooses charts $\Xi_{e_1'} = (U_{e_i'}, \Gamma_{e_i'}, \mathbb{B}_1 \times \mathbb{B}_2, \chi_{e_1'})$ about e_1' in E and $\Xi_{b_1} = (U_{b_1}, \Gamma_{b_1}, \mathbb{B}_2, \chi_{b_1})$ about b_1 in B and a homomorphism $\kappa_\pi : \Gamma_{e_i'} \to \Gamma_{b_1}$ such that π can be written in the local form:

$$\begin{tikzcd} \underbrace{\mathbb{B}_1 \times \mathbb{B}_2}_{\circlearrowleft} \arrow[r, "\widetilde{\pi}=proj"] & \underbrace{\mathbb{B}_2}_{\circlearrowleft} \\ \Gamma_{e_1'} \arrow[r, "\kappa_\pi"'] & \Gamma_{b_1} \end{tikzcd}$$

Since $\widehat{\gamma}$ is continuous, there exists $\varepsilon \in (0,1)$ such that $\widehat{\gamma}((\varepsilon,1]) \subset U_{e_1'}$. Then choose a local representation $\overline{\overline{\gamma}} = \left(\overline{\overline{\gamma}}_1, \overline{\overline{\gamma}}_2\right) : (\varepsilon,1] \to \mathbb{B}_1 \times \mathbb{B}_2$ (so that $\overline{\overline{\gamma}}_1(1) = 0$). Now choose $t > 0$ such that $\sigma(t) \in U_{e_1'}$ and lift the point $\sigma(t)$ to some preimage $\left(\overline{\sigma(t)}, 0\right)$ under the projection map $\mathbb{B}_1 \times \mathbb{B}_2 \xrightarrow{proj} \Gamma_{e_1'}\backslash\mathbb{B}_1 \times \mathbb{B}_2$. By altering the function $\overline{\overline{\gamma}}_1 : (\varepsilon,1] \to \mathbb{B}_1$ on some compact subset of $(\varepsilon,1]$, one can ensure that $\overline{\overline{\gamma}}_1(1) = \overline{\sigma(t)}$. Denote the new local representation $\left(\overline{\overline{\gamma}}_1, \overline{\overline{\gamma}}_2\right)$ by $\overline{\overline{\gamma}}'$. Projecting $\overline{\overline{\gamma}}'$ under the map:

$$\mathbb{B}_1 \times \mathbb{B}_2 \xrightarrow{proj} \Gamma_{e_1'}\backslash\mathbb{B}_1 \times \mathbb{B}_2$$

yields a new lift $\widehat{\gamma}'$ covering γ with the property that $\widehat{\gamma}'(1) = \sigma(t)$. Now iterate this argument, noting again that the process can be made to terminate in finite time by compactness of the domain of definition of σ.

$\square$

Lemma 4.3.11 ('Convergence in Measure')**.** *Let $\gamma : [0,1] \to B$ be a piecewise-C^1 path. Then there exists a sequence of piecewise-C^1 lifts $\widetilde{\gamma}_n$ of γ along π such that:*

$$\widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}} \to \widehat{\mathcal{L}}(\dot{\gamma})^{\frac{1}{2}} \text{ in measure as } n \to \infty.$$

Moreover, the endpoints of the lift, viz. $\widetilde{\gamma}_n(0)$ and $\widetilde{\gamma}_n(1)$, may be chosen to be any points in $\pi^{-1}(\gamma(0))$

and $\pi^{-1}(\gamma(1))$ *respectively.*

Proof. Firstly, note that it suffices to lift each C^1 portion of γ separately to a piecewise-C^1 curve in E and then use the freedom in specifying the endpoints of these separate lifts to ensure that the combined lift of γ is continuous over the non-differentiable points of γ. Thus without loss of generality one may assume that γ is everywhere C^1.

Write $I = [0,1]$. For each stratum B_i of B, recall the measurable subset:

$$I_i = \gamma^{-1}(B_i) \subseteq I.$$

Write $\mathscr{L}$ for the Lebesgue measure on I and define:

$$\dot{I}_i = \left\{ x \in I_i \cap (0,1) \;\middle|\; \lim_{r\to 0} \frac{\mathscr{L}\left[I_i \cap (x-r, x+r)\right]}{\mathscr{L}\left[(x-r,x+r)\right]} = 1 \right\} \subseteq I_i \cap (0,1).$$

By [108, Cor. 2.9, p. 20], $\mathscr{L}\left(I_i \backslash \dot{I}_i\right) = 0$. Define:

$$\widetilde{I}_i = \left\{ x \in \dot{I}_i \;\middle|\; \dot{\gamma} \in TB_i \right\}.$$

Then by [45, Lem. 3.1.7, p. 217], $\mathscr{L}\left(I_i \backslash \widetilde{I}_i\right) = 0$. For notational convenience later in the proof, define $\widetilde{I} = \bigcup_i \widetilde{I}_i$.

Fix $n \geqslant 1$. For each i and each $x \in \widetilde{I}_i$, choose $j = j(x,n)$ and $y = y(x,n) \in E_j \subseteq \pi^{-1}(B_i)$ such that for all lifts u of $\dot{\gamma}(x)$ to $\mathrm{T}_y E_j$ along $\mathrm{d}\pi$:

$$g_j(u)^{\frac{1}{2}} - \mathcal{L}_i\left(\dot{\gamma}(x)\right)^{\frac{1}{2}} < \frac{1}{n}.$$

(Note that the left-hand side is automatically non-negative, by definition of $\mathcal{L}_i$.) Choose a chart $\Xi_x = (U_x, \Gamma_x, \widetilde{U}_x, \chi_x)$ about $\gamma(x) \in B$ which is regular for the submanifold B_i, with regular subspace $\mathbb{I}$ (see Definition 2.1.4):

$$\widetilde{U}_x = \mathbb{A}_1 \times \mathbb{I} \to {}_{\Gamma_x}\backslash^{\mathbb{A}_1} \times \mathbb{I} \underbrace{\cong}_{\chi_x} U_x.$$

(Here, $\mathbb{A}_1$ and $\mathbb{I}$ are finite-dimensional real vector spaces and Γ_x acts on $\mathbb{A}_1$.) By Proposition 4.1.2, shrinking U_x if necessary one can choose a chart $\Xi_y = (U_y, \Gamma_y, \mathbb{A}_2 \times \mathbb{A}_1 \times \mathbb{I}, \chi_y)$ and a homomorphism $\kappa_\pi : \Gamma_y \to \Gamma_x$ such that π may be expressed in the charts Ξ_y and Ξ_x as the projection map:

$$\begin{array}{ccc} \underbrace{\mathbb{A}_2 \times \mathbb{A}_1}_{\circlearrowleft} \times \mathbb{I} & \xrightarrow{\widetilde{\pi} = proj} & \underbrace{\mathbb{A}_1}_{\circlearrowleft} \times \mathbb{I} \\ \Gamma_y & \xrightarrow[\kappa_\pi]{} & \Gamma_x \end{array} \tag{4.3.12}$$

Since $x \in \widetilde{I}_i \subseteq (0,1)$ and γ is continuous, one can choose $\eta(x,n) > 0$ such that $(x - \eta(x,n), x + \eta(x,n)) \subseteq (0,1)$ and:

$$\gamma\left[(x-\eta(x,n), x+\eta(x,n))\right] \subset U_x \cong \left({}_{\Gamma_x}\backslash^{\mathbb{A}_1}\right) \times \mathbb{I}.$$

Since $x \in \widetilde{I}_i \subseteq \dot{I}_i$, one has:

$$\lim_{r\to 0} \frac{\mathscr{L}\left[I_i \cap (x-r, x+r)\right]}{\mathscr{L}\left[(x-r,x+r)\right]} = \lim_{r\to 0} \frac{\mathscr{L}\left[\widetilde{I}_i \cap (x-r, x+r)\right]}{\mathscr{L}\left[(x-r,x+r)\right]} = 1$$

(where the first equality follows from the fact that $\mathscr{L}\left(I_i \backslash \widetilde{I}_i\right) = 0$) and so by reducing $\eta(x,n) > 0$ if necessary, one may assume that for all $0 < r \leqslant \eta(x,n)$:

$$\mathscr{L}\left[\widetilde{I}_i \cap (x-r,x+r)\right] \geqslant \frac{n-1}{n}\mathscr{L}\left[(x-r,x+r)\right] = 2r\frac{n-1}{n}. \tag{4.3.13}$$

Since γ defines a smooth map from the manifold I to the orbifold B, by reducing $\eta(x,n) > 0$ still further if necessary, one may assume that γ has a local lift $\overline{\gamma}$ with respect to the coordinate charts $(x-\eta(x,n), x+\eta(x,n))$ and Ξ_x:

$$\begin{tikzcd} & \mathbb{A}_1 \times \mathbb{I} \arrow[d] \\ (x-\eta(x,n), x+\eta(x,n)) \arrow[ur, "\overline{\gamma}"] \arrow[r, "\gamma"] & \Gamma_x \backslash \mathbb{A}_1 \times \mathbb{I} \cong U_x \end{tikzcd}$$

(Note that the equivariance of the lift $\overline{\gamma}$ is vacuous, since I is a manifold and so has trivial orbifold groups about every point.) Using the local representation of π given in eqn. (4.3.12), one can lift $\overline{\gamma}$ to the map $0 \times \overline{\gamma}$ as below:

$$\begin{tikzcd} & \mathbb{A}_2 \times \mathbb{A}_1 \times \mathbb{I} \arrow[d, "\overline{\pi}=proj"] \\ & \mathbb{A}_1 \times \mathbb{I} \arrow[d] \\ (x-\eta(x,n), x+\eta(x,n)) \arrow[uur, "0\times\overline{\gamma}"] \arrow[ur, "\overline{\gamma}"] \arrow[r, "\gamma"] & \Gamma_x \backslash \mathbb{A}_1 \times \mathbb{I} \cong U_x \end{tikzcd}$$

Projecting $0 \times \overline{\gamma}$ via $\mathbb{A}_2 \times \mathbb{A}_1 \times \mathbb{I} \to \left(\Gamma_y \backslash \mathbb{A}_2 \times \mathbb{A}_1\right) \times \mathbb{I}$ defines a local lift (C^1) of γ along π over the region $(x-\eta(x,n), x+\eta(x,n))$; denote this lift by $\widetilde{\gamma}(x,n)$.

Note that on the region $I_i \cap (x-\eta(x,n), x+\eta(x,n))$ (where $\gamma \in B_i$) the curve $\widetilde{\gamma}(x,n)$ lies in E_j. Indeed:

$$\widetilde{\gamma}(x,n)|_{I_i \cap (x-\eta(x,n), x+\eta(x,n))} \subset [0] \times \mathbb{I} \subseteq \left(\Gamma_y \backslash \mathbb{A}_2 \times \mathbb{A}_1\right) \times \mathbb{I}$$

and one may verify that $[0] \times \mathbb{I}$ lies in the stratum E_j. Hence by [45, Lem. 3.1.7, p. 217], $g_j\left(\dot{\widetilde{\gamma}}(x,n)_t\right)$ is well-defined for almost every $t \in I_i \cap (x-\eta(x,n), x+\eta(x,n))$ and thus for almost every $t \in \widetilde{I}_i \cap (x - \eta(x,n), x+\eta(x,n))$. Now consider the function:

$$g_j\left(\dot{\widetilde{\gamma}}(x,n)\right)^{\frac{1}{2}} - \mathcal{L}_i\left(\dot{\gamma}\right)^{\frac{1}{2}} \text{ where defined on } \widetilde{I}_i \cap (x-\eta(x,n), x+\eta(x,n)).$$

This is a continuous, non-negative map which is less than $\frac{1}{n}$ at the point x. Therefore, by reducing

$\eta(x,n) > 0$ if necessary, one may ensure that:

$$g_j\left(\dot{\widetilde{\gamma}}(x,n)\right)^{\frac{1}{2}} - \mathcal{L}_i\left(\dot{\gamma}\right)^{\frac{1}{2}} < \frac{2}{n} \text{ almost everywhere on } \widetilde{I}_i \cap (x-\eta(x,n), x+\eta(x,n)). \tag{4.3.14}$$

Now consider the collection of subsets of I given by:

$$\mathcal{S}_n = \left\{(x-r, x+r) \;\middle|\; x \in \widetilde{I}, r \in (0, \eta(x,n))\right\}.$$

By applying the Vitali Covering Theorem [102, Thm. 2.2, p. 26], there exist $x_p \in \widetilde{I}$ and $r_p \in (0, \eta(x_p, n))$, for $p \in \mathbb{N}$, such that:

- The sets $\{(x_p - r_p, x_p + r_p)\}_{p\in\mathbb{N}}$ are disjoint;
- $\mathscr{L}\left[I \backslash \bigcup_{p\in\mathbb{N}}(x_p - r_p, x_p + r_p)\right] = \mathscr{L}\left[\widetilde{I} \backslash \bigcup_{p\in\mathbb{N}}(x_p - r_p, x_p + r_p)\right] = 0.$

Choose $N = N(n)$ sufficiently large such that:

$$\mathscr{L}\left[I \Big\backslash \bigcup_{p=0}^{N(n)} (x_p - r_p, x_p + r_p)\right] < \frac{1}{n}. \tag{4.3.15}$$

Now construct the lift $\widetilde{\gamma}_n$ as follows:

- On each set $\left(x_p - \frac{n-1}{n} r_p, x_p + \frac{n-1}{n} r_p\right)$, $p = 0, ..., N(n)$, define:

  $$\widetilde{\gamma}_n = \widetilde{\gamma}(x_p, n)|_{\left(x_p - \frac{n-1}{n} r_p, x_p + \frac{n-1}{n} r_p\right)}.$$

- Since the open sets $\{(x_p - r_p, x_p + r_p)\}_{p\in\{0,\dots,N(n)\}}$ are disjoint, the complement of the union of the smaller open sets $\left\{\left(x_p - \frac{n-1}{n} r_p, x_p + \frac{n-1}{n} r_p\right)\right\}_{p\in\{0,\dots,N(n)\}}$ is a finite collection of closed intervals, including two intervals of the form $[0, \alpha]$ and $[\beta, 1]$. On each of these closed intervals, use Lemma 4.3.9 to choose some piecewise-C^1 lift of γ along π, with endpoints chosen so that the resulting lift $\widetilde{\gamma}_n$ is piecewise-C^1 and so that $\widetilde{\gamma}_n(0)$, $\widetilde{\gamma}_n(1)$ take the required values in $\pi^{-1}(\gamma(0))$ and $\pi^{-1}(\gamma(1))$ respectively.

I now claim that:

$$\mathscr{L}\left[\left\{x \in I \;\middle|\; \widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}}|_x - \widehat{\mathcal{L}}\left(\dot{\gamma}\right)^{\frac{1}{2}}|_x \geqslant \frac{2}{n}\right\}\right] < \frac{3}{n}, \tag{4.3.16}$$

a result which would imply the convergence of the functions $\widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}} \to \widehat{\mathcal{L}}\left(\dot{\gamma}\right)^{\frac{1}{2}}$ in measure. To verify eqn. (4.3.16), for each x_p choose $i(p)$ such that $x_p \in \widetilde{I}_{i(p)}$ and recall from eqn. (4.3.14) that:

$$\widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}} - \widehat{\mathcal{L}}\left(\dot{\gamma}\right)^{\frac{1}{2}} < \frac{2}{n} \text{ almost everywhere on } \widetilde{I}_{i(p)} \cap \left(x_p - \frac{n-1}{n} r_p, x_p + \frac{n-1}{n} r_p\right).$$

Therefore:

$$\begin{aligned}\mathscr{L}\left[\left\{x\in I \;\middle|\; \widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}}|_x - \widehat{\mathcal{L}}\left(\dot{\gamma}\right)^{\frac{1}{2}}|_x \geqslant \frac{2}{n}\right\}\right] &\leqslant 1-\mathscr{L}\left[\bigcup_{p=0}^{N(n)}\left\{\widetilde{I}_{i(p)}\cap\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right\}\right]\\ &= 1-\sum_{p=0}^{N(n)}\mathscr{L}\left[\widetilde{I}_{i(p)}\cap\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right],\end{aligned}$$

where the final equality follows from the fact that the union is disjoint. Now from eqn. (4.3.13), for each p:

$$\begin{aligned}\mathscr{L}\left[\widetilde{I}_{i(p)}\cap\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right] &\geqslant \frac{n-1}{n}\mathscr{L}\left[\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right]\\ &= \left(\frac{n-1}{n}\right)^2\mathscr{L}\left[(x_p-r_p, x_p+r_p)\right],\end{aligned}$$

and therefore:

$$\begin{aligned}\sum_{p=0}^{N(n)}\mathscr{L}\left[\widetilde{I}_{i(p)}\cap\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right] &\geqslant \left(\frac{n-1}{n}\right)^2\sum_{p=0}^{N(n)}\mathscr{L}\left[(x_p-r_p, x_p+r_p)\right]\\ &= \left(\frac{n-1}{n}\right)^2\mathscr{L}\left[\bigcup_{p=0}^{N(n)}(x_p-r_p, x_p+r_p)\right].\end{aligned}$$

By eqn. (4.3.15):

$$\mathscr{L}\left[\bigcup_{p=0}^{N(n)}(x_p-r_p, x_p+r_p)\right] > \frac{n-1}{n}$$

and hence:

$$\sum_{p=0}^{N(n)}\mathscr{L}\left[\widetilde{I}_{i(p)}\cap\left(x_p-\frac{n-1}{n}r_p, x_p+\frac{n-1}{n}r_p\right)\right] > \left(\frac{n-1}{n}\right)^3$$

and whence:

$$\mathscr{L}\left[\left\{x\in I \;\middle|\; \widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}}|_x - \widehat{\mathcal{L}}\left(\dot{\gamma}\right)^{\frac{1}{2}}|_x \geqslant \frac{2}{n}\right\}\right] < 1-\left(\frac{n-1}{n}\right)^3 \leqslant \frac{3}{n}.$$

This completes the proof of Lemma 4.3.11.

□

Using Lemmas 4.3.8, 4.3.9 and 4.3.11, I now prove Proposition 4.3.7:

Proof of Proposition 4.3.7. By the definition of $\widehat{\mathcal{L}}$, for all $e, e' \in E$ and all piecewise-C^1 paths $\gamma : e \to e'$:

$$\widehat{g}\left(\dot{\gamma}\right) \geqslant \widehat{\mathcal{L}}\left(\dot{\overline{\pi(\gamma)}}\right) \text{ almost everywhere,}$$

and hence:

$$\ell_{\widehat{g}}(\gamma) \geqslant \ell_{\widehat{\mathcal{L}}}(\pi(\gamma)) \geqslant d^{\widehat{\mathcal{L}}}(\pi(e), \pi(e')).$$

Taking the infimum over all such γ shows that:

$$d^{\widehat{g}}(e,e') \geqslant d^{\widehat{\mathcal{L}}}(\pi(e),\pi(e')).$$

Conversely, let $\gamma : [0,1] \to B$ be a piecewise-C^1 path $\pi(e) \to \pi(e')$. By Lemma 4.3.11, there exists a sequence of piecewise-C^1 lifts $\widetilde{\gamma}_n : e \to e'$ of γ along π such that:

$$\widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}} \to \widehat{\mathcal{L}}(\dot{\gamma})^{\frac{1}{2}} \text{ in measure as } n \to \infty.$$

By the *a priori* bound in Lemma 4.3.8 the Dominated Convergence Theorem (DCT) applies, and so:

$$d^{\widehat{g}}(e,e') \leqslant \lim_{n\to\infty} \ell_{\widehat{g}}(\widetilde{\gamma}_n) = \lim_{n\to\infty} \int_{[0,1]} \widehat{g}\left(\dot{\widetilde{\gamma}}_n\right)^{\frac{1}{2}} \mathrm{d}\mathscr{L} \underbrace{=}_{DCT} \int_{[0,1]} \widehat{\mathcal{L}}(\dot{\gamma})\, \mathrm{d}\mathscr{L}^{\frac{1}{2}} = \ell_{\widehat{\mathcal{L}}}(\gamma).$$

Taking the infimum over γ completes the proof.

□

4.4 Proof of Theorem 4.2.5: Part 2

The purpose of this section is to complete the proof of Theorem 4.2.5. By applying Proposition 2.4.3 and the results of §4.3, it suffices to prove the following result:

Proposition 4.4.1. *Let notation be as in Notation 4.2.1 and assume that conditions (i)–(iv) in Theorem 4.2.5 holds. Then:*

$$\mathfrak{D}\left[(E_1,d^{\mu}) \to (E_2,d^{\infty})\right] \to 0 \quad \textit{as} \quad \mu \to \infty.$$

The proof proceeds via a series of lemmas.

4.4.1 Convergence on the regions $E^{(r)}$

Consider the region $E^{(r)} \subset E_1\backslash S_1 \cong E_2\backslash S_2$. The restriction of each stratified Riemannian metric $\widehat{g}^{\mu}$ to $E^{(r)}$ induces a metric on $E^{(r)}$, denoted $d^{\mu,r}$. (Note that, in general, $d^{\mu,r} \neq d^{\mu}|_{E^{(r)}}$ since the metric on the left-hand side is intrinsic, defined by optimising over the length of paths contained only in $E^{(r)}$, while the metric on the right-hand side is extrinsic, defined by optimising over the length of paths in E_1.) Analogously, the restriction of the stratified Riemannian semi-metric $\widehat{g}^{\infty}$ to $E^{(r)}$ induces a semi-metric on $E^{(r)}$ denoted $d^{\infty,r}$ (where again $d^{\infty,r} \neq d^{\infty}|_{E^{(r)}}$ in general).

Lemma 4.4.2. *Assuming condition (i) from Theorem 4.2.5, for all fixed $r \in (0,1]$:*

$$d^{\mu,r} \to d^{\infty,r} \textit{ uniformly as } \mu \to \infty.$$

Proof. Fix $x,y \in E^{(r)}$ and let γ be any piecewise-C^1 path from x to y in $E^{(r)}$. Since $\widehat{g}^{\mu} \to \widehat{g}^{\infty}$ uniformly on $E^{(r)}$, it follows that $\ell_{\widehat{g}^{\mu}}(\gamma) \to \ell_{\widehat{g}^{\infty}}(\gamma)$ as $\mu \to \infty$. Moreover, one clearly has $d^{\mu,r}(x,y) \leqslant \ell_{\widehat{g}^{\mu}}(\gamma)$

for each μ. Taking first the limit superior over μ and then the infimum over all γ in this inequality, therefore, yields:

$$\limsup_{\mu} d^{\mu,r}(x,y) \leqslant d^{\infty,r}(x,y). \tag{4.4.3}$$

Conversely, for any γ as above, by eqn. (4.2.7) in condition (i), one has:

$$\ell_{\widehat{g}^{\mu}}(\gamma) \geqslant \Lambda_\mu \ell_{\widehat{g}^{\infty}}(\gamma) \geqslant \Lambda_\mu d^{\infty,r}(x,y).$$

Since $\Lambda_\mu \to 1$ as $\mu \to \infty$, taking firstly the infimum over all γ and then the limit inferior over all μ yields:

$$\liminf_{\mu} d^{\mu,r}(x,y) \geqslant d^{\infty,r}(x,y). \tag{4.4.4}$$

Combining eqns. (4.4.3) and (4.4.4) gives:

$$d^{\infty,r}(x,y) \leqslant \liminf_{\mu} d^{\mu,r}(x,y) \leqslant \limsup_{\mu} d^{\mu,r}(x,y) \leqslant d^{\infty,r}(x,y)$$

for all $x, y \in E^{(r)}$ and hence $d^{\mu,r} \to d^{\infty,r}$ pointwise on $E^{(r)}$.

Now fix some reference stratified Riemannian metric $\widehat{h}$ on E_1 and write d^r for the (intrinsic) metric on $E^{(r)}$ induced by $\widehat{h}|_{E^{(r)}}$. By assumption, for each stratum E_i of $E_1 \backslash S_1$:

$$\|g_i^\mu - g_i^\infty\|_{h_i} \to 0 \text{ uniformly on } E_i \cap E^{(r)} \text{ as } \mu \to \infty,$$

where $\| - \|_{h_i}$ denotes the pointwise norm on symmetric bilinear forms induced by h_i, as in Lemma 3.5.39. Next, note that since g_i^∞ may be continuously extended to a semi-metric $\overline{g}_i^\infty$ on all of E_2 (and likewise for h_i), then by compactness of E_2 there exists some constant $C_i > 0$ such that $\|g_i^\infty\|_{h_i} \leqslant \frac{C_i}{2}$ on all of $E_i \cap E^{(r)}$. It follows that for all sufficiently large μ:

$$\|g_i^\mu\|_{h_i} \leqslant C_i \text{ on } E_i \cap E^{(r)}.$$

Taking $C = \max_i C_i$, it follows from Lemma 3.5.39 that for all sufficiently large μ: $\widehat{g}^\mu \leqslant C\widehat{h}$ and in particular:

$$d^{\mu,r} \leqslant C^{\frac{1}{2}} d^r.$$

By applying the triangle inequality, it follows that for all pairs $(x,y), (x',y') \in E^{(r)} \times E^{(r)}$ and all sufficiently large μ (including $\mu = \infty$):

$$|d^{\mu,r}(x,y) - d^{\mu,r}(x',y')| \leqslant C^{\frac{1}{2}} \left(d^r(x,x') + d^r(y,y')\right)$$

and thus the family of functions $d^{\mu,r} : E^{(r)} \times E^{(r)} \to \mathbb{R}$ is uniformly Lipschitz (at least for all sufficiently large μ) and hence equicontinuous. By combining this equicontinuity with the pointwise convergence $d^{\mu,r} \to d^{\infty,r}$, the proof of Proposition 4.4.2 is now completed by the following variant of the well-known Ascoli–Arzelà theorem:

Theorem 4.4.5. *Let (X,d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be equicontinuous functions* *converging pointwise to a continuous function f. Then $f_n \to f$ uniformly.*

The proof is simple, and I omit it. This completes the proof of Lemma 4.4.2.

□

4.4.2 Combinatorial preliminaries

Recall that the 'singular' regions $S_1(j)$ are indexed by $j \in \{1, ..., N\}$ and likewise for E_2. Recall also the sets $\partial^{(r)}(j) = \partial U_1^{(r)}(j) \cong \partial U_2^{(r)}(j)$. Write $[N] = \{1, ..., N\}$ and given any $1 \leqslant k \leqslant N$, let $[N]^{(k)}$ denote the set of ordered tuples of k distinct elements of $[N]$, which will be denoted $(j_1, ..., j_k)$. For notational convenience, use $\wedge$ to denote the binary minimum of two numbers, i.e.: $a \wedge b = \min(a, b)$.

Lemma 4.4.6. *Fix $r \in (0,1]$ and let $x, y \in E^{(r)}$. Then for all $\mu \geqslant 1$ (including $\mu = \infty$):*

$$\begin{aligned} d^\mu(x,y) = d^{\mu,r}(x,y) \wedge \\ \min_{1 \leqslant k \leqslant N} \left[\min_{(j_1,...,j_k) \in [N]^{(k)}} \left(\inf_{\substack{x_i, y_i \in \partial^{(r)}(j_i) \\ i=1,...,k}} d^{\mu,r}(x, x_1) + d^{\mu,r}(y_n, y) + \sum_{i=1}^{k} d^\mu(x_i, y_i) + d^{\mu,r}(y_i, x_i) \right) \right]. \end{aligned} \tag{4.4.7}$$

In particular:

$$\begin{aligned} \sup_{x,y \in E^{(r)}} |d^\mu(x,y) - d^\infty(x,y)| \leqslant & (N+2) \sup_{x',y' \in E^{(r)}} |d^{\mu,r}(x',y') - d^{\infty,r}(x',y')| \\ & + N \max_{j \in [N]} \sup_{x'',y'' \in \partial^{(r)}(j)} |d^\mu(x'',y'') - d^\infty(x'',y'')| \,. \end{aligned}$$

Proof. It suffices to prove eqn. (4.4.7), the final claim being a direct consequence of this. I prove eqn. (4.4.7) in the case $\mu < \infty$, the case $\mu = \infty$ being identical.

Write $\Omega(x,y)$ for the set of all piecewise-C^1 paths γ from x to y in E_1 and $\Omega_0(x,y) \subseteq \Omega(x,y)$ for the set of all piecewise-C^1 paths γ from x to y which lie entirely within $E^{(r)}$. Let $\gamma \in \Omega(x,y) \backslash \Omega_0(x,y)$ and write $[a,b] \subset \mathbb{R}$ for the domain of γ. Assign to γ an index $j_1(\gamma) \in [N]$ and a number $t_1(\gamma) \in [a,b]$ as follows:

Define:

$$t_0(\gamma) = \inf \left\{ t \in [a,b] \;\middle|\; \gamma(t) \in U_1^{(r)}(j) \text{ for some } j \in [N] \right\},$$

the right-hand side being non-empty, precisely because $\gamma \notin \Omega_0(x,y)$. The by eqn. (4.2.2) in condition 3 of Notation 4.2.1, there is a unique $j \in [N]$ such that $\gamma(t_0(\gamma)) \in \overline{U_1^{(r)}}(j)$; denote this unique j by $j_1(\gamma)$. Now define:

$$t_1(\gamma) = \sup \left\{ t \in [a,b] \;\middle|\; \gamma(t) \in U_1^{(r)}(j_1(\gamma)) \right\}.$$

Now suppose that $t_1(\gamma) < b$ and that the path $\gamma|_{[t_1(\gamma),b]} \notin \Omega_0(\gamma(t_1(\gamma)), b)$. Then one may define:

$$j_2(\gamma) = j_1(\gamma|_{[t_1(\gamma),b]}) \quad \text{ and } \quad t_2(\gamma) = t_1(\gamma|_{[t_1(\gamma),b]}).$$

(Observe that $j_2(\gamma) \neq j_1(\gamma)$, since γ never lies in $U_1^{(r)}(j_1(\gamma))$ after time $t_1(\gamma)$.) One may continue in this vein, defining:

$$j_{k+1}(\gamma) = j_1(\gamma|_{[t_k(\gamma),b]}) \quad \text{and} \quad t_{k+1}(\gamma) = t_1(\gamma|_{[t_k(\gamma),b]}).$$

until either $t_k(\gamma) = b$ for some k or $\gamma|_{[t_k(\gamma),b]} \in \Omega_0(\gamma(t_k(\gamma)), b)$, one of these two conditions necessarily being reached for some $k \in [N]$ due to the fact that the $j_1(\gamma), j_2(\gamma), j_3(\gamma), \ldots$ are all distinct elements of the finite set $[N]$. Call:

$$(j_1(\gamma), \ldots, j_k(\gamma)) \in [N]^{(k)}$$

the characteristic tuple of γ.

Now for each $k \in [N]$ and each tuple $\mathbf{i} \in [N]^{(k)}$, define $\Omega_{\mathbf{i}}(x,y)$ to be the set of all $\gamma \in \Omega(x,y)$ with characteristic tuple $\mathbf{i}$. The above discussion shows that there is a disjoint union:

$$\Omega(x,y) = \Omega_0(x,y) \coprod \left(\coprod_{\substack{k \in [N] \\ \mathbf{i} \in [N]^{(k)}}} \Omega_{\mathbf{i}}(x,y) \right).$$

However, by definition:

$$\inf_{\gamma \in \Omega_0(x,y)} \ell_{\widehat{g}^\mu}(\gamma) = d^{\mu,r}(x,y).$$

Similarly, for each $k \in [N]$ and $\mathbf{i} \in [N]^{(k)}$, one may verify that:

$$\inf_{\gamma \in \Omega_{\mathbf{i}}(x,y)} \ell_{\widehat{g}^\mu}(x,y) = \inf_{\substack{x_i, y_i \in \partial^{(r)}(j_i) \\ i=1,\ldots,k}} d^{\mu,r}(x,x_1) + d^{\mu,r}(y_n,y) + \sum_{i=1}^{k} d^{\mu}(x_i,y_i) + d^{\mu,r}(y_i,x_i).$$

The result now follows.

□

4.4.3 Completing the proof of Theorem 4.4.1

Let $\widetilde{\Phi}$ be some fixed, possibly discontinuous, extension of the map $\Phi : E_1 \backslash S_1 \to E_2 \backslash S_2$ such that $\widetilde{\Phi}(S_1) \subseteq S_2$ and the following diagram:

$$\begin{tikzcd} U_1^{(1)}(j) \arrow[rr, "\widetilde{\Phi}"] \arrow[dr, "\mathfrak{f}_{1,j}"'] & & U_2^{(1)}(j) \arrow[dl, "\mathfrak{f}_{2,j}"] \\ & S(j) & \end{tikzcd}$$

commutes for each $j \in \{1, \ldots, N\}$. Explicitly, for each $j \in \{1, \ldots, N\}$ and each $x \in S_1(j)$, choose some point $y \in \mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{1,j}(x)\}) \cap S_2(j)$ and define $\widetilde{\Phi}(x) = y$. Recalling the definition of forwards discrepancy from §2.4.1, to prove Theorem 4.4.1 it suffices to prove that, given any $\eta > 0$, $\widetilde{\Phi}$ is an η-isometry $(E_1, d^\mu) \to (E_2, d^\infty)$ whenever μ is sufficiently large (depending on η). This is equivalent to the following two lemmas:

Lemma 4.4.8. *$\widetilde{\Phi}(E_1) \subseteq E_2$ is dense with respect to the semi-metric d^∞.*

Lemma 4.4.9.

$$\lim_{\mu\to\infty} \sup_{x,y\in E_1} \left| d^\mu(x,y) - d^\infty\left(\widetilde{\Phi}(x), \widetilde{\Phi}(y)\right)\right| = 0.$$

Proof of Lemma 4.4.8. Clearly $\widetilde{\Phi}(E_1) \supseteq \Phi(E_1 \backslash S_1) = E_2 \backslash S_2$. Thus, to prove that $\widetilde{\Phi}(E_1)$ is dense, it suffices to prove that for all $j \in \{1, ..., N\}$, all $w \in S_2(j)$ and all $\eta > 0$, there exists $x \in E_1$ such that:

$$d^\infty\left(w, \widetilde{\Phi}(x)\right) \leqslant \eta.$$

To this end, given $w \in S_2(j)$, choose x to be any point of $\mathfrak{f}_{1,j}^{-1}(\{\mathfrak{f}_{2,j}(w)\}) \cap S_1(j)$. Then by definition of $\widetilde{\Phi}$:

$$\begin{aligned}\widetilde{\Phi}(x) &\in \mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{2,j}(w)\}) \cap S_2(j)\\ &\subseteq \mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{2,j}(w)\}) \cap U_2^{(r)}(j) \text{ for any } r \in (0,1].\end{aligned}$$

Hence for all $r \in (0,1]$:

$$\begin{aligned}d^\infty\left(w, \widetilde{\Phi}(x)\right) &\leqslant \operatorname{diam}_{d^\infty}\left[\mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{2,j}(w)\}) \cap U_2^{(r)}(j)\right]\\ &\leqslant \max_{j\in\{1,\dots,N\}} \sup_{p\in S(j)} \operatorname{diam}_{d^\infty}\left[\mathfrak{f}_{2,j}^{-1}(p) \cap U_2^{(r)}(j)\right]\end{aligned}$$

and thus:

$$d^\infty\left(w, \widetilde{\Phi}(x)\right) \leqslant \limsup_{r\to 0} \max_{j\in\{1,\dots,N\}} \sup_{p\in S(j)} \operatorname{diam}_{d^\infty}\left[\mathfrak{f}_{2,j}^{-1}(p) \cap U_2^{(r)}(j)\right] = 0,$$

the final equality being condition (iii) in Theorem 4.2.5. The result follows.

□

Proof of Lemma 4.4.9. Pick $x, y \in E_1$ and let $r \in (0,1]$. Define (potentially) new points $x' = x'(r)$ and $y' = y'(r)$ in $E^{(r)}$ via:

$$x' = \begin{cases} x & \text{if } x \in E^{(r)};\\ \text{some point in } E^{(r)} \cap \overline{\mathfrak{f}_{1,j}^{-1}(\{\mathfrak{f}_{1,j}(x)\}) \cap U_1^{(r)}(j)} \subseteq \partial^{(r)}(j) & \text{if } x \in U_1^{(r)}(j)\end{cases}$$

and analogously for y'. Note that:

$$d^\mu(x,x'), d^\mu(y,y') \leqslant \max_{j\in[N]} \sup_{p\in S(j)} \operatorname{diam}_{d^\mu}\left[\mathfrak{f}_{1,j}^{-1}(\{p\}) \cap U_1^{(r)}(j)\right]. \tag{4.4.10}$$

Now let me bound $d^\infty\left(\widetilde{\Phi}(x), x'\right)$ and $d^\infty\left(\widetilde{\Phi}(y), y'\right)$, where x' and y' are identified with $\widetilde{\Phi}(x') = \Phi(x')$ and $\widetilde{\Phi}(y') = \Phi(y')$ in the usual way. Clearly if $x \in E^{(r)}$, then $\widetilde{\Phi}(x) = x = x'$ and $d^\infty\left(\widetilde{\Phi}(x), x'\right) =$

0. Thus suppose $x \in U_1^{(r)}(j)$ for some j. The commutative diagram:

$$\begin{tikzcd}
U_1^{(1)}(j)\Big\backslash S_1(j) \arrow[rr, "\Phi"] \arrow[dr, "\mathfrak{f}_{1,j}"'] & & U_2^{(1)}(j)\Big\backslash S_2(j) \arrow[dl, "\mathfrak{f}_{2,j}"] \\
& S(j) &
\end{tikzcd}$$

shows that:

$$x' \in E^{(r)} \cap \overline{\mathfrak{f}_{1,j}^{-1}(\{\mathfrak{f}_{1,j}(x)\}) \cap U_1^{(r)}(j)} \cong E^{(r)} \cap \overline{\mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{1,j}(x)\}) \cap U_2^{(r)}(j)} \subseteq \partial^{(r)}(j),$$

using the identification Φ in the usual way. Moreover, from the definition of $\widetilde{\Phi}$ one may verify that $\widetilde{\Phi}(x) \in \mathfrak{f}_{2,j}^{-1}(\{\mathfrak{f}_{1,j}(x)\}) \cap U_2^{(r)}(j)$. Thus:

$$d^{\infty}\left(\widetilde{\Phi}(x), x'\right) \leqslant \max_{j \in [N]} \sup_{p \in S(j)} \operatorname{diam}_{d^{\infty}}\left[\mathfrak{f}_{2,j}^{-1}(\{p\}) \cap U_2^{(r)}(j)\right]. \tag{4.4.11}$$

The same bound holds for $d^{\infty}\left(\widetilde{\Phi}(y), y'\right)$. Thus by the triangle inequality:

$$\begin{aligned}
\left|d^{\mu}(x,y) - d^{\infty}\left(\widetilde{\Phi}(x), \widetilde{\Phi}(y)\right)\right| &\leqslant |d^{\mu}(x',y') - d^{\infty}(x',y')| \\
&\qquad + d^{\mu}(x,x') + d^{\mu}(y,y') \\
&\qquad + d^{\infty}\left(\widetilde{\Phi}(x), x'\right) + d^{\infty}\left(y', \widetilde{\Phi}(y)\right) \\
&\leqslant |d^{\mu}(x',y') - d^{\infty}(x',y')| \\
&\qquad + 2 \max_{j \in [N]} \sup_{p \in S(j)} \operatorname{diam}_{d^{\mu}}\left[\mathfrak{f}_{1,j}^{-1}(\{p\}) \cap U_1^{(r)}(j)\right] \\
&\qquad + 2 \max_{j \in [N]} \sup_{p \in S(j)} \operatorname{diam}_{d^{\infty}}\left[\mathfrak{f}_{2,j}^{-1}(\{p\}) \cap U_2^{(r)}(j)\right],
\end{aligned}$$

where eqns. (4.4.10) and (4.4.11) have been used in passing to the final inequality. Taking supremum over x and y and applying Lemma 4.4.6 yields:

$$\begin{aligned}
\sup_{x,y \in E_1} \left|d^{\mu}(x,y) - d^{\infty}\left(\widetilde{\Phi}(x), \widetilde{\Phi}(y)\right)\right| &\leqslant (N+2) \sup_{x',y' \in E^{(r)}} |d^{\mu,r}(x',y') - d^{\infty,r}(x',y')| \\
&\qquad + N \max_{j \in [N]} \sup_{x'',y'' \in \partial^{(r)}(j)} |d^{\mu}(x'',y'') - d^{\infty}(x'',y'')| \\
&\qquad + 2 \max_{j \in [N]} \sup_{p \in S(j)} \operatorname{diam}_{d^{\mu}}\left[\mathfrak{f}_{1,j}^{-1}(\{p\}) \cap U_1^{(r)}(j)\right] \\
&\qquad + 2 \max_{j \in [N]} \sup_{p \in S(j)} \operatorname{diam}_{d^{\infty}}\left[\mathfrak{f}_{2,j}^{-1}(\{p\}) \cap U_2^{(r)}(j)\right].
\end{aligned}$$

By Proposition 4.4.2, taking the limit superior as $\mu \to \infty$ yields:

$$\begin{aligned}\limsup_{\mu\to\infty} \sup_{x,y\in E_1} \left|d^\mu(x,y) - d^\infty\left(\widetilde{\Phi}(x), \widetilde{\Phi}(y)\right)\right| \leqslant N \limsup_{\mu\to\infty} \max_{j\in[N]} \sup_{x'',y''\in\partial^{(r)}(j)} &|d^\mu(x'',y'') - d^\infty(x'',y'')| \\ + 2\limsup_{\mu\to\infty} \max_{j\in[N]} \sup_{p\in S(j)} &\operatorname{diam}_{d^\mu}\left[\mathfrak{f}_{1,j}^{-1}(\{p\}) \cap U_1^{(r)}(j)\right] \\ + 2 \max_{j\in[N]} \sup_{p\in S(j)} &\operatorname{diam}_{d^\infty}\left[\mathfrak{f}_{2,j}^{-1}(\{p\}) \cap U_2^{(r)}(j)\right].\end{aligned}$$

Moreover, by conditions (ii), (iii) and (iv) in Theorem 4.2.5, taking the limit as $r \to 0$ yields:

$$\limsup_{\mu\to\infty} \sup_{x,y\in E_1} \left|d^\mu(x,y) - d^\infty\left(\widetilde{\Phi}(x), \widetilde{\Phi}(y)\right)\right| = 0,$$

and hence the limit at $\mu \to \infty$ also equals zero, as required.

□

Combining Lemmas 4.4.8 and 4.4.9 shows that $\mathfrak{D}\left[(E_1, d^\mu) \to (E_2, d^\infty)\right] \to 0$ as $\mu \to \infty$, completing the proof of Proposition 4.4.1. The proof of Theorem 4.2.5 is now completed by combining Proposition 4.3.7 and Proposition 4.4.1, together with Theorem 2.4.3.

Chapter 5

The unboundedness above and below of $\mathcal{H}_4$, $\widetilde{\mathcal{H}}_3$, $\widetilde{\mathcal{H}}_4$ and the unboundedness below of $\mathcal{H}_3$

This chapter combines direct local calculations with the Vitali Covering Theorem to prove the unboundedness above and below (in a logarithmic sense) of the Hitchin functionals on G_2 4-forms, $\widetilde{\mathrm{G}}_2$ 3-forms and $\widetilde{\mathrm{G}}_2$ 4-forms, and the unboundedness below of the Hitchin functional on G_2 3-forms. As scholia, initial conditions of the Laplacian coflow which lead to non-convergent solutions are shown to be dense and the critical points of the Hitchin functionals on G_2 4-forms, $\widetilde{\mathrm{G}}_2$ 3-forms and $\widetilde{\mathrm{G}}_2$ 4-forms are shown to be saddles.

5.1 Local volume-altering perturbations of closed G_2 4-forms

View $\mathbb{R}^7$ as a manifold with canonical coordinates $(x^1, ..., x^7)$ and let B_η denote the ball of Euclidean radius $\eta > 0$ about 0. Given $\psi \in \Omega^4_+(\mathbb{R}^7)$, write $\mathcal{H}_{4,\mathrm{B}_1}(\psi) = \int_{\mathrm{B}_1} vol_\psi$. Initially consider the G_2 4-form on $\mathbb{R}^7$ defined by:

$$\psi_0 = \mathrm{d}x^{4567} + \mathrm{d}x^{2367} + \mathrm{d}x^{2345} + \mathrm{d}x^{1357} - \mathrm{d}x^{1346} - \mathrm{d}x^{1256} - \mathrm{d}x^{1247}. \tag{5.1.1}$$

The key result upon which this chapter's investigation of $\mathcal{H}_4$ is founded is:

Lemma 5.1.2.

1. *There exist* $\alpha^\pm \in \Omega^3(\mathbb{R}^7)$ *with* $\mathrm{supp}(\alpha^\pm) \Subset \mathrm{B}_1$ *such that*

$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^\pm, \mathrm{d}\alpha^\pm) \gtrless 0.$$

2. *There exist* $\beta^\pm \in \Omega^3(\mathbb{R}^7)$ *with* $\mathrm{supp}(\beta^\pm) \Subset \mathrm{B}_1$ *such that:*

$$\psi^\pm = \psi_0 + \mathrm{d}\beta^\pm \text{ is a } G_2 \text{ 4-form} \tag{5.1.3}$$

and

$$\mathcal{H}_{4,\mathrm{B}_1}(\psi^\pm) \gtrless \mathcal{H}_{4,\mathrm{B}_1}(\psi_0). \tag{5.1.4}$$

Proof. Using Proposition 2.2.17, the Hessian of $\mathcal{H}_{4,\mathrm{B}_1}$ is given by:

$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\gamma_1,\gamma_2) = \int_{\mathrm{B}_1}\frac{1}{4}\left(\frac{3}{4}g_{\psi_0}(\pi_1\gamma_1,\pi_1\gamma_2) + g_{\psi_0}(\pi_7\gamma_1,\pi_7\gamma_2) - g_{\psi_0}(\pi_{27}\gamma_1,\pi_{27}\gamma_2)\right)vol_{\psi_0}, \tag{5.1.5}$$

where $\pi_\bullet$ denotes the type decomposition with respect to ψ_0. For (1), consider the choices:

$$\alpha^+ = f(r)\cdot\bigstar_{\psi_0}\psi_0 \quad \text{and} \quad \alpha^- = f(r)\cdot \mathrm{d}x^{123},$$

where $r = \sqrt{(x^1)^2 + ... + (x^7)^2}$ and $f : [0,1] \to [0,1]$ is a smooth function such that $f \equiv 1$ on a neighbourhood of 0 and $f \equiv 0$ on a neighbourhood of 1. Then since $\mathrm{d}\bigstar_{\psi_0}\psi_0 = 0$, one finds $\mathrm{d}\alpha^+ = \mathrm{d}f\wedge\bigstar_{\psi_0}\psi_0 \in \Omega^4_7(\mathrm{B}_1)$ (cf. eqn. (2.2.10)). Hence using eqn. (5.1.5):

$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^+,\mathrm{d}\alpha^+) = \int_{\mathrm{B}_1}\frac{1}{4}\,\|\mathrm{d}\alpha^+\|^2_{\psi_0}\,vol_{\psi_0} > 0,$$

as required. For α^-, one computes that:

$$\mathrm{d}\alpha^- = \mathrm{d}\big(f(r)\cdot\mathrm{d}x^{123}\big) = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}(x^1\mathrm{d}x^1 + ... + x^7\mathrm{d}x^7)\wedge\mathrm{d}x^{123} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\cdot\nu,$$

where $\nu = (x^1\mathrm{d}x^1 + ... + x^7\mathrm{d}x^7)\wedge\mathrm{d}x^{123}$. One can verify directly that:

$$\|\pi_1(\nu)\|^2_{\psi_0} = 0, \qquad \|\pi_7(\nu)\|^2_{\psi_0} = \frac{1}{4}\left((x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\right)$$
$$\|\pi_{27}(\nu)\|^2_{\psi_0} = \frac{3}{4}\left((x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\right).$$

Thus, for this choice of α^-, one computes using eqn. (5.1.5) that:

$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^-,\mathrm{d}\alpha^-) = -\int_{\mathrm{B}_1}\frac{1}{8}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\right)^2\left((x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\right)vol_0 < 0.$$

For (2), take $\beta^\pm = t\alpha^\pm$ for some $t > 0$ to be determined later. Then $\psi^\pm(t)$ satisfies eqn. (5.1.3) for all t sufficiently small, since G_2 4-forms are stable. Using Proposition 2.2.17, one may Taylor expand:

$$\mathcal{H}_{4,\mathrm{B}_1}(\psi^\pm) = \mathcal{H}_{4,\mathrm{B}_1}(\psi_0) + \frac{t}{4}\int_{\mathrm{B}_1}\mathrm{d}\alpha^\pm\wedge\bigstar_{\psi_0}\psi_0 + \frac{t^2}{2}\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^\pm,\mathrm{d}\alpha^\pm) + \mathcal{O}(t^3).$$

Since $\mathrm{supp}(\alpha^\pm) \Subset \mathrm{B}_1$ and $\mathrm{d}\bigstar_{\psi_0}\psi_0 = 0$, by Stokes' Theorem one finds $\int_{\mathrm{B}_1}\mathrm{d}\alpha^\pm\wedge\bigstar_{\psi_0}\psi_0 = 0$. Thus:

$$\mathcal{H}_{4,\mathrm{B}_1}(\psi^\pm) = \mathcal{H}_{4,\mathrm{B}_1}(\psi_0) + \frac{t^2}{2}\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^\pm,\mathrm{d}\alpha^\pm) + \mathcal{O}(t^3).$$

Part (2) of the lemma now follows by taking $t > 0$ sufficiently small.

□

Next, I generalise the results of Lemma 5.1.2 to arbitrary closed G_2 4-forms. Let ψ be a closed G_2 4-form on $\mathbb{R}^7$ such that $\psi|_0 = \psi_0|_0$. (Note that this condition always holds in suitable coordinates on $\mathbb{R}^7$.) Let $\mathrm{B}_\eta(\psi)$ denote the geodesic ball of radius η centred at 0, defined by the metric g_ψ.

Proposition 5.1.6. *There exist* $\eta_0 > 0$ *(depending on* ψ*) and* $\varepsilon > 0$ *(independent of* ψ*) such that for all* $\eta \in (0, \eta_0]$*:*

1. *There exist* $\alpha_\eta^\pm \in \Omega^3(\mathbb{R}^7)$ *with* $\mathrm{supp}(\alpha_\eta^\pm) \Subset \mathrm{B}_\eta(\psi)$ *such that*
$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_\eta(\psi)}|_\psi(\mathrm{d}\alpha_\eta^\pm, \mathrm{d}\alpha_\eta^\pm) \gtrless 0.$$

2. *There exist* $\beta_\eta^\pm \in \Omega^3(\mathbb{R}^7)$ *with* $\mathrm{supp}(\beta_\eta^\pm) \Subset \mathrm{B}_\eta(\psi)$ *such that* $\psi_\eta^\pm = \psi + \mathrm{d}\beta_\eta^\pm$ *is a* G_2 *4-form and:*
$$\mathcal{H}_{4,\mathrm{B}_\eta(\psi)}(\psi_\eta^\pm) \gtreqless (1 \pm \varepsilon)\mathcal{H}_{4,\mathrm{B}_\eta(\psi)}(\psi).$$

Proof. The proof is a simple scaling argument. Firstly, note that the statements in (1) and (2) are diffeomorphism invariant and invariant under rescaling:
$$\eta \mapsto \lambda^{\frac{1}{4}}\eta, \quad \psi \mapsto \lambda\psi, \quad \alpha_\eta^\pm \mapsto \lambda\alpha_\eta^\pm \quad \text{and} \quad \beta_\eta^\pm \mapsto \lambda\beta_\eta^\pm$$
for any $\lambda > 0$ (in particular, note that $\mathrm{B}_{\lambda^{\frac{1}{4}}\eta}(\lambda\psi) = \mathrm{B}_\eta(\psi)$). For each $\eta > 0$, consider the diffeomorphism $\mu_\eta : x \in \mathbb{R}^7 \mapsto \eta x \in \mathbb{R}^7$ and define $\psi_\eta = \eta^{-4}\mu_\eta^*\psi$. Then by scale and diffeomorphism invariance, to prove the proposition it suffices to prove that for all $\eta > 0$ sufficiently small (depending on ψ) there exist $\alpha^\pm, \beta^\pm$ satisfying:
$$\mathrm{supp}(\alpha^\pm),\ \mathrm{supp}(\beta^\pm) \Subset \mathrm{B}_1(\psi_\eta) \tag{5.1.7}$$
such that:
$$\mathcal{D}^2\mathcal{H}_{4,\mathrm{B}_1(\psi_\eta)}|_{\psi_\eta}(\mathrm{d}\alpha^\pm, \mathrm{d}\alpha^\pm) \gtrless 0; \tag{5.1.8}$$
$$\psi_\eta + \mathrm{d}\beta^\pm \text{ is a } G_2 \text{ 4-form} \tag{5.1.9}$$
and:
$$\mathcal{H}_{4,\mathrm{B}_1(\psi_\eta)}(\psi_\eta + \mathrm{d}\beta^\pm) \gtrless (1 \pm \varepsilon)\mathcal{H}_{4,\mathrm{B}_1(\psi_\eta)}(\psi_\eta). \tag{5.1.10}$$
Fix $\alpha^\pm$, $\beta^\pm$ to be as in Lemma 5.1.2 and choose $\varepsilon > 0$ such that:
$$\mathcal{H}_{4,\mathrm{B}_1}(\psi^\pm) \gtrless (1 \pm \varepsilon)\mathcal{H}_{4,\mathrm{B}_1}(\psi_0).$$
Then by Lemma 5.1.2 each of eqns. (5.1.7), (5.1.8), (5.1.9) and (5.1.10) hold with ψ_0 in place of ψ_η (note in particular that $\mathrm{B}_1(\psi_0) = \mathrm{B}_1$). Now as $\eta \to 0$, $\psi_\eta \to \psi_0$ locally uniformly on $\mathbb{R}^7$ in all derivatives. As each of eqns. (5.1.7), (5.1.8), (5.1.9) and (5.1.10) are open conditions on the underlying G_2 4-form ψ_η, it follows that the equations are satisfied for all $\eta > 0$ sufficiently small. This completes the proof.

□

As an initial application of Proposition 5.1.6, I prove the first part of Theorem 1.0.5:

Theorem 5.1.11. *The critical points of* $\mathcal{H}_4$ *are always saddles. Specifically, let* M *be a closed, oriented 7-manifold (or, more generally, 7-orbifold) let* ψ *be a torsion-free* G_2 *4-form on* M *and consider the Hitchin functional* $\mathcal{H}_4 : [\psi]_+ \to (0, \infty)$*. Then there exist infinite-dimensional subspaces* $\mathcal{S}_4^{\pm}(\psi) \subset T_\psi[\psi]_+ \cong d\Omega^3(M)$ *along which* $\mathcal{D}^2\mathcal{H}_4|_\psi$ *is positive definite and negative definite respectively.*

Proof. Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable disjoint collection of open balls in M (in the case that M is an orbifold, require the balls to lie in the smooth locus of M). By Proposition 5.1.6, for each $i \in \mathbb{N}$ there exist 3-forms $\alpha_i^{\pm}$ on M with $\mathrm{supp}(\alpha_i^{\pm}) \Subset B_i$ such that $\mathcal{D}^2\mathcal{H}_4|_\psi$ is positive definite (respectively negative definite) along $d\alpha_i^{\pm}$. Now take $\mathcal{S}_4^{\pm}(\psi)$ to be the infinite-dimensional subspace of $d\Omega^3(M)$ given by all finite linear combinations of the $d\alpha_i^{\pm}$. It is simple to verify that $\mathcal{D}^2\mathcal{H}_4|_\psi$ is positive and negative definite along $\mathcal{S}_4^{\pm}(\psi)$ respectively, as required.

□

5.2 Laplacian coflow: density of initial conditions leading to non-convergent solutions

Let M be an oriented 7-manifold (not necessarily closed), let $\psi \in \Omega^4_+(M)$ be a closed G_2 4-form and recall the set $[\psi]_+ \subset \Omega^4(M)$. Given a Riemannian metric g on M and a countable exhaustion of M by compact subsets $K_0 \subseteq K_1 \subseteq ... \subseteq M$, the countable family of seminorms $\| - \|_{C^0_g(K_n)}$ on $\Omega^4(M)$ is separating and induces the C^0 topology on $\Omega^4(M)$ (and hence on $[\psi]_+$); this topology is independent of the choice of g and K_n.

Now recall that the Laplacian coflow of ψ is the solution of the evolution PDE:

$$\frac{\partial\psi(t)}{\partial t} = \Delta_{\psi(t)}\psi(t) = dd^*_{\psi(t)}\psi(t) \quad \text{and} \quad \psi(0) = \psi. \tag{5.2.1}$$

Using this terminology, I now prove:

Theorem 5.2.2. Let M be an oriented 7-manifold (not necessarily closed) and let $\psi \in \Omega^4_+(M)$ be a closed G_2 4-form. Consider the space:

$$\mathcal{O}_{[\psi]_+} = \left\{ \psi' \in [\psi]_+ \;\middle|\; \begin{array}{l} \text{no solution to the Laplacian coflow started} \\ \text{at } \psi' \text{ converges to a torsion-free } G_2 \text{ 4-form} \end{array} \right\}.$$

Then $\mathcal{O}_{[\psi]_+} \subset [\psi]_+$ is dense in the C^0 topology.

Proof. Begin by considering the open ball $B_\eta \subset \mathbb{R}^7$ equipped with the standard flat G_2 4-form ψ_0 and choose $\alpha_\eta^+ \in \Omega^3(B_\eta)$ compactly supported such that:

$$\mathcal{D}^2\mathcal{H}_{4,B_\eta}|_{\psi_0}(d\alpha_\eta^+, d\alpha_\eta^+) > 0 \tag{5.2.3}$$

according to Proposition 5.1.6. I begin by proving that for all $s > 0$ sufficiently small, the Laplacian coflow on B_η starting from $\psi_0 + s\mathrm{d}\alpha^+_\eta$ cannot converge to a torsion-free G_2 4-form.

Indeed, let $\widehat{\psi}$ be a torsion-free G_2 4-form on $\mathbb{R}^7$ such that:

$$\operatorname{supp}\left(\widehat{\psi} - \psi_0\right) \Subset \mathrm{B}_\eta. \tag{5.2.4}$$

Since $\widehat{\psi}$ is torsion-free, $g_{\widehat{\psi}}$ is Ricci-flat [114, Prop. 11.8]. Moreover, the mean curvature of $\partial\overline{\mathrm{B}_\eta} = S^6_\eta$ as a submanifold of $(\mathbb{R}^7, g_{\widehat{\psi}})$ with respect to the inwards pointing normal is $\frac{1}{\eta}$, since $g_{\widehat{\psi}}$ is simply the Euclidean metric in a neighbourhood of S^6_η by eqn. (5.2.4). Thus, using [67, Thm. 2.1], it follows that:

$$\begin{aligned}\mathcal{H}_{4,\mathrm{B}_\eta}\left(\widehat{\psi}\right) &\leqslant \int_0^\eta \left(1 - \frac{r}{\eta}\right)^7 \mathrm{d}r \cdot \mathrm{Vol}_{S^6_\eta}\left(\widehat{\psi}\right)\\ &= \frac{\eta}{7}\mathrm{Vol}_{S^6_\eta}(\psi_0),\end{aligned} \tag{5.2.5}$$

where $\mathrm{Vol}_{S^6_\eta}\left(\widehat{\psi}\right)$ is the volume of S^6_η with respect to the metric induced by $\widehat{\psi}$, which is the same as the metric induced by ψ_0 using eqn. (5.2.4). A direct calculation shows that eqn. (5.2.5) is saturated when $\widehat{\psi} = \psi_0$. Hence for all torsion-free $\widehat{\psi}$ as above:

$$\mathcal{H}_{4,\mathrm{B}_\eta}\left(\widehat{\psi}\right) \leqslant \mathcal{H}_{4,\mathrm{B}_\eta}(\psi_0). \tag{5.2.6}$$

Now let $\psi_s(t)$ denote a solution to the Laplacian coflow started at $\psi_0 + s\mathrm{d}\alpha^+_\eta$ and suppose that $\psi_s(t)$ existed for all t and converged to a torsion-free G_2 4-form $\widehat{\psi}$ as $t \to \infty$. Since Laplacian coflow preserves ψ_0, $\psi_s(t)$ is fixed on the region where $\psi_s(0) = \psi_0$ and hence:

$$\operatorname{supp}\left(\widehat{\psi} - \psi_0\right) \subseteq \operatorname{supp}(\psi_s(0) - \psi_0) \Subset \mathrm{B}_\eta.$$

Thus, using eqn. (5.2.6), one has $\mathcal{H}_{4,\mathrm{B}_\eta}\left(\widehat{\psi}\right) \leqslant \mathcal{H}_{4,\mathrm{B}_\eta}(\psi_0)$. However, the Laplacian coflow increases volume pointwise [59, eqn. (4.32)]. Hence:

$$\begin{aligned}\mathcal{H}_{4,\mathrm{B}_\eta}\left(\widehat{\psi}\right) &\geqslant \mathcal{H}_{4,\mathrm{B}_\eta}(\psi_s(0)) = \mathcal{H}_{4,\mathrm{B}_\eta}(\psi_0 + s\mathrm{d}\alpha^+_\eta)\\ &> \mathcal{H}_{4,\mathrm{B}_\eta}(\psi_0)\end{aligned}$$

where the last line follows from eqn. (5.2.3) for all $s > 0$ sufficiently small (cf. the proof of Lemma 5.1.2) contradicting eqn. (5.2.6).

Thus there are (uniformly) arbitrarily small compactly-supported perturbations $\psi_s(0) = \psi_0 + s\mathrm{d}\alpha^+_\eta$ of ψ_0 such that the Laplacian coflow started from $\psi_s(0)$ cannot converge to a torsion-free G_2 4-form. To complete the proof therefore, it suffices to prove that given any M, ψ as in the statement of the theorem and any $\psi' \in [\psi]_+$, there exists a closed G_2 4-form $\psi'' \in [\psi]_+$, arbitrarily close to ψ' in the C^0 topology, such that ψ'' is diffeomorphic to ψ_0 in some small neighbourhood of M. This follows from the subsequent local result:

Claim 5.2.7. *Let ψ' be a closed G_2 4-form on $\mathbb{R}^7$ such that $\psi|_0 = \psi_0|_0$. Then for all $\delta > 0$, there* *exists* $\eta > 0$ *and* $\alpha \in \Omega^3(\mathbb{R}^7)$ *with* $\operatorname{supp}(\alpha) \Subset \mathrm{B}_{2\eta}$ *such that:*

$$\psi' + \mathrm{d}\alpha = \psi_0 \text{ on } \mathrm{B}_\eta \quad \text{ and } \quad \|\mathrm{d}\alpha\|_{C^0_{\psi'}} < \delta.$$

(Note that $\psi' + \mathrm{d}\alpha$ *is automatically of* G_2*-type for* $\delta > 0$ *sufficiently small, by the stability of* G_2 *4-forms.)*

Proof of Claim. Consider the 4-form $\psi_0 - \psi'$. Since $\psi_0 - \psi'$ vanishes at 0, there is some constant $C_1 > 0$ such that for all η:

$$\|\psi_0 - \psi'\|_{C^0_{\psi'}(\mathrm{B}_{2\eta})} \leqslant C_1\eta. \tag{5.2.8}$$

Similarly, since $\mathrm{d}(\psi_0 - \psi') = 0$, one can choose a primitive $\varpi \in \Omega^3\left(\mathbb{R}^7\right)$ for $\psi_0 - \psi'$ such that for some constant $C_2 > 0$ and all η:

$$\|\varpi\|_{C^0_{\psi'}(\mathrm{B}_{2\eta})} \leqslant C_2\eta^2. \tag{5.2.9}$$

Indeed (cf. [48, p. 16]) identify $\mathbb{R}^7\backslash\{0\} \cong (0,\infty) \times S^6$ and write $\psi_0 - \psi' = \sigma_1 + \mathrm{d}t \wedge \sigma_2$, where t is the parameter along $(0,\infty)$, σ_i depends parametrically on t ($i = 1,2$) and $\mathrm{d}\sigma_1 = 0$ and $\frac{\partial\sigma_1}{\partial t} = \mathrm{d}\sigma_2$ (since $\mathrm{d}(\psi_0 - \psi') = 0$). Set $\varpi = \int_0^t \sigma_2 \mathrm{d}t$. Then:

$$\mathrm{d}\varpi = \int_0^t \frac{\partial\sigma_1}{\partial t}\mathrm{d}t + \mathrm{d}t \wedge \sigma_2 = \sigma_1 + \mathrm{d}t \wedge \sigma_2 = \psi_0 - \psi'$$

and ϖ clearly satisfies eqn. (5.2.9), as required.

Next, fix a smooth function $f : [0,1] \to [0,1]$ such that $f \equiv 1$ on a neighbourhood of 0 and $f \equiv 0$ on a neighbourhood of 1. Given any η, define:

$$f_\eta = \begin{cases} 1 & r \in [0,\eta], \\ f\left(\dfrac{r-\eta}{\eta}\right) & r \in [\eta, 2\eta] \end{cases}$$

and set $\alpha = f_\eta(r)\varpi$. Then $\psi' + \mathrm{d}\alpha = \psi' + \mathrm{d}\varpi = \psi_0$ on B_η, as required. Moreover:

$$\begin{aligned} \|\mathrm{d}\alpha\|_{C^0_{\psi'}} &= \left\| \frac{\mathrm{d}f_\eta}{\mathrm{d}r}\mathrm{d}r \wedge \varpi + f_\eta \mathrm{d}\varpi \right\|_{C^0_{\psi'}} \\ &\leqslant \frac{\sup|f'|}{\eta}\|\mathrm{d}r \wedge \varpi\|_{C^0_{\psi'}(\mathrm{B}_{2\eta})} + \|\psi_0 - \psi'\|_{C^0_{\psi'}(\mathrm{B}_{2\eta})} \\ &\leqslant C_3\eta, \end{aligned}$$

where the first inequality follows from the fact that $\operatorname{supp}(f_\eta) \Subset \mathrm{B}_{2\eta}$ and the second inequality follows from eqns. (5.2.8) and (5.2.9). The claim now follows by taking $\eta > 0$ sufficiently small. This in turn completes the proof of Theorem 1.0.7.

□

□

5.3 The unboundedness above and below of $\mathcal{H}_4$

The aim of this section is to prove the following result:

Theorem 5.3.1. *The functional $\mathcal{H}_4$ is always unbounded above and below. Specifically, let M be a closed 7-manifold (or, more generally, a closed orbifold) and let ψ be a closed G_2 4-form on M. Then:*

$$\inf_{\psi' \in [\psi]_+} \mathcal{H}_4(\psi') = 0 \quad \text{and} \quad \sup_{\psi' \in [\psi]_+} \mathcal{H}_4(\psi') = \infty.$$

Let (X, d) be a metric space and μ a Borel measure on X. Write $\overline{\mathrm{B}_r(x)}$ for the closed metric ball of radius $r > 0$ centred at $x \in X$. (X, d, μ) is termed a doubling metric measure space if there exist constants $C, R > 0$ such that for all $x \in X$ and $r \in (0, R]$:

$$\mu\left(\overline{\mathrm{B}_{2r}(x)}\right) \leqslant C\mu\left(\overline{\mathrm{B}_r(x)}\right). \tag{5.3.2}$$

Doubling metric measure spaces satisfy the following important property [66, Thm. 1.6] (N.B. whilst the statement *loc. cit.* requires that eqn. (5.3.2) hold for balls of arbitrarily large radius, the proof only uses the weaker condition stated above):

Theorem 5.3.3 (Vitali Covering Theorem)**.** *Let (X, d, μ) be a doubling metric measure space, let $A \subseteq X$ be Borel measurable and suppose that $\mathscr{F}$ is a family of closed balls in X centred at points of A such that $\inf\left\{r \,\middle|\, \overline{\mathrm{B}_r(a)} \in \mathscr{F}\right\} = 0$ for every $a \in A$. Then there exist disjoint balls $\left\{\overline{\mathrm{B}_{r_i}(a_i)}\right\}_{i=0}^{\infty} \in \mathscr{F}^{\mathbb{N}}$ such that:*

$$\mu\left(A \Big\backslash \bigcup_{i=0}^{\infty} \overline{\mathrm{B}_{r_i}(a_i)}\right) = 0.$$

Now let (M, ψ) be an oriented 7-manifold equipped with a closed G_2 4-form and for each $p \in \mathrm{M}$, let $\mathrm{B}_r(p)$ denote the geodesic ball of radius r centred at p defined by the metric g_ψ. By applying Proposition 5.1.6 about each point $p \in \mathrm{M}$, one immediately obtains the following result:

Lemma 5.3.4. *Let $\varepsilon > 0$ be as in Proposition 5.1.6. Then for each $p \in \mathrm{M}$, there exists $\eta_p > 0$ (depending on ψ) such that for all $\eta \in (0, \eta_p]$, there exist $\breve{\alpha}_\eta^{p/\pm} \in \Omega^3(\mathrm{M})$ with $\operatorname{supp}(\breve{\alpha}_\eta^{p/\pm}) \Subset \mathrm{B}_\eta(p)$ satisfying:*

$$\breve{\psi}_\eta^{p/\pm} = \psi + \mathrm{d}\breve{\alpha}_\eta^{p/\pm} \text{ is a } G_2 \text{ 4-form}$$

and:

$$\mathcal{H}_{4,\mathrm{B}_\eta(p)}\left(\breve{\psi}_\eta^{p/\pm}\right) \gtreqless (1 \pm \varepsilon)\mathcal{H}_{4,\mathrm{B}_\eta(p)}(\psi).$$

I now prove Theorem 5.3.1:

Proof. Firstly, choose $\nu > 0$ sufficiently small that:

$$(1+\varepsilon)(1-\nu) \geqslant \left(1 + \frac{\varepsilon}{2}\right) \quad \text{and} \quad 1 - \varepsilon + \nu\varepsilon \leqslant \left(1 - \frac{\varepsilon}{2}\right), \tag{5.3.5}$$

where $\varepsilon > 0$ was defined in Proposition 5.3.4.

Initially, let M be a closed, oriented 7-manifold and ψ a closed G_2 4-form on M. The Riemannian metric g_ψ and the volume form vol_ψ define a natural metric d_ψ and measure μ_ψ on M and, since M is compact, (M, d_ψ, μ_ψ) is a doubling metric measure space. Now take:

$$\mathscr{F} = \left\{ \overline{B_\eta(p)} \;\middle|\; p \in M,\ \eta \in (0, \eta_p] \right\}$$

for η_p as in Lemma 5.3.4, and choose $\left\{ \overline{B_{\eta_i}(p_i)} \right\}_{i=0}^{\infty}$ disjoint and measure-theoretically covering M, as in Theorem 5.3.3. Since:

$$\mu_\psi \left(\overline{B_{\eta_i}(p_i)} \middle\backslash B_{\eta_i}(p_i) \right) = 0$$

for all i, it follows that the open balls $\{B_{\eta_i}(p_i)\}_{i=0}^{\infty}$ also measure-theoretically cover M. Hence:

$$\mathcal{H}_4(\psi) = \sum_{i=0}^{\infty} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi).$$

In particular, the right-hand sum is convergent and so there is $N \geqslant 0$ such that:

$$\sum_{i=N+1}^{\infty} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi) < \nu \mathcal{H}_4(\psi). \tag{5.3.6}$$

Now let:

$$\psi_1^{\pm} = \psi + \sum_{i=0}^{N} \mathrm{d}\breve{\alpha}_{\eta_i}^{p_i/\pm},$$

where $\breve{\alpha}_{\eta_i}^{p_i/\pm}$ are defined according to Lemma 5.3.4. Then $\psi_1^{\pm} \in [\psi]_+$. Moreover, by the estimates obtained in Lemma 5.3.4 and eqn. (5.3.5):

$$\begin{aligned}
\mathcal{H}_4(\psi_1^-) &= \sum_{i=0}^{N} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi_1^-) + \sum_{i=N+1}^{\infty} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi_1^-) \\
&\leqslant (1-\varepsilon) \sum_{i=0}^{N} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi) + \sum_{i=N+1}^{\infty} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi) \\
&\leqslant (1-\varepsilon)\mathcal{H}_4(\psi) + \varepsilon \sum_{i=N+1}^{\infty} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi) \\
&\leqslant (1 - \varepsilon + \nu\varepsilon)\mathcal{H}_4(\psi) \leqslant \left(1 - \frac{\varepsilon}{2}\right) \mathcal{H}_4(\psi)
\end{aligned}$$

and

$$\mathcal{H}_4(\psi_1^+) \geqslant \sum_{i=0}^{N} \mathcal{H}_{4, B_{\eta_i}(p_i)}(\psi_1^+) \geqslant (1+\varepsilon)(1-\nu)\mathcal{H}_4(\psi) \geqslant \left(1 + \frac{\varepsilon}{2}\right) \mathcal{H}_4(\psi).$$

Now recursively define $\psi_n^{\pm}$ by applying the above argument to $\psi_{n-1}^{\pm}$ respectively for $n \geqslant 2$. Then since the value of ε in Proposition 5.3.4 is independent of the choice of closed G_2 4-form, it follows that for all $n \geqslant 0$:

$$\mathcal{H}_4(\psi_n^-) \leqslant \left(1 - \frac{\varepsilon}{2}\right)^n \mathcal{H}_4(\psi) \to 0 \text{ as } n \to \infty$$

and

$$\mathcal{H}_4(\psi_n^+) \geqslant \left(1 + \frac{\varepsilon}{2}\right)^n \mathcal{H}_4(\psi) \to +\infty \text{ as } n \to \infty,$$

as required.

In the case where M is a closed orbifold, consider the smooth locus M_{smooth} (note that the singular locus has positive codimension and thus measure zero). For each component M' of M_{smooth}, the Riemannian metric g_ψ and volume form vol_ψ again define a metric d_ψ and measure μ_ψ on M', and $(\mathrm{M}', d_\psi, \mu_\psi)$ is a doubling metric measure space by compactness of the original orbifold M. The rest of the argument now proceeds as before.

□

Remark 5.3.7. In [36, 37, 35], Donaldson explained how to extend the Hitchin functional $\mathcal{H}_3$ from closed 7-manifolds to compact 7-manifolds with boundary, by considering the data of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on the boundary of M. In the same way one can also extend the functional $\mathcal{H}_4$ to compact manifolds with boundary by considering suitable geometric data on the boundary. Theorem 5.3.1 is then also valid on compact manifolds with boundary; the argument is essentially the same as the orbifold case, with M_{smooth} replaced by $\mathrm{M}\backslash\partial\mathrm{M}$.

Likewise, the unboundedness of the functionals $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$ proved in this chapter also holds equally on compact manifolds with boundary.

5.4 The unboundedness below of $\mathcal{H}_3$

The deduction of Lemma 5.1.2(2) from Lemma 5.1.2(1), the proof of Proposition 5.1.6 and the arguments of §5.3 all apply equally to either $\mathcal{H}_3$ or $\mathcal{H}_4$. However Lemma 5.1.2(1) has no complete analogue for $\mathcal{H}_3$. Indeed, let $\mathrm{B}_1 \subset \mathbb{R}^7$ denote the open unit ball in $\mathbb{R}^7$ and let:

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{247} - \mathrm{d}x^{347} - \mathrm{d}x^{356},$$

be the ‘standard’ flat G_2 3-form on $\mathbb{R}^7$. Suppose there were α^+ such that $\mathcal{D}^2\mathcal{H}_{3,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^+, \mathrm{d}\alpha^+) > 0$. Then embedding $\mathrm{B}_1 \hookrightarrow \mathbb{R}^7 \big/ 3\mathbb{Z}^7 \cong \mathbb{T}^7$ and extending α by zero over all of $\mathbb{T}^7$, one would find:

$$\mathcal{D}\mathcal{H}_{3,\mathbb{T}^7}|_{\phi}(\mathrm{d}\alpha^+, \mathrm{d}\alpha^+) > 0.$$

However, in [71] Hitchin proved that on any closed manifold M with torsion-free G_2 3-form ϕ, the Hessian $\mathcal{D}^2\mathcal{H}_3|_\phi$ is negative definite (modulo diffeomorphisms), and thus a contradiction has been reached. In particular, the arguments presented in this chapter cannot be applied to prove that $\mathcal{H}_3$ is unbounded above. There is, however, a partial analogue of Lemma 5.1.2 for $\mathcal{H}_3$:

Lemma 5.4.1. *There exists* $\alpha^- \in \Omega^2(\mathbb{R}^7)$ *with* $\mathrm{supp}(\alpha^-) \Subset \mathrm{B}_1$ *such that:*

$$\mathcal{D}^2\mathcal{H}_{3,\mathrm{B}_1}|_{\psi_0}(\mathrm{d}\alpha^-, \mathrm{d}\alpha^-) < 0.$$

Proof. From Proposition 2.2.17 it suffices to construct $\alpha^- \in \Omega^2(\mathbb{R}^7)$ with $\operatorname{supp}\alpha^- \Subset \mathrm{B}_1$ such that:

$$\mathscr{I} = \int_{\mathrm{B}_1} \left(\frac{4}{3} \|\pi_1(\mathrm{d}\alpha^-)\|^2_{\phi_0} + \|\pi_7(\mathrm{d}\alpha^-)\|^2_{\phi_0} - \|\pi_{27}(\mathrm{d}\alpha^-)\|^2_{\phi_0} \right) vol_{\phi_0} < 0.$$

Consider $\alpha^- = f(r) \cdot \mathrm{d}x^{12}$, where $f : [0,1] \to [0,1]$ is a smooth function such that $f \equiv 1$ on a neighbourhood of 0 and $f \equiv 0$ on a neighbourhood of 1 (and $r = \sqrt{(x^1)^2 + ... + (x^7)^2}$). Then:

$$\mathrm{d}\alpha^- = \mathrm{d}\left(f(r) \cdot x^{12}\right) = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}(x^1\mathrm{d}x^1 + ... + x^7\mathrm{d}x^7) \wedge \mathrm{d}x^{12} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r} \cdot \nu,$$

where $\nu = (x^1\mathrm{d}x^1 + ... + x^7\mathrm{d}x^7) \wedge \mathrm{d}x^{12}$. A direct calculation yields:

$$\|\nu\|^2_{\phi_0} = (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2, \qquad \|\pi_1(\nu)\|^2_{\phi_0} = \frac{1}{7}(x^3)^2,$$
$$\|\pi_7(\nu)\|^2_{\phi_0} = \frac{1}{4}\left((x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\right).$$

Thus, for this choice of α^-

$$\mathscr{I} = -\int_{\mathrm{B}} \left(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\right)^2 \left(\frac{2}{3}(x^3)^2 + \frac{1}{2}\left((x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2\right)\right) vol < 0$$

as required. □

I thus obtain:

Theorem 5.4.2. *For any closed, oriented 7-manifold (or, more generally, 7-orbifold)* M *and any closed* G_2 *3-form* ϕ *on* M*:*

$$\inf_{\phi' \in [\phi]_+} \mathcal{H}_3(\phi') = 0.$$

5.5 The unboundedness above and below of $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$

The aim of this section is to prove the following result:

Theorem 5.5.1. *Let* M *be a closed 7-manifold (or, more generally, a closed orbifold) and let* $\widetilde{\phi}$ *and* $\widetilde{\psi}$ *be closed* $\widetilde{\mathrm{G}}_2$ *3-forms and* $\widetilde{\mathrm{G}}_2$ *4-forms on* M *respectively. Then:*

- $\inf_{\widetilde{\phi}' \in [\widetilde{\phi}]_\sim} \widetilde{\mathcal{H}}_3\left(\widetilde{\phi}'\right) = 0$ *and* $\sup_{\widetilde{\phi}' \in [\widetilde{\phi}]_\sim} \widetilde{\mathcal{H}}_3\left(\widetilde{\phi}'\right) = \infty$*;*
- $\inf_{\widetilde{\psi}' \in [\widetilde{\psi}]_\sim} \widetilde{\mathcal{H}}_4\left(\widetilde{\psi}'\right) = 0$ *and* $\sup_{\widetilde{\psi}' \in [\widetilde{\psi}]_\sim} \widetilde{\mathcal{H}}_4\left(\widetilde{\psi}'\right) = \infty$*.*

As in §5.4, by repeating the arguments of §§5.1 & 5.3, it suffices to prove the following result:

Lemma 5.5.2. *Let* $\mathrm{B}_1 \subset \mathbb{R}^7$ *denote the open unit ball in* $\mathbb{R}^7$ *and let:*

$$\widetilde{\phi}_0 = \mathrm{d}x^{123} - \mathrm{d}x^{145} - \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{247} - \mathrm{d}x^{347} - \mathrm{d}x^{356}$$
$$\widetilde{\psi}_0 = \mathrm{d}x^{4567} - \mathrm{d}x^{2367} - \mathrm{d}x^{2345} + \mathrm{d}x^{1357} - \mathrm{d}x^{1346} - \mathrm{d}x^{1256} - \mathrm{d}x^{1247}$$

be the 'standard' flat $\widetilde{\mathrm{G}}_2$ *3-form and* $\widetilde{\mathrm{G}}_2$ *4-form on* $\mathbb{R}^7$ *respectively. Then:*

1. *There exists* $\beta^{\pm} \in \Omega^2(\mathbb{R}^7)$ *with* $\mathrm{supp}(\beta^{\pm}) \Subset \mathrm{B}_1$ *such that:*

$$\mathscr{I}^{\pm} = \int_{\mathrm{B}_1} \left(\frac{4}{3} \|\pi_1(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} + \|\pi_7(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} - \|\pi_{27}(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} \right) vol_0 \gtrless 0. \tag{5.5.3}$$

2. *There exists* $\beta^{\pm} \in \Omega^3(\mathbb{R}^7)$ *with* $\mathrm{supp}(\beta^{\pm}) \Subset \mathrm{B}_1$ *such that:*

$$\mathscr{J}^{\pm} = \int_{\mathrm{B}_1} \left(\frac{3}{4} \|\pi_1(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} + \|\pi_7(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} - \|\pi_{27}(\mathrm{d}\beta^{\pm})\|^2_{\widetilde{\phi}_0} \right) vol_0 \gtrless 0. \tag{5.5.4}$$

Proof. (1) Again let $r = \sqrt{(x^1)^2 + ... + (x^7)^2}$ and $f : [0,1] \to [0,1]$ be a smooth function such that $f \equiv 1$ on a neighbourhood of 0 and $f \equiv 0$ on a neighbourhood of 1. I claim that:

$$\beta^+ = f(r)\mathrm{d}x^{12} \quad \text{and} \quad \beta^- = f(r)\mathrm{d}x^{14}$$

satisfy eqn. (5.5.3). A direct calculation yields:

$$\mathscr{I}^+ = \int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r} \frac{1}{r} \right)^2 \left(-\frac{2\left(x^3\right)^2}{3} + \frac{\left(x^4\right)^2}{2} + \frac{\left(x^5\right)^2}{2} + \frac{\left(x^6\right)^2}{2} + \frac{\left(x^7\right)^2}{2} \right) vol_0$$

and

$$\mathscr{I}^- = \int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r} \frac{1}{r} \right)^2 \left(\frac{\left(x^2\right)^2}{2} + \frac{\left(x^3\right)^2}{2} - \frac{2\left(x^5\right)^2}{3} - \frac{\left(x^6\right)^2}{2} - \frac{\left(x^7\right)^2}{2} \right) vol_0.$$

Observe also that, by symmetry, for any distinct $i, j \in \{1, ..., 7\}$:

$$\int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r} \frac{1}{r} \right)^2 \left(\left(x^i\right)^2 - \left(x^j\right)^2 \right) vol_0 = 0.$$

Thus, for β^+, one sees that:

$$\mathscr{I}^+ = \int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r} \frac{1}{r} \right)^2 \left(\frac{\left(x^5\right)^2}{3} + \frac{\left(x^6\right)^2}{2} + \frac{\left(x^7\right)^2}{2} \right) vol_0 > 0.$$

Similarly, for β^-, one sees that:

$$\mathscr{I}^- = -\int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r} \frac{1}{r} \right)^2 \frac{2\left(x^5\right)^2}{3} vol_0 < 0.$$

(2) Now take:

$$\beta^+ = f(r)\mathrm{d}x^{123} \quad \text{and} \quad \beta^- = f(r)\mathrm{d}x^{124}.$$

By direct calculation:

$$\mathscr{I}^{+} = \int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\right)^2 \left(\frac{\left(x^4\right)^2}{2} + \frac{\left(x^5\right)^2}{2} + \frac{\left(x^6\right)^2}{2} + \frac{\left(x^7\right)^2}{2}\right) vol_0 > 0$$

and:

$$\mathscr{I}^{-} = \int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\right)^2 \left(\frac{\left(x^3\right)^2}{2} - \frac{\left(x^5\right)^2}{2} - \frac{\left(x^6\right)^2}{2} - \frac{3\left(x^7\right)^2}{4}\right) vol_0 = -\int_{\mathrm{B}_1} \left(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r}\right)^2 \left(\frac{\left(x^6\right)^2}{2} + \frac{3\left(x^7\right)^2}{4}\right) vol_0 < 0,$$

using symmetry as above.

□

As in §5.1, as a corollary, one obtains:

Theorem 5.5.5. *Let* M *be a closed, oriented 7-manifold (or, more generally, 7-orbifold) and let* $\widetilde{\phi}$ *be a torsion-free* $\widetilde{\mathrm{G}}_2$ *3-form on* M*. Then there exist infinite-dimensional subspaces* $\mathcal{S}_3^{\pm}\left(\widetilde{\phi}\right)$ *along which* $\mathcal{D}^2\widetilde{\mathcal{H}}_3|_{\widetilde{\phi}}$ *is positive definite and negative definite respectively. Similarly, let* $\widetilde{\psi}$ *be a torsion-free* $\widetilde{\mathrm{G}}_2$ *4-form on* M*. Then there exist infinite-dimensional subspaces* $\mathcal{S}_4^{\pm}\left(\widetilde{\psi}\right)$ *along which* $\mathcal{D}^2\widetilde{\mathcal{H}}_4|_{\widetilde{\psi}}$ *is positive definite and negative definite respectively. In particular, the critical points of both* $\widetilde{\mathcal{H}}_3$ *and* $\widetilde{\mathcal{H}}_4$ *are always saddles.*

Chapter 6

Concluding remarks and open questions

Here, I briefly describe two implications of the results from Part I with impact beyond the research area of Hitchin functionals.

6.1 Gromov–Hausdorff convergence without assumptions on curvature or injectivity radius

It is well-known that for all $n \in \mathbb{N}$, $r > 0$, $V > 0$, $D > 0$, $\kappa \in \mathbb{R}$, the following two sets of manifolds are precompact in the Gromov–Hausdorff metric [25, p. 265]:

1. the set of all n-manifolds with volume at most V and injectivity radius at least r;
2. the set of all n-manifolds with diameter at most D and sectional (or Ricci) curvature at least κ.

As a consequence, theorems which prove Gromov–Hausdorff convergence typically require either a lower bound on either injectivity radius or sectional/Ricci curvature.

By contrast, Theorem 4.4.1 assumes no lower bound on curvature or injectivity radius, and therefore (combined with Theorem 4.3.7) can be used to prove Gromov–Hausdorff convergence even in the absence of such bounds.

It is therefore possible that Theorems 4.4.1 and 4.3.7 could provide useful tools for proving Gromov–Hausdorff convergence in other situations where lower bounds on curvature/injectivity radius are not available.

6.2 The structure of Laplacian coflow in a neighbourhood of a torsion-free G_2-structure

Lotay–Wei recently proved the following result on the structure of Laplacian flow near a torsion-free G_2 3-form:

Theorem ([96, Thm. 1.3])**.** *Let ϕ be a torsion-free* G_2 *3-form on a closed 7-manifold* M. *Then there exists a neighbourhood U of ϕ in $[\phi]_+$ such that for any $\phi' \in U$, the Laplacian flow:*

$$\frac{\partial \phi(t)}{\partial t} = \Delta_{\phi(t)} \phi(t), \qquad \phi(0) = \phi'$$

exists for all time and converges smoothly to some $\phi_\infty \in \mathrm{Diff}_0(\mathrm{M}) \cdot \phi$ as $t \to \infty$.

In contrast, currently nothing is known about the behaviour of the Laplacian coflow in a neighbourhood of a torsion-free G_2 4-form ψ.

Theorem 5.1.11 provides some potential insight into this problem. In Chapter 11 (specifically Proposition 11.4.1) it is proven that the saddle-like critical points of the functional $\mathcal{H}_4$ are non-degenerate modulo the action of diffeomorphisms, giving explicit formulae for subspaces $\mathcal{S}_4^{\pm}(\psi)$ along which $\mathcal{D}^2\mathcal{H}_4$ is positive and negative definite respectively. Thus, one can write:

$$\mathrm{T}_\psi[\psi]_+ = \mathrm{T}\left(\mathrm{Diff}_0(\mathrm{M}) \cdot \psi\right) \oplus \mathcal{S}_4^+(\psi) \oplus \mathcal{S}_4^-(\psi).$$

Motivated by this, it is natural to ask the following question:

Question 6.2.1. *Let* M *be a closed, oriented 7-manifold and let ψ be a torsion-free* G_2 *4-form on* M. *Does there exist an infinite-dimensional stable manifold $\mathscr{S}_4^-(\psi) \subset [\psi]_+$, tangent to $\mathcal{S}_4^-(\psi)$ at ψ, and an infinite-dimensional unstable manifold $\mathscr{S}_4^+(\psi) \subset [\psi]_+$, tangent to $\mathcal{S}_4^+(\psi)$ at ψ, such that for any sufficiently small neighbourhood U of ψ in $[\psi]_+$, the following is true:*

- *For all $\psi' \in U \cap \mathscr{S}_4^-(\psi)$, the Laplacian coflow:*

$$\frac{\partial \psi(t)}{\partial t} = \Delta_{\psi(t)} \psi(t), \qquad \psi(0) = \psi'$$

  exists for all time and converges smoothly to some $\psi_\infty \in \mathrm{Diff}_0(\mathrm{M}) \cdot \psi$ as $t \to \infty$.

- *For all $\psi' \in U \cap \mathscr{S}_4^+(\psi)$, the time reversed Laplacian coflow:*

$$\frac{\partial \psi(t)}{\partial t} = -\Delta_{\psi(t)} \psi(t), \qquad \psi(0) = \psi'$$

  exists for all time and converges smoothly to some $\psi_\infty \in \mathrm{Diff}_0(\mathrm{M}) \cdot \psi$ as $t \to \infty$.

- *No other trajectories of either the Laplacian coflow or the time-reversed Laplacian coflow in U converge to any $\psi_\infty \in \mathrm{Diff}_0(\mathrm{M}) \cdot \psi \cap \overline{U}$. (Here, $\overline{U}$ denotes the closure of U.)*

Further evidence to support the third point above is Theorem 5.2.2, which states that, at least in the C^0-topology, most trajectories of the Laplacian coflow in any neighbourhood of ψ fail to converge to any torsion-free G_2 4-form.

Question 6.2.1 seems to be far more challenging than Lotay–Wei's theorem, since it is known from Grigorian [59] that, unlike Laplacian flow, neither Laplacian coflow nor time-reversed Laplacian coflow is parabolic modulo diffeomorphisms, even when restricted to the direction of closed forms. This lack of parabolicity poses a significant technical hurdle to the study of Laplacian coflow.

Part II

h-principles for stable forms

Chapter 7

Relative h-principles for closed stable forms

This chapter uses convex integration to prove four new relative h-principles for closed, stable, exterior forms on manifolds, *viz.* the relative h-principle for co-symplectic forms, co-pseudoplectic forms and $\widetilde{\mathrm{G}}_2$ 3- and 4-forms. New proofs of the three previously known relative h-principles are also provided. The implication of these results for the unboundedness of Hitchin functionals is discussed.

7.1 Algebraic preliminaries: co-symplectic, pseudoplectic and co-pseudoplectic forms

The aim of this section is to establish the fundamental properties of stable 2-forms and $(n-2)$-forms which will be needed in this chapter. I begin with some general results on stable forms. Fix a volume form *vol* on $\mathbb{R}^n$; the following result is easily verified by direct calculation:

Lemma 7.1.1. *Consider the anti-isomorphism:*

$$\begin{aligned} \Phi : \mathrm{GL}_+(n;\mathbb{R}) &\to \mathrm{GL}_+(n;\mathbb{R}) \\ F &\mapsto \det(F)^{\frac{1}{n-p}} \cdot F^{-1}, \end{aligned} \tag{7.1.2}$$

with inverse given by $\Psi : F \mapsto \det(F)^{\frac{1}{p}} \cdot F^{-1}$. *Then the following* $\mathrm{GL}_+(n;\mathbb{R})$*-equivariant diagram commutes:*

$$\begin{tikzcd} \bigwedge^p \mathbb{R}^n \arrow[loop right] \arrow[d, "\sigma \mapsto \sigma \lrcorner vol"] & \mathrm{GL}_+(n;\mathbb{R}) \arrow[d, "\Phi"] \\ \bigwedge^{n-p} (\mathbb{R}^n)^* \arrow[loop right] & \mathrm{GL}_+(n;\mathbb{R}) \end{tikzcd} \tag{7.1.3}$$

where the left-hand vertical arrow is an isomorphism, the top action is a left action and the bottom action is a right action.

□

Next, given a p-form σ_0 on $\mathbb{R}^n$ define linear maps ι_{σ_0} and ε_{σ_0} by:

$$\begin{aligned} \iota_{\sigma_0} : \mathbb{R}^n &\to \bigwedge^{p-1} (\mathbb{R}^n)^* & \varepsilon_{\sigma_0} : (\mathbb{R}^n)^* &\to \bigwedge^{p+1} (\mathbb{R}^n)^* \\ u &\mapsto u \lrcorner \sigma_0 & \beta &\mapsto \beta \wedge \sigma_0. \end{aligned} \tag{7.1.4}$$

Recall from [93, §2] that a p-form σ_0 on $\mathbb{R}^n$ is termed multi-symplectic if ι_{σ_0} is injective.[1] Analogously, I term σ_0 multi-co-symplectic if ε_{σ_0} is injective. Write $\mathrm{GL}_-(n;\mathbb{R})$ for the set of orientation-reversing automorphisms of $\mathbb{R}^n$ and $\mathrm{Stab}_{\mathrm{GL}_-(n;\mathbb{R})}(\sigma_0)$ for the set of orientation-reversing automorphisms fixing a given p-form σ_0.

Lemma 7.1.5.
1. Let $\alpha \in \bigwedge^p (\mathbb{R}^n)^*$ be a multi-symplectic form, embed $\mathbb{R}^n \hookrightarrow \mathbb{R}^k \oplus \mathbb{R}^n \cong \mathbb{R}^{n+k}$ for some $k > 0$ and embed $\bigwedge^p (\mathbb{R}^n)^* \hookrightarrow \bigwedge^p \left(\mathbb{R}^{n+k}\right)^*$ in the canonical way. Then:

$$\begin{aligned} \mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\alpha) = &\left\{ \begin{pmatrix} A_{k\times k} & B_{k\times n} \\ 0_{n\times k} & C_{n\times n} \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}_+(k;\mathbb{R}),\ B \in \mathrm{End}(\mathbb{R}^n,\mathbb{R}^k), \\ C \in \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha) \end{array} \right\} \\ &\coprod \left\{ \begin{pmatrix} A_{k\times k} & B_{k\times n} \\ 0_{n\times k} & C_{n\times n} \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}_-(k;\mathbb{R}),\ B \in \mathrm{End}(\mathbb{R}^n,\mathbb{R}^k), \\ C \in \mathrm{Stab}_{\mathrm{GL}_-(n;\mathbb{R})}(\alpha) \end{array} \right\}. \end{aligned}$$

Thus if $\mathrm{Stab}_{\mathrm{GL}(n;\mathbb{R})}(\alpha)$ is connected, then $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\alpha)$ is also connected.
2. Now let $\alpha \in \bigwedge^p (\mathbb{R}^n)^*$ be a multi-co-symplectic form. Then:

$$\begin{aligned} \mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\theta^{12..k} \wedge \alpha) = &\left\{ \begin{pmatrix} A_{k\times k} & 0_{k\times n} \\ B_{n\times k} & \det(A)^{-\frac{1}{p}} C_{n\times n} \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}_+(k;\mathbb{R}),\ B \in \mathrm{End}(\mathbb{R}^k,\mathbb{R}^n), \\ C \in \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha) \end{array} \right\} \\ &\coprod \left\{ \begin{pmatrix} A_{k\times k} & 0_{k\times n} \\ B_{n\times k} & |\det(A)|^{-\frac{1}{p}} C_{n\times n} \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}_-(k;\mathbb{R}),\ B \in \mathrm{End}(\mathbb{R}^k,\mathbb{R}^n), \\ C \in J \cdot \mathrm{Stab}_{\mathrm{GL}_{-\operatorname{sign} J}(n;\mathbb{R})}(\alpha) \end{array} \right\}, \end{aligned} \tag{7.1.6}$$

where $J \in \mathrm{GL}(n;\mathbb{R})$ is any map such that $J^*\alpha = -\alpha$. If either:

- *α and $-\alpha$ lie in separate $\mathrm{GL}(n;\mathbb{R})$-orbits, or*
- *α and $-\alpha$ lie in the same $\mathrm{GL}_+(n;\mathbb{R})$-orbit and $\mathrm{Stab}_{\mathrm{GL}(n;\mathbb{R})}(\alpha) = \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha)$,*

then the second set on the right-hand side of eqn. (7.1.6) is empty. In particular, if either of these conditions holds and additionally $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha)$ is connected, then $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\theta^{12...k} \wedge \alpha)$ is also connected.

Proof. (1) Since $\alpha \in \bigwedge^p (\mathbb{R}^n)^*$ is multi-symplectic, the kernel of the linear map:

$$\begin{aligned} \mathbb{R}^{n+k} &\to \bigwedge^{p-1} \left(\mathbb{R}^{n+k}\right)^* \\ u &\mapsto u \lrcorner \alpha \end{aligned}$$

[1] *Caveat*: not every stable form is multi-symplectic, despite the claim to the contrary in [93, Cor. 2.3]. A counterexample is provided by pseudoplectic forms; see Proposition 7.1.14.

is precisely $\mathbb{R}^k \oplus 0$ and hence this subspace is invariant under $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\alpha)$. Thus any element of $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\alpha)$ has the form $F = \begin{pmatrix} A_{k\times k} & B_{k\times n} \\ 0_{n\times k} & C_{n\times n} \end{pmatrix}$ for some $A \in \mathrm{GL}(k;\mathbb{R})$, $B \in \mathrm{End}(\mathbb{R}^n, \mathbb{R}^k)$ and $C \in \mathrm{GL}(n;\mathbb{R})$. Therefore $F^*\alpha = C^*\alpha = \alpha$ and the result follows.

(2) Since $\alpha \in \bigwedge^p (\mathbb{R}^n)^*$ is multi-co-symplectic, the kernel of the linear map:

$$\begin{aligned} \left(\mathbb{R}^{n+k}\right)^* &\to \bigwedge\nolimits^{p+1} \left(\mathbb{R}^{n+k}\right)^* \\ \beta &\mapsto \beta \wedge \left(\theta^{12\dots k} \wedge \alpha\right) \end{aligned}$$

is precisely $\left(\mathbb{R}^k\right)^* \oplus 0$ and hence this subspace is invariant under $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\theta^{12\dots k} \wedge \alpha)$. Thus any element of $\mathrm{Stab}_{\mathrm{GL}_+(n+k;\mathbb{R})}(\theta^{12\dots k} \wedge \alpha)$ has the form $F = \begin{pmatrix} A_{k\times k} & 0_{k\times n} \\ B_{n\times k} & D_{n\times n} \end{pmatrix}$ for some $A \in \mathrm{GL}(k;\mathbb{R})$, $B \in \mathrm{End}(\mathbb{R}^k, \mathbb{R}^n)$ and $D \in \mathrm{GL}(n;\mathbb{R})$, hence $F^*\left(\theta^{12\dots k} \wedge \alpha\right) = \det(A)\theta^{12\dots k} \wedge D^*\alpha = \theta^{12\dots k} \wedge \alpha$ and whence:

$$\det(A) \cdot D^*\alpha = \alpha. \tag{7.1.7}$$

If $\det(A) > 0$, one can rewrite eqn. (7.1.7) as $\left(\det(A)^{\frac{1}{p}} \cdot D\right)^* \alpha = \alpha$. Since $\det(F) = \det(A)\det(D) > 0$, it follows that $\det\left(\det(A)^{\frac{1}{p}} \cdot D\right) > 0$ and thus $C = \det(A)^{\frac{1}{p}} \cdot D \in \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha)$ as claimed.

Now suppose $\det(A) < 0$ and rewrite eqn. (7.1.7) as $\left(|\det(A)|^{\frac{1}{p}} \cdot D\right)^* \alpha = -\alpha$, where now $C = |\det(A)|^{\frac{1}{p}} \cdot D$ has negative determinant. Then clearly α and $-\alpha$ lie in the same $\mathrm{GL}(n;\mathbb{R})$-orbit; let J be some fixed element of $\mathrm{GL}(n;\mathbb{R})$ (not necessarily equal to C) such that $J^*\alpha = -\alpha$. If α and $-\alpha$ lie in different $\mathrm{GL}_+(n;\mathbb{R})$-orbits, then J is automatically orientation-reversing and so $C \in J \cdot \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha)$ as required. Else, J may be chosen to be orientation-preserving and hence $C \cdot J$ is an orientation-reversing automorphism of α, implying that $\mathrm{Stab}_{\mathrm{GL}(n;\mathbb{R})}(\alpha) \neq \mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\alpha)$ and $C \in J \cdot \mathrm{Stab}_{\mathrm{GL}_-(n;\mathbb{R})}(\alpha)$, again as required. This completes the proof.

□

7.1.1 $(2k-2)$-forms in $2k$-dimensions, $k \geqslant 3$

I begin by recalling the following well-known result:

Proposition 7.1.8. *Let $k \geqslant 2$. The action of $\mathrm{GL}_+(2k;\mathbb{R})$ on $\bigwedge^2 \left(\mathbb{R}^{2k}\right)^*$ has exactly two open orbits, given by:*

$$\bigwedge\nolimits^2_+ \left(\mathbb{R}^{2k}\right)^* = \left\{\omega \in \bigwedge\nolimits^2 \left(\mathbb{R}^{2k}\right)^* \;\middle|\; \omega^k > 0\right\} \quad \textit{and} \quad \bigwedge\nolimits^2_- \left(\mathbb{R}^{2k}\right)^* = \left\{\omega \in \bigwedge\nolimits^2 \left(\mathbb{R}^{2k}\right)^* \;\middle|\; \omega^k < 0\right\} \tag{7.1.9}$$

which form a single orbit under $\mathrm{GL}(2k;\mathbb{R})$. The stabiliser in $\mathrm{GL}_+(2k;\mathbb{R})$ (equivalently $\mathrm{GL}(2k;\mathbb{R})$) of forms in either orbit is isomorphic to the real symplectic group $\mathrm{Sp}(2k;\mathbb{R})$. 'Standard' representatives of the two orbits are given by:

$$\omega_+(k) = \theta^{12} + \theta^{34} + \dots + \theta^{2k-1,2k} \quad \textit{and} \quad \omega_-(k) = -\theta^{12} + \theta^{34} + \dots + \theta^{2k-1,2k} \tag{7.1.10}$$

respectively.

Forms in $\bigwedge^2_+\left(\mathbb{R}^{2k}\right)^*$ are here termed emproplectic and forms in $\bigwedge^2_-\left(\mathbb{R}^{2k}\right)^*$ termed pisoplectic.[2] The $\mathrm{GL}(2k;\mathbb{R})$-orbit comprising of both emroplectic and pisoplectic forms is termed the orbit of symplectic forms ω and can be characterised by the property that the linear map ι_ω is an isomorphism (see eqn. (7.1.4)).

Using Proposition 7.1.8, I prove the following result (some aspects of which appear to be known; see, e.g. [72]):

Proposition 7.1.11. *Let $k \geqslant 3$. The action of $\mathrm{GL}_+(2k;\mathbb{R})$ on $\bigwedge^{2k-2}\left(\mathbb{R}^{2k}\right)^*$ has exactly two open orbits, given by:*

$$\textstyle\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^* = \left\{\omega^{k-1} \;\middle|\; \omega \text{ is emproplectic}\right\} \quad \text{and} \quad \bigwedge^{2k-2}_-\left(\mathbb{R}^{2k}\right)^* = \left\{-\omega^{k-1} \;\middle|\; \omega \text{ is pisoplectic}\right\}.$$

Call forms in $\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^*$ co-emproplectic and forms in $\bigwedge^{2k-2}_-\left(\mathbb{R}^{2k}\right)^*$ co-pisoplectic. Standard examples of co-emproplectic and co-pisoplectic forms are given by:

$$\varpi_+(k) = \left(\sum_{i=2}^{k}\theta^{12\ldots\widehat{2i-1,2i}\ldots 2k-1,2k}\right) + \theta^{34\ldots 2k} = \frac{\omega_+(k)^{k-1}}{(k-1)!} \tag{7.1.12}$$

and:

$$\varpi_-(k) = \left(\sum_{i=2}^{k}\theta^{12\ldots\widehat{2i-1,2i}\ldots 2k-1,2k}\right) - \theta^{34\ldots 2k} = -\frac{\omega_-(k)^{k-1}}{(k-1)!}, \tag{7.1.13}$$

where $\widehat{\ }$ denotes that the corresponding indices should be omitted. The stabiliser in $\mathrm{GL}_+(2k;\mathbb{R})$ of a co-emproplectic or co-pisoplectic form is isomorphic to $\mathrm{Sp}(2k;\mathbb{R})$. If k is odd, then $\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^*$ and $\bigwedge^{2k-2}_-\left(\mathbb{R}^{2k}\right)^*$ are both individually invariant under $\mathrm{GL}(n;\mathbb{R})$, while if k is even, the two $\mathrm{GL}_+(n;\mathbb{R})$-orbits form a single $\mathrm{GL}(n;\mathbb{R})$-orbit. I shall say that ϖ is co-symplectic if it lies in $\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^* \cup \bigwedge^{2k-2}_-\left(\mathbb{R}^{2k}\right)^*$. This is equivalent to the condition that ε_ϖ is an isomorphism.

Proof. Recall diagram (7.1.3) and let $n = 2k$ and $p = 2$. Let w denote an emproplectic (resp. pisoplectic) element of $\bigwedge^2\mathbb{R}^{2k}$ and let $\varpi = w \lrcorner\, vol$. Then one obtains the following diagram:

$$\begin{tikzcd}
\mathrm{GL}_+(2k;\mathbb{R})\cdot w \arrow[d, "\sigma\mapsto\sigma\lrcorner vol"] & \mathrm{GL}_+(2k;\mathbb{R}) \arrow[l, bend left] \arrow[d, "\Phi"] \\
\mathrm{GL}_+(2k;\mathbb{R})\cdot \varpi & \mathrm{GL}_+(2k;\mathbb{R}) \arrow[l, bend left]
\end{tikzcd}$$

Since the left-hand vertical arrow is an isomorphism, it follows that $\mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(\varpi) = \Phi\left(\mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(w)\right)$.

[2] I have created these non-standard terms, in keeping with the term 'symplectic' (literally, 'braided together'). 'Emproplectic' (literally, 'braided forwards') denotes that forms in the first orbit induce the 'correct' orientation on the underlying space, whilst 'pisoplectic' (literally, 'braided backwards') denotes that forms in the second orbit induce the 'opposite' orientation.

As $\mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(w) \cong \mathrm{Sp}(2k;\mathbb{R}) \subseteq \mathrm{SL}(2k;\mathbb{R})$, $\Phi(F) = F^{-1}$ on $\mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(w)$ (see eqn. (7.1.2)) and hence $\mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(\varpi) = \mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}(w) \cong \mathrm{Sp}(2k;\mathbb{R})$, as required.

Now take $w = e_{12} + e_{34} + ... + e_{2k-1,2k}$ and recall $\omega_+(k) = \theta^{12} + \theta^{34} + ... + \theta^{2k-1,2k}$ defined in eqn. (7.1.10). Taking $vol = \frac{\omega_+(k)^k}{k!}$, one finds that:

$$w \lrcorner vol = \frac{\omega_+(k)^{k-1}}{(k-1)!} = \varpi_+(k) \in \textstyle\bigwedge^{2k-2}_+ \left(\mathbb{R}^{2k}\right)^*.$$

Likewise, if one takes $w = -e_{12} + e_{34} + ... + e_{2k-1,2k}$ and $vol = \frac{\omega_+(k)^k}{k!} = -\frac{\omega_-(k)^k}{k!}$, one finds:

$$w \lrcorner vol = -\frac{\omega_-(k)^{k-1}}{(k-1)!} = \varpi_-(k) \in \textstyle\bigwedge^{2k-2}_- \left(\mathbb{R}^{2k}\right)^*$$

as required. To prove the statement regarding $\mathrm{GL}(2k;\mathbb{R})$-orbits, let ω be an emproplectic form, let $F \in \mathrm{GL}_-(2k;\mathbb{R})$ and consider $-F^*\omega$. If k is odd, $\left(-F^*\omega\right)^k = (-1)^k F^*\left(\omega^k\right) = -F^*\left(\omega^k\right) > 0$ and so $-F^*\omega$ is emproplectic. Thus $F^*\left(\omega^{k-1}\right) = \left(-F^*\omega\right)^{k-1}$ is co-emproplectic as claimed. Alternatively, if k is even, then $\left(-F^*\omega\right)^k < 0$, hence so $-F^*\omega$ is pisoplectic and whence $F^*\left(\omega^{k-1}\right) = -\left(-F^*\omega\right)^{k-1}$ is pisoplectic, as claimed. The final statement regarding the characterisation of co-symplectic forms is clear.

$\square$

I remark that the notions of co-emproplectic and co-pisoplectic forms are still valid in dimension 4, however in this case they coincide with emproplectic and pisoplectic forms respectively. Accordingly, I reserve the terms co-emproplectic and co-pisoplectic for dimension $2k$, $k \geqslant 3$. Also note that given a co-emproplectic form ϖ on an oriented $2k$-manifold, the form $-\varpi$ is a co-pisoplectic form on $\overline{\mathrm{M}}$, where the overline denotes orientation-reversal.

7.1.2 2- and $(2k-1)$-forms in $2k+1$-dimensions, $k \geqslant 2$

Although some aspects of the following result are known, to the author's knowledge, the complete statement does not appear in the literature:

Proposition 7.1.14. *Let $k \geqslant 2$. The action of $\mathrm{GL}_+(2k+1;\mathbb{R})$ on $\bigwedge^2\left(\mathbb{R}^{2k+1}\right)^*$ has a unique open orbit:*

$$\textstyle\bigwedge^2_P\left(\mathbb{R}^{2k+1}\right)^* = \left\{\mu \in \bigwedge^2\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \mu^k \neq 0\right\} \tag{7.1.15}$$

which is also an orbit of $\mathrm{GL}(2k+1;\mathbb{R})$. Equivalently, a 2-form μ lies in $\bigwedge^2_P\left(\mathbb{R}^{2k+1}\right)^*$ if and only if ι_μ has rank $2k$ (see eqn. (7.1.4)). Call forms in this orbit pseudoplectic; a standard representative of this orbit may be taken to be:

$$\mu_0(k) = \theta^{23} + \theta^{45} + ... + \theta^{2k,2k+1}. \tag{7.1.16}$$

The stabiliser of any pseudoplectic form is connected and is isomorphic to the group:

$$\left\{\begin{pmatrix} A_{1\times1} & B_{1\times2k} \\ 0_{2k\times1} & C_{2k\times2k}\end{pmatrix} \;\middle|\; A \in \mathbb{R}_{>0}, B \in \left(\mathbb{R}^{2k}\right)^* \text{ and } C \in \mathrm{Sp}(2k;\mathbb{R})\right\}. \tag{7.1.17}$$

Likewise, the action of $\mathrm{GL}_+(2k+1;\mathbb{R})$ *on* $\bigwedge^{2k-1}\left(\mathbb{R}^{2k+1}\right)^*$ *has a unique open orbit:*

$$\bigwedge^{2k-1}_{Co-P}\left(\mathbb{R}^{2k+1}\right)^* = \left\{\xi \in \bigwedge^{2k-1}\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \text{the linear map } \varepsilon_\xi \text{ has rank } 2k\right\} \tag{7.1.18}$$

which is also an orbit of $\mathrm{GL}(2k+1;\mathbb{R})$*. Term forms in this orbit co-pseudoplectic. A standard representative of this orbit may be taken to be:*

$$\xi_0(k) = \sum_{i=1}^{k} \theta^{123\ldots\widehat{2i,2i+1}\ldots 2k,2k+1} = \theta^1 \wedge \varpi_+(k), \tag{7.1.19}$$

where $\varpi_+(k)$ *is viewed as a form on* $\mathbb{R}^{2k+1}$ *via* $\mathbb{R}^{2k} \cong \langle e_2, \ldots, e_{2k+1}\rangle \subset \mathbb{R}^{2k+1}$ *(and formally* $\varpi_+(k) = \omega_+(k)$ *when* $k=2$*). The stabiliser of any co-pseuduoplectic form is isomorphic to the group:*

$$\left\{\begin{pmatrix} A_{1\times 1} & 0_{1\times 2k} \\ B_{2k\times 1} & A^{-\frac{1}{2k-2}}\cdot C_{2k\times 2k}\end{pmatrix} \;\middle|\; A \in \mathbb{R}_{>0}, B \in \mathbb{R}^{2k} \text{ and } C \in \mathrm{Sp}(2k;\mathbb{R})\right\}. \tag{7.1.20}$$

In particular, the stabiliser is connected.

Proof. The set $\left\{\mu \in \wedge^2\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \mu^k = 0\right\}$ is an affine subvariety of $\wedge^2\left(\mathbb{R}^{2k+1}\right)^*$, so if it is a proper subset of $\wedge^2\left(\mathbb{R}^{2k+1}\right)^*$ it must have positive codimension, and hence can contain no open orbits of $\mathrm{GL}_+(2k+1;\mathbb{R})$. Thus to prove eqn. (7.1.15) it suffices to prove that $\wedge^2_P\left(\mathbb{R}^{2k+1}\right)^* = \left\{\mu \in \wedge^2\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \mu^k \neq 0\right\}$ is non-empty and a single orbit.

Firstly, let me show that $\wedge^2_P\left(\mathbb{R}^{2k+1}\right)^*$ is a single orbit of $\mathrm{GL}_+(2k+1;\mathbb{R})$. Given $\mu \in \wedge^2_P\left(\mathbb{R}^{2k+1}\right)^*$, since the rank of an anti-symmetric bilinear form is always even (and the dimension of $\mathbb{R}^{2k+1}$ is odd) it follows that ι_μ has a non-trivial kernel. Pick a 1-dimensional subspace $\mathbb{L}$ of the kernel and let $\mathbb{B} \subset \mathbb{R}^{2k+1}$ be a $2k$-dimensional complement to $\mathbb{L}$ in $\mathbb{R}^{2k+1}$. Since $\mu^k \neq 0$, one may regard μ as an emproplectic 2-form on $\mathbb{B}$ for some suitable choice of orientation on $\mathbb{B}$. Thus one can pick a correctly-oriented basis $(f_2, \ldots, f_{2k+1})$ of $\mathbb{B}$ with dual basis $(f^2, \ldots, f^{2k+1})$ such that $\mu = f^{23} + \ldots + f^{2k,2k+1}$. Now define f_1 to be a non-zero vector in $\mathbb{L}$ such that $(f_1, \ldots, f_{2k+1})$ is a correctly oriented basis of $\mathbb{R}^{2k+1}$. Then with respect to this basis:

$$\mu = f^{23} + \ldots + f^{2k,2k+1}.$$

Thus $\wedge^2_P\left(\mathbb{R}^{2k+1}\right)^*$ is a single orbit under $\mathrm{GL}_+(2k+1;\mathbb{R})$. This also shows that $\wedge^2_P\left(\mathbb{R}^{2k+1}\right)^* \neq \varnothing$ (since $\left(f^{23} + \ldots + f^{2k,2k+1}\right)^k \neq 0$ in the above basis) and:

$$\bigwedge^2_P\left(\mathbb{R}^{2k+1}\right)^* = \left\{\mu \in \bigwedge^2\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \iota_\mu \text{ has rank } 2k\right\}.$$

Moreover, since this is the only open orbit of $\mathrm{GL}_+(2k+1;\mathbb{R})$, it follows that $\wedge^2_P\left(\mathbb{R}^{2k+1}\right)^*$ must also form a single $\mathrm{GL}(2k+1;\mathbb{R})$-orbit.

Now fix a (positive) volume element vol on $\mathbb{R}^{2k+1}$, let $\mu \in \wedge^2\mathbb{R}^{2k+1}$ and write $\xi = \mu \lrcorner vol$. Recall that ε_ξ and ι_μ denote the linear maps:

$$\varepsilon_\xi : \left(\mathbb{R}^{2k+1}\right)^* \to \bigwedge^{2k}\left(\mathbb{R}^{2k+1}\right)^* \qquad \iota_\mu : \left(\mathbb{R}^{2k+1}\right)^* \to \mathbb{R}^{2k+1}$$
$$\alpha \mapsto \alpha \wedge \xi \qquad\qquad\qquad\qquad \alpha \mapsto \alpha \lrcorner \mu$$

respectively. The first step is to understand how the maps ε_ξ and ι_μ are related:

Claim 7.1.21. *The maps ε_ξ and ι_μ satisfy the relation:*

$$\varepsilon_\xi = -\iota_\mu \lrcorner vol.$$

Proof of Claim. Firstly, given $\alpha \in (\mathbb{R}^n)^*$, $\beta \in \bigwedge^2\mathbb{R}^n$ and $\gamma \in \bigwedge^n(\mathbb{R}^n)^*$, a direct calculation verifies the following identity:

$$\alpha \wedge (\beta \lrcorner \gamma) = -(\alpha \lrcorner \beta) \lrcorner \gamma. \tag{7.1.22}$$

Thus given $\alpha \in \left(\mathbb{R}^{2k+1}\right)^*$, one computes that:

$$\begin{aligned} \varepsilon_\xi(\alpha) &= \alpha \wedge \xi \\ &= \alpha \wedge (\mu \lrcorner vol) \\ &= -(\alpha \lrcorner \mu) \lrcorner vol = -\iota_\mu(\alpha) \lrcorner vol, \end{aligned}$$

where eqn. (7.1.22) has been used in passing to the final line.

□

Thus ι_μ has the same rank as ε_ξ and so ι_μ has rank $2k$ if and only if ε_ξ has rank $2k$. It follows at once from Lemma 7.1.1 that the action of $\mathrm{GL}_+(2k+1;\mathbb{R})$ on $\bigwedge^{2k-1}\left(\mathbb{R}^{2k+1}\right)^*$ has a single open orbit (which is also an orbit of $\mathrm{GL}(2k+1;\mathbb{R})$) given by:

$$\bigwedge^{2k-1}_{Co-P}\left(\mathbb{R}^{2k+1}\right)^* = \left\{\xi \in \bigwedge^{2k-1}\left(\mathbb{R}^{2k+1}\right)^* \;\middle|\; \varepsilon_\xi \text{ has rank } 2k\right\}.$$

Likewise, the explicit formula for ξ_0 also follows at once.

The formula for the stabiliser of pseudoplectic forms is a simple application of Lemma 7.1.5(1). For the stabiliser of co-pseudoplectic forms, one uses Lemma 7.1.5(2), together with the observations that:

- If k is odd, then $-\varpi_+(k) = -\frac{(-\omega_+(k))^{k-1}}{(k-1)!}$ is co-pisoplectic (since $-\omega_+(k)$ is pisoplectic when k is odd) and since (when k is odd) the orbit of co-emproplectic and co-pisoplectic forms are each closed under the action of $\mathrm{GL}(2k;\mathbb{R})$ (and not just $\mathrm{GL}_+(2k;\mathbb{R})$) it follows that $\varpi_+(k)$ and $-\varpi_+(k)$ lie in separate $\mathrm{GL}_+(2k;\mathbb{R})$-orbits. This forces the second bracket in eqn. (7.1.6) to vanish, as claimed.

- If k is even then $-\varpi_+(k) = \frac{(-\omega_+(k))^{k-1}}{(k-1)!}$ is co-emproplectic (since $-\omega_+(k)$ is emproplectic when k is even) and thus $\varpi_+(k)$ and $-\varpi_+(k)$ lie in the same $\mathrm{GL}_+(2k;\mathbb{R})$-orbit. On the other hand, since k is even:

$$\mathrm{Stab}_{\mathrm{GL}(2k;\mathbb{R})}\left(\varpi_+(k)\right) = \mathrm{Stab}_{\mathrm{GL}_+(2k;\mathbb{R})}\left(\varpi_+(k)\right)$$

and thus once again the second bracket on the right-hand side of eqn. (7.1.6) vanishes. This completes the proof.

□

In particular, note from eqns. (7.1.17) and (7.1.20) that neither pseudoplectic forms nor co-pseudoplectic forms are Hitchin forms.

Remark 7.1.23. Let $\mu \in \Lambda^2_P\left(\mathbb{R}^{2k+1}\right)^*$. The kernel of the linear map ι_μ defines a 1-dimensional subspace of $\mathbb{R}^{2k+1}$ which I denote ℓ_μ. Moreover, the orientation on $\mathbb{R}^{2k+1}$ induces a natural orientation on ℓ_μ defined as follows: given a 1-form θ on $\mathbb{R}^{2k+1}$ which does not vanish on ℓ_μ, say that θ is positive on ℓ_μ if $\theta \wedge \mu^k > 0$. Likewise, let $\xi \in \Lambda^{2k-1}_{Co-P}\left(\mathbb{R}^{2k+1}\right)^*$ be co-pseudoplectic. Then the annihilator of the 1-dimensional subspace $\mathrm{Ker}(\varepsilon_\xi) \subset \left(\mathbb{R}^{2k+1}\right)^*$ defines a hyperplane in $\mathbb{R}^{2k+1}$ associated to ξ; denote this hyperplane by Π_ξ. By eqn. (7.1.20), every element F of $\mathrm{Stab}_{\mathrm{GL}_+(2k+1;\mathbb{R})}(\xi)$ restricts to an orientation preserving automorphism of Π_ξ; thus, it is again possible to orient the planes Π_ξ consistently for all ξ. Specifically, there is a unique orientation on Π_ξ such that $\xi = \theta \wedge \varpi = \theta \wedge \omega^{k-1}$, where θ is a compatibly oriented generator of $\mathrm{Ann}(\Pi|_\xi) = \mathrm{Ker}(\varepsilon_\xi)$ and ϖ is a co-emproplectic form on Π_ξ with respect to the given orientation (equivalently, ω is a emproplectic form on Π_ξ). Moreover, as θ varies, ω defines a conformal class of emproplectic forms on Π_ξ; thus ξ determines a co-oriented almost contact structure on $\mathbb{R}^{2k+1}$ (in an algebraic sense) [42, §10.1.B]. E.g. in the case of the standard co-pseudoplectic form $\xi_0(k) = \theta^1 \wedge \varpi_+(k)$, $\Pi_{\xi_0(k)} = \langle e_2, ..., e_{2k+1}\rangle$ and the corresponding conformal class of emproplectic forms on Π_ξ is just $\lambda \cdot \left(\theta^{23} + ... + \theta^{2k,2k+1}\right)$ for $\lambda > 0$.

7.1.3 Classification of stable forms

For the sake of completeness, I briefly recount the classification of stable forms; see [93] for further details of the 8-dimensional case (although note that the formulae for $\zeta_{c,s,n}$ differ from those *op. cit.*, and were computed by the author of this thesis in order to ensure that the corresponding metrics $g_{\zeta_{c,\,s,\,n}}$ assumed standard forms; likewise, the formulae for $\eta_{c,s,n}$ were also computed by the author of this thesis):

Theorem 7.1.24. *The action of* $\mathrm{GL}_+(n;\mathbb{R})$ *on* $\Lambda^p\left(\mathbb{R}^n\right)^*$ *has precisely the following open orbits for* $2 \leqslant p \leqslant n-2$*:*

- $\boldsymbol{n = 2k,\ k \geqslant 2}$*:* $\Lambda^2_\pm\left(\mathbb{R}^{2k}\right)^*$ *and* $\Lambda^{2k-2}_\pm\left(\mathbb{R}^{2k}\right)^*$*;*
- $\boldsymbol{n = 2k+1,\ k \geqslant 2}$*:* $\Lambda^2_P\left(\mathbb{R}^{2k+1}\right)^*$ *and* $\Lambda^{2k-1}_{Co-P}\left(\mathbb{R}^{2k+1}\right)^*$*;*
- $\boldsymbol{n = 6}$*:* $\Lambda^3_+\left(\mathbb{R}^6\right)^*$ *and* $\Lambda^3_+\left(\mathbb{R}^6\right)^*$*;*
- $\boldsymbol{n = 7}$*:* $\pm\Lambda^3_+\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^3_\sim\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^4_+\left(\mathbb{R}^7\right)^*$ *and* $\pm\Lambda^4_\sim\left(\mathbb{R}^7\right)^*$*;*
- $\boldsymbol{n = 8}$*: The action of* $\mathrm{GL}_+(8;\mathbb{R})$ *on* $\Lambda^3\left(\mathbb{R}^8\right)^*$ *has precisely three open orbits, represented by the*

3-forms:

$$\zeta_c = \theta^{123} + \frac{1}{2}\theta^{147} - \frac{1}{2}\theta^{156} + \frac{1}{2}\theta^{246} + \frac{1}{2}\theta^{257} + \frac{1}{2}\theta^{345} - \frac{1}{2}\theta^{367} + \frac{\sqrt{3}}{2}\theta^{458} + \frac{\sqrt{3}}{2}\theta^{678};$$
$$\zeta_s = \frac{\sqrt{3}}{2}\theta^{147} - \frac{\sqrt{3}}{2}\theta^{156} + \theta^{238} + \frac{1}{2}\theta^{246} - \frac{1}{2}\theta^{257} + \frac{1}{2}\theta^{347} + \frac{1}{2}\theta^{356} + \frac{1}{2}\theta^{458} - \frac{1}{2}\theta^{678};$$
$$\zeta_n = -\theta^{123} - \frac{1}{2}\theta^{156} - \frac{1}{2}\theta^{178} + \frac{1}{2}\theta^{257} - \frac{1}{2}\theta^{268} - \frac{1}{2}\theta^{358} - \frac{1}{2}\theta^{367} - \frac{\sqrt{3}}{2}\theta^{458} + \frac{\sqrt{3}}{2}\theta^{467},$$

with stabilisers $\mathbb{P}\mathrm{SU}(3)$, $\mathrm{SL}(3;\mathbb{R})$ *and* $\mathbb{P}\mathrm{SU}(1,2)$ *respectively. Here,* $\mathbb{P}\mathrm{SU}(3)$ *acts on* $\mathbb{R}^8 \cong \mathfrak{su}(3)$ *via the diagram:*

$$\begin{tikzcd}
\mathrm{SU}(3) \arrow[d, "quot"] \arrow[dr, "Ad"] & \\
\mathbb{P}\mathrm{SU}(3) \arrow[r, hook] & \mathrm{GL}_+(\mathfrak{su}(3))
\end{tikzcd}$$

and preserves the inner-product $g_{\zeta_c} = \sum_{i=1}^{8}\left(\theta^i\right)^{\otimes 2}$, $\mathrm{SL}(3;\mathbb{R})$ *acts on* $\mathbb{R}^8 \cong \mathfrak{sl}(3;\mathbb{R})$ *(faithfully) via its adjoint representation and preserves the indefinite inner-product* $g_{\zeta_s} = \sum_{i=1}^{5}\left(\theta^i\right)^{\otimes 2} - \sum_{i=6}^{8}\left(\theta^i\right)^{\otimes 2}$*, and* $\mathbb{P}\mathrm{SU}(1,2)$ *acts on* $\mathbb{R}^8 \cong \mathfrak{su}(1,2)$ *via the diagram:*

$$\begin{tikzcd}
\mathrm{SU}(1,2) \arrow[d, "quot"] \arrow[dr, "Ad"] & \\
\mathbb{P}\mathrm{SU}(1,2) \arrow[r, hook] & \mathrm{GL}_+(\mathfrak{su}(1,2))
\end{tikzcd}$$

and preserves the indefinite inner-product $g_{\zeta_n} = \sum_{i=1}^{4}\left(\theta^i\right)^{\otimes 2} - \sum_{i=5}^{8}\left(\theta^i\right)^{\otimes 2}$*. Likewise, the action of* $\mathrm{GL}_+(8;\mathbb{R})$ *on* $\bigwedge^5\left(\mathbb{R}^8\right)^*$ *has precisely three open orbits, represented by the 5-forms:*

$$\eta_c = -\frac{\sqrt{3}}{2}\theta^{12345} - \frac{\sqrt{3}}{2}\theta^{12367} - \frac{1}{2}\theta^{12458} + \frac{1}{2}\theta^{12678} + \frac{1}{2}\theta^{13468} + \frac{1}{2}\theta^{13578} - \frac{1}{2}\theta^{23478} + \frac{1}{2}\theta^{23568} + \theta^{45678};$$
$$\eta_s = \frac{1}{2}\theta^{12345} - \frac{1}{2}\theta^{12367} + \frac{1}{2}\theta^{12478} + \frac{1}{2}\theta^{12568} - \frac{1}{2}\theta^{13468} + \frac{1}{2}\theta^{13578} - \theta^{14567} - \frac{\sqrt{3}}{2}\theta^{23478} + \frac{\sqrt{3}}{2}\theta^{23568};$$
$$\eta_n = -\frac{\sqrt{3}}{2}\theta^{12358} + \frac{\sqrt{3}}{2}\theta^{12367} - \frac{1}{2}\theta^{12458} - \frac{1}{2}\theta^{12467} - \frac{1}{2}\theta^{13457} + \frac{1}{2}\theta^{13468} - \frac{1}{2}\theta^{23456} - \frac{1}{2}\theta^{23478} - \theta^{45678},$$

with stabilisers again given by $\mathbb{P}\mathrm{SU}(3)$, $\mathrm{SL}(3;\mathbb{R})$ *and* $\mathbb{P}\mathrm{SU}(1,2)$*, preserving the inner products* $g_{\eta_c} = \sum_{i=1}^{8}\left(\theta^i\right)^{\otimes 2}$, $g_{\eta_s} = \sum_{i=1}^{5}\left(\theta^i\right)^{\otimes 2} - \sum_{i=6}^{8}\left(\theta^i\right)^{\otimes 2}$ *and* $g_{\eta_n} = \sum_{i=1}^{4}\left(\theta^i\right)^{\otimes 2} - \sum_{i=5}^{8}\left(\theta^i\right)^{\otimes 2}$ *respectively. In particular, note that for any stable form* $\sigma_0 \in \bigwedge^p\left(\mathbb{R}^n\right)^*$ *with* $2 \leqslant p \leqslant n-2$, $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}$ *is connected.*

The results of Theorem 7.1.24 are summarised in Table 7.1, which also provides explicit formulae for the Hitchin duality maps Ξ (when defined).

Dim.	Stable Forms and Hitchin dualities	
$2k$ $k \geqslant 2$ (resp. $k \geqslant$ 3)	$\left\{\begin{array}{c}\text{emproplectic}\\ \text{forms}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{co-emproplectic}\\ \text{forms}\end{array}\right\}$ $\omega \xmapsto{\quad\Xi\quad} \omega^{k-1}$ $\omega \xleftarrow[\quad\Xi\quad]{} \omega^{k-1}$	$\left\{\begin{array}{c}\text{pisoplectic}\\ \text{forms}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{co-pisoplectic}\\ \text{forms}\end{array}\right\}$ $\omega \xmapsto{\quad\Xi\quad} -\omega^{k-1}$ $\omega \xleftarrow[\quad\Xi\quad]{} -\omega^{k-1}$
$2k+1$ $k \geqslant 2$	{pseudoplectic forms} (Ξ undefined)	{co-pseudoplectic forms} (Ξ undefined)
6	$\left\{\mathrm{SL}(3;\mathbb{R})^2\text{-structures}\right\}$ $\rho \xmapsto{\Xi} -I_\rho^* \rho$	$\{\mathrm{SL}(3;\mathbb{C})\text{-structures}\}$ $\rho \xmapsto{\Xi} J_\rho^* \rho$
7	$\{\mathrm{G}_2 \text{ 3-forms}\} \longleftrightarrow \{\mathrm{G}_2 \text{ 4-forms}\}$ $\phi \xmapsto{\quad\Xi\quad} \Theta(\phi)$ $\Sigma(\psi) \xleftarrow[\quad\Xi\quad]{} \psi$ $\left\{\widetilde{\mathrm{G}}_2 \text{ 3-forms}\right\} \longleftrightarrow \left\{\widetilde{\mathrm{G}}_2 \text{ 4-forms}\right\}$ $\widetilde{\phi} \xmapsto{\quad\Xi\quad} \Theta\left(\widetilde{\phi}\right)$ $\Sigma\left(\widetilde{\psi}\right) \xleftarrow[\quad\Xi\quad]{} \widetilde{\psi}$	$-\{\mathrm{G}_2 \text{ 3-forms}\} \longleftrightarrow -\{\mathrm{G}_2 \text{ 4-forms}\}$ $-\phi \xmapsto{\quad\Xi\quad} -\Theta(\phi)$ $-\Sigma(\psi) \xleftarrow[\quad\Xi\quad]{} -\psi$ $-\left\{\widetilde{\mathrm{G}}_2 \text{ 3-forms}\right\} \longleftrightarrow -\left\{\widetilde{\mathrm{G}}_2 \text{ 4-forms}\right\}$ $-\widetilde{\phi} \xmapsto{\quad\Xi\quad} -\Theta\left(\widetilde{\phi}\right)$ $-\Sigma\left(\widetilde{\psi}\right) \xleftarrow[\quad\Xi\quad]{} -\widetilde{\psi}$
8	$\{\mathbb{P}\mathrm{SU}(3) \text{ 3-forms}\} \longleftrightarrow \{\mathbb{P}\mathrm{SU}(3) \text{ 5-forms}\}$ $\zeta \xmapsto{\quad\Xi\quad} \bigstar_\zeta \zeta$ $\bigstar_\eta \eta \xleftarrow[\quad\Xi\quad]{} \eta$ $\{\mathbb{P}\mathrm{SU}(1,2) \text{ 3-forms}\} \longleftrightarrow \{\mathbb{P}\mathrm{SU}(1,2) \text{ 5-forms}\}$ $\zeta \xmapsto{\quad\Xi\quad} \bigstar_\zeta \zeta$ $\bigstar_\eta \eta \xleftarrow[\quad\Xi\quad]{} \eta$	$\{\mathrm{SL}(3;\mathbb{R}) \text{ 3-forms}\} \longleftrightarrow \{\mathrm{SL}(3;\mathbb{R}) \text{ 5-forms}\}$ $\zeta \xmapsto{\quad\Xi\quad} -\bigstar_\zeta \zeta$ $\bigstar_\eta \eta \xleftarrow[\quad\Xi\quad]{} \eta$

Table 7.1: Classification of Stable Forms and Hitchin Dualities

7.2 Relative h-principles for stable forms: precise formulation and corollaries

Let $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)^*$ and let M be an oriented n-manifold. Recall that a p-form $\sigma \in \Omega^p(\mathrm{M})$ is a σ_0-form if, for each $x \in \mathrm{M}$, there exists an orientation-preserving isomorphism $\alpha : \mathrm{T}_x\mathrm{M} \to \mathbb{R}^n$ satisfying $\alpha^*\sigma_0 = \sigma$. Write $\bigwedge^p_{\sigma_0}\mathrm{T}^*\mathrm{M}$ for the bundle of σ_0-forms over M and $\Omega^p_{\sigma_0}$ for the corresponding sheaf of sections.

Let $A \subset \mathrm{M}$ be a polyhedron, let D^q denote the q-dimensional disc ($q \geqslant 0$), let $\alpha : D^q \to H^p_{\mathrm{dR}}(\mathrm{M})$ be a continuous map and let $\mathfrak{F}_0 : D^q \to \Omega^p_{\sigma_0}(\mathrm{M})$ be a continuous map such that:

1. For all $s \in \partial D^q$: $\mathrm{d}\mathfrak{F}_0(s) = 0$ and $[\mathfrak{F}_0(s)] = \alpha(s) \in H^p_{\mathrm{dR}}(\mathrm{M})$;
2. For all $s \in D^q$: $\mathrm{d}\left(\mathfrak{F}_0(s)|_{\mathcal{O}pA}\right) = 0$ and $\left[\mathfrak{F}_0(s)|_{\mathcal{O}p(A)}\right] = \alpha(s)|_{\mathcal{O}p(A)} \in H^p_{\mathrm{dR}}(\mathcal{O}p(A))$.

(Recall that, from §2.5, one can always assume that $\mathcal{O}p(A)$ deformation retracts onto A and hence $\mathcal{O}p(A)$ and A have identical cohomology rings. Thus condition (2) is independent of the choice of $\mathcal{O}p(A)$.) The following definition combines the h-principles defined in [42, §6.2.C] and [32, Thm. 5.3].

Definition 7.2.1. σ_0-forms shall be said to satisfy the relative h-principle if for every M, A, q, α and $\mathfrak{F}_0$ as above, there exists a homotopy $\mathfrak{F}_\bullet : [0,1] \times D^q \to \Omega^p_{\sigma_0}(\mathrm{M})$, constant over ∂D^q, satisfying:

3. For all $s \in D^q$ and $t \in [0,1]$: $\mathfrak{F}_t(s)|_{\mathcal{O}p(A)} = \mathfrak{F}_0(s)|_{\mathcal{O}p(A)}$;
4. For all $s \in D^q$: $\mathrm{d}\mathfrak{F}_1(s) = 0$ and $[\mathfrak{F}_1(s)] = \alpha(s) \in H^p_{\mathrm{dR}}(\mathrm{M})$.

Definition 7.2.1 has two notable consequences. Firstly, given $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)^*$, an oriented n-manifold M and a fixed cohomology class $\alpha \in H^p_{\mathrm{dR}}(\mathrm{M})$, write $\mathcal{C}l^p_{\sigma_0}(\mathrm{M})$ for the set of closed σ_0-forms on M and $\mathcal{C}l^p_{\sigma_0}(\alpha)$ for the set of closed σ_0-forms representing the cohomology class α. More generally, given a polyhedron $A \subset \mathrm{M}$, let σ_r be a closed σ_0-form on $\mathcal{O}p(A)$ such that $[\sigma_r] = \alpha|_{\mathcal{O}p(A)} \in H^p_{\mathrm{dR}}(\mathcal{O}p(A))$ and write:

$$\Omega^p_{\sigma_0}(\mathrm{M};\sigma_r) = \left\{\sigma \in \Omega^p_{\sigma_0}(\mathrm{M}) \;\middle|\; \sigma|_{\mathcal{O}p(A)} = \sigma_r\right\}$$
$$\mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) = \left\{\sigma \in \Omega^p_{\sigma_0}(\mathrm{M};\sigma_r) \;\middle|\; \mathrm{d}\sigma = 0\right\}$$
$$\mathcal{C}l^p_{\sigma_0}(\alpha;\sigma_r) = \left\{\sigma \in \mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) \;\middle|\; [\sigma] = \alpha \in H^p_{\mathrm{dR}}(\mathrm{M})\right\}.$$

Standard homotopy-theoretic arguments (see [42, §6.2.A]) then yield:

Theorem 7.2.2. *Suppose that σ_0-forms satisfy the relative h-principle. Then for every* M, *A, α and σ_r, the inclusions:*

$$\mathcal{C}l^p_{\sigma_0}(\alpha;\sigma_r) \hookrightarrow \mathcal{C}l^p_{\sigma_0}(\mathrm{M};\sigma_r) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M};\sigma_r)$$

are homotopy equivalences (where $p = 3, 4, 2k-2, 2k-1$ as appropriate). In particular, taking $A = \varnothing$, the inclusions:

$$\mathcal{C}l^p_{\sigma_0}(\alpha) \hookrightarrow \mathcal{C}l^p_{\sigma_0}(\mathrm{M}) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M})$$

are also homotopy equivalences. Thus if M *admits any σ_0-form, then every degree p cohomology class on* M *can be represented by a (closed) σ_0-form.*

Secondly, let σ_0 be as above and suppose additionally that σ_0 is a Hitchin form.

Theorem 7.2.3. *For any closed, oriented n-manifold* M *admitting σ_0-forms and any $\alpha \in H^p_{\mathrm{dR}}(\mathrm{M})$, the Hitchin functional:*

$$\mathcal{H} : \mathcal{C}l^p_{\sigma_0}(\alpha) \to (0, \infty)$$

is unbounded above. More generally, if M *is a closed, oriented n-orbifold and $\mathcal{C}l^p_{\sigma_0}(\alpha) \neq \emptyset$, then the same conclusion applies.*

Proof. Begin with the case where M is a manifold. Since $\Omega^p_{\sigma_0}(\mathrm{M}) \neq \emptyset$ and $\mathcal{C}l^p_{\sigma_0}(\alpha) \hookrightarrow \Omega^p_{\sigma_0}(\mathrm{M})$ is a homotopy equivalence, $\mathcal{C}l^p_{\sigma_0}(\alpha) \neq \emptyset$. Thus pick $\sigma \in \mathcal{C}l^p_{\sigma_0}(\alpha)$. Let $f : \mathrm{B}^n_1(0) \hookrightarrow \mathrm{M}$ be an embedding, write $W = f\left(\mathrm{B}^n_1(0)\right)$ and $U = f\left(\mathrm{B}^n_{\frac{1}{2}}(0)\right)$ and consider the polyhedron $A \subset \mathrm{M}$ given by $A = \overline{U} \cup \mathrm{M} \backslash W$. Let $\chi : \mathrm{M} \to [0,1]$ be a smooth function on M such that $\chi|_{\mathcal{O}p\left(\overline{U}\right)} \equiv 1$ and $\chi|_{\mathcal{O}p(\mathrm{M}\backslash W)} \equiv 0$, and for each $\lambda \in (0, \infty)$ define $\sigma_\lambda \in \Omega^p_{\sigma_0}(\mathrm{M})$ by:

$$\sigma_\lambda = (1 + \lambda \cdot \chi)\, \sigma.$$

Then $\mathrm{d}\sigma_\lambda = 0$ on $\mathcal{O}p(A)$ since $\mathrm{d}\sigma = 0$ and χ is locally constant on $\mathcal{O}p(A)$, and hence $\sigma_\lambda \in \Omega_{\sigma_0}(\mathrm{M}; \sigma_\lambda|_{\mathcal{O}p(A)})$ for all $\lambda > 0$. Moreover, the restrictions $\sigma|_{\mathcal{O}p(A)}$ and $\sigma_\lambda|_{\mathcal{O}p(A)}$ both lie in $\alpha|_{\mathcal{O}p(A)} \in H^p_{\mathrm{dR}}(\mathcal{O}p(A))$ (for this point, it is useful to recall that $\overline{U}$ is contractible), so by the relative h-principle, $\mathcal{C}l^p_{\sigma_0}(\alpha; \sigma_\lambda|_{\mathcal{O}p(A)}) \hookrightarrow \Omega_{\sigma_0}(\mathrm{M}; \sigma_\lambda|_{\mathcal{O}p(A)})$ is a homotopy equivalence and thus one can continuously deform σ_λ relative to $\mathcal{O}p(A)$ into $\sigma'_\lambda \in \mathcal{C}l^p_{\sigma_0}(\alpha)$ such that:

$$\sigma'_\lambda = (1 + \lambda)\sigma \text{ on } U \quad \text{ and } \quad \sigma'_\lambda = \sigma \text{ on } \mathrm{M}\backslash W.$$

One now computes that:

$$\mathcal{H}(\sigma'_\lambda) \geqslant \int_U vol_{\sigma'_\lambda} = \int_U vol_{(1+\lambda)\sigma} = (1 + \lambda)^{\frac{n}{p}} \int_U vol_\sigma \to \infty \quad \text{ as } \quad \lambda \to \infty,$$

as required. In the case where M is an orbifold, provided $\mathcal{C}l^p_{\sigma_0}(\alpha) \neq \emptyset$, one can apply the above argument to the smooth locus of M, leading to the same conclusion.

□

As a simple application, note that Theorem 7.2.3 can be used to prove that emproplectic forms do not satisfy the relative h-principle. Indeed, let M be a $2k$-dimensional closed manifold and let ω be an emproplectic form on M. For every emproplectic form $\omega' \in \mathcal{C}l^2_{\omega_+(k)}([\omega])$:

$$\mathcal{H}(\omega') = \int_{\mathrm{M}} (\omega')^k = \langle [\omega']^{\cup k}, [\mathrm{M}] \rangle = \langle [\omega]^{\cup k}, [\mathrm{M}] \rangle,$$

independently of ω', where $[\mathrm{M}] \in \mathrm{H}^{2k}(\mathrm{M}; \mathbb{R})$ denotes the fundamental class of M and $\langle , \rangle$ denotes the usual pairing between cohomology and homology. In particular, the Hitchin functional on $\mathcal{C}l^2_{\omega_+(k)}([\omega])$ is constant, and thus not unbounded above (see §10.2 for further discussion of this).

Remark 7.2.4. Recall from the discussion at the end of §7.1.1 that given a co-emproplectic form ϖ on an oriented $2k$-manifold M, the form $-\varpi$ is a co-pisoplectic form on $\overline{\mathrm{M}}$. It follows that co-emproplectic forms satisfy the relative h-principle if and only if co-pisoplectic forms satisfy the relative h-principle. Analogous remarks apply to five other pairs of orbits of stable forms, *viz.* $\Lambda^2_\pm\left(\mathbb{R}^{2k}\right)^*$, $\pm\Lambda^3_+\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^3_\sim\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^4_+\left(\mathbb{R}^7\right)^*$ and $\pm\Lambda^4_\sim\left(\mathbb{R}^7\right)^*$. Thus, for the purpose of considering which stable forms satisfy the relative h-principle, the 22 types of stable forms described in Theorem 7.1.24 can be further grouped into 16 classes (as claimed in the introduction to this thesis) where each of $\Lambda^2_\pm\left(\mathbb{R}^{2k}\right)^*$, $\Lambda^{2k-2}_\pm\left(\mathbb{R}^{2k}\right)^*$, $\pm\Lambda^3_+\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^3_\sim\left(\mathbb{R}^7\right)^*$, $\pm\Lambda^4_+\left(\mathbb{R}^7\right)^*$ and $\pm\Lambda^4_\sim\left(\mathbb{R}^7\right)^*$ is considered a single class.

7.3 Relative h-principle for stable, ample forms

The aim of this section is to prove the following theorem:

Theorem 7.3.1. *Let $\sigma_0 \in \Lambda^p\left(\mathbb{R}^n\right)^*$ be stable. Given an arbitrary p form τ on $\mathbb{R}^{n-1}$, define:*

$$\mathcal{N}_{\sigma_0}(\tau) = \left\{\nu \in \Lambda^{p-1}\left(\mathbb{R}^{n-1}\right)^* \;\middle|\; \theta\wedge\nu + \tau \in \Lambda^p_{\sigma_0}\left(\mathbb{R}\oplus\mathbb{R}^{n-1}\right)^*\right\} \subset \Lambda^{p-1}\left(\mathbb{R}^{n-1}\right)^*$$

where θ is the standard annihilator of $\mathbb{R}^{n-1} \subset \mathbb{R}\oplus\mathbb{R}^{n-1}$. Suppose that, for every τ, the set $\mathcal{N}_{\sigma_0}(\tau)$ is ample in the sense of affine geometry, i.e. $\mathcal{N}_{\sigma_0}(\tau)$ is either empty, or the convex hull of every path component of $\mathcal{N}_{\sigma_0}(\tau)$ equals $\Lambda^{p-1}\left(\mathbb{R}^{n-1}\right)^*$ (in such cases, say that σ_0 itself is ample). Then σ_0-forms satisfy the relative h-principle.

Let M be an oriented n-manifold. Recall that the symbol of the exterior derivative on $(p-1)$-forms is the unique vector-bundle homomorphism $\mathcal{D} : \Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)} \to \Lambda^p\mathrm{T}^*\mathrm{M}$ such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma\left(\mathrm{M}, \Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}\right) \arrow[rr, "\mathcal{D}"] & & \Omega^p(\mathrm{M}) \\
& \Omega^{p-1}(\mathrm{M}) \arrow[ul, "j_1"] \arrow[ur, "\mathrm{d}"'] &
\end{tikzcd}$$

Explicitly, identifying $\Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)} \cong \Lambda^{p-1}\mathrm{T}^*\mathrm{M} \oplus \left(\mathrm{T}^*\mathrm{M}\otimes\Lambda^{p-1}\mathrm{T}^*\mathrm{M}\right)$ as usual, $\mathcal{D}$ is simply the composite map:

$$\Lambda^{p-1}\mathrm{T}^*\mathrm{M} \oplus \left(\mathrm{T}^*\mathrm{M}\otimes\Lambda^{p-1}\mathrm{T}^*\mathrm{M}\right) \xrightarrow{proj_2} \mathrm{T}^*\mathrm{M}\otimes\Lambda^{p-1}\mathrm{T}^*\mathrm{M} \xrightarrow{\wedge} \Lambda^p\mathrm{T}^*\mathrm{M}.$$

It follows that $\mathcal{D} : \Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)} \to \Lambda^p\mathrm{T}^*\mathrm{M}$ is a fibrewise surjective linear map.

Definition 7.3.2. Let $a : D^q \to \Omega^p(\mathrm{M})$ be a continuous map. Define fibred differential relation $\mathscr{R}_{\sigma_0}(a) \subset \Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}_{D^q}$ via:

$$\mathscr{R}_{\sigma_0}(a) = \left\{(s,T) \in \Lambda^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}_{D^q} \;\middle|\; \mathcal{D}(T) + a(s) \in \Lambda^p_{\sigma_0}\mathrm{T}^*\mathrm{M}\right\}.$$

Equivalently, $\mathscr{R}_{\sigma_0}(a)$ is the preimage of $\bigwedge^p_{\sigma_0}\mathrm{T}^*\mathrm{M}_{D^q}$ under the fibred map:

$$\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}_{D^q} \xrightarrow{\mathcal{D}+a} \bigwedge^{p}\mathrm{T}^*\mathrm{M}_{D^q}.$$

Lemma 7.3.3. *Suppose that, for every $q \geqslant 0$ and every continuous $a : D^q \to \Omega^p(\mathrm{M})$, the relation $\mathscr{R}_{\sigma_0}(a)$ satisfies the relative h-principle. Then σ_0-forms satisfy the relative h-principle.*

Proof. Let $A \subset \mathrm{M}$ be a (possibly empty) polyhedron, let $\alpha : D^q \to H^p_{\mathrm{dR}}(\mathrm{M})$ be a continuous map and let $\mathfrak{F}_0 : D^q \to \Omega^p_{\sigma_0}(\mathrm{M})$ be a continuous map such that:

1. For all $s \in \partial D^q$: $\mathrm{d}\mathfrak{F}_0(s) = 0$ and $[\mathfrak{F}_0(s)] = \alpha(s) \in H^p_{\mathrm{dR}}(\mathrm{M})$;

2. For all $s \in D^q$: $\mathrm{d}\left(\mathfrak{F}_0(s)|_{\mathcal{O}pA}\right) = 0$ and $\left[\mathfrak{F}_0(s)|_{\mathcal{O}p(A)}\right] = \alpha(s)|_{\mathcal{O}p(A)} \in H^p_{\mathrm{dR}}(\mathcal{O}p(A))$.

Then $\mathfrak{F}_0$ is a D^q-indexed family of p-forms on M, and thus one may regard $\mathfrak{F}_0$ as a section of the bundle $(\bigwedge^p\mathrm{T}^*\mathrm{M})_{D^q}$. Let $a : D^q \to \Omega^p(\mathrm{M})$ be a continuous map such that $a(s)$ represents the cohomology class $\alpha(s)$ for each $s \in D^q$ and consider the diagram:

$$\begin{tikzcd}
\left(\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}\right)^{(1)}_{D^q} \arrow[rr, "\mathcal{D}+a"] \arrow[dr] & & (\bigwedge^{p}\mathrm{T}^*\mathrm{M})_{D^q} \arrow[dl, shift left] \\
& D^q \times \mathrm{M} \arrow[ur, shift left, "\mathfrak{F}_0"'] &
\end{tikzcd} \tag{7.3.4}$$

The task is to lift the section $\mathfrak{F}_0$ along the map $\mathcal{D}+a$ to a section F_0 of $\left(\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}\right)^{(1)}_{D^q}$ which is holonomic over the region $(\partial D^q \times \mathrm{M}) \cup [D^q \times \mathcal{O}p(A)]$, and then apply the relative h-principle for the fibred relation $\mathscr{R}_{\sigma_0}(a)$.

Firstly, note that the map $\left(\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}\right)^{(1)}_{D^q} \xrightarrow{\mathcal{D}+a} (\bigwedge^{p}\mathrm{T}^*\mathrm{M})_{D^q}$ is an affine linear and surjective map of vector bundles; thus the preimage of the section $\mathfrak{F}_0$ under $\mathcal{D}+a$ defines an affine bundle over $D^q \times \mathrm{M}$ denoted $\mathcal{A}$. For each $s \in D^q$, $\mathfrak{F}_0(s)|_{\mathcal{O}p(A)} - a(s)|_{\mathcal{O}p(A)}$ is exact, since $\mathfrak{F}_0(s)$ and $a(s)$ both represent the cohomology class $\alpha(s)$ when restricted to $\mathcal{O}p(A)$. Pick $\eta : D^q \to \Omega^{p-1}(\mathcal{O}p(A))$ such that $\mathrm{d}\eta(s) = \mathfrak{F}_0(s)|_{\mathcal{O}p(A)} - a(s)|_{\mathcal{O}p(A)}$ for all $s \in D^q$, view η as a section of $\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}_{D^q}$ over $D^q \times \mathcal{O}p(A)$ and write $G_0(s) = j_1\eta(s)$ for the corresponding 1-jet. Then G_0 defines a section of the bundle $\mathcal{A}$ over the region $D^q \times \mathcal{O}p(A)$; choose some extension of G_0 to the whole of $D^q \times \mathrm{M}$ (which is possible since the extension problem for sections of affine bundles is trivial). Next, note that for each $s \in \partial D^q$, $\mathfrak{F}_0(s) - a(s)$ is exact and thus one can choose $\zeta : \partial D^q \to \Omega^p(\mathrm{M})$ such that $\mathrm{d}\zeta(x) = \mathfrak{F}_0(x) - a$ for each $s \in \partial D^q$. Write $H_0 = j_1\zeta$ as above and extend H_0 to a section of $\mathcal{A}$ over all of $D^q \times \mathrm{M}$. Now let $\chi : D^q \to [0,1]$ be a continuous function such that $\chi|_{\partial D^q} \cong 0$ and consider the section:

$$F_0 = \chi G_0 + (1-\chi)H_0$$

of $\mathcal{A}$. I claim that F_0 is holonomic over $(\partial D^q \times \mathrm{M}) \cup [D^q \times \mathcal{O}p(A)]$. Indeed, on $D^q \times \mathcal{O}p(A)$:

$$F_0 = \chi \cdot j_1\eta + (1-\chi)\cdot j_1\zeta = j_1(\chi\eta + (1-\chi)\zeta),$$

while on $\partial D^q \times \mathrm{M}$:

$$F_0 = (1-\chi)H_0 = (1-\chi)j_1\zeta = j_1((1-\chi)\zeta).$$

Since $\mathcal{A} \subset \mathscr{R}_{\sigma_0}(a)$, one can apply the relative h-principle for the relation $\mathscr{R}_{\sigma_0}(a)$ to obtain a homotopy of sections F_t of $\mathscr{R}_{\sigma_0}(a)$, constant over $(\partial D^q \times \mathrm{M}) \cup [D^q \times \mathcal{O}p(A)]$, such that F_1 is holonomic. Then $\mathfrak{F}_t = \mathcal{D}F_t + a$ defines the required homotopy of $\mathfrak{F}_0$, showing that σ_0-forms satisfy the h-principle.

□

Note that the homotopy of sections $\mathfrak{F}_t : [0,1] \times D^q \times \mathrm{M} \to (\bigwedge^p \mathrm{T}^*\mathrm{M})_{D^q}$ cannot be taken to be arbitrarily C^0-small, due to the well-known consequence of Stokes' Theorem that $\Omega^p_{\text{closed}}(\mathrm{M}) \subset \Omega^p(\mathrm{M})$ is closed in the C^0-topology and not just the C^1-topology. Nevertheless, for the choices of σ_0 considered in this paper, the relation $\mathscr{R}_{\sigma_0}(a)$ satisfies the C^0-dense relative h-principle and thus the homotopy p_1F_t of sections of $\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}_{D^q}$ arising in the above proof can be taken to be arbitrarily C^0-small. This is not a contradiction, however, since $\mathfrak{F}_t$ depends on the full 1-jet F_t, and not just on the underlying section p_1F_t.

In view of Lemma 7.3.3, to prove Theorem 7.3.1, it suffices to prove that if σ_0 is stable and ample, then $\mathscr{R}_{\sigma_0}(a)$ satisfies the relative h-principle for any $a : D^q \to \Omega^p(\mathrm{M})$. This follows by combining the Convex Integration Theorem 2.5.8 with the following result:

Proposition 7.3.5. *Fix $q \geqslant 0$ and $a : D^q \to \Omega^p(\mathrm{M})$.*

1. *$\mathscr{R}_{\sigma_0}(a)$ is an open subset of $\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}$ if and only if σ_0 is stable.*
2. *$\mathscr{R}_{\sigma_0}(a)$ is an ample fibred differential relation if and only if σ_0 is ample.*

Proof. 1 is clear, since $\bigwedge^p\mathrm{T}^*\mathrm{M}_{D^q} \xrightarrow{+a} \bigwedge^p\mathrm{T}^*\mathrm{M}_{D^q}$ is a homeomorphism and $\bigwedge^{p-1}\mathrm{T}^*\mathrm{M}^{(1)}_{D^q} \xrightarrow{\mathcal{D}} \bigwedge^p\mathrm{T}^*\mathrm{M}_{D^q}$ is continuous and open (being a fibrewise linear surjection). For 2, note that, as in the discussion after Definition 2.5.7:

$$\mathscr{R}_{\sigma_0}(a) = \bigwedge^{p-1}\mathrm{T}^*\mathrm{M}_{D^q} \times_{(D^q\times\mathrm{M})} \mathscr{R}'_{\sigma_0}(a) \subset \bigwedge^{p-1}\mathrm{T}^*\mathrm{M}_{D^q} \times_{(D^q\times\mathrm{M})} \left(\mathrm{T}^*\mathrm{M} \otimes \bigwedge^{p-1}\mathrm{T}^*\mathrm{M}\right)_{D^q}$$

where:

$$\mathscr{R}'_{\sigma_0}(a) = \left\{(s,T) \in \left(\mathrm{T}^*\mathrm{M} \otimes \bigwedge^{p-1}\mathrm{T}^*\mathrm{M}\right)_{D^q} \;\middle|\; \wedge(T) + a(s) \in \bigwedge^p_{\sigma_0}\mathrm{T}^*\mathrm{M}\right\}.$$

Then, in the notation introduced after Definition 2.5.7, for each $s \in D^q$:

$$\mathscr{R}'_{\sigma_0}(a)_s = \left\{T \in \mathrm{T}^*\mathrm{M} \otimes \bigwedge^{p-1}\mathrm{T}^*\mathrm{M} \;\middle|\; \wedge(T) + a(s) \in \bigwedge^p_{\sigma_0}\mathrm{T}^*\mathrm{M}\right\}.$$

As explained after Definition 2.5.7, $\mathscr{R}_{\sigma_0}(a)$ is ample if and only if for all $x \in \mathrm{M}$, $\mathbb{B} \subset \mathrm{T}_x\mathrm{M}$ a hyperplane and $\lambda \in \mathbb{B}^* \otimes \bigwedge^{p-1}\mathrm{T}^*_x\mathrm{M}$: $\mathscr{R}'_{\sigma_0}(a)_s \cap \Pi(\mathbb{B},\lambda) \subseteq \Pi(\mathbb{B},\lambda)$ is ample.

Choose a splitting $\mathrm{T}_x\mathrm{M} = \mathbb{L} \oplus \mathbb{B}$ and pick an orientation on $\mathbb{B}$. This choice canonically orients $\mathbb{L}$; choose a correctly oriented generator θ of $\mathbb{L}^*$. Using this data, one may write:

$$\mathrm{T}^*_x\mathrm{M} = \mathbb{R}\cdot\theta \oplus \mathbb{B}^* \quad \text{and} \quad \bigwedge^{p-1}\mathrm{T}^*_x\mathrm{M} = \theta \wedge \bigwedge^{p-2}\mathbb{B}^* \oplus \bigwedge^{p-1}\mathbb{B}^*.$$

Hence there is an isomorphism:

$$\begin{aligned}\textstyle\bigwedge^{p-2}\mathbb{B}^* \oplus \bigwedge^{p-1}\mathbb{B}^* \oplus \left(\mathbb{B}^* \otimes \bigwedge^{p-1}\mathrm{T}^*_x\mathrm{M}\right) &\xrightarrow{\cong} \mathrm{T}^*_x\mathrm{M} \otimes \textstyle\bigwedge^{p-1}\mathrm{T}^*_x\mathrm{M}\\ \alpha \oplus \nu \oplus \lambda &\longmapsto \theta \otimes (\theta \wedge \alpha + \nu) + \lambda.\end{aligned}$$

(For completeness, in the case $p = 1$ one simply treats the space $\bigwedge^{p-2}\mathbb{B}^*$ as 0, although I shall only be concerned with the case $p \geqslant 2$.) Using this isomorphism, one obtains the explicit description:

$$\Pi(\mathbb{B},\lambda) \cong \bigwedge^{p-2}\mathbb{B}^* \times \bigwedge^{p-1}\mathbb{B}^* \times \{\lambda\}.$$

Now define $\nu_0 \in \bigwedge^{p-1}\mathbb{B}^*$ and $\tau_0 \in \bigwedge^{p}\mathbb{B}^*$ by the equation:

$$\wedge(\lambda) + a(s) = \theta \wedge \nu_0 + \tau_0. \tag{7.3.6}$$

Then given $(\alpha,\nu) \in \bigwedge^{p-2}\mathbb{B}^* \oplus \bigwedge^{p-1}\mathbb{B}^*$, one can compute that:

$$\wedge\left[\theta \otimes (\theta \wedge \alpha + \nu) + \lambda\right] + a(s) = \theta \wedge (\nu + \nu_0) + \tau_0,$$

which is a σ_0-form if and only if $\nu + \nu_0 \in \mathcal{N}_{\sigma_0}(\tau_0)$. Thus:

$$\mathscr{R}'_{\sigma_0}(a)_s \cap \Pi(\mathbb{B},\lambda) \cong \bigwedge^{p-2}\mathbb{B}^* \times (\mathcal{N}_{\sigma_0}(\tau_0) - \nu_0) \times \{\lambda\} \subseteq \bigwedge^{p-2}\mathbb{B}^* \times \bigwedge^{p-1}\mathbb{B}^* \times \{\lambda\} \cong \Pi(\mathbb{B},\lambda).$$

Thus $\mathscr{R}'_{\sigma_0}(a)_s \cap \Pi(\mathbb{B},\lambda) \subseteq \Pi(\mathbb{B},\lambda)$ is ample if and only if $\mathcal{N}_{\sigma_0}(\tau_0) - \nu_0 \subseteq \bigwedge^{p-1}\mathbb{B}^*$ is ample, which in turn is equivalent to $\mathcal{N}_{\sigma_0}(\tau_0) \subseteq \bigwedge^{p-1}\mathbb{B}^*$ being ample.

Finally, note that, for fixed $a(s)$, the assignment $\lambda \mapsto (\nu_0, \tau_0)$ described in eqn. (7.3.6) is surjective, and thus as λ varies, τ_0 realises all possible values in $\bigwedge^p\mathbb{B}^*$. Hence $\mathscr{R}_{\sigma_0}(a)$ is ample if and only if σ_0 is ample, as claimed.

□

7.4 Faithful, connected and abundant p-forms

The aim of this section is to develop theoretical tools for effectively verifying whether a given stable form is ample. Let $Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right)$ denote the space of linear embeddings $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$. Given $\sigma_0 \in \bigwedge^p\left(\mathbb{R}^n\right)^*$, there is a natural map:

$$\begin{aligned}Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right) &\xrightarrow{\mathscr{T}_{\sigma_0}} \textstyle\bigwedge^p\left(\mathbb{R}^{n-1}\right)^*\\ \iota &\longmapsto \iota^*\sigma_0.\end{aligned}$$

$\mathrm{GL}_+(n-1;\mathbb{R})$ acts on $Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right)$ via precomposition, and the quotient $Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right)\big/\mathrm{GL}_+(n-1;\mathbb{R})$ can naturally be identified with the oriented Grassmannian $\widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right)$. Given $f \in \mathrm{GL}_+(n-1;\mathbb{R})$, a direct computation shows:

$$\mathscr{T}_{\sigma_0}(\iota \circ f) = f^* \iota^* (\sigma_0) = f^* \mathscr{T}_{\sigma_0}(\iota).$$

Thus $\mathscr{T}_{\sigma_0}$ descends to a map:

$$\widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \longrightarrow \left. \Lambda^p \left(\mathbb{R}^{n-1}\right)^* \middle/ \mathrm{GL}_+(n-1;\mathbb{R}) \right. .$$

Write $\mathcal{S}(\sigma_0)$ for the stabiliser of σ_0 in $\mathrm{GL}_+(n;\mathbb{R})$ and note that $\mathcal{S}(\sigma_0)$ acts on $Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right)$ (and hence on $\widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n)$) on the left via post-composition. Clearly $\mathscr{T}_{\sigma_0}$ is invariant under this action and thus $\mathscr{T}_{\sigma_0}$ descends further to a map:

$$\mathcal{T}_{\sigma_0} : \left. \mathcal{S}(\sigma_0) \middle\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \right. \longrightarrow \left. \Lambda^p \left(\mathbb{R}^{n-1}\right)^* \middle/ \mathrm{GL}_+(n-1,\mathbb{R}) \right. .$$

Using this notation, I make the following definition:

Definition 7.4.1. Let $\sigma_0 \in \Lambda^p(\mathbb{R}^n)^*$.

1. σ_0 is termed faithful if $\mathcal{T}_{\sigma_0}$ is injective.

2. σ_0 is termed connected if for each orbit $\mathcal{O} \in \left. \mathcal{S}(\sigma_0) \middle\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \right.$ the stabiliser of some (equivalently any) $\tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$ is path-connected.

3. σ_0 is termed abundant if for all $\mathcal{O} \in \left. \mathcal{S}(\sigma_0) \middle\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \right.$ and some (equivalently any) $\tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$:

$$0 \in \mathrm{Conv}\, \mathcal{N}_{\sigma_0}(\tau) \subseteq \Lambda^{p-1} \left(\mathbb{R}^{n-1}\right)^* .$$

In words, σ_0 is faithful if for all oriented hyperplanes $\mathbb{B}$, $\mathbb{B}' \subset \mathbb{R}^n$, the restrictions $\sigma_0|_{\mathbb{B}}$ and $\sigma_0|_{\mathbb{B}'}$ are isomorphic if and only if $\mathbb{B}$ and $\mathbb{B}'$ lie in the same orbit of $\mathcal{S}(\sigma_0)$, σ_0 is connected if for every oriented hyperplane $\mathbb{B} \subset \mathbb{R}^n$, the stabiliser of $\sigma_0|_{\mathbb{B}}$ in $\mathrm{GL}_+(\mathbb{B})$ is connected, and σ_0 is abundant if for every $\tau \in \Lambda^p \left(\mathbb{R}^{n-1}\right)^*$, either $\mathcal{N}_{\sigma_0}(\tau)$ is empty, or the convex hull of $\mathcal{N}_{\sigma_0}(\tau)$ contains 0.

Verifying the above three properties in practice is greatly helped by the following three results:

Proposition 7.4.2. *Let $\sigma_0 \in \Lambda^p(\mathbb{R}^n)^*$ be stable and equip the spaces $\left. \mathcal{S}(\sigma_0) \middle\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \right.$ and $\left. \Lambda^p \left(\mathbb{R}^{n-1}\right)^* \middle/ \mathrm{GL}_+(n-1,\mathbb{R}) \right.$ with their natural quotient topologies. Then the map $\mathcal{T}_{\sigma_0}$ is an open map. In particular, if $\mathcal{O} \in \left. \mathcal{S}(\sigma_0) \middle\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n) \right.$ is an open orbit, then $\mathcal{T}_{\sigma_0}(\mathcal{O})$ is also an open orbit, i.e. the orbit of a stable p-form on $\mathbb{R}^{n-1}$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right) \arrow[r, "\mathscr{T}_{\sigma_0}"] \arrow[d, "quot"] & \bigwedge^p\left(\mathbb{R}^{n-1}\right)^* \arrow[d, "quot"] \\
\mathcal{S}(\sigma_0)\Big\backslash\widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right) \arrow[r, "\mathcal{T}_{\sigma_0}"] & \bigwedge^p\left(\mathbb{R}^{n-1}\right)^*\Big/\mathrm{GL}_+(n-1,\mathbb{R})\cdot
\end{tikzcd}$$

Since the left hand map is a continuous surjection and the right hand map is open, to prove $\mathcal{T}_{\sigma_0}$ is open it suffices to prove that $\mathscr{T}_{\sigma_0}$ is open. To this end, let $\iota:\mathbb{R}^{n-1}\to\mathbb{R}^n$ be an embedding and fix a splitting β of the exact sequence:

$$\textstyle\bigwedge^p\left(\mathbb{R}^n\right)^* \underset{\beta}{\overset{\iota^*}{\rightleftarrows}} \bigwedge^p\left(\mathbb{R}^{n-1}\right)^* \longrightarrow 0.$$

Consider $\tau\in\bigwedge^p\left(\mathbb{R}^{n-1}\right)^*$. Since σ_0 is stable, for all τ sufficiently small there is $F\in\mathrm{GL}_+(n;\mathbb{R})$ (close to Id) such that:

$$F^*\sigma_0=\sigma_0+\beta(\tau).$$

Set $\iota'=F\circ\iota\in Emb\left(\mathbb{R}^{n-1},\mathbb{R}^n\right)$. For τ small enough, ι' can be taken arbitrarily close to ι. Moreover:

$$\iota'^*\sigma_0=\iota^*F^*\sigma_0=\iota^*\left(\sigma_0+\beta(\tau)\right)=\iota^*\sigma_0+\tau,$$

since $\iota^*\circ\beta=\mathrm{Id}$. Thus $\mathscr{T}_{\sigma_0}$ is an open map and the result follows.

□

Proposition 7.4.3. *Let σ_0 be a stable p-form on $\mathbb{R}^n$ and suppose that $\mathcal{S}(\sigma_0)$ acts transitively on $\widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right)$. Then σ_0 is faithful and connected. In particular, if $\mathcal{S}(\sigma_0)$ contains a subgroup which preserves an inner-product and acts transitively on the corresponding unit sphere in $\mathbb{R}^n$ (equivalently in $\left(\mathbb{R}^n\right)^*$) then σ_0 is faithful and connected.*

Proof. Faithfulness is clear. Since $\mathcal{S}(\sigma_0)$ acts transitively on $\widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right)$, the unique orbit must be open and hence by Proposition 7.4.2, it must map under $\mathcal{T}_{\sigma_0}$ to the orbit of a stable form. However, using the results of Theorem 7.1.24, the stabiliser in $\mathrm{GL}_+(n-1;\mathbb{R})$ of every stable form on $\mathbb{R}^{n-1}$ is connected. The final statement follows since $\widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right)$ is isomorphic to the unit sphere in $\left(\mathbb{R}^n\right)^*$.

□

Proposition 7.4.4. *If there exists an orientation-reversing automorphism of $\mathbb{R}^n$ which preserves σ_0, then σ_0 is abundant. In particular, if $n=2k+1$ is odd and $2\leqslant p\leqslant 2k$ is even, then any $\sigma_0\in\bigwedge^p\left(\mathbb{R}^n\right)^*$ is abundant.*

Proof. Fix $\tau\in\bigwedge^p\left(\mathbb{R}^{n-1}\right)^*$ and suppose $\mathcal{N}_{\sigma_0}(\tau)\neq\varnothing$. Choose some $\nu\in\mathcal{N}_{\sigma_0}(\tau)$. I claim that $-\nu$ also lies in $\mathcal{N}_{\sigma_0}(\tau)$. Indeed since $\theta\wedge\nu+\tau\in\bigwedge^p\left(\mathbb{R}\oplus\mathbb{R}^{n-1}\right)$ is a σ_0-form, by assumption there exists an orientation reversing map $F \in \mathrm{GL}(\mathbb{R} \oplus \mathbb{R}^{n-1})$ preserving $\theta \wedge \nu + \tau$. Now consider the map:

$$\begin{aligned} \mathcal{I} : \mathbb{R} \oplus \mathbb{R}^{n-1} &\longrightarrow \mathbb{R} \oplus \mathbb{R}^{n-1} \\ v \oplus w &\longmapsto -v \oplus w. \end{aligned}$$

Since $\mathcal{I}$ is also orientation-reversing, the composite $F \circ \mathcal{I}$ is orientation preserving and thus $(F \circ \mathcal{I})^*(\tau + \theta \wedge \nu)$ is a σ_0-form. On the other hand:

$$\begin{aligned} (F \circ \mathcal{I})^*(\tau + \theta \wedge \nu) &= \mathcal{I}^*(F)^*(\tau + \theta \wedge \nu) \\ &= \mathcal{I}^*(\tau + \theta \wedge \nu) \\ &= \tau - \theta \wedge \nu \end{aligned}$$

and thus $-\nu \in \mathcal{N}_{\sigma_0}(\tau)$ as claimed. The proof is completed by noting that if $n = 2k+1$ and $2 \leqslant p \leqslant 2k$ is even, then $-\mathrm{Id}$ is an orientation-reversing automorphism preserving σ_0.

□

The significance of the notions of faithfulness, connectedness and abundance lies in the following result:

Theorem 7.4.5. *Let $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)^*$ be stable, faithful, abundant and connected. Then σ_0 is ample; in particular, σ_0-forms satisfy the h-principle.*

Remark 7.4.6. The converse need not hold: see §7.6.

The proof proceeds by a series of lemmas.

Lemma 7.4.7. *Let $\mathbb{A}$ be a (real) finite-dimensional vector space and let $A \subseteq \mathbb{A}$ be a path-connected, open subset such that:*

- $0 \in \mathrm{Conv}(A)$*;*
- *A is scale-invariant, i.e. for all $\lambda \in (0, \infty)$, $\lambda \cdot A = A$.*

Then $\mathrm{Conv}(A) = \mathbb{A}$*, i.e. A is ample.*

Proof. Since A is open and scale-invariant, so too is $\mathrm{Conv}(A)$. However by assumption $\mathrm{Conv}(A)$ contains 0, and thus by openness it contains a small open ball about 0 in $\mathbb{A}$. The scale-invariance of $\mathrm{Conv} A$ then implies that $\mathrm{Conv}(A) = \mathbb{A}$.

□

Lemma 7.4.8. *Let $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)$, $\tau \in \bigwedge^p(\mathbb{R}^{n-1})$ and suppose $\mathcal{N}_{\sigma_0}(\tau) \neq \varnothing$. Then $\mathcal{N}_{\sigma_0}(\tau)$ is scale-invariant, i.e. for all $\lambda \in (0, \infty)$: $\lambda \cdot \mathcal{N}_{\sigma_0}(\tau) = \mathcal{N}_{\sigma_0}(\tau)$.*

Proof. Suppose $\nu \in \mathcal{N}_{\sigma_0}(\tau)$, i.e. $\theta \wedge \nu + \tau \in \bigwedge^p (\mathbb{R} \oplus \mathbb{R}^{n-1})^*$ is a σ_0-form. Consider the orientation-preserving isomorphism:

$$\begin{aligned} F : \mathbb{R} \oplus \mathbb{R}^{n-1} &\longrightarrow \mathbb{R} \oplus \mathbb{R}^{n-1} \\ v \oplus w &\longmapsto \lambda v \oplus w. \end{aligned}$$

Then $F^*\sigma = \theta \wedge (\lambda\nu) + \tau$ is a σ_0-form, as required.

$\square$

Lemma 7.4.9. *Let $\sigma_0 \in \bigwedge^p (\mathbb{R}^n)^*$ and suppose that $\mathcal{O} \in \mathcal{S}(\sigma_0)\Big\backslash \widetilde{\mathrm{Gr}}_{n-1}(\mathbb{R}^n)$ satisfies $\mathcal{T}_{\sigma_0}^{-1}(\mathcal{T}_{\sigma_0}(\mathcal{O})) = \{\mathcal{O}\}$. Suppose moreover that some (equivalently every) $\tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$ has path-connected stabiliser in* $\mathrm{GL}_+(n-1;\mathbb{R})$. *Then for all $\tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$, the space $\mathcal{N}_{\sigma_0}(\tau) \subset \bigwedge^{p-1}\left(\mathbb{R}^{n-1}\right)^*$ is path-connected. In particular, if σ_0 is faithful and connected, then for every $\tau \in \bigwedge^p\left(\mathbb{R}^{n-1}\right)^*$, either $\mathcal{N}_{\sigma_0}(\tau) = \varnothing$ or $\mathcal{N}_{\sigma_0}(\tau)$ is path-connected.*

Proof. Let $\mathcal{O}$ be as in the statement of the lemma, let $\tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$ and let $\nu_1, \nu_2 \in \mathcal{N}_{\sigma_0}(\tau)$. Then by definition, the two p-forms:

$$\sigma_i = \theta \wedge \nu_i + \tau \in \bigwedge^p\left(\mathbb{R} \oplus \mathbb{R}^{n-1}\right)^*,\ i = 1, 2$$

are both σ_0-forms on $\mathbb{R} \oplus \mathbb{R}^{n-1}$. Thus, there is $F \in \mathrm{GL}_+\left(\mathbb{R} \oplus \mathbb{R}^{n-1}\right)$ such that $F^*\sigma_2 = \sigma_1$.

Claim 7.4.10. *The oriented hyperplanes $\mathbb{R}^{n-1}$ and $F\left(\mathbb{R}^{n-1}\right)$ in $\mathbb{R} \oplus \mathbb{R}^{n-1}$ lie in the same orbit of the stabiliser of σ_2.*

Proof of Claim. Since σ_2 is a σ_0-form, there is an oriented isomorphism $\mathscr{I} : \mathbb{R} \oplus \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that $\mathscr{I}^*\sigma_0 = \sigma_2$. Hence it is equivalent prove that the oriented hyperplanes $\mathscr{I}\left(\mathbb{R}^{n-1}\right)$ and $\mathscr{I} \circ F\left(\mathbb{R}^{n-1}\right)$ in $\mathbb{R}^n$ lie in the same $\mathcal{S}(\sigma_0)$-orbit.

Consider the commutative diagram:

$$\begin{array}{ccc} Emb\left(\mathbb{R}^{n-1}, \mathbb{R}^n\right) & \xrightarrow{\quad \mathscr{T}_{\sigma_0} \quad} & \bigwedge^p\left(\mathbb{R}^{n-1}\right)^* \\ \big\downarrow{\scriptstyle quot} & & \big\downarrow{\scriptstyle quot} \\ \mathcal{S}(\sigma_0)\Big\backslash \widetilde{\mathrm{Gr}}_{n-1}\left(\mathbb{R}^n\right) & \xrightarrow{\mathcal{T}_{\sigma_0}} & \bigwedge^p\left(\mathbb{R}^{n-1}\right)^*\Big/\mathrm{GL}_+(n-1, \mathbb{R}) \end{array}.$$

Since $\mathcal{T}_{\sigma_0}^{-1}(\mathcal{T}_{\sigma_0}(\mathcal{O})) = \{\mathcal{O}\}$, it suffices to prove that both $\mathscr{T}_{\sigma_0}\left(\mathscr{I}|_{\mathbb{R}^{n-1}}\right)$ and $\mathscr{T}_{\sigma_0}\left((\mathscr{I} \circ F)|_{\mathbb{R}^{n-1}}\right)$ lie in the orbit $\mathcal{T}_{\sigma_0}(\mathcal{O}) \in \bigwedge^p\left(\mathbb{R}^{n-1}\right)^*\Big/\mathrm{GL}_+(n-1, \mathbb{R})$. But this result is clear, since:

$$\left(\mathscr{I}|_{\mathbb{R}^{n-1}}\right)^* \sigma_0 = \sigma_2|_{\mathbb{R}^{n-1}} = \tau \in \mathcal{T}_{\sigma_0}(\mathcal{O})$$

and

$$\left((\mathscr{I} \circ F)|_{\mathbb{R}^{n-1}}\right)^* \sigma_0 = (F^*\sigma_2)|_{\mathbb{R}^{n-1}} = \sigma_1|_{\mathbb{R}^{n-1}} = \tau \in \mathcal{T}_{\sigma_0}(\mathcal{O}).$$

$\square$

Thus choose $G \in \mathrm{GL}_+\left(\mathbb{R} \oplus \mathbb{R}^{n-1}\right)$ stabilising σ_2 such that $G \circ F\left(\mathbb{R}^{n-1}\right) = \mathbb{R}^{n-1}$ (and $G \circ F$ identifies the orientations). By replacing F with $G \circ F$, one may assume without loss of generality that F fixes the space $\mathbb{R}^{n-1}$. Then:

$$\tau = \sigma_1|_{\mathbb{R}^{n-1}} = (F^*\sigma_2)|_{\mathbb{R}^{n-1}} = \left(F|_{\mathbb{R}^{n-1}}\right)^* \sigma_2|_{\mathbb{R}^{n-1}} = \left(F|_{\mathbb{R}^{n-1}}\right)^* \tau.$$

Thus F lies in the space:

$$\mathscr{K} = \left\{ F' \in \mathrm{GL}_+\left(\mathbb{R} \oplus \mathbb{R}^{n-1}\right) \;\middle|\; F'\left(\mathbb{R}^{n-1}\right) = \mathbb{R}^{n-1} \text{ and } F'|_{\mathbb{R}^{n-1}} \in \mathrm{Stab}(\tau) \subseteq \mathrm{GL}_+(n-1,\mathbb{R}) \right\}.$$

Since $\mathrm{Stab}(\tau) \subseteq \mathrm{GL}_+(n-1,\mathbb{R})$ is path-connected, so too is $\mathscr{K} \subset \mathrm{GL}_+\left(\mathbb{R} \oplus \mathbb{R}^{n-1}\right)$, so one can choose a smooth 1-parameter family $(F_t)_{t\in[0,1]} \in \mathscr{K}$ such that $F_0 = \mathrm{Id}$, $F_1 = F$. Then for each t:

$$F_t^*\sigma_2 = \theta \wedge \nu(t) + \tau$$

for some $\nu(t) \in \mathcal{N}_{\sigma_0}(\tau)$ (note that $F_t^*\sigma_2$ is evidently a σ_0-form for each t) such that $\nu(1) = \nu_1$ and $\nu(0) = \nu_2$. Thus $\mathcal{N}_{\sigma_0}(\tau)$ is path-connected.

□

I now prove Theorem 7.4.5:

Proof of Theorem 7.4.5. Let $\sigma_0 \in \Lambda^p(\mathbb{R}^n)^*$ be stable, faithful, connected and abundant, let $\tau \in \Lambda^p\left(\mathbb{R}^{n-1}\right)^*$ and suppose $\mathcal{N}_{\sigma_0}(\tau) \neq \varnothing$. Since σ_0 is stable, $\mathcal{N}_{\sigma_0}(\tau) \subseteq \Lambda^{p-1}\left(\mathbb{R}^{n-1}\right)^*$ is open. Moreover $\mathcal{N}_{\sigma_0}(\tau)$ is scale invariant by Lemma 7.4.8, path-connected by Lemma 7.4.9 and $0 \in \mathrm{Conv}\left(\mathcal{N}_{\sigma_0}(\tau)\right)$ since τ is abundant. Hence $\mathcal{N}_{\sigma_0}(\tau)$ is ample by Lemma 7.4.7.

□

7.5 Initial applications: G_2 4-forms, $\mathrm{SL}(3;\mathbb{C})$ 3-forms and pseudoplectic forms

This section illustrates the results of §§7.3 and 7.4, by providing new, unified proofs of the three previously established relative h-principles, *viz.* the relative h-principles for G_2 4-forms [32], $\mathrm{SL}(3;\mathbb{C})$ 3-forms [37] and pseudoplectic forms [104].

7.5.1 G_2 4-forms

Theorem 7.5.1 ([32, Thm. 5.3])**.** G_2 *4-forms satisfy the relative h-principle.*

Remark 7.5.2. In [32], Crowley–Nordström only state the non-relative version of the h-principle for G_2 4-forms (corresponding to $A = \varnothing$, in the notation of the introduction), however their proof can easily be generalised to the case $A \neq \varnothing$.

Proof. Let $\sigma_0 = \psi_0 \in \Lambda^4(\mathbb{R}^7)^*$ be the standard G_2 4-form (see eqn. (2.2.3)) and write g_0 for the corresponding metric. Since G_2 preserves an inner-product and acts transitively on S^6, by Proposition 7.4.3 ψ_0 is faithful and connected. Moreover, since ψ_0 has even degree on an odd-dimensional space, ψ_0 is abundant by Proposition 7.4.4. Thus the result follows by Theorem 7.4.5.

□

7.5.2 $\mathrm{SL}(3;\mathbb{C})$ 3-forms

Theorem 7.5.3 ([37, §4]). $\mathrm{SL}(3;\mathbb{C})$ *3-forms satisfy the relative h-principle.*

Proof. Let $\sigma_0 = \rho_- \in \bigwedge^3 \left(\mathbb{R}^6\right)^*$ be the standard $\mathrm{SL}(3;\mathbb{C})$ 3-form (see eqn. (2.3.5)). The stabiliser $\mathrm{SL}(3;\mathbb{C})$ of ρ_- contains the subgroup $\mathrm{SU}(3)$ which preserves an inner-product and acts transitively on S^5; thus ρ_- is faithful and connected by Proposition 7.4.3. Moreover, by Proposition 2.3.4, ρ_- is fixed by an orientation-reversing automorphism of $\mathbb{R}^6$ and thus ρ_- is abundant by Lemma 7.4.4. Thus the result follows by Theorem 7.4.5.

□

7.5.3 Pseudoplectic forms

Theorem 7.5.4 ([104, Thm. 2.5]). *Pseudoplectic forms satisfy the relative h-principle.*

The proof in this case is slightly more involved.

Proof of Theorem 7.5.4. Let $\mu_0(k) = \theta^{23} + \theta^{45} + ...\theta^{2k,2k+1}$ be the standard pseudoplectic form on $\mathbb{R}^{2k+1}$ (see eqn. (7.1.16)), write $\mathcal{S}$ for the stabiliser of $\mu_0(k)$ in $\mathrm{GL}_+(2k+1;\mathbb{R})$ and recall the 1-dimensional subspace $\ell_{\mu_0(k)} = \langle e_1 \rangle$ defined in Remark 7.1.23. Given an oriented hyperplane $\mathbb{B} \subset \mathbb{R}^{2k+1}$, on dimensional grounds, either $\dim(\mathbb{B} \cap \ell_{\mu_0(k)}) = 1$, in which case $\ell_{\mu_0(k)} \subset \mathbb{B}$ and $\mu_0(k)|_{\mathbb{B}}$ is a degenerate bilinear form, or $\dim(\mathbb{B} \cap \ell_{\mu_0(k)}) = 0$ and $\mathbb{B}$ is transverse to $\ell_{\mu_0(k)}$, in which case $\mu_0(k)|_{\mathbb{B}}$ is either emproplectic or pisoplectic on $\mathbb{B}$. Thus the image of the map:

$$\mathcal{T}_{\mu_0(k)} : \mathcal{S} \Big\backslash \widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right) \longrightarrow \bigwedge^2 \left(\mathbb{R}^{2k}\right)^* \Big/ \mathrm{GL}_+(2k,\mathbb{R})$$

contains at least three distinct orbits and thus the action of $\mathcal{S}$ on $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)$ has at least 3 orbits.

Therefore, to prove that $\mu_0(k)$ is faithful, it suffices to prove that the action of $\mathcal{S}$ on $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)$ has exactly three orbits. Recall from Proposition 7.1.14 that, with respect to the splitting $\mathbb{R}^{2k+1} = \ell_{\mu_0(k)} \oplus \langle e_2, ..., e_{2k+1} \rangle$, the stabiliser $\mathcal{S}$ consists precisely of those $(2k+1) \times (2k+1)$-matrices of the form:

$$\begin{pmatrix} \lambda & G_{2k\times 1} \\ 0_{1\times 2k} & F_{2k\times 2k} \end{pmatrix}$$

where $F \in \mathrm{Sp}(2k;\mathbb{R})$ and $\lambda > 0$. Next, note that oriented hyperplanes in $\mathbb{R}^{2k+1}$ containing $\ell_{\mu_0(k)}$ are in 1-1 correspondence with oriented hyperplanes in $\langle e_2, ..., e_{2k+1} \rangle$ and since $\mathrm{SU}(k) \subset \mathrm{Sp}(2k;\mathbb{R})$ acts transitively on oriented hyperplanes in $\langle e_2, ..., e_{2k+1} \rangle$ (see Proposition 7.4.3) it follows that $\mathcal{S}$ acts transitively on the set of oriented hyperplanes in $\mathbb{R}^{2k+1}$ containing $\ell_{\mu_0(k)}$. Similarly, the (unoriented) hyperplanes in $\mathbb{R}^{2k+1}$ transverse to $\ell_{\mu_0(k)}$ are in 1-1 correspondence with linear maps $\langle e_2, ..., e_{2k+1} \rangle \to \ell_{\mu_0(k)}$ and $\mathcal{S}$ acts transitively on this space. Thus the action of $\mathcal{S}$ on oriented hyperplanes in $\mathbb{R}^{2k+1}$ transverse to $\ell_{\mu_0(k)}$ has at most 2 orbits. It follows that the action of $\mathcal{S}$ on $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)$ has exactly three orbits and $\mu_0(k)$ is faithful. To see that $\mu_0(k)$ is also connected, firstly note that the stabiliser of both emproplectic and pisoplectic forms on $\mathbb{R}^{2k}$ is connected, being isomorphic to $\mathrm{Sp}(2k;\mathbb{R})$. For the remaining case, suppose $\ell_{\mu_0(k)} \subset \mathbb{B}$; then the restriction $\mu_0(k)|_{\mathbb{B}}$ may be written in some basis $(f_1, ..., f_{2k})$ as:

$$\mu_0(k)|_{\mathbb{B}} = f^{12} + ... + f^{2k-3,2k-2}.$$

This is an emproplectic form on $\langle f_1, ..., f_{2k-2}\rangle$. Splitting $\mathbb{B} \cong \langle f_1, ..., f_{2k-2}\rangle \oplus \langle f_{2k-1}, f_{2k}\rangle$ and applying Lemma 7.1.5, it follows that the stabiliser of $\mu_0(k)|_{\mathbb{B}}$ in $\mathrm{GL}_+(\mathbb{B})$ is connected. Thus $\mu_0(k)$ is connected.

Finally, since pseudoplectic forms constitute a single $\mathrm{GL}(2k+1;\mathbb{R})$-orbit, it follows from Lemma 7.4.4 that pseudoplectic forms are abundant. Thus the result follows by Theorem 7.4.5.

□

Remark 7.5.5. Crowley–Nordström and Donaldson used a technique known as 'Hirsch's microextension trick' (after its use by Hirsch in [69]) to prove the h-principles for G_2 4-forms and $\mathrm{SL}(3;\mathbb{C})$ 3-forms respectively. E.g. for G_2 4-forms, the argument may be sketched as follows: given any 8-manifold N, define a subset $\mathcal{S}(\mathrm{N}) \subset \bigwedge^4\mathrm{T}^*\mathrm{N}$ by declaring $\alpha \in \bigwedge^4\mathrm{T}^*_p\mathrm{N}$ to lie in $\mathcal{S}(\mathrm{N})$ if and only if the restriction of α to every hyperplane in $\mathrm{T}_p\mathrm{N}$ is a G_2 4-form. Then $\mathcal{S}(\mathrm{N})$ is an open, $\mathrm{Diff}_0(\mathrm{N})$-invariant subbundle of $\bigwedge^4\mathrm{T}^*\mathrm{N}$. Given an oriented 7-manifold M, it can be shown that every G_2 4-form ψ on M extends to a 4-form Ψ on the open (i.e. non-closed) manifold $(-\varepsilon,\varepsilon)\times\mathrm{M}$ such that $\Psi \in \mathcal{S}\big((-\varepsilon,\varepsilon)\times\mathrm{M}\big)$ for some $\varepsilon > 0$ sufficiently small. The h-principle for coclosed G_2-structures then follows from Gromov's h-principle for open, diffeomorphism-invariant relations on open manifolds; cf. [42, Thm. 10.2.1]. This method is limited in scope, however, since for a general stable p-form σ_0 on $\mathbb{R}^n$, there are no p-forms σ on $\mathbb{R}^{n+1}$ such that the restriction $\sigma|_{\mathbb{A}}$ is a σ_0-form for every hyperplane $\mathbb{A} \subset \mathbb{R}^{n+1}$. As a simple example of this phenomenon, suppose that $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0) = \mathrm{Stab}_{\mathrm{GL}(n;\mathbb{R})}(\sigma_0)$, i.e. σ_0 has no orientation-reversing automorphisms. If there were some $\sigma \in \bigwedge^p\left(\mathbb{R}^{n+1}\right)^*$ such that for all $\mathbb{A} \in \mathrm{Gr}_n\left(\mathbb{R}^{n+1}\right)$ the restriction $\sigma|_{\mathbb{A}}$ was a σ_0-form for some choice of orientation on $\mathbb{A}$, then this choice of orientation would be unique (since the stabilisers of σ_0 in $\mathrm{GL}_+(n;\mathbb{R})$ and $\mathrm{GL}(n;\mathbb{R})$ coincide) and would thus define a section of the 'forgetful' degree 2 covering map $\widetilde{\mathrm{Gr}}_n\left(\mathbb{R}^{n+1}\right) \to \mathrm{Gr}_n\left(\mathbb{R}^{n+1}\right)$, yielding a contradiction, as claimed.

By contrast, the techniques introduced in this thesis can be used to prove h-principles for stable forms σ_0 satisfying $\mathrm{Stab}_{\mathrm{GL}_+(n;\mathbb{R})}(\sigma_0) = \mathrm{Stab}_{\mathrm{GL}(n;\mathbb{R})}(\sigma_0)$; indeed, this property is satisfied by both $\widetilde{\mathrm{G}}_2$ 3-forms and co-emproplectic forms in dimension $2k$, where k is odd, both of which are shown to satisfy the h-principle in this thesis.

7.6 Co-emproplectic and co-pseudoplectic forms

The aim of this section is to prove the relative h-principles for co-emproplectic and co-pseudoplectic forms. The proofs proceed via a series of lemmas:

Lemma 7.6.1. *Let $k \geqslant 2$ and let $\omega_+(k) \in \bigwedge^2_+\left(\mathbb{R}^{2k}\right)^*$ be the standard emproplectic form on $\mathbb{R}^{2k}$ defined in eqn. (7.1.10). Identify $\mathbb{R}^{2k} \cong \mathbb{R} \oplus \mathbb{R}^{2k-1}$ in the usual way and fix $\tau \in \bigwedge^2\left(\mathbb{R}^{2k-1}\right)^*$. Then $\mathcal{N}_{\omega_+(k)}(\tau) \neq \varnothing$ if and only if τ is pseudoplectic. Moreover in this case:*

$$\mathcal{N}_{\omega_+(k)}(\tau) = \left\{\nu \in \bigwedge^1\left(\mathbb{R}^{2k-1}\right)^* \;\middle|\; \nu|_{\ell_\tau} > 0\right\}$$

(see Remark 7.1.23 for the definition of ℓ_τ).

Proof. Write θ for the standard annihilator of $\mathbb{R}^{2k-1}$ in $\mathbb{R}^{2k}$ and define $\omega = \theta \wedge \nu + \tau$. Then:

$$\omega^k = \theta \wedge \nu \wedge \tau^{k-1}.$$

Thus ω is emproplectic if and only if $\tau^{k-1} \neq 0$ (i.e. τ is pseudoplectic) and $\nu|_{\ell_\tau} > 0$ (by definition of the choice of orientation on ℓ_τ).

$\square$

Corollary 7.6.2. *Let $k \geqslant 3$, let $\varpi_+(k) \in \bigwedge^{2k-2}_+ \left(\mathbb{R}^{2k}\right)^*$ be the standard co-emproplectic form on $\mathbb{R}^{2k}$ defined in eqn. (7.1.12), identify $\mathbb{R}^{2k} \cong \mathbb{R} \oplus \mathbb{R}^{2k-1}$ and fix $\tau \in \bigwedge^{2k-2}\left(\mathbb{R}^{2k-1}\right)^*$. Then:*

$$\mathcal{N}_{\varpi_+(k)}(\tau) = \left\{\nu \in \bigwedge^{2k-3}\left(\mathbb{R}^{2k-1}\right)^* \;\middle|\; \nu \text{ is co-pseudoplectic and } \tau|_{\Pi_\nu} > 0\right\}$$

(see Remark 7.1.23 for the definition of the hyperplane Π_ν). In particular, if $\tau = 0$ then $\mathcal{N}_{\varpi_+(k)}(\tau) = \varnothing$.

Proof. The proof uses the duality described in Lemma 7.1.1. Write $e = 1 \oplus 0 \in \mathbb{R} \oplus \mathbb{R}^{2k-1}$, choose $\sigma > 0 \in \bigwedge^{2k-1}\mathbb{R}^{2k-1}$ and set $\upsilon = e \wedge \sigma > 0 \in \bigwedge^{2k}\mathbb{R}^{2k}$. Then as in Lemma 7.1.1, $\theta \wedge \nu + \tau$ is co-emproplectic if and only if:

$$(\theta \wedge \nu + \tau) \lrcorner \upsilon = \nu \lrcorner \sigma + e \wedge (\tau \lrcorner \sigma)$$

is emproplectic on $\left(\mathbb{R}^{2k}\right)^*$, which by Lemma 7.6.1 is equivalent to $\nu \lrcorner \sigma$ being pseudoplectic on $\left(\mathbb{R}^{2k-1}\right)^*$ and $(\tau \lrcorner \sigma)|_{\ell_{\nu \lrcorner \sigma}} > 0$. However using the duality as in Lemma 7.1.1 again, $\nu \lrcorner \sigma$ is pseudoplectic on $\left(\mathbb{R}^{2k-1}\right)^*$ if and only if ν is co-pseudoplectic on $\mathbb{R}^{2k-1}$. Moreover, one may verify that:

$$\ell_{\nu \lrcorner \sigma} = \mathrm{Ann}(\Pi_\nu)$$

compatibly with orientations, and thus $(\tau \lrcorner \sigma)|_{\ell_{\nu \lrcorner \sigma}} > 0$ if and only if $\tau|_{\Pi_\nu} > 0$.

$\square$

Lemma 7.6.3. *Let $k \geqslant 2$. Then:*

$$\mathrm{Conv}\left(\bigwedge^{2k-2}_+ \left(\mathbb{R}^{2k}\right)^*\right) = \bigwedge^{2k-2}\left(\mathbb{R}^{2k}\right)^*.$$

(Note that when $k = 2$, $\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^*$ is simply the orbit of emproplectic 2-forms.)

Proof. Since $\bigwedge^{2k-2}_+\left(\mathbb{R}^{2k}\right)^* \subset \bigwedge^{2k-2}\left(\mathbb{R}^{2k}\right)^*$ is open, path-connected and scale-invariant, by Lemma 7.4.7, it suffices to prove that:

$$0 \in \mathrm{Conv}\left(\bigwedge^{2k-2}_+ \left(\mathbb{R}^{2k}\right)^*\right) = \bigwedge^{2k-2}\left(\mathbb{R}^{2k}\right)^*.$$

Write $\left(\theta^1, ..., \theta^{2k}\right)$ for the canonical basis of $\left(\mathbb{R}^{2k}\right)^*$ and recall:

$$\varpi_+(k) = \sum_{i=1}^{k} \theta^{12...\widehat{2i-1,2i}...2k-1,2k}.$$

(Recall also that formally $\varpi_+(2) = \omega_+(2)$.) Choose $r \geqslant 1$ such that $r + \frac{1}{r} = k \geqslant 2$. For each ordered pair of distinct $p, q \in \{1, ..., k\}$, let $F(p,q)$ denote the orientation-preserving automorphism of $\mathbb{R}^{2k}$ given by:

$$F(p,q) = \begin{array}{cc} \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && -r &&&& \\ &&& \ddots &&& \\ &&&& -\frac{1}{r} && \\ &&&&& \ddots & \\ &&&&& \ddots & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow 2p^{\text{th}} \text{ row} \\ \\ \leftarrow 2q^{\text{th}} \text{ row} \\ \\ \\ \end{array} \\ \begin{array}{ccc} \uparrow & & \uparrow \\ 2p^{\text{th}} \text{ col.} & & 2q^{\text{th}} \text{ col.} \end{array} & \end{array}$$

Then for all p, q and r:

$$F(p,q)^* \varpi_+(k) \in \bigwedge\nolimits_+^{2k-2} \left(\mathbb{R}^{2k}\right)^*$$

and thus:

$$2(k-1)\varpi_+(k) + \sum_{p \neq q \in \{1,...,k\}} F(p,q)^* \varpi_+(k) \in \operatorname{Conv}\left(\bigwedge\nolimits_+^{2k-2} \left(\mathbb{R}^{2k}\right)^*\right).$$

(N.B. The coefficients in the linear-combination on the left-hand side of the above expression are all positive and thus, even though they do not sum to 1, the above expression is valid since $\bigwedge_+^{2k-2} \left(\mathbb{R}^{2k}\right)^*$ is scale-invariant.) A direct calculation shows that:

$$2(k-1)\varpi_+(k) + \sum_{p \neq q \in \{1,...,k\}} F(p,q)^* \varpi_+(k) = (k-1)\left(k - r - \frac{1}{r}\right)\varpi_+(k) = 0,$$

as required.

$\square$

Theorem 7.6.4. *Co-symplectic forms satisfy the relative h-principle.*

Proof. Recall that the stabiliser of $\varpi_+(k)$ in $\mathrm{GL}_+(2k;\mathbb{R})$ is isomorphic to $\mathrm{Sp}(2k;\mathbb{R})$. Since $\mathrm{SU}(k) \subset \mathrm{Sp}(2k;\mathbb{R})$, it follows by Proposition 7.4.3 that $\varpi_+(k)$ is faithful and connected. Thus, it suffices to prove that $\varpi_+(k)$ is abundant. When k is even, this follows from Lemma 7.4.4 since $\varpi_+(k)$ admits an orientation-reversing automorphism (see Proposition 7.1.11). However, in general, by Corollary 7.6.2 one must prove directly that 0 lies in the convex hull of the set:

$$\mathcal{N}_{\varpi_+(k)}(\tau) = \left\{\nu \in \bigwedge\nolimits^{2k-3} \left(\mathbb{R}^{2k-1}\right)^* \;\middle|\; \nu \text{ is co-pseudoplectic and } \tau|_{\Pi_\nu} > 0\right\}.$$

Choose a correctly-oriented basis $(e_2, ..., e_{2k})$ of $\mathbb{R}^{2k-1}$ with dual basis $\left(\theta^2, ..., \theta^{2k}\right)$ such that $\tau = \theta^{34...2k}$. Then for all $\varpi \in \bigwedge_+^{2k-4} \left(\mathbb{R}^{2k-2}\right)$, observe that $\nu = \theta^2 \wedge \varpi \in \mathcal{N}_{\varpi_+(k)}(\tau)$ and thus:

$$\theta^2 \wedge \bigwedge\nolimits_+^{2k-4} \left(\mathbb{R}^{2k-2}\right)^* \subseteq \mathcal{N}_{\varpi_+(k)}(\tau).$$

But by Lemma 7.6.3, $0 \in \mathrm{Conv}\left(\Lambda^{2k-4}_{+}\left(\mathbb{R}^{2k-2}\right)^*\right)$, completing the proof.

□

Now fix $k \geqslant 2$ and consider the standard co-pseudoplectic form $\xi_0(k) = \theta^1 \wedge \sum_{i=1}^k \theta^{23...\widehat{2i,2i+1}...2k,2k+1} \in \Lambda^{2k-1}_{Co-P}\left(\mathbb{R}^{2k+1}\right)^*$. Theorem 7.4.5 does not apply to co-pseudoplectic forms, since $\xi_0(k)$ is not faithful. Indeed, recall from Remark 7.1.23 that $\xi_0(k)$ canonically defines an oriented hyperplane $\Pi_{\xi_0(k)} = \langle e_2, ..., e_{2k}\rangle$. Then both $\left\{\Pi_{\xi_0(k)}\right\}$ and $\left\{\overline{\Pi}_{\xi_0(k)}\right\}$ (where the overline denotes orientation-reversal) form singleton orbits for the action of $\mathcal{S}(\xi_0(k))$ on $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)$, however:

$$\xi_0(k)|_{\Pi_{\xi_0(k)}} = \xi_0(k)|_{\overline{\Pi}_{\xi_0(k)}} = 0.$$

Despite this observation, Theorem 7.3.1 does apply to co-pseudoplectic forms:

Theorem 7.6.5. $\xi_0(k)$ *is ample; hence co-pseudoplectic forms satisfy the relative h-principle.*

Proof. Write $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen} = \widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right) \setminus \left\{\Pi_{\xi_0(k)}, \overline{\Pi}_{\xi_0(k)}\right\}$. Then $\mathcal{S}(\xi_0(k))$ acts transitively on $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen}$: indeed, let $\Pi \in \widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen}$. Π intersects $\Pi_{\xi_0(k)}$ transversely and thus $\Pi \cap \Pi_{\xi_0(k)}$ has dimension $2k-1$. Moreover $\Pi \cap \Pi_{\xi_0(k)}$ can be canonically oriented as follows: choose any $u \in \Pi$ such that $\theta^1(u) > 0$. Then the chosen orientation on Π together with the decomposition:

$$\Pi = \langle u \rangle \oplus \left(\Pi \cap \Pi_{\xi_0(k)}\right)$$

orients $\Pi \cap \Pi_{\xi_0(k)}$ and thus $\Pi \cap \Pi_{\xi_0(k)}$ defines an element of $\widetilde{\mathrm{Gr}}_{2k-1}\left(\Pi_{\xi_0(k)}\right)$. Since $\mathrm{Sp}(2k;\mathbb{R})$ acts transitively on $\widetilde{\mathrm{Gr}}_{2k-1}\left(\mathbb{R}^{2k}\right)$, by eqn. (7.1.20) it follows that $\mathcal{S}(\xi_0(k))$ acts transitively on $\widetilde{\mathrm{Gr}}_{2k-1}\left(\Pi_{\xi_0(k)}\right)$ and thus without loss of generality one may assume that:

$$\Pi \cap \Pi_{\xi_0(k)} = \langle e_3, ..., e_{2k+1}\rangle,$$

compatibly with its orientation. Thus:

$$\Pi = \langle e_1 + te_2, e_3, ..., e_{2k+1}\rangle$$

for some $t \in \mathbb{R}$. Now consider the automorphism of $\mathbb{R}^{2k+1}$ given by:

$$F = \left(\begin{array}{c|ccc} 1 & & & \\ \hline -t & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right)$$

By examining eqn. (7.1.20), $F \in \mathcal{S}(\xi_0(k))$ and clearly $F(\Pi) = \langle e_1, e_3, ..., e_{2k+1}\rangle$; thus $\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen}$ forms a single orbit, as claimed. Moreover, $\mathcal{T}_{\xi_0(k)}\left(\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen}\right)$ is precisely the orbit of non-zero $(2k-1)$-forms on $\mathbb{R}^{2k}$.

Now let $\tau \in \mathcal{T}_{\xi_0(k)}\left(\widetilde{\mathrm{Gr}}_{2k}\left(\mathbb{R}^{2k+1}\right)_{gen}\right)$. Clearly the stabiliser of τ in $\mathrm{GL}_+(2k;\mathbb{R})$ is connected. Also, since:

$$\mathcal{T}^{-1}_{\xi_0(k)}\left(\mathcal{T}_{\xi_0(k)}\left[\widetilde{\mathrm{Gr}}_{2k}(\mathbb{R}^{2k+1})_{gen}\right]\right) = \widetilde{\mathrm{Gr}}_{2k}(\mathbb{R}^{2k+1})_{gen}$$

the set $\mathcal{N}_{\xi_0(k)}(\tau)$ is path-connected for any $\tau \neq 0$ by Lemma 7.4.9. Moreover, since the $\mathrm{GL}_+(2k+1;\mathbb{R})$-orbit of co-pseudoplectic forms is closed under the action of $\mathrm{GL}(2k+1;\mathbb{R})$, by Lemma 7.4.4 it follows that $\xi_0(k)$ is abundant. Thus by Lemma 7.4.7, $\mathcal{N}_{\xi_0(k)}(\tau)$ is ample for all $\tau \neq 0$.

Now consider $\tau = 0$. Note that $\theta^1 \wedge \nu$ is co-pseudoplectic if and only if ν is co-symplectic and thus:

$$\mathcal{N}_{\xi_0(k)}(0) = \bigwedge\nolimits^{2k-2}_{+}\left(\mathbb{R}^{2k}\right)^* \cup \bigwedge\nolimits^{2k-2}_{-}\left(\mathbb{R}^{2k}\right)^*.$$

This space has two path components, and thus abundance alone is not sufficient to deduce that $\mathcal{N}_{\xi_0(k)}(0)$ is ample. However by Lemma 7.6.3, the convex hull of each path component of $\mathcal{N}_{\xi_0(k)}(0)$ equals $\bigwedge^{2k-2}\left(\mathbb{R}^{2k}\right)^*$ and thus $\xi_0(k)$ is ample, as claimed.

□

7.7 $\widetilde{\mathrm{G}}_2$ 3- and 4-forms

The aim of this section is to prove the relative h-principles for $\widetilde{\mathrm{G}}_2$ 3- and 4-forms. Let $\mathbb{R}^7$ have basis $(e_1, ..., e_7)$ and dual basis $\left(\theta^1, ..., \theta^7\right)$ as usual. Recall the standard $\widetilde{\mathrm{G}}_2$ 3- and 4-forms defined in eqns. (2.2.5) and (2.2.3) respectively by:

$$\widetilde{\varphi}_0 = \theta^{123} - \theta^{145} - \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}$$
$$\widetilde{\psi}_0 = \theta^{4567} - \theta^{2367} - \theta^{2345} + \theta^{1357} - \theta^{1346} - \theta^{1256} - \theta^{1247}$$

inducing the metric $\widetilde{g}_0 = \sum_{i=1}^{3}(\theta^i)^{\otimes 2} - \sum_{i=4}^{7}(\theta^i)^{\otimes 2}$ and volume form $\theta^{12...7}$. For the purposes of calculations, it is advantageous to have a second 'standard representative' of $\widetilde{\mathrm{G}}_2$ 3- and 4-forms:

Proposition 7.7.1. *The 3-form:*

$$\widetilde{\varphi}_1 = \frac{1}{2}\left(\theta^{147} + \theta^{156} - \theta^{237} + \theta^{246} - \theta^{345}\right) \tag{7.7.2}$$

is of $\widetilde{\mathrm{G}}_2$-type. It induces the metric and volume-form:

$$\widetilde{g}_1 = -\theta^1 \odot \theta^7 + \theta^2 \odot \theta^6 - \theta^3 \odot \theta^5 - \theta^4 \odot \theta^4 \quad \text{and} \quad vol_1 = \frac{1}{8}\theta^{1234567}.$$

Moreover:

$$\widetilde{\psi}_1 = \Theta(\widetilde{\varphi}_1) = \frac{1}{4}\left(\theta^{2356} + 2\theta^{2347} - 2\theta^{1456} + \theta^{1357} - \theta^{1267}\right).$$

(To prove this result, one simply calculates the bilinear form $Q_{\widetilde{\varphi}_1} = \frac{1}{6}\left(- \lrcorner\, \widetilde{\varphi}_1\right)^2 \wedge \widetilde{\varphi}_1$ explicitly, from which the metric and volume form can simply be written down.)

Now consider the space $\widetilde{\mathrm{G}}_2 \backslash \widetilde{\mathrm{Gr}}_6(\mathbb{R}^7)$. Since $\widetilde{g}_0$ is non-degenerate, taking orthocomplement with respect to $\widetilde{g}_0$ establishes a $\widetilde{\mathrm{G}}_2$-equivariant isomorphism:

$$\begin{aligned} \mathrm{Gr}_1(\mathbb{R}^7) &\cong \mathrm{Gr}_6(\mathbb{R}^7) \\ \mathbb{L} &\mapsto \mathbb{L}^\perp \\ \mathbb{B}^\perp &\leftarrow\!\shortmid \mathbb{B}. \end{aligned} \tag{7.7.3}$$

Motivated by this, I term a hyperplane $\mathbb{B} \subset \mathbb{R}^7$ spacelike, timelike or null according to whether its orthocomplement is spacelike, timelike or null and write $\mathrm{Gr}_{6,+}(\mathbb{R}^7)$, $\mathrm{Gr}_{6,-}(\mathbb{R}^7)$ and $\mathrm{Gr}_{6,0}(\mathbb{R}^7)$ for the corresponding Grassmannians. (Recall that a 1-dimensional subspace ℓ is spacelike, timelike or null according to whether $\widetilde{g}_0(u,u) > 0$, < 0 or $= 0$ respectively for some (equivalently all) $u \in \ell\backslash\{0\}$.)

Lemma 7.7.4.

$$\widetilde{\mathrm{G}}_2 \Big\backslash \widetilde{\mathrm{Gr}}_6(\mathbb{R}^7) = \left\{ \widetilde{\mathrm{Gr}}_{6,i}(\mathbb{R}^7) \;\middle|\; i = +,-,0 \right\}.$$

Proof. If $\mathbb{B}$ is either spacelike or timelike, then $\mathbb{R}^7 = \mathbb{B}^\perp \oplus \mathbb{B}$ and so eqn. (7.7.3) can be lifted to an isomorphism $\widetilde{\mathrm{Gr}}_{1,\pm}(\mathbb{R}^7) \cong \widetilde{\mathrm{Gr}}_{6,\pm}(\mathbb{R}^7)$. Thus since $\widetilde{\mathrm{G}}_2$ acts transitively on each of $\widetilde{\mathrm{Gr}}_{1,\pm}(\mathbb{R}^7)$ [84, Prop. 2.2], each of $\widetilde{\mathrm{Gr}}_{6,\pm}(\mathbb{R}^7)$ are orbits of $\widetilde{\mathrm{G}}_2$.

For null planes, $\mathbb{B}^\perp \subset \mathbb{B}$ and so a different approach is required. Since $\widetilde{\mathrm{G}}_2$ acts transitively on $\mathrm{Gr}_{1,0}(\mathbb{R}^7)$ [68, Prop. 5.4], $\widetilde{\mathrm{G}}_2$ also acts transitively on $\mathrm{Gr}_{6,0}(\mathbb{R}^7)$ by eqn. (7.7.3) and thus the action of $\widetilde{\mathrm{G}}_2$ on $\widetilde{\mathrm{Gr}}_{6,0}(\mathbb{R}^7)$ has at most two orbits. Now recall the $\widetilde{\mathrm{G}}_2$ 4-form $\widetilde{\psi}_0$ and consider the oriented null 6-plane $\mathbb{B} = \langle e_1, e_2, e_4, e_5, e_6, e_3 + e_7 \rangle$. Consider $F \in \widetilde{\mathrm{G}}_2$ given by:

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \mapsto (e_1, -e_2, -e_3, e_4, e_5, -e_6, -e_7).$$

Then F preserves $\mathbb{B}$ and $F|_{\mathbb{B}}$ is orientation reversing. Thus $\widetilde{\mathrm{Gr}}_{6,0}(\mathbb{R}^7)$ forms a single orbit of $\widetilde{\mathrm{G}}_2$ as claimed.

□

7.7.1 h-principle for $\widetilde{\mathrm{G}}_2$ 4-forms

Theorem 7.7.5. *$\widetilde{\mathrm{G}}_2$ 4-forms satisfy the relative h-principle.*

Proof. By Lemma 7.4.4, $\widetilde{\mathrm{G}}_2$ 4-forms are automatically abundant. Thus it suffices to prove that $\widetilde{\mathrm{G}}_2$ 4-forms are faithful and connected. Initially, consider the $\widetilde{\mathrm{G}}_2$ 4-form $\widetilde{\psi}_0$ and the hyperplanes:

$$\begin{aligned} \mathbb{B}_+ &= \langle e_2, e_3, e_4, e_5, e_6, e_7 \rangle \in \widetilde{\mathrm{Gr}}_{6,+}(\mathbb{R}^7) \\ \mathbb{B}_- &= \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \in \widetilde{\mathrm{Gr}}_{6,-}(\mathbb{R}^7) \end{aligned}$$

Then:

$$\widetilde{\psi}_0|_{\mathbb{B}_+} = \theta^{4567} - \theta^{2367} - \theta^{2345} = \frac{1}{2}\left(\theta^{23} - \theta^{45} - \theta^{67}\right)^2$$

is a co-emproplectic form, with connected stabiliser in $\mathrm{GL}_+(\mathbb{B}_+)$ isomorphic to $\mathrm{Sp}(6;\mathbb{R})$, while:

$$\widetilde{\psi}_0|_{\mathbb{B}_-} = -\theta^{2345} - \theta^{1346} - \theta^{1256} = -\frac{1}{2}\left(\theta^{16} + \theta^{25} + \theta^{34}\right)^2$$

is co-pisoplectic (also with connected stabiliser in $\mathrm{GL}_+(\mathbb{B}_-)$).

Now turn to the null case. Consider the $\widetilde{\mathrm{G}}_2$ 4-form $2\widetilde{\psi}_1$ and the null-hyperplane $\mathbb{B}_0 = \langle e_1, ..., e_6 \rangle$. Then:

$$2\widetilde{\psi}_1|_{\mathbb{B}_0} = \frac{1}{2}\theta^{2356} - \theta^{1456} = \left(\frac{1}{2}\theta^{23} - \theta^{14}\right) \wedge \theta^{56}$$

is a degenerate 4-form (i.e. neither co-emproplectic nor co-pisoplectic) and hence $\widetilde{\mathrm{G}}_2$ 4-forms are faithful. To verify that the stabiliser of $2\widetilde{\psi}_1|_{\mathbb{B}_0}$ in $\mathrm{GL}_+(\mathbb{B}_0)$ is connected, split $\mathbb{B}_0 = \langle e_5, e_6 \rangle \oplus \langle e_1, e_2, e_3, e_4 \rangle \cong \mathbb{R}^2 \oplus \mathbb{R}^4$ and apply Lemma 7.1.5 to the pisoplectic (and hence multi-co-symplectic) 2-form:

$$\alpha = \frac{1}{2}\theta^{23} - \theta^{14} \in \bigwedge^2 \left(\mathbb{R}^4\right)^*,$$

noting that $-\alpha$ and α lie in the same $\mathrm{GL}_+(4;\mathbb{R})$-orbit (since $(-\alpha)^2 = \alpha^2 = -\frac{1}{4}\theta^{1234}$) and that the stabilisers of α in $\mathrm{GL}_+(2k;\mathbb{R})$ and $\mathrm{GL}(2k;\mathbb{R})$ coincide and are connected. Thus $\widetilde{\mathrm{G}}_2$ 4-forms are also connected.

□

7.7.2 Faithfulness of $\widetilde{\mathrm{G}}_2$ 3-forms

Proposition 7.7.6. $\widetilde{\varphi}_0$ *is faithful. More specifically, the orbits* $\mathcal{T}_{\widetilde{\varphi}_0}\left(\widetilde{\mathrm{Gr}}_{6,\mp}(\mathbb{R}^7)\right) \in \bigwedge^3\left(\mathbb{R}^6\right) \big/ \mathrm{GL}_+(6;\mathbb{R})$ *are precisely the orbits* $\bigwedge^3_\pm\left(\mathbb{R}^6\right)$ *of* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms and* $\mathrm{SL}(3;\mathbb{C})$ *3-forms respectively, while the orbit* $\mathcal{T}_{\widetilde{\varphi}_0}\left(\widetilde{\mathrm{Gr}}_{6,0}(\mathbb{R}^7)\right) \in \bigwedge^3\left(\mathbb{R}^6\right) \big/ \mathrm{GL}_+(6;\mathbb{R})$ *is not open, i.e. forms in this orbit are not stable.*

Proof. I consider each orbit in turn. For the timelike case, it suffices to prove that for some $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\varphi}$ on $\mathbb{R}^7$ and some oriented timelike subspace $\mathbb{B} \subset \mathbb{R}^7$, the restriction $\widetilde{\varphi}|_{\mathbb{B}}$ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form. Consider $2\widetilde{\varphi}_1$ (see eqn. (7.7.2)) and let $\mathbb{B} \subset \mathbb{R}^7$ be the oriented timelike hyperplane $\langle e_1, e_5, e_6, -e_2, e_3, e_7 \rangle$. Then $2\widetilde{\varphi}_1|_{\mathbb{B}} = \theta^{156} - \theta^{237}$ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form on $\mathbb{B}$. For the spacelike case, consider $\widetilde{\varphi}_0$ and let $\mathbb{B} \subset \mathbb{R}^7$ be the oriented spacelike hyperplane $\langle e_2, e_3, e_4, e_5, e_6, e_7 \rangle$. Then $\widetilde{\varphi}_0|_{\mathbb{B}} = \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}$ is an $\mathrm{SL}(3;\mathbb{C})$ 3-form on $\mathbb{B}$.

Finally, for the null case, consider $2\widetilde{\varphi}_1$ and let $\mathbb{B} \subset \mathbb{R}^7$ be the oriented null hyperplane $\langle e_2, e_3, e_4, e_5, e_6, e_7 \rangle$. Define:

$$\rho_0 = 2\widetilde{\varphi}_1|_{\mathbb{B}} = -\theta^{237} + \theta^{246} - \theta^{345}. \tag{7.7.7}$$

The 'Hitchin map' $K_{\rho_0} : \mathbb{B} \to \mathbb{B} \otimes \bigwedge^6\left(\mathbb{R}^6\right)^*$ defined in §2.3.1 is given by:

$$K_{\rho_0}(e_i) = \begin{cases} e_5 \otimes \theta^{234567} & \text{if } i = 2; \\ e_6 \otimes \theta^{234567} & \text{if } i = 3; \\ e_7 \otimes \theta^{234567} & \text{if } i = 4; \\ 0 & \text{otherwise.} \end{cases} \tag{7.7.8}$$

In particular, $K_{\rho_0}^2 = 0$, and so by the results of §2.3.1, ρ_0 is not stable.

□

The space $\mathcal{T}_{\widetilde{\varphi}_0}\left(\widetilde{\mathrm{Gr}}_{6,0}(\mathbb{R}^7)\right)$ shall be termed the orbit of parabolic 3-forms, and denoted $\bigwedge^3_0\left(\mathbb{R}^6\right)^*$. (The motivation for this name derives from the fact that the stabiliser in $\widetilde{\mathrm{G}}_2$ of a non-zero null vector is a maximal parabolic subgroup of $\widetilde{\mathrm{G}}_2$: see [68, §5].) In Djoković's classification of 3-forms in dimensions $n \leqslant 8$, parabolic 3-forms correspond to the real form of the complex orbit 'IV'; see [34, §9].

I remark that Proposition 7.7.6 also shows, for all $\rho \in \mathcal{T}_{\widetilde{\varphi}_0}\left(\widetilde{\mathrm{Gr}}_{6,\pm}\left(\mathbb{R}^7\right)\right)$, that the stabiliser of ρ in $\mathrm{GL}_+(6;\mathbb{R})$ is connected, being isomorphic to $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ respectively. By Theorem 7.4.5, proving the relative h-principle for $\widetilde{\mathrm{G}}_2$ 3-forms, is thus reduced to the following three lemmas:

Lemma 7.7.9. *For each (equivalently any)* $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$*:* $0 \in \mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho)\right)$.

Lemma 7.7.10. *For each (equivalently any)* $\rho \in \bigwedge^3_-\left(\mathbb{R}^6\right)^*$*:* $0 \in \mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho)\right)$.

Lemma 7.7.11. *For each (equivalently any)* $\rho \in \bigwedge^3_0\left(\mathbb{R}^6\right)^*$, $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho)$ *is connected and* $0 \in \mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho)\right)$.

The rest of this chapter is devoted to proving each lemma in turn.

7.7.3 Timelike case: Lemma 7.7.9

Let $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$ be an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form. The decomposition $\mathbb{R}^6 = E_+ \oplus E_-$ gives rise to a decomposition $\left(\mathbb{R}^6\right)^* \cong E_+^* \oplus E_-^*$ and hence:

$$\bigwedge^p\left(\mathbb{R}^6\right)^* \cong \bigoplus_{r+s=p} \bigwedge^r E_+^* \otimes \bigwedge^s E_-^* = \bigoplus_{r+s=p} \bigwedge^{r,s}\left(\mathbb{R}^6\right)^*. \tag{7.7.12}$$

ρ defines an element of $\bigwedge^{3,0}\left(\mathbb{R}^6\right)^* \oplus \bigwedge^{0,3}\left(\mathbb{R}^6\right)^*$; thus the map:

$$\begin{aligned} \mathbb{R}^6 &\longrightarrow \bigwedge^{2,0}\left(\mathbb{R}^6\right)^* \oplus \bigwedge^{0,2}\left(\mathbb{R}^6\right)^* \\ u &\longmapsto u \lrcorner \rho \end{aligned} \tag{7.7.13}$$

is an $\mathrm{SL}(3;\mathbb{R})^2$-equivariant isomorphism. Moreover, I_ρ defines a map:

$$\begin{aligned} \mathscr{I}_\rho : \bigwedge^2\left(\mathbb{R}^6\right)^* &\longrightarrow \bigodot^2\left(\mathbb{R}^6\right)^* \\ \omega &\longmapsto \left\{(a,b) \mapsto \tfrac{1}{2}\left[\omega(I_\rho a, b) + \omega(I_\rho b, a)\right]\right\} \end{aligned} \tag{7.7.14}$$

(where $\bigodot^2$ denotes the symmetric square) which vanishes on the subspace $\bigwedge^{2,0}\left(\mathbb{R}^6\right)^* \oplus \bigwedge^{0,2}\left(\mathbb{R}^6\right)^*$ and satisfies $\mathscr{I}_\rho\omega(u_1,u_2) = \omega(I_\rho u_1, u_2)$ for $\omega \in \bigwedge^{1,1}\left(\mathbb{R}^6\right)^*$. In particular, $\mathscr{I}_\rho$ defines an injection $\bigwedge^{1,1}\left(\mathbb{R}^6\right)^* \hookrightarrow \bigodot^2\left(\mathbb{R}^6\right)^*$.

Proposition 7.7.15. *Let* ρ *be an* $\mathrm{SL}(3;\mathbb{R})^2$ *3-form on* $\mathbb{R}^6$*. Then:*

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) = \left\{\omega \in \bigwedge^2\left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{I}_\rho\omega \textit{ has signature } (3,3) \textit{ and } \omega^3 < 0\right\}. \tag{7.7.16}$$

Proof. Firstly, I claim:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) = \left[\mathcal{N}_{\widetilde{\varphi}_0}(\rho) \cap \bigwedge^{1,1}\left(\mathbb{R}^6\right)^*\right] \oplus \bigwedge^{2,0}\left(\mathbb{R}^6\right)^* \oplus \bigwedge^{0,2}\left(\mathbb{R}^6\right)^*. \tag{7.7.17}$$

Indeed, let $\omega \in \wedge^2\left(\mathbb{R}^6\right)^*$ and define a 3-form on $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{R}^6$ via:

$$\widetilde{\phi} = \theta \wedge \omega + \rho. \tag{7.7.18}$$

Let $u \in \mathbb{R}^6$ and consider the orientation-preserving automorphism of $\mathbb{R}^7$ given by:

$$F = \begin{pmatrix} 1_{1\times1} & 0_{1\times6} \\ u_{6\times1} & \mathrm{Id}_{6\times6} \end{pmatrix}$$

Then $F^*\widetilde{\phi} = \theta \wedge (\omega + u \lrcorner \rho) + \rho$. Thus $\omega \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho)$ if and only if $\omega + u \lrcorner \rho \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho)$ for all $u \in \mathbb{R}^6$ and eqn. (7.7.17) follows by eqn. (7.7.13). Moreover, given $\omega \in \wedge^2\left(\mathbb{R}^6\right)^*$, $\mathscr{I}_\rho\omega$ and ω^3 only depend on the $(1,1)$-part of ω. Thus, to prove Proposition 7.7.15, it suffices to prove:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) \cap \bigwedge^{1,1}\left(\mathbb{R}^6\right)^* = \left\{\omega \in \bigwedge^{1,1}\left(\mathbb{R}^6\right)^* \,\middle|\, \mathscr{I}_\rho\omega \text{ has signature } (3,3) \text{ and } \omega^3 < 0\right\}. \tag{7.7.19}$$

Recall the invariant quadratic form $Q_{\widetilde{\phi}}$ defined in Proposition 2.3.6. Using eqn. (7.7.18), for $a \in \mathbb{R}$ and $u \in \mathbb{R}^6$ one may compute:

$$(ae_1 + u) \lrcorner \widetilde{\phi} = a\omega - \theta \wedge (u \lrcorner \omega) + u \lrcorner \rho$$

and hence:

$$\begin{aligned} 6Q_{\widetilde{\phi}}(ae_1 + u) &= \left[(ae_1 + u) \lrcorner \widetilde{\phi}\right]^2 \wedge \widetilde{\phi} \\ &= a^2\theta \wedge \omega^3 + \underbrace{\theta \wedge \omega \wedge (u \lrcorner \rho)^2}_{(1)} - \underbrace{2a\theta \wedge (u \lrcorner \omega) \wedge \omega \wedge \rho}_{(2)} + \underbrace{2a\theta \wedge \omega^2 \wedge (u \lrcorner \rho)}_{(3)} \\ &\quad - \underbrace{2\theta \wedge (u \lrcorner \omega) \wedge (u \lrcorner \rho) \wedge \rho}_{(4)}. \end{aligned} \tag{7.7.20}$$

Term (2) vanishes since $\omega \in \wedge^{1,1}\left(\mathbb{R}^6\right)^*$ and $\rho \in \wedge^{3,0}\left(\mathbb{R}^6\right)^* \oplus \wedge^{0,3}\left(\mathbb{R}^6\right)^*$ and hence $\omega \wedge \rho = 0$. To simplify the remaining terms, I utilise the following lemma:

Lemma 7.7.21. *For $\alpha \in \wedge^p(\mathbb{R}^n)^*$, $\beta \in \wedge^q(\mathbb{R}^n)^*$ with $p + q = n + 1$:*

$$\forall u \in \mathbb{R}^n : (u \lrcorner \alpha) \wedge \beta = (-1)^{p-1}\alpha \wedge (u \lrcorner \beta).$$

(To prove this lemma, by linearity, it suffices to consider $u = e_1$, $\alpha = \theta^{i_1\dots i_p}$, $\beta = \theta^{j_1\dots j_q}$ with $1 \leqslant i_1 < \dots < i_p \leqslant n$ and $1 \leqslant j_1 < \dots < j_q \leqslant n$. The result then follows by direct calculation.)

Returning to eqn. (7.7.20), firstly consider term (3). Applying Lemma 7.7.21 on $\mathbb{R}^6$ yields $\omega^2 \wedge (u \lrcorner \rho) = -(u \lrcorner \omega^2) \wedge \rho = -2(u \lrcorner \omega) \wedge \omega \wedge \rho$, which vanishes as above. Similarly, for term (1), since

$\omega \wedge (u \lrcorner \rho)^2 = \omega \wedge [u \lrcorner ((u \lrcorner \rho) \wedge \rho)]$, Lemma 7.7.21 yields:

$$\omega \wedge (u \lrcorner \rho)^2 = -(u \lrcorner \omega) \wedge (u \lrcorner \rho) \wedge \rho$$

and hence terms (1) and (4) may be combined to give $-3\theta \wedge (u \lrcorner \omega) \wedge (u \lrcorner \rho) \wedge \rho$. However $(u \lrcorner \rho) \wedge \rho = I_\rho(u) \lrcorner vol_\rho$, by definition of I_ρ, and thus by Lemma 7.7.21 again:

$$-(u \lrcorner \omega) \wedge (u \lrcorner \rho) \wedge \rho = \mathscr{I}_\rho \omega(u,u) \cdot vol_\rho.$$

Hence terms (1) and (4) may collectively be written as $3\mathscr{I}_\rho \omega(u,u) \cdot \theta \wedge vol_\rho$ and whence:

$$6Q_{\widetilde{\phi}}(ae_1 + u) = a^2 \theta \wedge \omega^3 + 3\mathscr{I}_\rho \omega(u,u) \theta \wedge vol_\rho. \tag{7.7.22}$$

In particular, $\mathbb{L} = \mathbb{R} \oplus 0$ and $\mathbb{B} = 0 \oplus \mathbb{R}^6$ are orthogonal with respect to $Q_{\widetilde{\phi}}$.

Recall that $\widetilde{\phi}$ is a $\widetilde{\mathrm{G}}_2$ 3-form if and only if $Q_{\widetilde{\phi}}$ has signature $(3,4)$. Moreover, since $\widetilde{\phi}|_{\mathbb{B}} = \rho$ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form, by Proposition 7.7.6 whenever $\widetilde{\phi}$ is a $\widetilde{\mathrm{G}}_2$ 3-form, the hyperplane $\mathbb{B} \subset \mathbb{R}^7$ is timelike. Since $\mathbb{L}$ and $\mathbb{B}$ are orthogonal, it follows that $\widetilde{\phi}$ is a $\widetilde{\mathrm{G}}_2$ 3-form if and only if $Q_{\widetilde{\phi}}$ has signature $(3,3)$ on $\mathbb{B}$ and signature $(0,1)$ on $\mathbb{L}$. But by eqn. (7.7.22), this is precisely the statement that $\mathscr{I}_\rho \omega$ has signature $(3,3)$ and $\omega^3 < 0$, as required.

□

I now prove Lemma 7.7.9:

Proof of Lemma 7.7.9. Without loss of generality take $\rho = \rho_+$ (see eqn. (2.3.5)) and consider the 2-forms:

$$\omega_1 = 2\theta^{14} - \theta^{25} - \theta^{36}, \quad \omega_2 = -\theta^{14} + 2\theta^{25} - \theta^{36} \quad \text{and} \quad \omega_3 = -\theta^{14} - \theta^{25} + 2\theta^{36}.$$

Then:

$$\mathscr{I}_{\rho_+}\omega_1 = 4\theta^1 \odot \theta^4 - 2\theta^2 \odot \theta^5 - 2\theta^3 \odot \theta^6, \quad \mathscr{I}_{\rho_+}\omega_2 = -2\theta^1 \odot \theta^4 + 4\theta^2 \odot \theta^5 - 2\theta^3 \odot \theta^6$$
$$\text{and} \quad \mathscr{I}_{\rho_+}\omega_3 = -2\theta^1 \odot \theta^4 - 2\theta^2 \odot \theta^5 + 4\theta^3 \odot \theta^6$$

which all have signature $(3,3)$. Moreover $\omega_i^3 = -12\theta^{12\dots6}$ for $i = 1,2,3$. Thus by Proposition 7.7.15 $\omega_i \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho_+)$ for all $i = 1,2,3$. Therefore:

$$\mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho_+)\right) \ni \frac{1}{3}(\omega_1 + \omega_2 + \omega_3) = 0,$$

as required.

□

7.7.4 Spacelike case: Lemma 7.7.10

The spacelike case is closely analogous to the timelike case; accordingly, the exposition in this subsection will be brief. Let $\rho \in \Lambda^3_-\left(\mathbb{R}^6\right)^*$ be an $\mathrm{SL}(3;\mathbb{C})$ 3-form. The complex structure J_ρ induces a type-decomposition $\wedge^p \left(\mathbb{R}^6\right)^* \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{r+s=p} \wedge^{r,s}\left(\mathbb{R}^6\right)^*$. As in [114, p. 32], for $r \neq s$, write $\left[\!\left[\wedge^{r,s}\left(\mathbb{R}^6\right)^*\right]\!\right] = \left(\wedge^{r,s}\left(\mathbb{R}^6\right)^* \oplus \wedge^{s,r}\left(\mathbb{R}^6\right)^*\right) \cap \wedge^{r+s}\left(\mathbb{R}^6\right)^*$ for the set of real forms of type $(r,s)+(s,r)$; likewise, write $\left[\wedge^{r,r}\left(\mathbb{R}^6\right)^*\right] = \wedge^{r,r}\left(\mathbb{R}^6\right)^* \cap \wedge^{2r}\left(\mathbb{R}^6\right)^*$ for the set of real forms of type (r,r). Then for all p:

$$\bigwedge^{2p}\left(\mathbb{R}^6\right)^* = \left(\bigoplus_{\substack{r+s=2p\\ r<s}} \left[\!\left[\bigwedge^{r,s}\left(\mathbb{R}^6\right)^*\right]\!\right]\right) \oplus \left[\bigwedge^{p,p}\left(\mathbb{R}^6\right)^*\right] \quad \text{and} \quad \bigwedge^{2p+1}\left(\mathbb{R}^6\right)^* = \bigoplus_{\substack{r+s=2p\\ r<s}} \left[\!\left[\bigwedge^{r,s}\left(\mathbb{R}^6\right)^*\right]\!\right].$$

As in the timelike case, ρ defines an element of $\left[\!\left[\wedge^{3,0}\left(\mathbb{R}^6\right)^*\right]\!\right]$ and $u \in \mathbb{R}^6 \mapsto u \lrcorner \rho \in \left[\!\left[\wedge^{2,0}\left(\mathbb{R}^6\right)^*\right]\!\right]$ defines an $\mathrm{SL}(3;\mathbb{C})$-equivariant isomorphism. Moreover, J_ρ defines a map:

$$\begin{aligned} \mathscr{J}_\rho : \wedge^2\left(\mathbb{R}^6\right)^* &\longrightarrow \odot^2\left(\mathbb{R}^6\right)^* \\ \omega &\longmapsto \left\{(a,b) \mapsto -\tfrac{1}{2}\left[\omega(J_\rho a, b) + \omega(J_\rho b, a)\right]\right\} \end{aligned} \tag{7.7.23}$$

(note the difference in sign convention from the timelike case) which vanishes on the subspace $\left[\!\left[\wedge^{2,0}\left(\mathbb{R}^6\right)^*\right]\!\right]$ and satisfies $\mathscr{J}_\rho\omega(u_1,u_2) = -\omega(J_\rho u_1, u_2)$ for $\omega \in \left[\wedge^{1,1}\left(\mathbb{R}^6\right)^*\right]$. In particular, $\mathscr{J}_\rho$ defines an injection $\left[\wedge^{1,1}\left(\mathbb{R}^6\right)^*\right] \hookrightarrow \odot^2\left(\mathbb{R}^6\right)^*$.

Proposition 7.7.24. *Let ρ be an $\mathrm{SL}(3;\mathbb{C})$ 3-form on $\mathbb{R}^6$. Then:*

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) = \left\{\omega \in \bigwedge^2\left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{J}_\rho\omega \textit{ has signature } (2,4)\right\}. \tag{7.7.25}$$

Proof. As in the proof of Proposition 7.7.15, it suffices to prove that:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) \cap \left[\bigwedge^{1,1}\left(\mathbb{R}^6\right)^*\right] = \left\{\omega \in \left[\bigwedge^{1,1}\left(\mathbb{R}^6\right)^*\right] \;\middle|\; \mathscr{J}_\rho\omega \text{ has signature } (2,4)\right\}. \tag{7.7.26}$$

Given $\omega \in \left[\wedge^{1,1}\left(\mathbb{R}^6\right)^*\right]$, writing $\widetilde{\phi} = \theta \wedge \omega + \rho \in \wedge^3\left(\mathbb{R} \oplus \mathbb{R}^6\right)^*$, the calculations from the proof of Proposition 7.7.15 yield:

$$6Q_{\widetilde{\phi}}(ae_1 + u) = a^2\theta \wedge \omega^3 + 6\mathscr{J}_\rho\omega(u,u)\theta \wedge vol_\rho, \tag{7.7.27}$$

where the final term has a factor of 6 now (rather than a factor of 3) since:

$$(u \lrcorner \rho) \wedge \rho = -2J_\rho(u) \lrcorner vol_\rho \quad \text{and} \quad \mathscr{J}_\rho\omega(u_1,u_2) = -\frac{1}{2}\left[\omega(J_\rho u_1, u_2) + \omega(J_\rho u_2, u_1)\right]$$

when ρ is an $\mathrm{SL}(3;\mathbb{C})$ 3-form, as opposed to:

$$(u \lrcorner \rho) \wedge \rho = I_\rho(u) \lrcorner vol_\rho \quad \text{and} \quad \mathscr{I}_\rho\omega(u_1,u_2) = +\frac{1}{2}\left[\omega(I_\rho u_1, u_2) + \omega(I_\rho u_2, u_1)\right]$$

when ρ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form. In particular, $\mathbb{L} = \mathbb{R} \oplus 0$ and $\mathbb{B} = 0 \oplus \mathbb{R}^6$ are again orthogonal with respect to $Q_{\widetilde{\phi}}$.

Since $\widetilde{\phi}|_{\mathbb{B}} = \rho$ is an $\mathrm{SL}(3;\mathbb{C})$ 3-form, by Proposition 7.7.6 whenever $\widetilde{\phi}$ is a $\widetilde{\mathrm{G}}_2$ 3-form, the hyperplane $\mathbb{B} \subset \mathbb{R}^7$ is spacelike and thus $Q_{\widetilde{\phi}}$ must have signature $(2,4)$ upon restriction to $\mathbb{B}$ and $(1,0)$ upon restriction to $\mathbb{L}$. Thus by eqn. (7.7.27), one sees that $\widetilde{\phi}$ is a $\widetilde{\mathrm{G}}_2$ 3-form if and only if $\mathscr{J}_\rho\omega$ has signature $(2,4)$ and $\omega^3 > 0$. However now (unlike the timelike case), the condition that $\mathscr{J}_\rho\omega$ has signature $(2,4)$ automatically forces $\omega^3 > 0$. Thus:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho) = \left\{\omega \in \bigwedge\nolimits^2 \left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{J}_\rho\omega \text{ has signature } (2,4)\right\}$$

as required.

□

I now prove Lemma 7.7.10:

Proof of Lemma 7.7.10. Without loss of generality take $\rho = \rho_-$ (see eqn. (2.3.5)) and consider the 2-forms:

$$\omega_1 = 2\theta^{12} - \theta^{34} - \theta^{56}, \quad \omega_2 = -\theta^{12} + 2\theta^{34} - \theta^{56} \quad \text{and} \quad \omega_3 = -\theta^{12} - \theta^{34} + 2\theta^{56}.$$

Then:

$$\begin{aligned}
\mathscr{J}_{\rho_-}\omega_1 &= 2\left(\theta^1\right)^{\otimes 2} + 2\left(\theta^2\right)^{\otimes 2} - \left(\theta^3\right)^{\otimes 2} - \left(\theta^4\right)^{\otimes 2} - \left(\theta^5\right)^{\otimes 2} - \left(\theta^6\right)^{\otimes 2} \\
\mathscr{J}_{\rho_-}\omega_2 &= -\left(\theta^1\right)^{\otimes 2} - \left(\theta^2\right)^{\otimes 2} + 2\left(\theta^3\right)^{\otimes 2} + 2\left(\theta^4\right)^{\otimes 2} - \left(\theta^5\right)^{\otimes 2} - \left(\theta^6\right)^{\otimes 2} \\
\mathscr{J}_{\rho_-}\omega_3 &= -\left(\theta^1\right)^{\otimes 2} - \left(\theta^2\right)^{\otimes 2} - \left(\theta^3\right)^{\otimes 2} - \left(\theta^4\right)^{\otimes 2} + 2\left(\theta^5\right)^{\otimes 2} + 2\left(\theta^6\right)^{\otimes 2}
\end{aligned}$$

which all have signature $(2,4)$. Thus by Proposition 7.7.15 $\omega_i \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho_-)$ for all $i = 1,2,3$. Therefore:

$$\mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho_+)\right) \ni \frac{1}{3}\left(\omega_1 + \omega_2 + \omega_3\right) = 0,$$

as required.

□

7.7.5 Null case: Lemma 7.7.11 – connectedness of $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho)$

As usual, without loss of generality assume that $\rho = \rho_0$ (see eqn. (7.7.7)). To prove Lemma 7.7.11 – which is manifestly invariant under the natural $\mathrm{GL}_+(6;\mathbb{R})$ action on ρ_0 – it is beneficial to reduce this 'gauge freedom' to $\mathrm{SL}(6;\mathbb{R})$; the 'gauge' is (partially) fixed by defining $vol_{\rho_0} = \theta^{234567}$. One can then define a linear map $H_{\rho_0} : \mathbb{R}^6 \to \mathbb{R}^6$ via:

$$K_{\rho_0} = H_{\rho_0} \otimes vol_{\rho_0}. \tag{7.7.28}$$

(The need to arbitrarily fix a volume form arises since K_{ρ_0}, being nilpotent, has no non-trivial $\left(\bigwedge^6 \left(\mathbb{R}^6\right)^*\right)^n$-valued invariants for any n and thus parabolic 3-forms do not canonically define volume forms as $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms do.)

To compute $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$, I begin by identifying a convenient subgroup:

Proposition 7.7.29. *Identify* $\mathbb{R}^6 = \langle e_2, e_3, e_4 \rangle \oplus \langle e_5, e_6, e_7 \rangle \cong \mathbb{R}^3 \oplus \mathbb{R}^3$ *and let* $\mathrm{SL}(3;\mathbb{R})$ *act diagonally on* $\mathbb{R}^6$ *according to this splitting. Write* $\xi : \mathrm{SL}(3;\mathbb{R}) \to \mathrm{SL}(6;\mathbb{R})$ *for the corresponding group homomorphism. Then* $\xi(\mathrm{SL}(3;\mathbb{R}))$ *preserves* ρ_0, vol_{ρ_0} *and* H_{ρ_0}.

Proof. Clearly $\xi(\mathrm{SL}(3;\mathbb{R}))$ preserves vol_{ρ_0}. Moreover, with respect to the basis $\langle e_2, ..., e_7 \rangle$:

$$H_{\rho_0} = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix}$$

and thus for $A \in \mathrm{SL}(3;\mathbb{R})$:

$$\xi(A) \circ H_{\rho_0} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = H_{\rho_0} \circ \xi(A),$$

as required. Now consider the map:

$$\begin{aligned} j : \textstyle\bigwedge^3 \left(\mathbb{R}^6\right)^* &\longrightarrow \textstyle\bigwedge^3 \left(\mathbb{R}^6\right)^* \\ \theta^{qrs} &\longmapsto H^*_{\rho_0}(\theta^{qr}) \wedge \theta^s + H^*_{\rho_0}(\theta^q) \wedge \theta^r \wedge H^*_{\rho_0}(\theta^s) + \theta^q \wedge H^*_{\rho_0}(\theta^{rs}). \end{aligned}$$

Since $\xi(\mathrm{SL}(3;\mathbb{R}))$ preserves H_{ρ_0}, it also preserves j. However $-j(\theta^{567}) = -\theta^{237} + \theta^{246} - \theta^{345} = \rho_0$ and thus $\xi(\mathrm{SL}(3;\mathbb{R}))$ also preserves ρ_0.

□

Note that the subspace $\langle e_5, e_6, e_7 \rangle$ can be invariantly defined as the kernel of the map K_{ρ_0}. By applying Proposition 7.7.29, one obtains:

Corollary 7.7.30. *Every* $F \in \mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$ *preserves the subspace* $\langle e_5, e_6, e_7 \rangle$. *Moreover* $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$ *acts transitively on non-zero vectors and on ordered pairs of linearly independent vectors in* $\langle e_5, e_6, e_7 \rangle$.

I now prove the first half of Lemma 7.7.11. Specifically:

Lemma 7.7.31. $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$ *is connected. Explicitly:*

$$\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0) = \xi\left(\mathrm{SL}(3;\mathbb{R})\right) \cdot \mathscr{G}$$

where ξ *was defined in Proposition 7.7.29,* $\mathscr{G}$ *is the contractible subgroup of* $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$ *defined by:*

$$\mathscr{G} = \left\{ \left(\begin{array}{ccc|ccc} d & & & & & \\ e & d^{-1} & & & & \\ f & & d^{-1} & & & \\ \hline k & l & m & d^2 & & \\ n & o & p & de & 1 & \\ q & r & s & df & & 1 \end{array} \right) \middle| \; d > 0 \text{ and } d^{-1}el + d^{-1}fm - d^{-2}k - s - o = 0 \right\} \tag{7.7.32}$$

and $\xi\,(\mathrm{SL}(3;\mathbb{R}))\cap\mathscr{G}=\xi(\mathcal{G})$*, where* $\mathcal{G}\subset\mathrm{SL}(3;\mathbb{R})$ *consists of the set of* 3×3*-matrices of the form:*

$$\begin{pmatrix}1 & & \\ \lambda & 1 & \\ \mu & & 1\end{pmatrix}\ \textit{for } \lambda,\mu\in\mathbb{R}.$$

Proof. Define:

$$\mathscr{G}=\left\{F\in\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)\;\middle|\;F(e_6)=e_6\text{ and }F(e_7)=e_7\right\}.$$

Then since (by Corollary 7.7.30) $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)$ preserves $\langle e_5,e_6,e_7\rangle$ and $\xi(\mathrm{SL}(3;\mathbb{R}))$ acts transitively on ordered pairs of linearly independent vectors in $\langle e_5,e_6,e_7\rangle$, it follows that:

$$\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(\rho_0)=\xi(\mathrm{SL}(3;\mathbb{R}))\cdot\mathscr{G}.$$

The next task is to verify eqn. (7.7.32). Let $F\in\mathscr{G}$. Since $\theta^{23}=-e_7\lrcorner\rho_0$, and F preserves ρ_0 and e_7, it follows that $F^*\theta^{23}=\theta^{23}$ and similarly $F^*\theta^{24}=\theta^{24}$, since $\theta^{24}=e_6\lrcorner\rho_0$. Since F also preserves $\langle e_5,e_6,e_7\rangle$, with respect to the decomposition $\mathbb{R}^6=\langle e_2,e_3,e_4\rangle\oplus\langle e_5,e_6,e_7\rangle$ one can write:

$$F=\left(\begin{array}{c|ccc} F_1 & & & \\ \hline & a & & \\ F_2 & b & 1 & \\ & c & & 1\end{array}\right),$$

where $a,b,c\in\mathbb{R}$ with $a\neq 0$, $F_2\in\mathrm{End}(\mathbb{R}^3,\mathbb{R}^3)$, $F_1\in\mathrm{GL}(3;\mathbb{R})$ is such that $F_1^*\theta^{23}=\theta^{23}$ and $F_1^*\theta^{24}=\theta^{24}$, and $a\cdot\det(F_1)>0$.

To better understand the map F_1, let $\mathbb{B}=\langle e_2,e_3,e_4\rangle$ and temporarily restrict attention to $\mathbb{B}$. Since $\langle e_4\rangle\subset\mathbb{B}$ is the kernel of the linear map $u\in\mathbb{B}\mapsto u\lrcorner\theta^{23}\in\mathbb{B}^*$, the space $\langle e_4\rangle$ must be preserved by F_1. Likewise $\langle e_3\rangle$ must also be preserved by F_1 since F_1 preserves θ^{24}. Thus:

$$F_1=\begin{pmatrix}d & & \\ e & \lambda & \\ f & & \mu\end{pmatrix}$$

for some $d,\mu,\lambda\in\mathbb{R}\backslash\{0\}$ and $e,f\in\mathbb{R}$. The conditions $F_1^*\theta^{23}=\theta^{23}$ and $F_1^*\theta^{24}=\theta^{24}$ then force $\lambda=d^{-1}$ and $\mu=d^{-1}$.

Returning now to $\mathbb{R}^6$, it has been shown that:

$$F=\left(\begin{array}{ccc|ccc} d & & & & & \\ e & d^{-1} & & & & \\ f & & d^{-1} & & & \\ \hline k & l & m & a & & \\ n & o & p & b & 1 & \\ q & r & s & c & & 1\end{array}\right)$$

for $k, l, m, n, o, p, q, r, s \in \mathbb{R}$. One may then compute that $F^* \rho_0 = \rho_0$ is equivalent to $a = d^2$, $c = df$, $b = de$, together with the condition:

$$d^{-1}el + d^{-1}fm - d^{-2}k - s - o = 0.$$

Moreover, given $a = d^2$ one has $\det(F) = d > 0$. Thus, it has been established that:

$$\mathscr{G} = \left\{ \left(\begin{array}{ccc|ccc} d & & & & & \\ e & d^{-1} & & & & \\ f & & d^{-1} & & & \\ \hline k & l & m & d^2 & & \\ n & o & p & de & 1 & \\ q & r & s & df & & 1 \end{array} \right) \middle| \; d > 0 \text{ and } d^{-1}el + d^{-1}fm - d^{-2}k - s - o = 0 \right\}$$

as claimed.

The expression for $\xi(\mathrm{SL}(3;\mathbb{R})) \cap \mathscr{G}$ is now manifest. To see that $\mathscr{G}$ is contractible, consider the projection:

$$\begin{aligned} \mathscr{G} &\xrightarrow{\quad\pi\quad} (0,\infty) \times \mathbb{R}^6 \\ (d,e,f,k,l,m,n,o,p,q,r,s) &\longmapsto (d,e,f,n,p,q,r). \end{aligned}$$

Then π is surjective, with fibre over (d,e,f,n,p,q,r) given by:

$$\left\{ (k,m,l,o,s) \in \mathbb{R}^5 \;\middle|\; d^{-2}k - d^{-1}fm - d^{-1}el + o + s = 0 \right\}.$$

Thus $\mathscr{G}$ is topologically a rank-4 vector bundle over the contractible space $(0,\infty) \times \mathbb{R}^6$, hence contractible. This completes the proof.

$\square$

7.7.6 Null case: Lemma 7.7.11 – $0 \in \mathrm{ConvConv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho)\right)$

By analogy with eqns. (7.7.14) and (7.7.23), define:

$$\begin{aligned} \mathscr{H}_{\rho_0} : \textstyle\bigwedge^2 \left(\mathbb{R}^6\right)^* &\longrightarrow \odot^2 \left(\mathbb{R}^6\right)^* \\ \omega &\longmapsto \left\{ (a,b) \mapsto \tfrac{1}{2} \left[\omega(H_{\rho_0} a, b) + \omega(H_{\rho_0} b, a) \right] \right\}. \end{aligned}$$

Proposition 7.7.33.

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0) = \left\{ \omega \in \textstyle\bigwedge^2 \left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{H}_{\rho_0}\omega \text{ has signature } (2,1,3) \right\}. \qquad (7.7.34)$$

(Here signature $(2,1,3)$ means that $\mathscr{H}_{\rho_0}$ has a maximal positive definite subspace of dimension 2, a maximal negative definite subspace of dimension 3 and a 1-dimensional kernel.)

The proof proceeds via a series of lemmas:

Lemma 7.7.35. *For $n \geqslant 1$, $x_1, ..., x_n \in \mathbb{R}\backslash\{0\}$ and $y \in \mathbb{R}$, any symmetric, 'forward-triangular' $(2n+1)\times(2n+1)$ bilinear form:*

$$M = \begin{pmatrix} * & * & \cdots & * & \cdots & * & x_1 \\ * & * & \cdots & * & \cdots & x_2 & \\ \vdots & \vdots & & & \iddots & & \\ * & * & & y & & & \\ \vdots & \vdots & \iddots & & & & \\ * & x_2 & & & & & \\ x_1 & & & & & & \end{pmatrix}$$

is non-degenerate if and only if $y \neq 0$, having signature $(n+1, n)$ if $y > 0$ and signature $(n, n+1)$ if $y < 0$. Moreover, the $2n \times 2n$ bilinear form:

$$M' = \begin{pmatrix} * & * & \cdots & * & x_1 \\ * & * & \cdots & x_2 & \\ \vdots & \vdots & \iddots & & \\ * & x_2 & & & \\ x_1 & & & & \end{pmatrix}$$

has signature (n, n).

Proof. Start with the matrix M, call the diagonal running from the $(2n+1, 1)$-entry to the $(1, 2n+1)$-entry the counter diagonal and call the elements in front of the counter diagonal the strictly forward entries. Clearly the bilinear form is degenerate when $y = 0$. Moreover, when $y \neq 0$, M is non-singular for any values of the strictly forward entries and thus it suffices to compute the signature of M when all of the strictly forward entries vanish. However, in this case, M has eigenvalues y, $\pm x_1$, $\pm x_2$,...,$\pm x_n$, with corresponding eigenvectors:

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ \pm 1 \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \pm 1 \\ \vdots \\ 0 \end{pmatrix}.$$

The case of M' is similar.

□

Lemma 7.7.36.

$$\mathrm{Ker}(\mathscr{H}_{\rho_0}) = \mathbb{R}^6 \lrcorner \rho_0.$$

Proof. Take a basis of $\mathbb{R}^6 \lrcorner \rho_0$ as follows:

$$(e_2 \lrcorner \rho_0, e_3 \lrcorner \rho_0, ..., e_7 \lrcorner \rho_0) = \left(-\theta^{37} + \theta^{46}, \theta^{27} - \theta^{45}, -\theta^{26} + \theta^{35}, -\theta^{34}, \theta^{24}, -\theta^{23}\right).$$

Extend this to a basis of $\wedge^2 \left(\mathbb{R}^6\right)^*$ via:

$$\left(\theta^{25}, \theta^{36}, \theta^{47}, \theta^{56}, \theta^{57}, \theta^{67}, \theta^{26} + \theta^{35}, \theta^{27} + \theta^{45}, \theta^{37} + \theta^{46}\right).$$

(By analogy with the spacelike and timelike cases, denote the span of this latter set of vectors by $\wedge^{1,1} \left(\mathbb{R}^6\right)^*$.) Then:

$$\bigwedge^2 \left(\mathbb{R}^6\right)^* = \bigwedge^{1,1} \left(\mathbb{R}^6\right)^* \oplus \mathbb{R}^6 \lrcorner \rho_0, \tag{7.7.37}$$

although the reader should note that I only define this splitting for ρ_0; no attempt is made to define $\wedge^{1,1} \left(\mathbb{R}^6\right)^*$ for an arbitrary parabolic 3-form. Then $\mathscr{H}_{\rho_0}$ vanishes identically on $\mathbb{R}^6 \lrcorner \rho_0$: indeed:

$$\begin{aligned} -\theta^{37} + \theta^{46} &\xmapsto{\mathscr{H}_{\rho_0}} \theta^3 \odot \theta^4 - \theta^3 \odot \theta^4 = 0 \\ \theta^{27} - \theta^{45} &\xmapsto{\mathscr{H}_{\rho_0}} -\theta^2 \odot \theta^4 + \theta^2 \odot \theta^4 = 0 \end{aligned}$$

and similarly for the other basis vectors. Moreover, one may verify that:

$$\begin{pmatrix} \theta^{25} \\ \theta^{36} \\ \theta^{47} \\ \theta^{56} \\ \theta^{57} \\ \theta^{67} \\ \theta^{26} + \theta^{35} \\ \theta^{27} + \theta^{45} \\ \theta^{37} + \theta^{46} \end{pmatrix} \xmapsto{\mathscr{H}_{\rho_0}} \begin{pmatrix} -\theta^2 \odot \theta^2 \\ -\theta^3 \odot \theta^3 \\ -\theta^4 \odot \theta^4 \\ \theta^2 \odot \theta^6 - \theta^3 \odot \theta^5 \\ \theta^2 \odot \theta^7 - \theta^4 \odot \theta^5 \\ \theta^3 \odot \theta^7 - \theta^4 \odot \theta^6 \\ -2\theta^2 \odot \theta^3 \\ -2\theta^2 \odot \theta^4 \\ -2\theta^3 \odot \theta^4 \end{pmatrix}. \tag{7.7.38}$$

Since these images are linearly independent, the map $\mathscr{H}_{\rho_0}$ is injective when restricted to $\wedge^{1,1} \left(\mathbb{R}^6\right)^*$. This completes the proof.

$\square$

Lemma 7.7.39. *For all* $\omega \in \wedge^2 \left(\mathbb{R}^6\right)^*$*:*

$$\mathrm{Ker}(\mathscr{H}_{\rho_0}\omega) \cap \langle e_5, e_6, e_7 \rangle \neq 0.$$

Here $\mathrm{Ker}(\mathscr{H}_{\rho_0}\omega)$ *denotes the kernel of the map:*

$$\begin{aligned} \flat : \mathbb{R}^6 &\to \left(\mathbb{R}^6\right)^* \\ u &\mapsto \mathscr{H}_{\rho_0}\omega(u, -). \end{aligned}$$

Proof. It is equivalent to show that $\flat|_{\langle e_5,e_6,e_7\rangle}$ is not injective. Thus fix $\omega \in \bigwedge^2 \left(\mathbb{R}^6\right)^*$. Recalling that $\mathscr{H}_{\rho_0}$ vanishes on $\mathbb{R}^6 \lrcorner \rho_0$ and inspecting eqn. (7.7.38), one sees that $\flat|_{\langle e_5,e_6,e_7\rangle}$ only depends on the component of ω in the subspace $\langle \theta^{56}, \theta^{57}, \theta^{67}\rangle \subset \bigwedge^{1,1}\left(\mathbb{R}^6\right)^*$, so without loss of generality assume that:

$$\omega = \lambda_1\theta^{56} + \lambda_2\theta^{57} + \lambda_3\theta^{67}.$$

Thus:

$$\mathscr{H}_{\rho_0}\omega = \lambda_1\left(\theta^2\odot\theta^6 - \theta^3\odot\theta^5\right) + \lambda_2\left(\theta^2\odot\theta^7 - \theta^4\odot\theta^5\right) + \lambda_3\left(\theta^3\odot\theta^7 - \theta^4\odot\theta^6\right).$$

Hence $\flat|_{\langle e_5,e_6,e_7\rangle}$ maps $\langle e_5, e_6, e_7\rangle$ into $\langle e_2, e_3, e_4\rangle$ and is represented by the matrix:

$$\frac{1}{2}\begin{pmatrix} 0 & \lambda_1 & \lambda_2 \\ -\lambda_1 & 0 & \lambda_3 \\ -\lambda_2 & -\lambda_3 & 0 \end{pmatrix} \tag{7.7.40}$$

which has determinant 0, as required.

□

I now prove Proposition 7.7.33:

Proof of Proposition 7.7.33. As in the proof of Proposition 7.7.15:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0) = \left[\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0) \cap \textstyle\bigwedge^{1,1}\left(\mathbb{R}^6\right)^*\right] \oplus \mathbb{R}^6 \lrcorner \rho_0$$

and therefore, since $\mathscr{H}_{\rho_0}$ vanishes on $\mathbb{R}^6 \lrcorner \rho_0$, it suffices to prove that:

$$\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0) \cap \textstyle\bigwedge^{1,1}\left(\mathbb{R}^6\right)^* = \left\{\omega \in \bigwedge^{1,1}\left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{H}_{\rho_0}\omega \text{ has signature } (2,1,3)\right\}. \tag{7.7.41}$$

Let $\omega \in \bigwedge^{1,1}\left(\mathbb{R}^6\right)^*$ and define $\widetilde{\phi} = \theta\wedge\omega + \rho_0$. Proceeding as in the proof of Proposition 7.7.15, one obtains:

$$6Q_{\widetilde{\phi}}(ae_1 + u) = a^2\theta^1\wedge\omega^3 - 6a\theta^1\wedge(u\lrcorner\omega)\wedge\omega\wedge\rho_0 + 3\mathscr{H}_{\rho_0}\omega(u,u)\theta^1\wedge vol_{\rho_0} \tag{7.7.42}$$

where now, unlike for $\mathrm{SL}(3;\mathbb{R})^2$ and $\mathrm{SL}(3;\mathbb{C})$ 3-forms, $\omega\wedge\rho_0$ need not vanish. Initially suppose that $\widetilde{\phi}$ is of $\widetilde{\mathrm{G}}_2$-type and write $\mathbb{B} = 0\oplus\mathbb{R}^6 \subset \mathbb{R}^7$. Since $\widetilde{\phi}|_{\mathbb{B}} = \rho_0$ is parabolic, it follows that $\mathbb{B}$ is null, hence $Q_{\widetilde{\phi}}$ has signature $(2,1,3)$ upon restriction to $\mathbb{B}$ and whence by eqn. (7.7.42) $\mathscr{H}_{\rho_0}\omega$ has signature $(2,1,3)$, as required.

Conversely, suppose that $\mathscr{H}_{\rho_0}\omega$ has signature $(2,1,3)$. Then $\mathscr{H}_{\rho_0}\omega$ has a 1-dimensional kernel which by Lemma 7.7.39 must be contained in $\langle e_5, e_6, e_7\rangle$. By applying a suitable $\mathrm{SL}(3;\mathbb{R})$-symmetry (see Proposition 7.7.29), without loss of generality one can assume that:

$$\mathrm{Ker}(\mathscr{H}_{\rho_0}\omega) = \langle e_7\rangle.$$

Since $\omega \in \bigwedge^{1,1}\left(\mathbb{R}^6\right)^*$, by examining the matrix for $\flat|_{\langle e_5,e_6,e_7\rangle}$ in eqn. (7.7.40), it follows that ω has the form:

$$\omega = A\theta^{25} + B\theta^{36} + C\theta^{47} + D\theta^{56} + E\left(\theta^{26} + \theta^{35}\right) + F\left(\theta^{27} + \theta^{45}\right) + G\left(\theta^{37} + \theta^{46}\right),$$

for some constants $A, B, C, D, E, F, G \in \mathbb{R}$ with $D \neq 0$. Upon restriction to $\langle e_2, ..., e_6\rangle$ the bilinear form $\mathscr{H}_{\rho_0}\omega$ is represented by the matrix:

$$\begin{array}{c} \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{pmatrix}
-A & -E & -F & 0 & \frac{D}{2} \\
-E & -B & -G & -\frac{D}{2} & \\
-F & -G & -C & & \\
0 & -\frac{D}{2} & & & \\
\frac{D}{2} & & & &
\end{pmatrix}
\end{array}.$$

By assumption, this bilinear form is non-degenerate with signature $(2,3)$, and thus it follows from Lemma 7.7.35 that $C > 0$.

Next, observing $\omega \wedge \rho_0 = -D\theta^{23567}$ yields:

$$\left((-) \lrcorner\, \omega\right) \wedge \omega \wedge \rho_0 = D\left(C\theta^7 + F\theta^5 + G\theta^6\right) \otimes vol_{\rho_0}.$$

Substituting this result into eqn. (7.7.42) and polarising shows that $2Q_{\widetilde{\phi}}$ is represented by the symmetric 7×7-matrix:

$$\begin{array}{c|ccccccc}
 & e_1 & e_7 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
e_1 & \frac{H}{3} & -DC & & & & -DF & -DG \\
e_7 & -DC & & & & & & \\
e_2 & & & -A & -E & -F & 0 & \frac{D}{2} \\
e_3 & & & -E & -B & -G & -\frac{D}{2} & \\
e_4 & & & -F & -G & -C & & \\
e_5 & -DF & & 0 & -\frac{D}{2} & & & \\
e_6 & -DG & & \frac{D}{2} & & & &
\end{array}.$$

where $H \in \mathbb{R}$ is such that $\theta^1 \wedge \omega^3 = H\theta^1 \wedge vol_{\rho_0}$. Thus to complete the proof, it suffices to prove that this matrix has signature $(3,4)$. In fact, I show that for any $h, r, s, t \in \mathbb{R}$, $r \neq 0$, the matrix:

$$M_{h,r,s,t} = \begin{array}{c|ccccccc}
 & e_1 & e_7 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
e_1 & h & r & & & & s & t \\
e_7 & r & & & & & & \\
e_2 & & & -A & -E & -F & 0 & \frac{D}{2} \\
e_3 & & & -E & -B & -G & -\frac{D}{2} & \\
e_4 & & & -F & -G & -C & & \\
e_5 & s & & 0 & -\frac{D}{2} & & & \\
e_6 & t & & \frac{D}{2} & & & &
\end{array}$$

has signature $(3,4)$ (which completes the proof, as $-DC \neq 0$). Since $\det M_{h,r,s,t} = -Cr^2\left(\frac{D}{2}\right)^4 \neq 0$, $M_{h,r,s,t}$ is non-singular for all values of h, s, t and thus it suffices to consider $M_{0,r,0,0}$. However $M_{0,r,0,0}$ is block diagonal with blocks:

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -A & -E & -F & 0 & \frac{D}{2} \\ -E & -B & -G & -\frac{D}{2} & \\ -F & -G & -C & & \\ 0 & -\frac{D}{2} & & & \\ \frac{D}{2} & & & & \end{pmatrix}.$$

By Lemma 7.7.35 the former block has signature $(1,1)$ and the latter block has signature $(2,3)$, and thus $M_{0,r,0,0}$ has signature $(3,4)$, as claimed.

$\square$

I now prove the second part of Lemma 7.7.11. Specifically:

Lemma 7.7.43. *For each (equivalently any)* $\rho \in \wedge^3_0\left(\mathbb{R}^6\right)^*$, $0 \in \mathrm{Conv}\left(\mathcal{N}_{\widetilde{\varphi}_0}(\rho)\right)$.

Proof. As usual, without loss of generality let $\rho = \rho_0$. Consider the 2-form $\omega_0 = -2\theta^{47} + 2\varepsilon(\theta^{67} + \theta^{25} - \theta^{36})$ for some $\varepsilon \in \mathbb{R}\backslash\{0\}$ to be specified later. One may compute using eqn. (7.7.38) that:

$$\mathscr{H}_{\rho_0}\omega_0 = 2\theta^4 \otimes \theta^4 + 2\varepsilon(\theta^3 \odot \theta^7 - \theta^4 \odot \theta^6 - \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3).$$

This may be represented by the 6×6-matrix:

$$\begin{array}{c|c} & \begin{matrix} e_2 & e_3 & e_7 & e_4 & e_6 & e_5 \end{matrix} \\ \begin{matrix} e_2 \\ e_3 \\ e_7 \\ e_4 \\ e_6 \\ e_5 \end{matrix} & \begin{pmatrix} -2\varepsilon & & & & & \\ & 2\varepsilon & \varepsilon & & & \\ & \varepsilon & & & & \\ & & & 2 & -\varepsilon & \\ & & & -\varepsilon & & \\ & & & & & 0 \end{pmatrix} \end{array}$$

which has signature $(2,1,3)$, by applying Lemma 7.7.35 to each matrix along the (block) diagonal. Thus $\omega_0 \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho_0)$.

Now consider $\omega_\pm = \theta^{47} \pm \theta^{56}$. Then one may verify again using eqn. (7.7.38) that:

$$\mathscr{H}_{\rho_0}\omega_\pm = -\theta^4 \otimes \theta^4 \pm (\theta^2 \odot \theta^6 - \theta^3 \odot \theta^5)$$

both of which have signature $(2,1,3)$. Thus $\omega_\pm \in \mathcal{N}_{\widetilde{\varphi}_0}(\rho_0)$. However $\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0) \subset \wedge^2\left(\mathbb{R}^6\right)^*$ is open, so it follows that for all $\varepsilon \in \mathbb{R}\backslash\{0\}$ with $|\varepsilon|$ sufficiently small, the 3-forms:

$$\omega'_\pm = \omega_\pm - \varepsilon(\theta^{67} + \theta^{25} - \theta^{36})$$

also lie in $\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0)$. Fix some suitable choice of ε; then the three 2-forms $\omega_0, \omega'_\pm$ all lie in $\mathcal{N}_{\widetilde{\varphi}_0}(\rho_0)$. Moreover, one may compute that:

$$\begin{aligned}\frac{1}{3}\left(\omega_0 + \omega'_+ + \omega'_-\right) =& \frac{1}{3}\Big(-2\theta^{47} + 2\varepsilon(\theta^{67} + \theta^{25} - \theta^{36}) \\ &+ \theta^{47} + \theta^{56} - \varepsilon(\theta^{67} + \theta^{25} - \theta^{36}) \\ &+ \theta^{47} - \theta^{56} - \varepsilon(\theta^{67} + \theta^{25} - \theta^{36})\Big) \\ =&0,\end{aligned}$$

completing the proof.

$\square$

Thus, by Theorem 7.4.5, it has been proven:

Theorem 7.7.44. $\widetilde{\mathrm{G}}_2$ *3-forms satisfy the relative h-principle.*

$\square$

Chapter 8

h-principle for $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms

This chapter uses convex integration with avoidance, together with careful analysis of the rank 3 distributions induced by $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, to prove the relative h-principle for $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on oriented 6-manifolds.

8.1 Lack of ampleness of $\mathrm{SL}(3;\mathbb{R})^2$ 3-form

Consider the standard $\mathrm{SL}(3;\mathbb{R})^2$ 3-form $\rho_+ = e^{123} + e^{456}$ on $\mathbb{R}^6$ and recall the ± 1-eigenspaces of the para-complex structure I_{ρ_+}:

$$E_+ = \langle e_1, e_2, e_3 \rangle \quad \text{and} \quad E_- = \langle e_4, e_5, e_6 \rangle.$$

Given a hyperplane $\mathbb{B} \subset \mathbb{R}^6$, on dimensional grounds one of the following statements holds:

1. $\dim(\mathbb{B} \cap E_\pm) = 2$;
2. $\dim(\mathbb{B} \cap E_+) = 2$ but $\dim(\mathbb{B} \cap E_-) = 3$ (equivalently $E_- \subset \mathbb{B}$);
3. $\dim(\mathbb{B} \cap E_-) = 2$ but $\dim(\mathbb{B} \cap E_+) = 3$ (equivalently $E_+ \subset \mathbb{B}$).

Denote the sets of oriented hyperplances corresponding to conditions 1, 2 and 3 above by $\widetilde{\mathrm{Gr}}_{5,gen}(\mathbb{R}^6)$, $\widetilde{\mathrm{Gr}}_{5,-}(\mathbb{R}^6)$ and $\widetilde{\mathrm{Gr}}_{5,+}(\mathbb{R}^6)$ respectively.

Proposition 8.1.1.

$$\mathrm{SL}(3;\mathbb{R})^2 \backslash \widetilde{\mathrm{Gr}}_5(\mathbb{R}^6) = \left\{ \widetilde{\mathrm{Gr}}_{5,gen}(\mathbb{R}^6), \widetilde{\mathrm{Gr}}_{5,-}(\mathbb{R}^6), \widetilde{\mathrm{Gr}}_{5,+}(\mathbb{R}^6) \right\}.$$

Proof. Firstly note that there is an isomorphism:

$$\begin{aligned} \widetilde{\mathrm{Gr}}_{5,+}\left(\mathbb{R}^6\right) &\longrightarrow \widetilde{\mathrm{Gr}}_2\left(E_-\right) \\ \Pi &\longmapsto \Pi \cap E_- \end{aligned}$$

where $\Pi \cap E_-$ is oriented via the decomposition $\Pi = E_+ \oplus (\Pi \cap E_-)$. Recalling that $\mathrm{SL}(3;\mathbb{R})^2$ acts on $\mathbb{R}^6$ diagonally via the decomposition $\mathbb{R}^6 = E_+ \oplus E_-$, and that $\mathbf{1} \times \mathrm{SL}(3;\mathbb{R})$ acts transitively on

$\widetilde{\mathrm{Gr}}_2\,(E_-)$, it follows that $\widetilde{\mathrm{Gr}}_{5,+}\left(\mathbb{R}^6\right)$ is a single orbit for the action of $\mathrm{SL}(3;\mathbb{R})^2$. Likewise $\widetilde{\mathrm{Gr}}_{5,-}\left(\mathbb{R}^6\right)$ is a single orbit.

In the remaining case, firstly note that $\mathrm{Gr}_{5,gen}\left(\mathbb{R}^6\right)$ forms a single orbit for $\mathrm{SL}(3;\mathbb{R})^2$. Indeed, there is a natural line bundle $\mathcal{L}_+$ over $\mathrm{Gr}_2(E_+)$ with fibre over $\pi_+ \in \mathrm{Gr}_2(E_+)$ given by:

$$\mathcal{L}_+|_{\pi_+} = {E_+}\big/{\pi_+}\,.$$

The action of $\mathrm{SL}(3;\mathbb{R}) \times \mathbf{1}$ on $\mathrm{Gr}_2(E_+)$ lifts naturally to define an action on $\mathcal{L}_+$ which one may verify acts transitively on $\mathcal{L}_+\backslash\mathrm{Gr}_2(E_+)$, the complement of the zero section. The analogous statement holds for $\mathcal{L}_-\backslash\mathrm{Gr}_2(E_-)$. Now note that there is a surjective map:

$$\begin{aligned} \mathcal{L}_+\backslash\mathrm{Gr}_2(E_+) \times \mathcal{L}_-\backslash\mathrm{Gr}_2(E_-) &\longrightarrow \mathrm{Gr}_{5,gen}\left(\mathbb{R}^6\right) \\ \left(u_+ + \pi_+ \in {E_+}\big/{\pi_+}, u_- + \pi_- \in {E_-}\big/{\pi_-}\right) &\longmapsto \pi_+ \oplus \pi_- \oplus \langle u_+ + u_-\rangle. \end{aligned}$$

Since $\mathrm{SL}(3;\mathbb{R})^2$ acts transitively on $\mathcal{L}_+\backslash\mathrm{Gr}_2(E_+) \times \mathcal{L}_-\backslash\mathrm{Gr}_2(E_-)$, it follows that $\mathrm{Gr}_{5,gen}\left(\mathbb{R}^6\right)$ forms a single $\mathrm{SL}(3;\mathbb{R})^2$-orbit as claimed. To verify that moreover $\widetilde{\mathrm{Gr}}_{5,gen}\left(\mathbb{R}^6\right)$ forms a single orbit, it suffices to consider $\mathbb{B} \in \widetilde{\mathrm{Gr}}_{5,gen}\left(\mathbb{R}^6\right)$ with oriented basis $\langle e_1, e_2, e_4, e_5, e_3 + e_6\rangle$ and note that:

$$F = \begin{pmatrix} -1 &&&&& \\ & 1 &&&& \\ && -1 &&& \\ &&& -1 && \\ &&&& 1 & \\ &&&&& -1 \end{pmatrix} \in \mathrm{SL}(3;\mathbb{R})^2$$

preserves $\mathbb{B}$ and $F|_{\mathbb{B}}$ is orientation-reversing.

$\square$

Clearly $\widetilde{\mathrm{Gr}}_{5,gen}\left(\mathbb{R}^6\right) \subset \widetilde{\mathrm{Gr}}_5\left(\mathbb{R}^6\right)$ is open and dense. By Proposition 7.4.2, it follows that $\mathcal{T}_{\rho_+}\left(\widetilde{\mathrm{Gr}}_{5,gen}(\mathbb{R}^6)\right)$ must be the (unique) open orbit of 3-forms on $\mathbb{R}^5$, i.e. $\bigwedge^3_{Co\text{-}P}\left(\mathbb{R}^5\right)$. Now consider the orbit $\widetilde{\mathrm{Gr}}_{5,+}(\mathbb{R}^6)$. Taking $\mathbb{B} = \langle e_1, ..., e_5\rangle \in \widetilde{\mathrm{Gr}}_{5,+}(\mathbb{R}^6)$ yields:

$$\rho_+|_{\mathbb{B}} = \theta^{123}.$$

It follows that $\mathcal{T}_{\rho_+}\left(\widetilde{\mathrm{Gr}}_{5,+}(\mathbb{R}^6)\right)$ is the orbit of non-zero, decomposable 3-forms on $\mathbb{R}^5$. By considering $\mathbb{B} = \langle e_2, ..., e_6\rangle \in \widetilde{\mathrm{Gr}}_{5,-}\left(\mathbb{R}^6\right)$, one sees that $\mathcal{T}_{\rho_+}\left(\widetilde{\mathrm{Gr}}_{5,-}(\mathbb{R}^6)\right)$ is precisely the same orbit.

Proposition 8.1.2. *Let $\tau \in \bigwedge^3_{Co\text{-}P}\left(\mathbb{R}^5\right)$. Then $\mathcal{N}_{\rho_+}(\tau)$ is ample. In contrast, now let τ be a non-zero decomposable 3-form on $\mathbb{R}^5$. Then $\mathcal{N}_{\rho_+}(\tau)$ consists of two convex, connected components; in particular, it is not ample.*

Proof. Let $\tau \in \bigwedge^3_{Co-P}\left(\mathbb{R}^5\right)^*$. Then $\mathrm{Stab}_{\mathrm{GL}_+(5;\mathbb{R})}(\tau)$ is connected by Proposition 7.1.14 and:

$$\mathcal{T}_{\rho_+}^{-1}\left(\mathcal{T}_{\rho_+}\left[\widetilde{\mathrm{Gr}}_{5,gen}(\mathbb{R}^6)\right]\right) = \left\{\widetilde{\mathrm{Gr}}_{5,gen}\left(\mathbb{R}^6\right)\right\}$$

by the above discussion. Since ρ_+ admits the orientation-reversing automorphism:

$$e_1 \leftrightarrow e_4, \quad e_2 \leftrightarrow e_5, \quad e_3 \leftrightarrow e_6$$

it follows from Proposition 7.4.4 that $\mathcal{N}_{\rho_+}(\tau)$ is ample.

Now let τ be a non-zero, decomposable 3-form. Identify $\mathbb{R}^5$ with the subspace $\langle e_2, ..., e_6\rangle$ of $\mathbb{R}^6$ and take $\tau = \theta^{456}$. Then:

$$\mathcal{N}_{\rho_+}(\tau) = \left\{\omega \in \bigwedge^2\langle\theta^2, ..., \theta^6\rangle \;\middle|\; \theta^1 \wedge \omega + \theta^{456} \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*\right\}.$$

Recall that a 3-form $\rho \in \bigwedge^3\left(\mathbb{R}^6\right)^*$ is of $\mathrm{SL}(3;\mathbb{R})^2$-type if and only if the quadratic invariant Λ defined in §2.3.1 is positive. A direct calculation shows that:

$$\Lambda(\theta^1 \wedge \omega + \theta^{456}) = \omega(e_2, e_3)^2 \cdot (\theta^{123456})^{\otimes 2}.$$

Thus:

$$\mathcal{N}_{\rho_+}(\tau) = \left\{\omega \in \bigwedge^2\langle\theta^2, ..., \theta^6\rangle \;\middle|\; \omega(e_2, e_3) \neq 0\right\},$$

which has the form claimed.

□

8.2 Defining an avoidance template for $\mathscr{R}_{\rho_+}(a)$

The remainder of this chapter is devoted to proving:

Theorem 8.2.1. $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms satisfy the relative h-principle.*

Recall from Lemma 7.3.3 that in order to prove Theorem 8.2.1, it suffices to show that for all oriented 6-manifolds M, $q \geqslant 0$ and continuous maps $a : D^q \to \Omega^3(\mathrm{M})$, the fibred differential relation $\mathscr{R}_{\rho_+}(a)$ over M satisfies the relative h-principle. Since ρ_+ is not ample, by Proposition 7.3.5 the relation $\mathscr{R}_{\rho_+}(a)$ is not ample, and thus convex integration cannot be used to establish the h-principle for $\mathscr{R}_{\rho_+}(a)$. Instead, I employ Theorem 2.5.10. Therefore, to prove Theorem 8.2.1, it suffices to show the existence of an avoidance template for $\mathscr{R}_{\rho_+}(a)$. The aim of this section, therefore, is to define an avoidance template $\mathscr{A}$ for $\mathscr{R}_{\rho_+}(a)$ and prove that it satisfies conditions (1)–(3) in Definition 2.5.9.

Definition 8.2.2. Let $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$ be an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form and let $\{\mathbb{B}_1, ..., \mathbb{B}_k\} \in \mathrm{Gr}_5^{(k)}(\mathbb{R}^6)$ be a configuration of hyperplanes in $\mathbb{R}^6$. Say that $\{\mathbb{B}_1, ..., \mathbb{B}_k\}$ is generic with respect to ρ if for all $i \in \{1, ..., k\}$: $\mathbb{B}_i \in \mathrm{Gr}_{5,gen}(\mathbb{R}^6)$ and if for all distinct $i, j \in \{1, ..., k\}$, at least one of the conditions:

$$\mathbb{B}_i \cap E_{+,\rho} \neq \mathbb{B}_j \cap E_{+,\rho} \quad \text{or} \quad \mathbb{B}_i \cap E_{-,\rho} \neq \mathbb{B}_j \cap E_{-,\rho} \quad \text{holds.}$$

Write $\mathrm{Gr}^{(\infty)}_{5,gen}(\mathbb{R}^6)_\rho$ for the collection of all generic configurations of hyperplanes in $\mathbb{R}^6$ with respect to ρ, or simply $\mathrm{Gr}^{(\infty)}_{5,gen}(\mathbb{R}^6)$, when ρ is clear from context. (Note that formally $\mathrm{Gr}^{(1)}_{5,gen}\left(\mathbb{R}^6\right) = \mathrm{Gr}_{5,gen}\left(\mathbb{R}^6\right)$.)

The appellation 'generic' is justified by the following proposition:

Proposition 8.2.3. *Let $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$ be an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form. Then:*

$$\mathrm{Gr}^{(\infty)}_{5,gen}(\mathbb{R}^6) \subset \mathrm{Gr}^{(\infty)}_5(\mathbb{R}^6)$$

is an open and dense subset.

Proof. Recall from above that $\mathrm{Gr}_{5,gen}\left(\mathbb{R}^6\right) \subset \mathrm{Gr}_5\left(\mathbb{R}^6\right)$ is open and dense. Thus it is equivalent to prove that $\mathrm{Gr}^{(k)}_{5,gen}(\mathbb{R}^6) \subset \mathrm{Gr}^{(k)}_5(\mathbb{R}^6)$ is open and dense for every $k \geqslant 2$.

Fix $k \geqslant 2$ and recall the open, dense subset:

$$\widetilde{\prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)} = \left\{(\mathbb{B}_1, ..., \mathbb{B}_k) \in \prod_1^k \mathrm{Gr}_5(\mathbb{R}^6) \;\middle|\; \text{for all } i \neq j\text{: } \mathbb{B}_i \neq \mathbb{B}_j\right\} \subset \prod_1^k \mathrm{Gr}_5(\mathbb{R}^6),$$

whose complement $\left(\prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)\right)_{sing}$ is a stratified submanifold of codimension 5 in the space $\prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)$. Define $\mathcal{G} \subset \widetilde{\prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)} \subset \prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)$ by:

$$\mathcal{G} = \left\{(\mathbb{B}_1, ..., \mathbb{B}_k) \in \prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6) \;\middle|\; \text{for all } i \neq j\text{: } \mathbb{B}_i \cap E_+ \neq \mathbb{B}_j \cap E_+ \text{ or } \mathbb{B}_i \cap E_- \neq \mathbb{B}_j \cap E_-\right\}.$$

Then $\mathcal{G}$ is precisely the preimage of $\mathrm{Gr}^{(k)}_{5,gen}\left(\mathbb{R}^6\right)$ under the quotient map:

$$\widetilde{\prod_1^k \mathrm{Gr}_5\left(\mathbb{R}^6\right)} \to \widetilde{\prod_1^k \mathrm{Gr}_5\left(\mathbb{R}^6\right)} \Big/ Sym_k \cong \mathrm{Gr}^{(k)}_5\left(\mathbb{R}^6\right).$$

Since the quotient is open and surjective, to prove Proposition 8.2.3 it suffices to prove that $\mathcal{G} \subset \prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)$ is open and dense, or equivalently that $\mathcal{G} \subset \prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6)$ is open and dense (since $\prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6) \subset \prod_1^k \mathrm{Gr}_5(\mathbb{R}^6)$ is also open and dense).

To this end, note that there is an inclusion:

$$\prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6) \Big\backslash \mathcal{G} \subset \left\{(\mathbb{B}_1, ..., \mathbb{B}_k) \in \prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6) \;\middle|\; \text{for some } i \neq j\text{: } \mathbb{B}_i \cap E_+ = \mathbb{B}_j \cap E_+\right\} = \mathcal{S}. \tag{8.2.4}$$

However $\mathcal{S}$ is a stratified submanifold of $\prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6)$ of codimension 2. Indeed, there is a $\mathrm{SL}(3;\mathbb{R})^2$-equivariant map:

$$\begin{aligned} \cap^+ : \mathrm{Gr}_{5,gen}(\mathbb{R}^6) &\longrightarrow \mathrm{Gr}_2(E_+) \\ \mathbb{B} &\longmapsto \mathbb{B} \cap E_+ \end{aligned}$$

which is submersive since $\mathrm{SL}(3;\mathbb{R})^2$ acts transitively on $\mathrm{Gr}_2(E_+)$. Taking the Cartesian product yields a submersion:

$$\prod_1^k \cap^+ : \prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6) \to \prod_1^k \mathrm{Gr}_2(E_+).$$

By definition:

$$\mathcal{S} = \left(\prod_1^k \cap^+\right)^{-1} \left(\prod_1^k \mathrm{Gr}_2(E_+)\right)_{sing}.$$

From §2.5.2, the set $\left(\prod_1^k \mathrm{Gr}_2(E_+)\right)_{sing} \subset \prod_1^k \mathrm{Gr}_2(E_+)$ is a stratified submanifold of codimension $\dim \mathrm{Gr}_2(E_+) = 2$. Using the Preimage Theorem (which applies equally well to stratified submanifolds; see e.g. [42, p. 17]) it follows that $\mathcal{S}$ is a stratified submanifold of codimension 2. The openness and density of $\mathcal{G}$ in $\prod_1^k \mathrm{Gr}_{5,gen}(\mathbb{R}^6)$ now follows from eqn. (8.2.4), completing the proof.

□

Definition 8.2.5. Let M be an oriented 6-manifold, fix $q \geqslant 0$ and let $a : D^q \to \Omega^3(\mathrm{M})$ be a continuous map. Define:

$$\mathscr{A} = \left\{ [(s,T),(s,\Xi)] \in \mathscr{R}_{\rho_+}(a) \times_{(D^q \times \mathrm{M})} \mathrm{Gr}_5^{(\infty)}(\mathrm{TM}_{D^q}) \;\middle|\; \Xi \in \mathrm{Gr}_{5,gen}^{(\infty)}(\mathrm{TM})_{\mathcal{D}(T)+a(s)} \right\}.$$

Proposition 8.2.6. *$\mathscr{A}$ is a pre-template for $\mathscr{R}_{\rho_+}(a)$. Moreover, for each $s \in D^q$, $x \in \mathrm{M}$ and $(s,T) \in \mathscr{R}_{\rho_+}(a)_{(s,x)}$:*

$$\mathscr{A}(s,T) \subset \mathrm{Gr}_5^{(\infty)}(\mathrm{T}_x\mathrm{M})$$

is a(n open and) dense subset.

Proof. It is clear that $\mathscr{A} \subset \mathscr{R}_{\rho_+}(a) \times_{(D^q \times \mathrm{M})} \mathrm{Gr}_5^{(\infty)}(\mathrm{TM}_{D^q})$ is open, since for $\rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^*$ and $\Xi \in \mathrm{Gr}_5^{(\infty)}(\mathbb{R}^6)$, the condition $\Xi \in \mathrm{Gr}_{5,gen}^{(\infty)}(\mathbb{R}^6)_\rho$ is open in both ρ and Ξ. Now fix $s \in D^q$ and $x \in \mathrm{M}$, consider $\Xi' \subseteq \Xi \in \mathrm{Gr}_5^{(\infty)}(\mathrm{T}_x\mathrm{M})$ and suppose $T \in \mathscr{A}(s,\Xi) \subseteq E_x^{(1)}$. Write $\rho = \mathcal{D}(T) + a(s)$. Then $\Xi \in \mathrm{Gr}_5^{(\infty)}(\mathrm{T}_x\mathrm{M})_{\rho,gen}$ and so since $\Xi' \subseteq \Xi$, it follows that $\Xi' \in \mathrm{Gr}_{5,gen}^{(\infty)}(\mathrm{T}_x\mathrm{M})_\rho$ and hence that $T \in \mathscr{A}(s,\Xi')$. Thus $\mathscr{A}(s,\Xi) \subseteq \mathscr{A}(s,\Xi')$ and hence $\mathscr{A}$ is a pre-template for $\mathscr{R}_{\rho_+}(a)$, as claimed. The final claim follows immediately from Proposition 8.2.3.

□

Note that the pre-template $\mathscr{A}$ has the form described in the discussion after Definition 2.5.9. Thus to prove that $\mathscr{A}$ is an avoidance template for $\mathscr{R}_{\rho_+}(a)$, and hence complete the proof of Theorem 8.2.1, it suffices to prove that for all $s \in D^q$, $x \in \mathrm{M}$, $\Xi \in \mathrm{Gr}_5^{(\infty)}(\mathrm{T}_x\mathrm{M})$, $\mathbb{B} \in \Xi$, $\lambda \in \mathrm{Hom}(\mathbb{B}, \bigwedge^2\mathrm{T}_x^*\mathrm{M})$ and $e \in \bigwedge^2\mathrm{T}_x^*\mathrm{M}$, the subset:

$$\mathscr{A}'(s,\Xi) \cap \Pi(\mathbb{B},\lambda) \subseteq \Pi(\mathbb{B},\lambda)$$

is ample. Fix $\mathbb{B} \in \Xi$, choose an orientation on $\mathbb{B}$, fix an oriented splitting $\mathrm{T}_x\mathrm{M} = \mathbb{L} \oplus \mathbb{B}$ and choose an oriented generator θ of the 1-dimensional oriented vector space $\mathrm{Ann}(\mathbb{B}) \subset \mathrm{T}_x^*\mathrm{M}$. Then there is an isomorphism:

$$\begin{aligned} \mathbb{B}^* \oplus \textstyle\bigwedge^2\mathbb{B}^* \oplus \left(\mathbb{B}^* \otimes \bigwedge^2\mathrm{T}_x^*\mathrm{M}\right) &\longleftrightarrow \mathrm{T}_x^*\mathrm{M} \otimes \textstyle\bigwedge^2\mathrm{T}_x^*\mathrm{M} \\ \alpha \oplus \nu \oplus \lambda &\longmapsto \theta \otimes (\theta \wedge \alpha + \nu) + \lambda. \end{aligned}$$

Using this identification:

$$\Pi(\mathbb{B},\lambda) \cong \mathbb{B}^* \times \bigwedge^2\mathbb{B}^* \times \{\lambda\}$$

and thus:

$$\mathscr{A}'(s,\Xi) \cap \Pi(\mathbb{B},\lambda) \cong \mathbb{B}^* \times \left\{ \nu \in \bigwedge^2\mathbb{B}^* \;\middle|\; \begin{array}{c} \theta \wedge \nu + \wedge(\lambda) + a(s)|_x \in \bigwedge^3_+ \mathrm{T}^*_x\mathrm{M} \text{ and} \\ \Xi \text{ is generic for } \theta \wedge \nu + \wedge(\lambda) + a(s)|_x \end{array} \right\} \times \{\lambda\}.$$

In particular, the amplitude of $\mathscr{A}'(s,\Xi) \cap \Pi(\mathbb{B},\lambda)$ depends only on $\wedge(\lambda)$ (for a fixed choice of a). Thus, writing $\tau = \wedge(\lambda) + a(s)|_x$, the task is to prove that for each $\tau \in \wedge^3\mathbb{B}^*$, the subset:

$$\mathcal{N}(\tau;\Xi,\mathbb{B}) = \left\{ \nu \in \bigwedge^2\mathbb{B}^* \;\middle|\; \theta \wedge \nu + \tau \in \bigwedge^3_+ \mathrm{T}^*_p\mathrm{M} \text{ and } \Xi \text{ is generic for } \theta \wedge \nu + \tau \right\} \subset \bigwedge^2\mathbb{B}^*$$

is ample. If this set is empty, the result is trivial, so without loss of generality one may assume that there exists $\nu_0 \in \wedge^2\mathbb{B}^*$ such that $\rho = \theta \wedge \nu_0 + \tau$ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form on $\mathrm{T}_p\mathrm{M}$ with respect to which Ξ is generic. Since $\mathcal{N}(\tau;\Xi,\mathbb{B}) = \mathcal{N}(\rho;\Xi,\mathbb{B}) + \nu_0$, one sees that to prove Theorem 8.2.1, it suffices to prove:

Proposition 8.2.7. *Let $\rho \in \wedge^3_+\left(\mathbb{R}^6\right)$ be an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form, let $\Xi \in \mathrm{Gr}_5^{(\infty)}(\mathbb{R}^6)$ be a generic configuration of hyperplanes with respect to ρ, let $\mathbb{B} \in \Xi$, choose an orientation on $\mathbb{B}$, fix an oriented splitting $\mathbb{R}^6 = \mathbb{L} \oplus \mathbb{B}$ and choose an oriented generator θ of the 1-dimensional oriented vector space $\mathrm{Ann}(\mathbb{B}) \subset \left(\mathbb{R}^6\right)^*$. Define:*

$$\mathcal{N}(\rho;\Xi,\mathbb{B}) = \left\{ \nu \in \bigwedge^2\mathbb{B}^* \;\middle|\; \theta \wedge \nu + \rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^* \textit{ and } \Xi \textit{ is generic for } \theta \wedge \nu + \rho \right\}.$$

Then $\mathcal{N}(\rho;\Xi,\mathbb{B}) \subset \wedge^2\mathbb{B}^*$ is ample.

I begin with an elementary lemma:

Lemma 8.2.8. *Let X be a connected topological space and let $Y \subset X$ have empty interior. Suppose that for every $y \in Y$, there exists an open neighbourhood U_y of y in X such that $U_y \backslash Y$ is connected. Then $X\backslash Y$ is connected.*

Proof. The proof is a simple exercise in point-set topology. Suppose that $A, B \subseteq X\backslash Y$ are open, disjoint subsets such that $X\backslash Y = A \cup B$. For each $y \in Y$, since $U_y\backslash Y$ is connected, it follows that either:

$$U_y\backslash Y \subseteq A \quad \text{ or } \quad U_y\backslash Y \subseteq B. \tag{8.2.9}$$

Thus define:

$$A' = A \cup \left\{ y \in Y \;\middle|\; \begin{array}{c} \text{there exists some open neighbourhood} \\ W_y \text{ of } y \text{ in } X \text{ such that } W_y\backslash Y \subseteq A \end{array} \right\} \tag{8.2.10}$$

and let B' be defined analogously. Then by eqn. (8.2.9), clearly $A' \cup B' = A \cup B \cup Y = X$. Next, note that $A' \subseteq X$ is open. Indeed, since $A \subseteq X\backslash Y$ is open, there exists an open subset $\mathcal{O} \subseteq X$ such that $A = \mathcal{O} \cap (X\backslash Y)$. Then clearly every $y \in \mathcal{O} \cap Y$ also lies in A' (just take $W_y = \mathcal{O}$) so $A \subseteq \mathcal{O} \subseteq A'$. Now let $y \in Y \cap A'$ and let W_y be as in eqn. (8.2.10). Then every $y' \in W_y \cap Y$ also lies in A' (just take

$W_{y'} = W_y$) and so $y \in W_y \subseteq A'$. Thus:

$$A' \subseteq \mathcal{O} \cup \bigcup_{y \in Y \cap A'} W_y \subseteq A',$$

hence equality holds, and whence A' is open. Similarly $B' \subseteq X$ is also open.

Now suppose there exists $y \in A' \cap B'$. Then clearly $y \in Y$ (since $A' \cap B' \cap (X \backslash Y) = A \cap B = \varnothing$). By definition, there exist neighbourhoods W_y and W'_y of y in X such that $W_y \backslash Y \subseteq A$ and $W'_y \backslash Y \subseteq B$. Then:

$$(W_y \cap W'_y) \cap (X \backslash Y) \subseteq A \cap B = \varnothing,$$

which contradicts the density of $X \backslash Y$ (since $W_y \cap W'_y$ is an open neighbourhood of y in X). Thus $A' \cap B' = \varnothing$. Since X is connected, it follows that one of A' and B' must be empty, and hence so must one of A and B.

□

Now let $\mathbb{A}$ be an affine space and $X \subseteq \mathbb{A}$ an open subset. I term a subset $Y \subset X$ macilent if it is closed and if, for every point $y \in Y$, there exists an open neighbourhood U_y of y in X and a submanifold $S_y \subset U_y$ of codimension at least 2 such that:

$$Y \cap U_y \subseteq S_y. \tag{8.2.11}$$

Say that a subset $Y \subset X$ is scarce if it is a finite union of macilent subspaces.

Lemma 8.2.12. *Let $X \subseteq \mathbb{A}$ be open and path-connected, and suppose that* $\mathrm{Conv}(X) = \mathbb{A}$. *Let $Y \subset X$ be scarce. Then $X \backslash Y$ is path-connected and* $\mathrm{Conv}(X \backslash Y) = \mathbb{A}$. *In particular, if $X' \subseteq \mathbb{A}$ is open and ample and $Y' \subset X'$ is scarce, then $X' \backslash Y'$ is open and ample.*

Remark 8.2.13. A related result concerning so-called 'thin' sets was stated without proof in [42, §18.1], however to the author's knowledge, the formulation used in this paper cannot be found in the literature.

Proof. Begin with the first statement. By writing Y as the union of n macilent subsets and inducting on n, without loss of generality assume that Y is macilent. Since S_y has codimension at least 2 in U_y, it follows that Y has empty interior in X and that $U_y \backslash S_y$ is connected for all $y \in Y$. But $U_y \backslash S_y$ is dense in U_y, hence certainly dense in $U_y \backslash Y$ and whence $U_y \backslash Y$ is also connected for all $y \in Y$. It follows from Lemma 8.2.8 that $X \backslash Y$ is connected. Since $X \backslash Y$ is open in X and X is open in $\mathbb{A}$, it follows that $X \backslash Y$ is also locally path-connected and hence it is path-connected, as claimed. To see that $\mathrm{Conv}(X \backslash Y) = \mathbb{A}$, note that for each $y \in Y$, by eqn. (8.2.11):

$$y \in \mathrm{Conv}(U_y \backslash Y) \subseteq \mathrm{Conv}(X \backslash Y)$$

and hence:

$$\mathrm{Conv}(X \backslash Y) = \mathrm{Conv}(X) = \mathbb{A},$$

as required. The final claim now follows by applying the above result to each path-component of X'. $\square$

Now return to Proposition 8.2.7. The proof of this result is broken into two stages. Firstly, define the larger set:

$$\mathcal{N}(\rho;\Xi,\mathbb{B})_1 = \left\{\nu \in \bigwedge^2\mathbb{B}^* \;\middle|\; \theta\wedge\nu + \rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*\right\} \subset \bigwedge^2\mathbb{B}^*.$$

Since Ξ is generic for ρ and $\mathbb{B} \in \Xi$, it follows that $\tau = \rho|_{\mathbb{B}}$ is a co-pseudoplectic form on $\mathbb{B}$. Noting that $\mathcal{N}(\rho;\Xi,\mathbb{B})_1$ is just a translated copy of $\mathcal{N}_{\rho_+}(\tau)$, by Proposition 8.1.2 it follows that $\mathcal{N}(\rho;\Xi,\mathbb{B})_1 \subset \bigwedge^2\mathbb{B}^*$ is ample.

Next, for each $\mathbb{B}' \in \Xi$, define a closed subset $\Sigma_{\mathbb{B}'} \subset \mathcal{N}(\rho;\Xi,\mathbb{B})_1$ by:

$$\Sigma_{\mathbb{B}'} = \{\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_1 \mid \mathbb{B}' \text{ is not generic for } \theta\wedge\nu+\rho\}$$

and define:

$$\mathcal{N}(\rho;\Xi,\mathbb{B})_2 = \mathcal{N}(\rho;\Xi,\mathbb{B})_1 \Big\backslash \bigcup_{\mathbb{B}'\in\Xi} \Sigma_{\mathbb{B}'}.$$

Explicitly:

$$\mathcal{N}(\rho;\Xi,\mathbb{B})_2 = \left\{\nu \in \bigwedge^2\mathbb{B}^* \;\middle|\; \theta\wedge\nu+\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^* \text{ and every } \mathbb{B}'\in\Xi \text{ is generic for } \theta\wedge\nu+\rho\right\}.$$

Finally, for each pair $\{\mathbb{B}',\mathbb{B}''\} \subseteq \Xi$ define a closed subset $\Sigma_{\{\mathbb{B}',\mathbb{B}''\}} \subset \mathcal{N}(\rho;\Xi,\mathbb{B})_2$ by:

$$\Sigma_{\{\mathbb{B}',\mathbb{B}''\}} = \left\{\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_2 \;\middle|\; \mathbb{B}'\cap E_{\pm,\theta\wedge\nu+\rho} = \mathbb{B}''\cap E_{\pm,\theta\wedge\nu+\rho}\right\}.$$

Then:

$$\mathcal{N}(\rho;\Xi,\mathbb{B}) = \mathcal{N}(\rho;\Xi,\mathbb{B})_2 \Big\backslash \bigcup_{\{\mathbb{B}',\mathbb{B}''\}\subseteq\Xi} \Sigma_{\{\mathbb{B}',\mathbb{B}''\}}.$$

By applying Lemma 8.2.12 twice, to prove Proposition 8.2.7, it suffices to prove the following two lemmas:

Lemma 8.2.14. *For all $\mathbb{B}' \in \Xi$, the subset $\Sigma_{\mathbb{B}'} \subset \mathcal{N}(\rho;\Xi,\mathbb{B})_1$ is scarce.*

Lemma 8.2.15. *For all $\{\mathbb{B}',\mathbb{B}''\} \subseteq \Xi$, the subset $\Sigma_{\{\mathbb{B}',\mathbb{B}''\}} \subset \mathcal{N}(\rho;\Xi,\mathbb{B})_2$ is scarce.*

8.3 A preparatory result: computing the derivatives of the maps $E_\pm$

Given $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$, recall that there is a decomposition $\mathbb{R}^6 = E_{+,\rho}\oplus E_{-,\rho}$. Thus, there is also a decomposition:

$$\bigwedge^p\left(\mathbb{R}^6\right)^* \cong \bigoplus_{r+s=p} \bigwedge^r E^*_{+,\rho}\otimes\bigwedge^s E^*_{-,\rho} = \bigoplus_{r+s=p}\bigwedge^{r,s}\left(\mathbb{R}^6\right)^*.$$

Define $\mathrm{SL}(3;\mathbb{R})^2$-equivariant isomorphisms $\kappa_\rho^+ : \bigwedge^{2,0}\left(\mathbb{R}^6\right)^* \to E_{+,\rho}$ and $\kappa_\rho^- : \bigwedge^{0,2}\left(\mathbb{R}^6\right)^* \to E_{-,\rho}$ as the inverses to the maps:

$$\begin{aligned} E_{+,\rho} &\longrightarrow \textstyle\bigwedge^{2,0}\left(\mathbb{R}^6\right)^* \\ w &\longmapsto w \lrcorner \left(\rho|_{E_{+,\rho}}\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} E_{-,\rho} &\longrightarrow \textstyle\bigwedge^{0,2}\left(\mathbb{R}^6\right)^* \\ w &\longmapsto w \lrcorner \left(\rho|_{E_{-,\rho}}\right) \end{aligned} \qquad \text{respectively.}$$

Proposition 8.3.1. *Consider the smooth maps:*

$$\begin{aligned} E_\pm : \textstyle\bigwedge^3_+\left(\mathbb{R}^6\right)^* &\longrightarrow \mathrm{Gr}_3\left(\mathbb{R}^6\right) \\ \rho &\longmapsto E_{\pm,\rho}. \end{aligned}$$

Fix $\rho \in \bigwedge^3_+\left(\mathbb{R}^6\right)^*$. *Then:*

$$\begin{aligned} \mathcal{D}E_+|_\rho : \textstyle\bigwedge^3\left(\mathbb{R}^6\right)^* &\longrightarrow \left(E_{+,\rho}\right)^* \otimes E_{-,\rho} \cong \mathrm{Hom}(E_{+,\rho}, E_{-,\rho}) \\ \alpha &\longmapsto -(\mathrm{Id} \otimes \kappa_\rho^-)(\pi_{1,2}(\alpha)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}E_-|_\rho : \textstyle\bigwedge^3\left(\mathbb{R}^6\right)^* &\longrightarrow E_{+,\rho} \otimes \left(E_{-,\rho}\right)^* \cong \mathrm{Hom}(E_{-,\rho}, E_{+,\rho}) \\ \alpha &\longmapsto (\kappa_\rho^+ \otimes \mathrm{Id})(\pi_{2,1}(\alpha)) \end{aligned}$$

respectively, where $\pi_{r,s}$ *denotes the projection onto forms of type* (r,s).

Proof. Start with the first statement. Since $\bigwedge^3_+\left(\mathbb{R}^6\right)^* \subset \bigwedge^3\left(\mathbb{R}^6\right)^*$ is open, one has $\mathrm{T}_\rho \bigwedge^3_+\left(\mathbb{R}^6\right)^* = \bigwedge^3\left(\mathbb{R}^6\right)^*$. Likewise, the decomposition $\mathbb{R}^6 = \mathbb{E}_{+,\rho} \oplus E_{-,\rho}$ yields $\mathrm{T}_{E_{+,\rho}}\mathrm{Gr}_3\left(\mathbb{R}^6\right) \cong \mathrm{Hom}(E_{+,\rho}, E_{-,\rho})$. Since the only simple $\mathrm{SL}(3;\mathbb{R})^2$-submodule of $\bigwedge^3\left(\mathbb{R}^6\right)^*$ which is isomorphic to $\mathrm{Hom}(E_{+,\rho}, E_{-,\rho}) \cong \left(E_{+,\rho}\right)^* \otimes E_{-,\rho}$ is $\bigwedge^{1,2}\left(\mathbb{R}^6\right)^*$, it follows that:

$$\mathcal{D}E_+|_\rho(\alpha) = C\,\mathrm{Id} \otimes \kappa_\rho^-(\pi_{1,2}(\alpha))$$

for some constant C.

The value of C may be computed directly. Consider $\rho = \rho_+ = \theta^{123} + \theta^{456}$ and write:

$$\rho_t = \rho_+ + t\theta^{145}.$$

A direct calculation shows that:

$$E_{+,\rho_t} = \langle e_1 - te_6, e_2, e_3 \rangle$$

so that:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{+,\rho_t}\bigg|_{t=0} = -\theta^1 \otimes e_6.$$

By comparison:

$$(\mathrm{Id} \otimes \kappa_{\rho_+}^-)(\pi_{1,2}(\theta^{145})) = \theta^1 \otimes e_6,$$

forcing $C = -1$, as claimed. The calculation of $\mathcal{D}E_-|_\rho$ is similar.

$\square$

8.4 Lemma 8.2.14: the scarcity of $\Sigma_{\mathbb{B}'}$

Recall the subsets:

$$\mathcal{N}(\rho;\Xi,\mathbb{B})_1 = \left\{\nu \in \textstyle\bigwedge^2\mathbb{B}^* \;\middle|\; \theta\wedge\nu+\rho \in \textstyle\bigwedge^3_+\left(\mathbb{R}^6\right)^*\right\} \subset \textstyle\bigwedge^2\mathbb{B}^*$$

and:

$$\Sigma_{\mathbb{B}'} = \{\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_1 \mid \mathbb{B}' \text{ is not generic for } \theta\wedge\nu+\rho\}.$$

Lemma 8.4.1.

$$\Sigma_{\mathbb{B}} = \varnothing.$$

Proof. Indeed, let $\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_1$, i.e. suppose that $\theta\wedge\nu+\rho$ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form. Then:

$$(\theta\wedge\nu+\rho)|_{\mathbb{B}} = \rho|_{\mathbb{B}}.$$

Since $\mathbb{B}$ is generic for ρ, $\rho|_{\mathbb{B}}$ is a co-pseudoplectic 3-form and thus $\mathbb{B}$ must also be generic for $\theta\wedge\nu+\rho$ (else $(\theta\wedge\nu+\rho)|_{\mathbb{B}}$ would be decomposable).

$\square$

Remark 8.4.2. The above proof also shows that if $\mathbb{B}$ is non-generic for ρ (equivalently if $\rho|_{\mathbb{B}}$ is decomposable) then it is also non-generic for all $\theta\wedge\nu+\rho$. At first sight, this result seems surprising, since one expects non-genericity to be destroyed by pertubations. On closer examination, however, the result is less surprising, since the space of perturbations of ρ of the form $\theta\wedge\nu+\rho$ is $\binom{5}{2} = 10$-dimensional, whereas the space of all perturbations of ρ is instead $\binom{6}{3} = 20$-dimensional.

Lemma 8.4.3. *Let $\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_1$ and write:*

$$\rho' = \theta\wedge\nu+\rho \in \textstyle\bigwedge^3_+\left(\mathbb{R}^6\right)^*.$$

Then:

$$(\mathbb{B}\cap E_{+,\rho}) \oplus (\mathbb{B}\cap E_{-,\rho}) = (\mathbb{B}\cap E_{+,\rho'}) \oplus (\mathbb{B}\cap E_{-,\rho'}).$$

Proof. By applying a suitable orientation-preserving automorphism of $\mathbb{R}^6$ one can always assume that:

$$\rho = \theta^{123}+\theta^{456} \quad \text{and} \quad \mathbb{B} = \langle e_1, e_2, e_4, e_5, e_3+e_6\rangle.$$

Hence:

$$(\mathbb{B}\cap E_{+,\rho}) \oplus (\mathbb{B}\cap E_{-,\rho}) = \langle e_1, e_2\rangle \oplus \langle e_4, e_5\rangle = \langle e_1, e_2, e_4, e_5\rangle. \tag{8.4.4}$$

Now take $\mathbb{L} = \langle e_3 - e_6\rangle$, $\theta = \theta^3 - \theta^6$ and write:

$$\rho' = \theta^{123}+\theta^{456}+(\theta^3-\theta^6)\wedge\nu.$$

Write $I_{\rho'}$ for the para-complex structure induced by ρ'.

Claim 8.4.5.
$$I_{\rho'}\left(\langle e_1, e_2, e_4, e_5\rangle\right) \subseteq \langle e_1, e_2, e_4, e_5\rangle.$$

Proof of Claim. Recall the map:

$$\begin{aligned} \mathfrak{i}_{\rho'} : \mathbb{R}^6 &\longrightarrow \Lambda^5\left(\mathbb{R}^6\right)^* \\ v &\longmapsto (v \lrcorner \rho') \wedge \rho'. \end{aligned}$$

Then, by the definition of $I_{\rho'}$, it is equivalent to prove that:

$$\mathfrak{i}_{\rho'}\left(\langle e_1, e_2, e_4, e_5\rangle\right) \subseteq \theta^{36} \wedge \bigwedge^3 \left(\mathbb{R}^6\right)^*.$$

Consider the subgroup $\mathrm{SL}(2;\mathbb{R})^2 \subset \mathrm{SL}(3;\mathbb{R})^2$ acting block diagonally on $\langle e_1, e_2\rangle \oplus \langle e_4, e_5\rangle$ and trivially on $\langle e_3, e_6\rangle$. Clearly $\mathrm{SL}(2;\mathbb{R})^2$ preserves ρ, $\mathbb{B}$, $\mathbb{L}$ and θ as described above, and acts transitively on the set of non-zero vectors in both $\langle e_1, e_2\rangle$ and $\langle e_4, e_5\rangle$. By exploiting this freedom, it suffices to prove:

$$\mathfrak{i}_{\rho'}(e_1), \mathfrak{i}_{\rho'}(e_4) \in \theta^{36} \wedge \bigwedge^3 \left(\mathbb{R}^6\right)^*.$$

However, a direct calculation shows that:

$$\begin{aligned} (e_1 \lrcorner \rho') \wedge \rho' &= \left(\theta^{23} - \theta^3 \wedge (e_1 \lrcorner \nu) + \theta^6 \wedge (e_1 \lrcorner \nu)\right) \wedge \left(\theta^{123} + \theta^{456} + (\theta^3 - \theta^6) \wedge \nu\right) \\ &= \left(\theta^{245} - \theta^2 \wedge \nu + \theta^{12} \wedge (e_1 \lrcorner \nu) + \theta^{45} \wedge (e_1 \lrcorner \nu)\right) \wedge \theta^{36} \end{aligned}$$

while:
$$\begin{aligned} (e_4 \lrcorner \rho') \wedge \rho' &= \left(\theta^{56} - \theta^3 \wedge (e_4 \lrcorner \nu) + \theta^6 \wedge (e_4 \lrcorner \nu)\right) \wedge \left(\theta^{123} + \theta^{456} + (\theta^3 - \theta^6) \wedge \nu\right) \\ &= \left(-\theta^{125} - \theta^5 \wedge \nu + \theta^{45} \wedge (e_4 \lrcorner \nu) + \theta^{12} \wedge (e_4 \lrcorner \nu)\right) \wedge \theta^{36}, \end{aligned}$$

as required.

□

Using the claim, $\left(I_{\rho'}|_{\langle e_1 e_2, e_4, e_5\rangle}\right)^2 = \mathrm{Id}$ and thus:

$$\langle e_1, e_2, e_4, e_5\rangle = e_+ \oplus e_-$$

where $e_\pm$ are the ±1-eigenspaces of $I_{\rho'}|_{\langle e_1, e_2, e_4, e_5\rangle}$. Since $\langle e_1, e_2, e_4, e_5\rangle \subset \mathbb{B}$, it follows that $e_\pm \subseteq \mathbb{B} \cap E_{\pm,\rho'}$ and hence:
$$\langle e_1, e_2, e_4, e_5\rangle = e_+ \oplus e_- \subseteq \left(\mathbb{B} \cap E_{+,\rho'}\right) \oplus \left(\mathbb{B} \cap E_{-,\rho'}\right).$$

However $\mathbb{B}$ is generic for ρ' by Lemma 8.4.1 and hence:

$$\dim\left[\left(\mathbb{B} \cap E_{+,\rho'}\right) \oplus \left(\mathbb{B} \cap E_{-,\rho'}\right)\right] = 4.$$

Therefore (see eqn. (8.4.4)):

$$(\mathbb{B} \cap E_{+,\rho'}) \oplus (\mathbb{B} \cap E_{-,\rho'}) = \langle e_1, e_2, e_4, e_5 \rangle = (\mathbb{B} \cap E_{+,\rho}) \oplus (\mathbb{B} \cap E_{-,\rho}),$$

as required. $\square$

Lemma 8.4.6. *Let $\nu \in \mathcal{N}(\rho; \Xi, \mathbb{B})_1$ and write $\rho' = \theta \wedge \nu + \rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^*$. Suppose a hyperplane $\mathbb{B}' \neq \mathbb{B}$ satisfies:*

$$\mathbb{B} \cap E_{+,\rho'} \subseteq \mathbb{B}' \cap E_{+,\rho'} \quad \textit{and} \quad \mathbb{B} \cap E_{-,\rho'} \subseteq \mathbb{B}' \cap E_{-,\rho'}. \tag{8.4.7}$$

Then eqn. (8.4.7) also holds with respect to ρ, i.e.:

$$\mathbb{B} \cap E_{+,\rho} \subseteq \mathbb{B}' \cap E_{+,\rho} \quad \textit{and} \quad \mathbb{B} \cap E_{-,\rho} \subseteq \mathbb{B}' \cap E_{-,\rho}. \tag{8.4.8}$$

In particular, $\{\mathbb{B}, \mathbb{B}'\}$ is non-generic for ρ.

Proof. Firstly, note that:

$$\begin{aligned}
\mathbb{B} \cap E_{\pm,\rho} &= \left[(\mathbb{B} \cap E_{+,\rho}) \oplus (\mathbb{B} \cap E_{-,\rho})\right] \cap E_{\pm,\rho} \\
&= \left[(\mathbb{B} \cap E_{+,\rho'}) \oplus (\mathbb{B} \cap E_{-,\rho'})\right] \cap E_{\pm,\rho} \quad \text{by Lemma 8.4.3} \\
&\subseteq \left[(\mathbb{B}' \cap E_{+,\rho'}) \oplus (\mathbb{B}' \cap E_{-,\rho'})\right] \cap E_{\pm,\rho} \quad \text{by eqn. (8.4.7)} \\
&\subseteq \mathbb{B}' \cap E_{\pm,\rho},
\end{aligned}$$

as required. For the final statement, note that either $\mathbb{B}'$ is non-generic for ρ, or else $\dim(\mathbb{B}' \cap E_{+,\rho}) = \dim(\mathbb{B}' \cap E_{-,\rho}) = 2$ together with eqn. (8.4.8) forces:

$$\mathbb{B} \cap E_{+,\rho} = \mathbb{B}' \cap E_{+,\rho} \quad \text{and} \quad \mathbb{B} \cap E_{-,\rho} = \mathbb{B}' \cap E_{-,\rho}.$$

$\square$

Remark 8.4.9. If both $\mathbb{B}$ and $\mathbb{B}'$ are individually generic for ρ, it is clear that $\{\mathbb{B}, \mathbb{B}'\}$ is non-generic for ρ if and only if eqn. (8.4.8) is satisfied.

I now prove Lemma 8.2.14. Recall the statement of the lemma:

Lemma 8.2.14. *For all $\mathbb{B}' \in \Xi$, the subset $\Sigma_{\mathbb{B}'} \subset \mathcal{N}(\rho; \Xi, \mathbb{B})_1$ is scarce. More precisely, it is either empty, or the union of two closed submanifolds, each of codimension 3.*

Proof. By Lemma 8.4.1, it suffices to consider $\mathbb{B}' \neq \mathbb{B}$. Consider the maps:

$$\begin{aligned}
\mathbb{E}_\pm : \mathcal{N}(\rho; \Xi, \mathbb{B})_1 &\longrightarrow \mathrm{Gr}_3\left(\mathbb{R}^6\right) \\
\nu &\longmapsto E_{\pm,\theta\wedge\nu+\rho}.
\end{aligned}$$

(Note that, unlike the maps $E_\pm$, the arguments of the maps $\mathbb{E}_\pm$ are 2-forms, and not $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms.) Consider the submanifold $\mathrm{Gr}_3(\mathbb{B}') \subset \mathrm{Gr}_3\left(\mathbb{R}^6\right)$ and recall that $\mathbb{B}'$ is non-generic for $\theta \wedge \nu + \rho$ if and only if either $\mathbb{E}_+(\nu)$ or $\mathbb{E}_-(\nu)$ lies in $\mathrm{Gr}_3(\mathbb{B}')$. Thus:

$$\Sigma_{\mathbb{B}'} = \left[(\mathbb{E}_+)^{-1}\,\mathrm{Gr}_3(\mathbb{B}')\right] \cup \left[(\mathbb{E}_-)^{-1}\,\mathrm{Gr}_3(\mathbb{B}')\right].$$

Claim 8.4.10. *The maps $\mathbb{E}_\pm$ are transverse to the submanifold $\mathrm{Gr}_3(\mathbb{B}')$.*

Proof. I consider $\mathbb{E}_+$, the case of $\mathbb{E}_-$ being essentially identical. Suppose that $\nu \in \mathcal{N}(\rho;\Xi,\mathbb{B})_1$ satisfies $\mathbb{E}_+(\nu) \in \mathrm{Gr}_3(\mathbb{B}')$. Write $\rho' = \theta \wedge \nu + \rho$ and after applying a suitable orientation-preserving automorphism of $\mathbb{R}^6$, one may assume that:

- $\rho' = \theta^{123} + \theta^{456}$;
- $\mathbb{B}' = \langle e_1, e_2, e_3, e_4, e_5 \rangle$.

(Note that there is a residual $\mathrm{SL}(3;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$ freedom in choosing such an automorphism, acting diagonally on $\langle e_1, e_2, e_3\rangle \oplus \langle e_4, e_5\rangle$ and trivially on $\langle e_6\rangle$, a fact which will be exploited below.) Then one may identify $\mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3(\mathbb{B}') \cong \mathrm{Hom}(\langle e_1,e_2,e_3\rangle, \langle e_4, e_5\rangle)$ and moreover:

$$\begin{aligned}\mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3\left(\mathbb{R}^6\right)\Big/\mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3(\mathbb{B}') &\cong \mathrm{Hom}(\langle e_1,e_2,e_3\rangle, \langle e_4,e_5,e_6\rangle)\Big/\mathrm{Hom}(\langle e_1,e_2,e_3\rangle, \langle e_4,e_5\rangle)\\ &\cong \mathrm{Hom}(\langle e_1,e_2,e_3\rangle, \langle e_6\rangle).\end{aligned}$$

Next recall that $\mathrm{Ann}(\mathbb{B}) = \langle \theta \rangle$ and write:

$$\theta = \sum_{i=1}^{6} \lambda_i \theta^i = \sum_{i=1}^{3} \lambda_i \theta^i + \sum_{i=4}^{5} \lambda_i \theta^i + \lambda_6 \theta^6.$$

By exploiting the residual $\mathrm{SL}(3;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$ freedom described above, without loss of generality one can assume that:

$$\theta = \lambda_1\theta^1 + \lambda_4\theta^4 + \lambda_6\theta^6.$$

I claim that $\lambda_4 \neq 0$. Indeed suppose $\theta = \lambda_1\theta^1 + \lambda_6\theta^6$. If $\lambda_6 = 0$, then $E_{-,\rho'} = \langle e_4, e_5, e_6\rangle \subset \mathrm{Ker}(\theta) = \mathbb{B}$, hence $\mathbb{B}$ is non-generic for ρ' and whence $\nu \in \Sigma_{\mathbb{B}}$, contradicting Lemma 8.4.1. Thus $\lambda_6 \neq 0$ and:

$$\mathbb{B} \cap E_{-,\rho'} = \langle e_4, e_5 \rangle = \mathbb{B}' \cap E_{-,\rho'}.$$

However, since $E_{+,\rho'} \subset \mathbb{B}'$, one trivially has that $\mathbb{B} \cap E_{+,\rho'} \subset \mathbb{B}' \cap E_{+,\rho'}$. Thus using Lemma 8.4.6, the pair $\{\mathbb{B}, \mathbb{B}'\} \subseteq \Xi$ is not generic for ρ, which contradicts the assumption that Ξ is generic for ρ. Thus $\lambda_4 \neq 0$, as claimed.

Finally, note that $\mathrm{T}_\nu\mathcal{N}(\rho;\Xi,\mathbb{B})_1 = \bigwedge^2\mathbb{B}^*$ since $\mathcal{N}(\rho;\Xi,\mathbb{B})_1 \subset \bigwedge^2\mathbb{B}^*$ is open (since $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms are stable). Choose $\nu_i \in \bigwedge^2\mathbb{B}^*$ for $i = 1,2,3$ such that:

$$\theta \wedge \nu_i = \theta \wedge \theta^{i5}.$$

(Such ν_i exists, since $(\theta \wedge \theta^{i5})|_{\mathbb{B}} = 0$.) Then:

$$\begin{aligned}\mathcal{D}E_+|_{\rho'}(\nu_i) &= -\operatorname{Id} \otimes \kappa^-_{\rho'}(\pi_{1,2}(\theta \wedge \theta^{i5}))\\ &= \lambda_4 \theta^i \otimes e_6 - \lambda_6 \theta^i \otimes e_4\end{aligned}$$

which projects to the element $\lambda_4 \theta^i \otimes e_6$ in $\operatorname{Hom}(\langle e_1, e_2, e_3 \rangle, \langle e_6 \rangle) \cong \mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3\left(\mathbb{R}^6\right) \big/ \mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3(\mathbb{B}')$. Since $\lambda_4 \neq 0$, this proves the surjectivity of the composite:

$$\textstyle\bigwedge^2 \mathbb{B}^* \xrightarrow{\mathcal{D}\mathbb{E}_+|_\nu} \mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3\left(\mathbb{R}^6\right) \longrightarrow \mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3\left(\mathbb{R}^6\right) \big/ \mathrm{T}_{\mathbb{E}_+(\nu)}\mathrm{Gr}_3(\mathbb{B}').$$

Thus $\mathbb{E}_+$ is transverse to $\mathrm{Gr}_3(\mathbb{B}')$.

□

Resuming the main proof, since $\mathrm{Gr}_3(\mathbb{B}')$ is closed and has codimension $9 - 6 = 3$ in $\mathrm{Gr}_3\left(\mathbb{R}^6\right)$, by Claim 8.4.10 it follows that the submanifolds $(\mathbb{E}_+)^{-1}\mathrm{Gr}_3(\mathbb{B}')$ and $(\mathbb{E}_-)^{-1}\mathrm{Gr}_3(\mathbb{B}')$ of $\mathcal{N}(\rho; \Xi, \mathbb{B})_1$ are closed and each have codimension 3, and hence:

$$\Sigma_{\mathbb{B}'} = (\mathbb{E}_+)^{-1}\mathrm{Gr}_3(\mathbb{B}') \cup (\mathbb{E}_-)^{-1}\mathrm{Gr}_3(\mathbb{B}')$$

is scarce. This completes the proof.

□

8.5 Lemma 8.2.15: the scarcity of $\Sigma_{\{\mathbb{B}', \mathbb{B}''\}}$

Recall the set:

$$\mathcal{N}(\rho; \Xi, \mathbb{B})_2 = \left\{ \nu \in \textstyle\bigwedge^2 \mathbb{B}^* \;\middle|\; \theta \wedge \nu + \rho \in \textstyle\bigwedge^3_+ \left(\mathbb{R}^6\right)^* \text{ and every } \mathbb{B}' \in \Xi \text{ is generic for } \theta \wedge \nu + \rho \right\}.$$

For each $\{\mathbb{B}', \mathbb{B}''\} \subseteq \Xi$, recall further the subset $\Sigma_{\{\mathbb{B}', \mathbb{B}''\}} \subset \mathcal{N}(\rho; \Xi, \mathbb{B})_2$ defined by:

$$\Sigma_{\{\mathbb{B}', \mathbb{B}''\}} = \left\{ \nu \in \mathcal{N}(\rho; \Xi, \mathbb{B})_2 \;\middle|\; \mathbb{B}' \cap E_{\pm, \theta\wedge\nu+\rho} = \mathbb{B}'' \cap E_{\pm, \theta\wedge\nu+\rho} \right\}.$$

Lemma 8.5.1. *For all* $\{\mathbb{B}, \mathbb{B}'\} \subseteq \Xi$*:*

$$\Sigma_{\{\mathbb{B}, \mathbb{B}'\}} = \varnothing.$$

Proof. Suppose $\nu \in \Sigma_{\{\mathbb{B}, \mathbb{B}'\}}$ and write $\rho' = \theta \wedge \nu + \rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^*$. Then:

$$\mathbb{B} \cap E_{\pm, \rho'} = \mathbb{B}' \cap E_{\pm, \rho'}.$$

Applying Lemma 8.4.6, it follows that $\{\mathbb{B}, \mathbb{B}'\} \subseteq \Xi$ is not generic for ρ, contradicting the fact that Ξ is generic for ρ. Thus $\Sigma_{\{\mathbb{B}, \mathbb{B}'\}} = \varnothing$ for all $\{\mathbb{B}, \mathbb{B}'\} \subseteq \Xi$.

□

Now suppose that $\mathbb{B}' \neq \mathbb{B} \neq \mathbb{B}''$. Define three new closed subsets of $\Sigma_{\{\mathbb{B}',\mathbb{B}''\}}$ by:

$$\begin{aligned}
\Sigma^+_{\mathbb{B}'} &= \left\{\nu \in \Sigma_{\{\mathbb{B}',\mathbb{B}''\}} \,\middle|\, \mathbb{B}' \cap E_{+,\theta\wedge\nu+\rho} = \mathbb{B} \cap E_{+,\theta\wedge\nu+\rho}\right\} \\
\Sigma^-_{\mathbb{B}'} &= \left\{\nu \in \Sigma_{\{\mathbb{B}',\mathbb{B}''\}} \,\middle|\, \mathbb{B}' \cap E_{-,\theta\wedge\nu+\rho} = \mathbb{B} \cap E_{-,\theta\wedge\nu+\rho}\right\} \\
\Sigma'_{\{\mathbb{B}',\mathbb{B}''\}} &= \left\{\nu \in \Sigma_{\{\mathbb{B}',\mathbb{B}''\}} \,\middle|\, \mathbb{B}' \cap E_{\pm,\theta\wedge\nu+\rho} \neq \mathbb{B} \cap E_{\pm,\theta\wedge\nu+\rho}\right\}.
\end{aligned}$$

Then clearly:

$$\Sigma_{\{\mathbb{B}',\mathbb{B}''\}} = \Sigma^+_{\mathbb{B}'} \cup \Sigma^-_{\mathbb{B}'} \cup \Sigma'_{\{\mathbb{B}',\mathbb{B}''\}}. \tag{8.5.2}$$

Lemma 8.5.3. *Let $\{\mathbb{B}',\mathbb{B}''\} \subseteq \Xi$ satisfy $\mathbb{B}' \neq \mathbb{B} \neq \mathbb{B}''$. Then $\Sigma^\pm_{\mathbb{B}'} \subset \mathcal{N}(\rho;\Xi,\mathbb{B})_2$ are both macilent.*

Proof. Write $\mathfrak{C} = \mathbb{B} \cap \mathbb{B}'$, a 4-dimensional subspace of $\mathbb{R}^6$ (since $\mathbb{B} \neq \mathbb{B}'$). Using $\mathfrak{C}$, one may stratify the manifold $\mathrm{Gr}_3\left(\mathbb{R}^6\right)$ as:

$$\mathrm{Gr}_3\left(\mathbb{R}^6\right) = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$$

where:

$$\Sigma_i = \left\{E \in \mathrm{Gr}_3\left(\mathbb{R}^6\right) \mid \dim(\mathfrak{C} \cap E) = i\right\}.$$

Explicitly, Σ_1 is the open and dense subset of 3-planes intersecting $\mathfrak{C}$ transversally, while $\Sigma_3 = \mathrm{Gr}_3(\mathfrak{C})$. To understand the submanifold structure on Σ_2, it is useful to describe its tangent space as a subspace of the tangent space of $\mathrm{Gr}_3\left(\mathbb{R}^6\right)$. Specifically, fix $E \in \Sigma_2$ and write $\mathfrak{E} = E \cap \mathfrak{C}$. Choose splittings:

$$E = \mathfrak{E}^2 \oplus \mathfrak{L}^1, \qquad \mathfrak{C} = \mathfrak{E}^2 \oplus \mathfrak{F}^2 \qquad \text{and} \qquad \mathbb{R}^6 = \mathfrak{E}^2 \oplus \mathfrak{L}^1 \oplus \mathfrak{F}^2 \oplus \mathfrak{K}^1, \tag{8.5.4}$$

where the superscripts denote the dimension of the respective subspaces. Then, $\mathrm{T}_E\mathrm{Gr}_3\left(\mathbb{R}^6\right)$ may be identified with the space:

$$\mathrm{Hom}(\mathfrak{E} \oplus \mathfrak{L}, \mathfrak{F} \oplus \mathfrak{K}) \cong \mathrm{Hom}(\mathfrak{E},\mathfrak{F}) \oplus \mathrm{Hom}(\mathfrak{E},\mathfrak{K}) \oplus \mathrm{Hom}(\mathfrak{L},\mathfrak{F}) \oplus \mathrm{Hom}(\mathfrak{L},\mathfrak{K}).$$

Using this description, $\mathrm{T}_E\Sigma_2$ is given by the subspace:

$$\mathrm{T}_E\Sigma_2 = \mathrm{Hom}(\mathfrak{E},\mathfrak{F}) \oplus \mathrm{Hom}(\mathfrak{L},\mathfrak{F}) \oplus \mathrm{Hom}(\mathfrak{L},\mathfrak{K}),$$

and hence $\mathrm{T}_E\mathrm{Gr}_3\left(\mathbb{R}^6\right)\big/\mathrm{T}_E\Sigma_2$ may be identified with:

$$\mathrm{T}_E\mathrm{Gr}_3\left(\mathbb{R}^6\right)\big/\mathrm{T}_E\Sigma_2 \cong \mathrm{Hom}(\mathfrak{E},\mathfrak{K}).$$

In particular, the codimension of Σ_2 in $\mathrm{Gr}_3\left(\mathbb{R}^6\right)$ is $\dim\mathrm{Hom}(\mathfrak{E},\mathfrak{K}) = 2$.

Now consider the smooth maps:

$$\begin{aligned}
\mathbb{E}_\pm : \mathcal{N}(\rho;\Xi,\mathbb{B})_2 &\longrightarrow \mathrm{Gr}_3\left(\mathbb{R}^6\right) \\
\nu &\longmapsto E_{\pm,\theta\wedge\nu+\rho}.
\end{aligned}$$

Since $\mathfrak{C} = \mathbb{B} \cap \mathbb{B}'$, one has:

$$\mathbb{E}_+(\nu) \cap \mathfrak{C} = (\mathbb{E}_+(\nu) \cap \mathbb{B}) \cap (\mathbb{E}_+(\nu) \cap \mathbb{B}').$$

Since both $\mathbb{E}_+(\nu) \cap \mathbb{B}$ and $\mathbb{E}_+(\nu) \cap \mathbb{B}'$ are 2-dimensional, it follows that $\dim \mathbb{E}_+(\nu) \cap \mathfrak{C} \leqslant 2$, with equality if and only if $\mathbb{E}_+(\nu) \cap \mathbb{B} = \mathbb{E}_+(\nu) \cap \mathbb{B}'$. Thus $\mathbb{E}_+ (\mathcal{N}(\rho; \Xi, \mathbb{B})_2) \subseteq \Sigma_1 \cup \Sigma_2$ and:

$$\Sigma^+_{\mathbb{B}'} \subseteq (\mathbb{E}_+)^{-1} (\Sigma_2).$$

Likewise $\Sigma^-_{\mathbb{B}'} \subseteq (\mathbb{E}_-)^{-1} (\Sigma_2)$. Therefore (since $\Sigma^\pm_{\mathbb{B}'}$ are both closed) to prove that $\Sigma^\pm_{\mathbb{B}'}$ are macilent, it suffices to prove that for all $\nu \in \Sigma^\pm_{\mathbb{B}'}$, the maps $\mathbb{E}_\pm$ respectively are transversal to the submanifold $\Sigma_2 \subset \mathrm{Gr}_3 \left(\mathbb{R}^6\right)$ at ν. (Note that I do not claim $\mathbb{E}_\pm$ are transverse to Σ_2 at all points of $(\mathbb{E}_\pm)^{-1} (\Sigma_2)$ and thus I do not claim that $(\mathbb{E}_\pm)^{-1} (\Sigma_2)$ themselves are submanifolds of $\mathcal{N}(\rho; \Xi, \mathbb{B})_2$. The fact that $\mathbb{E}_\pm$ are transverse to Σ_2 at (and hence also near) each point of $\Sigma^\pm_{\mathbb{B}'}$ shows that $(\mathbb{E}_\pm)^{-1} (\Sigma_2)$ are submanifolds of codimension 2 near each point of $\Sigma^\pm_{\mathbb{B}'}$, which is sufficient to establish the macilence of $\Sigma^\pm_{\mathbb{B}'}$.)

Firstly consider the case of $\Sigma^-_{\mathbb{B}'}$. Let $\nu \in \Sigma^-_{\mathbb{B}'}$ and define $\rho' = \theta \wedge \nu + \rho \in \bigwedge^3_+ \left(\mathbb{R}^6\right)^*$. After applying a suitable orientation-preserving automorphism of $\mathbb{R}^6$, one may assume that:

$$\rho' = \theta^{123} + \theta^{456} \quad \text{and} \quad \mathbb{B} = \langle e_1, e_2, e_4, e_5, e_3 + e_6 \rangle.$$

Since $\nu \in \Sigma^-_{\mathbb{B}'}$ one has $\mathbb{B}' \cap E_{-,\rho'} = \mathbb{B} \cap E_{-,\rho'} = \langle e_4, e_5 \rangle$. If additionally $\mathbb{B}' \cap E_{+,\rho'} = \mathbb{B} \cap E_{+,\rho'}$, then $\nu \in \Sigma_{\{\mathbb{B}, \mathbb{B}'\}}$, contradicting Lemma 8.5.1. Thus $\mathbb{B}' \cap E_{+,\rho'}$ and $\mathbb{B} \cap E_{+,\rho'}$ intersect along a 1-dimensional subspace of $\mathbb{B} \cap E_{+,\rho'} = \langle e_1, e_2 \rangle$ which, by applying a suitable $\mathrm{SL}(2; \mathbb{R})$ symmetry to the subspace $\langle e_1, e_2 \rangle$, can be taken to be $\langle e_1 \rangle$. Therefore $\mathbb{B}' \cap E_{+,\rho'} = \langle e_1, \lambda e_2 + e_3 \rangle$ for some $\lambda \in \mathbb{R}$. Now consider $F \in \mathrm{SL}(3; \mathbb{R})^2$ given by:

$$(e_1, e_2, e_3, e_4, e_5, e_6) \mapsto (e_1, e_2, e_3 - \lambda e_2, e_4, e_5, e_6).$$

Then F preserves ρ' and $\mathbb{B}$ (and hence $\mathbb{B}' \cap E_{-,\rho'} = \mathbb{B} \cap E_{-,\rho'}$) and maps:

$$\langle e_1, \lambda e_2 + e_3 \rangle \mapsto \langle e_1, e_3 \rangle.$$

Thus without loss of generality one can take $\mathbb{B}' \cap E_{+,\rho'} = \langle e_1, e_3 \rangle$. Therefore:

$$\mathbb{B}' = \langle e_1, e_3, e_4, e_5, \mu e_2 + \nu e_6 \rangle$$

for some $\mu, \nu \in \mathbb{R}$. Note that $\mu \neq 0$ (as else $E_{-,\rho'} \subset \mathbb{B}'$ and so $\mathbb{B}'$ is non-generic for ρ', contradicting $\nu \in \mathcal{N}(\rho; \Xi, \mathbb{B})_2$) and similarly $\nu \neq 0$ (as else $E_{+,\rho'} \subset \mathbb{B}'$). Thus, by rescaling μ and ν, one may assume without loss of generality that $\nu = 1$. Now consider $G \in \mathrm{SL}(3; \mathbb{R})^2$ given by:

$$G : (e_1, e_2, e_3, e_4, e_5, e_6) \mapsto (\mu e_1, \mu^{-1} e_2, e_3, e_4, e_5, e_6).$$

Then G preserves ρ', $\mathbb{B}$ and preserves $\mathbb{B}' \cap E_{+,\rho'} = \langle e_1, e_3 \rangle$ and maps:

$$\langle e_1, e_3, e_4, e_5, \mu e_2 + e_6 \rangle \mapsto \langle \mu^{-1} e_1, e_3, e_4, e_5, e_2 + e_6 \rangle = \langle e_1, e_3, e_4, e_5, e_2 + e_6 \rangle.$$

Thus without loss of generality one can take $\mathbb{B}' = \langle e_1, e_3, e_4, e_5, e_2 + e_6 \rangle$ and thus:

$$\mathbb{B} \cap \mathbb{B}' = \langle e_1, e_4, e_5, e_2 + e_3 + e_6 \rangle.$$

One can then choose:

$$\mathfrak{E} = \langle e_4, e_5 \rangle, \quad \mathfrak{L} = \langle e_6 \rangle, \quad \mathfrak{F} = \langle e_1, e_2 + e_3 + e_6 \rangle \quad \text{and} \quad \mathfrak{K} = \langle e_2 - e_3 \rangle.$$

Note that $\theta = \theta^3 - \theta^6$ (up to rescaling).

The proof now proceeds by direct calculation. Choose $\nu_1, \nu_2 \in \bigwedge^2 \mathbb{B}^*$ such that:

$$\theta \wedge \nu_1 = \theta \wedge \theta^{14} \quad \text{and} \quad \theta \wedge \nu_2 = \theta \wedge \theta^{15}.$$

(Such ν_i exists, since $(\theta \wedge \theta^{14})|_{\mathbb{B}} = (\theta \wedge \theta^{15})|_{\mathbb{B}} = 0$.) Using the identification:

$$\mathrm{T}_{E_{-,\rho'}} \mathrm{Gr}_3\left(\mathbb{R}^6\right) \cong \mathrm{Hom}\left(E_{-,\rho'}, E_{+,\rho'}\right) = \mathrm{Hom}\left(\langle e_4, e_5, e_6 \rangle, \langle e_1, e_2, e_3 \rangle\right) \tag{8.5.5}$$

and using Proposition 8.3.1, one computes that:

$$\begin{aligned} \mathcal{D}\mathbb{E}_-|_\nu(\nu_1) &= \kappa_\rho^+ \otimes \mathrm{Id}(\pi_{2,1}[(\theta^3 - \theta^6) \wedge \theta^{14}]) \\ &= \theta^4 \otimes e_2 \end{aligned}$$

and:

$$\begin{aligned} \mathcal{D}\mathbb{E}_-|_\nu(\nu_2) &= \kappa_\rho^+ \otimes \mathrm{Id}(\pi_{2,1}[(\theta^3 - \theta^6) \wedge \theta^{15}]) \\ &= \theta^5 \otimes e_2. \end{aligned}$$

Replacing the identification in eqn. (8.5.5) with the identification:

$$\mathrm{T}_{E_{-,\rho'}} \mathrm{Gr}_3\left(\mathbb{R}^6\right) = \mathrm{Hom}(\mathfrak{E} \oplus \mathfrak{L}, \mathfrak{F} \oplus \mathfrak{K}) = \mathrm{Hom}(\langle e_4, e_5, e_6 \rangle, \langle e_1, e_2 - e_3, e_2 + e_3 + e_6 \rangle)$$

the above results become:

$$\mathcal{D}\mathbb{E}_-|_\nu(\nu_1) = \theta^4 \otimes \left(e_2 + \frac{1}{2} e_6\right) \quad \text{and} \quad \mathcal{D}\mathbb{E}_-|_\nu(\nu_2) = \theta^5 \otimes \left(e_2 + \frac{1}{2} e_6\right)$$

and hence:

$$\mathcal{D}\mathbb{E}_-\left(\mathrm{T}_\nu \mathcal{N}(\rho; \Xi, \mathbb{B})_2\right) \supseteq \mathrm{Hom}\left(\langle e_4, e_5 \rangle, \left\langle e_2 + \frac{1}{2} e_6 \right\rangle\right).$$

Thus:

$$\begin{aligned} \mathcal{D}\mathbb{E}_-\left(\mathrm{T}_\nu \mathcal{N}(\rho; \Xi, \mathbb{B})_2\right) + \mathrm{T}_{E_{-,\rho'}} \Sigma_2 \supseteq \mathrm{Hom}\left(\langle e_4, e_5 \rangle, \left\langle e_2 + \frac{1}{2} e_6 \right\rangle\right) + \mathrm{Hom}(\mathfrak{E}, \mathfrak{F}) \\ + \mathrm{Hom}(\mathfrak{L}, \mathfrak{F}) + \mathrm{Hom}(\mathfrak{L}, \mathfrak{K}). \end{aligned}$$

Substituting the formulae for $\mathrm{Hom}(\mathfrak{E},\mathfrak{F})$, $\mathrm{Hom}(\mathfrak{L},\mathfrak{F})$ and $\mathrm{Hom}(\mathfrak{L},\mathfrak{K})$, it follows that:

$$\mathcal{D}\mathbb{E}_-\left(\mathrm{T}_\nu\mathcal{N}(\rho;\Xi,\mathbb{B})_2\right) + \mathrm{T}_{E_{-,\rho'}}\Sigma_2 \supseteq \mathrm{Hom}(\langle e_4,e_5,e_6\rangle,\langle e_1,e_2-e_3,e_2+e_3+e_6\rangle) = \mathrm{T}_{E_{-,\rho'}}\mathrm{Gr}_3\left(\mathbb{R}^6\right).$$

Thus $\mathbb{E}_-$ is transverse to Σ_2 as required.

The case of $\Sigma^+_{\mathbb{B}'}$ is analogous. In a similar fashion to above, one argues that without loss of generality:

$$\rho' = \theta^{123}+\theta^{456}, \quad \mathbb{B} = \langle e_1,e_2,e_4,e_5,e_3+e_6\rangle, \quad \mathbb{B}' = \langle e_1,e_2,e_4,e_6,e_3+e_5\rangle \quad \text{and} \quad \theta = \theta^3-\theta^6,$$

takes:

$$\mathfrak{E} = \langle e_1,e_2\rangle, \quad \mathfrak{L} = \langle e_3\rangle, \quad \mathfrak{F} = \langle e_4,e_3+e_5+e_6\rangle \quad \text{and} \quad \mathfrak{K} = \langle e_5-e_6\rangle$$

and identifies:

$$\mathrm{T}_{E_{+,\rho'}}\mathrm{Gr}_3\left(\mathbb{R}^6\right) = \mathrm{Hom}(\mathfrak{E}\oplus\mathfrak{L},\mathfrak{F}\oplus\mathfrak{K}) = \mathrm{Hom}(\langle e_1,e_2,e_3\rangle,\langle e_4,e_5-e_6,e_3+e_5+e_6\rangle).$$

By considering the derivative in the θ^{14} and θ^{24} directions, one verifies that:

$$\mathcal{D}\mathbb{E}_+\left(\mathrm{T}_\nu\mathcal{N}(\rho;\Xi,\mathbb{B})_2\right) \supseteq \mathrm{Hom}\left(\langle e_1,e_2\rangle,\left\langle \frac{1}{2}e_3+e_5\right\rangle\right)$$

from which the result follows.

□

Lemma 8.5.6. *Let $\{\mathbb{B}',\mathbb{B}''\}\subseteq\Xi$ satisfy $\mathbb{B}'\neq\mathbb{B}\neq\mathbb{B}''$. Then $\Sigma'_{\{\mathbb{B}',\mathbb{B}''\}}\subset\mathcal{N}(\rho;\Xi,\mathbb{B})_2$ is macilent.*

Proof. Since $\mathbb{B}'\neq\mathbb{B}''$, defining $\mathfrak{C}' = \mathbb{B}'\cap\mathbb{B}''$ one finds that once again $\mathfrak{C}'\subset\mathbb{R}^6$ is 4-dimensional and induces a stratification:

$$\mathrm{Gr}_3\left(\mathbb{R}^6\right) = \Sigma'_1\cup\Sigma'_2\cup\Sigma'_3$$

where:

$$\Sigma'_i = \{E\in\mathrm{Gr}_3\left(\mathbb{R}^6\right) \mid \dim(\mathfrak{C}'\cap E) = i\}.$$

Consider the map:

$$\begin{aligned}\mathbb{E}_+ : \mathcal{N}(\rho;\Xi,\mathbb{B})_2 &\longrightarrow \mathrm{Gr}_3\left(\mathbb{R}^6\right)\\ \nu &\longmapsto E_{+,\theta\wedge\nu+\rho}.\end{aligned}$$

Since $\mathfrak{C}' = \mathbb{B}'\cap\mathbb{B}''$, one has:

$$\mathbb{E}_+(\nu)\cap\mathfrak{C}' = \left(\mathbb{E}_+(\nu)\cap\mathbb{B}'\right)\cap\left(\mathbb{E}_+(\nu)\cap\mathbb{B}''\right). \tag{8.5.7}$$

Since both $\mathbb{E}_+(\nu)\cap\mathbb{B}'$ and $\mathbb{E}_+(\nu)\cap\mathbb{B}''$ are 2-dimensional, it follows that $\dim\mathbb{E}_+(\nu)\cap\mathfrak{C}'\leqslant 2$, with equality if and only if $\mathbb{E}_+(\nu) \cap \mathbb{B}' = \mathbb{E}_+(\nu) \cap \mathbb{B}''$. Thus $\mathbb{E}_+ \left(\mathcal{N}(\rho; \Xi, \mathbb{B})_2\right) \subseteq \Sigma_1 \cup \Sigma_2$ and:

$$\Sigma'_{\{\mathbb{B}', \mathbb{B}''\}} \subseteq (\mathbb{E}_+)^{-1} (\Sigma_2).$$

(Likewise $\Sigma'_{\{\mathbb{B}', \mathbb{B}''\}} \subseteq (\mathbb{E}_-)^{-1} (\Sigma_2)$, a fact which will prove useful below.) Since Σ'_2 has codimension 2 in $\mathrm{Gr}_3\left(\mathbb{R}^6\right)$, to complete the proof of the thinness of $\Sigma'_{\{\mathbb{B}', \mathbb{B}''\}}$, it suffices to prove the following claim:

Claim 8.5.8. *For all $\nu \in \Sigma'_{\{\mathbb{B}', \mathbb{B}''\}}$, the map $\mathbb{E}_+$ is transverse to the submanifold $\Sigma'_2 \subset \mathrm{Gr}_3\left(\mathbb{R}^6\right)$ at ν.*

(Again, it is not claimed that $\mathbb{E}_+$ is transverse to Σ'_2 at all points of $(\mathbb{E}_+)^{-1} (\Sigma'_2)$.)

Proof of Claim. Suppose that $\nu \in \Sigma'_{\{\mathbb{B}', \mathbb{B}''\}}$ and write $\rho' = \theta \wedge \nu + \rho$. After applying a suitable orientation-preserving automorphism of $\mathbb{R}^6$, one may assume that $\rho' = \theta^{123} + \theta^{456}$, $\mathbb{B} = \langle e_1, e_2, e_4, e_5, e_3 + e_6 \rangle$ and $\theta = \theta^3 - \theta^6$. Recall from eqn. (8.5.7) that:

$$E_{\pm,\rho'} \cap \mathfrak{C}' = E_{\pm,\rho'} \cap \mathbb{B}' = E_{\pm,\rho'} \cap \mathbb{B}''.$$

Hence by definition of $\Sigma'_{\{\mathbb{B}', \mathbb{B}''\}}$, since $\nu \in \Sigma'_{\{\mathbb{B}', \mathbb{B}''\}}$, it follows that $E_{\pm,\rho'} \cap \mathfrak{C}' \neq \mathbb{B} \cap E_{\pm,\rho'}$ for both '+' and '−'. Therefore $E_{+,\rho'} \cap \mathfrak{C}'$ must intersect $\mathbb{B} \cap E_{+,\rho'} = \langle e_1, e_2 \rangle$ in a 1-dimensional subspace, which without loss of generality may be taken to be $\langle e_1 \rangle$. Thus:

$$E_{+,\rho'} \cap \mathfrak{C}' = \langle e_1, \lambda e_2 + e_3 \rangle \text{ for some } \lambda \in \mathbb{R}.$$

Analogously, one can assume without loss of generality that:

$$E_{-,\rho'} \cap \mathfrak{C}' = \langle e_4, \mu e_5 + e_6 \rangle \text{ for some } \mu \in \mathbb{R}.$$

Since $\mathfrak{C}'$ is itself 4-dimensional, it follows that:

$$\mathfrak{C}' = \langle e_1, \lambda e_2 + e_3, e_4, \mu e_5 + e_6 \rangle.$$

Thus, using notation analogous to eqn. (8.5.4), one has:

$$\mathfrak{E}' = \mathbb{E}_+(\nu) \cap \mathfrak{C}' = \langle e_1, \lambda e_2 + e_3 \rangle$$

and one may then choose $\mathfrak{L}', \mathfrak{F}', \mathfrak{K}'$ as:

$$\mathfrak{L}' = \langle e_2 \rangle, \quad \mathfrak{F}' = \langle e_4, \mu e_5 + e_6 \rangle \quad \text{and} \quad \mathfrak{K}' = \langle e_5 \rangle.$$

Now choose $\nu_1, \nu_2 \in \bigwedge^2 \mathbb{B}^*$ such that:

$$\theta \wedge \nu_1 = \theta \wedge \theta^{46} \quad \text{and} \quad \theta \wedge \nu_2 = \theta \wedge \theta^{14}.$$

(Such ν_i exists, since $(\theta \wedge \theta^{46})|_{\mathbb{B}} = (\theta \wedge \theta^{14})|_{\mathbb{B}} = 0$.) Now compute:

$$\begin{aligned} \mathcal{D}E_+|_{\rho'}(\theta \wedge \nu_1) &= -\operatorname{Id} \otimes \kappa^-_{\rho'}(\pi_{1,2}((\theta^3 - \theta^6) \wedge \theta^{46})) \\ &= \theta^3 \otimes e_5 \end{aligned}$$

while:

$$\begin{aligned} \mathcal{D}E_+|_{\rho'}(\theta \wedge \nu) &= -\operatorname{id} \otimes \kappa^-_{\rho'}(\pi_{1,2}((\theta^3 - \theta^6) \wedge \theta^{14})) \\ &= -\theta^1 \otimes e_5. \end{aligned}$$

Thus:

$$\mathcal{D}\mathbb{E}_+ \left(\mathrm{T}_\nu \mathcal{N}(\rho; \Xi, \mathbb{B})_2 \right) \supseteq \operatorname{Hom}(\langle e_1, e_3 \rangle, \langle e_5 \rangle)$$

and thus:

$$\begin{aligned} \mathcal{D}\mathbb{E}_+ \left(\mathrm{T}_\nu \mathcal{N}(\rho; \Xi, \mathbb{B})_2 \right) + \mathrm{T}_{E_{+,\rho'}} \Sigma_2 \supseteq \operatorname{Hom}(\langle e_1, e_3 \rangle, \langle e_5 \rangle) &\oplus \operatorname{Hom}(\mathfrak{E}', \mathfrak{F}) \\ &\oplus \operatorname{Hom}(\mathfrak{L}', \mathfrak{F}') \oplus \operatorname{Hom}(\mathfrak{L}', \mathfrak{K}') \\ = \operatorname{Hom}(\langle e_1, e_2, e_3 \rangle, \langle e_4, e_5, e_6 \rangle) &= \mathrm{T}_{E_{+,\rho'}} \mathrm{Gr}_3 \left(\mathbb{R}^6 \right), \end{aligned}$$

which is the required statement of transversality, completing the proof of the claim and hence of Lemma 8.5.6.

□

□

Thus $\Sigma_{\{\mathbb{B}', \mathbb{B}''\}}$ is the union of three macilent subsets of $\mathcal{N}(\rho; \Xi, \mathbb{B})_2$, and hence is scarce. This completes the proof of Lemma 8.2.15 and hence of Theorem 8.2.1.

Chapter 9

Topological properties of closed $\widetilde{\mathrm{G}}_2$, $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ forms on manifolds

This chapter uses characteristic classes and obstruction theory, together with the h-principles for $\widetilde{\mathrm{G}}_2$ and $\mathrm{SL}(3;\mathbb{R})^2$ forms established in Chapters 7, 8, to prove various theorems on the topological properties of closed $\widetilde{\mathrm{G}}_2$, $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ forms on oriented 6- and 7-manifolds. Results obtained include a criterion for an arbitrary oriented 7-manifold to admit a closed $\widetilde{\mathrm{G}}_2$-structure (in the process, proving a conjecture of Lê), a generalisation of Donaldson's 'G_2-cobordisms' to $\widetilde{\mathrm{G}}_2$, $\mathrm{SL}(3;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{R})^2$ forms, and a complete classification of closed $\mathrm{SL}(3;\mathbb{C})$ 3-forms up to homotopy. A lower bound on the number of homotopy classes of closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on a given manifold is also obtained.

9.1 A vanishing result for natural cohomology classes

The aim of this section is to prove the following result:

Lemma 9.1.1. *Suppose there is an assignment to each n-manifold M (with, possibly empty, boundary) of a degree p cohomology class $\nu(\mathrm{M}) \in \mathrm{H}^p(\mathrm{M};G)$, where G is either a field or a finite Abelian group, which is natural, in the sense that for each embedding $f : \mathrm{M} \hookrightarrow \mathrm{M}'$ of n-manifolds with boundary:*

$$\nu(\mathrm{M}) = f^*\nu(\mathrm{M}').$$

Then if ν vanishes on every closed (resp. closed, oriented) n-manifold, it vanishes on every (resp. every oriented) n-manifold with boundary.

Examples of such classes ν are any cohomology class which is constructed only from Stiefel–Whitney classes, or only from the reduction of the Chern, Pontryagin and Euler classes to real coefficients. More generally, for any cohomology operation $\Theta : \mathrm{H}^p(-;G) \to \mathrm{H}^q(-;G')$ (see [65, p. 448]), if $\nu(\mathrm{M}) \in \mathrm{H}^p(\mathrm{M};G)$ is natural, then $\Theta \circ \nu(\mathrm{M}) \in \mathrm{H}^q(\mathrm{M};G')$ is also natural. Note also that only the case $G = \mathbb{Z}/2\mathbb{Z}$ will be used in this chapter, however I allow more general G in Lemma 9.1.1 since the proof for all such G is essentially the same.

Proof of Lemma 9.1.1. By assumption $\nu(\mathrm{M}) = 0$ for all closed (resp. closed, oriented) n-manifolds M. The proof proceeds by considering three cases of increasing generality.

Case 1: M **is compact with boundary.** Consider the double $\mathscr{D}\mathrm{M} = \mathrm{M} \cup_{\partial \mathrm{M}} \overline{\mathrm{M}}$ formed by gluing M to a second copy of itself $\overline{\mathrm{M}}$ (now with the opposite orientation, if appropriate) along the boundary ∂M. Then $\mathscr{D}$M is a closed (resp. closed, oriented) n-manifold and thus $\nu(\mathscr{D}\mathrm{M}) = 0$, by assumption. Writing $\iota : \mathrm{M} \hookrightarrow \mathscr{D}\mathrm{M}$ for the natural inclusion, the naturality of ν implies that:

$$\nu(\mathrm{M}) = \iota^* \nu(\mathscr{D}\mathrm{M}) = 0.$$

Case 2: M **is non-compact and without boundary.** Let $f : \mathrm{M} \to \mathbb{R}$ be a proper Morse function (see, e.g. [106, Thm. 6.6]) and choose increasing unbounded sequences $i_k \in \mathbb{R}_{>0}$ and $j_k \in \mathbb{R}_{>0}$ such that both i_k and $-j_k$ are regular values of f for all $k \in \mathbb{N}$. Then for each k the subset $f^{-1}[-j_k, i_k] = \mathrm{M}_k$ is a compact submanifold-with-boundary of M (see [63, Lem., p. 62] for a similar result). Moreover, each M_{k+1} is obtained from M_k by attaching a finite number of m-cells, for suitable choices of m, and thus the function f gives M the structure of a CW complex such that each M_k is a subcomplex of M. Define:

$$\varprojlim \mathrm{H}^p(\mathrm{M}_k; G) = \left\{ (m_k)_k \in \prod_{i=0}^{\infty} \mathrm{H}^p(\mathrm{M}_k; G) \;\middle|\; \text{for all } k \geqslant 0 : m_{k+1}|_{\mathrm{M}_k} = m_k \right\}.$$

Suppose initially that $G = \mathbb{Q}$, or $G = \mathbb{Z}/_{q\mathbb{Z}}$ for some prime q. Then by [65, Prop. 3F.5], the natural map:

$$\begin{aligned} \mathrm{H}^p(\mathrm{M}; G) &\to \varprojlim \mathrm{H}^p(\mathrm{M}_k; G) \\ m &\mapsto (m|_{\mathrm{M}_k})_k \end{aligned} \tag{9.1.2}$$

is an isomorphism. Thus $\nu(\mathrm{M}) = 0$ if and only if $\nu(\mathrm{M})|_{\mathrm{M}_k} = 0$ for each k. However by naturality $\nu(\mathrm{M})|_{\mathrm{M}_k} = \nu(\mathrm{M}_k)$, which vanishes by case 1, yielding $\nu(\mathrm{M}) = 0$, as required.

For more general G, eqn. (9.1.2) is replaced by [105, Lem. 2] the short exact sequence:

$$0 \to \varprojlim{}^1 \mathrm{H}^{p-1}(\mathrm{M}_k; G) \to \mathrm{H}^p(\mathrm{M}; G) \to \varprojlim \mathrm{H}^p(\mathrm{M}_k; G) \to 0,$$

where $\varprojlim{}^1$ is the first right-derived functor of $\varprojlim$ (see [105] for a more explicit definition). Thus, to prove the lemma when G is an arbitrary field or finite Abelian group, it suffices to prove that $\varprojlim{}^1 \mathrm{H}^{p-1}(\mathrm{M}_k; G) = 0$ in this case. However this is clear: since each M_k is a finite cell-complex, the spaces $\mathrm{H}^{p-1}(\mathrm{M}_k; G)$ are finite-dimensional G vector spaces if G is a field, and are finite Abelian groups if G is a finite Abelian group. The result now follows by [124, Exercise 3.5.2].

Case 3: M **is non-compact with boundary.** By considering the double $\mathcal{D}$M of M and using case 2, it follows that $\nu(\mathrm{M}) = 0$.

□

9.2 Existence of $\widetilde{\mathrm{G}}_2$-structures

The aim of this section is to prove the following result, conjectured by Lê in [92]:

Theorem 9.2.1. *Let* M *be an oriented 7-manifold (not necessarily closed). Then* M *admits* $\widetilde{\mathrm{G}}_2$*-structures if and only if it is spin.*

Combining Theorem 9.2.1 with the h-principles established in Theorems 7.7.44 and 7.7.5 yields the following corollary:

Theorem 9.2.2. *Let* M *be an oriented 7-manifold. If* M *is spin, then every degree 3 de Rham class can be represented by a* $\widetilde{\mathrm{G}}_2$ *3-form and every degree 4 de Rham class can be represented by a* $\widetilde{\mathrm{G}}_2$ *4-form.*

I begin by recalling the following definition, taken from [64]:

Definition 9.2.3. Let $\phi \in \bigwedge^3_+ \left(\mathbb{R}^7\right)^*$. An oriented 3-plane $C \in \widetilde{\mathrm{Gr}}_3(\mathbb{R}^7)$ is called calibrated with respect to ϕ if, writing vol_C for the volume form on C induced by the metric $g_\phi|_C$ and the orientation on C, one has:

$$\phi|_C = vol_C.$$

Analogously, let $\widetilde{\phi} \in \bigwedge^3_\sim \left(\mathbb{R}^7\right)^*$. I call an oriented 3-plane $C \in \widetilde{\mathrm{Gr}}_3(\mathbb{R}^7)$ positively calibrated if $g_{\widetilde{\phi}}$ is positive definite on C and, writing vol_C for the volume form on C induced by the metric $g_{\widetilde{\phi}}|_C$ and the orientation on C, one has:

$$\widetilde{\phi}|_C = vol_C.$$

It is well-known that G_2 acts transitively on the set of calibrated planes and that the stabiliser of any calibrated plane is isomorphic to $\mathrm{SO}(4)$ (see [78, §10.8]). Similarly:

Proposition 9.2.4. $\widetilde{\mathrm{G}}_2$ *acts transitively on the set of positively calibrated planes and the stabiliser of any positively calibrated plane is a maximal compact subgroup of* $\widetilde{\mathrm{G}}_2$ *isomorphic to* $\mathrm{SO}(4)$.

Proof. To prove transitivity of the action, consider the standard $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\varphi}_0$ on $\mathbb{R}^7$ and let C be positively calibrated with respect to $\widetilde{\varphi}_0$. Pick an oriented orthonormal basis (c_1, c_2, c_3) of C with respect to $\widetilde{g}_0|_C$ (which exists since $\widetilde{g}_0$ is positive definite on C). By [84, Prop. 2.3], $\widetilde{\mathrm{G}}_2$ acts transitively on ordered pairs of orthonormal, spacelike vectors in $\mathbb{R}^7$, so without loss of generality $c_i = e_i$ $(i = 1, 2)$. Since $\widetilde{\varphi}_0|_C = vol_C$ and (c_1, c_2, c_3) is an oriented orthonormal basis of C, one has:

$$\widetilde{\varphi}_0(c_1, c_2, c_3) = 1.$$

It follows that $c_3 = e_3 + u$ for some $u \in \langle e_4, ..., e_7\rangle$. Since $\widetilde{g}_0(c_3, c_3) = 1$, u satisfies $\widetilde{g}_0(u, u) = 0$ and hence $u = 0$, since $\widetilde{g}_0$ is negative definite on $\langle e_4, ..., e_7\rangle$. Thus $C = \langle e_1, e_2, e_3\rangle$ up to the action of $\widetilde{\mathrm{G}}_2$ and hence $\widetilde{\mathrm{G}}_2$ acts transitively on positively calibrated planes. The statement regarding stabilisers is proven in [68, Prop. 4.4].

$\square$

(Positively) calibrated planes have the following desirable property:

Lemma 9.2.5. *1. Let ϕ be a G_2 3-form on $\mathbb{R}^7$ and let C be a calibrated plane. Then:*

$$\phi_C = 2\phi|_C - \phi$$

defines a $\widetilde{\mathrm{G}}_2$ 3-form on $\mathbb{R}^7$ and C is positively calibrated with respect to ϕ_C (here $\phi|_C$ is interpreted as a 3-form on $\mathbb{R}^7$ using the splitting $\mathbb{R}^7 = C \oplus C^\perp$, where the orthocomplement is taken with respect to g_ϕ).

2. Let $\widetilde{\phi}$ be a $\widetilde{\mathrm{G}}_2$ 3-form on $\mathbb{R}^7$ and let C be a positively calibrated plane. Then:

$$\widetilde{\phi}_C = 2\widetilde{\phi}|_C - \widetilde{\phi}$$

defines a G_2 3-form on $\mathbb{R}^7$ and C is calibrated with respect to $\widetilde{\phi}_C$ (again $\widetilde{\phi}|_C$ is interpreted as a 3-form on $\mathbb{R}^7$ using the splitting $\mathbb{R}^7 = C \oplus C^\perp$, where the orthocomplement is taken with respect to $g_{\widetilde{\phi}}$).

Proof. The proof is by direct calculation. For 1, since G_2 acts transitively on the set of calibrated planes, without loss of generality one may assume that $\phi = \varphi_0$ and $C = \langle e_1, e_2, e_3 \rangle$. Then:

$$\phi_C = 2\theta^{123} - \left(\theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}\right) = \theta^{123} - \theta^{145} - \theta^{167} - \theta^{246} + \theta^{257} + \theta^{347} + \theta^{356}$$

which is easily seen to be of $\widetilde{\mathrm{G}}_2$-type and have C as a positively calibrated plane. The converse is similar.

□

Since $\mathrm{SO}(4) \subset \widetilde{\mathrm{G}}_2$ is a maximal compact subgroup, the quotient $\widetilde{\mathrm{G}}_2 / \mathrm{SO}(4)$ is contractible. Thus, given any oriented 7-manifold M equipped with a $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi}$, there exists a positively calibrated rank 3 distribution C on M. The corresponding result in the G_2 case is non-trivial, since $\mathrm{G}_2 / \mathrm{SO}(4)$ is not contractible.

Proposition 9.2.6. *Let* M *be an oriented 7-manifold and let ϕ be a G_2 3-form on* M*. Then* M *admits a 3-plane distribution C which is calibrated with respect to ϕ.*

Proof. The proof is a generalisation of Friedrich–Kath–Moroianu–Semmelmann's proof of the existence of $\mathrm{SU}(2)$-structures on closed 7-manifolds with G_2-structures (see [51, Thm. 3.2]). Define a cross-product $\times$ on M by the equation:

$$g(u_1 \times u_2, u_3) = \phi(u_1, u_2, u_3)$$

for all $p \in \mathrm{M}$ and $u_i \in \mathrm{T}_p\mathrm{M}$, $(i = 1, 2, 3)$. An easy calculation then shows that if u_1 and u_2 are linearly independent, then:

$$\mathrm{Span}\langle u_1, u_2, u_1 \times u_2 \rangle$$

together with its natural orientation induced by the above ordering of basis vectors defines a calibrated plane in $\mathrm{T}_p\mathrm{M}$. If M is closed, M admits a pair of everywhere linearly independent vector fields by [120].

To prove Proposition 9.2.6 for open (i.e. non-closed) manifolds, therefore, it suffices to prove every open orientable 7-manifold M also admits two everywhere linearly independent vector fields. By [107, Thm. 12.1] (see also the preceding discussion *op. cit.*), the condition $w_6(\mathrm{M}) = 0$ is necessary and sufficient to ensure the existence of two vector fields X and Y defined over the 6-skeleton of M which are everywhere linearly independent. Moreover, since M is open, M deformation retracts onto a subcomplex of its 6-skeleton (cf. [42, Prop. 4.3.1]) and thus M itself admits two globally defined vector fields X and Y if and only if $w_6(\mathrm{M}) = 0$. By [99, Thm. III], w_6 vanishes on every closed oriented 7-manifold. Thus by Lemma 9.1.1 it follows that w_6 vanishes on every oriented 7-manifold, completing the proof.

□

Using Proposition 9.2.6, I now prove Theorem 9.2.1:

Proof of Theorem 9.2.1. By a well-known result of Gray ([22, Remark 3]; cf. [58]) M admits a G_2-structure ϕ if and only if M is orientable and spin. By Proposition 9.2.6, M admits a pair (ϕ, C) of a G_2 3-form ϕ together with a calibrated distribution C if and only if M admits a G_2-structure. By Lemma 9.2.5, M admits a pair $\left(\widetilde{\phi}, C\right)$ with $\widetilde{\phi}$ a $\widetilde{\mathrm{G}}_2$ 3-form and C a positively calibrated distribution if and only if M admits a pair (ϕ, C) with ϕ a G_2 3-form and C a calibrated distribution. Finally – as discussed above – since $\mathrm{SO}(4) \subset \widetilde{G}_2$ is maximal compact, the quotient space $\widetilde{G}_2 / \mathrm{SO}(4)$ is contractible and thus a manifold M admits a $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi}$ if and only if it admits a pair $\left(\widetilde{\phi}, C\right)$ with $\widetilde{\phi}$ a $\widetilde{\mathrm{G}}_2$ 3-form and C a positively calibrated distribution. The result follows by combining these four logical equivalences.

□

9.3 $\widetilde{\mathrm{G}}_2$-cobordisms

The aim of this section is to introduce a $\widetilde{\mathrm{G}}_2$-analogue of Donaldson's theory of G_2-cobordisms between closed $\mathrm{SL}(3;\mathbb{C})$ 3-forms. For brevity of notation, I shall use the term 'SL form' to refer to either a $\mathrm{SL}(3;\mathbb{C})$ 3-form or a $\mathrm{SL}(3;\mathbb{R})^2$ 3-form, as appropriate.

Definition 9.3.1. Let N be an oriented 6-manifold. Let $\bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N}$ denote the pullback of the bundle $\bigwedge^2\mathrm{T}^*\mathrm{N} \to \mathrm{N}$ along the bundle $\bigwedge^3\mathrm{T}^*\mathrm{N} \to \mathrm{N}$ and write $\mathfrak{p} : \bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N} \to \bigwedge^2\mathrm{T}^*\mathrm{N}$ for the natural projection:

$$\begin{array}{ccc} \bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N} & \xrightarrow{\mathfrak{p}} & \bigwedge^2\mathrm{T}^*\mathrm{N} \\ \downarrow & & \downarrow \\ \bigwedge^3\mathrm{T}^*\mathrm{N} & \longrightarrow & \mathrm{N} \end{array}$$

Write $\bigwedge^3_+\mathrm{T}^*\mathrm{N}$ for the bundle of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on N and define a subbundle $\mathcal{E}_+$ of $\bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N}$ over $\bigwedge^3_+\mathrm{T}^*\mathrm{N}$ by declaring the fibre over $\rho \in \bigwedge^3_+\mathrm{T}^*\mathrm{N}$ to be:

$$\mathcal{E}_+|_\rho = \left\{ \omega \in \bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N}|_\rho \;\middle|\; \mathscr{I}_\rho(\mathfrak{p}\omega) \text{ has signature } (3,3) \text{ and } (\mathfrak{p}\omega)^3 < 0 \right\}$$

(see eqn. (7.7.14) for the definition of $\mathscr{I}_\rho$). Likewise, define a subbundle $\mathcal{E}_-$ of $\bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N}$ over $\bigwedge^3_-\mathrm{T}^*\mathrm{N}$ by:

$$\mathcal{E}_-|_\rho = \left\{\omega \in \bigwedge^{\binom{2}{3}}\mathrm{T}^*\mathrm{N}|_\rho \;\middle|\; \mathscr{J}_\rho(\mathfrak{p}\omega) \text{ has signature } (2,4)\right\}$$

(see eqn. (7.7.23) for the definition of $\mathscr{J}_\rho$).

Now let ρ be an SL form, i.e. a section of $\bigwedge^3_\pm\mathrm{T}^*\mathrm{N}$ as appropriate. I term ρ extendible if there exists a lift ω of the section ρ along the map $\mathcal{E}_\pm \to \bigwedge^3_\pm\mathrm{T}^*\mathrm{N}$:

$$\begin{tikzcd} & \mathcal{E}_\pm \arrow[d] \\ \mathrm{N} \arrow[r, "\rho"] \arrow[ur, dashed, "\omega"] & \bigwedge^3_\pm\mathrm{T}^*\mathrm{M} \end{tikzcd}$$

Proposition 9.3.2. *Let* N *be an oriented 6-manifold and let* ρ *be a (closed)* SL *form on* N. *Then the following are equivalent:*

- *There exists an oriented 7-manifold-with-boundary* M *together with a (closed)* $\widetilde{\mathrm{G}}_2$ *3-form* $\widetilde{\phi}$ *such that* N *is a connected component of* $\partial\mathrm{M}$ *and* $\widetilde{\phi}|_\mathrm{N} = \rho$*;*

- ρ *is extendible.*

Proof. Suppose that ρ is extendible and let ω be a lift of ρ along $\mathcal{E}_\pm \to \bigwedge^3_\pm\mathrm{T}^*\mathrm{N}$. Let $f : \mathrm{N} \to (0,\infty)$ be chosen later and consider the manifold:

$$\mathrm{M} = \{(t,p) \in [0,\infty) \times \mathrm{N} \mid 0 \leqslant t < f(p)\}.$$

Define a 3-form $\widetilde{\phi}$ on M via:

$$\widetilde{\phi} = \mathrm{d}t \wedge (\mathfrak{p}\omega) + \rho + t\mathrm{d}(\mathfrak{p}\omega).$$

By Propositions 7.7.24 and 7.7.15, $\widetilde{\phi}$ is of $\widetilde{\mathrm{G}}_2$-type along $\{0\}\times\mathrm{N}$ and hence, by the stability of $\widetilde{\mathrm{G}}_2$ 3-forms, it is of $\widetilde{\mathrm{G}}_2$-type on all of M if $f(p)$ is sufficiently small, depending on $p \in \mathrm{N}$. Moreover $\mathrm{d}\widetilde{\phi} = \mathrm{d}\rho$ and thus if ρ is closed on N, then $\widetilde{\phi}$ is closed on M, as claimed.

Conversely if N is a connected component of $\partial\mathrm{M}$, then by the Collar Neighbourhood Theorem ([94, Thm. 9.25]; cf. also [19, Lem. 5]) there is an open neighbourhood of N in M which is diffeomorphic to $[0,1)\times\mathrm{N}$. One now simply applies the above argument in reverse.

□

I now prove the main result of this section:

Theorem 9.3.3. *Let* N *be a 6-manifold and let* ρ, ρ' *be closed, extendible* $\mathrm{SL}(3;\mathbb{C})$ *(resp.* $\mathrm{SL}(3;\mathbb{R})^2$*) 3-forms on* N. *Suppose that* ρ *and* ρ' *are homotopic and lie in the same cohomology class. Then* (N,ρ) *and* (N,ρ') *are* $\widetilde{\mathrm{G}}_2$*-cobordant.*

Proof. Let ρ_t denote a homotopy of sections of $\bigwedge^3_\pm\mathrm{T}^*\mathrm{N}$ over N such that $\rho_0 = \rho$ and $\rho_1 = \rho'$, and choose a lift ω of ρ along $\mathcal{E}_\pm \to \bigwedge^3_\pm\mathrm{T}^*\mathrm{N}$. Using the covering homotopy property for fibre bundles [75, Ch. III, Thm. 4.1], there is a homotopy of sections $\omega_t : \mathrm{N} \to \mathcal{E}_\pm$ such that for each $t \in [0,1]$, ω_t is a lift of ρ_t along $\mathcal{E}_\pm \to \Lambda^3_\pm \mathrm{T}^*\mathrm{N}$.

Write α for the common cohomology class defined by ρ and ρ' and consider the space $\mathrm{M} = [0,1]_t \times \mathrm{N}$. Since M and N have identical cohomology, α also defines a cohomology class on M. Let $f_1 : \mathrm{M} \to [0,\infty)$ be a smooth function which is identically 1 on an open neighbourhood of $\{0\} \times \mathrm{N}$, but which vanishes outside some larger neighbourhood of $\{0\} \times \mathrm{N}$. Likewise, let $f_2 : \mathrm{M} \to [0,\infty)$ be identically 1 on a small open neighbourhood of $\{1\} \times \mathrm{N}$ and vanish outside some larger open neighbourhood. Define a 3-form $\widetilde{\phi}$ on M via:

$$\widetilde{\phi} = \mathrm{d}t \wedge (\mathfrak{p}\omega_t) + \rho_t + [t f_1 + (t-1) f_2] \mathrm{d}_\mathrm{N}(\mathfrak{p}\omega_t),$$

where d_N denotes the exterior derivative on N. Then by Propositions 7.7.24 and 7.7.15 $\mathrm{d}t \wedge (\mathfrak{p}\omega_t) + \rho_t$ is of $\widetilde{\mathrm{G}}_2$-type on M; hence, by the stability of $\widetilde{\mathrm{G}}_2$ 3-forms, if the supports of f_1 and f_2 are chosen to be sufficiently small, then $\widetilde{\phi}$ too is of $\widetilde{\mathrm{G}}_2$-type. Moreover, a direct calculation shows that $\mathrm{d}\widetilde{\phi} = 0$ on $\{f_1 \equiv 1\} \cup \{f_2 \equiv 1\}$ and that $\widetilde{\phi}$ represents the restriction of the class α to $\{f_1 \equiv 1\} \cup \{f_2 \equiv 1\}$. Without loss of generality assume $\mathcal{O}p(\partial\mathrm{M}) \subseteq \{f_1 \equiv 1\} \cup \{f_2 \equiv 1\}$ and recall the sets:

$$\Omega^3_{\widetilde{\varphi}_0}\left(\mathrm{M}; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right) = \left\{\widetilde{\phi}' \in \Omega^3_{\widetilde{\varphi}_0}(\mathrm{M}) \;\middle|\; \widetilde{\phi}'|_{\mathcal{O}p(A)} = \widetilde{\phi}|_{\mathcal{O}p(A)}\right\}$$
$$\mathcal{C}l^3_{\widetilde{\varphi}_0}\left(\alpha; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right) = \left\{\widetilde{\phi}' \in \Omega^3_{\widetilde{\varphi}_0}(\mathrm{M}; \widetilde{\phi}'|_{\mathcal{O}p(\partial\mathrm{M})}) \;\middle|\; \mathrm{d}\widetilde{\phi}' = 0 \text{ and } [\widetilde{\phi}'] = \alpha \in H^3_{\mathrm{dR}}(\mathrm{M})\right\}.$$

Note that $\widetilde{\phi}$ defines an element of $\Omega^3_{\widetilde{\varphi}_0}\left(\mathrm{M}; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right)$. By Theorem 7.7.44 (see also Theorem 7.2.2), $\mathcal{C}l^3_{\widetilde{\varphi}_0}\left(\alpha; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right) \hookrightarrow \Omega^3_{\widetilde{\varphi}_0}\left(\mathrm{M}; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right)$ is a homotopy equivalence. Thus, one can deform $\widetilde{\phi} \in \Omega^p_{\widetilde{\varphi}_0}\left(\mathrm{M}; \widetilde{\phi}|_{\mathcal{O}p(\partial\mathrm{M})}\right)$ relative to $\mathcal{O}p(\partial\mathrm{M})$ into a closed $\widetilde{\mathrm{G}}_2$ 3-form $\widetilde{\phi}'$ on M (representing the class α). The pair $(\mathrm{M}, \widetilde{\phi}')$ then gives the required cobordism from (N, ρ) to (N, ρ').

□

9.4 Topological properties of $\mathrm{SL}(3;\mathbb{C})$ 3-forms

The aim of this section is to investigate when two $\mathrm{SL}(3;\mathbb{C})$ 3-forms are homotopic, and when a single $\mathrm{SL}(3;\mathbb{C})$ 3-form is extendible.

9.4.1 Homotopic $\mathrm{SL}(3;\mathbb{C})$ 3-forms

In this subsection, I prove Theorem 1.0.16. Recall the statement of the theorem:

Theorem 1.0.16. *There is a 1-1 correspondence between homotopy classes of* $\mathrm{SL}(3;\mathbb{C})$ *3-forms on* N *(equivalently closed* $\mathrm{SL}(3;\mathbb{C})$ *3-forms, or* $\mathrm{SL}(3;\mathbb{C})$ *3-forms in any fixed degree 3 de Rham class) and spin structures on* N *which in turn correspond non-canonically with elements of* $\mathrm{H}^1\left(\mathrm{N}, \mathbb{Z}/2\mathbb{Z}\right)$. *(More precisely, the set of spin structures, and hence the set* $\mathcal{SL}_\mathbb{C}(\mathrm{N})$ *of homotopy classes of* $\mathrm{SL}(3;\mathbb{C})$ *3-forms, possibly closed or in any given de Rham class, is a torsor for the group* $\mathrm{H}^1\left(\mathrm{N}, \mathbb{Z}/2\mathbb{Z}\right)$, *i.e. it admits a faithful, transitive action by the group* $\mathrm{H}^1\left(\mathrm{N}, \mathbb{Z}/2\mathbb{Z}\right)$.*)*

I begin by remarking that the existence of the faithful, transitive action of $\mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$ on $\mathcal{SL}_{\mathbb{C}}(\mathrm{N})$ can be proven directly via classical Obstruction Theory, without any reference to spin structures. Indeed, the fibre of the bundle $\wedge^3_- \mathrm{T}^*\mathrm{N}$ of $\mathrm{SL}(3;\mathbb{C})$ 3-forms over N is homeomorphic to $\mathrm{GL}_+(6;\mathbb{R}) \big/ \mathrm{SL}(3;\mathbb{C})$, which deformation retracts onto the space $\mathrm{SO}(6) \big/ \mathrm{SU}(3) \cong \mathbb{RP}^7$. Since $\pi_n\left(\mathbb{RP}^7\right) = 0$ for $n = 2, \ldots, 6$, classical Obstruction Theory (see [125, Thm. 6.13]) implies that the set of homotopy classes of sections of $\wedge^3_- \mathrm{T}^*\mathrm{N}$ over N is a torsor for the group $\mathrm{H}^1\left(\mathrm{N}; \pi_1\left(\wedge^3_- \mathrm{T}^*_\bullet\mathrm{N}\right)\right)$, where $\pi_1\left(\wedge^3_- \mathrm{T}^*_\bullet\mathrm{N}\right)$ denotes the bundle of groups over N given by the first fundamental groups of the fibres of $\wedge^3_- \mathrm{T}^*\mathrm{N}$. Since $\pi_1\left(\mathbb{RP}^7\right) \cong \mathbb{Z}/2\mathbb{Z}$ has no non-trivial automorphisms, the bundle $\pi_1\left(\wedge^3_- \mathrm{T}^*_\bullet\mathrm{N}\right)$ itself must be trivial (or simple, in the terminology of [125]; see p. 263 *op. cit.*) and thus $\mathrm{H}^1\left(\mathrm{N}; \pi_1\left(\wedge^3_- \mathrm{T}^*_\bullet\mathrm{N}\right)\right)$ is simply the usual cohomology group $\mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$. Thus, the set of homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms is a torsor for $\mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$, as claimed.

The action of $\chi \in \mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$ on $\mathcal{SL}_{\mathbb{C}}(\mathrm{N})$ admits a very explicit description in the case that χ lies in the image of the natural map $r_2 : \mathrm{H}^1(\mathrm{N};\mathbb{Z}) \to \mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$. Indeed, firstly note that, given any $\rho \in \wedge^3_-\left(\mathbb{R}^6\right)^*$, the map:

$$
\begin{aligned}
\gamma_\rho : \mathrm{U}(1) &\longrightarrow \wedge^3_-\left(\mathbb{R}^6\right)^* \\
e^{i\theta} &\longmapsto \cos(\theta)\rho + \sin(\theta) J^*_\rho \rho
\end{aligned}
$$

generates the first fundamental group of $\wedge^3_-\left(\mathbb{R}^6\right)^*$. Next, recall that the cohomology group $\mathrm{H}^1(\mathrm{N};\mathbb{Z})$ can be identified with the space of homotopy classes of maps $\mathrm{N} \to \mathrm{U}(1)$. Thus, let ρ be an $\mathrm{SL}(3;\mathbb{C})$ 3-form on N representing the homotopy class $[\rho] \in \mathcal{SL}_{\mathbb{C}}(\mathrm{N})$, pick some $\chi' \in r_2^{-1}(\chi) \subseteq \mathrm{H}^1(\mathrm{N};\mathbb{Z})$ and choose some $f : \mathrm{N} \to \mathrm{U}(1)$ representing the class χ'. Then, $\chi \cdot [\rho] \in \mathcal{SL}_{\mathbb{C}}(\mathrm{N})$ can be explicitly represented by the $\mathrm{SL}(3;\mathbb{C})$ 3-form $\rho' = \mathfrak{Re}(f)\rho + \mathfrak{Im}(f) J^*_\rho \rho$.

I now return to the full statement of Theorem 1.0.16. Fix a Riemannian metric g on N and write $\mathcal{P}$ for the SO(6)-structure on N induced by g. Recall that a spin structure on N is a principal Spin(6)-bundle $\mathcal{Q}$ together with a 2-sheeted covering map $q : \mathcal{Q} \to \mathcal{P}$, such that the following diagram commutes:

$$
\begin{tikzcd}
& \mathcal{Q} \arrow[dl] \arrow[dd, "q"] & \circlearrowleft \ \mathrm{Spin}(6) \arrow[dd] \\
\mathrm{N} & & \\
& \mathcal{P} \arrow[ul] & \circlearrowleft \ \mathrm{SO}(6)
\end{tikzcd}
$$

(The reader may wish to note that the bundle $\mathcal{Q}$ alone does not determine the map q; see, e.g. [91, p. 84, Remark 1.14].) I now prove Theorem 1.0.16.

Proof of Theorem 1.0.16. Firstly, note that homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms (equivalently, $\mathrm{SL}(3;\mathbb{C})$-structures) on N correspond bijectively to homotopy classes of principal SU(3)-subbundles of $\mathcal{P}$. Indeed, writing $\mathcal{F}_+\mathrm{N}$ for the oriented frame bundle of N, $\mathrm{SL}(3;\mathbb{C})$-structures on N are equivalent to sections of the bundle $\mathcal{F}_+\mathrm{N} \big/ \mathrm{SL}(3;\mathbb{C})$, and likewise principal SU(3)-subbundles $\mathcal{P}$ are equivalent to sections of the bundle ${}^{\mathcal{P}}\!\big/_{\mathrm{SU}(3)}$. The equivalence now follows from the observation that the fibres of the natural map ${}^{\mathcal{F}_+\mathrm{N}}\!\big/_{\mathrm{SL}(3;\mathbb{C})} \to {}^{\mathcal{P}}\!\big/_{\mathrm{SU}(3)}$ are contractible. Thus, to complete the proof of Theorem 1.0.16, it suffices to prove that there exists a map σ from homotopy classes of $\mathrm{SU}(3)$-subbundles of $\mathcal{P}$ to spin structures on (N, g) and that σ is bijective.

The existence of the map σ is essentially well-known (see, e.g. [78, Prop. 3.6.2] for a related result). Indeed, let $\mathcal{R} \subset \mathcal{P}$ be an $\mathrm{SU}(3)$-subbundle. Consider the diagram:

$$\begin{tikzcd} & \mathrm{Spin}(6) \arrow[d, "\pi"] \\ \mathrm{SU}(3) \arrow[ur, hook, dashed, "\varrho"] \arrow[r, hook, "\iota"] & \mathrm{SO}(6) \end{tikzcd} \tag{9.4.1}$$

Since $\mathrm{SU}(3)$ is simply connected, Covering Space Theory implies that there is a unique homomorphism $\mathrm{SU}(3) \xrightarrow{\varrho} \mathrm{Spin}(6)$ lifting the inclusion $\mathrm{SU}(3) \overset{\iota}{\hookrightarrow} \mathrm{SO}(6)$ along the homomorphism $\pi : \mathrm{Spin}(6) \to \mathrm{SO}(6)$. Diagram (9.4.1) induces a diagram of bundles:

$$\begin{tikzcd} & \mathcal{R} \times_\varrho \mathrm{Spin}(6) \arrow[d, "q"] \\ \mathcal{R} \arrow[ur, hook] \arrow[r, hook] & \mathcal{R} \times_\iota \mathrm{SO}(6) \cong \mathcal{P} \end{tikzcd}$$

and thus, setting $\mathcal{Q} = \mathcal{R} \times_\varrho \mathrm{Spin}(6)$ (together with the natural map q induced by $\pi : \mathrm{Spin}(6) \to \mathrm{SO}(6)$), it has been shown that every $\mathrm{SU}(3)$-subbundle $\mathcal{R} \subset \mathcal{P}$ canonically induces a spin structure on N, which clearly depends only on the homotopy class of $\mathcal{R}$, thus defining the map σ.

Before proving that σ is bijective, it is useful to note that the spin structure induced by $\mathcal{R}$ may alternatively be characterised as follows. Given any choice of spin structure $(\mathcal{Q}, q)$ on (N, g), the bundle $\mathcal{R}' = q^{-1}(\mathcal{R})$ defines an $(\mathrm{SU}(3) \times \{\pm 1\})$-subbundle of $\mathcal{Q}$. Clearly, if $(\mathcal{Q}, q)$ is the spin structure induced by $\mathcal{R}$, then $q : \mathcal{R}' \to \mathcal{R}$ is a disconnected 2-1 cover, i.e. $\mathcal{R}' \cong \mathcal{R} \times \{\pm 1\}$. Conversely, if $\mathcal{R}' \cong \mathcal{R} \times \{\pm 1\}$, then:

$$\mathcal{Q} \cong \mathcal{R}' \times_{(\mathrm{SU}(3)\times\{\pm 1\})} \mathrm{Spin}(6) \cong (\mathcal{R} \times \{\pm 1\}) \times_{(\mathrm{SU}(3)\times\{\pm 1\})} \mathrm{Spin}(6) \cong \mathcal{R} \times_{\mathrm{SU}(3)} \mathrm{Spin}(6)$$

with q defined accordingly, and thus $(\mathcal{Q}, q)$ is precisely the spin structure on N induced by $\mathcal{R}$.

Using this observation, I now prove that σ is bijective. Given a choice of spin structure $(\mathcal{Q}, q)$ on N, consider the bundle ${}^{\mathcal{Q}}\!\big/_{\mathrm{SU}(3)}$, where one identifies $\mathrm{SU}(3) \subset \mathrm{Spin}(6)$ via ϱ, and observe that sections of the bundle ${}^{\mathcal{Q}}\!\big/_{\mathrm{SU}(3)}$ correspond to $\mathrm{SU}(3)$-subbundles of $\mathcal{Q}$. Since $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ (see [91, Ch. I, Thm. 8.1]), ${}^{\mathrm{Spin}(6)}\!\big/_{\mathrm{SU}(3)} \cong {}^{\mathrm{SU}(4)}\!\big/_{\mathrm{SU}(3)} \cong S^7$ and thus it follows from classical Obstruction Theory (see [125, Thms. 6.11 & 6.12]) that $\mathcal{Q}$ admits an $\mathrm{SU}(3)$-subbundle and that any two such subbundles are homotopic. Given such a subbundle $\mathcal{R}'$, the image $\mathcal{R} = q(\mathcal{R}') \subset \mathcal{P}$ defines an $\mathrm{SU}(3)$-subbundle of $\mathcal{P}$ and since $q^{-1}(\mathcal{R}) \cong \mathcal{R} \times \{\pm 1\}$, $(\mathcal{Q}, q)$ is precisely the spin structure induced by $\mathcal{R}$; thus the map σ is surjective. Moreover, since homotopic $\mathrm{SU}(3)$-subbundles of $\mathcal{Q}$ give rise to homotopic $\mathrm{SU}(3)$-subbundles of $\mathcal{P}$, the injectivity of σ is now clear.

$\square$

Remarks 9.4.2.

1. The above argument provides an alternative proof that $\mathrm{H}^1\left(\mathrm{N};\mathbb{Z}/_{2\mathbb{Z}}\right)$ acts faithfully and transitively on $\mathcal{SL}_{\mathbb{C}}(\mathrm{N})$, by using the well known result (see [91, p. 82, Thm. 1.7]) that the space of spin structures on a spin manifold N forms a torsor for the group $\mathrm{H}^1\left(\mathrm{N};\mathbb{Z}/_{2\mathbb{Z}}\right)$.

2. Returning to the perspective of classical Obstruction Theory, recall from [125, Thm. 6.11] that the primary (and, in this case, only) obstruction to the existence of a section of $\bigwedge^3_-\mathrm{T}^*\mathrm{N}$ is determined by an obstruction class:

$$\gamma \in \mathrm{H}^2\left(\mathrm{N};\pi_1\left(\textstyle\bigwedge^3_-\mathrm{T}^*_\bullet\mathrm{N}\right)\right) \cong \mathrm{H}^2\left(\mathrm{N};\mathbb{Z}/_{2\mathbb{Z}}\right).$$

Theorem 1.0.16 shows that γ is simply the second Stiefel–Whitney class of N.

Note that the bundle $\mathcal{Q}/_{\mathrm{SU}(3)}$ arising in the above proof is essentially the unit sphere bundle in the spinor bundle $\mathcal{S}(\mathrm{N}) = \mathcal{Q}\times_{\mathrm{Spin}(6)}\mathbb{C}^4$ associated with $(\mathcal{Q},q)$, where $\mathrm{Spin}(6)$ acts on $\mathbb{C}^4$ via the identification $\mathrm{Spin}(6)\cong\mathrm{SU}(4)$. Using this observation, it is possible to provide a very explicit description of the correspondence between homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms and spin structures. Indeed, fix a choice of spin structure $(\mathcal{Q},q)$ on N and recall that the rank 10 complex vector bundle $\odot^2_{\mathbb{C}}\mathcal{S}(\mathrm{N})$ is isomorphic to the bundle $\bigwedge^3_{\mathbb{C}SD}\mathrm{T}^*\mathrm{N}$ of complex self-dual 3-forms, i.e. 3-forms α satisfying $\star\alpha = i\alpha$, where $\star$ denotes the Hodge star induced by the metric g. Given a non-zero section ς of $\mathcal{S}(\mathrm{N})$, the section $\varsigma\otimes\varsigma \in \odot^2_{\mathbb{C}}\mathcal{S}(\mathrm{N})$ corresponds to a complex 3-form α_ς and thus to a real 3-form $\rho_\varsigma = \alpha_\varsigma + \overline{\alpha_\varsigma}$. Since the stabiliser of ς in $\mathrm{Spin}(6)$ at each point of N is isomorphic to $\mathrm{SU}(3)$, the stabiliser of ρ_ς in $\mathrm{SO}(6)$ at each point of N is also isomorphic to $\mathrm{SU}(3)$, and thus ρ_ς is an $\mathrm{SL}(3;\mathbb{C})$ 3-form such that the metric g is Hermitian with respect to J_ρ. Since all non-zero sections of $\mathcal{S}(\mathrm{N})$ are homotopic, it is immediately clear that all $\mathrm{SL}(3;\mathbb{C})$ 3-forms obtained in this way are likewise homotopic.

Conversely, given an $\mathrm{SL}(3;\mathbb{C})$ 3-form ρ, choose a Hermitian metric g on N (with respect to J_ρ). For each spin structure $(\mathcal{Q},q)$ on (N,g), there is a unique section ς of the bundle $\mathcal{S}(\mathrm{N})\backslash\mathrm{N}/_{\{\pm1\}}$ such that $\rho=\rho_\varsigma$ (note that ρ_ς is well-defined, since $\rho_{-s}=\rho_s$ for each non-zero spinor s). It follows from the proof of Theorem 1.0.16, that there is a unique spin structure $(\mathcal{Q},q)$ such that the section ς of $\mathcal{S}(\mathrm{N})\backslash\mathrm{N}/_{\{\pm1\}}$ can be lifted to a section of $\mathcal{S}(\mathrm{N})\backslash\mathrm{N}$ and this is precisely the spin structure induced by ρ.

I end this subsection by providing some explicit examples.

Examples 9.4.3.

1. Consider the torus $\mathrm{N}=\mathbb{T}^6$ and let ρ_- denote the 'standard' $\mathrm{SL}(3;\mathbb{C})$ 3-form on $\mathbb{T}^6$ defined by identifying $\mathrm{T}\left(\mathbb{T}^6\right)$ with $\mathbb{T}^6\times\mathbb{R}^6$. Since $\mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}/_{2\mathbb{Z}}\right)\cong\left(\mathbb{Z}/_{2\mathbb{Z}}\right)^6$, $\mathbb{T}^6$ admits $2^6=64$ distinct homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms. Moreover, since the map $\mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}\right)\to\mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}/_{2\mathbb{Z}}\right)$ is surjective, one may provide an explicit description of all 64 classes as follows. Let $\left(x^1,...,x^6\right)$ denote the canonical periodic coordinates on $\mathbb{T}^6\cong\mathbb{R}^6/_{\mathbb{Z}^6}$ and, for each $a=(a_1,...,a_6)\in\left(\mathbb{Z}/_{2\mathbb{Z}}\right)^6\cong\mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}/_{2\mathbb{Z}}\right)$,

consider the map:

$$\begin{aligned} f_a : \mathbb{T}^6 &\longrightarrow \mathrm{U}(1) \\ \left(x^1, ..., x^6\right) &\longmapsto \exp\left(i \textstyle\sum_{j=1}^6 a_j x^j\right). \end{aligned}$$

The map f_a represents a cohomology class in $\mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}\right)$ which maps to a under $r_2 : \mathrm{H}^1\left(\mathbb{T}^6;\mathbb{Z}\right) \to \mathrm{H}^1\left(\mathbb{T}^6; \mathbb{Z}/2\mathbb{Z}\right)$. It follows that the 64 homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on $\mathbb{T}^6$ can be explicitly represented by the 3-forms:

$$\rho_a = \cos\left(\sum_{j=1}^6 a_j x^j\right)\rho_- + \sin\left(\sum_{j=1}^6 a_j x^j\right) J_{\rho_-}\rho_-.$$

2. Let (N, J, g) be a compact, Hermitian manifold. By [6, Prop. 3.2] (see also [70, Thm. 2.2]), there is a bijective correspondence between spin structures on N and holomorphic square roots of the canonical bundle $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$. Thus, given an $\mathrm{SL}(3;\mathbb{C})$ 3-form ρ on N such that:

$$J_\rho = J, \tag{9.4.4}$$

Theorem 9.4.1 predicts that ρ defines a holomorphic square root of $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$.

In the case where $\wedge^{3,0}\mathrm{T}^*\mathrm{N} \cong \mathcal{O}$ (i.e. (N, J, g) has trivial canonical bundle) this may be seen directly as follows. Initially, let ρ be an $\mathrm{SL}(3;\mathbb{C})$ 3-form on N satisfying eqn. (9.4.4) such that $\mathrm{d}\rho = \mathrm{d}J^*\rho = 0$. Then, $\Omega = \rho + iJ^*\rho$ defines a non-zero holomorphic $(3,0)$-form on N, hence a holomorphic trivialisation of $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$ and whence a natural square root of $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$, *viz.* $\mathcal{O}$.

Now, let ρ' be an arbitrary $\mathrm{SL}(3;\mathbb{C})$ 3-form satisfying eqn. (9.4.4). Firstly, note that ρ' canonically defines a class $\delta_{\rho'}$ in $\mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$. Indeed, ρ' defines a unique map $f_{\rho'} : \mathrm{N} \to \mathbb{C}\backslash\{0\}$ via $\rho' + iJ^*\rho' = f\Omega$. Define $\delta_{\rho'}$ to be the reduction modulo 2 of the pullback of the canonical generator $\mathbf{1} \in \mathrm{H}^1\left(\mathbb{C}\backslash\{0\};\mathbb{Z}\right)$ along the map $f_{\rho'}$. Next, by [70, p. 15], the space of holomophic square roots of $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$ is naturally a torsor for the group $\mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$. Indeed, the short exact sequence of sheaves:

$$\begin{array}{ccccccccc} \mathbf{1} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathcal{O}^* & \longrightarrow & \mathcal{O}^* & \longrightarrow & \mathbf{1} \\ & & & & f & \longmapsto & f^2 & & \end{array}$$

induces a sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^1\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right) & \longrightarrow & \underbrace{\mathrm{H}^1\left(\mathrm{N};\mathcal{O}^*\right)}_{\cong Pic(\mathrm{N})} & \longrightarrow & \underbrace{\mathrm{H}^1\left(\mathrm{N};\mathcal{O}^*\right)}_{\cong Pic(\mathrm{N})} \\ & & & & L & \longmapsto & L^{\otimes 2} \end{array}$$

as claimed. The square root of $\wedge^{3,0}\mathrm{T}^*\mathrm{N}$ defined by the $\mathrm{SL}(3;\mathbb{C})$ 3-form ρ' is then simply $\delta_{\rho'} \cdot \mathcal{O}$.

I end by remarking on one interesting aspect of this case. By Theorem 9.4.1, two $\mathrm{SL}(3;\mathbb{C})$ 3-forms ρ' and ρ'' satisfying eqn. (9.4.4) are homotopic through arbitrary $\mathrm{SL}(3;\mathbb{C})$ 3-forms if and only if $\delta_{\rho'} = \delta_{\rho''}$. However, clearly ρ' and ρ'' are homotopic through $\mathrm{SL}(3;\mathbb{C})$ 3-forms satisfying eqn. (9.4.4) if and only if the induced maps $f_{\rho'}, f_{\rho''} : \mathrm{N} \to \mathbb{C}\backslash\{0\}$ are homotopic, which occurs if and only if the classes $f_{\rho'}^*\mathbf{1}$ and $f_{\rho''}^*\mathbf{1}$ in $\mathrm{H}^1(\mathrm{N};\mathbb{Z})$ coincide. Thus, in general, there exist pairs of homotopic $\mathrm{SL}(3;\mathbb{C})$ 3-forms, each satisfying eqn. (9.4.4), which nevertheless cannot be connected by any path of $\mathrm{SL}(3;\mathbb{C})$ 3-forms satisfying eqn. (9.4.4).

9.4.2 Extendibility of $\mathrm{SL}(3;\mathbb{C})$ 3-forms

By Definition 9.3.1, ρ is extendible if and only if the almost complex manifold (N, J_ρ) admits a pseudo-Hermitian metric of (real) signature $(2,4)$. In general, the existence of metrics of indefinite signature is an open problem and thus completely classifying when ρ is extendible appears unfeasible. Nevertheless, much insight into the extendibility of ρ can be gained by the following proposition:

Proposition 9.4.5. *Let* N *be an oriented 6-manifold and* ρ *an* $\mathrm{SL}(3;\mathbb{C})$ *3-form on* N*. Then* ρ *is extendible if and only if the (complex) projectivised tangent bundle of* N*,* $\mathbb{P}_{\mathbb{C}}(\mathrm{TN}, J_\rho)$*, admits a global section, i.e.* (TN, J_ρ) *admits a complex line subbundle.*

Proof. Initially, suppose that $\mathscr{L} \subset (\mathrm{TN}, J_\rho)$ is a complex line subbundle. Let g be any Hermitian metric (of real signature $(3,0)$) on N and write:

$$\mathrm{TN} = \mathscr{L} \oplus \mathscr{L}^\perp, \quad g = g|_{\mathscr{L}} + g|_{\mathscr{L}^\perp},$$

where the orthocomplement is defined with respect to g. Then $g_{\mathscr{L}} - g_\perp$ is a pseudo-Hermitian metric of (real) signature $(2,4)$.

Conversely, let $\widetilde{g}$ be a pseudo-Hermitian metric of real signature $(2,4)$. Define a subbundle $\Pi_{\widetilde{g}} \subset \mathbb{P}_{\mathbb{C}}(\mathrm{TN}, J_\rho)$ via:

$$\Pi_{\widetilde{g}}\big|_p = \left\{\mathscr{L} \in \mathbb{P}_{\mathbb{C}}(\mathrm{TN}, J_\rho)|_p \;\middle|\; \widetilde{g} \text{ is positive definite on } \mathscr{L} \subset \mathrm{T}_p\mathrm{N}\right\}.$$

Given $\mathscr{L} \in \Pi_{\widetilde{g}}\big|_p$, every other $\mathcal{L}' \in \Pi_{\widetilde{g}}\big|_p$ can be written as a graph over $\mathscr{L}$. Thus $\Pi_{\widetilde{g}}$ has contractible fibres and hence admits a global section, and whence so does $\mathbb{P}_{\mathbb{C}}(\mathrm{TN}, J_\rho)$.

□

I now prove the main result of this subsection:

Theorem 9.4.6. *Let* N *be an oriented 6-manifold. If the Euler class* $e(\mathrm{N}) = 0$*, then any* $\mathrm{SL}(3;\mathbb{C})$ *3-form on* N *is extendible. In particular:*

- *If* N *is open, then any* $\mathrm{SL}(3;\mathbb{C})$ *3-form on* N *is extendible.*
- *If* N *is closed and the Euler characteristic* $\chi(\mathrm{N}) = 0$*, then any* $\mathrm{SL}(3;\mathbb{C})$ *3-form on* N *is extendible.*

Conversely, if $e(\mathrm{N}) \neq 0$ *and in addition* $b^2 = 0$ *(i.e.* $\mathrm{H}^2(\mathrm{N};\mathbb{Z})$ *and* $\mathrm{H}^4(\mathrm{N};\mathbb{Z})$ *are pure torsion), then no* $\mathrm{SL}(3;\mathbb{C})$ *3-form on* N *is extendible.*

Proof. Firstly note that if $e(\mathrm{N}) = 0$, then N admits a nowhere vanishing vector field. Indeed if N is closed this follows from [74], whereas if N is open, this follows since N deformation retracts onto a subcomplex of its 5-skeleton and every rank 6 vector bundle over a 5-dimensional cell-complex admits a nowhere vanishing section. Thus let X be a nowhere vanishing vector field on N and let ρ be an $\mathrm{SL}(3;\mathbb{C})$ 3-form on N. Then the real rank 2 distribution on N generated by X and $J_\rho X$ defines a complex line subbundle of (TN, J_ρ) and hence ρ is extendible, by Proposition 9.4.5.

Conversely, suppose $e(\mathrm{N}) \neq 0$ (in particular, N must be closed) and let ρ be an extendible $\mathrm{SL}(3;\mathbb{C})$ 3-form on N. By Proposition 9.4.5, one can write $(\mathrm{TN}, J_\rho) = \mathscr{L}_1 \oplus \mathscr{L}_2$ with $\mathscr{L}_{1,2}$ complex subbundles of (TN, J_ρ) of (complex) ranks 1 and 2 respectively. Then:

$$e(\mathrm{N}) = e(\mathscr{L}_1) \cup e(\mathscr{L}_2) \in \mathrm{H}^6(\mathrm{N};\mathbb{Z}) \cong \mathbb{Z},$$

where $\cup$ denotes the usual cup-product on cohomology. Since $e(\mathrm{N})$ is non-zero, neither $e(\mathscr{L}_1)$ nor $e(\mathscr{L}_2)$ can have finite order (since $\mathbb{Z}$ is torsion-free). Thus $b^2(\mathrm{N}) \neq 0$, as claimed.

□

Using Theorem 9.4.6, it is possible to give many examples of extendible and non-extendible $\mathrm{SL}(3;\mathbb{C})$ 3-forms:

Examples.

1. Let K be any closed, oriented (connected) spin 5-manifold and set $\mathrm{N} = S^1 \times \mathrm{K}$. Then N is also oriented and spin. Thus since $\chi(\mathrm{N}) = 0$ and:

$$\mathrm{H}^1\left(S^1 \times \mathrm{K}; \mathbb{Z}/2\mathbb{Z}\right) \cong \mathrm{H}^1\left(\mathrm{K}; \mathbb{Z}/2\mathbb{Z}\right) \oplus \mathrm{H}^0\left(\mathrm{K}; \mathbb{Z}/2\mathbb{Z}\right) \cong \mathrm{H}^1\left(\mathrm{K}; \mathbb{Z}/2\mathbb{Z}\right) \oplus \mathbb{Z}/2\mathbb{Z},$$

   by Theorem 1.0.16 the manifold N admits $2^{1+b^1\left(\mathrm{N};\mathbb{Z}/2\right)} \geqslant 2$ distinct homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms, all of which are extendible by Theorem 9.4.6. As a special case, $\mathbb{T}^6$ admits $2^6 = 64$ distinct homotopy classes of extendible $\mathrm{SL}(3;\mathbb{C})$ 3-forms.

2. Consider the sphere S^6. Clearly S^6 is orientable and spin, and $\mathrm{H}^1\left(S^6; \mathbb{Z}/2\mathbb{Z}\right) = 0$. Thus S^6 admits a unique homotopy class of $\mathrm{SL}(3;\mathbb{C})$ 3-forms, which is not extendible since $\chi(S^6) = 2$. In particular, recall that the embedding $S^6 \hookrightarrow \mathbb{R}^7$, where $\mathbb{R}^7$ is equipped with its standard (flat) G_2 3-form ϕ_0, induces the 'standard' $\mathrm{SL}(3;\mathbb{C})$ 3-form on S^6; then $\phi_0|_{S^6}$ is not extendible (to a $\widetilde{\mathrm{G}}_2$ 3-form).

3. Consider the manifold $Y_1 = \mathbb{RP}^3 \times \mathbb{RP}^3$ and let $Y_n = \underbrace{Y_1 \# \ldots \# Y_1}_{n \text{ times}}$, where $\#$ denotes connected sum. Then Y_n is spin and $\mathrm{H}^1\left(Y_n; \mathbb{Z}/2\mathbb{Z}\right) \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^{2n}$, so Y_n admits 2^{2n} distinct homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms. However, the Betti numbers of Y_n are:

$$\left(b^0, b^1, b^2, b^3, b^4, b^5, b^6\right) = (1, 0, 0, 2n, 0, 0, 1)$$

and so $\chi(Y_n) = 2 - 2n$, whilst $b^2(Y_n) = 0$. Thus for $n > 1$, none of the 2^{2n} homotopy classes of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on Y_n are extendible.

9.5 Topological properties of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms

9.5.1 Homotopic $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms

Let N be an oriented 6-manifold.

Proposition 9.5.1. *Write $\mathcal{SL}_{\mathbb{R}}(\mathrm{N})$ for the set of homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on N and $\widetilde{\mathcal{GR}}_3(\mathrm{N})$ for the set of homotopy classes of sections of $\widetilde{\mathrm{Gr}}_3(\mathrm{N})$. Then there is a 1-1 correspondence:*

$$\mathcal{E} : \mathcal{SL}_{\mathbb{R}}(\mathrm{N}) \to \widetilde{\mathcal{GR}}_3(\mathrm{N})$$

given by $[\rho] \mapsto [E_{+,\rho}]$. In particular, N admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms if and only if it admits oriented rank 3 distributions. The same conclusion applies to homotopy classes of closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, or to homotopy classes of closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms representing a fixed de Rham class.

Proof. Write $\Lambda^3_+\mathrm{T}^*\mathrm{N}$ for the bundle of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms over N and consider the diagram:

$$\begin{tikzcd}
\Lambda^3_+\mathrm{T}^*\mathrm{N} \arrow[d, "\rho\mapsto E_{+,\rho}"'] \arrow[dr] & \\
\widetilde{\mathrm{Gr}}_3(\mathrm{TN}) \arrow[r] & \mathrm{N}
\end{tikzcd}$$

Then the left-hand map is a fibration, with fibre diffeomorphic to $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(E_+)\big/\mathrm{SL}(3;\mathbb{R})^2$, where $E_+ = \langle e_1, e_2, e_3\rangle$ denotes the +1-eigenspace of the standard para-complex structure I_0 on $\mathbb{R}^6$. Explicitly:

$$\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(E_+) = \left\{ \begin{pmatrix} A & C \\ & B \end{pmatrix} \;\middle|\; A, B \in \mathrm{GL}_+(3;\mathbb{R}), C \in \mathfrak{gl}(3;\mathbb{R}) \right\}.$$

The result follows, since the quotient $\mathrm{Stab}_{\mathrm{GL}_+(6;\mathbb{R})}(E_+)\big/\mathrm{SL}(3;\mathbb{R})^2$ is contractible. The final remark now follows from Theorem 8.2.1.

$\square$

Explicitly, the inverse to $\mathcal{E}$ can be described as follows: given an oriented rank 3 distribution E on N, choose a distribution E' such that $\mathrm{TN} = E \oplus E'$. Since E and TN are oriented, so is E' and thus one can pick volume elements $\varpi_\pm \in \Lambda^3 E^*_\pm$. Using the inclusion:

$$\bigwedge^3 E^*_+ \oplus \bigwedge^3 E^*_- \hookrightarrow \bigwedge^3 (E_+ \oplus E_-)^* \cong \bigwedge^3 \mathrm{T}^*(\mathbb{T}^6)$$

one may regard $\rho = \varpi_+ + \varpi_-$ as a 3-form on $\mathbb{T}^6$. It is then simple to verify that ρ is an $\mathrm{SL}(3;\mathbb{R})^2$ 3-form on $\mathbb{T}^6$ such that $E_\pm$ are the ±1-eigenbundles of the para-complex structure I_ρ.

Note that in order for N to admit an oriented rank 3 distribution, the Euler class $e(\mathrm{N})$ must vanish (cf. [93, Prop. 5.1.1]). I now prove the main result of this subsection:

Theorem 9.5.2. *Let* N *be a closed, oriented, 6-manifold with* $e(\mathrm{N}) = 0$ *and suppose* $w_2(\mathrm{N})^2 = 0$. *Write* $\rho_2 : \mathrm{H}^4(\mathrm{N};\mathbb{Z}) \to \mathrm{H}^4\left(\mathrm{N}; \mathbb{Z}/2\mathbb{Z}\right)$ *for reduction modulo 2 and define:*

$$\mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2} = \left\{u \in \mathrm{H}^4(\mathrm{N};\mathbb{Z}) \;\middle|\; \rho_2 u \cup w_2(\mathrm{N}) = 0\right\}.$$

Then there is an injection from $\mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2} \big/ \text{2-torsion}$ *into the set of homotopy classes of* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms on* N *(equivalently closed* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms, or* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms in any fixed degree 3 de Rham class). In particular, if* N *is spin and* $b^4(\mathrm{N}) > 0$, *then each of these sets is infinite.*

Proof. Recall the first spin characteristic class q_1 defined by Thomas in [118, Thm. 1.2], which is related to the first Pontryagin class p_1 by $p_1 = 2q_1$. Now assume $e(\mathrm{N}) = 0$ and $w_2(\mathrm{N})^2 = 0$. By applying [119, Cor. 1.7], for every $u \in \mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2}$, there exists an oriented, spin, rank 3 distribution E on N with $q_1(E) = 2u$. Given classes $u, u' \in \mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2}$ with corresponding bundles E and E', note that if E and E' are homotopic as sections of $\widetilde{\mathrm{Gr}}_3(\mathrm{TN})$, then $q_1(E) = q_1(E')$, and hence $2(u - u') = 0$. The result follows.

□

Corollary 1.0.19 now follows at once, by restricting attention to the case $w_2(\mathrm{N}) = 0$. I remark that $w_2(\mathrm{N}) = 0$ is actually necessary for N to admit any extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms; indeed, this result follows from Proposition 9.3.2 and Theorem 9.2.1, together with the fact that the boundary of a spin manifold is also spin. Thus the condition that N be spin is very natural from the perspective taken in this paper.

Using the above results, one can give many examples of manifolds admitting multiple homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms:

Examples 9.5.3.

1. As a simple example, take $\mathrm{N} = \mathbb{T}^6$. $\mathbb{T}^6$ is parallelisable and so trivially it is orientable, spin and has vanishing Euler class. Since $\mathrm{H}^4\left(\mathbb{T}^6;\mathbb{Z}\right) \cong \mathbb{Z}^{15}$, it follows that $\mathbb{T}^6$ admits infinitely many distinct homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms.

2. Now consider $\mathrm{N} = S^6$. Since $\chi(S^6) = 2$, S^6 admits no $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms. Likewise, the manifolds Y_n $(n > 1)$ considered in §9.4.2 admit no $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms.

3. Let $\mathrm{N} = S^1 \times \mathrm{K}$ where K is any closed, orientable, spin 5-manifold. Then $S^1 \times \mathrm{K}$ is also spin and has vanishing Euler class. Thus $S^1 \times \mathrm{K}$ admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms. Moreover, if $b^4(\mathrm{N}) = 0$, then the relation $b^n(\mathrm{N}) = b^n(\mathrm{K}) + b^{n-1}(\mathrm{K})$ forces $b^3(\mathrm{K}) = b^4(\mathrm{K}) = 0$, and hence $b^1(\mathrm{K}) = b^2(\mathrm{K}) = 0$ too, by Poincaré duality, i.e. K is a rational homology sphere. Thus, unless K is a rational homology sphere, N admits infinitely many distinct homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms.

4. Let $\mathcal{E}$ denote the Enriques surface – i.e. the quotient of a K3-surface by a fixed-point-free holomorphic involution – viewed as a real 4-manifold. Then the 6-manifold $\mathrm{N} = \mathbb{T}^2 \times \mathcal{E}$ provides

an example of a non-spin manifold which admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms; in fact, N admits an infinite number of distinct homotopy classes of $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, none of which can be extendible, by the above remark.

To verify this, it is necessary to recall some results on the topology of $\mathcal{E}$. Recall that [13, Lem. 15.1], [33, §6.10][1]:

$$\mathrm{H}^2(\mathcal{E};\mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$$

where the $\mathbb{Z}/2\mathbb{Z}$-factor is generated by $c_1(\mathcal{E})$ and recall also that the Betti numbers of $\mathcal{E}$ are $b^0(\mathcal{E}) = b^4(\mathcal{E}) = 1$, $b^1(\mathcal{E}) = b^3(\mathcal{E}) = 0$ and $b^2(\mathcal{E}) = 10$. The restriction of $c_1(\mathcal{E})$ modulo 2 is non-zero and thus $w_2(\mathcal{E}) \neq 0$ and $\mathcal{E}$ (and hence N) is not spin. Nevertheless $\chi(\mathcal{E}) = 12$ which vanishes modulo 2 and thus $w_2(\mathcal{E})^{\cup 2} = w_4(\mathcal{E}) = 0$. Hence by Theorem 9.5.2 $\mathrm{N} = \mathbb{T}^2 \times \mathcal{E}$ admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms. Moreover, given a class $u \in \mathrm{H}^4(\mathrm{N};\mathbb{Z})$, identifying $c_1(\mathcal{E})$ with an integral degree 2 cohomology class on N in the natural way, one finds that:

$$2(u \cup c_1(\mathcal{E})) = u \cup 2c_1(\mathcal{E}) = 0$$

and hence $u \cup c_1(\mathcal{E}) = 0$, since $\mathrm{H}^6(\mathrm{N};\mathbb{Z})$ has no 2-torsion. Since $w_2(\mathrm{N}) = \rho_2 c_1(\mathcal{E})$ it follows that:

$$\rho_2 u \cup w_2(\mathrm{N}) = \rho_2(u \cup c_1(\mathcal{E})) = 0,$$

and thus $\mathrm{H}^4(\mathrm{N};\mathbb{Z})_{\perp w_2} = \mathrm{H}^4(\mathrm{N};\mathbb{Z})$. Since $b^4(\mathrm{N}) = 11$, it follows that $\mathcal{SL}_{\mathbb{R}}(\mathrm{N})$ is infinite, as claimed.

9.5.2 Examples of extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms

The previous subsection saw examples of non-extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms, however so far no explicit examples of extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms have been provided. This section aims to provide such examples. I begin with a preparatory result:

Proposition 9.5.4. *Let* N *be an oriented 6-manifold and let* ρ *be an* $\mathrm{SL}(3;\mathbb{R})^2$ *3-form on* N*. Then* ρ *is extendible if and only if the bundle* $\mathrm{Iso}(E_{+,\rho}, E_{-,\rho})$ *of isomorphisms from* $E_{+,\rho}$ *to* $E_{-,\rho}$ *has a global section.*

Proof. By Definition 9.3.1, ρ is extendible if and only if N admits a 2-form ω satisfying $\omega^3 < 0$ such that $\mathscr{I}_\rho\omega$ is a pseudo-Riemannian metric of signature $(3,3)$. Thus suppose ρ is extendible and define $\widetilde{g} = \mathscr{I}_\rho\omega$. One may verify that $\widetilde{g}(I_\rho\cdot, I_\rho\cdot) = -\widetilde{g}(\cdot,\cdot)$. Thus given $u \in E_{+,\rho}$, for any other $w \in E_{+,\rho}$ one has:

$$\widetilde{g}(u,w) = \widetilde{g}(I_\rho u, I_\rho w) = -\widetilde{g}(u,w) = 0 \tag{9.5.5}$$

and hence $\widetilde{g}(v,\cdot)$ may naturally be identified with an element of $(E_{-,\rho})^*$.

Now choose a positive definite metric h on $E_{-,\rho}$ and define $L(v) \in E_{-,\rho}$ to be the unique element such that:

$$h(L(v),\cdot) = \widetilde{g}(v,\cdot) \in (E_{-,\rho})^*. \tag{9.5.6}$$

[1]Note a slight error in this second reference: $h^{2,0}(\mathcal{E}) = 0$, not 1 as stated *loc. cit.*.

Then L defines an isomorphism $E_{+,\rho} \to E_{-,\rho}$.

Conversely, let $L : E_{+,\rho} \to E_{-,\rho}$ be a fibrewise isomorphism. Define subbundles $F_\pm \subset \mathrm{TN}$ by:

$$F_\pm = (\mathrm{Id} \pm L)(F_{+,\rho}).$$

Then $\mathrm{TN} = F_+ \oplus F_-$ and I_ρ maps F_+ isomorphically onto F_- and *vice versa*.

Now choose any positive definite metric l on F_+, extend it to a symmetric bilinear form on TN by setting $l(F_-, \cdot) = 0$ and define:

$$\widetilde{g}(\cdot,\cdot) = l(\cdot,\cdot) - l(I_\rho\cdot, I_\rho\cdot).$$

Then $\omega(\cdot,\cdot) = \widetilde{g}(I_\rho\cdot,\cdot)$ is a 2-form and $\mathscr{I}_\rho\omega = \widetilde{g}$ has signature $(3,3)$. In particular, it follows that $\omega^3 \neq 0$ and hence, by replacing ω by $-\omega$ if necessary, one may assume that $\omega^3 < 0$. Thus ρ is extendible.

$\square$

I remark that Proposition 9.5.4 has the following, curious result. Recall that if ρ is an $\mathrm{SL}(3;\mathbb{C})$ 3-form, then ρ is always homotopic to $-\rho$ [37, §4]. By applying Proposition 9.5.4, a partial analogue for $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms may be obtained:

Corollary 9.5.7. *Let* N *be an oriented 6-manifold and let ρ be an extendible* $\mathrm{SL}(3;\mathbb{R})^2$ *3-form on* N. *Then ρ is homotopic (through* $\mathrm{SL}(3;\mathbb{R})^2$ *3-forms) to $-\rho$.*

In particular, by Theorem 8.2.1, if ρ is closed and extendible, then ρ is homotopic to $-\rho$ through closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms and, likewise, if ρ is exact and extendible, then ρ is homotopic to $-\rho$ through exact $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms.

Proof. By Proposition 9.5.1, it is equivalent to prove that the sections of $\widetilde{\mathrm{Gr}}_3(\mathrm{N})$ induced by $E_{+,\rho}$ and $\overline{E}_{+,\rho}$ (the same bundle equipped with the opposite orientation) are homotopic. Since ρ is extendible, one can choose $L \in \mathrm{Iso}(E_{+,\rho}, E_{-,\rho})$. For $t \in [0,1]$, define:

$$\begin{aligned} \iota_t : E_{+,\rho} &\longrightarrow \mathrm{TN} \\ v &\longmapsto \cos(\pi t)v + \sin(\pi t)L(v). \end{aligned}$$

Then $\iota_t(E_{+,\rho})$ defines the required homotopy from $E_{+,\rho}$ to $\overline{E}_{+,\rho}$.

$\square$

The converse of Corollary 9.5.7 does not hold. Indeed, consider the manifold $\mathrm{N} = \mathbb{T}^2 \times \mathcal{E}$ of Example 9.5.3(4). It is known [13, Lem. 15.1], [33, §6.10] that the intersection form of $\mathcal{E}$ is indefinite with signature $\sigma(\mathcal{E}) = 8$; in particular, since $\chi(\mathcal{E}) = 12$ the equations:

$$\sigma(\mathcal{E}) \pm \chi(\mathcal{E}) \equiv 0 \mod 4$$

hold. Using [101, Thm. 2(A)], it follows that there exists an oriented rank 2 distribution $\mathcal{E}$, which I shall denote by π. Write ∂_1, ∂_2 for the standard basis of constant vector fields on $\mathbb{T}^2$ and define an oriented rank 3 distribution E on $\mathrm{N} = \mathbb{T}^2 \times \mathcal{E}$ via:

$$E = \langle \partial_1 \rangle \oplus \pi$$

where E is oriented according to the ordering of the summands. Then E is manifestly homotopic to $\overline{E}$: an explicit homotopy is given by:

$$E_t = \langle \cos(\pi t)\partial_1 + \sin(\pi t)\partial_2 \rangle \oplus \pi.$$

However, as observed above, none of the $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on N are extendible.

I now return to the task of constructing explicit examples of extendible forms. Recall the following definition:

Definition 9.5.8 (See [11, §2])**.** Let M be an arbitrary manifold. A vector bundle $\mathbb{E} \to \mathrm{M}$ is termed flat if it admits some connection of curvature 0. Equivalently, write $\widetilde{\mathrm{M}}$ for the universal covering space of M and view $\widetilde{\mathrm{M}} \to \mathrm{M}$ as a principal $\pi_1(\mathrm{M})$-bundle over M. Then a vector bundle $\mathbb{E} \to \mathrm{M}$ of rank k is flat if and only if it can be written as:

$$\mathbb{E} = \widetilde{\mathrm{M}} \times \mathbb{R}^k \big/ \pi_1(\mathrm{M})$$

for some representation $\pi_1(\mathrm{M}) \to \mathrm{GL}(k;\mathbb{R})$.

This definition can be made very explicit for $\mathbb{T}^6$: identify $\mathbb{T}^6 \cong \mathbb{R}^6 \big/ \mathbb{Z}^6$ so that the quotient map $\mathbb{R}^6 \to \mathbb{T}^6$ is also the universal cover. Then for every representation $\varrho : \mathbb{Z}^6 \to \mathrm{GL}(k;\mathbb{R})$, one obtains the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^6 \times \mathbb{R}^k \arrow[r, "\text{quot}"] \arrow[d, "proj_1"] & \mathbb{R}^6 \times \mathbb{R}^k \big/ (\mathbb{Z}^6, \varrho) = \mathbb{E} \arrow[d] \\
\mathbb{R}^6 \arrow[r, "\text{quot}"] & \mathbb{R}^6 \big/ \mathbb{Z}^6 \cong \mathbb{T}^6
\end{tikzcd}$$

where $\mathbb{Z}^6$ acts on $\mathbb{R}^6$ by translation and $\mathbb{R}^k$ via ϱ. Then $\mathbb{E} \to \mathbb{T}^6$ is the flat bundle corresponding to the representation ϱ. Moreover, note that $\mathrm{T}(\mathbb{T}^6)$ is simply the flat bundle corresponding to the trivial representation $\varrho : \mathbb{Z}^6 \to \mathrm{GL}(6;\mathbb{R})$.

The following result was proved in [11, Thm. 3.3]:

Theorem 9.5.9 (Auslander–Szczarba)**.** *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be flat, rank k vector bundles over $\mathbb{T}^6$. Then $\mathbb{E}_1$ and $\mathbb{E}_2$ are isomorphic if and only if their first and second Stiefel–Whitney classes coincide.*

I restrict attention to flat, orientable bundles; these correspond to representations ϱ whose image lies in $\mathrm{GL}_+(k,\mathbb{R})$. By Theorem 9.5.9, two flat, orientable, rank k vector bundles over $\mathbb{T}^6$ are isomorphic if and only if their second Stiefel–Whitney classes coincide. Using this result, one can construct a large number of (non-homotopic) extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on $\mathbb{T}^6$.

Construction 9.5.10. By Proposition 9.5.1, it suffices to construct oriented rank 3 distributions $E_+ \subset \mathrm{T}(\mathbb{T}^6)$ such that one can write $\mathrm{T}(\mathbb{T}^6) = E_+ \oplus E_-$ for some $E_- \cong E_+$. Let $\mathbb{E}$ be a flat, orientable, rank 3 vector bundle over $\mathbb{T}^6$ and define $\mathbb{T} = \mathbb{E}_+ \oplus \mathbb{E}_-$, where $\mathbb{E}_+ = \mathbb{E}_- = \mathbb{E}$ are all identical and the subscripts only serve to keep track of the summands. Then $\mathbb{T}$ is automatically orientable and so $w_1(\mathbb{T}) = 0$. Moreover:

$$w_2(\mathbb{T}) = 2 \cdot w_2(\mathbb{E}) + w_1(\mathbb{E})^{\cup 2} = 0 \in \mathrm{H}^2\left(\mathbb{T}^6; \mathbb{Z}/2\mathbb{Z}\right),$$

where $\cup$ denotes the usual cup-product on cohomology; thus by Theorem 9.5.9, $\mathbb{T}$ is isomorphic to $\mathrm{T}(\mathbb{T}^6)$. Write $E_\pm$ for the images of $\mathbb{E}_\pm$ respectively under this isomorphism and note that the bundle $\mathrm{Iso}(E_+, E_-)$ has a natural global section, corresponding to $\mathrm{Id} \in \mathrm{Iso}(\mathbb{E}_+, \mathbb{E}_-)$.

Given flat, orientable, rank 3 bundles $\mathbb{E}$ and $\mathbb{E}'$ with corresponding $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms ρ and ρ', by Proposition 9.5.1 ρ and ρ' are homotopic if and only if $\mathbb{E}$ and $\mathbb{E}'$ are homotopic. If $\mathbb{E}$ and $\mathbb{E}'$ are homotopic, then clearly $w_2(\mathbb{E}) = w_2(\mathbb{E}')$; moreover, the converse holds by Theorem 9.5.9. The following result classifies which classes in $\mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$ can arise as the second Stiefel–Whitney class of a flat, orientable, rank 3 vector bundle:

Proposition 9.5.11. *Let $w \in \mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$. Then w is the second Stiefel–Whitney class of a flat, orientable, rank 3 vector bundle over $\mathbb{T}^6$ if and only if it can be written as $w = a \cup b$ for some classes $a, b \in \mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$.*

Proof. Firstly, from the Künneth formula ([65, Thm. 3.15]), one may prove by induction that for any n:

$$\mathrm{H}^*\left(\mathbb{T}^n; \mathbb{Z}/2\mathbb{Z}\right) \cong \bigwedge\nolimits^*\left(\mathbb{Z}/2\mathbb{Z}\right)^n,$$

where:

$$\bigwedge\nolimits^*\left(\mathbb{Z}/2\mathbb{Z}\right)^n = \bigotimes\nolimits^*\left(\mathbb{Z}/2\mathbb{Z}\right)^n \Big/ \left\{\gamma \otimes \gamma = 0 \;\middle|\; \gamma \in \left(\mathbb{Z}/2\mathbb{Z}\right)^n\right\}$$

as usual. (Here, the tensor product is taken over $\mathbb{Z}/2\mathbb{Z}$.)

Now, by [11, Thm. 3.2], every flat vector bundle over a torus is isomorphic to a Whitney sum of flat line bundles. (The statement of Thm. 3.2 in [11] does not include the fact that the line bundles are themselves flat, however this result follows from the proof given on p. 273 *op. cit.*.) Thus consider $\mathbb{E} = \ell_1 \oplus \ell_2 \oplus \ell_3$ for ℓ_i flat line bundles over $\mathbb{T}^6$. The requirement that $\mathbb{E}$ be orientable is equivalent to the requirement that $w_1(\mathbb{E}) = 0$, i.e. that:

$$w_1(\ell_3) = w_1(\ell_1) + w_1(\ell_2).$$

Using this relation, one may compute that:

$$\begin{aligned} w_2(\mathbb{E}) &= w_1(\ell_1) \cup w_1(\ell_2) + w_1(\ell_1) \cup [w_1(\ell_1) + w_1(\ell_2)] + w_1(\ell_2) \cup [w_1(\ell_1) + w_1(\ell_2)] \\ &= w_1(\ell_1) \cup w_1(\ell_2) \end{aligned}$$

since $w_1(\ell_1)^{\cup 2} = w_1(\ell_2)^{\cup 2} = 0$, and so $w_2(\mathbb{E})$ has the form claimed.

Conversely, suppose given a class $w \in \mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$ such that $w = a \cup b$ for $a, b \in \mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$. Using the diagram:

$$\begin{tikzcd}
\mathrm{Hom}\left(\pi_1(\mathbb{T}^6), O(1)\right) \cong \mathrm{Hom}\left(H_1(\mathbb{T}^6;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}\right) \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
& \mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right) \\
\text{Flat bundles}/\text{isomorphism} \arrow[ur, "w_1", "\cong"'] &
\end{tikzcd}$$

there exist flat line bundles ℓ_1, ℓ_2 and ℓ_3 over $\mathbb{T}^6$ such that $w_1(\ell_1) = a$, $w_1(\ell_2) = b$ and $w_1(\ell_3) = a + b$. Then $w_1(\ell_1 \oplus \ell_2 \oplus \ell_3) = a + b + (a + b) = 0$ and thus $\mathbb{E} = \ell_1 \oplus \ell_2 \oplus \ell_3$ is orientable. Moreover, the earlier calculation also shows that $w_2(\mathbb{E}) = a \cup b$, as required.

□

Using Proposition 9.5.11, one can count the number of distinct homotopy classes of extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms produced by Construction 9.5.10. Firstly, note that there is a 1-1 correspondence between the non-zero elements of $\mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)$ and $\mathbb{P}_{\mathbb{Z}/2}\left(\mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)\right)$. Since:

$$\mathrm{H}^2\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right) \cong \textstyle\bigwedge^2 \mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right),$$

where the exterior-square is taken over the base field $\mathbb{Z}/2\mathbb{Z}$, it follows that the set of non-zero second Stiefel–Whitney classes of flat orientable rank 3 bundles over $\mathbb{T}^6$ is precisely the image of the 'Plücker-type' embedding:

$$\begin{aligned}
\mathrm{Gr}_2\left(\mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)\right) &\hookrightarrow \mathbb{P}_{\mathbb{Z}/2}\left(\textstyle\bigwedge^2 \mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)\right) \\
\Pi &\longmapsto \textstyle\bigwedge^2 \Pi.
\end{aligned}$$

Since $\mathrm{Gr}_2\left(\mathrm{H}^1\left(\mathbb{T}^6, \mathbb{Z}/2\mathbb{Z}\right)\right)$ contains 651 elements (see Appendix B) Construction 9.5.10 generates $652 = 651 + 1$ distinct homotopy classes of extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms over $\mathbb{T}^6$ (the extra case arising from $w_2(\mathbb{E}) = 0$.) Moreover, by applying the h-principle for closed $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms (Theorem 8.2.1) and since extendibility is a homotopy invariant, Construction 9.5.10 implies the existence of 652 distinct homotopy classes of closed, extendible $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms on $\mathbb{T}^6$.

Chapter 10

Concluding remarks and open questions

In this chapter, I make some brief remarks regarding the 7 classes of stable form for which the relative h-principle is still unverified, *viz.* G_2 3-forms, $\mathbb{P}\mathrm{SU}(3)$ 3- and 5-forms, $\mathbb{P}\mathrm{SU}(1,2)$ 3- and 5-forms and $\mathrm{SL}(3;\mathbb{R})$ 3- and 5-forms. I explain that, whilst the techniques developed in Part II cannot be straightforwardly applied to prove the relative h-principle for these 7 classes of stable forms, the relative h-principle should reasonably be expected to hold in all remaining 7 cases. The chapter ends with some brief comments regarding other partial differential relations on stable forms.

10.1 Limitations of convex integration

To understand why G_2 3-forms cannot be investigated using convex integration, I record the following result:

Proposition 10.1.1. G_2 *3-forms are not ample.*

Proof. Recall from §7.5.1 that G_2 acts transitively on $\widetilde{\mathrm{Gr}}_6\left(\mathbb{R}^7\right)$. The image of this single orbit under the map $\mathcal{T}_{\varphi_0}$ is well-known to be the orbit of $\mathrm{SL}(3;\mathbb{C})$ 3-forms on $\mathbb{R}^6$; see e.g. [37]. Moreover, for $\rho \in \bigwedge^3_-\left(\mathbb{R}^6\right)^*$, it follows from p. 106 *op. cit.* that:

$$\mathcal{N}_{\varphi_0}(\rho) = \left\{\nu \in \bigwedge^2\left(\mathbb{R}^6\right)^* \;\middle|\; \mathscr{J}_\rho\nu \text{ is a (positive definite) Hermitian form }\right\}.$$

However this is a convex subset of $\bigwedge^2\left(\mathbb{R}^6\right)^*$ which does not contain 0.

□

Thus convex integration cannot be used to prove the h-principle for G_2 3-forms. Moreover, since G_2 3-forms are not ample along any hyperplane, convex integration with avoidance cannot be used either.

By contrast, in 8-dimensions, it is not known whether stable forms are ample or not. The difficulty in verifying amplitude can be understood via the following result:

Proposition 10.1.2. $\mathbb{P}\mathrm{SU}(3)$ *3- and 5-forms are not faithful. In particular, the action of* $\mathbb{P}\mathrm{SU}(3)$ *on* $\widetilde{\mathrm{Gr}}_7\left(\mathbb{R}^8\right)$ *has an infinite number of orbits.*

Proof. As usual, since $\mathbb{P}\mathrm{SU}(3) \subset \mathrm{SO}(7)$, there is a $\mathbb{P}\mathrm{SU}(3)$-equivariant isomorphism:

$$S^7 \cong \widetilde{\mathrm{Gr}}_1\left(\mathbb{R}^8\right) \to \widetilde{\mathrm{Gr}}_7\left(\mathbb{R}^8\right)$$

and thus it suffices to understand the orbit space $\mathbb{P}\mathrm{SU}(3)\backslash S^7$.

Recall that $\mathbb{P}\mathrm{SU}(3)$ acts on $\mathbb{R}^8 \cong \mathfrak{su}(3)$ via the adjoint representation. It is well-known that for a compact Lie group G with Lie algebra $\mathfrak{g}$, after choosing a Cartan subalgebra $\mathfrak{h}$ (in this case, equivalently a maximal Abelian subalgebra) there is an isomorphism:

$$\mathrm{Ad}(G)\backslash\mathfrak{g} \cong W_G\backslash\mathfrak{h},$$

where W_G denotes the Weyl group of G (this result is essentially the infinitesimal version of [86, Thm. 4.44]). Thus:

$$\mathrm{Ad}(G)\backslash S_{\mathfrak{g}} \cong W_G\backslash S_{\mathfrak{h}},$$

where $S_{\mathfrak{g}}$ denotes the unit sphere in $\mathfrak{g}$ and $S_{\mathfrak{h}} = S_{\mathfrak{g}} \cap \mathfrak{h}$. In the specific case of $\mathrm{SU}(3)$, $\dim \mathfrak{h} = 2$ and $W_G = D_6$, the dihedral group of order 6, and thus:

$$\mathbb{P}\mathrm{SU}(3)\backslash S^7 \cong D_6\backslash S^1$$

is infinite, as claimed. The final comment follows from the fact that action of $\mathrm{GL}_+(8;\mathbb{R})$ on $\Lambda^3\left(\mathbb{R}^8\right)^*$ (and hence on $\Lambda^5\left(\mathbb{R}^8\right)^*$) has only finitely many orbits [34].

$\square$

It follows that Theorem 7.4.5 does not apply to $\mathbb{P}\mathrm{SU}(3)$-forms. Moreover, since $\mathbb{P}\mathrm{SU}(3)\backslash\widetilde{\mathrm{Gr}}_7\left(\mathbb{R}^8\right)$ is infinite, explicit 'orbit-by-orbit' calculations as performed for $\widetilde{\mathrm{G}}_2$ 3-forms, co-pseudoplectic forms and $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms do not appear feasible for $\mathbb{P}\mathrm{SU}(3)$-forms. Moreover, for $\mathbb{P}\mathrm{SU}(1,2)$-forms and $\mathrm{SL}(3;\mathbb{R})$-forms, the situation is further complicated by the need to distinguish between orbits of spacelike, timelike and null hyperplanes. Thus stable forms in 8-dimensions appear to lie outside the scope of the techniques developed in this thesis.

10.2 Biclosed forms and conjectural h-principles

Recall that emproplectic forms do not satisfy the relative h-principle [42, Ch. 11.1.C]. My first observation is that, since emproplectic forms are closed if and only if they are biclosed, this failure of the h-principle is a special case of the following result:

Proposition 10.2.1. *Biclosed stable forms never satisfy the h-principle. Specifically, given any stable form $\sigma_0 \in \Lambda^p\left(\mathbb{R}^n\right)^*$, there exist oriented n-manifolds admitting σ_0-forms but no biclosed σ_0-forms.*

Proof. Let M be a closed, oriented n-manifold and let $\sigma \in \Omega^p_{\sigma_0}(\mathrm{M})$ be biclosed. Then by eqn. (2.3.2):

$$\langle ([\sigma] \cup [\Xi(\sigma)]), [\mathrm{M}] \rangle = \int_{\mathrm{M}} \sigma \wedge \Xi(\sigma) > 0, \tag{10.2.2}$$

where $[\sigma]$ and $[\Xi(\sigma)]$ denote the cohomology classes of σ and $\Xi(\sigma)$ respectively, $[\mathrm{M}]$ denotes the fundamental class of M and $\langle , \rangle$ denotes the usual pairing between cohomology and homology. In particular $[\sigma] \neq 0$ and hence $H^p_{\mathrm{dR}}(\mathrm{M}) \neq 0$. The proof is now completed by the following list of explicit counterexamples:

- **(co)-emproplectic forms:** Consider the manifold $\mathrm{M} = \left(S^6\right)^n$. Since S^6 admits an almost complex structure, so too does M, and hence M admits emproplectic and co-emproplectic forms. However $H^2_{\mathrm{dR}}(\mathrm{M}) = H^{6n-2}_{\mathrm{dR}}(\mathrm{M}) = 0$.

- **$\mathrm{SL}(3;\mathbb{C})$ 3 -forms:** Consider $\mathrm{M} = S^6$. Then M admits $\mathrm{SL}(3;\mathbb{C})$ 3-forms since it is orientable and spin (see Theorem 1.0.16) however $H^3_{\mathrm{dR}}(\mathrm{M}) = 0$.

- **$\mathrm{SL}(3;\mathbb{R})^2$ 3-forms:** Consider $\mathrm{M} = S^1 \times S^5$. Then M admits $\mathrm{SL}(3;\mathbb{R})^2$ 3-forms by Example 9.5.3.2, however $H^3_{\mathrm{dR}}(\mathrm{M}) = 0$.

- **G_2- and $\widetilde{\mathrm{G}}_2$-structures:** Consider $\mathrm{M} = S^7$. M admits both G_2- and $\widetilde{\mathrm{G}}_2$-structures since it is orientable and spin (see [22, Remark 3] and [92] respectively), however $H^3_{\mathrm{dR}}(\mathrm{M}) = H^4_{\mathrm{dR}}(\mathrm{M}) = 0$.

- **$\mathbb{P}\mathrm{SU}(3)$-, $\mathrm{SL}(3;\mathbb{R})$- and $\mathbb{P}\mathrm{SU}(1,2)$-structures:** Consider $\mathrm{M} = S^1 \times S^7$. M admits $\mathbb{P}\mathrm{SU}(3)$-, $\mathrm{SL}(3;\mathbb{R})$- and $\mathbb{P}\mathrm{SU}(1,2)$-structures since it is parallelisable, however $H^3_{\mathrm{dR}}(\mathrm{M}) = H^5_{\mathrm{dR}}(\mathrm{M}) = 0$.

□

(A similar topological obstruction exists for the 'extension problem' for biclosed forms, the simplest form of relative h-principle; for a discussion of this in the symplectic case, I refer the reader to [42, Ch. 11.1.C].)

Significantly, emproplectic (and pisoplectic) forms are the only classes of stable forms for which closedness and biclosedness coincide. Thus, in the author's opinion, the failure of the relative h-principle for emproplectic forms should be regarded as anomalous and should not be used to predict the validity of the h-principle for the remaining 7-classes of stable forms.

Therefore, the fact that the relative h-principle has been shown to hold for every class of stable forms (other than emproplectic forms) for which the answer is known, together with the recent result of Bertelson–Meigniez [17] that the h-principle for emproplectic forms does hold if the condition of biclosedness is weakened to 'conformal closedness' leads me to the following conjecture:

Conjecture 10.2.3. *All the remaining 7 classes of closed, stable forms satisfy all forms of the h-principle. Specifically:*

- *Closed* G_2 *3-forms;*

- *Closed* $\mathbb{P}\mathrm{SU}(3)$ *3- and 5-forms;*

- *Closed* $\mathrm{SL}(3;\mathbb{R})$ *3- and 5-forms;*

- *Closed* $\mathbb{P}\mathrm{SU}(1,2)$ *3- and 5-forms,* *in any given cohomology class satisfy all forms of the h-principle. In particular, the Hitchin functional on each of these 7 classes of closed, stable forms is always unbounded above.*

In particular, this result (if proven) would completely answer Bryant's 2005 question of whether $\mathcal{H}_3$ is unbounded above or not [22, Remark 17].

10.3 Other partial differential relations

In the past, h-principles for stable forms have mostly been considered on an individual *ad hoc* basis [104, 32, 37, 17]. This thesis has sought to provide the first systematic investigation of h-principles for stable forms, by making a thorough study of the closed partial differential relation (PDR) on stable forms. However, there are many other natural PDR's which can be imposed on stable forms, for example by demanding the vanishing of certain irreducible components of the intrinsic torsion of the principle bundle induced by the stable form (defined in e.g. [78, §2.6]). In light of the flexibility of the closed PDR demonstrated in this thesis, it is an interesting question to ask whether other natural PDR's on stable forms might also satisfy the h-principle.

Part III

Spectral invariants of torsion-free G_2-structures

Chapter 11

Spectral Morse indices and the definition of the μ-invariants

This chapter introduces two new spectral invariants of torsion-free G_2-structures on closed orbifolds and computes their values on all Joyce orbifolds. The invariants may be viewed as regularisations of the classical Morse indices of the Hitchin functionals on closed and coclosed G_2-structures respectively. In the case of Joyce orbifolds, an interesting link with twisted Epstein ζ-functions is also observed.

11.1 The moduli space of torsion-free G_2 3-forms on Joyce orbifolds

Let:

$$\phi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}$$

denote the standard, flat G_2 3-form on $\mathbb{R}^7$ (viewed as a manifold) and consider the orbifold $\mathrm{M}_\Gamma = \Gamma\backslash\mathbb{T}^7$ for $\Gamma \subset \mathrm{SL}(7;\mathbb{Z}) \ltimes \mathbb{T}^7$ a finite subgroup of automorphisms of $\mathbb{T}^7$. If ϕ is a torsion-free G_2 3-form on M_Γ, then ϕ lifts to define a Γ-invariant torsion-free G_2 3-form $\overline{\phi}$ on $\mathbb{T}^7$ which by Theorem 2.2.19 is necessarily constant (with respect to the usual parallelism of $\mathbb{T}^7$), since $b^1(\mathbb{T}^7) = 7 = \dim(\mathbb{T}^7)$. Thus $\overline{\phi} = F^*\phi_0$ for some $F \in \mathrm{GL}_+(7;\mathbb{R})$. Conversely, given $F \in \mathrm{GL}_+(7;\mathbb{R})$, the G_2 3-form $F^*\phi_0$ descends to M_Γ if and only if Γ preserves $F^*\phi_0$. This is equivalent to the condition that for all $\mathcal{A} = (A,t) \in \Gamma \subset \mathrm{SL}(7;\mathbb{Z}) \ltimes \mathbb{T}^7$, $A^*F^*\phi_0 = F^*\phi_0$, i.e. that $FAF^{-1} \in \mathrm{G}_2$. Thus, writing $\mathfrak{p}_1 : \mathrm{SL}(7;\mathbb{Z}) \ltimes \mathbb{T}^7 \to \mathrm{SL}(7;\mathbb{Z})$ for the projection homomorphism and defining:

$$G_\Gamma^{\mathrm{G}_2} = \left\{F \in \mathrm{GL}_+(n;\mathbb{R}) \;\middle|\; F\mathfrak{p}_1(\Gamma)F^{-1} \subset \mathrm{G}_2\right\}$$

it has been established that the map:

$$\begin{aligned} \beta : G_\Gamma^{\mathrm{G}_2} &\longrightarrow \mathscr{G}_2^{TF}(\mathrm{M}_\Gamma) \\ F &\longmapsto F^*\phi_0 \end{aligned}$$

is surjective. Call M_Γ a Joyce orbifold if $G_\Gamma^{\mathrm{G}_2} \neq \varnothing$, equivalently if M_Γ admits torsion-free G_2-structures.

Next, note that G_2 acts on $G_\Gamma^{\mathrm{G}_2}$ on the left, and that the map β is invariant under this action. Moreover, the automorphism group of M_Γ is $\mathrm{Norm}_{\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7}(\Gamma)$, the normaliser of Γ in $\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7$, where $\mathcal{A}\in\mathrm{Norm}_{\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7}(\Gamma)\subseteq\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7$ acts via the diagram:

$$\begin{tikzcd}
\mathbb{T}^7 \arrow[r, "\mathcal{A}"] \arrow[d, "quot"] & \mathbb{T}^7 \arrow[d, "quot"] \\
\Gamma\backslash\mathbb{T}^7 \arrow[r] & \Gamma\backslash\mathbb{T}^7
\end{tikzcd}$$

Then $\mathfrak{p}_1\left(\mathrm{Norm}_{\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7}(\Gamma)\right)$ acts on $G_\Gamma^{\mathrm{G}_2}$ on the right, and the map β is invariant under this action. It follows that the moduli space of torsion-free G_2-structures on M_Γ is given by:

$$\mathscr{G}_2^{TF}(\mathrm{M}_\Gamma)\cong\mathrm{G}_2\backslash G_\Gamma^{\mathrm{G}_2}\big/\mathfrak{p}_1\left(\mathrm{Norm}_{\mathrm{SL}(7;\mathbb{Z})\ltimes\mathbb{T}^7}(\Gamma)\right). \tag{11.1.1}$$

(Cf. [126, p. 314] for a similar discussion of flat metrics on tori.)

11.2 A spectral generalisation of Morse indices in infinite dimensions

Recall the following classical definition [106, §2]:

Definition 11.2.1. Let $b\in\bigodot^2\mathbb{A}^*$ be a symmetric bilinear form on a finite-dimensional real vector space $\mathbb{A}$. The index of b is the dimension of any maximal subspace $\mathbb{B}\subseteq\mathbb{A}$ such that $b|_\mathbb{B}$ is negative definite. Equivalently, using a choice of inner-product on $\mathbb{A}$, one may regard b as a self-adjoint linear map $b^\sharp:\mathbb{A}\to\mathbb{A}$; then the index of b is simply the number of negative eigenvalues of $b^\sharp$.

Now let N be a finite-dimensional manifold, let $f:\mathrm{N}\to\mathbb{R}$ be a Morse function (i.e. a function with only non-degenerate critical points) and let $p\in\mathrm{N}$ be a critical point of f. The Morse index of f at p is the index of the symmetric bilinear form $D^2f|_p\in\bigodot^2\mathrm{T}^*_p\mathrm{N}$.

In this section, I use the results of [85, 44] (see also [115, 8, 9, 10]) to propose an extension of this definition to infinite dimensions, resulting in the notion of spectral Morse indices.

Let (N,h) be a closed, oriented, Riemannian orbifold of odd dimension n equipped with a real orbifold vector bundle E with metric h^E and let A be a smooth, elliptic, real, self-adjoint pseudodifferential operator of positive order m acting on sections of E. (See [44, Defn. 1.2] for the definition of pseudodifferential operators on orbifolds.) Then A defines a densely-defined, closed, self-adjoint linear operator on $L^2(\mathrm{N},E)$, where the L^2-norm is defined using the metrics h, h^E. Define the spectral ζ- and η- functions of A to be the partial functions:

$$\begin{aligned}
\zeta_A:\mathbb{C}&\longrightarrow\mathbb{C} & \qquad \eta_A:\mathbb{C}&\longrightarrow\mathbb{C}\\
s&\longmapsto\textstyle\sum_{\lambda\in\mathrm{Spec}(A)\backslash\{0\}}|\lambda|^s & s&\longmapsto\textstyle\sum_{\lambda\in\mathrm{Spec}(A)\backslash\{0\}}\mathrm{sign}\,\lambda|\lambda|^s,
\end{aligned} \tag{11.2.2}$$

defined wherever the sums converge absolutely and locally uniformly. Using [85, 44], it follows that:

Theorem 11.2.3. *For* N, h, E, h^E *and* A *as above, the spectral* ζ*- and* η*-functions* ζ_A *and* η_A *converge absolutely and locally uniformly on the region:*

$$\left\{ s \in \mathbb{C} \;\middle|\; \mathscr{R}e(s) > \frac{n}{m} \right\}$$

and admit meromorphic continuations to all of $\mathbb{C}$ *which are holomorphic on a neighbourhood 0; let* $\zeta(A)$ *and* $\eta(A)$ *denote their respective values at* 0*. Then* $\zeta(A), \eta(A) \in \mathbb{R}$*, and for any* $\ell > 0$*:*

$$\zeta(\ell A) = \zeta(A) \quad \text{and} \quad \eta(\ell A) = \eta(A).$$

Moreover, the maps:

$$\begin{aligned} \zeta : \Psi^{>0}_{\textit{inv-sa}}(\mathrm{N}; E) &\longrightarrow \mathbb{R} \\ A &\longmapsto \zeta_A(0) \end{aligned} \qquad \begin{aligned} \eta : \Psi^{>0}_{\textit{inv-sa}}(\mathrm{N}; E) &\longrightarrow \mathbb{R} \\ A &\longmapsto \eta_A(0) \end{aligned}$$

are smooth, where $\Psi^{>0}_{\textit{inv-sa}}(\mathrm{N}; E)$ *denotes the space of (smooth) invertible, real, self-adjoint pseudodifferential operators of positive order acting on* E*.*

Using Theorem 11.2.3, I make the following definition:

Definition 11.2.4. Let N, h, E, h^E and A be as above. I define the spectral Morse index of A to be:

$$\mathscr{I}\mathrm{nd}(A) = \frac{\zeta(A) - \eta(A)}{2}.$$

Then $\mathscr{I}\mathrm{nd}(A)$ is real and invariant under rescalings $A \mapsto \ell A$ for $\ell > 0$. Moreover, $\mathscr{I}\mathrm{nd}$ defines a smooth map:

$$\mathscr{I}\mathrm{nd} : \Psi^{>0}_{\text{inv-sa}}(\mathrm{N}; E) \to \mathbb{R}.$$

The motivation for Definition 11.2.4 can be understood as follows: define the spectral Morse function of A to be:

$$\begin{aligned} \mu_A : \left\{ s \in \mathbb{C} \;\middle|\; \mathfrak{Res} > \tfrac{n}{m} \right\} &\longrightarrow \mathbb{C} \\ s &\longmapsto \sum_{\substack{\lambda \in \mathrm{Spec}(A) \\ \lambda < 0}} |\lambda|^{-s}. \end{aligned}$$

Then by Theorem 11.2.3, μ_A admits an analytic continuation to all of $\mathbb{C}$ and $\mu_A(0) = \mathscr{I}\mathrm{nd}(A)$. If A has only finitely many negative eigenvalues, then the sum defining μ_A converges on all of $\mathbb{C}$ and $\mu_A(0)$ is simply the number of negative eigenvalues of A. Thus in general, one should think of $\mathscr{I}\mathrm{nd}(A)$ as a regularised measure of the 'number of negative eigenvalues of A'.

11.3 μ_3: Morse indices of the critical points of $\mathcal{H}_3$

The aim of this section is to prove that the critical points of the functional $\mathcal{H}_3$ have well-defined spectral Morse indices. Let M be a closed, oriented 7-orbifold and let ϕ be a torsion-free G_2 3-form on M. Since $\mathcal{H}_3$ is diffeomorphism invariant, it induces a functional $\mathcal{H}'_3 : {}^{[\phi]_+}\big/_{\mathrm{Diff}_0(\mathrm{M})} \to (0, \infty)$.

The following result generalises [71, Thm. 19 and Prop. 21] to the case of orbifolds, as well as rephrasing the argument *op. cit.* to obtain an explicit expression for $\mathcal{D}^2\mathcal{H}_3'$ transverse to the action of diffeomorphisms:

Proposition 11.3.1. *The tangent space* $\mathrm{T}_\phi\left([\phi]_+ \big/ \mathrm{Diff}_0(\mathrm{M}) \right)$ *can formally be identified with the space:*

$$\mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M}).$$

Moreover, using the natural L^2 *inner-product on* $\mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M})$ *induced by* ϕ*, the Hessian* $\mathcal{D}^2\mathcal{H}_3'|_\phi$ *can formally be identified with the invertible, linear map* $\mathcal{E}(\phi) = -\frac{1}{3}\mathrm{d}^*\mathrm{d}$*. In particular, the critical points of* $\mathcal{H}_3'$ *are non-degenerate.*

Proof. Recall that $\mathrm{T}_\phi[\phi]_+$ is simply $\mathrm{d}\Omega^2(\mathrm{M})$, by the stability of G_2 3-forms. Let $X \in \Gamma(\mathrm{M}, \mathrm{TM})$ be a vector-field on M. The Lie derivative of ϕ along X may be computed using Cartan's formula [123, Prop. 2.25(d)] to be:

$$\mathscr{L}_X\phi = X \lrcorner \mathrm{d}\phi + \mathrm{d}(X \lrcorner \phi) = \mathrm{d}(X \lrcorner \phi),$$

since $\mathrm{d}\phi = 0$. Thus, as X varies, the space of Lie derivatives $\mathscr{L}_X\phi$, and hence the tangent space to the orbit of $\mathrm{Diff}_0(\mathrm{M})$ through ϕ, is precisely the space $\mathrm{d}\Omega^2_7(\mathrm{M})$.

Next, I describe the tangent space $\mathrm{T}_\phi\left([\phi]_+ \big/ \mathrm{Diff}_0(\mathrm{M}) \right)$. Recall that the usual Hodge decomposition:

$$\Omega^p(\mathrm{M}) = \mathscr{H}^p(\mathrm{M}) \oplus \mathrm{d}\Omega^{p-1}(\mathrm{M}) \oplus \mathrm{d}^*\Omega^{p+1}(\mathrm{M})$$

is valid on closed orbifolds. Using the isomorphism:

$$\mathrm{d}^*\Omega^3(\mathrm{M}) \underset{\mathrm{d}^*G}{\overset{\mathrm{d}}{\rightleftarrows}} \mathrm{d}\Omega^2(\mathrm{M})$$

(where G is the Green's operator for the Hodge Laplacian Δ induced by ϕ) I identify $\mathrm{T}_\phi[\phi]_+ \cong \mathrm{d}^*\Omega^3(\mathrm{M})$ and:

$$\mathrm{T}_\phi\,\mathrm{Diff}_0(\mathrm{M}) \cong \mathrm{d}^*G\mathrm{d}\Omega^2_7(\mathrm{M}) = \mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M}) \subset \mathrm{d}^*\Omega^3(\mathrm{M})$$

where the middle equality uses that G commutes with type-decomposition, since ϕ is torsion-free (see Theorem 2.2.22). Thus, one can identify $\mathrm{T}_\phi\left([\phi]_+ \big/ \mathrm{Diff}_0(\mathrm{M}) \right)$ with the L^2-orthocomplement of $\mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M})$ in $\mathrm{d}^*\Omega^3(\mathrm{M})$. Explicitly, writing $\langle , \rangle$ for the L^2 inner-product on forms induced by g_ϕ, given $\gamma \in \mathrm{d}^*\Omega^3(\mathrm{M})$ and $\delta \in \Omega^2_7(\mathrm{M})$, one computes that:

$$\langle \gamma, \mathrm{d}^*\mathrm{d}\delta \rangle = \langle \mathrm{d}^*\mathrm{d}\gamma, \delta \rangle = \langle \Delta\gamma, \delta \rangle = \langle \gamma, \Delta\delta \rangle$$

and thus $\gamma \in \mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M})^\perp$ if and only if $\gamma \perp \Delta\Omega^2_7(\mathrm{M})$. Using the refined Hodge decomposition (see Theorem 2.2.22):

$$\Omega^2_7(\mathrm{M}) = \mathscr{H}^2_7(\mathrm{M}) \oplus \Delta\Omega^2_7(\mathrm{M})$$

and since $\gamma \in \mathrm{d}^*\Omega^3(\mathrm{M})$ is automatically orthogonal to $\mathscr{H}^2_7(\mathrm{M})$, it follows that $\gamma \in \mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M})^\perp$ if and only if $\gamma \perp \Omega^2_7(\mathrm{M})$ and thus:

$$\mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M})^\perp = \mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M}).$$

Using this description, together with Proposition 2.2.17, the second functional derivative of $\mathcal{H}'_3$ at ϕ is:

$$\begin{aligned} \mathcal{D}^2\mathcal{H}'_3|_\phi : \left(\mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M})\right)^2 &\longrightarrow \mathbb{R} \\ (\gamma_1, \gamma_2) &\longmapsto \tfrac{1}{3}\int_\mathrm{M} \gamma_1 \wedge \bigstar_\phi(\mathrm{d}^* I \mathrm{d}\gamma_2). \end{aligned} \tag{11.3.2}$$

Using eqn. (A.0.3), one may compute that for $\gamma \in \mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M})$:

$$\mathrm{d}^* I \mathrm{d}\gamma = -\mathrm{d}^*\mathrm{d}\gamma.$$

Thus, writing $\langle , \rangle$ for the L^2-inner product on $\mathrm{d}^*\Omega^3(\mathrm{M})$ induced by ϕ, it follows that:

$$\mathcal{D}^2\mathcal{H}'_3|_\phi(\gamma_1, \gamma_2) = -\frac{1}{3}\langle \mathrm{d}\gamma_1, \mathrm{d}\gamma_2 \rangle,$$

as claimed.

□

In light of Proposition 11.3.1, and motivated by Morse theory, it is natural to ask whether the critical point ϕ has a well-defined notion of Morse index. Clearly the classical Morse index of ϕ is infinite, since $\mathcal{D}^2\mathcal{H}'_3|_\phi$ is negative definite. Nevertheless, ϕ has a well-defined spectral Morse index. In particular, consider the second-order pseudodifferential operator acting on $\Omega^2(\mathrm{M})$ via:

$$\mathscr{E}(\phi) = \pi_{harm,\phi} + \Delta + 2\mathrm{d}^* I \mathrm{d},$$

where $\pi_{harm,\phi}$ denotes the L^2-orthogonal projection onto ϕ-harmonic forms. With respect to the decomposition:

$$\Omega^2(\mathrm{M}) = \mathscr{H}^2(\mathrm{M}) \oplus \mathrm{d}\Omega^1(\mathrm{M}) \oplus \mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M}) \oplus \left[\mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M})\right]$$

obtained in the proof of Proposition 11.3.1, the operator $\mathscr{E}(\phi)$ acts diagonally, given explicitly by:

$$\mathscr{E}(\phi) = \begin{cases} \mathrm{id} & \text{on } \mathscr{H}^2(\mathrm{M}); \\ \mathrm{d}\mathrm{d}^* & \text{on } \mathrm{d}\Omega^1(\mathrm{M}); \\ \mathrm{d}^*\mathrm{d} & \text{on } \mathrm{d}^*\mathrm{d}\Omega^2_7(\mathrm{M}); \\ -\mathrm{d}^*\mathrm{d} & \text{on } \mathrm{d}^*\Omega^3(\mathrm{M}) \cap \Omega^2_{14}(\mathrm{M}). \end{cases} \tag{11.3.3}$$

In particular, the operator $\mathscr{E}(\phi)$ is invertible and self-adjoint, and has the same negative spectrum as the operator $\mathcal{E}(\phi)$ defined in Proposition 11.3.1 (up to a factor of $\frac{1}{3}$, which is irrelevant by the scale-invariance of spectral Morse indices). Thus, by Definition 11.2.4, the sum:

$$\begin{aligned} \mu_\phi : \left\{ s \in \mathbb{C} \;\middle|\; \mathfrak{Res} > \tfrac{7}{2} \right\} &\longrightarrow \mathbb{C} \\ s &\longmapsto \sum_{\substack{\lambda \in \mathrm{Spec}(\mathscr{E}(\phi)) \\ \lambda < 0}} |\lambda|^{-s} \end{aligned}$$

converges absolutely and locally uniformly, and admits a meromorphic continuation to all of $\mathbb{C}$ which is holomorphic at 0. Moreover, the value at 0 is simply $\mathscr{I}\mathrm{nd}(\mathscr{E}(\phi))$ and since $\mathscr{E}(\phi)$ depends smoothly on ϕ and $\mathscr{I}\mathrm{nd} : \Psi^{>0}_{inv-sa} \to \mathbb{R}$ is smooth, it follows that $\mu_\phi(0)$ depends smoothly on ϕ. Thus, I obtain:

Theorem 11.3.4. *Let* M *be a closed, oriented 7-orbifold and let* $\mathscr{G}_2^{TF}(\mathrm{M})$ *denote the moduli space of torsion-free* G_2*-structures on* M*. Define the* μ_3*-invariant of a torsion-free* G_2 *3-form* ϕ *to be the value of the meromorphic function* μ_ϕ *at* 0*. Then* μ_3 *is diffeomorphism invariant, invariant under rescaling* $\phi \mapsto \ell^3\phi$ *for* $\ell > 0$ *and defines a smooth map:*

$$\mu_3 : \mathscr{G}_2^{TF}(\mathrm{M}) \to \mathbb{R}.$$

Proof. The only statement which remains to be proven is that $\mu_3(\ell^3\phi) = \mu_3(\phi)$. However, $g_{\ell^3\phi} = \ell^2 g_\phi$ and thus by [7, p. 306] $\mathrm{d}^* \mapsto \ell^{-2}\mathrm{d}^*$. Hence, whilst it is not true that $\mathscr{E}(\ell^3\phi) = \ell^{-2}\mathscr{E}(\phi)$ (due to the presence of $\pi_{harm,\phi}$ in the definition of $\mathscr{E}(\phi)$) it is true that the negative spectrum of $\mathscr{E}(\ell^3\phi)$ coincides with the negative spectrum of $\ell^{-2}\mathscr{E}(\phi)$; the result now follows from the scale-invariance of $\mathscr{I}\mathrm{nd}$. □

11.4 μ_4: Morse indices of the critical points of $\mathcal{H}_4$

The aim of this section is to prove that the critical points of the functional $\mathcal{H}_4$ also have well-defined spectral Morse indices. Let M be a closed oriented 7-orbifold, let ψ be a torsion-free G_2 4-form on M and write $\mathcal{H}'_4$ for the functional ${[\psi]_+}\big/{\mathrm{Diff}_0(\mathrm{M})} \to (0,\infty)$ induced by $\mathcal{H}_4$. The Hessian $\mathcal{D}^2\mathcal{H}'_4|_\psi$ is completely described via the following result:

Proposition 11.4.1. *Write* $\Omega^3_{1\oplus 27}(\mathrm{M})$ *as a shorthand for* $\Omega^3_1(\mathrm{M}) \oplus \Omega^3_{27}(\mathrm{M})$*. Then the tangent space* $\mathrm{T}_\psi\left({[\psi]_+}\big/{\mathrm{Diff}_0(\mathrm{M})}\right)$ *can formally be identified with the space:*

$$\mathrm{d}^*\Omega^4(\mathrm{M}) \cap \Omega^3_{1\oplus 27}(\mathrm{M}).$$

Moreover, there is an L^2*-orthogonal decomposition:*

$$\begin{aligned} \mathrm{d}^*\Omega^4(\mathrm{M}) \cap \Omega^3_{1\oplus 27}(\mathrm{M}) &= \left\{ \omega \in \mathrm{d}^*\Omega^4(\mathrm{M}) \cap \Omega^3_{1\oplus 27}(\mathrm{M}) \;\middle|\; \pi_{27}\mathrm{d}\omega = 0 \right\} \oplus \left(\mathrm{d}^*\Omega^4(\mathrm{M}) \cap \Omega^3_{27}(\mathrm{M}) \right) \\ &= \mathcal{S}^+_4(\psi) \oplus \mathcal{S}^-_4(\psi) \end{aligned}$$

and, using the L^2 *inner-product,* $\mathcal{D}^2\mathcal{H}'_4|_\psi$ *can formally be identified with the invertible linear map* $\mathcal{F}(\psi) = \mathrm{d}^*\mathrm{d} \oplus -\mathrm{d}^*\mathrm{d}$ *on* $\mathcal{S}^+_4(\psi) \oplus \mathcal{S}^-_4(\psi)$*. In particular,* $\mathcal{D}^2\mathcal{H}'_4|_\psi$ *is positive/negative definite on* $\mathcal{S}^\pm_4(\psi)$ *respectively and the critical points of* $\mathcal{H}'_4$ *are non-degenerate.*

Proof. As in the proof of Proposition 11.3.1, one can identify $\mathrm{T}_\psi[\psi]_+$ and $\mathrm{T}_\psi(\mathrm{Diff}_0(\mathrm{M})\cdot\psi)$ with the spaces $\mathrm{d}^*\Omega^4(\mathrm{M})$ and $\mathrm{d}^*\mathrm{d}\Omega^3_7(\mathrm{M})$ respectively. Hence one can identify $\mathrm{T}_\psi\left({}^{[\psi]_+}\big/_{\mathrm{Diff}_0(\mathrm{M})}\right)$ with the L^2-orthocomplement of $\mathrm{d}^*\mathrm{d}\Omega^3_7(\mathrm{M})$ in $\mathrm{d}^*\Omega^4(\mathrm{M})$, *viz.*:

$$\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M}).$$

Using this description, together with Proposition 2.2.17, the second functional derivative of $\mathcal{H}'_4$ at ψ is:

$$\begin{aligned}\mathcal{D}^2\mathcal{H}'_4|_\psi : \left(\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})\right)^2 &\longrightarrow \mathbb{R}\\ (\omega_1,\omega_2) &\longmapsto \frac{1}{4}\int_\mathrm{M}\omega_1\wedge\bigstar_\psi(\mathrm{d}^*J\mathrm{d}\omega_2)\end{aligned} \tag{11.4.2}$$

where $J = \frac{3}{4}\pi_1+\pi_7-\pi_{27}$ was defined in eqn. (2.2.18). To further analyse $\mathcal{D}^2\mathcal{H}'_4|_\psi$, I prove:

Claim 11.4.3. *There is an L^2-orthogonal decomposition:*

$$\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M}) = \underbrace{\left\{\omega\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})\;\middle|\;\pi_{27}\mathrm{d}\omega=0\right\}}_{=\mathcal{S}^+_4(\psi)}\oplus\underbrace{\left\{\omega\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})\;\middle|\;\pi_7\mathrm{d}\omega=0\right\}}_{=\mathcal{S}^-_4(\psi)}.$$

Moreover:

$$\left\{\omega\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})\;\middle|\;\pi_7\mathrm{d}\omega=0\right\} = \mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{27}(\mathrm{M}).$$

Proof of Claim. Recall that in the statement of Theorem 2.2.25, there are no operators of the form d^1_1 and d^{27}_1. This implies, in particular, that:

$$\mathrm{d}\Big(\Omega^3_{1\oplus27}(\mathrm{M})\Big)\subset\Omega^4_7(\mathrm{M})\oplus\Omega^4_{27}(\mathrm{M}) \tag{11.4.4}$$

and hence the spaces $\mathrm{d}\mathcal{S}^+_4(\psi)\subset\Omega^4_7(\mathrm{M})$ and $\mathrm{d}\mathcal{S}^-_4(\psi)\subset\Omega^4_{27}(\mathrm{M})$ are L^2-orthogonal. Using Theorem 2.2.22, one can also verify that $\mathrm{d}^*\mathrm{d}\mathcal{S}^\pm_4(\psi) = \Delta\mathcal{S}^\pm_4(\psi) = \mathcal{S}^\pm_4(\psi)$. Thus:

$$\begin{aligned}\mathcal{S}^+_4(\psi)\text{ and }\mathcal{S}^-_4(\psi)\text{ are }L^2\text{-orthogonal} &\Leftrightarrow \mathrm{d}^*\mathrm{d}\mathcal{S}^+_4(\psi)\text{ and }\mathcal{S}^-_4(\psi)\text{ are }L^2\text{-orthogonal}\\ &\Leftrightarrow \mathrm{d}\mathcal{S}^+_4(\psi)\text{ and }\mathrm{d}\mathcal{S}^-_4(\psi)\text{ are }L^2\text{-orthogonal,}\end{aligned}$$

so $\mathcal{S}^+_4(\psi)$ and $\mathcal{S}^-_4(\psi)$ are indeed L^2-orthogonal as claimed. To prove the claim, therefore, it suffices to prove that each $\omega\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})$ can be written as $\omega=\omega^++\omega^-$, for some $\omega^\pm\in\mathcal{S}^\pm_4(\psi)$.

Given $\omega\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M})$, write $\omega = f\phi+\gamma$ for some unique $f\in\Omega^0(\mathrm{M})$ and $\gamma\in\Omega^3_{27}(\mathrm{M})$. Note that one may trivially write:

$$\omega = \left(f\phi+\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7G\gamma\right)+\left(\gamma-\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7G\gamma\right) = \omega^++\omega^-, \tag{11.4.5}$$

where G denotes the Green's operator for the Hodge Laplacian defined by ψ. I claim that $\omega^\pm\in\mathcal{S}^\pm_4(\psi)$, i.e.:

$$\omega^\pm\in\mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus27}(\mathrm{M}),\qquad \pi_{27}\mathrm{d}\omega^+=0\quad\text{and}\quad\pi_7\mathrm{d}\omega^-=0.$$

Begin with the first of these points. Since clearly $\omega^\pm \in \Omega^3_{1\oplus 27}(\mathrm{M})$, it suffices to prove $\omega^\pm \in \mathrm{d}^*\Omega^4(\mathrm{M})$. Since $\omega = f\phi + \gamma \in \mathrm{d}^*\Omega^4(\mathrm{M}) \subset \Delta\Omega^3(\mathrm{M})$, it follows that $f\phi \in \Delta\Omega^3_1(\mathrm{M})$ by Theorem 2.2.22 and hence it is orthogonal to $\mathscr{H}^3_1(\mathrm{M})$; likewise $\gamma \in \Delta\Omega^3_{27}(\mathrm{M})$ is orthogonal to $\mathscr{H}^3_{27}(\mathrm{M})$. Moreover:

$$\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma = \frac{7}{12} G\mathrm{d}^7_{27}\mathrm{d}^{27}_7\gamma \in \Delta\Omega^3_{27}(\mathrm{M})$$

and hence $\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma$ is also orthogonal to $\mathscr{H}^3_{27}(\mathrm{M})$. It follows that $\omega^\pm$ are each orthogonal to $\mathscr{H}^3(\mathrm{M})$ and so to prove that $\omega^\pm \in \mathrm{d}^*\Omega^4(\mathrm{M})$, it suffices to prove that $\mathrm{d}^*\omega^\pm = 0$.

In general, given $f' \in \Omega^0(\mathrm{M})$ and $\gamma' \in \Omega^3_{27}(\mathrm{M})$, by eqns. (A.0.1) and (A.0.4) the condition $\mathrm{d}^*(f'\phi + \gamma') = 0$ is equivalent to the pair of equations:

$$\mathrm{d}^{27}_7\gamma' = 3\mathrm{d}f' \quad \text{and} \quad \mathrm{d}^{27}_{14}\gamma' = 0. \tag{11.4.6}$$

Since $\omega = f\phi + \gamma \in \mathrm{d}^*\Omega^4(\mathrm{M})$, it follows that $\mathrm{d}^{27}_7\gamma = 3\mathrm{d}f$ and $\mathrm{d}^{27}_{14}\gamma = 0$. Therefore:

$$\begin{aligned}
\mathrm{d}^{27}_7\left(\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma\right) &= \mathrm{d}^{27}_7\left(\Delta \mathrm{G}\gamma - (\mathrm{d}^{27}_{27})^2 G\gamma\right) && \text{(using eqn. (A.0.9) and } \mathrm{d}^{27}_{14}\gamma = 0\text{)}\\
&= \mathrm{d}^{27}_7 \underbrace{\Delta \mathrm{G}\gamma}_{=\gamma} - \underbrace{\left(\mathrm{d}^{27}_7\mathrm{d}^{27}_{27}\right)}_{=\frac{1}{2}\mathrm{d}^7_7\mathrm{d}^{27}_7} \mathrm{d}^{27}_{27} G\gamma && \text{(using } \gamma \perp \mathscr{H}^3_{27}(\mathrm{M}) \text{ and eqn. (A.0.5))}\\
&= \mathrm{d}^{27}_7\gamma - \frac{1}{2}\mathrm{d}^7_7 \underbrace{\left(\mathrm{d}^{27}_7\mathrm{d}^{27}_{27}\right)}_{=-\frac{3}{2}\mathrm{d}^{14}_7\mathrm{d}^{27}_{14}} G\gamma && \text{(using two subequations from eqn. (A.0.5))}\\
&= \mathrm{d}^{27}_7\gamma && \text{(using } \mathrm{d}^{27}_{14}G\gamma = G\mathrm{d}^{27}_{14}\gamma = 0\text{).}
\end{aligned}$$

Likewise:

$$\begin{aligned}
\mathrm{d}^{27}_{14}\left(\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma\right) &= \underbrace{\mathrm{d}^{27}_{14}\gamma}_{=0} - \underbrace{\left(\mathrm{d}^{27}_{14}\mathrm{d}^{27}_{27}\right)}_{=-\frac{1}{4}\mathrm{d}^7_{14}\mathrm{d}^{27}_7} \mathrm{d}^{27}_{27} G\gamma && \text{(using eqn. (A.0.5))}\\
&= \frac{1}{4}\mathrm{d}^7_{14} \underbrace{\left(\mathrm{d}^{27}_7\mathrm{d}^{27}_{27}\right)}_{=-\frac{3}{2}\mathrm{d}^{14}_7\mathrm{d}^{27}_{14}} G\gamma && \text{(using two subequations from eqn. (A.0.5))}\\
&= 0 && \text{(using } \mathrm{d}^{27}_{14}G\gamma = G\mathrm{d}^{27}_{14}\gamma = 0\text{).}
\end{aligned}$$

It follows from these last two calculations, together with the conditions $\mathrm{d}^{27}_7\gamma = 3\mathrm{d}f$ and $\mathrm{d}^{27}_{14}\gamma = 0$, that $\omega^\pm$ each satisfy eqn. (11.4.6) and hence $\mathrm{d}^*\omega^\pm = 0$. Thus, all that remains is to prove $\pi_{27}\mathrm{d}\omega^+ = 0$ and $\pi_7\mathrm{d}\omega^- = 0$.

Using eqn. (A.0.4), one computes:

$$\begin{aligned}
\pi_{27}\mathrm{d}\omega^+ &= \frac{7}{12}\bigstar_\phi \underbrace{\left(\mathrm{d}^{27}_{27}\mathrm{d}^7_{27}\right)}_{=\frac{1}{2}\mathrm{d}^7_{27}\mathrm{d}^7_7}\mathrm{d}^{27}_7 G\gamma \qquad \text{(by eqn. (A.0.5))}\\
&= \frac{7}{24}\bigstar_\phi \mathrm{d}^7_{27}\underbrace{\left(\mathrm{d}^7_7\mathrm{d}^{27}_7\right)}_{=-3\mathrm{d}^{14}_7\mathrm{d}^{27}_{14}} G\gamma \qquad \text{(by eqn. (A.0.5))}\\
&= -\frac{7}{8}\bigstar_\phi \mathrm{d}^7_{27}\mathrm{d}^{14}_7\mathrm{d}^{27}_{14}G\gamma = 0,
\end{aligned}$$

since $\mathrm{d}^{27}_{14}G\gamma = G\mathrm{d}^{27}_{14}\gamma = 0$. Similarly:

$$\pi_7\mathrm{d}\omega^- = \frac{1}{4}\mathrm{d}^{27}_7\left[\gamma - \frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma\right]\wedge\phi = 0,$$

since $\mathrm{d}^{27}_7\left(\frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7 G\gamma\right) = \mathrm{d}^{27}_7\gamma$, as above. Thus $\omega^\pm \in \mathcal{S}^\pm_4(\psi)$, as claimed.

Finally, to verify that $\mathcal{S}^-_4(\psi) = \mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{27}(\mathrm{M})$, note that $\mathcal{S}^-_4(\psi) \subseteq \mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{27}(\mathrm{M})$ follows by eqn. (11.4.5). Conversely, if $\gamma \in \mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{27}(\mathrm{M})$, then $\mathrm{d}^*\gamma = 0$ forces $\mathrm{d}^{27}_7\gamma = 0$ by eqn. (11.4.6) and hence $\pi_7\mathrm{d}\gamma = 0$ (see eqn. (A.0.4)). Thus $\gamma \in \mathcal{S}^-_4(\psi)$ as claimed.

□

Given this claim, Theorem 11.4.1 follows swiftly. Indeed, recalling the definition of J in eqn. (2.2.18), it follows that for $\omega \in \mathrm{d}^*\Omega^4(\mathrm{M})\cap\Omega^3_{1\oplus 27}(\mathrm{M})$:

$$\mathrm{d}^*J\mathrm{d}\omega = \begin{cases} +\mathrm{d}^*\mathrm{d}\omega \in \mathcal{S}^+_4(\psi) & \text{if } \omega\in\mathcal{S}^+_4(\psi);\\ -\mathrm{d}^*\mathrm{d}\omega \in \mathcal{S}^-_4(\psi) & \text{if } \omega\in\mathcal{S}^-_4(\psi).\end{cases}$$

Thus, the symmetric bilinear form $D^2\mathcal{H}'_4|_\psi$ is given by:

$$D^2\mathcal{H}'_4|_\psi(\omega_1,\omega_2) = \begin{cases} +\langle\mathrm{d}\omega_1,\mathrm{d}\omega_2\rangle & \text{if } \omega_1,\omega_2\in\mathcal{S}^+_4(\psi);\\ -\langle\mathrm{d}\omega_1,\mathrm{d}\omega_2\rangle & \text{if } \omega_1,\omega_2\in\mathcal{S}^-_4(\psi).\end{cases}$$

□

It follows from Theorem 5.1.11 that both $\mathcal{S}^\pm$ are infinite dimensional. In particular, the classical Morse index of ψ is, as for $\mathcal{H}_3$, infinite. Nevertheless, it is again possible to define the regularised Morse index of ψ. Consider the second-order pseudodifferential operator:

$$\mathscr{F}(\psi) = \pi_{harm,\psi} + \Delta + 2\mathrm{d}^*J\mathrm{d}$$

where $\pi_{harm,\psi}$ denotes the L^2-orthogonal projection onto ψ-harmonic forms. With respect to the decomposition:

$$\Omega^3 = \mathscr{H}^3(\mathrm{M})\oplus\mathrm{d}\Omega^2(\mathrm{M})\oplus\mathrm{d}^*\mathrm{d}\Omega^3_7(\mathrm{M})\oplus\mathcal{S}^+_4(\psi)\oplus\mathcal{S}^-_4(\psi)$$

obtained in the proof of Proposition 11.4.1, the operator $\mathscr{F}(\psi)$ acts diagonally, given explicitly by:

$$\mathscr{F}(\phi) = \begin{cases} \mathrm{id} & \text{on } \mathscr{H}^3(\mathrm{M}); \\ \mathrm{dd}^* & \text{on } \mathrm{d}\Omega^2(\mathrm{M}); \\ \mathrm{d}^*\mathrm{d} & \text{on } \mathrm{d}^*\mathrm{d}\Omega^3_7(\mathrm{M}); \\ 3\mathrm{d}^*\mathrm{d} & \text{on } \mathcal{S}^+_4(\psi); \\ -\mathrm{d}^*\mathrm{d} & \text{on } \mathcal{S}^-_4(\psi). \end{cases} \tag{11.4.7}$$

In particular, the operator $\mathscr{F}(\psi)$ is invertible and self-adjoint, and has the same negative spectrum as the operator $\mathcal{F}(\psi)$ defined in Proposition 11.4.1 (up to a factor of $\frac{1}{4}$, which is irrelevant by the scale-invariance of spectral Morse indices). Thus, by Definition 11.2.4, the sum:

$$\begin{aligned} \mu_\psi : \left\{s \in \mathbb{C} \;\middle|\; \mathfrak{Res} > \tfrac{7}{2}\right\} &\longrightarrow \mathbb{C} \\ s &\longmapsto \textstyle\sum_{\substack{\lambda\in\mathrm{Spec}(\mathcal{F}(\psi)) \\ \lambda<0}} |\lambda|^{-s} \end{aligned}$$

converges absolutely and locally uniformly, and admits a meromorphic continuation to all of $\mathbb{C}$ which is holomorphic at 0. Moreover, the value at 0 is simply $\mathscr{I}\mathrm{nd}(\mathscr{F}(\psi))$ and since $\mathscr{F}(\psi)$ depends smoothly on ψ and $\mathscr{I}\mathrm{nd} : \Psi^{>0}_{inv-sa} \to \mathbb{R}$ is smooth, it follows that $\mu_\psi(0)$ depends smoothly on ψ. Moreover $\mu_\psi(0)$ is scale-invariant by the same argument as for μ_3. Thus, it has been shown that:

Theorem 11.4.8. *Let* M *be a closed, oriented 7-orbifold and let* $\mathscr{G}_2^{TF}(\mathrm{M})$ *denote the moduli space of torsion-free* G_2*-structures on* M*. Define the* μ_4*-invariant of a torsion-free* G_2 *4-form* ψ *to be the value of the meromorphic function* μ_ψ *at* 0*. Then* μ_4 *is diffeomorphism invariant, invariant under rescaling* $\psi \mapsto \ell^4\psi$ *for* $\ell > 0$ *and defines a smooth map:*

$$\mu_4 : \mathscr{G}_2^{TF}(\mathrm{M}) \to \mathbb{R}.$$

□

11.5 Computing the eigenvalues of $\mathcal{E}(\phi)$ and $\mathcal{F}(\psi)$ on Joyce orbifolds

This is the first of two sections which aim to compute μ_3 and μ_4 on an arbitrary Joyce orbifold M_Γ. Let ϕ be a (constant) torsion-free G_2 3-form on M_Γ and let ψ denote the corresponding G_2 4-form. Recall from §11.3 that $\mu_3(\phi)$ is the value at 0 of the meromorphic extension of:

$$\begin{aligned} \mu_\phi : \left\{s \in \mathbb{C} \;\middle|\; \mathfrak{Res} > \tfrac{7}{2}\right\} &\longrightarrow \mathbb{C} \\ s &\longmapsto \textstyle\sum_{\substack{\lambda\in\mathrm{Spec}(\mathcal{E}(\phi)) \\ \lambda<0}} |\lambda|^{-s} \end{aligned}$$

where $\mathcal{E}(\phi)$ acts on $\mathrm{d}^*\Omega^3(\mathrm{M}_\Gamma) \cap \Omega^2_{14}(\mathrm{M}_\Gamma)$ via $-\mathrm{d}^*\mathrm{d}$. Thus the task is to explicitly compute the spectrum of $\mathcal{E}(\phi)$. Since exterior forms on M_Γ are equivalent to Γ-invariant exterior forms on $\mathbb{T}^7$ and $-\mathrm{d}^*\mathrm{d}$ is a real operator, and using elliptic regularity, this is equivalent to computing the spectrum of $-\mathrm{d}^*\mathrm{d}$ acting on the complex Hilbert space:

$$\mathbb{H}^\Gamma = \left(\mathrm{d}^* H^1\Omega^3(\mathbb{T}^7)_\mathbb{C} \cap L^2\Omega^2_{14}(\mathbb{T}^7)_\mathbb{C}\right)^\Gamma,$$

where $(-)_\mathbb{C} = (-) \otimes_\mathbb{R} \mathbb{C}$, L^2 and H^1 denote Lebesgue and Sobolev spaces of sections respectively and $(-)^\Gamma$ denotes the Γ-invariant subspace.

To this end, identify $\left(\mathrm{T}_0\mathbb{T}^7\right)_\mathbb{C} \cong \left(\mathbb{R}^7\right)_\mathbb{C}$ and recall that every $\omega \in \bigwedge^\bullet \left(\mathbb{R}^7\right)^*_\mathbb{C}$ defines a left-invariant, complex exterior form on $\mathbb{T}^7$ which I also denote by ω. This defines a natural embedding $\bigwedge^\bullet \left(\mathbb{R}^7\right)^*_\mathbb{C} \hookrightarrow \Omega^\bullet(\mathbb{T}^7)_\mathbb{C}$ onto the space of constant (equivalently, ϕ-harmonic) complex exterior forms on $\mathbb{T}^7$. Given an $l \in \mathbb{Z}^7$, define a smooth $\mathbb{C}$-valued function $\chi_l : \mathbb{T}^7 \to \mathbb{C}$ by:

$$\begin{aligned} \chi_l &: \mathbb{T}^7 \to \mathbb{C} \\ x + \mathbb{Z}^7 &\mapsto e^{2\pi i g(l,x)}, \end{aligned}$$

where $g = g_\phi$ denotes the metric induced by ϕ. Define:

$$\mathbb{H}_l = \left\{\chi_l \cdot \alpha \;\middle|\; \alpha \in \textstyle\bigwedge^2_{14} \left(\mathbb{R}^7\right)^*_\mathbb{C} \text{ satisfies } l \lrcorner \alpha = 0\right\}$$

and finally define:

$$\mathcal{L} = \left\{-4\pi^2\|l\|^2_g \;\middle|\; l \in \mathbb{Z}^7\right\}.$$

Proposition 11.5.1. *For each $\lambda \in \mathcal{L}\backslash\{0\}$, define:*

$$\mathbb{H}(\lambda) = \bigoplus_{l \in \mathbb{Z}^7 \,:\, -4\pi^2\|l\|^2_g = \lambda} \mathbb{H}_l.$$

Then there is a decomposition:

$$\mathbb{H} = \overline{\bigoplus_{\lambda \in \mathcal{L}\backslash\{0\}} \mathbb{H}(\lambda)}$$

of $\mathbb{H}$ into eigenspaces of $\mathcal{E}(\phi) = -\mathrm{d}^*\mathrm{d}$, where $\mathcal{E}(\phi)$ acts on $\mathbb{H}(\lambda)$ via $\lambda\,\mathrm{Id}$.

Proof. By the Peter–Weyl theorem [113]:

$$L^2\Omega^2_{14}(\mathbb{T}^7)_\mathbb{C} = \overline{\bigoplus_{l \in \mathbb{Z}^7} \left\{\chi_l\alpha \;\middle|\; \alpha \in \textstyle\bigwedge^2_{14} (\mathbb{R}^7)^*_\mathbb{C}\right\}}.$$

Given $\alpha = \sum_{l\in\mathbb{Z}^7} \chi_l \alpha^l \in L^2\Omega^2_{14}(\mathbb{T}^7)_\mathbb{C}$, observe that:

$$\pi_{harm}\alpha = \alpha^0 \in \textstyle\bigwedge^2_{14} \left(\mathbb{R}^7\right)^*_\mathbb{C}.$$

Similarly, using the identity:

$$\mathrm{d}\chi_l = 2\pi i \chi_l l^\flat,$$

where $^\flat : \mathbb{R}^7 \to \left(\mathbb{R}^7\right)^*$ denotes the musical isomorphism induced by g, one computes that for $\alpha \in \Omega^2_{14}(\mathbb{T}^7)_{\mathbb{C}}$ smooth:

$$\mathrm{d}^*\alpha = -2\pi i \sum_{l \in \mathbb{Z}^7} \chi_l\big(l \lrcorner \alpha^l\big).$$

Since the space $\mathbb{H} = \mathrm{d}^* H^1\Omega^3(\mathbb{T}^7)_{\mathbb{C}} \cap L^2\Omega^2_{14}(\mathbb{T}^7)_{\mathbb{C}}$ is simply the closure of the space:

$$\left\{\alpha \in \Omega^2_{14}(\mathbb{T}^7)_{\mathbb{C}} \;\middle|\; \pi_{harm}\alpha = 0 \text{ and } \mathrm{d}^*\alpha = 0\right\}$$

in the L^2-norm, it follows that:

$$\begin{aligned}\mathbb{H} &= \overline{\bigoplus_{l \in \mathbb{Z}^7\backslash\{0\}} \mathbb{H}_l} \\ &= \overline{\bigoplus_{\lambda \in \mathcal{L}\backslash\{0\}} \mathbb{H}(\lambda)}.\end{aligned}$$

Thus to complete the proof, it suffices to note that, for $\chi_l\alpha^l \in \mathbb{H}_l$:

$$-\mathrm{d}^*\mathrm{d}(\chi_l\alpha^l) = -4\pi^2\|l\|^2_g\chi_l\alpha^l,$$

which follows from $l \lrcorner \alpha = 0$ (see [15, p. 363]). Thus $\mathbb{H}(\lambda)$ is in fact the λ-eigenspace of $-\mathrm{d}^*\mathrm{d}$, as required.

□

Since Γ commutes with the action of $-\mathrm{d}^*\mathrm{d}$, it follows that:

$$\mathbb{H}^\Gamma = \overline{\bigoplus_{\lambda \in \mathcal{L}\backslash\{0\}} \mathbb{H}(\lambda)^\Gamma}.$$

Thus, for $\mathfrak{Re}(s) > \frac{7}{2}$, one finds that:

$$\mu_{\mathcal{E}(\phi)}(s) = \sum_{\lambda \in \mathcal{L}\backslash\{0\}} \frac{\dim \mathbb{H}(\lambda)^\Gamma}{|\lambda|^s}. \tag{11.5.2}$$

The calculation for μ_4 is closely analogous: firstly note that the negative spectrum of $\mathcal{F}(\psi)$ is the same as the spectrum of $-\mathrm{d}^*\mathrm{d}$ acting on the space:

$$(\mathbb{H}')^\Gamma = \left(\mathrm{d}^* H^1\Omega^4(\mathbb{T}^7)_{\mathbb{C}} \cap L^2\Omega^3_{27}(\mathbb{T}^7)_{\mathbb{C}}\right)^\Gamma.$$

For $l \in \mathbb{Z}^7$ and $\lambda \in \mathcal{L}$, define:

$$\mathbb{H}'_l = \left\{\chi_l \cdot \alpha \;\middle|\; \alpha \in {\textstyle\bigwedge}^3_{27}\left(\mathbb{R}^7\right)^*_{\mathbb{C}} \text{ satisfies } l \lrcorner \alpha = 0\right\}$$

and:

$$\mathbb{H}'(\lambda) = \bigoplus_{l \in \mathbb{Z}^7 \,:\, -4\pi^2\|l\|^2_g = \lambda} \mathbb{H}'_l.$$

Then as for μ_3:

$$(\mathbb{H}')^\Gamma = \overline{\bigoplus_{\lambda \in \mathcal{L}\backslash\{0\}} \mathbb{H}'(\lambda)^\Gamma}$$

where $-\mathrm{d}^*\mathrm{d}$ acts on each $\mathbb{H}'(\lambda)$ by $\lambda\,\mathrm{Id}$. It follows that for $\mathfrak{Re}(s) > \frac{7}{2}$:

$$\mu_{\mathcal{F}'(\psi)}(s) = \sum_{\lambda \in \mathcal{L}\backslash\{0\}} \frac{\dim \mathbb{H}'(\lambda)^\Gamma}{|\lambda|^s}.$$

Thus, the computation of $\mu_3(\phi)$ and $\mu_4(\psi)$ has been reduced to the representation-theoretic problem of computing $\dim \mathbb{H}(\lambda)^\Gamma$ and $\dim \mathbb{H}'(\lambda)^\Gamma$ for each $\lambda \in \mathcal{L}\backslash\{0\}$. This will occupy the next section.

11.6 Multiplicities of the eigenvalues of $\mathcal{E}(\phi)$ and $\mathcal{F}(\psi)$

Write $\rho^{(\lambda)}$ for the representation of Γ on $\mathbb{H}(\lambda) = \bigoplus_{l \in \mathbb{Z}^7 : -4\pi^2\|l\|_g^2 = \lambda} \mathbb{H}_l$. Recall that the character $\chi^{(\lambda)} : \Gamma \to \mathbb{R}$ of $\rho^{(\lambda)}$ is defined by:

$$\chi^{(\lambda)}(\mathcal{A}) = \mathrm{Tr}_{\mathbb{H}(\lambda)}(\rho^{(\lambda)}(\mathcal{A})), \;\; \mathcal{A} \in \Gamma.$$

The dimension of $\mathbb{H}(\lambda)^\Gamma$ can be computed using $\chi^{(\lambda)}$ via the formula [52, eqn. (2.9)]:

$$\dim \mathbb{H}(\lambda)^\Gamma = \frac{1}{|\Gamma|} \sum_{\mathcal{A} \in \Gamma} \chi^{(\lambda)}(\mathcal{A}). \tag{11.6.1}$$

Thus, the task is to compute the character $\chi^{(\lambda)}$. This is accomplished by the following proposition:

Proposition 11.6.2. *Given $A \in \mathrm{End}(\mathbb{R}^7)$, define:*

$$\mathrm{Tr}_8^{\mathrm{SU}(3)}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2\,\mathrm{Tr}_{\mathbb{R}^7}(A) + 1. \tag{11.6.3}$$

Moreover, given $\lambda \in \mathcal{L}\backslash\{0\}$ and $\mathcal{A} = (A,t) \in \mathrm{SL}(7;\mathbb{Z}) \ltimes \mathbb{T}^7$, define:

$$\mathcal{G}(\lambda, \mathcal{A}) = \left\{ l \in \mathbb{Z}^7 \;\middle|\; -4\pi^2\|l\|_g^2 = \lambda, Al = l \right\}.$$

Then:

$$\chi^{(\lambda)}(\mathcal{A}) = \sum_{l \in \mathcal{G}(\lambda,\mathcal{A})} e^{2\pi i g(l,t)}\,\mathrm{Tr}_8^{\mathrm{SU}(3)}(A).$$

In particular, by eqn. (11.6.1):

$$\dim \mathbb{H}(\lambda)^\Gamma = \frac{1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \sum_{l \in \mathcal{G}(\lambda,\mathcal{A})} e^{2\pi i g(l,t)}\,\mathrm{Tr}_8^{\mathrm{SU}(3)}(A).$$

The proof proceeds by a series of lemmas:

Lemma 11.6.4. *For each $\mathbb{H}_l \subset \mathbb{H}(\lambda)$, define a representation ρ_l of Γ on $\mathbb{H}_l$ via:*

$$\rho_l(\mathcal{A})[u] = \mathrm{proj}_{\mathbb{H}_l}\left\{\rho^{(\lambda)}(\mathcal{A})[u]\right\}, \quad \mathcal{A} \in \Gamma, u \in \mathbb{H}_l,$$

where $\mathrm{proj}_{\mathbb{H}_l}$ denotes the projection $\mathbb{H}(\lambda) = \bigoplus_{l' \in \mathbb{Z}^7 : -4\pi^2\|l'\|_g^2 = \lambda} \mathbb{H}_{l'} \to \mathbb{H}_l$, and write χ_l for the corresponding character. Then for each $\mathcal{A} \in \Gamma$:

$$\chi^{(\lambda)}(\mathcal{A}) = \sum_{l \in \mathcal{G}(\lambda,\mathcal{A})} \chi_l(\mathcal{A}). \tag{11.6.5}$$

Proof. For all $\mathcal{A} \in \Gamma$, the linear map $\rho^{(\lambda)}(\mathcal{A})$ is represented by the block matrix:

$$\begin{pmatrix} \rho_{l_1}(\mathcal{A}) & * & \cdots \\ * & \rho_{l_2}(\mathcal{A}) & \\ \vdots & & \ddots \end{pmatrix}$$

where $\mathbb{H}(\lambda) = \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \ldots$. In particular:

$$\chi^{(\lambda)} = \sum_{l \in \mathbb{Z}^7 : -4\pi^2\|l\|_g^2 = \lambda} \chi_l. \tag{11.6.6}$$

Now given $\mathcal{A} = (A, t) \in \Gamma \subset \mathrm{SL}(7;\mathbb{Z}) \times \mathbb{T}^7$, for $\chi_l \alpha^l = e^{2\pi i g(l,x)}\alpha^l \in \mathbb{H}_l$:

$$\mathcal{A}^*\left(e^{2\pi i g(l,x)}\alpha^l\right) = e^{2\pi i g(l,t)} e^{2\pi i g(A^T l,x)} A^*\alpha^l \in \mathbb{H}_{A^T l}. \tag{11.6.7}$$

Thus $\rho_l(\mathcal{A})$ is non-zero if and only if $A^T l = l$, which is equivalent to $Al = l$ (since A preserves g). This completes the proof.

□

Lemma 11.6.8. *Given $l \in \mathcal{G}(\lambda, \mathcal{A})$, define:*

$$\mathbb{A}_l = \left\{\alpha \in \textstyle\bigwedge^2_{14}\left(\mathbb{R}^7\right) \;\middle|\; l \lrcorner \alpha = 0\right\}.$$

Then the trace $\mathrm{Tr}_{(\mathbb{A}_l)_\mathbb{C}}(A)$ of $A = \mathfrak{p}_1(\mathcal{A})$ acting on $(\mathbb{A}_l)_\mathbb{C}$ via pullback is:

$$\mathrm{Tr}_{(\mathbb{A}_l)_\mathbb{C}}(A) = \mathrm{Tr}_8^{\mathrm{SU}(3)}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2\,\mathrm{Tr}_{\mathbb{R}^7}(A) + 1.$$

Proof. Firstly, note that since complexification does not affect the trace of a real operator, it is equivalent to compute the trace of A acting on $\mathbb{A}_l$. Identify $\mathrm{Stab}_{\mathrm{GL}_+(7;\mathbb{R})}(\phi)$ with the group G_2 and recall that $\mathrm{Stab}_{\mathrm{G}_2}(l) \cong \mathrm{SU}(3)$ [68, Prop. 2.7]. Let $\mathbb{B} = \langle l \rangle^\perp$ with its natural orientation, let $\theta \in \left(\mathbb{R}^7\right)^*$ be a correctly oriented annihilator of $\mathbb{B}$ and using the splitting $\mathbb{R}^7 = \langle l \rangle \oplus \mathbb{B}$ write:

$$\phi = \theta \wedge \omega + \rho.$$

Since $\mathrm{SU}(3) \subset \mathrm{GL}(3;\mathbb{C})$, $\mathbb{B}$ inherits a natural complex structure J with respect to which ω is a positive $(1,1)$-form on $\mathbb{B}$ [35] and there is an $\mathrm{SU}(3)$-invariant decomposition:

$$\bigwedge^2 \mathbb{B}^* = \mathbb{R}\cdot\omega \oplus \left[\bigwedge_8^{1,1}\mathbb{B}^*\right] \oplus \left[\!\left[\bigwedge^{2,0}\mathbb{B}^*\right]\!\right]$$

where $\left[\bigwedge_8^{1,1}\mathbb{B}^*\right]$ is the orthocomplement to $\mathbb{R}\cdot\omega$ in $\left[\bigwedge^{1,1}\mathbb{B}^*\right]$ and $\left[\!\left[\bigwedge^{2,0}\mathbb{B}^*\right]\!\right] = \{u \lrcorner \rho \mid u \in \mathbb{B}\} \cong \mathbb{B}$. (In fact, the 3-form ρ is an $\mathrm{SL}(3;\mathbb{C})$ 3-form, and the complex structure may be written explicitly in terms of ρ; see [71]). Define an isomorphism:

$$\begin{aligned} \chi_6 : \mathbb{B}^* &\xrightarrow{\sim} \left[\!\left[\bigwedge^{2,0}\mathbb{B}^*\right]\!\right] \\ v \lrcorner \omega &\mapsto v \lrcorner \rho. \end{aligned}$$

Then by [35, Lem. 1]:

$$\bigwedge^2_{14}\left(\mathbb{R}^7\right)^* = \left[\bigwedge_8^{1,1}\mathbb{B}^*\right] \oplus \{2\theta\wedge\alpha + \chi_6(\alpha) \mid \alpha \in \mathbb{B}^*\}.$$

In particular:

$$\mathbb{A}_l = \left\{\alpha \in \bigwedge^2_{14}\left(\mathbb{R}^7\right) \;\middle|\; l \lrcorner \alpha = 0\right\} = \left[\bigwedge_8^{1,1}\mathbb{B}^*\right]$$

and thus $\mathrm{Tr}_{\mathbb{A}_l}(A)$ is simply the trace of A acting on the space $\left[\bigwedge_8^{1,1}\mathbb{B}^*\right]$.

Using [52, Prop. 2.1], the trace of A acting on $\bigwedge^2\left(\mathbb{R}^7\right)^*$ is:

$$\mathrm{Tr}_{\bigwedge^2(\mathbb{R}^7)^*}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2}. \tag{11.6.9}$$

Hence, using $\bigwedge^2\left(\mathbb{R}^7\right)^* = \bigwedge^2_7\left(\mathbb{R}^7\right)^* \oplus \bigwedge^2_{14}\left(\mathbb{R}^7\right)^* \cong \mathbb{R}^7 \oplus \bigwedge^2_{14}\left(\mathbb{R}^7\right)^*$, one finds that:

$$\mathrm{Tr}_{\bigwedge^2_{14}(\mathbb{R}^7)^*}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - \mathrm{Tr}_{\mathbb{R}^7}(A). \tag{11.6.10}$$

Next, note that $\mathrm{Tr}_{\mathbb{B}}(A) = \mathrm{Tr}_{\mathbb{R}^7}(A) - 1$, since $Al = l$. Thus $\bigwedge^2_{14}\left(\mathbb{R}^7\right)^* \cong \mathbb{B} \oplus \left[\bigwedge_8^{1,1}\mathbb{B}^*\right]$, yields:

$$\mathrm{Tr}_{\left[\bigwedge_8^{1,1}\mathbb{B}^*\right]}(A) = \mathrm{Tr}_{\bigwedge^2_{14}(\mathbb{R}^7)^*}(A) - \mathrm{Tr}_{\mathbb{R}^7}(A) + 1 = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^2 - \mathrm{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2\,\mathrm{Tr}_{\mathbb{R}^7}(A) + 1,$$

as required.

□

Proof of Proposition 11.6.2. By eqn. (11.6.7), it follows that for $\mathcal{A} = (A,t) \in \Gamma$, $l \in \mathcal{G}(\lambda,\mathcal{A})$:

$$\chi_l(\mathcal{A}) = e^{2\pi i g(l,t)}\,\mathrm{Tr}_{(\mathbb{A}_l)_{\mathbb{C}}}(A) = e^{2\pi i g(l,t)}\,\mathrm{Tr}_8^{\mathrm{SU}(3)}(A).$$

The result now follows from Lemma 11.6.4.

□

Using Proposition 11.6.2, it follows that for all $\mathfrak{Re}(s) > \frac{7}{2}$:

$$\mu_{\mathcal{E}(\phi)}(s) = \sum_{\lambda \in \mathcal{L}\backslash\{0\}} \frac{1}{|\lambda|^s} \left(\frac{1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \sum_{l \in \mathcal{G}(\lambda,\mathcal{A})} e^{2\pi i g(l,t)} \operatorname{Tr}_8^{\mathrm{SU}(3)}(A) \right). \tag{11.6.11}$$

Write $\mathcal{G}(\mathcal{A}) = \left\{ l \in \mathbb{Z}^7 \;\middle|\; Al = l \right\}$, a lattice in the 1-eigenspace of A. Note that:

- The 1-eigenspace of A is non-zero, since A is orientation-preserving and preserves the metric g (and the dimension of $\mathbb{R}^7$ is odd);
- Whilst the lattice $\mathcal{G}(\mathcal{A})$ need not have rank equal to the dimension of the 1-eigenspace of A, it must certainly be non-zero. Indeed, since $A \in \mathrm{SL}(7;\mathbb{Z})$, A defines a linear map on $\mathbb{Q}^7 \subset \mathbb{R}^7$ which also has a (non-zero) eigenvector $u \in \mathbb{Q}^7$ with eigenvalue 1 and by rescaling u appropriately, one may ensure that $u \in \mathbb{Z}^7\backslash\{0\}$.

Then by rearranging eqn. (11.6.11), one finds:

$$\mu_{\mathcal{E}(\phi)}(s) = \frac{1}{(2\pi)^{2s}|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \operatorname{Tr}_8^{\mathrm{SU}(3)}(A) \left(\sum_{l \in \mathcal{G}(\mathcal{A})\backslash\{0\}} \frac{e^{2\pi i g(l,t)}}{\|l\|_g^{2s}} \right).$$

The sum of the form:

$$\sum_{l \in \mathcal{G}(\mathcal{A})\backslash\{0\}} \frac{e^{2\pi i g(l,t)}}{\|l\|_g^{2s}}$$

is an example of an Epstein ζ-function, and hence the value at $s = 0$ of its meromorphic extension to $\mathbb{C}$ is always -1, independent of t or the rank of the lattice [43, p. 627]. Thus:

$$\mu_3(\mathrm{M}_\Gamma, \phi) = \frac{-1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \operatorname{Tr}_8^{\mathrm{SU}(3)}(A);$$

in particular, this formula is independent of ϕ. Thus it has been established:

Theorem 11.6.12. *Let $\mathrm{M}_\Gamma = \Gamma\backslash\mathbb{T}^7$ be a Joyce orbifold. The μ_3-invariant $\mu_3 : \mathscr{G}_2^{TF}(\mathrm{M}_\Gamma) \to \mathbb{R}$ is constant, taking the value:*

$$\mu_3(\mathrm{M}_\Gamma) = \frac{-1}{|\Gamma|} \sum_{\mathcal{A}=(A,t)\in\Gamma} \operatorname{Tr}_8^{\mathrm{SU}(3)}(A).$$

Now consider μ_4. All of the above analysis is easily adapted to the case of μ_4 except Lemma 11.6.8, which must be replaced by the following result:

Lemma 11.6.13. *Given $l \in \mathcal{G}(\lambda, \mathcal{A})$, define:*

$$\mathbb{A}'_l = \left\{ \alpha \in \textstyle\bigwedge^3_{27}\left(\mathbb{R}^7\right)_{\mathbb{C}} \;\middle|\; l \lrcorner \alpha = 0 \right\}.$$

Then the trace $\operatorname{Tr}_{(\mathbb{A}'_l)_{\mathbb{C}}}(A)$ of $A = \mathfrak{p}_1(\mathcal{A})$ acting on $\left(\mathbb{A}'_l\right)_{\mathbb{C}}$ via pullback is:

$$\operatorname{Tr}_{(\mathbb{A}'_l)_{\mathbb{C}}}(A) = \operatorname{Tr}_{12}^{\mathrm{SU}(3)}(A) = \frac{\operatorname{Tr}_{\mathbb{R}^7}(A)^3 + 2\operatorname{Tr}_{\mathbb{R}^7}(A^3) - 3\operatorname{Tr}_{\mathbb{R}^7}(A^2)\operatorname{Tr}_{\mathbb{R}^7}(A)}{6} - \frac{\operatorname{Tr}_{\mathbb{R}^7}(A)^2 - \operatorname{Tr}_{\mathbb{R}^7}(A^2)}{2} - 2.$$

Proof. As before, note that $\mathrm{Tr}_{(\mathbb{A}'_l)_{\mathbb{C}}}(A) = \mathrm{Tr}_{\mathbb{A}'_l}(A)$. Let $\mathbb{B}$, θ, ρ, ω and J be as in the proof of Lemma 11.6.8. Then there is a decomposition:

$$\bigwedge^3\mathbb{B}^* = \underbrace{\mathbb{R}\cdot\langle\rho, J^*\rho\rangle}_{\left[\!\left[\wedge^{(3,0)}\mathbb{B}^*\right]\!\right]} \oplus \underbrace{\left[\!\left[\bigwedge^{2,1}_6\mathbb{B}^*\right]\!\right] \oplus \left[\!\left[\bigwedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]}_{\left[\!\left[\wedge^{(2,1)}\mathbb{B}^*\right]\!\right]}$$

into simple SU(3)-modules, where $\left[\!\left[\wedge^{2,1}_6\mathbb{B}^*\right]\!\right] = \{\vartheta\wedge\omega \mid \vartheta\in\mathbb{B}^*\} \cong \mathbb{B}$ and $\left[\!\left[\wedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]$ denotes the ortho-complement to $\left[\!\left[\wedge^{2,1}_6\mathbb{B}^*\right]\!\right]$ in $\left[\!\left[\wedge^{2,1}\mathbb{B}^*\right]\!\right]$. Define an isomorphism:

$$\begin{aligned} \widetilde{\chi}_6 : \mathbb{B}^* &\xrightarrow{\sim} \left[\!\left[\bigwedge^{2,0}\mathbb{B}^*\right]\!\right] \\ u \lrcorner\, \omega &\mapsto u \lrcorner\, J^*_\rho \rho. \end{aligned}$$

Then one may verify that:

$$\bigwedge^3_{27}\mathbb{A}^* = \mathbb{R}\cdot(4\theta\wedge\omega - 3\rho) \oplus \{\theta\wedge\widetilde{\chi}_6(\alpha) - \alpha\wedge\omega \mid \alpha\in\mathbb{B}^*\} \oplus \theta\wedge\left[\bigwedge^{1,1}_8\mathbb{B}^*\right] \oplus \left[\!\left[\bigwedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]$$

and hence:

$$\mathbb{A}'_l = \left\{\alpha\in\bigwedge^2_{14}\left(\mathbb{R}^7\right) \,\middle|\, l\lrcorner\,\alpha = 0\right\} = \left[\!\left[\bigwedge^{2,1}_{12}\mathbb{B}^*\right]\!\right].$$

One may compute directly that:

$$\mathrm{Tr}_{\wedge^3(\mathbb{R}^7)^*}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^3 + 2\,\mathrm{Tr}_{\mathbb{R}^7}(A^3) - 3\,\mathrm{Tr}_{\mathbb{R}^7}(A^2)\,\mathrm{Tr}_{\mathbb{R}^7}(A)}{6}. \tag{11.6.14}$$

Hence, using the G_2-invariant decomposition $\wedge^3\left(\mathbb{R}^7\right)^* \cong \mathbb{R}\oplus\mathbb{R}^7\oplus\wedge^3_{27}\left(\mathbb{R}^7\right)^*$, one finds that:

$$\mathrm{Tr}_{\wedge^3_{27}(\mathbb{R}^7)^*}(A) = \frac{\mathrm{Tr}_{\mathbb{R}^7}(A)^3 + 2\,\mathrm{Tr}_{\mathbb{R}^7}(A^3) - 3\,\mathrm{Tr}_{\mathbb{R}^7}(A^2)\,\mathrm{Tr}_{\mathbb{R}^7}(A)}{6} - \mathrm{Tr}_{\mathbb{R}^7}(A) - 1. \tag{11.6.15}$$

Now, since there is an SU(3)-invariant decomposition $\wedge^3_{27}\left(\mathbb{R}^7\right)^* \cong \mathbb{R}\oplus\mathbb{B}\oplus\left[\wedge^{1,1}_8\mathbb{B}^*\right]\oplus\left[\!\left[\wedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]$, it follows that:

$$\begin{aligned} \mathrm{Tr}_{\left[\!\left[\wedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]}(A) &= \mathrm{Tr}_{\wedge^3_{27}(\mathbb{R}^7)^*}(A) - 1 - \mathrm{Tr}_{\mathbb{B}}(A) - \mathrm{Tr}_{\left[\wedge^{1,1}_8\mathbb{B}^*\right]}(A) \\ &= \mathrm{Tr}_{\wedge^3_{27}(\mathbb{R}^7)^*}(A) - \mathrm{Tr}_{\mathbb{R}^7}(A) - \mathrm{Tr}^{\mathrm{SU}(3)}_8(A). \end{aligned} \tag{11.6.16}$$

The result follows.

$\square$

Arguing as in for μ_3, one obtains:

Theorem 11.6.17. *Let* $\mathrm{M}_\Gamma = \Gamma\backslash\mathbb{T}^7$ *be a Joyce orbifold. Then the* μ_4*-invariant* $\mu_4 : \mathscr{G}_2^{TF}(\mathrm{M}_\Gamma)\to\mathbb{R}$ *is constant, taking the value:*

$$\mu_4(\mathrm{M}_\Gamma) = \frac{-1}{|\Gamma|}\sum_{\mathcal{A}=(A,t)\in\Gamma}\mathrm{Tr}^{\mathrm{SU}(3)}_{12}(A).$$

11.7 Examples

Using the explicit formulae for μ_3 and μ_4 given in Theorems 11.6.12 and 11.6.17, many explicit examples of μ_3 and μ_4 can be computed in practice. I give a few examples below:

Example 11.7.1 (Flat Tori)**.** Firstly consider the case $\Gamma = \mathbf{1}$. Then:

$$\mu_3(\mathbb{T}^7) = -\operatorname{Tr}_8^{\mathrm{SU}(3)}(\mathrm{Id}) = -8$$

and:

$$\mu_4(\mathbb{T}^7) = -\operatorname{Tr}_{12}^{\mathrm{SU}(3)}(\mathrm{Id}) = -12.$$

(Note that $\operatorname{Tr}_8^{\mathrm{SU}(3)}(\mathrm{Id}) = \dim\left[\bigwedge^{1,1}_8\mathbb{B}^*\right]$ and $\operatorname{Tr}_{12}^{\mathrm{SU}(3)}(\mathrm{Id}) = \dim\left[\!\left[\bigwedge^{2,1}_{12}\mathbb{B}^*\right]\!\right]$, as expected.)

For the first non-trivial case, let me consider a family of examples in [77, §3.1]. Consider $\widetilde{\Gamma} = \langle\alpha,\beta,\gamma\rangle \subset (\mathrm{G}_2 \cap \mathrm{SL}(7;\mathbb{Z})) \ltimes \mathbb{T}^7$ where:

$$\begin{aligned}
\alpha &: (x^1,x^2,x^3,x^4,x^5,x^6,x^7) \mapsto (-x^1,-x^2,-x^3,-x^4,x^5,x^6,x^7)\\
\beta &: (x^1,x^2,x^3,x^4,x^5,x^6,x^7) \mapsto \left(b^1-x^1,b^2-x^2,x^3,x^4,-x^5,-x^6,x^7\right)\\
\gamma &: (x^1,x^2,x^3,x^4,x^5,x^6,x^7) \mapsto \left(c^1-x^1,x^2,c^3-x^3,x^4,c^5-x^5,x^6,-x^7\right)
\end{aligned}$$

where $b^1,b^2,c^1,c^3,c^5 \in \left\{0,\frac{1}{2}\right\}$. Then it is shown in [77] that $\widetilde{\Gamma} \cong \left(\mathbb{Z}/2\right)^3$, generated by α, β and γ. One may compute that for all $\mathcal{A} = (A,t) \in \widetilde{\Gamma}\backslash\{\mathrm{Id}\}$, A is diagonal, with diagonal entries (in some order):

$$1,1,1,-1,-1,-1,-1.$$

Thus, one may verify that for all $\mathcal{A} = (A,t) \in \widetilde{\Gamma}\backslash\{\mathrm{Id}\}$:

$$\operatorname{Tr}_8^{\mathrm{SU}(3)}(A) = 0 \quad\text{and}\quad \operatorname{Tr}_{12}^{\mathrm{SU}(3)}(A) = 4.$$

Using this, one can compute further examples:

Example 11.7.2 (K3 Orbifold)**.** Take $\Gamma_1 = \langle\alpha\rangle \subset \widetilde{\Gamma}$. Then $\mathrm{M}_1 = \mathrm{M}_{\Gamma_1} \cong \left(\mathbb{Z}/2\right)\backslash\mathbb{T}^4 \times \mathbb{T}^3$ where $\left(\mathbb{Z}/2\right)\backslash\mathbb{T}^4$ is the standard orbifold used in the Kummer construction of the K3 surface. Using Theorems 11.6.12 and 11.6.17, one may compute that:

$$\mu_3(\mathrm{M}_1) = \frac{-1}{2}(8+0) = -4$$

and:

$$\mu_4(\mathrm{M}_1) = \frac{-1}{2}(12+4) = -8.$$

Example 11.7.3 (Calabi-Yau Orbifold)**.** Set $(b^1,b^2) = \left(\frac{1}{2},0\right)$ and take $\Gamma_2 = \langle\alpha,\beta\rangle \subset \widetilde{\Gamma}$. Then $\mathrm{M}_2 = \mathrm{M}_{\Gamma_2} \cong \left(\mathbb{Z}/2\right)^2\backslash\mathbb{T}^6 \times S^1$, where $\left(\mathbb{Z}/2\right)^2\backslash\mathbb{T}^6$ is an SU(3)-orbifold admitting a smooth Calabi-

Yau 3-fold as a crepant resolution. Then:

$$\mu_3(\mathrm{M}_2) = \frac{-1}{4}(8 + 3 \times 0) = -2$$

and:

$$\mu_4(\mathrm{M}_2) = \frac{-1}{4}(12 + 3 \times 4) = -6.$$

Example 11.7.4 (G$_2$ Orbifold)**.** Now consider the full group $\Gamma_3 = \widetilde{\Gamma}$. Then for suitable choices of b^i and c^j, the orbifold $\mathrm{M}_3 = \mathrm{M}_{\Gamma_3}$ may be resolved to form a smooth G$_2$-manifold (see [76, 77]). Then:

$$\mu_3(\mathrm{M}_3) = \frac{-1}{8}(8 + 7 \times 0) = -1$$

and:

$$\mu_4(\mathrm{M}_3) = \frac{-1}{8}(12 + 7 \times 4) = -5.$$

Using similar methods, many further explicit examples can be computed.

Remark 11.7.5. In [31], Crowley–Goette–Nordström defined a different spectral invariant of torsion-free G$_2$-structures on manifolds, denoted $\overline{\nu}$. By [44, Thm. 7.7], $\overline{\nu}$ is equally well-defined on closed G$_2$-orbifolds. Moreover, for any closed G$_2$-orbifold (M, ϕ) which admits an orientation-reversing isometry, $\overline{\nu}(\phi) = 0$ (cf. [31, Prop. 1.5(iii)]).

Now consider the torsion-free G$_2$-structure ϕ_0 on the orbifolds $\mathbb{T}^7$, M_1 and M_2 above. Each of $(\mathbb{T}^7, \phi_0)$, (M_1, ϕ_0) and (M_2, ϕ_0) admits an orientation-reversing isometry, since each orbifold is the Riemannian product of S^1 with a 6-orbifold. Thus:

$$\overline{\nu}(\mathbb{T}^7, \phi_0) = \overline{\nu}(\mathrm{M}_1, \phi_0) = \overline{\nu}(\mathrm{M}_2, \phi_0) = 0;$$

in particular, the $\overline{\nu}$-invariant alone cannot distinguish between these three non-diffeomorphic G$_2$-orbifolds. By contrast:

$$\mu_3(\mathbb{T}^7) = -8, \quad \mu_3(\mathrm{M}_1) = -4 \quad \text{and} \quad \mu_3(\mathrm{M}_2) = -2$$
$$\mu_4(\mathbb{T}^7) = -12, \quad \mu_4(\mathrm{M}_1) = -8 \quad \text{and} \quad \mu_4(\mathrm{M}_2) = -6$$

and thus either of μ_3 or μ_4 alone is sufficient to distinguish the orbifolds $\mathbb{T}^7$, M_1 and M_2. This provides some evidence that the μ_3 and μ_4 might be better suited than the $\overline{\nu}$-invariant to studying Joyce orbifolds, and thus perhaps also to studying Joyce manifolds.

Chapter 12

Concluding remarks and open questions

Prior to this thesis, the $\overline{\nu}$-invariant discussed above was the only known invariant of torsion-free G_2-structures. Whilst $\overline{\nu}$ has been effectively computed for the extra-twisted connect-sum G_2-manifolds constructed in [30, 110], it is not known how to compute $\overline{\nu}$ for Joyce manifolds.

The μ-invariants introduced above aim to address this issue. In particular, Chapter 11 lays the foundations for a larger project for obtaining formulae for the μ-invariants μ_3 and μ_4 on an arbitrary Joyce manifold, as constructed in [76, 77, 78]. I now briefly outline the proposed shape of such a project.

Recall that, given a Joyce orbifold M_Γ with torsion-free G_2 3-form ϕ equipped with a choice of resolution data (see [78, Defn. 11.4.1]) there is a smooth resolution $\widetilde{\mathrm{M}}_\Gamma$ of M together with a family of torsion-free G_2 3-forms ϕ_t for $t > 0$ sufficiently small such that $\left(\widetilde{\mathrm{M}}_\Gamma, \phi_t\right)$ tends to the orbifold $(\mathrm{M}_\Gamma, \phi)$ in the Gromov–Hausdorff sense as $t \to 0$ (cf. [78, Thm. 11.6.2]). The first stage in the project would be to verify that the value $\mu_3(\phi_t)$ was independent of t, and similarly for μ_4. Specifically, I conjecture that:

Conjecture 12.0.1. *Let* M *be a closed, oriented 7-orbifold. Then the μ-invariants:*

$$\mu_{3,4} : \mathscr{G}_2^{TF}(\mathrm{M}) \to \mathbb{R}$$

are locally constant (i.e. constant on each connected component of the moduli space).

Given Conjecture 12.0.1, to compute the value of $\mu_{3,4}(\phi_t)$ at any fixed value of t, it would suffice to compute the limiting value of $\mu_{3,4}(\phi_t)$. It is then hoped that this limiting value will be closely related to $\mu_{3,4}(\phi)$, the $\mu_{3,4}$-invariant of the orbifold $(\mathrm{M}_\Gamma, \phi)$, which can be explicitly calculated using the results of Chapter 11.

The proof of Conjecture 12.0.1 is anticipated to proceed as follows: restricting for simplicity to the case of manifolds, it follows from the results of [10] that given a 1-parameter family $\phi(s) \in \mathscr{G}_2^{TF}(\mathrm{M})$ ($s \in (-\varepsilon, \varepsilon)$), the derivative $\frac{\mathrm{d}}{\mathrm{d}s}\mu_{3,4}(\phi(s))\big|_{s=0}$ is local, meaning that it can be written in the form:

$$\int_{\mathrm{M}} \alpha_0\left(\phi(0), \dot{\phi}(0)\right)$$

for some 7-form α_0 depending linearly on $\dot{\phi}(0)$. By exploiting G_2-invariance as in [7], one can verify that α_0 is in fact a polynomial in the derivatives of $\dot{\phi}(0)$ and the derivatives of the Riemann tensor R of $\phi(0)$, which is linear in $\dot{\phi}(0)$. Moreover, the possible monomials occurring in this polynomial can be explicitly computed (although computing the coefficients is impractical). Thus, to prove Conjecture 12.0.1, the task is to prove that each monomial vanishes when integrated over M, a result which is expected to follow from the fact that $\dot{\phi}(0)$ may be taken to be harmonic with respect to $\phi(0)$ (since $\mathrm{T}_{\phi(0)}\mathscr{G}_2^{TF}(\mathrm{M}) = \mathscr{H}^3_{\phi(0)}(\mathrm{M})$; see [78, Thm. 10.4.4]). Verifying this result will form the basis of a future project.

Appendix A

Formulae for refined exterior derivatives on G_2-manifolds

Recall that on a G_2-manifold (M, ϕ), the usual exterior derivative may be decomposed according to type, yielding the 'refined' exterior differential operators:

$$\begin{aligned} \mathrm{d}^1_7 : \Omega^0(\mathrm{M}) &\to \Omega^1(\mathrm{M}) & \mathrm{d}^7_7 : \Omega^1(\mathrm{M}) &\to \Omega^1(\mathrm{M}) & \mathrm{d}^7_{14} : \Omega^1(\mathrm{M}) &\to \Omega^2_{14}(\mathrm{M}) \\ f &\mapsto \mathrm{d}f & \alpha &\mapsto \bigstar_\phi \mathrm{d}(\alpha \wedge \psi) & \alpha &\mapsto \pi_{14}(\mathrm{d}\alpha) \end{aligned}$$

$$\begin{aligned} \mathrm{d}^7_{27} : \Omega^1(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) & \mathrm{d}^{14}_{27} : \Omega^2_{14}(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) & \mathrm{d}^{27}_{27} : \Omega^3_{27}(\mathrm{M}) &\to \Omega^3_{27}(\mathrm{M}) \\ \alpha &\mapsto \pi_{27}\mathrm{d}\bigstar_\phi(\alpha \wedge \psi) & \beta &\mapsto \pi_{27}(\mathrm{d}\beta) & \gamma &\mapsto \bigstar_\phi \pi_{27}(\mathrm{d}\beta). \end{aligned}$$

Analogously, define $\mathrm{d}^7_1 = (\mathrm{d}^1_7)^*$, $\mathrm{d}^{14}_7 = (\mathrm{d}^7_{14})^*$, $\mathrm{d}^{27}_7 = (\mathrm{d}^7_{27})^*$ and $\mathrm{d}^{27}_{14} = (\mathrm{d}^{14}_{27})^*$, where * denotes the formal L^2 adjoint (d^7_7 and d^{27}_{27} are both formally L^2 self-adjoint). Then the main result of [22, §5] is:

Theorem. *All exterior and co-exterior derivatives on the* G_2*-manifold* (M, ϕ) *can be expressed purely in terms of the operators* d^1_7, d^7_1, d^7_7, d^7_{14}, d^{14}_7, d^7_{27}, d^{27}_7, d^{14}_{27}, d^{27}_{14} *and* d^{27}_{27}. *Explicitly:*

- *For* $f \in \Omega^0(\mathrm{M})$*:*

$$\mathrm{d}f = \mathrm{d}^1_7 f, \quad \mathrm{d}(f\phi) = \mathrm{d}^1_7 f \wedge \phi \quad \textit{and} \quad \mathrm{d}(f\psi) = \mathrm{d}^1_7 f \wedge \psi; \tag{A.0.1}$$

- *For* $\alpha \in \Omega^1(\mathrm{M})$*:*

$$\begin{gathered} \mathrm{d}\alpha = \frac{1}{3}\bigstar_\phi\left(\mathrm{d}^7_7\alpha \wedge \bigstar_\phi\phi\right) + \mathrm{d}^7_{14}\alpha, \quad \mathrm{d}(\alpha\wedge\phi) = \frac{2}{3}\mathrm{d}^7_7\alpha\wedge\psi - \bigstar_\phi \mathrm{d}^7_{14}\alpha, \\ \mathrm{d}\bigstar_\phi(\alpha\wedge\phi) = \frac{4}{7}(\mathrm{d}^7_1\alpha)\psi + \frac{1}{2}\mathrm{d}^7_7\alpha\wedge\phi + \bigstar_\phi\mathrm{d}^7_{27}\alpha, \\ \mathrm{d}\left(\bigstar_\phi(\alpha\wedge\bigstar_\phi\phi)\right) = -\frac{3}{7}\left(\mathrm{d}^7_1\alpha\right)\phi - \frac{1}{2}\bigstar_\phi\left(\mathrm{d}^7_7\alpha\wedge\phi\right) + \mathrm{d}^7_{27}\alpha, \\ \mathrm{d}(\alpha\wedge\psi) = \bigstar_\phi\mathrm{d}^7_7\alpha^{1} \quad \textit{and} \quad \mathrm{d}(\bigstar_\phi\alpha) = -\left(\mathrm{d}^7_1\alpha\right)vol_\phi. \end{gathered} \tag{A.0.2}$$

[1] This is incorrectly stated in [22, §5] as $\mathrm{d}(\alpha\wedge\psi) = -\bigstar_\phi\mathrm{d}^7_7$. The error was pointed out by Bryant–Xu in [24].

- *For* $\beta \in \Omega^2_{14}(\mathrm{M})$*:*

$$\mathrm{d}\beta = \frac{1}{4}\bigstar_\phi(\mathrm{d}^{14}_7\beta \wedge \phi) + \mathrm{d}^{14}_{27}\beta \quad \textit{and} \quad \mathrm{d}^*\beta = \mathrm{d}^{14}_7\beta; \tag{A.0.3}$$

- *For* $\gamma \in \Omega^3_{27}(\mathrm{M})$*:*

$$\mathrm{d}\gamma = \frac{1}{4}\mathrm{d}^{27}_7\gamma \wedge \phi + \bigstar_\phi \mathrm{d}^{27}_{27}\gamma \quad \textit{and} \quad \mathrm{d}^*\gamma = \frac{1}{3}\bigstar_\phi(\mathrm{d}^{27}_7\gamma \wedge \psi) + \bigstar_\phi \mathrm{d}^{27}_{14}\gamma. \tag{A.0.4}$$

The condition $\mathrm{d}^2 = 0$ *corresponds to the 14 identities:*

$$\begin{gathered}
\mathrm{d}^7_7\mathrm{d}^1_7 = 0, \quad \mathrm{d}^7_{14}\mathrm{d}^1_7 = 0, \quad \mathrm{d}^7_1\mathrm{d}^7_7 = 0, \quad \mathrm{d}^{14}_7\mathrm{d}^7_{14} = \frac{2}{3}\left(\mathrm{d}^7_7\right)^2, \quad \mathrm{d}^{27}_7\mathrm{d}^7_{27} = \left(\mathrm{d}^7_7\right)^2 + \frac{12}{7}\mathrm{d}^1_7\mathrm{d}^7_1, \\
\mathrm{d}^7_{14}\mathrm{d}^7_7 + 2\mathrm{d}^{27}_{14}\mathrm{d}^7_{27} = 0, \quad 3\mathrm{d}^{14}_{27}\mathrm{d}^7_{14} + \mathrm{d}^7_{27}\mathrm{d}^7_7 = 0, \quad 2\mathrm{d}^{27}_{27}\mathrm{d}^7_{27} - \mathrm{d}^7_{27}\mathrm{d}^7_7 = 0, \quad \mathrm{d}^7_1\mathrm{d}^{14}_7 = 0, \\
\mathrm{d}^7_7\mathrm{d}^{14}_7 + 2\mathrm{d}^{27}_7\mathrm{d}^{14}_{27} = 0, \quad \mathrm{d}^7_{27}\mathrm{d}^{14}_7 + 4\mathrm{d}^{27}_{27}\mathrm{d}^{14}_{27} = 0, \quad 3\mathrm{d}^{14}_7\mathrm{d}^{27}_{14} + \mathrm{d}^7_7\mathrm{d}^{27}_7 = 0, \\
2\mathrm{d}^{27}_7\mathrm{d}^{27}_{27} - \mathrm{d}^7_7\mathrm{d}^{27}_7 = 0, \quad \mathrm{d}^7_{14}\mathrm{d}^{27}_7 + 4\mathrm{d}^{27}_{14}\mathrm{d}^{27}_{27} = 0.
\end{gathered} \tag{A.0.5}$$

Finally, all Hodge Laplacians can be expressed in terms of the same operators. Explicitly:

- *For* $f \in \Omega^0(\mathrm{M})$*:*

$$\Delta f = \mathrm{d}^7_1\mathrm{d}^1_7 f. \tag{A.0.6}$$

- *For* $\alpha \in \Omega^1(\mathrm{M})$*:*

$$\Delta\alpha = (\mathrm{d}^7_7)^2\alpha + \mathrm{d}^1_7\mathrm{d}^7_1\alpha. \tag{A.0.7}$$

- *For* $\beta \in \Omega^2_{14}(\mathrm{M})$*:*

$$\Delta\beta = \frac{5}{4}\mathrm{d}^7_{14}\mathrm{d}^{14}_7\beta + \mathrm{d}^{27}_{14}\mathrm{d}^{14}_{27}\beta. \tag{A.0.8}$$

- *For* $\gamma \in \Omega^3_{27}(\mathrm{M})$*:*

$$\Delta\gamma = \frac{7}{12}\mathrm{d}^7_{27}\mathrm{d}^{27}_7\gamma + \mathrm{d}^{14}_{27}\mathrm{d}^{27}_{14}\gamma + (\mathrm{d}^{27}_{27})^2\gamma. \tag{A.0.9}$$

Formulae for the Hodge Laplacian acting on sections of the remaining bundles $\Lambda^p_q\mathrm{T}^*\mathrm{M}$ *are obtained by identifying* $\Lambda^p_q\mathrm{T}^*\mathrm{M}$ *with either* $\Lambda^0\mathrm{T}^*\mathrm{M}$*,* $\Lambda^1\mathrm{T}^*\mathrm{M}$*,* $\Lambda^2_{14}\mathrm{T}^*\mathrm{M}$ *or* $\Lambda^3_{27}\mathrm{T}^*\mathrm{M}$ *as appropriate, and noting that, since* ϕ *is torsion-free,* Δ *commutes with the identification (so that, e.g.* $\Delta(f\phi) = (\Delta f)\phi$*).*

Appendix B

Enumerating k-planes in $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$

The aim of this appendix is to prove the following result.

Proposition B.0.1. *Let $\mathbb{F}$ be a finite field. Recall the q-Pochhammer symbol:*

$$(a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i),$$

where $a \in \mathbb{R}$, $q \in (0,1)$ and $n \in \mathbb{N}$. Then:

$$|\mathrm{Gr}_k(\mathbb{F}^n)| = N^{k(n-k)} \frac{\left(\frac{1}{N};\frac{1}{N}\right)_n}{\left(\frac{1}{N};\frac{1}{N}\right)_k \left(\frac{1}{N};\frac{1}{N}\right)_{(n-k)}}.$$

Initially, let $\mathbb{F}$ be an arbitrary field.

Lemma B.0.2.

$$\mathrm{Gr}_k(\mathbb{F}^n) \cong \mathrm{GL}(n;\mathbb{F}) \Big/ \left(\mathrm{GL}(k;\mathbb{F}) \times \mathrm{GL}(n-k;\mathbb{F})\right) \ltimes \mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)$$

where the multiplication on $\left(\mathrm{GL}(k;\mathbb{F}) \times \mathrm{GL}(n-k;\mathbb{F})\right) \ltimes \mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)$ is given by:

$$(A, B; C) \cdot (A', B'; C') = (AA', BB', AC'B^{-1} + C).$$

Here $A, A' \in \mathrm{GL}(k;\mathbb{F})$, $B, B' \in \mathrm{GL}(n-k,\mathbb{F})$ and $C, C' \in \mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)$.

Proof. Clearly $\mathrm{GL}(n,\mathbb{F})$ acts transitively on $\mathrm{Gr}_k(\mathbb{F}^n)$. Thus fix $\Pi \in \mathrm{Gr}_k(\mathbb{F}^n)$ and choose an algebraic complement Π' to Π in $\mathbb{F}^n$. With respect to the splitting $\Pi \oplus \Pi' \cong \mathbb{F}^n$, the stabiliser in $\mathrm{GL}(n,\mathbb{F})$ of Π consists precisely of those linear maps of the form:

$$\begin{pmatrix} A & D \\ & B \end{pmatrix},$$

where $A \in \mathrm{GL}(k;\mathbb{F})$, $B \in \mathrm{GL}(n-k;\mathbb{F})$ and $D \in \mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)$. The map $(A, B, D) \mapsto (A, B, DB^{-1})$ defines an isomorphism from the stabiliser of Π to the group $\left(\mathrm{GL}(k;\mathbb{F}) \times \mathrm{GL}(n-k;\mathbb{F})\right) \ltimes \mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)$

as defined above.

∎

Now restrict attention to the case where $\mathbb{F}$ is a finite field, say $|\mathbb{F}| = N$.

Lemma B.0.3. *Write* $\mathbb{FP}^{n-1} = \mathrm{Gr}_1(\mathbb{F}^n)$. *Then:*

$$\left|\mathbb{FP}^{n-1}\right| = \frac{N^n - 1}{N - 1}.$$

Proof. Every non-zero element in $\mathbb{F}^n$ (of which there are $N^n - 1$) determines a unique line through the origin, however each line through the origin contains precisely $N - 1$ non-zero points. The result follows.

∎

Lemma B.0.4.

$$|\mathrm{GL}(n,\mathbb{F})| = N^{(n^2)} \prod_{i=1}^{n} \left(1 - \left(\frac{1}{N}\right)^i\right) = N^{n^2} \left(\frac{1}{N}; \frac{1}{N}\right)_n. \tag{B.0.5}$$

Proof. Proceed by induction. In the case $n = 1$, $\mathrm{GL}(1,\mathbb{F})$ consists of the non-zero elements of $\mathbb{F}$ and thus has size $(N - 1)$, as required. In general, using Lemma B.0.2, one sees that:

$$\frac{|\mathrm{GL}(n+1;\mathbb{F})|}{|\mathrm{GL}(1;\mathbb{F})| \times |\mathrm{GL}(n;\mathbb{F})| \times \left|(\mathbb{F}^n)^*\right|} = |\mathbb{FP}^n|.$$

Thus, by using Lemma B.0.3 together with $\left|(\mathbb{F}^n)^*\right| = N^n$, one sees inductively that:

$$\begin{aligned}
|\mathrm{GL}(n+1;\mathbb{F})| &= \underbrace{N^n}_{|(\mathbb{F}^n)^*|} \cdot \underbrace{(N-1)}_{|\mathrm{GL}(1;\mathbb{F})|} \cdot \underbrace{\frac{N^{n+1}-1}{N-1}}_{|\mathbb{FP}^n|} \cdot \underbrace{N^{(n^2)} \left(\frac{1}{N}; \frac{1}{N}\right)_n}_{|\mathrm{GL}(n;\mathbb{F})|} \\
&= N^{n^2+2n+1} \left(1 - \left(\frac{1}{N}\right)^{n+1}\right) \left(\frac{1}{N}; \frac{1}{N}\right)_n \\
&= N^{(n+1)^2} \left(\frac{1}{N}; \frac{1}{N}\right)_{n+1},
\end{aligned}$$

as required.

∎

I now prove Proposition B.0.1.

Proof. Using Lemma B.0.2, one computes that:

$$|\mathrm{Gr}_k(\mathbb{F}^n)| = \frac{|\mathrm{GL}(n;\mathbb{F})|}{|\mathrm{GL}(k;\mathbb{F})| \times |\mathrm{GL}(n-k;\mathbb{F})| \times \left|\mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^k\right)\right|}.$$

Substituting the result of Lemma B.0.4 together with $\left|\mathrm{End}\left(\mathbb{F}^{n-k}, \mathbb{F}^{k}\right)\right| = N^{k(n-k)}$ yields:

$$|\mathrm{Gr}_k(\mathbb{F}^n)| = \frac{N^{n^2}\left(\frac{1}{N};\frac{1}{N}\right)_n}{N^{k^2}\left(\frac{1}{N};\frac{1}{N}\right)_k \times N^{(n-k)^2}\left(\frac{1}{N};\frac{1}{N}\right)_{(n-k)} \cdot N^{k(n-k)}}.$$

The result follows from the identity $k^2 + (n-k)^2 + k(n-k) = n^2 - k(n-k)$.

□

In particular, the number of 2-planes in 6-dimensional space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ is:

$$2^{2\cdot 4}\frac{\left(\frac{1}{2};\frac{1}{2}\right)_6}{\left(\frac{1}{2};\frac{1}{2}\right)_2\left(\frac{1}{2};\frac{1}{2}\right)_4} = 651,$$

as claimed in Construction 9.5.10.

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