Two-dimensional Discrete Gaussian model at high temperature

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I would like to dedicate this thesis to my parents and my brother

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified here and in the text. The entire thesis is a mixture of [12, 13], which are written in collaboration with Roland Bauerschmidt and Pierre-François Rodriguez with equal contributions, and [81]. Appendix 1.A originates from [13, Appendix A]. While the structure of Chapter 2–7 is based on [12] with some modifications, there are additional inputs from the other two papers. Most of the inputs from [13] have been simplified and integrated with those of [81]. Chapter 8 is due to [81] alone, and Chapter 9 is again a mixture of [12, 13, 81].

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Abstract

The Discrete Gaussian model is a Gaussian free field on lattice restricted to take integer values. In dimension two, it was proved by the seminal work of Fröhlich-Spencer that the Discrete Gaussian model exhibits localisation-delocalisation phase transition. The phase transition is ubiquitous in two-dimensional statistical physics models, intriguing the need for a unified framework for studying these phenomena.

The goal of this thesis is to apply rigorous renormalisation group method to study the two-dimensional discrete Gaussian model in the delocalised phase, thereby obtaining central limit theorems in long-distance limit—in physics literature, the renormalisation group is a standard apparatus used to study scaling phenomena, in particular computing critical exponents and proving scaling limits and universality.

We study the central limit theorem in three different regimes, first on macroscopic scale, second on mesoscopic scale and the third on microscopic scale. The first two amount to studying the scaling limits of the spin model under different limit regimes, while the final one discusses both pointwise and limit results. The final results have in particular prolific by-products, producing analogues of a number of results proved for different interface models.

The entire thesis is devoted to solving these problems, but the strategy of the proof we develop is expected to have general applicability. Indeed, we develop renormalisation technology in the first half (Chapter 2–4) that only has weak requirements on the model. Then in the rest of the thesis, we develop an analysis specific to our model to prove the main theorems.

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Chapter 1

Introduction

The aim of this chapter is to motivate the study of the two-dimensional Discrete Gaussian model and to give an outline of the forthcoming chapters. In Section 1.1, we give an overview of the two-dimensional Discrete Gaussian model and present the main results of this thesis. In Section 1.2, we discuss some implications of the main results and also introduce related open problems. In Section 1.3, we introduce the method of the proofs and given an outline of how these are implemented in the rest of the thesis.

1.1 Motivation and main results

1.1.1 Motivation

The main player of this thesis is the two-dimensional *Discrete Gaussian* (DG) model. In any dimension, the Discrete Gaussian model is an effective model for discrete interfaces, while if we only focus on the two-dimensional case, then the Discrete Gaussian model can also be thought of as a dual representation of the two-dimensional lattice Coulomb gas model with charge symmetry (see Section 1.3.3). The two-dimensional Coulomb gas model can be used to represent a broad range of two-dimensional field models, thus giving direct and indirect connections with various two-dimensional statistical and quantum physical models. An incomplete list of such models includes the Ising model, three and four-states Potts model (see [60]), Solid-On-Solid model, the sine-Gordon model, the dimer model, the square ice models, the XY model, each of which has been a central object of study in the mathematical physics and probability community, partially because of their interesting physics properties and partially because of their exact solvability (for a subset of models above such as the Ising model, the dimer model, and some specific 6 and 8 vertex models, see [16], for example). The Discrete Gaussian model and height functions associated with some of these models, such as dimer model, the square ice model, Solid-on-Solid model, the sine-Gordon model and the dual of the XY model (as a discrete height function), also exhibit localisation-delocalisation phase transition. The definition and detailed exposition of the phase transition will be discussed in Section 1.1.2. Here, we just mention that, for the case of the Discrete Gaussian model, the localisation-delocalisation transition can be interpreted as a type of Kosterlitz-Thouless phase transition, which was originally observed in the charge-symmetric Coulomb gas model by Berezinskiĭ [18] and Kosterlitz–Thouless [66].

With this motivation, the aim of this thesis is to study the delocalised phase of the two-dimensional Discrete Gaussian model in detail, using the renormalisation group, explained in Section 1.3.1.

1.1.2 The localisation-delocalisation phase transition

The model

Given $N \in 2\mathbb{N}+1$, $N \in \mathbb{N}$, let Λ_N be the $L^N \times L^N$ discrete torus with distinguished $0 \in \Lambda_N$. It can be realised as $\Lambda_N = [-\frac{L^N-1}{2}, \frac{L^N-1}{2}]^2 \cap \mathbb{Z}^2$ with toroidal graph structure, i.e., we let $\{(x_1,x_2),(y_1,y_2)\} \subset \Lambda_N$ be an edge of Λ_N if $x_1 - y_1 \equiv \pm 1 (\mathrm{mod}\ L^N)$ and $x_2 = y_2$ or $x_2 - y_2 \equiv \pm 1 (\mathrm{mod}\ L^N)$ and $x_1 = y_1$. We also denote $(x_1,x_2) \sim (y_1,y_2)$ in this case. Then the Discrete Gaussian model on Λ_N with *temperature* β is the random height function $\sigma \sim \mathbb{P}^{\Lambda_N}_{\beta,\mathrm{DG}}$ given by

$$\begin{aligned} \mathbb{P}^{\Lambda_N}_{\beta,\mathrm{DG}}(\sigma) &= e^{-\frac{1}{4\beta}\sum_{x\sim y}(\sigma(x)-\sigma(y))^2} / Z^{\Lambda_N}_{\beta,\mathrm{DG}} = e^{-\frac{1}{2\beta}(\sigma,-\Delta\sigma)} / Z^{\Lambda_N}_{\beta,\mathrm{DG}} \\ Z^{\Lambda_N}_{\beta,\mathrm{DG}} &= \sum_{\sigma\in\Omega^{\Lambda_N}} e^{-\frac{1}{4\beta}\sum_{x\sim y}(\sigma)x_-\sigma(y))^2} \end{aligned} \tag{1.1}$$

where $\sum_{x\sim y}$ counts each edge twice, Δ is the Laplacian $\Delta f(x) = \sum_{y:y\sim x} f(y) - f(x)$ and the state space is

$$\Omega^{\Lambda_N} = \{\sigma \in (2\pi\mathbb{Z})^{\Lambda_N} : \sigma(0) = 0\}. \tag{1.2}$$

The expectation with respect to $\mathbb{P}^{\Lambda}_{\beta,\mathrm{DG}}$ is also denoted $\langle\cdot\rangle^{\Lambda_N}_{\beta,\mathrm{DG}}$. Note that, although $\sigma \in \Omega^{\Lambda}$ is pinned to take $\sigma(0) = 0$, we really want to study distribution of the gradient of σ. Thus any function $F : (2\pi\mathbb{Z})^{\Lambda_N} \to \mathbb{R}$ satisfying $F(\sigma) = F(\sigma + c)$ for any constant $c \in 2\pi\mathbb{Z}$ can be tested against $\langle\cdot\rangle^{\Lambda_N}_{\beta,\mathrm{DG}}$.

We use $\frac{1}{\beta}$ instead of β in (1.1) (thus β is the temperature, not the inverse temperature) to relate better to the Coulomb gas literature—by duality, β corresponds to an inverse temperature for the dual Coulomb gas.

For comparison, we also state the definition of the Gaussian free field. With the state space $\{\phi \in \mathbb{R}^{\Lambda_N} : \phi(0) = 0\}$, the real-valued discrete Gaussian free field on Λ_N is the random height function ϕ with measure

$$\mathbb{P}^{\Lambda_N}_{\mathrm{GFF}}(d\phi) \propto e^{-\frac{1}{2}(\phi,-\Delta\phi)} \prod_{x\in\Lambda_N\setminus\{0\}} d\phi_x. \tag{1.3}$$

The moment generating function of the gradient of ϕ can be computed explicitly by

$$\langle e^{(f,\phi)}\rangle^{\Lambda_N}_{\mathrm{GFF}} = \exp\Big(\frac{1}{2}(f,(-\Delta_{\Lambda_N})^{-1}f)\Big), \tag{1.4}$$

for any $f \in \mathbb{R}^{\Lambda_N}$ such that $\sum_{x\in\Lambda_N} f(x) = 0$.

Phase transition

The phase transition is observed when β is varied. When β is sufficiently small, then the Peierls' argument shows that the variance of the height function is bounded uniformly in the volume, so

$$\sup_{N>0}\sup_{x\in\Lambda_N} \mathrm{Var}^{\Lambda_N}_{\beta,\mathrm{DG}}(\sigma_x - \sigma_0) < \infty. \tag{1.5}$$

In fact, a sophisticated Peierls' argument [19] even shows the exponential clustering at small β, that is, the correlation function between any two spin sets decays exponentially in their distance. While, if β is sufficiently large, then (1.5) is violated, and it is said the height function is delocalised, or exhibits roughening transition. The existence of the delocalisation phase was first proved by Fröhlich–Spencer [41]. In fact, they prove a stronger statement that the Discrete Gaussian model exhibits approximate Gaussianity when combined with a Gaussian domination inequality [38]: when β is sufficiently large, and f is a function on Λ_N such that $\sum_{x\in\Lambda_N} f(x) = 0$,

$$\langle e^{(1-s_0)(f,\phi)}\rangle^{\Lambda_N}_{\mathrm{GFF}} \underset{[41]}{\leqslant} \langle e^{\beta^{-1/2}(f,\sigma)}\rangle^{\Lambda_N}_{\beta,\mathrm{DG}} \underset{[38]}{\leqslant} \langle e^{(f,\phi)}\rangle^{\Lambda_N}_{\mathrm{GFF}} \tag{1.6}$$

(see also [65] for a detailed review) for some $s_0 \equiv s_0(\beta) \geqslant 0$ and $\phi \sim \mathbb{P}^{\Lambda_N}_{\mathrm{GFF}}$. If we set $f(x) = t(\delta_0(x) - \delta_z(x))$ applying (1.4), then differentiating twice the first inequality of (1.6) in t proves

$$\mathrm{Var}^{\Lambda_N}_{\beta,\mathrm{DG}}(\sigma_z - \sigma_0) \geqslant C\beta \log \mathrm{dist}_2(z,0) \tag{1.7}$$

for some $C > 0$, where dist_2 is the ℓ^2-distance, thus the first inequality of (1.6) implies delocalisation. Moreover, it is proved that $s_0(\beta) \to 0$ as $\beta \to \infty$, so (1.6) shows that the scaling limit of the Discrete Gaussian model is a Gaussian Free Field in the limit $\beta \to \infty$.

The proof of the lower bound of (1.6) in [41] is based on a multi-scale expansion of spin-waves, whose name originates from the XY model. The spin-wave is also what is responsible for the lack of spontaneous magnetisation in the XY model at all temperature, as proved by Mermin–Wagner [76, 75]. The Mermin-Wagner-type argument has a wide range of application in statistical physics, see [40, 82, 31]. Also, applications of the multi-scale expansion of [41] can be found in [46, 91]. However, the spin-wave expansion is not the only approach to delocalisation. For example, some recent alternative accounts on the delocalistion includes [67, 68, 70, 3, 88], all of them using percolation arguments. The percolation arguments can be generalised to different settings more flexibly, for example to various types of planar graphs (not necessarily square lattice), but relatively lack quantitative control.

The main results of this thesis consists of improving (1.6) in certain limit regimes. In particular, by considering the infrared limit on the torus (Theorem 1.1.1) or on the infinite plane $\mathbb{R}^2$ (Theorem 1.1.3), we prove that the scaling limit of the Discrete Gaussian model for any sufficiently high, but finite temperature is a multiple of the Gaussian free field. Proving the scaling limits involves testing a (discretised version of) smooth function against the Discrete Gaussian model and taking 0-mesh size limit. Thus the scaling limit does not see the microscopic detail of the model. A complementing result on multi-point functions is also proven (Theorem 1.1.5). The multi-point functions will contain the information about the microscopic structure, but it can be seen that these only contribute as a constant in the large-distance limit, thus we will also be able to prove a type of central limit theorem for the two-point function when scaled appropriately (Corollary 1.2.5) and compute the scaling dimension of the fractional charge correlation (Corollary 1.2.6).

The delocalisation is actually a widely observed phenomenon in two-dimensional height function models, and also had been extensively studied under different settings, see [20, 79, 59, 28, 93], for example. More detailed analysis had been performed for the gradient models with sufficiently smooth uniformly convex potential, with some classical results including [80, 44, 43]. There are also works [27, 1] using the renormalisation group approach. Discrete height functions exhibiting delocalisation transition includes dimer models [71, 63, 54, 53, 32, 83] and six-vertex models [55, 33, 34, 92]. Reviews [90, 43] contain more extensive list of models and problems related to the delocalisation.

1.1.3 Finite-range interaction

We present our result for Discrete Gaussian models with general finite-range interactions. In the rest of the thesis, we only use finite-range step distribution $J \subset \mathbb{Z}^2 \backslash \{0\}$ that respects lattice rotations and lattice reflection symmetries and contains the nearest neighbour of 0. In this case, define the (normalised) J-Laplacian Δ_J by

$$(\Delta_J f)(x) = \frac{1}{|J|} \sum_{y \in J} (f(x+y) - f(x)), \tag{1.8}$$

for $f : \mathbb{Z}^d \to \mathbb{R}$, where $|J|$ is the cardinality of J. The Green's function of Δ_J satisfies asymptotics

$$(-\Delta_J)^{-1}(x,y) \sim -\frac{1}{2\pi v_J^2} \log|x-y|, \qquad \text{as } |x-y| \to \infty \tag{1.9}$$

where

$$v_J^2 = \frac{1}{2|J|} \sum_{x \in J} |x_1|^2, \qquad x = (x_1, x_2) \tag{1.10}$$

see [72, Theorem 4.4.4], for example.

If we let the normalised standard nearest-neighbour Laplacian to be given by $J = J_{\text{nn}} = \{(1,0),(0,1),(-1,0),(0,-1)\}$, then the usual Laplacian is related by $\Delta = 4\Delta_{J_{\text{nn}}}$. Another particular case is the range-ρ Laplacian, for $\rho \geqslant 1$, defined by

$$J_\rho = \{x \in \mathbb{Z}^2 \backslash \{0\} : \|x\|_\infty \leqslant \rho\}. \tag{1.11}$$

We do not treat them as special cases in our main theorems, but it will be convenient to consider them as model examples where J is allowed to vary. One may also see Theorem 1.2.1 to motivate the use of J-DG model—we can study near-critical points by taking J sufficiently long-ranged.

The Discrete Gaussian model with periodic boundary condition will be extended to general J-range interactions. Let $L, N > 0$, $\Lambda = \Lambda_N$ and Ω^{Λ_N} be as in Section 1.1.2 and recall that there was a distinguished point $0 \in \Lambda$. Then for J as above, the J-Discrete Gaussian (or just J-DG) model on Λ_N at temperature $\beta > 0$ is the probability measure

$$\begin{aligned}
\mathbb{P}^{\Lambda_N}_{J,\beta}(\sigma) &= \frac{1}{Z^{\Lambda_N}_{J,\beta}} e^{-\frac{1}{2\beta}(\sigma,-\Delta_J\sigma)} = \frac{1}{Z^{\Lambda_N}_{J,\beta}} e^{-\frac{1}{4\beta|J|}\sum_{x-y\in J}(\sigma_x-\sigma_y)^2}, \\
Z^{\Lambda_N}_{J,\beta} &= \sum_{\sigma\in\Omega^{\Lambda_N}} e^{-\frac{1}{4\beta|J|}\sum_{x-y\in J}(\sigma_x-\sigma_y)^2}
\end{aligned} \tag{1.12}$$

and expectation is written either $\mathbb{E}^{\Lambda_N}_{J,\beta}$ or $\langle\cdot\rangle^{\Lambda_N}_{J,\beta}$, where we again use the convention that $\sum_{x-y\in J}$ sums over each pair $\{x,y\}$ twice. Remarks on $\mathbb{P}^{\Lambda_N}_{\beta,\mathrm{DG}}$ in Section 1.1.2 apply the same for $\mathbb{P}^{\Lambda_N}_{J,\beta}$.

1.1.4 Summary of the main results

The main results of this thesis have three parts, which can be considered as quantitative refinements of (1.6) in three different regimes. In particular, under certain scaling limits, we see that the DG model at high temperature converges to a continuum Gaussian free field.

- The first is on the convergence of the torus scaling limit (Theorem 1.1.1 [12]) on $\mathbb{T}^2$, the unit square torus, to a Gaussian free field. We study the limit of the canonical ensembles (closed system with fixed temperature) on a sequence of discrete tori, so the limit is also called the ensemble scaling limit. Since the tested observable is smooth on $\mathbb{T}^2$, the ensemble scaling limit describes the thermal system in the macroscopic scale.

- The second is on the convergence of the $\mathbb{R}^2$ scaling limit (Theorem 1.1.3 [13]) to a Gaussian free field. We first define the infinite volume J-DG measure in Proposition 1.1.2, and then a smooth function on $\mathbb{R}^2$ is tested against the infinite volume measure with varying mesh sizes. Then the 0-mesh size limit is studied. The macroscopic scale is not visible in the infinite volume measure, thus it is necessary to think of observables that live on multiple scales.

  We also mention Theorem 1.1.4, the mesoscopic scaling limit, for comparison. While Theorem 1.1.3 first takes volume to infinite and then manipulates the scales of the observables, Theorem 1.1.4 manipulates these two scales simultaneously.

- The third is on the multipoint functions (Theorem 1.1.5 [81]). We still consider the infinite volume measure, but the observables are more singular under the scalings of Theorem 1.1.3. Thus one has to consider different scalings to obtain meaningful limits. These results are summarised in Section 1.2.3, see Corollaries 1.2.5 and 1.2.6 for example, which matches with Gaussian computations.

The three results all indicate that the DG model is in the *Gaussian universality class* (see Section 1.3.1). The main theorems will be stated in Sections 1.1.5, 1.1.6, and 1.1.7. Results in this chapter without proof are proved in Chapter 9, if not mentioned otherwise. The technical differences of the three regimes will be discussed in Section 1.3.4.

1.1.5 Torus scaling limit

For the first result, let $\mathbb{T}^2$ be the (continuous) unit square torus, i.e., $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2 \simeq ([-\frac{1}{2}, \frac{1}{2}]/\mathbb{Z})^2$, with geometry inherited from $\mathbb{R}^2$. The torus scaling limit considers a sufficiently smooth function $f \in C^\infty(\mathbb{T}^2)$ (or any sufficiently smooth function) tested against the J-DG measure, scaled to fit in $\mathbb{T}^2$. More precisely, we take f such that $\int_{\mathbb{T}^2} f(x)dx = 0$ and for each $N > 0$, we let f_N be the discretised version of f given by

$$f_N(x) = \frac{1}{|\Lambda_N|}\Big(f(L^{-N}x) - \frac{1}{|\Lambda_N|}\sum_{y\in\Lambda_N} f(L^{-N}y)\Big), \qquad x \in \Lambda_N, \tag{1.13}$$

so that $\sum_{x\in\Lambda_N} f_N(x) = 0$. Then f_N is tested against J-DG measure.

Theorem 1.1.1. *[12] Let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. Then there exists $\beta_0(J) > 0$ and an integer $L = L(J)$ such that for the J-DG model on the torus Λ_N of side length L^N at temperature $\beta \geqslant \beta_0(J)$, there is $\beta_{\mathrm{eff}}(J,\beta) > 0$ such that for any $f \in C^\infty(\mathbb{T}^2)$ with $\int f\,dx = 0$, as $N \to \infty$,*

$$\log\langle e^{(f_N,\sigma)_{\Lambda_N}}\rangle^{\Lambda_N}_{J,\beta} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}. \tag{1.14}$$

Moreover, $\beta_{\mathrm{eff}}(J,\beta) = \beta + O_J(e^{-c\beta})$ for some $c > 0$ (independent of J).

On the left-hand side of (1.14), $(f,g)_{\Lambda_N}$ is $\frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N} f(x)g(x)$. On the right-hand side, $(f,g)_{\mathbb{T}^2} = \int_{x\in\mathbb{T}^2} f(x)g(x)dx$, $\Delta_{\mathbb{T}^2}$ is the Laplace-Beltrami operator on $\mathbb{T}^2$ and the domain of $(-\Delta_{\mathbb{T}^2})^{-1}$ is $\{f : \int f = 0\}$. The constant v_J^2 was defined in (1.9).

It can be shown that (see Lemma 9.1.3 for a related result)

$$(f_N,(-\Delta_J)^{-1}f_N)_{\Lambda_N} \to \frac{1}{v_J^2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2} \tag{1.15}$$

and we may recall that $\frac{1}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2} = \log\langle e^{(f,\phi)}\rangle^{\mathrm{GFF}}_{\mathbb{T}^2}$ when ϕ is the standard Gaussian free field on $\mathbb{T}^2$. Then we see that there is a multiplicative correction on the scaling limit of the J-DG model compared to the standard Gaussian free field by a factor $\frac{\beta_{\mathrm{eff}}(J,\beta)}{\beta} = 1 + O_J(e^{-c\beta})$.

This was partially predicted by (1.6) (in particular, we need $\beta_{\text{eff}} \leqslant \beta$). There is also a quantitative estimate in [45] saying $\frac{\beta_{\text{eff}}(J,\beta)}{\beta} - 1 < -c_1 \beta e^{-c_2 \beta}$ for some $c_1, c_2 > 0$ when $J = J_{\text{nn}}$, the nearest neighbourhood interaction, ruling out the possibility of $\beta_{\text{eff}}(J_{\text{nn}}, \beta) = \beta$. This bound is due to the spin-vortex contribution (a terminology again originating from the XY model), which can be taken into account rigorously by using a coupling of the Villain XY model with discrete random variables and map it to the lattice Coulomb gas. Then each charge in the lattice Coulomb gas contributes as a spin-vortex. The exact value of β_{eff}/β can be understood to arise from the competition between the spin-waves (the mechanism used to prove (1.6)) and the spin-vortices.

1.1.6 $\mathbb{R}^2$ scaling limit

The second result considers the scaling limit when a smooth function is tested against the DG model on the whole plane. For this purpose, we extend the DG measure to $\mathbb{Z}^2$. For the case of the nearest-neighbour interaction (when $J = J_{\text{nn}}$), it is proved using cluster-swapping argument [86] that there exists a unique (translation) ergodic gradient Gibbs measure $\mathbb{P}^{\mathbb{Z}^2}_{\beta, J_{\text{nn}}}$ with tilt 0. Having tilt 0 means $\mathbb{E}^{\mathbb{Z}^2}_{\beta, J_{\text{nn}}}[\sigma_x - \sigma_y] = 0$ for any $x, y \in \mathbb{Z}^2$. However, this is not known for J-DG with general finite-range interaction J. Instead, we make a specific choice of infinite-volume measure obtained as a limit of DG measures on a specific sequence of tori. For the case of the nearest-neighbourhood interaction DG model, the infinite volume measure corresponds to the infinite volume Gibbs measure.

Proposition 1.1.2. *Let $\overline{\Lambda}_N$ be a sequence of tori with side lengths 3^N. Then the limit $\langle \cdot \rangle^{\mathbb{Z}^2}_{J,\beta} := \lim_{N\to\infty} \langle \cdot \rangle^{\overline{\Lambda}_N}_{J,\beta}$ exists and has tilt 0. Moreover,*

$$\langle e^{(f,\sigma)} \rangle^{\mathbb{Z}^2}_{J,\beta} = \lim_{N\to\infty} \langle e^{(f,\sigma)} \rangle^{\overline{\Lambda}_N}_{J,\beta} \tag{1.16}$$

for any $f : \mathbb{Z}^2 \to \mathbb{R}$ with $\sum_x f(x) = 0$ and when $\overline{\Lambda}_N$ is realised as a subset $[-\frac{3^N-1}{2}, \frac{3^N-1}{2}] \cap \mathbb{Z}^2$ of $\mathbb{Z}^2$.

The proof is given in Appendix 1.A.

We now define the discretisation of given $f \in C_c^\infty(\mathbb{R}^2)$ (smooth function with compact support) with $\int_{\mathbb{R}^2} f(x)\,dx = 0$. For each $\varepsilon > 0$, let the discretisation of f with mesh size ε be $f_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ such that $\sum_{x\in\mathbb{Z}^2} f_\varepsilon(x) = 0$ and, with $d = 2$,

$$\begin{gathered}
\max_{0\leqslant k\leqslant 2} \max_{x\in\mathbb{Z}^d} |(\varepsilon^{-1}\nabla)^k f_\varepsilon(x)| \leqslant C_f \varepsilon^{1+d/2}, \qquad \operatorname{supp} f_\varepsilon \subset [-R_f\varepsilon^{-1}, R_f\varepsilon^{-1}]^d, \\
\max_{x\in\mathbb{Z}^d} \left| \varepsilon^{-1-d/2} f_\varepsilon(x) - f(\varepsilon x) \right| \to 0,
\end{gathered} \tag{1.17}$$

for some constants $C_f, R_f > 0$ (that do not depend on ε) and ∇ is the vector of discrete gradients on $\mathbb{Z}^2$. For example, if $f = \nabla_i g$ for some $g \in C_c^\infty(\mathbb{R}^2)$ and $i \in \{1,2\}$ then one can take $f_\varepsilon(x) = \varepsilon^{d/2}(g(\varepsilon x + \varepsilon e_i) - g(\varepsilon x))$. In the statements, we take $\varepsilon \to 0$ to obtain the scaling limit.

In the next theorem, we test the discretised field f against J-DG model on $\mathbb{Z}^2$.

Theorem 1.1.3. *[13] Let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. Then there exists $\beta_0(J) > 0$ such that the following holds for $\beta \geqslant \beta_0(J)$ and $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$ such that there exists discretisation f_ε as in* (1.17)*. As $\varepsilon \to 0$,*

$$\log \big\langle e^{(f_\varepsilon, \sigma)_{\mathbb{Z}^2}} \big\rangle_{J,\beta}^{\mathbb{Z}^2} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2 v_J^2} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2}, \tag{1.18}$$

where β_{eff} is the same as Theorem 1.1.1 and $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ is the infinite volume measure defined by Proposition 1.1.2.

The order of taking limit $N \to \infty$ and $\varepsilon \to 0$ does not matter too much, as long as $L^N \varepsilon \to \infty$. This is stated as a separate theorem, which can be understood as the mesoscopic limit of the J-DG model.

Theorem 1.1.4. *Under the setting of Theorem 1.1.3, there exists $L \equiv L(J)$ such that, whenever $(\varepsilon_N)_{N \geqslant 0}$ is a sequence such that $\varepsilon_N \to 0$ and $L^N \varepsilon_N \to \infty$,*

$$\log \big\langle e^{(f_{\varepsilon_N}, \sigma)_{\Lambda_N}} \big\rangle_{J,\beta}^{\Lambda_N} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2 v_J^2} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2} \tag{1.19}$$

as $N \to \infty$.

1.1.7 Multi-point functions

Before stating our result, we characterise the most general class of multi-point observables in the following assumption, where we also take care of multi-cluster observables $\mathfrak{f}$, expressed as a linear combination of clusters $\mathfrak{f}_\alpha$. In the following, we think of $\mathfrak{n}$ as the number of clusters (allowed to be 1) and $M\rho^2$ as the L^1 size of each cluster, so $\mathfrak{n} M \rho^2$ is the L^1 size of the observable. For $y \in \mathbb{Z}^2$, T_y is the translation by y, i.e., for any $f : \mathbb{Z}^2 \to \mathbb{R}$, we have $T_y f(x) = f(x-y)$.

(A_f) $\mathfrak{f}$ is decomposed as $\mathfrak{f} = \sum_{\alpha=1}^{\mathfrak{n}} T_{y_\alpha} \mathfrak{f}_\alpha$ where each $\mathfrak{f}_\alpha \in \mathbb{R}^{\mathbb{Z}^2}$ is a function with compact support, $0 \in \mathrm{supp}(\mathfrak{f}_\alpha)$ and $\sum_{\alpha=1}^{\mathfrak{n}} \sum_x \mathfrak{f}_\alpha(x) = 0$. Also,

$$\max_{\alpha=1,\cdots,\mathfrak{n}} \{\|\mathfrak{f}_\alpha\|_{L^\infty(\mathbb{Z}^2)}\} \leqslant M, \qquad \max_{\alpha=1,\cdots,\mathfrak{n}} \{\mathrm{diam}(\mathrm{supp}(\mathfrak{f}_\alpha))\} \leqslant \rho \tag{1.20}$$

with $\mathfrak{n} M \rho^2 \leqslant 1$.

We briefly explain these assumptions. Each y_α can be thought of as the centre of each cluster $T_{y_\alpha} f_\alpha$. For convenience, we denote $\vec{\mathfrak{f}} = (\mathfrak{f}_1, \cdots, \mathfrak{f}_\mathfrak{n})$ and $\vec{y} = (y_1, \cdots, y_\mathfrak{n})$ when such $\mathfrak{f}$ is given. Also, let

$$d_{\vec{y}} = \min\{\|y_{\alpha_1} - y_{\alpha_2}\|_2 : \alpha_1 \neq \alpha_2\}. \tag{1.21}$$

Since $\|y_{\alpha_1} - y_{\alpha_2}\|_2 \geqslant d_{\vec{y}}$, each pair of clusters are separated by distance at least $d_{\vec{y}} - 2\rho$, and we will be interested in situations where $d_{\vec{y}} \to \infty$. However, the clusters do not have to be disjoint in principle, so we do not impose any restrictions on y_α's. We are restricting the L^1 size of the observable because we treat the observable as a perturbation to the renormalisation group flow (explained in Section 1.3.1), and the perturbation only has finite radius of convergence. In fact, we have an extra parameter h_ω that restricts the observable size in the next theorem, but thanks to $\mathfrak{n} M \rho^2 \leqslant 1$, we can choose h_ω independent of the other parameters.

We test $\omega \tilde{\mathfrak{f}}$ against $\sigma \sim \mathbb{P}_{J,\beta}^{\mathbb{Z}^2}$ for $\omega \in \mathbb{D}_{h_\omega}$ in the following theorem, where we use the notation $\mathbb{D}_r = \{\omega \in \mathbb{C} : |\omega| < r\}$.

Theorem 1.1.5. *[81] Let $J \subset \mathbb{Z}^2 \backslash \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. There exists a translation invariant covariance matrix $\mathfrak{C}_\beta \equiv \mathfrak{C}_{J,\beta}$, $\beta_0 \equiv \beta_0(J)$ and $h_\omega \equiv h_\omega(J)$ such that the following holds. Let $\omega \in \mathbb{D}_{h_\omega}$ with $h_\omega > 0$, $\beta \geqslant \beta_0$ and $\sigma \sim \mathbb{P}_{J,\beta}^{\mathbb{Z}^2}$. Then for $\mathfrak{f} = \sum_{i=1}^{\mathfrak{n}} T_{y_i} \mathfrak{f}_i$ satisfying (A_f),*

$$\log \langle e^{\beta^{-1/2} \omega(\mathfrak{f}, \sigma)} \rangle_{J,\beta}^{\mathbb{Z}^2} = \frac{1}{2} \omega^2 (\mathfrak{f}, \mathfrak{C}_\beta \mathfrak{f}) + \sum_{i=1}^{\mathfrak{n}} h_\beta^{(1)}[\mathfrak{f}_i](\omega) + h_\beta^{(2)}[\vec{y}, \vec{\mathfrak{f}}](\omega) \tag{1.22}$$

where $h_\beta^{(a)}$ ($a \in \{1,2\}$) are analytic functions in $\mathbb{D}_{h_\omega} \ni \omega$ satisfying the following.

- $|h_\beta^{(2)}[\vec{y}, \vec{\mathfrak{f}}](\omega)| = O_\beta(d_{\vec{y}}^{-\alpha})$ *uniformly in $\omega \in \mathbb{D}_{h_\omega}$ and some $\alpha > 0$.*
- $h_\beta^{(1)}[\mathfrak{f}_1] = h_\beta^{(1)}[-\mathfrak{f}_1]$, $h_\beta^{(1)}[0] = 0$ *and* $h_\beta^{(2)}[\vec{y}, \mathfrak{f}_1, 0, \cdots, 0] = 0$.

We will see that $\mathfrak{C}_\beta$ roughly behaves like the Green's function for the Laplacian in long distances as in Lemma 1.2.3, for example. Thus in the theorem, $\frac{1}{2}\omega^2(\mathfrak{f},\mathfrak{C}_\beta\mathfrak{f})$ consists of the effect of the Gaussian part of the field, while each $h_\beta^{(1)}[\mathfrak{f}_i]$ is a y_i-independent correction coming from each cluster. The final term $h_\beta^{(2)}$ are corrections due to the interactions between the clusters, decaying polynomially in the distance between the clusters. Some interesting corollaries of this theorem are summarized in Section 1.2.3, where the interpretations will get more clear.

1.2 Remarks and related problems

1.2.1 Novelties of this work

Compared to earlier works on the renormalisation group method, we develop a new finite-range covariance decomposition that admits a 0^{th} scale part covariance with range 0, see Chapter 2. This permits integrating a preliminary renormalisation group which smooths out the discreteness in the model. Once combined with the renormalisation group analysis of [9, 7, 10], we expect this to have applications in the analysis of the Ising model or the spin $O(n)$ model or strictly self-avoiding walks in dimension $\geqslant 4$, and even near the critical point if we use the spread-out interactions.

In the renormalisation group analysis, we provide a novel systematic way to study the mesoscopic and microscopic observables by investigating how the tilted expectation interacts with the fluctuation integrations of the RG analysis. Complex tilting is allowed, and tracing the complex analyticity along the renormalisation group flow gives the analyticity statements of Theorem 1.1.5.

1.2.2 Near-critical scaling limit

If we consider the nearest-neighbourhood interaction $J = J_{\mathrm{nn}}$, our proof requires choosing $\beta_0(J_{\mathrm{nn}})$ sufficiently large, well above the expected critical point. However, if we consider interaction J with a sufficiently large range, then $\beta_0(J)$ is allowed to approach the critical temperature as much as desired. At the level of our first-order renormalisation group analysis, we cannot determine the critical temperature constructively, so we instead state the result with reference temperature

$$\beta_{\mathrm{free}}(J) := 8\pi v_J^2, \quad \text{any step distribution } J. \tag{1.23}$$

If we let $\beta_c(J)$ be the conjectured critical point, then it is expected that $\beta_{\text{free}}(J_\rho) \sim \beta_c(J_\rho)$ as $\rho \to \infty$ and $\beta_{\text{free}}(J_\rho) < \beta_c(J_\rho)$, so we are only allowed $\beta_0 \geqslant (1+\delta)\beta_{\text{free}}(J_\rho)$ for some $\delta > 0$.

Theorem 1.2.1. *Consider $\beta_0(J)$ of Theorem 1.1.1. For the standard range-ρ distribution $J = J_\rho$, there exists $C > 0$ such that for any $\delta > 0$, $\rho^2 \geqslant C|\log\delta|$ and $L = L(J_\rho, \delta)$, the conclusion of Theorem 1.1.1 holds with $\beta_0(J_\rho) = (1+\delta)\beta_{\text{free}}(J_\rho)$.*

This theorem was the motivation for studying the DG model with interaction J in the first place. This result actually also applies to $\beta_0(J)$'s of Theorem 1.1.3 and Theorem 1.1.5, but we will not mention this explicitly.

In fact, we expect that for sufficiently large ρ, the critical point is also admissible to our analysis. In the following conjecture, we state what we expect could be proved.

Conjecture 1.2.2. *Let $J = J_\rho = \{x \in \mathbb{Z}^2 \setminus 0 : |x|_\infty \leqslant \rho\}$ be the standard range-ρ step distribution. Then there is $\rho_0 \in [1,\infty)$ such that for $\rho \geqslant \rho_0$, one can choose $\beta_0(J) = \beta_c(\rho)$ in the above theorem where $\beta_c(\rho)$ is such that, as $\beta \downarrow \beta_c = \beta_c(\rho)$,*

$$\beta_{\text{eff}}(J_\rho, \beta_c) = \beta_{\text{free}}(J_\rho) = 8\pi v_{J_\rho}^2, \tag{1.24}$$

As $\rho \to \infty$, the critical temperature satisfies

$$\beta_c(\rho) \sim \beta_{\text{free}}(J_\rho) \sim \frac{4\pi}{3}\rho^2. \tag{1.25}$$

and the criticality can be detected from logarithmic corrections on pointwise correlation functions.

The final sentence is verified for the 2-component critical lattice Coulomb gas in [36] by computing the fractional charge correlation. Although we also expect the conjecture to be true for any finite-range step distribution, so in particular also for the nearest-neighbourhood interaction $J = J_{\text{nn}}$, this proof currently seems to be out of reach.

1.2.3 Corollaries of Theorem 1.1.5

Multi-point functions

Theorem 1.1.5 has a number of interesting applications, for which we first need to present the role of $\mathfrak{C}_\beta$.

Lemma 1.2.3. *With $\mathfrak{C}_\beta$ as in Theorem 1.1.5 and $f_y = \delta_0 - \delta_y$,*

$$(f_y, \mathfrak{C}_\beta f_y) = C(\beta) + \frac{\beta_{\mathrm{eff}}(J,\beta)/\beta}{\pi} \log\|y\|_2 + O(\|y\|_2^{-1}) \tag{1.26}$$

for some $C(\beta) \in \mathbb{R}$ and $\beta_{\mathrm{eff}}(J,\beta)$ as in Theorem 1.1.1.

The lemma says that $\mathfrak{C}_\beta$ only behaves like the usual lattice Green's function in long distances. Whenever $\mathfrak{f}_i = c_i\delta_0$ and $\mathfrak{f} = \sum_{i=1}^{\mathfrak{n}} c_i \delta_{y_i}$ is a multi-point function such that y_i's are well-separated, and then we can disentangle $(\mathfrak{f}, \mathfrak{C}_\beta \mathfrak{f})$ using the lemma.

Corollary 1.2.4. *Consider the setting of Theorem 1.1.5 and suppose $(c_i)_{i=1}^{\mathfrak{n}}$ are such that $\sum_{i=1}^{\mathfrak{n}} |c_i| \leqslant 1$ and $\sum_{i=1}^{\mathfrak{n}} c_i = 0$. Then for $\omega \in \mathbb{D}_{h_\omega}$,*

$$\log \left\langle e^{\beta^{-1/2}\omega \sum_{i=1}^{\mathfrak{n}} c_i \sigma_{y_i}} \right\rangle_{J,\beta}^{\mathbb{Z}^2} = \frac{\beta_{\mathrm{eff}}/\beta}{2\pi} \omega^2 \mathfrak{L}(\vec{c},\vec{y}) + f_{J,\beta}[\vec{y},\vec{c}](\omega) \tag{1.27}$$

where

$$\mathfrak{L}(\vec{c},\vec{y}) = -\sum_{i<j} c_i c_j \log\|y_i - y_j\|_2 \tag{1.28}$$

and $f_{J,\beta}$ is such that $f_{J,\beta}(0) = \partial_\omega f_{J,\beta}(0) = 0$ and

$$\sup_{\omega \in \mathbb{D}_{h_\omega}} \left| f_{J,\beta}[\vec{c},\vec{y}](\omega) - f_{J,\beta}[\vec{c},\infty](\omega) \right| \leqslant O(d_{\vec{y}}^{-\alpha}) \tag{1.29}$$

for some $f_{J,\beta}[\vec{c},\infty]$. Both $f_{J,\beta}[\vec{c},\vec{y}]$ and $f_{J,\beta}[\vec{c},\infty]$ are analytic functions of $\omega \in \mathbb{D}_{h_\omega}$.

Since $f_{J,\beta}$ is bounded uniformly in the corollary, we see that $\mathfrak{L}(\vec{c},\vec{y})$ is the dominant term in the limit $d_{\vec{y}} \to \infty$. We have an intuitive interpretation when we scale the variables.

Corollary 1.2.5 (Central limit theorem)**.** *Consider the setting of Theorem 1.1.5 and let $(z_i)_{i=1}^{\mathfrak{n}}$ be distinct points in $\mathbb{Z}^2$, $(w_i)_{i=1}^{\mathfrak{n}}$ be such that $\sum w_i = 0$. Then*

$$\log \left\langle \exp\left(t(\log k)^{-1/2} \sum_{i=1}^{\mathfrak{n}} w_i \sigma_{kz_i} \right) \right\rangle_{\beta,J}^{\mathbb{Z}^2} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{4\pi} \sum_{i=1}^{\mathfrak{n}} w_i^2 t^2 \tag{1.30}$$

as $k \to \infty$, for any $t \in \mathbb{C}$. Thus we have convergence in distribution

$$\frac{\sum_{i=1}^{\mathfrak{n}} w_i \sigma_{kz_i}}{\sqrt{\log k}} \Rightarrow_d \mathcal{N}\left(0, \frac{\beta_{\mathrm{eff}}(J,\beta)}{2\pi} \sum_{i=1}^{\mathfrak{n}} w_i^2\right) \tag{1.31}$$

Proof. Take $y_i = kz_i$ and $\omega = \beta^{1/2}(\log k)^{-1/4}t$ and $c_i = (\log k)^{-1/4}w_i$ in Corollary 1.2.4. Then ω and c_i satisfy the conditions for sufficiently large k and

$$\mathfrak{L}(\vec{c}, k\vec{z}) = \frac{1}{2}\sum_{i=1}^{\mathfrak{n}} c_i^2 \log k + \mathfrak{L}(\vec{c}, \vec{z}) \tag{1.32}$$

using $-\sum_{i<j} c_i c_j = -\frac{1}{2}(\sum_i c_i)^2 + \frac{1}{2}\sum_i c_i^2$. $\square$

The central limit theorem for the multi-point functions is in a similar flavour to that of Theorem 1.1.3, as they both show the Gaussianity of scaled variables in the long-distance limit. However, the multi-point functions are microscopic observables, making the control more delicate.

When the field is allowed to take continuous values and the interaction potential satisfies uniform convexity condition, the same observable as in Corollary 1.2.5 was shown to satisfy the central limit theorem in [28], and recently, the local central limit theorem was also proved in [93]. These results are obtained by applying stochastic homogenisation on the Helffer-Sjöstrand representation of the field. This technique has wide-ranging applications in this context, but it is not directly applicable to integer-valued systems. There are also results on some specific integer-valued models. The height function for the dimer model was studied extensively in [64] and the central limit theorem was proved in [71]. Also, a central limit theorem (with unknown scaling factors) was obtained for the square ice model in [92] using the Russo-Seymour-Welsh estimate on the level lines.

We obtain another application of Corollary 1.2.4 when $\omega = i\eta$ is purely imaginary.

Corollary 1.2.6 (Cosine correlation)**.** *Under the setting of Corollary 1.2.4, whenever $\eta \in (-h_\omega, h_\omega)$,*

$$\left\langle e^{i\beta^{-1/2}\eta\sum_{i=1}^{\mathfrak{n}} c_i\sigma_{y_i}}\right\rangle_{\beta,J}^{\mathbb{Z}^2} = C(\beta,\eta)\prod_{i<j}\|y_i - y_j\|_2^{\frac{\eta^2}{2\pi}(\beta_{\mathrm{eff}}/\beta)c_ic_j}\left(1 + O\left(d_{\vec{y}}^{-\alpha}\right)\right) \tag{1.33}$$

for some $C(\beta,\eta)$ smooth in η.

This is exactly like the Gaussian free field prediction (see [61] for an in-depth study and [32, (2.6)] for a brief introduction). The interpretation is not so clear for the DG model. But using the analogue for the sine-Gordon model, the cosine correlation with $\mathfrak{n} = 2$, $c_i = \pm 1$ corresponds to the correlation function of two test charges $\pm\eta$ inserted into the 2D lattice multi-component Coulomb gas system dual to the generalised sine-Gordon model. For this reason, this observable is also called the fractional charge correlation or the electric correlator.

The polynomial decay of the fractional charge correlation also characterises the delocalisation, and the polynomial lower and upper bounds were established in [41, 42]. In the localised phase, the Debye screening (cf. [24, 94] for the lattice sine-Gordon model and [19] for the Discrete Gaussian model) induces exponential decay of the truncated charge correlation. The same type of result was studied for the dimer model in [83, 32]. A similar result was also obtained for the interacting dimer model in [53] but only after properly smoothing the point observables. We mention again that, for the lattice sine-Gordon model at the critical point, the fractional charge correlation was computed by Falco [36].

Falco's method of using observable fields also extends the range of η to $(0,\beta^{1/2})$ when β is near the critical value. (It was seen in [12, 13] that the near-critical values of these models are within the scope.) It does not give the required bounds for the usual Discrete Gaussian model where we can implement the renormalisation group method only for large values of β (well above the critical point), but by Theorem 1.2.1, this extension has a better chance to hold when J has sufficiently long range.

Conjecture 1.2.7. *When $J = J_\rho$ with sufficiently large ρ and β is sufficiently close to $\beta_{\mathrm{free}}(J)$, Corollary 1.2.6 holds for $\eta \in (0,\beta^{1/2}]$ (with α depending on η), but with $C(\beta,\eta)$ discontinuous at $\eta = \frac{1}{2}\beta^{1/2}$.*

Note that we have only included $\eta \leqslant \frac{1}{2}\beta^{1/2}$ because

$$\left\langle e^{i\beta^{-1/2}\eta(\sigma_0-\sigma_y)}\right\rangle_{\beta,J}^{\mathbb{Z}^2} = \left\langle e^{i\beta^{-1/2}(\beta^{1/2}-\eta)(\sigma_0-\sigma_y)}\right\rangle_{\beta,J}^{\mathbb{Z}^2}, \tag{1.34}$$

since σ takes $2\pi\mathbb{Z}$-values, so $\beta^{-1/2}\eta$-charge correlation for $\eta \in (\frac{1}{2}\beta^{1/2},\beta^{1/2})$ can also be obtained from the statement. Then (1.33) does not hold anymore, but η on the right-hand side has to be replaced by $\beta^{1/2}-\eta$. Thus one may also expect that $C(\eta,\beta)$ has a point of discontinuity at $\eta = \frac{1}{2}\beta^{1/2}$. This is not weird, because there is actually a subleading term of the fractional charge correlation coming from fractional charge $1-\beta^{-1/2}\eta$. This conjecture can be compared with [36, Theorem 2.1].

Using the observable fields is more common in this context, for example, as in [8, 87, 58], but we do not use it here.

Gradient correlation of the Villain XY model

Finally, we mention an application of Theorem 1.1.5 to the 2D Villain XY model. For $M \in \mathbb{N}$, let $\square_M = [-M,M]^2 \cap \mathbb{Z}^2$ and we define the Villain XY model with free boundary condition on $\square_M$ and inverse temperature β by

$$\mathbb{P}^M_{\beta,\mathrm{Vil}}(d\theta) \propto \prod_{\{x,y\}\in E(\square_M)} \sum_{m\in 2\pi\mathbb{Z}} e^{-\frac{\beta}{2}(\theta_x-\theta_y-m)^2} \prod_{x\in\square_M} d\theta_x \tag{1.35}$$

for $\theta \in [0,2\pi)^{\square_M}$, where $d\theta_x$ is the uniform measure on $[0,2\pi)$ and $E(\square_M)$ is the set of edges. The infinite volume correlation functions are given by

$$\mathbb{E}^{\mathbb{Z}^2}_{\beta,\mathrm{Vil}}\big[\exp(i\sum_{x\in A} n_x\sigma_x)\big] := \lim_{M\to\infty} \mathbb{E}^{M}_{\beta,\mathrm{Vil}}\big[\exp(i\sum_{x\in A} n_x\sigma_x)\big] \tag{1.36}$$

for a finite set A and $n_x \in \mathbb{Z}$ such that $\sum_{x\in A} n_x = 0$. In fact, $\mathbb{E}^{\mathbb{Z}^2}_{\beta,\mathrm{Vil}}$ is a (translation invariant) Gibbs measure on $\mathbb{Z}^2$, whose existence and uniqueness are proved by [21, 77] (their works are on the 2D XY model, but they can be extended to the Villain model using the fact that the Villain model can be represented as a limit of XY model on the cable graph of $\mathbb{Z}^2$).

In 2D, the Villain XY model and the (nearest-neighbourhood interaction) DG model are dual to each other. The Villain XY model on $\mathbb{Z}^2$ is mapped to the DG model on the dual of $\mathbb{Z}^2$ (the faces of $\mathbb{Z}^2$), denoted $F(\mathbb{Z}^2)$ and isomorphic to $\mathbb{Z}^2$, according to the following procedure: if $A \subset \mathbb{Z}^2$ is a finite set and $(n_x)_{x\in A}$ sums to 0, then let $\gamma : E(\mathbb{Z}^2) \to \mathbb{Z}$ (the set of 1-forms) be such that $d\gamma = n$ (which is non-unique), where $d : \mathbb{Z}^{E(\mathbb{Z}^2)} \to \mathbb{Z}^{\mathbb{Z}^2}$ is the exterior derivative mapping the set of 1-forms to 0-forms. Also letting $d^* : \mathbb{Z}^{E(\mathbb{Z}^2)} \to \mathbb{Z}^{F(\mathbb{Z}^2)}$ be the exterior coderivative, we have

$$\langle e^{i(n,\theta)_{\mathbb{Z}^2}}\rangle^{\mathbb{Z}^2}_{\beta/4\pi^2,\mathrm{Vil}} = \langle e^{\beta^{-1}(d^*\gamma,\sigma)_{F(\mathbb{Z}^2)}}\rangle^{F(\mathbb{Z}^2)}_{\beta,\mathrm{DG}}, \tag{1.37}$$

where now the DG model is defined on $F(\mathbb{Z}^2)$ instead (cf., [6, Section 5]).

We use this duality to study the gradient correlation of the Villain XY model. Suppose $n_1, n_2 : \mathbb{Z}^2 \to \mathbb{Z}$ have compact support and $\sum_x n_1(x) = \sum_x n_2(x) = 0$. Then by (1.37), there exists $\mathfrak{f}_1, \mathfrak{f}_2$ such that $\sum_x \mathfrak{f}_1(x) = \sum_x \mathfrak{f}_2(x) = 0$ and

$$\langle e^{i\sum_x (n_1(x)+n_2(x-y))\theta_x}\rangle^{\mathbb{Z}^2}_{\beta/4\pi^2,\mathrm{Vil}} = \langle e^{\beta^{-1}\sum_x(\mathfrak{f}_1(x)+\mathfrak{f}_2(x-y))\sigma_x}\rangle^{\mathbb{Z}^2}_{\beta,\mathrm{DG}} \tag{1.38}$$

where $(\theta_x)_x \sim \mathbb{P}^{\mathbb{Z}^2}_{\beta',\mathrm{Vil}}$ is the unique translation invariant infinite volume Gibbs state of the Villain XY model at temperature β and $\langle\cdot\rangle^{\mathbb{Z}^2}_{\beta',\mathrm{Vil}}$ is the expectation. Thus for sufficiently large β, by Theorem 1.1.5,

$$\begin{aligned}&\log\langle e^{i\sum_x (n_1(x)+n_2(x-y))\theta_x}\rangle^{\mathbb{Z}^2}_{\beta/4\pi^2,\mathrm{Vil}}\\ &= \frac{\beta^{-1}}{2}\big((\mathfrak{f}_1+T_y\mathfrak{f}_2), \mathfrak{C}_\beta(\mathfrak{f}_1+T_y\mathfrak{f}_2)\big) + h^{(1)}_\beta[\mathfrak{f}_1](\beta^{-1/2}) + h^{(1)}_\beta[y,\mathfrak{f}_1,\mathfrak{f}_2](\beta^{-1/2}) + O(\|y\|_2^{-\alpha})\end{aligned} \tag{1.39}$$

while

$$\log\langle e^{i\sum_x n_1(x)\theta_x}\rangle^{\mathbb{Z}^2}_{\beta/4\pi^2,\mathrm{Vil}} = \frac{\beta^{-1}}{2}\big(\mathfrak{f}_1, \mathfrak{C}_\beta \mathfrak{f}_1\big) + h^{(1)}_\beta[\mathfrak{f}_1](\beta^{-1/2}) + O(\|y\|_2^{-\alpha}), \tag{1.40}$$

$$\log\langle e^{i\sum_x n_2(x-y)\theta_x}\rangle^{\mathbb{Z}^2}_{\beta/4\pi^2,\mathrm{Vil}} = \frac{\beta^{-1}}{2}\big(\mathfrak{f}_2, \mathfrak{C}_\beta \mathfrak{f}_2\big) + h^{(1)}_\beta[\mathfrak{f}_2](\beta^{-1/2}) + O(\|y\|_2^{-\alpha}). \tag{1.41}$$

But $(T_y\mathfrak{f}_2, \mathfrak{C}_\beta \mathfrak{f}_1)$ decays polynomially in $\|y\|$.

Lemma 1.2.8. *If $J = J_{\mathrm{nn}}$ and $\sum_x f_1(x) = \sum_x f_2(x) = 0$, then*

$$(T_y\mathfrak{f}_2, \mathfrak{C}_\beta \mathfrak{f}_1) = O(\|y\|_2^{-2}) \tag{1.42}$$

This lemma is proved in Chapter 9. Thus we have the following.

Corollary 1.2.9. *Let $n_1, n_2 : \mathbb{Z}^2 \to \mathbb{Z}$ have compact support and $\sum_x n_1(x) + \sum_x n_2(x) = 0$. Then for sufficiently large $\beta > 0$, there exists $\alpha \equiv \alpha(\beta) > 0$ such that*

$$\mathrm{Cov}^{\mathbb{Z}^2}_{\beta,\mathrm{Vil}}\Big[\prod_x e^{in_1(x)\theta_x}; \prod_x e^{in_2(x)\theta_{x+y}}\Big] = O(\|y\|_2^{-\alpha}). \tag{1.43}$$

Proof. The case $\sum_x n_1(x) = \sum_x n_2(x) = 0$ is covered by the argument above. The case $\sum_x n_1(x) = -\sum_x n_2(x) \neq 0$ is dealt with by the McBryan–Spencer inequality [73] (which works for any $\beta > 0$). □

1.2.4 Potential extensions

As was explained, the Discrete Gaussian model is related to several important problems in two-dimensional statistical physics. We list some interesting problems together with potential extensions or refinements of our method.

1. Due to the exact duality relationship (1.37), refinement of the method is expected to give correlation functions of the Villain XY model at low temperatures. Up to date, unmatching upper and lower bounds polynomially decaying in the distance are known via [41] and [73].

   One may also see [89] for more about this duality and it is proven by Lammers [69] that the point of phase transitions exactly coincide in the two models.

2. Currently, our proof strongly relies on the quadratic form of the interaction potential, but there are also models with non-quadratic interaction, such as the Solid-On-Solid model and the dual height function of the XY model [41]. It would be of considerable interest if the renormalisation group method can be extended to those models.

3. Studying the effect of the presence of the boundary on the DG model is also of interest. For example, it is most convenient to study the level line when we impose Dirichlet boundary conditions on the system, cf., [84, 78]. A serious obstacle to applying the renormalisation group analysis to the DG model with Dirichlet boundary condition is that the renormalisation group flow is not stable near the boundary, so we need a new method to control this.

1.3 Methodology

1.3.1 Renormalisation group

The *renormalisation group* (RG) method is a systematic apparatus in statistical and quantum physics that enables the study of limits. In statistical physics, one usually studies the infrared limit, which corresponds to the long-distance limit where macroscopic physics arises from microscopic physics laws.

Suppose we are working on a lattice $\Lambda \subset \mathbb{Z}^d$ and a spin system $\phi \in \mathbb{R}^\Lambda$ is described by Hamiltonian

$$\begin{aligned} H_0(\phi;A) &= \frac{1}{2}(\phi,(-\Delta+m^2)\phi) + h_0(\phi;A) \\ h_0(\phi;A) &= -\sum_{x\in\Lambda} A_x\phi(x) + \lambda \sum_x V_0(\phi(x),\nabla\phi(x)) \end{aligned} \tag{1.44}$$

for an external source field $A \in \mathbb{R}^\Lambda$, some potential function V and a probability measure

$$\mathbb{E}_A[F(\phi)] = \frac{1}{Z(A)}\int d\phi F(\phi)e^{-H_0(\phi;A)}, \qquad Z(A) = \int d\phi e^{-H_0(\phi;A)}. \tag{1.45}$$

The truncated correlation function is given by $\mathbb{E}_A^T[\phi(x_1);\cdots;\phi(x_n)] = \frac{\partial^n}{\partial A_{x_1}\cdots\partial A_{x_n}}\log Z(A)$, so $Z(A)$ will be the primary object of study.

The renormalisation group can be used to study the limit of $Z(A)$ in the limit $|\Lambda| \to \infty$. For example, let Λ be a d-dimensional cube parametrised by a scale parameter N, thus $|\Lambda| = L^{dN}$ for some $L > 0$, and we study $N \to \infty$. $Z(A)$ can be reformulated in terms of a Gaussian integral

$$Z(A) = N(m^2)\mathbb{E}^\varphi[e^{-h_0(\varphi;A)}], \qquad \varphi \sim \mathscr{N}(0,(-\Delta+m^2)^{-1}) \tag{1.46}$$

where $\mathscr{N}(0,C)$ is the Gaussian distribution with covariance matrix C and $N(m^2)$ is a normalisation constant. The renormalisation group analysis decomposes the Gaussian integral into a multiple number of progressive integrals of successive scales, each associated with coarse-graining and rescaling. Each progressive integral can be viewed as a decomposed component of the Gaussian integral. Equivalently, if $(-\Delta+m^2)^{-1} = \sum_j C_j$ where C_j contains the information of $(-\Delta+m^2)^{-1}$ at length scale L^j and $\zeta_j \sim \mathscr{N}(0, C_j)$ are independent Gaussian random variables representing the fluctuation of the field at scale j, then $\varphi =_d \sum_j \zeta_j$ (where $=_d$ means that they have the same distribution). Then progressive integral at scale j means integrating against the variable ζ_j. After scale j progressive integral, we group L^{dj} neighbouring lattice sites into blocks $B \in \mathscr{B}_j(\Lambda)$ and the rescaling reparametrises the field by multiplying $L^{\frac{d-2}{2}}$ so that the size of the field fluctuation stays normalised at each scale j. These steps are called coarse-graining and rescaling, respectively. The resulting function would have a generic form

$$Z_{j+1}(\varphi_{j+1};A) := \exp(-h_j(\varphi_{j+1};A)) := \mathbb{E}^{\zeta_{\leqslant j}}[e^{-h_0(\varphi;A)}] \tag{1.47}$$

where $\varphi_{j+1} = \sum_{k>j} \zeta_k$ and $\mathbb{E}^{\zeta_{\leqslant j}}$ only integrates the variables $\zeta_1, \cdots, \zeta_j$ (note that the rescaling is not reflected in this representation). Then the fundamental postulate of the renormalisation group states that there exists a family of local interaction functions $(V_\alpha)_{\alpha\in\mathbb{N}}$ and a family of *coupling constants* $(\lambda_j^\alpha)_{\alpha\in\mathbb{N}}$ such that

$$h_j(\varphi;0) \approx \sum_{\alpha\in\mathbb{N}} \lambda_j^\alpha \sum_{B\in\mathscr{B}_j} V_\alpha(\varphi|_B, \nabla\varphi|_B, \cdots). \tag{1.48}$$

(The notation for V_α indicates that it can have higher-derivatives dependence, but not infinitely many, to keep the locality of the interaction.) Then $h_j(\varphi;0)$ is the effective potential describing the renormalised theory at scale j. When $A \neq 0$, there should also be extra coupling constants associated with the linear potential terms. Putting aside the problem of how to justify the postulate, we focus on the dynamical system of the coupling constants $(\lambda_j^\alpha)_{j\geqslant 0}$ constructed from the renormalisation group steps. By our hypothesis (1.48), Z_{j+1} blurs out the microscopic detail of the system, thus the limit $j\to\infty$ will effectively describe the macroscopic limit of the probability measure, and the limit is parametrised in terms of $j\to\infty$ limit of the coordinates $(\lambda_j^\alpha)_\alpha$. The interaction functions can be divided into three classes according to how they evolve: V_α is called (i) *irrelevant* if $(\lambda_j^\alpha)_{j\geqslant 0}$ is contracting, (ii) *relevant* if $(\lambda_j^\alpha)_{j\geqslant 0}$ is blowing up and (iii) *marginal* if it is neither irrelevant nor relevant. Usually (but not always), the cases of interest occur when all but a finite number of the interactions are irrelevant and the dynamical system is stable. In this case, the limiting theory is described in terms of only a finite number of coordinates. Moreover, due to the stability of the dynamical system, it does not respond sensitively to the microscopic detail, leading the limiting measure of any perturbed system to be effectively described by the same coordinates, giving emergence to the *universality*.

In conclusion, the strength of the renormalisation group analysis is in the presence of a systematic way to study scaling limits and the potential to detect universality. However, in general, a postulate of type (1.48) is not easy to verify, nor is it easy to prove the convergence of the coupling constants $(\lambda_j^\alpha)_\alpha$ as $j \to \infty$. One of the problems making the rigorous implementation difficult is that it is not easy to control the divergence of the series sum of (1.48) when they are exponentiated, and it easily happens that not even the integrability of (1.47) is easily justified. This is the *problem of large fields* (cf. [23, 47]). Thus, instead of using the full expansion of (1.48), it is desirable to find a way to reduce the mode of complexity, for example as in Section 1.3.2 using polymer gas representation, even if it means losing track of detailed information.

We also mention a partial list of incidents where the rigorous renormalisation group method has been applied. The ϕ^4-model is a fundamental field-theoretic model taking $V_0(\phi) \propto |\phi|^4$ [56]. It had long been a subject of the renormalisation group analysis, especially at the upper critical dimension $d = 4$ [37, 48, 7, 57, 87, 14]. Also, due to the supersymmetric representation, the weakly self-avoiding walk (WSAW) is a subject of a similar type of analysis obtained in [15, 8, 9]. Some (continuous-valued) interface models had also been studied, when $V_0(\phi; \nabla\phi)$ is only a function of $\nabla\phi$. Classical examples include [47] when $V_0(\nabla\phi) \propto |\nabla\phi|^4$, [27] when $V_0(\nabla\phi) \propto \cos(\alpha\nabla\phi)$, and [2, 1, 58] when $V_0(\nabla\phi)$ is a more general sufficiently smooth function. There are also a number of renormalisation group works on the dimer and the Ising model using fermionic variables—their partition functions have (relatively simple) exact formulae that can be represented using fermionic variables [62], [85] (also see [16, 74]). Based on these observations, the interacting dimer model [53, 52] and non-integrable Ising model [51], also with boundary condition [4], were studied. One may also see [49, 17] for more examples and a pedagogical display of the method. The most related to our analysis are the works by Dimock–Hurd [30, 29] and Falco [35, 36] studying the lattice sine-Gordon model, where $V_0(\phi) = z\cos(\alpha\phi)$ and z is the activity. Their implementation of the renormalisation group method can be traced back to [27] where the two-dimensional dipole gas had been studied, and we would also have to mention [23] where [35] is built. Our implementation also imports important inputs from [23].

1.3.2 Polymer gas representation

The polymer gas representation of the renormalisation group writes the renormalised theory $Z_j(\varphi)$ in terms of normalising constant E_j, finite number of leading terms, combined into U_j, and a remainder term K_j, put together by polymer expansion. The leading terms inside U_j

are chosen so that it reflects the key structure of the model (or its limit), while the remainder K_j is chosen to be irrelevant in the renormalisation group flow. If we recall $\mathscr{B}_j$ is the coarse-graining of the lattice Λ (for example, take $(L^j\mathbb{Z})^d$-translations of $[-(L^j-1)/2,(L^j-1)/2]^d$), we call $X \subset \mathscr{B}_j$ a polymer of Λ. Then we seek for a representation

$$Z_j(\varphi) = e^{-E_j|\Lambda|} \sum_{X \subset \mathscr{B}_j} e^{\sum_{B\in X}\sum_{x\in B} U_j(\varphi(x),\nabla\varphi(x))} K_j(\Lambda\backslash X,\varphi) \tag{1.49}$$

by letting $K_j(Y,\varphi)$ to be a function of polymers. The polymer gas representation can be considered as a multiscale version of the cluster expansion, which were proved to be useful in various mathematical physics contexts, see [22, 56] for example. It can also be motivated by the expansion of the effective potentials in the hierarchical models [23, 11].

The restriction on K_j will be specified in Chapter 3 by defining the normed space in which such polymer functions reside. These normed spaces should be large enough to embrace sufficient amount of freedom, but it should also restrict the mode of divergence as the field φ tends to infinity, as we need to take guaranteed the convergence of the progressive integrals (1.47). Also, the space needs to encode the essential symmetries of the system. These properties are used in Chapter 5 to prove crucial contraction inequalities. Compared to this, the choice of U_j is usually relatively straightforward, and is composed of the leading terms in the expansion. The leading terms can be identified by testing the relevance and irrelevance. In the long run, only E_j survives, as we will prove $(U_j,K_j) \to (0,0)$ as $j\to\infty$. Thus $E_\infty := \lim_{j\to\infty} E_j$ is the negative (infinite volume) free energy per volume. It will not be a subject of interest in this thesis, but free energy also arises from the variational characterisation of the Gibbs measure [50, 86], and plays important role in the abstract formulation of statistical physics. It is also an important tool for studying phase transitions, as the point of phase transition usually coincides with the point of non-analyticity of E_∞. More detailed outline will be given in Section 1.3.4.

1.3.3 Renormalisation group on the Discrete Gaussian model and mapping to a Coulomb gas

The general theme of this thesis is to develop a method of how to rigorously implement the renormalisation group analysis using the polymer gas expansion. We achieve this goal by considering the Discrete Gaussian model as a multi-component lattice Coulomb gas with charge symmetry: on the level of partition functions and with extra mass $m^2 > 0$,

$$\begin{aligned} Z^{\Lambda_N}_{\beta,m^2,\mathrm{DG}} := \sum_{\sigma \in (2\pi\mathbb{Z})^{\Lambda_N}} e^{-\frac{1}{2\beta}(\sigma,(-\Delta+m^2)\sigma)} &= \int_{\mathbb{R}^{\Lambda_N}} d\phi e^{-\frac{1}{2\beta}(\phi,(-\Delta+m^2)\phi)} \prod_{x\in\Lambda_N} \sum_{n\in 2\pi\mathbb{Z}} \delta_n(\phi(x)) \\ &\propto \int_{\mathbb{R}^{\Lambda_N}} d\phi e^{-\frac{1}{2\beta}(\phi,(-\Delta+m^2)\phi)} \prod_{x\in\Lambda_N} \sum_{q\in\mathbb{Z}} e^{iq\phi(x)} \\ &= \sum_{\vec{q}\in\mathbb{Z}^{\Lambda_N}} \int_{\mathbb{R}^{\Lambda_N}} d\phi e^{-\frac{1}{2\beta}(\phi,(-\Delta+m^2)\phi)} e^{i(\vec{q},\phi)} \\ &= \sum_{\vec{q}\in\mathbb{Z}^{\Lambda_N}} e^{-\frac{\beta}{2}(\vec{q},(-\Delta+m^2)^{-1}\vec{q})} \end{aligned} \tag{1.50}$$

with the constant of proportionality only depending on $|\Lambda_N|$. In the limit $m^2 \to 0$, the sum on the right-hand side concentrates on $\{\vec{q} : \sum_x q(x) = 0\}$, thus we see that the (nearest-neighbourhood interaction) DG model is mapped to a system of 'charges' $\vec{q}$ that are subject to Coulomb interaction $(-\Delta)^{-1}$ with inverse temperature β and neutral in the sense that $\sum_x q(x) = 0$. Similarly, if we allow $\vec{q}$ to talk only value 0 or ± 1 (so the Coulomb gas has two components) and add *activity* $z \in \mathbb{R}$, then this system is mapped to the lattice sine-Gordon model:

$$\sum_{\vec{q}\in\{\pm 1,0\}^{\Lambda_N}} \frac{(z/2)^{|\vec{q}|}}{|\vec{q}|!} e^{-\frac{\beta}{2}(\vec{q},(-\Delta+m^2)^{-1}\vec{q})} \propto \int_{\mathbb{R}^{\Lambda_N}} d\phi e^{-\frac{1}{2\beta}(\phi,(-\Delta+m^2)\phi)+z\sum_{x\in\Lambda_N}\cos(\phi(x))} \tag{1.51}$$

where $|\vec{q}| = \sum_x |q(x)|$. For the lattice sine-Gordon model with small activity, the renormalisation group method was used in [35] to study the critical line. The small parameter assumptions are often necessary for renormalisation group arguments because renormalisation group flows are treated as perturbations of the Gaussian free field. However, since the activity of any charge is of order 1 for the Discrete Gaussian model, we need an extra step that puts the activity small prior to the analysis. In this thesis, this is done by trading the 'ultra-local' part of the interaction with the activity. The ultra-local part of the interaction can be as large as desired by taking the temperature high, so the activity can also be suppressed as much as desired. This also has the effect of smoothing the discreteness present in spin values of the Discrete Gaussian, thus making it feasible for the usual analysis. This reformulation to the model will be explained in the first half of Chapter 6. As a result, this will give rise to a new covariance

$$C(s,m^2) = \big(((-\Delta_J + m^2)^{-1} - \gamma)^{-1} - s\Delta\big)^{-1} \tag{1.52}$$

studied in Chapter 2 and the leading terms of the expansion (1.49) will be given by

$$U_j(\varphi(x), \nabla\varphi(x)) = \frac{1}{2}s_j|\nabla\varphi(x)|^2 + \sum_{q\geqslant 1} z_j^{(q)} \cos(q\beta^{1/2}\varphi(x)) \tag{1.53}$$

The first term of (1.53) reflects the gradient interaction and is marginal. The second term reflects the periodicity of the integer-valued restriction of the Discrete Gaussian model and is irrelevant or marginal depending on the temperature β in the delocalised phase.

1.3.4 Outline of the thesis

We now explain the strategies for proving the different scaling limits and give an overview of the thesis.

Difference between the scalings

To understand the scaling limits, it is essential to understand how observables in different scales affect the renormalised theories Z_j in various scales. Suppose f is the observable of interest. After the change of variable in the Gaussian measure with covariance $C(s,m^2)$, the observable roughly amounts to perturbing the field by $C(s,m^2)f \simeq (-\Delta)^{-1}f$. First, we consider the case of the torus scaling limit, so $\Delta \equiv \Delta_{\Lambda_N}$, the Laplace operator on the discrete torus, and f is smooth on the torus. Then both the discretisation f_N and $(-\Delta)^{-1}f_N$ are smooth in scale N, the scale of the torus. It turns out that both functions U_j and K_j in (1.49) are 'smooth' functions of their field variables, for fields that are smooth in scale j, therefore $Z_N(\varphi + (-\Delta)^{-1}f_N)$ is also a smooth in f. Moreover, since $(U_N, K_N) \to 0$ in appropriate norms, we see that the perturbation made by the external field f_N vanishes as $N \to \infty$, which is an indication that we can compute the scaling limit.

Next, we consider the $\mathbb{R}^2$ scaling limit, so $\Delta \equiv \Delta_{\mathbb{Z}^2}$ and f_ε is smooth in scale ε^{-1} in the sense of (1.17). Then $(-\Delta)^{-1}f_\varepsilon$ is not smooth in the macroscopic scale anymore–it is rather very singular. Thus we decompose the perturbations by letting $f_\varepsilon = \sum_{j\geqslant 0} f_{\varepsilon,j}$ and $f_{\varepsilon,j} = C_j f_\varepsilon$, where we recall that $(-\Delta)^{-1} = \sum_{j\geqslant 0} C_j$ (when $m^2 = 0$). We will then see that $f_{\varepsilon,j}$ is smooth in scale $\max\{\varepsilon^{-1}, j\}$, so each $f_{\varepsilon,j}$ can only contribute to a bounded amount of perturbation on $Z_j(\varphi + f_{\varepsilon,j})$ if $j \geqslant \varepsilon^{-1}$ (and by a negligible amount if $j < \varepsilon^{-1}$). In fact, the amount of perturbation is also controlled by the norm of (U_j, K_j), so the total amount of accumulated perturbations is roughly bounded by $\sum_{j\geqslant \varepsilon^{-1}} \|(U_j, K_j)\|$ (again in some norm), which vanishes as $\varepsilon \to 0$, if $\|(U_j, K_j)\|$ is summable.

Finally, we consider the multipoint functions. We still have $\Delta \equiv \Delta_{\mathbb{Z}^2}$, but f is not scaled, so we do not have a parameter that plays the role of ε above. However, as long as L^1-norm on

f is controlled, we will see that the perturbation generated by $C_j f$ is still an analytic function of f. Also, the size of the perturbations decay like $\|(U_j, K_j)\|$, so if $\|(U_j, K_j)\|$ is summable, we see that the accumulated perturbation is still an analytic function of f.

Structure of the thesis

As explained in Section 1.3.1, the renormalisation group analysis proceeds from a covariance decomposition of a Gaussian field, which is used to approximate the Gaussian free field with renormalised stiffness in the scaling limit. We use a particular form of covariance decomposition where each covariance has finite range, whose existence and regularity are proved in Chapter 2. Expectation in each decomposed covariance is the fluctuation integral.

In Chapter 3, we define polymer activities, which become the building blocks of the polymer expansion. The large field problem is taken care by introducing large field regulators and bounding the polymer activities with them. A large part of the chapter is devoted to studying how the polymer activities or the large field regulators interact with the fluctuation integral. We start discussing about how polymer expansions are handled in Chapter 4. We define various polymer expansion operations that are used to define the renormalisation group map on the polymer expansion. We also obtain estimates on these operations.

Chapter 2–4 are designed to apply to generic type of renormalisation group analysis, but Chapter 5 restricts the type of interactions to those relevant to the discussion of the high-temperature Discrete Gaussian model. The potentials are characterised by periodicity in global constant field addition, and the polymer activities also inherit this periodicity. We collect estimates that are ultimately responsible for the contractiveness of the renormalisation group map for periodic polymer activities.

In Chapter 6, we define the (bulk) renormalisation group map that applies to systems with periodic potentials and reformulate the Discrete Gaussian model that is admissible to this analysis. If the initial condition is tuned correctly, the sequence of (bulk) renormalisation group maps defines a convergent dynamical system of coupling constants (parametrising U_j) and polymer activities. This tuning is equivalent to constructing the stable manifold of the dynamical system, which is done in Chapter 7. The equivalent results for the observable renormalisation group are then proved in Chapter 8, using the bulk renormalisation group as the reference point.

We finally conclude the proof of our main theorems in Chapter 9. These are not of particular surprise once we have the (bulk and observable) renormalisation group flows, but certain technical points have to be verified, such as removing the coarse-grained structure of the polymer expansion for the final result.

Last but not least, we mention that there are various constants and parameters used to define the renormalisation group map. We have tried to summarize them in Section 3.2.3 and list of frequently used assumptions also appear in (A_f),(A'_f), (A'_u), (A_v) and (Φ_{IC}).

Appendix 1.A Existence of infinite-volume limit

We prove Proposition 1.1.2 in this appendix.

We recall the Fröhlich–Park–Ginibre inequalities. Let Λ be finite, let C be a positive definite matrix, and let $\langle\cdot\rangle_C$ be the expectation of the associated (generalised) Discrete Gaussian model:

$$\langle F\rangle_C \propto \sum_{\sigma\in\mathbb{Z}^\Lambda} e^{-\frac{1}{2}(\sigma,C^{-1}\sigma)}F(\sigma). \tag{1.54}$$

By taking limits, the definition of $\langle\cdot\rangle_C$ can also be extended to C positive semidefinite. Then the finite volume states $\langle\cdot\rangle^\Lambda_{J,\beta}$ given by (1.12) correspond to $C=\beta(-\Delta_J)^{-1}$ when σ is identified up to constants (as we do). The results of [39, Section 3] (see also [65, Proposition 1.2]) then imply that for $f:\Lambda\to\mathbb{R}$ with $\sum f=0$:

$$\langle e^{(f,\sigma)}\rangle^\Lambda_{J,\beta} \leqslant e^{\frac{1}{2}(f,(-\Delta_J)^{-1}_\Lambda f)}, \tag{1.55}$$

$$\langle (f,\sigma)^2\rangle^\Lambda_{J,\beta} \leqslant (f,(-\Delta_J)^{-1}_\Lambda f). \tag{1.56}$$

Moreover, [39, Corollary 3.2 (1)] implies that

$$\langle e^{i(\varphi,f)}\rangle_{C_1} \leqslant \langle e^{i(\varphi,f)}\rangle_{C_2} \qquad \text{if } C_2\leqslant C_1. \tag{1.57}$$

Proposition 1.A.1. *Let $L>1$ be an integer. For any finite-range step distribution J and any sequence of discrete tori Λ_N with side lengths L^N, with $N\in\mathbb{N}$, the measures $\langle\cdot\rangle^{\Lambda_N}_{J,\beta}$ converge weakly as $N\to\infty$ (when the field is identified up to constants). For any $f:\mathbb{Z}^d\to\mathbb{R}$ with compact support and $\sum f=0$, one also has $\langle e^{(f,\sigma)}\rangle^{\Lambda_N}_{J,\beta}\to\langle e^{(f,\sigma)}\rangle$ where $\langle\cdot\rangle=\lim_{N\to\infty}\langle\cdot\rangle^{\Lambda_N}_{J,\beta}$ is the weak limit.*

Proof. We consider the Laplacian $-\Delta^{\Lambda_N}$ as an operator on $\ell^2(\mathbb{Z}^d)$ with domain

$$D(-\Delta^{\Lambda_N}) = \{f\in\ell^2(\mathbb{Z}^d): f(0)=0,\ f(x)=f(x+L^Ny) \text{ for any } y\in\mathbb{Z}^d\}. \tag{1.58}$$

Then clearly $D(-\Delta^{\Lambda_N})\subset D(-\Delta^{\Lambda_{N+1}})$ and $-\Delta^{\Lambda_N}=-\Delta^{\Lambda_{N+1}}$ on $D(-\Delta^{\Lambda_N})$. This implies $-\Delta^{\Lambda_N}\geqslant-\Delta^{\Lambda_{N+1}}$ and hence $(-\Delta^{\Lambda_N})^{-1}\leqslant(-\Delta^{\Lambda_{N+1}})^{-1}$. From (1.57), it follows that for any $f:\mathbb{Z}^d\to\mathbb{R}$ compactly supported and with $\sum f=0$, $S_N(f)=\langle e^{i(f,\varphi)}\rangle^{\Lambda_N}_{J,\beta}$ is decreasing in N. In particular, since also $S_N(f)\leqslant 1$, the limit $S(f)=\lim_{N\to\infty}S_N(f)$ exists. To show $S(f)$ is the characteristic function of a probability measure on $(2\pi\mathbb{Z})^{\mathbb{Z}^2}/\text{constants}$ to which $\langle\cdot\rangle^{\Lambda_N}_{J,\beta}$ converges weakly, we will apply Minlos' theorem. To this end, we consider $(2\pi\mathbb{Z})^{\mathbb{Z}^2}/\text{constants}$ as a topological vector space with the topology defined by the condition that $\varphi_k\to\varphi$ in $(2\pi\mathbb{Z})^{\mathbb{Z}^2}/\text{constants}$ if $(\varphi_k,g)\to(\varphi,g)$ for all compactly supported $g:\mathbb{Z}^d\to\mathbb{R}$ with $\sum g=0$.

In particular, $(2\pi\mathbb{Z})^{\mathbb{Z}^2}/\text{constants}$ is the dual of a nuclear space. To apply Minlos' theorem we need to check that S is continuous in this topology. But this is immediate from the correlation inequality (1.56) which implies that for any $g : \mathbb{Z}^2 \to \mathbb{R}$ with compact support and $\sum g = 0$,

$$|S(f+g) - S(f)| = \lim_{N\to\infty} |S_N(f+g) - S_N(f)| \leqslant \lim_{N\to\infty} (g, (-\Delta_J^{\Lambda_N})^{-1} g) = (g, (-\Delta_J)^{-1} g), \tag{1.59}$$

from which the continuity is clear.

The final statement about the convergence of $\langle e^{(f,\sigma)} \rangle_{J,\beta}^{\Lambda_N}$ follows from the weak convergence and (1.55) which implies that the random variables $e^{(f,\sigma)}$ are uniformly integrable. □

It is also standard, see [50] and analogous extensions to the gradient Gibbs setting as in [43, 44], that any limit as in the previous proposition is translation invariant and satisfies the gradient Gibbs property. Moreover, the limit satisfies the analogous correlation inequalities.

Proposition 1.A.2. *The measure $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ has tilt 0, i.e., for each gradient Gibbs state in the ergodic decomposition of $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ the gradient field has mean 0.*

Proof. The proof is analogous to that of [44, Theorem 3.2]. The correlation decay can be replaced by the following application of the Riemann–Lebesgue lemma. For $g : \mathbb{Z}^2 \to \mathbb{R}^d$ with compact support, where now $\nabla\sigma : \mathbb{Z}^d \to \mathbb{R}^d$ denotes the vector of discrete forward derivatives, (1.56) implies

$$\langle (g, \nabla\sigma)^2 \rangle_{J,\beta}^{\mathbb{Z}^2} \leqslant C \int_{[-\pi,\pi]^2} \frac{|\hat{g}(p) \cdot p|^2}{|p|^2} \, dp. \tag{1.60}$$

Thus the distributional Fourier transform of $\langle \nabla_{e_i}\sigma(0) \nabla_{e_i}\sigma(x) \rangle$ is integrable in the Fourier variable. From this, the Riemann-Lebesgue lemma implies that

$$\langle \nabla_{e_i}\sigma(x) \nabla_{e_i}\sigma(y) \rangle_{J,\beta}^{\mathbb{Z}^2} \to 0 \qquad (|x-y| \to \infty). \tag{1.61}$$

In particular, for every $i = 1, \dots, d$, with $Q_R = [-R,R]^2 \cap \mathbb{Z}^2$,

$$\left\langle \left(\liminf_{R\to\infty} \frac{1}{|Q_R|} \sum_{x\in Q_R} \nabla_{e_i}\sigma(x) \right)^2 \right\rangle_{J,\beta}^{\mathbb{Z}^2} \leqslant \liminf_{R\to\infty} \frac{1}{|Q_R|^2} \sum_{x,y\in Q_R} |\langle \nabla_{e_i}\sigma(x) \nabla_{e_i}\sigma(y) \rangle_{J,\beta}^{\mathbb{Z}^2}| = 0. \tag{1.62}$$

This implies that every measure μ in the ergodic decomposition of $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ has mean 0 for $\nabla\sigma$ (see e.g. [44, Theorem 3.2] for a similar argument): indeed, for any such μ, by (1.62) and ergodicity, one deduces that $|Q_R|^{-1} \sum_{x\in Q_R} \nabla_{e_i}\sigma(x)$ converges μ-a.s. and that the limit vanishes, whence $E_\mu[\nabla_{e_i}\sigma(x)] = 0$. □

Proof of Proposition 1.1.2. We obtain $\langle \cdot \rangle_{J,\beta}^{\mathbb{Z}^2}$ when we take limit $N \to \infty$ in Proposition 1.A.1. The measure has tilt 0 due to Proposition 1.A.2. $\square$

Chapter 2

Finite-range decomposition

The starting point for our analysis is a finite-range decomposition for a covariance modified from the lattice Green's function for Laplacian. We expect that this construction will be useful for the analysis of other models where an initial renormalisation step can be carried out. Since it comes at no additional cost, we formulate the decomposition in any dimension $d \geqslant 2$. The main results of this section are Propositions 2.1.2 and 2.1.4 below, which exhibit the desired decomposition, first on $\mathbb{Z}^d$ and then on $\Lambda_N = (\mathbb{Z}/L^N\mathbb{Z})^d$, respectively.

2.1 Existence of the finite-range decomposition

First, recall our convenient convention that Δ denotes the standard *unnormalised* nearest neighbour lattice Laplacian while Δ_J is the *normalised* Laplacian with step distribution $J \subset \mathbb{Z}^d \setminus \{0\}$. As was introduced in Section 1.3.4 (cf. (1.52)), we will be interested in modified versions of the lattice Green's functions. First, we will see that for any small $\gamma > 0$ and any finite-range step distribution J, we can then decompose

$$(-\Delta_J + m^2)^{-1} = \gamma + C(m^2), \tag{2.1}$$

with $C(m^2) = C_J(m^2)$ a J-dependent positive-definite symmetric matrix. We also let

$$C(s,m^2) := (C(m^2)^{-1} - s\Delta)^{-1} = C(m^2)(1 - s\Delta C(m^2))^{-1}, \tag{2.2}$$

which makes sense and is positive definite for $|s|$ small (depending on J; see Proposition 2.1.2 below), as can be seen from the second representation. The main result of this section yields a decomposition of $C(s,m^2)$ into an integral of covariances with a finite-range property. Available results on such decompositions (see Remark 2.1.5 for some reference) apply to

$s = 0$ or without subtraction of the constant γ, i.e., to $(-\Delta + m^2)^{-1}$ instead of $C(s, m^2)$, but for our purposes it is important to permit both $s \neq 0$ and the subtraction of γ.

Recall that the step distribution $J \subset \mathbb{Z}^d \setminus \{0\}$ is assumed to satisfy the conditions above (1.8). The estimates for the resulting decomposition depend on the following parameters specific to J:

$$\rho_J = \sup\{|x|_\infty : x \in J\} \qquad \text{(range)}, \tag{2.3}$$

$$v_J^2 = \frac{1}{2|J|} \sum_{x \in J} |x_1|^2 \qquad \text{(variance)}, \tag{2.4}$$

$$\theta_J = \inf_{p \neq 0} (\lambda_J(p)/\lambda(p)) \qquad \text{(spectral lower bound)}. \tag{2.5}$$

In the spectral lower bound, $\lambda_J(p)$ and $\lambda(p)$ are the Fourier multipliers of $-\Delta_J$ and $-\Delta$, defined precisely in Section 2.2 below.

Example 2.1.1. For the standard range-ρ step distribution $J_\rho = \{x \in \mathbb{Z}^d \setminus \{0\} : |x|_\infty \leqslant \rho\}$,

$$\rho_{J_\rho} = \rho, \qquad v_{J_\rho}^2 \sim \frac{1}{6}\rho^2, \qquad \theta_{J_\rho} \geqslant 3^{-d}, \tag{2.6}$$

see Lemma 2.2.2 below.

We now state the main results of this section. We refer to Remark 2.1.5 below regarding a version of these findings for the (simpler) choice $\gamma = 0$ in (2.1), which implies various known results of this type (notably for the usual Green's function in the nearest-neighbour case). In the following proposition, we first consider (2.1) and (2.2) as operators on $\mathbb{Z}^d$; the inverses are then well-defined as bounded operators on $\ell^2(\mathbb{Z}^d)$ if $m^2 > 0$. Thus the proposition considers the infinite-volume case of $\mathbb{Z}^d$ rather than the finite torus relevant for our application to the Discrete Gaussian model. The torus case is treated thereafter in Proposition 2.1.4. Proposition 2.1.4 can be obtained in large part as a corollary of Proposition 2.1.2. Indeed the contributions to the torus decomposition comprising ranges smaller than the torus size are directly inherited from the decomposition on $\mathbb{Z}^d$, cf. (2.7) and (2.15) below. Scales which 'feel' the periodic boundary condition however must be treated separately. In what follows, a multi-index α is a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ with each $\alpha_i \in \hat{e} = \{\pm e_1, \pm e_2\}$, where e_1, e_2 is the standard basis of $\mathbb{Z}^2$. Then we denote $|\alpha| = n$ and $\nabla^\alpha = \nabla^{\alpha_1} \cdot \nabla^{\alpha_n}$.

Proposition 2.1.2. *Let $d \geqslant 2$. There exist absolute constants $\gamma > 0$ and $\varepsilon_s > 0$ (both purely numerical) such that for any finite-range step distribution $J \subset \mathbb{Z}^d \setminus \{0\}$ as specified above* (1.8)*, the following holds. For all $|s| \leqslant \varepsilon_s \theta_J$ and $m^2 \in (0, 1]$, one has a decomposition of the* *form*

$$C^{\mathbb{Z}^d}(s,m^2) := \big(C^{\mathbb{Z}^d}(m^2)^{-1} - s\Delta\big)^{-1}_{\mathbb{Z}^d} = \int_{\rho_J}^{\infty} D_t^{\mathbb{Z}^d}(s,m^2)\,dt, \tag{2.7}$$

where the $D_t^{\mathbb{Z}^d}(s,m^2)$ are positive-definite symmetric kernels with range less than t, i.e.,

$$D_t^{\mathbb{Z}^d}(x,y;s,m^2) := (\delta_x, D_t^{\mathbb{Z}^d}(s,m^2)\delta_y) = 0 \;\; \textit{whenever} \;\; |x-y|_\infty \geqslant t. \tag{2.8}$$

The left-hand side depends only on $x-y \in \mathbb{Z}^d$ and is invariant under lattice rotations. Moreover,

(i) uniformly in $(s,m^2) \in [-\varepsilon_s\theta_J, \varepsilon_s\theta_J] \times (0,1]$, all multi-indices α (including $|\alpha|=0$), $t \geqslant \rho_J$,

$$|\nabla^\alpha D_t^{\mathbb{Z}^d}(0,x;s,m^2)| \leqslant C_\alpha \rho_J^{-2} t \Big(\frac{\rho_J}{v_J t}\Big)^{d+|\alpha|} + C_\alpha \theta_J^{-d-|\alpha|} \rho_J^{-2} t \Big(\frac{\rho_J}{t^2}\Big)^{d+|\alpha|} e^{-c(\theta_J^{1/2} t)^{1/4}} \tag{2.9}$$

(note: if θ_J is bounded from below by a positive value, the second term can be omitted since $v_J \leqslant \rho_J/2 \leqslant t/2$);

(ii) for all $|s| \leqslant \varepsilon_s\theta_J$ and all t, the map $m^2 \mapsto D_t^{\mathbb{Z}^d}(s,m^2)$ is continuous and has a limit

$$D_t^{\mathbb{Z}^d}(s) \equiv D_t^{\mathbb{Z}^d}(s,0) = \lim_{m^2 \downarrow 0} D_t^{\mathbb{Z}^d}(s,m^2), \tag{2.10}$$

and the map $s \mapsto D_t^{\mathbb{Z}^d}(s)$ is analytic in $|s| \leqslant \varepsilon\theta_J$;

(iii) if $d=2$, then for all $|s| \leqslant \varepsilon_s\theta_J$,

$$D_t^{\mathbb{Z}^d}(0,0;s) = \frac{1}{2\pi t(v_J^2+s)}\Big(1 + O\Big(\frac{\rho_J}{t} + \frac{\rho_J^4}{v_J^2 t^2} + \frac{\theta_J^{-2} v_J^2}{t^2}\Big)\Big),$$
$$\textit{as} \;\; \frac{\rho_J}{t} + \frac{\rho_J^4}{v_J^2 t^2} + \frac{\theta_J^{-2} v_J^2}{t^2} \to 0. \tag{2.11}$$

In the above estimates, all constants are independent of J (and s).

In the particular case of the standard range-ρ step distribution the conclusions simplify as follows.

Remark 2.1.3. For $J = J_\rho$ (see Example 2.1.1) the uniform lower bound on θ_{J_ρ} in (2.6) implies that the domain of s can be chosen independently of ρ. For such s, using the bounds from (2.6), the estimates in items *(i)* and *(iii)* above become (see also the note below (2.9)), for $t \geqslant \rho$,

$$|\nabla^\alpha D_t^{\mathbb{Z}^d}(0,x;s,m^2)| \leqslant C_\alpha \rho^{-2} t^{1-d-|\alpha|}, \tag{2.12}$$

and (in $d=2$)

$$D_t^{\mathbb{Z}^d}(0,0;s) = \frac{1}{2\pi t(v_{J_\rho}^2+s)}\Big(1+O\Big(\frac{\rho}{t}\Big)\Big), \tag{2.13}$$

with all constants independent of ρ, s and m^2.

Proposition 2.1.2 applies to covariances defined on all of $\mathbb{Z}^d$. By periodisation, Proposition 2.1.2 and its proof also imply an analogous statement for the discrete torus, which we state next. Since this is the decomposition we will use in the present thesis, we consider the torus Λ_N of side length L^N (even though an analogous statement holds for any side length). For $t < \frac{1}{4}L^N$, the covariances $D_t^{\mathbb{Z}^d}$ from Proposition 2.1.2 are translation invariant and have range less than half the diameter of the torus. They can thus naturally be identified with covariances on the torus Λ_N by projection. More precisely, with $\pi_N : \mathbb{Z}^d \to \Lambda_N$ denoting the canonical projection and for any $f : \Lambda_N \to \mathbb{R}$, $t < \frac{1}{4}L^N$, one sets $D_t f(\pi_N(x)) := D_t^{\mathbb{Z}^d}(f \circ \pi_N)(x)$, for $x \in \mathbb{Z}^d$, and readily verifies that this is well-defined, i.e. the right-hand side does not depend on the choice of representative x in the equivalence class. In the sequel, we normally do not distinguish between D_t and $D_t^{\mathbb{Z}^d}$ for $t < \frac{1}{4}L^N$ and often omit the superscript. On the other hand, for $t \geqslant \frac{1}{4}L^N$, the periodisation of the covariance $D_t^{\mathbb{Z}^d}$ does depend on the torus.

Proposition 2.1.4. *Let $d \geqslant 2$, $L \geqslant 1$, $N \geqslant 1$, and $\Lambda_N = (\mathbb{Z}/L^N\mathbb{Z})^d$. With the same constants $\gamma > 0$ and $\varepsilon_s > 0$ as in Proposition 2.1.2, $m^2 \in (0,1]$, the matrix*

$$(-\Delta_J + m^2)^{-1}_{\Lambda_N} - \gamma = C^{\Lambda_N}(m^2) \tag{2.14}$$

is positive definite and for all $|s| \leqslant \varepsilon_s \theta_J$,

$$(C^{\Lambda_N}(m^2) - s\Delta)^{-1}_{\Lambda_N} = \int_{\rho_J}^{\frac{1}{4}L^{N-1}} D_t^{\mathbb{Z}^d}(s,m^2)\,dt + \int_{\frac{1}{4}L^{N-1}}^{\infty} \tilde{D}_t^{\Lambda_N}(s,m^2)\,dt + t_N(s,m^2)Q_N \tag{2.15}$$

where the $D_t^{\mathbb{Z}^d}(s,m^2)$ are the same as in (2.7) *(with the identification discussed above), for all $t > \frac{1}{4}L^{N-1}$, the covariances $\tilde{D}_t^{\Lambda_N}(s,m^2)$ satisfy the same upper bounds as $D_t^{\mathbb{Z}^d}$ in* (2.9)*, the same analyticity in s, and the same continuity in m^2 (including as $m^2 \downarrow 0$). Finally, Q_N denotes the matrix with all entries equal to $1/|\Lambda_N| = L^{-dN}$ and $t_N(s,m^2) \in (0,m^{-2})$ is a constant satisfying*

$$|t_N(s,m^2) - m^{-2}| \leqslant C\rho_J^{-2}L^{2N}. \tag{2.16}$$

Remark 2.1.5. Analogues of Propositions 2.1.2 and 2.1.4 continue to hold for the choice $\gamma = 0$ in (2.1), yielding for $|s| \leqslant \varepsilon_s \theta_J$ and $m^2 \in (0,1]$ the decomposition

$$(-\Delta_J + m^2 - s\Delta)^{-1}_{\mathbb{Z}^d} = \int_0^\infty D_t^{\mathbb{Z}^d}(s, m^2)\, dt, \tag{2.17}$$

(along with a corresponding analogue on Λ_N); the properties (2.8)–(2.11) remain valid for all $t \geqslant \rho_J$. Moreover, the range of $D_t^{\mathbb{Z}^d}$ is 0 for $t \leqslant \rho_J$, i.e., $D_t^{\mathbb{Z}^d}(0,x) = 1_{x=0} D_t^{\mathbb{Z}^d}(0,0)$ and (2.9) is complemented by the fact that $D_t^{\mathbb{Z}^d}(0,0) > 0$ is constant for all $t \leqslant \rho_J$. The decomposition (2.17) is obtained by following the arguments below, which simplify when $\gamma = 0$ (essentially boiling down to [11, Section 3]). In particular, for J the usual nearest-neighbour interaction and $s = 0$, (2.17) recovers a well-known decomposition for the Green's function $(-\Delta + m^2)^{-1}$, see e.g. [5, 11, 25].

The rest of this chapter is devoted to proving Propositions 2.1.2 and 2.1.4.

2.2 Preliminaries on Fourier transforms

Before proving Proposition 2.1.2, we collect some preliminaries and conventions about normalisation of Fourier transforms and of the lattice Laplacian Δ and its generalised version Δ_J with step distribution J. Throughout, Λ is a discrete d-dimensional torus of integer period R with Fourier dual

$$\Lambda^* = \{2\pi R^{-1} k : k \in \{-\lceil (R-2)/2 \rceil, \dots, \lfloor R/2 \rfloor\}^d\} \subset (-\pi, \pi]^d. \tag{2.18}$$

For an integrable function $\hat{f} : (-\pi,\pi]^d \to \mathbb{R}$, we define

$$f^{\mathbb{Z}^d}(x) = \int_{(-\pi,\pi]^d} \hat{f}(p) e^{ip\cdot x} \frac{dp}{(2\pi)^d}, \tag{2.19}$$

$$f^{\Lambda}(x) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \hat{f}(p) e^{ip\cdot x}, \tag{2.20}$$

where $|\Lambda| = R^d$ denotes the number of points in Λ. Then by the Poisson summation formula,

$$f^{\Lambda}(x) = \sum_{y \in \mathbb{Z}^d} f^{\mathbb{Z}^d}(x + yR). \tag{2.21}$$

This notation also applies for translation invariant covariances, i.e., when a function $f(x,y)$ depends only on $x - y$ we will usually identify it with the function $f(0,x)$.

We write $\lambda = \lambda(p)$ and $\lambda_{J,m^2} = \lambda_{J,m^2}(p) \geqslant 0$ for the Fourier multipliers of $-\Delta$ and $-\Delta_J + m^2$:

$$
\begin{aligned}
\lambda(p) &= \sum_{|e|=1} (1-\cos(p\cdot e)), \\
\lambda_J(p) &= \frac{1}{|J|}\sum_{x\in J}(1-\cos(p\cdot x)), \qquad \lambda_{J,m^2}(p) = \lambda_J(p) + m^2
\end{aligned}
\tag{2.22}
$$

(recall our convention regarding normalisation of Δ and Δ_J). The following elementary lemma provides some comparison estimates for $\lambda_J(p)$ and $\lambda(p)$, which will be useful in the sequel.

Lemma 2.2.1. *For any step distribution J as above* (1.8) *(with implicit constants independent of J),*

$$
\lambda_J(p) = v_J^2|p|^2 + O(\rho_J^2 v_J^2 |p|^4) \qquad (p \to 0) \tag{2.23}
$$

$$
\lambda_J(p) \leqslant \min\{1, v_J^2|p|^2\} \qquad (p \in (-\pi,\pi]^d), \tag{2.24}
$$

with ρ_J and v_J defined by (2.3) *and* (2.4). *Moreover, $\lambda(p) = |p|^2 + O(|p|^4)$ as $p \to 0$ and $\lambda(p) \in [\frac{4}{\pi^2}|p|^2, |p|^2]$ for $p \in (-\pi,\pi]^d$, hence in particular $\theta_J \leqslant \frac{\pi}{4}v_J^2$.*

Proof. Let v_J^2 be as defined by (2.4). Then as $p \to 0$, substituting $1-\cos x = \frac{x^2}{2} + O(x^4)$ in (2.22), one finds that

$$
\begin{aligned}
\lambda_J(p) &= \frac{1}{|J|}\sum_{y\in J}\Big(1-\cos\big(\sum_{i=1}^d p_i y_i\big)\Big) \\
&= \frac{1}{2|J|}\sum_{y\in J}|p|^2 y_1^2 + O\big(\frac{1}{|J|}\sum_{y\in J}|y|^4|p|^4\big) = v_J^2|p|^2 + O(\rho_J^2 v_J^2|p|^4).
\end{aligned}
\tag{2.25}
$$

The upper bound in (2.24) follows similarly, using the inequality $1-\cos x \le x^2/2$ valid for all $x \in \mathbb{R}$ instead.

To see the lower bound for $\lambda(p)$, consider the function $g(q) = 1-\cos(q) - \frac{2}{\pi^2}q^2$ on $q \in [0,\pi]$. Then $g(0) = g(\pi) = 0$ while $g'(q) = \sin(q) - \frac{4}{\pi^2}q$ has only one non-zero zero, hence $g(q)$ does not attain 0 on $(0,\pi)$, *i.e.*, $g(q) \geqslant 0$ on $[0,\pi]$. Therefore

$$
\lambda(p) = 2\sum_{i=1}^d (1-\cos(p_i)) \geqslant \frac{4}{\pi^2}|p|^2 \tag{2.26}
$$

which is the claimed lower bound. □

Lemma 2.2.2. *For the step distribution* $J = J_\rho = \{x \in \mathbb{Z}^d \setminus \{0\} : |x|_\infty \leqslant \rho\}$,

$$\rho_{J_\rho} = \rho, \qquad v_{J_\rho}^2 \equiv v_\rho^2 \sim \frac{1}{6}\rho^2, \quad \text{as } \rho \to \infty, \tag{2.27}$$

and with $\lambda_\rho \equiv \lambda_{J_\rho}$,

$$\lambda(p) \leqslant 3^d \lambda_\rho(p), \tag{2.28}$$

i.e., $\theta_{J_\rho} \equiv \theta_\rho = \inf_{p \neq 0} \lambda_\rho(p)/\lambda(p) \geqslant 3^{-d}$.

Proof. Using that $\sum_{j=1}^{\rho} j^2 \sim \frac{1}{3}\rho^3$ as $\rho \to \infty$,

$$v_\rho^2 = \frac{(2\rho+1)^{d-1}}{(2\rho+1)^d - 1} \sum_{j=1}^{\rho} j^2 \sim \frac{\rho^2}{6}. \tag{2.29}$$

To show $\lambda \leqslant 3^d \lambda_\rho$, first note that since $\sum_{a=1}^{\rho} \cos(ax) = \frac{\sin((\rho+1/2)x)}{2\sin(x/2)} - \frac{1}{2}$,

$$\begin{aligned}
\lambda_\rho(p) &= \frac{1}{(2\rho+1)^d - 1} \sum_{|y|_\infty \leqslant \rho} (1 - \prod_{i=1}^{d} e^{ip_iy_i}) \\
&= \frac{(2\rho+1)^d}{(2\rho+1)^d - 1} \Big(1 - \prod_{i=1}^{d} \sum_{x=-\rho}^{\rho} \frac{e^{ip_ix}}{(2\rho+1)})\Big) \\
&= \frac{(2\rho+1)^d}{(2\rho+1)^d - 1} \Big(1 - \prod_{i=1}^{d} \frac{2}{2\rho+1} (\frac{1}{2} + \sum_{x=1}^{\rho} \cos(p_ix)))\Big) \\
&= \frac{(2\rho+1)^d}{(2\rho+1)^d - 1} \Big(1 - \prod_{i=1}^{d} \frac{\sin((2\rho+1)p_i/2)}{(2\rho+1)\sin(p_i/2)}\Big).
\end{aligned} \tag{2.30}$$

But

$$\sup_{\rho \geqslant 1} \frac{\sin((2\rho+1)p_1/2)}{(2\rho+1)\sin(p_1/2)} = \frac{\sin(3p_1/2)}{3\sin(p_1/2)} \tag{2.31}$$

and so $\lambda_\rho(p) \geqslant (1 - 3^{-d})\lambda_{\rho=1}(p)$. But

$$(3^d - 1)\lambda_{\rho=1}(p) = \sum_{|y|_\infty = 1} (1 - \cos(\sum_{i=1}^{d} p_iy_i)) \geqslant \sum_{|y|_1 = 1} (1 - \cos(\sum_{i=1}^{d} p_iy_i)) = \lambda(p) \tag{2.32}$$

so the claim holds since $(1 - 3^{-d})/(3^d - 1) = 3^{-d}$. □

2.3 Proof of Proposition 2.1.2: finite-range property

The starting point for the construction of the finite-range decomposition is the following lemma, from which one can directly obtain the finite-range decomposition when $s = 0$. The lemma originated in [5], but we obtain here a better decay estimate, which is important for our construction of the finite-range decomposition for $s \neq 0$.

Lemma 2.3.1. *For $t > 0$, there exist polynomials P_t of degree at most t such that for $\lambda \in (0,3]$,*

$$\frac{1}{\lambda} = \int_0^\infty t^2 P_t(\lambda) \frac{dt}{t}. \tag{2.33}$$

For $\lambda \in (0,3]$ and $t > 1$, the polynomials satisfy $P_t(\lambda) \geqslant 0$ and there is an entire function f that is non-negative on the real axis and satisfies $\int_0^\infty t^2 f(t) \frac{dt}{t} = 1$ such that

$$P_t(\lambda) \leqslant Ce^{-c(\lambda t^2)^{1/4}} \tag{2.34}$$

$$|P_t(\lambda) - f(\sqrt{\lambda} t)| \leqslant Ct^{-1} e^{-c(\lambda t^2)^{1/4}}. \tag{2.35}$$

For $t \leqslant 1$, $P_t(\lambda) = \gamma/t$ for some constant $\gamma > 0$.

Proof. Let $f : \mathbb{R} \to [0,\infty)$ be any non-negative function with the following properties: the Fourier transform of f is smooth, symmetric and has support in $[-1,1]$, and $\int_0^\infty t^2 f(t) \frac{dt}{t} = 1$. Then by [11, Lemma 3.3.3], (2.33) holds for $\lambda \in [0,4]$ with the function P_t given by

$$P_t(\lambda) = f_t^*(\arccos(1 - \frac{1}{2}\lambda)) \tag{2.36}$$

where

$$f_t^*(x) = \sum_{n \in \mathbb{Z}} f(xt - 2\pi n t). \tag{2.37}$$

By [11, Lemma 3.3.5], (2.36) defines a polynomial $P_t(\cdot)$ on $(0,3]$, of degree bounded by t. We will now choose f as follows. Let

$$\kappa(s) = e^{-(1-(2s)^2)^{-1}} 1_{|s|<1/2} \tag{2.38}$$

be the standard bump function with support $[-1/2, 1/2]$. By Proposition 2.A.1, $|\hat{\kappa}(x)| \leqslant Ce^{-|x|^{1/2}}$ for all $x \in \mathbb{R}$. We set $\hat{f}(s) = c(\kappa * \kappa)(s)$, with $c > 0$ chosen as to ensure the normalisation $\int_0^\infty t f(t)\, dt = 1$. Then f has all the required properties. In particular, since its Fourier transform has compact support, it extends to an entire function, as easily seen by expanding the exponential in the Fourier integral, yielding an absolutely convergent power series. Also, $f(x) = c\hat{\kappa}(x)^2 \leqslant C' e^{-2|x|^{1/2}}$ for $x \in \mathbb{R}$. For $t \geqslant 1$,

$$f_t^*(x) \leqslant C' e^{-(2t|x|)^{1/2}} \sum_{n \geqslant 0} e^{-\sqrt{4\pi n}} \leqslant C'' e^{-(2t|x|)^{1/2}}. \tag{2.39}$$

Since $\arccos(1 - \frac{1}{2}\lambda) \geqslant \sqrt{\lambda}$, the estimate (2.34) follows immediately from (2.36) and (2.39). The bound (2.35) follows similarly using $|f'(x)| \leqslant C''' e^{-|x|^{1/2}}$ (which follows from the explicit form of f and that κ has compact support) and using that $\arccos(1 - \frac{1}{2}\lambda) - \sqrt{\lambda} = O(\lambda)$; see [5, Proposition 3.1] for a similar argument. The constant γ is given by $\hat{f}(0)/2\pi$, see [11, Lemma 3.3.6]. □

By applying the previous lemma, we first construct a finite-range decomposition for $s = 0$. To this end, insert $\lambda_{J,m^2}(p)$ into (2.33) for $m^2 \leqslant 1$ (so that $\lambda_{J,m^2} \leqslant 3$ and Lemma 2.3.1 is in force). Since λ_{J,m^2} has range ρ_J, in the sense that it is the Fourier multiplier of an operator with range ρ_J, and since P_t is a polynomial of degree at most t, it follows that $P_t(\lambda_{J,m^2})$ has range $\rho_J t$. We therefore set in Fourier space

$$\hat{D}_t(p; m^2) = \rho_J^{-2} t P_{\rho_J^{-1} t}(\lambda_{J,m^2}(p)), \quad p \in (-\pi, \pi]^d. \tag{2.40}$$

By (2.33) and the explicit form of P_t for $t \leqslant 1$, it follows with $\hat{D}_t(m^2) \equiv \hat{D}_t(\cdot; m^2)$ that

$$\frac{1}{\lambda_{J,m^2}(\cdot)} = \int_0^\infty \hat{D}_t(m^2)\, dt = \gamma + \int_{\rho_J}^\infty \hat{D}_t(m^2)\, dt = \gamma + \hat{C}(m^2), \tag{2.41}$$

with the last equality defining $\hat{C}(m^2) = \hat{C}(p; m^2)$, $p \in (-\pi, \pi]^d$, and we used that

$$\int_0^{\rho_J} \rho_J^{-2} t P_{\rho_J^{-1} t}\, dt = \int_0^1 t P_t\, dt = \gamma. \tag{2.42}$$

By (2.41), the function $\hat{C}(m^2)$ thus defined in terms of $\hat{D}_t(m^2)$ is indeed the Fourier transform of $C(m^2)$ appearing in (2.1). With a view towards our aim (2.7), we expand for $|s| < \theta_J = \inf_{m^2} \inf(\lambda_{J,m^2}/\lambda)$ and $m^2 \in (0, 1]$,

$$\begin{aligned} \hat{C}(s, m^2) &\overset{\text{def.}}{=} (\hat{C}(m^2)^{-1} + s\lambda)^{-1} = \hat{C}(m^2)(1 + s\lambda \hat{C}(m^2))^{-1} \\ &= \sum_{l=0}^\infty s^{2l} \lambda^{2l} \hat{C}(m^2)^{2l+1}(1 - s\lambda \hat{C}(m^2)). \end{aligned} \tag{2.43}$$

The expansion is absolutely convergent since $|s|\lambda\hat{C}(m^2) \leqslant |s|\lambda/\lambda_{J,m^2} \leqslant |s|/\theta_J < 1$. Moreover, this condition implies that the following integrand is positive:

$$1 - s\lambda\hat{C}(m^2) = \frac{\lambda_{J,m^2}}{\lambda_{J,m^2}} - s\lambda\hat{C}(m^2) = \int_0^\infty (\lambda_{J,m^2} - s\lambda 1_{t>\rho_J})\hat{D}_t(m^2)\,dt. \tag{2.44}$$

This motivates the following definition of the finite-range decomposition for $|s| < \theta_J$.

Definition 2.3.2. *For $m^2 \in (0,1]$, all $|s| < \theta_J$ and $t > 0$, let*

$$\hat{D}_t(s,m^2) = \\ \frac{1}{4\lambda}\sum_{l=0}^\infty s^{2l} \int_{[0,\infty)\times[\rho_J,\infty)^{2l+1}:\sum t_i=(t-\rho_J)/4} (\lambda_{J,m^2} - s\lambda 1_{t_0>\rho_J})\hat{D}_{t_0}(m^2)\prod_{i=1}^{2l+1}\lambda\hat{D}_{t_i}(m^2)\,dt_i\,dt_0. \tag{2.45}$$

In this definition, the integral $\int_{\sum t_i=T}\prod_{i=0}^{2l+1} dt_i$ over the simplex is the push-forward of the Lebesgue measure on $\mathbb{R}^{2l+1}$ along the map $(t_1,\dots,t_{2l+1}) \mapsto (T - \sum_{k=1}^{2l+1} t_i, t_1, \dots, t_{2l+1})$, i.e.,

$$\int_{[0,\infty)\times[\rho_J,\infty)^{2l+1}:\sum t_i=T} f(t_0,\dots,t_{2l+1})\prod_{i=0}^{2l+1} dt_i = \int_{[\rho_J,\infty)^{2l+1}:\sum t_i\leqslant T} f(T-\sum t_i, t_1,\dots,t_{2l+1})\prod_{i=1}^{2l+1} dt_i \tag{2.46}$$

for $T > (2l+1)\rho_J$, and the left-hand side is interpreted as 0 when $T \leqslant (2l+1)\rho_J$. The same remark applies to various similar quantities below. In particular, $\hat{D}_t(s,m^2) = 0$ for $t \leqslant 5\rho_J$, and if $\hat{D}_t(s,m^2)$ is nonzero, then $T = (t-\rho_J)/4$ in (2.45) satisfies $T \in [\frac{1}{5}t, \frac{1}{4}t]$.

Proof of Proposition 2.1.2: finite-range property. We will show that the covariances $D_t^{\mathbb{Z}^d}(s,m^2)$ with Fourier transforms given by (2.45) define the desired decomposition (2.7). First, it is clear from (2.40) and Lemma 2.3.1 that $D_t^{\mathbb{Z}^d}(s,m^2)$ is positive definite. That the decomposition (2.7) holds follows by substituting (2.41) and (2.44) into (2.43) and using the change of variables

$$\int_{[0,\infty)^{2l+2}} dt_0\cdots dt_{2l+1}\, f(t_0,\dots,t_{2l+1}) = \int_0^\infty dT \int_{[0,\infty)^{2l+2}:\sum t_i=T} dt_0\cdots dt_{2l+1}\, f(t_0,\dots,t_{2l+1}), \tag{2.47}$$

with $T = (t-\rho_J)/4$.

Next we verify the finite-range property. Since λ has range 1 and $D_{t_i}(m^2)$ has range t_i, we see that $\lambda D_{t_i}(m^2)$ has range at most $1+t_i \leqslant 2t_i$ for $t_i \geqslant \rho_J \geqslant 1$ and $\lambda_{J,m^2}D_{t_0}(m^2)$ has range $\rho_J + t_0 \leqslant \rho_J + 2t_0$. Since $\sum t_i = \frac{1}{4}(t-\rho_J)$, from the definition (2.45), it follows that the range of $D_t(s,m^2)$ is at most $\rho_J + \frac{1}{2}(t-\rho_J) = \frac{1}{2}(t+\rho_J) \leqslant t$ for $t > \rho_J$. On the other hand, $D_t(s,m^2) = 0$ for $t \leqslant \rho_J$. $\square$

We now proceed to prove the estimates asserted in items (i)-(iii) of Proposition 2.1.2 for the above finite-range decomposition.

2.4 Proof of Proposition 2.1.2: (i) and (ii)

To prove the estimates (i) and (ii), we begin with the following lemma which we will use repeatedly.

Lemma 2.4.1. *Let* $g(x) = e^{-c\sqrt{x}}$ *and* $\tilde{C} = \max\{\|xg\|_\infty, \|xg\|_1\} \in (0,\infty)$. *Then for all integers* $k \geqslant 1$,

$$\int_{[0,\infty)^k : \sum_{i=1}^k t_i = t} \prod_{i=1}^k \lambda g(\sqrt{\lambda} t_i) t_i dt_i \leqslant \sqrt{\lambda} \min\Big\{\tilde{C}^k, \frac{(\sqrt{\lambda} t)^{2k-1}}{(2k-1)!} g(\sqrt{\lambda} t)\Big\}. \tag{2.48}$$

In fact, the same estimate applies to any supermultiplicative $g : [0,\infty) \to \mathbb{R}$, *i.e.,* $g(x)g(y) \leqslant g(x+y)$.

Proof. We bound the left-hand side in two ways. First, the left-hand side equals

$$\sqrt{\lambda} \int_{[0,\infty)^k : \sum_{i=1}^k u_i = \sqrt{\lambda} t} \prod_{i=1}^k g(u_i) u_i du_i \leqslant \sqrt{\lambda} \|gu\|_\infty \|gu\|_1^{k-1} \leqslant \sqrt{\lambda} \tilde{C}^k. \tag{2.49}$$

On the other hand, since $g(x)g(y) \leqslant g(x+y)$, we can also bound it by

$$\begin{aligned} \int_{[0,\infty)^k : \sum_{i=1}^k t_i = t} \prod_{i=1}^k \lambda g(\sqrt{\lambda} t_i) t_i dt_i &\leqslant \lambda^k g(\sqrt{\lambda} t) \int_{[0,\infty)^k : \sum_{i=1}^k t_i = t} \prod_{i=1}^k t_i dt_i \\ &= \sqrt{\lambda} \frac{(\sqrt{\lambda} t)^{2k-1}}{(2k-1)!} g(\sqrt{\lambda} t) \end{aligned} \tag{2.50}$$

where we used

$$h_k(t) := \int_{[0,\infty)^k : \sum_{i=1}^k t_i = t} \prod_{i=1}^k t_i dt_i = t^{2k-1} \int_{[0,\infty)^k : \sum_{i=1}^k u_i = 1} \prod_{i=1}^k u_i du_i = \frac{t^{2k-1}}{(2k-1)!}. \tag{2.51}$$

The last equality can be seen by induction: $h_2(1) = 1/6$ and

$$h_k(1) = \int_0^1 s h_{k-1}(1-s) ds = \int_0^1 s(1-s)^{2k-3} h_{k-1}(1) ds = \frac{h_{k-1}(1)}{(2k-2)(2k-1)} \tag{2.52}$$

advances the induction. □

Lemma 2.4.2. *Let $g(x) = e^{-c\sqrt{x}}$. Then there is $\varepsilon_s > 0$ and constants $\tilde{C}, \tilde{c}$ such that for $|s| \leqslant \varepsilon_s$,*

$$\frac{1}{\lambda}\sum_{l=0}^{\infty} s^{2l}\int_{[0,\infty)^{2l+2}:\sum t_i = t}\prod_{i=0}^{2l+1}\lambda t_i g(\sqrt{\lambda}t_i)\,dt_i \leqslant \tilde{C}te^{-\tilde{c}(\sqrt{\lambda}t)^{1/4}}. \tag{2.53}$$

Proof. By Lemma 2.4.1, the left-hand side is bounded by

$$\frac{1}{\sqrt{\lambda}}\sum_{l=0}^{\infty} s^{2l}\min\left\{\tilde{C}^{2l}, \frac{(\sqrt{\lambda}t)^{4l+3}}{(4l+3)!}g(\sqrt{\lambda}t)\right\} \tag{2.54}$$

where $\tilde{C} = \max\{1, \|xg\|_1, \|xg\|_\infty\}$. We assume $\tilde{C}|s| \leqslant 1/4$, i.e., set $\varepsilon_s = 1/(4\tilde{C})$. For $\lambda t^2 \leqslant 2$ by using the second term in the minimum, this immediately gives the desired estimate since

$$t(\sqrt{\lambda}t)^2 g(\sqrt{\lambda}t)\sum_{l=0}^{\infty}\frac{(\lambda t^2 s)^{2l}}{(4l+3)!} \leqslant t(\sqrt{\lambda}t)^2 g(\sqrt{\lambda}t)\sum_{l=0}^{\infty} 2^{-l} \leqslant 4tg(\sqrt{\lambda}t). \tag{2.55}$$

Thus assume $\lambda t^2 \geqslant 2$. By switching between the two terms in the minimum at $l = l_0$, the left-hand side of the claim is bounded by the sum of the following two contributions:

$$\frac{1}{\sqrt{\lambda}}\sum_{l=l_0+1}^{\infty}(\tilde{C}s)^{2l} \leqslant \frac{1}{\sqrt{\lambda}}\sum_{l=l_0+1}^{\infty} 16^{-l} \leqslant \frac{1}{\sqrt{\lambda}}16^{-l_0} \tag{2.56}$$

and

$$\frac{g(\sqrt{\lambda}t)}{\sqrt{\lambda}}\sum_{l=0}^{l_0}\frac{(\sqrt{s\lambda}t)^{4l}(\sqrt{\lambda}t)^3}{(4l+3)!} \leqslant \lambda t^3 g(\sqrt{\lambda}t)\sum_{l=0}^{l_0}(\sqrt{\lambda}t)^{4l} \leqslant tg(\sqrt{\lambda}t)(\sqrt{\lambda}t)^{4l_0+2}. \tag{2.57}$$

Choosing $l_0 = (c/16)(\sqrt{\lambda}t)^{1/2}(\log(\sqrt{\lambda}t))^{-1}$ gives the upper bound

$$\begin{aligned} t\left(\frac{e^{-\log(16)l_0}}{\sqrt{\lambda t^2}} + g(\sqrt{\lambda}t)e^{8\log(\sqrt{\lambda}t)l_0}\right) &\leqslant t\left(e^{-\log(16)l_0} + e^{-c(\sqrt{\lambda}t)^{1/2}}e^{8\log(\sqrt{\lambda}t)l_0}\right) \\ &\leqslant t\left(e^{-\log(16)(c/16)(\sqrt{\lambda}t)^{1/2}(\log(\sqrt{\lambda}t))^{-1}} + e^{-\frac{c}{2}(\sqrt{\lambda}t)^{1/2}}\right) \\ &\leqslant 2te^{-\tilde{c}(\sqrt{\lambda}t)^{1/2}(\log(\sqrt{\lambda}t))^{-1}} \end{aligned} \tag{2.58}$$

which is less than the claimed bound. $\square$

Proposition 2.4.3. *There are constants $\tilde{C}, \tilde{c}, \varepsilon_s > 0$ independent of J, m^2, s such that for $C|s| \leqslant \varepsilon_s \theta_J$ and $m^2 \in (0,1]$,*

$$0 \leqslant \hat{D}_t(p;s,m^2) \leqslant \tilde{C}\rho_J^{-2} t \exp\Big(-\tilde{c}\Big(\rho_J^{-1}\sqrt{\lambda_{J,m^2}}\,t\Big)^{1/4}\Big). \tag{2.59}$$

Proof. Let $g(x) = e^{-c\sqrt{x}}$ so that $P_t(\lambda) \leqslant Cg(\sqrt{\lambda} t)$ by (2.34). By the definition (2.45), then

$$\begin{aligned}
\hat{D}_t(s,m^2) &\leqslant \frac{1}{4\lambda_{J,m^2}} \sum_{l=0}^{\infty} s^{2l} \sup\big(\frac{\lambda}{\lambda_{J,m^2}}\big)^{2l} \int_{[0,\infty)^{2l+2}:\sum t_i = T} \prod_{i=0}^{2l+1} \rho_J^{-2} \lambda_{J,m^2} t_i P_{\rho_J^{-1} t_i}(\lambda_{J,m^2})\, dt_i \\
&\leqslant \frac{1}{4\rho_J \lambda_{J,m^2}} \sum_{l=0}^{\infty} s^{2l}\, \theta_J^{-2l} \int_{[0,\infty)^{2l+2}:\sum t_i = \rho_J^{-1} T} \prod_{i=0}^{2l+1} \lambda_{J,m^2} t_i P_{t_i}(\lambda_{J,m^2})\, dt_i \\
&\leqslant \frac{C^2}{4\rho_J \lambda_{J,m^2}} \sum_{l=0}^{\infty} (Cs/\theta_J)^{2l} \int_{[0,\infty)^{2l+2}:\sum t_i = \rho_J^{-1} T} \prod_{i=0}^{2l+1} \lambda_{J,m^2} t_i g(\sqrt{\lambda_{J,m^2}} t_i)\, dt_i,
\end{aligned} \tag{2.60}$$

where $T = (t-\rho_J)/4 \in [\frac{1}{5}t, \frac{1}{4}t]$. Thus the claim follows from Lemma 2.4.2 with s replaced by Cs/θ_J, with t replaced by T/ρ_J, and with λ replaced by λ_{J,m^2}. □

Proof of Proposition 2.1.2 (i) and (ii). We will show (2.7), i.e.,

$$|\nabla^\alpha D_t^{\mathbb{Z}^d}(0,x;s,m^2)| \leqslant C_\alpha \rho_J^{-2} t \Big(\frac{\rho_J}{v_J t}\Big)^{d+|\alpha|} + C_\alpha \theta_J^{-d-|\alpha|} \rho_J^{-2} t \Big(\frac{\rho_J}{t^2}\Big)^{d+|\alpha|} e^{-c(\theta_J^{1/2} t)^{1/4}}. \tag{2.61}$$

Clearly, (2.59) implies that $|\nabla^\alpha D_t(0,x;s,m^2)|$ is bounded, uniformly in s and m^2, by

$$\tilde{C} \int_{[-\pi,\pi]^d} \lambda^{|\alpha|/2} \rho_J^{-2} t e^{-\tilde{c}(t\rho_J^{-1}\sqrt{\lambda_J(p)})^{1/4}} \frac{dp}{(2\pi)^d}. \tag{2.62}$$

To apply the lower bound on λ_{J,m^2} from Lemma 2.2.1, i.e., $\lambda_J = v_J^2 |p|^2 (1 + O(\rho_J^2 |p|^2))$, we will split the above integral into integrals over $|p| \geqslant 1/\rho_J$ and $|p| \leqslant 1/\rho_J$. The former integral is bounded by (with other constants C, c)

$$C_\alpha \int_{[-\rho_J^{-1},\rho_J^{-1}]^d} |p|^{|\alpha|} \rho_J^{-2} t e^{-c(\rho_J^{-1} v_J t |p|)^{1/4}} dp, \tag{2.63}$$

which yields the main term in (2.7), as can be seen by substituting $p \to \rho_J v_J^{-1} t^{-1} p$. For the integral over $|p| \geqslant 1/\rho_J$ we use $\lambda_J \geqslant \theta_J \lambda \geqslant \frac{4}{\pi^2} \theta_J |p|^2$ on $[-\pi,\pi]^d$ to obtain the bound (again with possibly different constants)

$$C_\alpha \int_{[-\pi,\pi]^d \setminus [-\rho_J^{-1},\rho_J^{-1}]^d} |p|^{|\alpha|} \rho_J^{-2} t e^{-c(\rho_J^{-1}\theta_J^{1/2} t|p|)^{1/4}} dp, \tag{2.64}$$

which by substituting $p \to \rho_J p$ is seen to be bounded by

$$C_\alpha \rho_J^{d-2+|\alpha|} t \int_{\mathbb{R}^d \setminus [-1,1]^d} |p|^{|\alpha|} e^{-c(t\theta_J^{1/2}|p|)^{1/4}} dp \leqslant C_\alpha \theta_J^{-(|\alpha|+d)/2} \rho_J^{d-2+|\alpha|} t^{1-|\alpha|-d} e^{-c'(\theta_J^{1/2} t)^{1/4}}. \tag{2.65}$$

Also using $e^{-c(t\theta_J^{1/2})^{1/4}} \leqslant C_n (t\theta_J^{1/2})^{-n} e^{-\frac{1}{2}c(t\theta_J^{1/2})^{1/4}}$ with $n = d + |\alpha|$,

$$C_\alpha \rho_J^{d-2+|\alpha|} t \int_{\mathbb{R}^d \setminus [-1,1]^d} |p|^{|\alpha|} e^{-c(t\theta_J^{1/2}|p|)^{1/4}} dp \leqslant C'_\alpha \theta_J^{-(d+|\alpha|)} \rho_J^{-2} t \Big(\frac{\rho_J}{t^2}\Big)^{d+|\alpha|} e^{-\frac{1}{2}c'(\theta_J^{1/2} t)^{1/4}}. \tag{2.66}$$

Now using this bound and assuming θ_J^{-1} bounded, $t \geqslant \rho_J$, we directly have the required bound.

Since all estimates above are uniform in m^2, the continuity claim of Proposition 2.1.2 (ii) is immediate. $\square$

2.5 Proof of Proposition 2.1.2: (iii)

Next we collect the last piece of our proof of Proposition 2.1.2, which are the asymptotics of the covariances in two dimensions.

Proposition 2.5.1. *Let $d = 2$. Then for $|s| \leqslant \varepsilon_s \theta_J$, as $\rho_J/t + (\rho_J^4 v_J^{-2} + \theta_J^{-2} v_J^2)/t^2 \to 0$,*

$$D_t^{\mathbb{Z}^2}(0,0;s) = \int_{[-\pi,\pi]^2} \hat{D}(p;s) \frac{dp}{(2\pi)^2} = \frac{1}{2\pi t(v_J^2 + s)} \Big(1 + O\Big(\frac{\rho_J}{t} + \frac{\rho_J^4 v_J^{-2}}{t^2} + \frac{\theta_J^{-2} v_J^2}{t^2}\Big)\Big). \tag{2.67}$$

Proof. To estimate the integral over (2.45), we will approximate

$$\frac{1}{4\lambda} \int_{[0,\infty)\times[\rho_J,\infty)^{2l+1} : \sum t_i = (t-\rho_J)/4} (\lambda_J - s\lambda 1_{t_0 > \rho_J}) \hat{D}_{t_0} \prod_{i=1}^{2l+1} \lambda \hat{D}_{t_i} \, dt_i \, dt_0 \tag{2.68}$$

as follows: replace $P_t(\lambda)$ by $f(\sqrt{\lambda} t)$ using (2.35); replace λ by $|p|^2$ and λ_J by $v_J^2 |p|^2$ using (2.23); remove the constraints $t_i > \rho_J$ from the integration domain and similarly $t_0 \geqslant \rho_J$ from the integrand; and replace $(t - \rho_J)/4$ by $t/4$. After these approximations (which we will justify afterwards, in inverse order), we are left with

$$
\begin{aligned}
&\frac{1}{4|p|^2}(v_J^2-s)\int_{[0,\infty)^{2l+2}:\sum t_i=t/4}\prod_{i=0}^{2l+1}|p|^2\rho_J^{-2}t_i f\big(\rho_J^{-1}t_i v_J|p|\big)\,dt_i\\
&=\frac{1}{4|p|^2}(v_J^2-s)\int_{[0,\infty)^{2l+2}:\sum t_i=t/4}\prod_{i=0}^{2l+1}|p|^2\rho_J^{-2}t_i f\big(\rho_J^{-1}t_i v_J|p|\big)\,dt_i\\
&=\frac{1}{|p|^2}(v_J^2-s)t^{-1}\int_{[0,\infty)^{2l+2}:\sum u_i=1}\prod_{i=0}^{2l+1}\frac{1}{4^2}\rho_J^{-2}t^2|p|^2u_i f\big(\tfrac{1}{4}u_i\rho_J^{-1}t\,v_J|p|\big)\,du_i\\
&=\frac{1}{|p|^2}(v_J^2-s)v_J^{-4l-4}t^{-1}\tilde f_{2l}\big(\tfrac{1}{4}\rho_J^{-1}t\,v_J|p|\big),
\end{aligned}
\tag{2.69}
$$

with

$$
\tilde f_{2l}(y)=\int_{[0,\infty)^{2l+2}:\sum u_i=1}\prod_{i=0}^{2l+1}f(yu_i)y^2u_i\,du_i,\qquad (y\in[0,\infty)).
\tag{2.70}
$$

Note that $\tilde f_{2l}(0)=0$, that $\tilde f_{2l}$ decays rapidly, and that for all $l\in\mathbb{N}$ and $t>0$,

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{dp}{|p|^2}\tilde f_{2l}(t|p|)&=\int_{\mathbb{R}^2}\frac{dp}{|p|^2}\tilde f_{2l}(|p|)\\
&=2\pi\int_0^\infty\frac{dy}{y}\tilde f_{2l}(y)\\
&=2\pi\int_0^\infty\frac{dy}{y}\int_{[0,\infty)^{2l+2}:\sum u_i=1}\prod_{i=0}^{2l+1}f(yu_i)y^2u_i\,du_i\\
&=2\pi\int_{[0,\infty)^{2l+2}}\prod_{i=0}^{2l+1}f(u_i)u_i\,du_i=2\pi\left(\int_0^\infty f(u)u\,du\right)^{2l+2}=2\pi
\end{aligned}
\tag{2.71}
$$

where the last equality follows from Lemma 2.3.1. By definition, $D_t^{\mathbb{Z}^2}(0,0;s)$ is the integral of (2.45) over $p\in[-\pi,\pi]^2$ with respect to $dp/(2\pi)^2$, and (2.45) is the sum over (2.68) multiplied by s^{2l}. Using the above approximation (2.69) for (2.68) and then replacing the integration domain $[-\pi,\pi]^2$ by $\mathbb{R}^2$, we obtain the main contribution to $D_t^{\mathbb{Z}^d}(0,0;s)$ as

$$
\frac{1}{2\pi t}(v_J^2-s)v_J^{-4}\sum_{l=0}^{\infty}(v_J^{-2}s)^{2l}=\frac{1}{2\pi t}(v_J^2-s)v_J^{-4}(1-v_J^{-4}s^2)^{-1}=\frac{1}{2\pi t}(v_J^2+s)^{-1}.
\tag{2.72}
$$

In the remainder of the proof, we show that the approximations we made above are smaller than the claimed error term.

Error from replacing $(t-\rho_J)/4$ *by* $t/4$*:* the same computation with $t/4$ instead of $(t-\rho_J)/4$ gives

$$\frac{1}{2\pi(t-\rho_J)}(v_J^2+s)^{-1}=\frac{1}{2\pi t}(v_J^2+s)^{-1}(1+\frac{\rho_J}{t-\rho_J})=\frac{1}{2\pi t}(v_J^2+s)^{-1}\Big(\big(1+O(\frac{\rho_J}{t})\big)\Big), \quad (2.73)$$

so the error is smaller than claimed.

Error from extending the integral from $p\in[-\pi,\pi]^2$ *to* $p\in\mathbb{R}^2$*:* By changing to polar coordinates, this error is of order

$$\begin{aligned}&\frac{1}{t}(v_J^2-s)v_J^{-4}\sum_{l=0}^{\infty}(v_J^{-2}s)^{2l}\int_{\mathbb{R}^2\setminus[-\pi,\pi]^2}\frac{dp}{|p|^2}\tilde f_{2l}(\frac{1}{4}\rho_J^{-1}tv_J|p|)\\&\leqslant\frac{2\pi}{t}(v_J^2-s)v_J^{-4}\int_{\rho_J^{-1}tv_J\pi/4}^{\infty}\frac{dy}{y}\sum_{l=0}^{\infty}(v_J^{-2}s)^{2l}\int_{[0,\infty)^{2l+2}:\sum u_i=1}\prod_{i=0}^{2l+1}r^2u_if(ru_i)du_i \qquad (2.74)\end{aligned}$$

By Lemma 2.4.2, the right-hand side is bounded, up to some absolute multiplicative factor, by

$$\frac{1}{t}(v_J^2-s)v_J^{-4}\int_{\rho_J^{-1}tv_J\pi/4}^{\infty}dy\,e^{-cy^{1/4}}=O\Big(\frac{e^{-c'(\rho_J^{-1}tv_J)^{1/4}}}{v_J^2t}\Big)=\frac{1}{2\pi tv_J^2}O\Big(\frac{\rho_Jv_J^{-1}}{t}\Big)=\frac{1}{2\pi tv_J^2}O\Big(\frac{\rho_J}{t}\Big) \qquad (2.75)$$

where we used that $v_J\geqslant 1/4$ for all J.

Error from removing the restriction on $t_0\geqslant\rho_J$ *from the integration:* The error is bounded by

$$\begin{aligned}&\int_{\mathbb{R}^2}dp\,\frac{1}{\lambda}\sum_{l=0}^{\infty}s^{2l+1}\int_{[0,\infty)^{2l+2}:\sum t_i=t/4}1_{t_0\leqslant\rho_J}\prod_{i=0}^{2l+1}|p|^2\rho_J^{-2}t_if(v_J\rho_J^{-1}|p|t_i)dt_i\\&\leqslant\int_{\mathbb{R}^2}dp\,\frac{v_J^{-2}}{\lambda}\sum_{l=0}^{\infty}(v_J^{-2}s)^{2l+1}\int_0^{\rho_J}\Big(\int_{[0,\infty)^{2l+1}:\sum t_i=t/4-t_0}\prod_{i=0}^{2l+1}v_J^2\rho_J^{-2}|p|^2t_if(v_J\rho_J^{-1}|p|t_i)\,dt_i\Big)dt_0\\&\leqslant Cs\int_{\mathbb{R}^2}dp\,\rho_J^{-4}|p|^2\int_0^{\rho_J}t_0(t/4-t_0)e^{-c'(v_J\rho_J^{-1}|p|(t/4-t_0))^{1/4}}f(v_J\rho_J^{-1}|p|t_0)dt_0 \qquad (2.76)\end{aligned}$$

where the final inequality follows from Lemma 2.4.2 and the fact that $f(x)\leqslant Ce^{-c|x|^{1/2}}$ which follows from (2.34)–(2.35). But since $e^{-c'(v_J\rho_J^{-1}|p|(t/4-t_0))^{1/4}}f(v_J\rho_J^{-1}|p|t_0)\leqslant C'e^{-c''(v_J\rho_J|p|t)^{1/4}}$ for some $c'',C'>0$ and $t/4-t_0\geqslant t/20$ because $t\geqslant 5\rho_J\geqslant 5t_0$, the last integral is bounded by

$$Cs\int_{\mathbb{R}^2}\rho_J^{-2}|p|^2te^{-c''(v_J\rho_J^{-1}|p|t)^{1/4}}dp\leqslant\frac{C|s|\rho_J^2}{v_J^4t^3}\leqslant\frac{1}{2\pi v_J^2t}O\Big(\frac{\rho_J^2}{t^2}\Big)\leqslant\frac{1}{2\pi v_J^2t}O\Big(\frac{\rho_J}{t}\Big), \qquad (2.77)$$

where the second inequality holds because $|s| \leqslant \varepsilon_s \theta_J \leqslant c\varepsilon_s v_J^2$ and the final inequality because $t \geqslant 5\rho_J$.

Error from removing the restriction on $t_j \geqslant \rho_J$ from the integration: Similarly as above, the error for removing the restriction on t_j ($j \geqslant 1$) in the integral $\int_{t_j \geqslant \rho_J, \sum_i t_i = t/4}$ is bounded by

$$\int_{[0,\infty)^{2l+2}:\sum t_i = t/4} 1_{t_j \leqslant \rho_J} v_J^2 \rho_J^{-2} |p|^2 t_0 f(v_J \rho_J^{-1}|p|t_0) \prod_{i=1}^{2l+1} |p|^2 \rho_J^{-2} t_i f(v_J \rho_J^{-1}|p|t_i) dt_i dt_0$$
$$\leqslant (Cv_J^{-2})^{2l+1} \int_{[0,\infty)^{2l+2}:\sum t_i = t/4} 1_{t_j \leqslant \rho_J} \prod_{i=0}^{2l+1} v_J^2 |p|^2 \rho_J^{-2} t_i e^{-c(v_J \rho_J^{-1}|p|t_i)^{1/2}} dt_i. \tag{2.78}$$

But since the last expression is symmetric in j, we can just replace $1_{t_j \leqslant \rho_J}$ by $1_{t_0 \leqslant \rho_J}$, so summing the errors over $j \in \{1, \cdots, 2l+1\}$ and applying $\int_{\mathbb{R}^2} dp\, \lambda^{-1} \sum_{l=0}^{\infty} s^{2l+1}$ gives the bound

$$\int_{\mathbb{R}^2} dp \frac{1}{|p|^2} \sum_{l=0}^{\infty} (2l+1)(Cv_J^{-2}s)^{2l+1} \int_{[0,\infty)^{2l+2}:\sum t_i = t/4} 1_{t_0 \leqslant \rho_J} \prod_{i=0}^{2l+1} v_J^2 |p|^2 \rho_J^{-2} t_i e^{-c(v_J \rho_J^{-1}|p|t_i)^{1/2}} dt_i$$
$$\leqslant C \int_{\mathbb{R}^2} dp\, v_J^4 \rho_J^{-4} \int_0^{\rho_J} t_0 (t/4 - t_0) e^{-c'(v_J \rho_J^{-1}|p|(t/4 - t_0))^{1/4}} f(v_J \rho_J^{-1}|p|t_0) dt_0. \tag{2.79}$$

Comparing this with (2.76), this integral is bounded by $\frac{1}{2\pi v_J^2 t} O(\frac{\rho_J^2}{t^2}) = \frac{1}{2\pi v_J^2 t} O(\frac{\rho_J}{t})$ because $t \geqslant 5\rho_J$.

Error from replacement of $\lambda_J(p)$ by $v_J^2|p|^2$ and $\lambda(p)$ by $|p|^2$: As in the argument following (2.62), the contribution from $|p| \geqslant \rho_J^{-1}$ is bounded by

$$\int_{|p| \geqslant \rho_J^{-1}} dp\, \hat{D}_t(p; s, m^2) \leqslant C_0 \theta_J^{-2} t^{-3} e^{-c(\theta_J^{1/2} t)^{1/4}} \leqslant \frac{1}{2\pi v_J^2 t} O(\frac{\theta_J^{-2} v_J^2}{t^2}). \tag{2.80}$$

It remains to consider the contribution coming from $|p| \leqslant \rho_J^{-1}$. But then by Lemma 2.2.1,

$$0 \leqslant |p|^2 - \lambda(p) \leqslant O(|p|^4) \leqslant O(|p|^2) \tag{2.81}$$
$$0 \leqslant v_J^2 |p|^2 - \lambda_J(p) \leqslant O(\rho_J^2 v_J^2 |p|^4) \leqslant O(v_J^2 |p|^2). \tag{2.82}$$

With $\hat{\kappa}$ as in in the proof of Lemma 2.3.1, we have $f = c\hat{\kappa}^2$, so

$$
\begin{aligned}
|f(\rho_J^{-1}t\sqrt{\lambda_J}) - f(v_J\rho_J^{-1}t|p|)| &\leqslant C\rho_J^{-1}t(v_J|p| - \sqrt{\lambda_J})\max\{\hat{\kappa}(\rho_J^{-1}t\lambda_J), \hat{\kappa}(v_J\rho_J^{-1}t|p|)\} \\
&\leqslant C\rho_J^{-1}t\min\left\{1, \frac{\rho_J^2 v_J^2}{2v_J}|p|^3\right\}e^{-\frac{1}{2}(\sqrt{\lambda_J}\rho_J^{-1}t)^{1/2}} \\
&\leqslant C\rho_J^4 v_J^{-2}t^{-2}e^{-c(v_J\rho_J^{-1}t|p|)^{1/2}} \qquad (2.83)
\end{aligned}
$$

where the first inequality holds because $\|\hat{\kappa}'\|_\infty < \infty$ and the second because $\kappa(x)$ is decreasing in $|x|$ and $|\hat{\kappa}(x)| \leqslant Ce^{-\frac{1}{2}|x|^{1/2}}$. Thus the error from this approximation is, up to an absolute multiplicative factor, bounded by

$$
\frac{\rho_J^4 v_J^{-2}}{t^2}\int_{|p|\leqslant\rho_J^{-1}}\frac{dp}{|p|^2}\sum_{l=0}^{\infty}(2l+2)(C's)^{2l+1}\int_{[0,\infty)^{2l+2}:\sum t_i = t/4}(v_J^2+s)\prod_{i=0}^{2l+1}|p|^2\rho_J^{-2}t_i e^{-c(v_J\rho_J^{-1}|p|t_i)^{1/2}}. \qquad (2.84)
$$

Since $|s| \leqslant \varepsilon_s\theta_J \leqslant O(v_J^2)$, this error is again of order $\frac{1}{2\pi v_J^2 t}O(\frac{\rho_J^4 v_J^{-2}}{t^2})$, comparing this expression with (2.69).

Error from replacement $P_t(x)$ by $f(\sqrt{x}t)$: We consider the difference between

$$
(1-s)\int_{[0,\infty)^{2l+2}:\sum t_i = t/4}\lambda_J\rho_J^{-2}t_0 P_{\rho_J^{-1}t_0}(\lambda_J)\prod_{i=1}^{2l+1}\lambda\rho_J^{-2}t_i P_{\rho_J^{-1}t_i}(\lambda_J)\,dt_i \qquad (2.85)
$$

and

$$
(1-s)\int_{[0,\infty)^{2l+2}:\sum t_i = t/4}\lambda_J\rho_J^{-2}t_0 f(\sqrt{\lambda_J}\rho_J^{-1}t_0)\prod_{i=1}^{2l+1}\lambda\rho_J^{-2}t_i f(\sqrt{\lambda_J}\rho_J^{-1}t_i)\,dt_i. \qquad (2.86)
$$

By (2.35), one has $P_{\rho_J^{-1}t}(\lambda_J) - f(\sqrt{\lambda_J}\rho_J^{-1}t) = (\rho_J/t)g(\sqrt{\lambda_J}t)$ with $g(x) = Ce^{-c\sqrt{x}}$. This is essentially the same bound as $P_t(\lambda_J)$ or $f(\sqrt{\lambda_J}t)$ except for an additional factor ρ_J/t. Therefore, again using Lemma 2.4.1, the difference between the above two displays is bounded by

$$
O\left(\frac{\rho_J}{t}C^l v_J^{-4l+2}\frac{1}{t}\tilde{g}_{2l}(\sqrt{\lambda_J}\rho_J^{-1}t)\right), \qquad (2.87)
$$

when $\tilde{g}_{2l}$ is defined analogously to $\tilde{f}_{2l}$. As in (2.71) the integral of $\tilde{g}_{2l}(t|p|)$ over $dp/|p|^2$ is bounded by $2\pi C^{2l}$ with $C \geqslant \int_0^\infty g(u)u\,du$, for all $t > 0$. Hence possibly decreasing $|s|$ relative to C we obtain the claimed relative error $O(\rho_J/t)$.

Summing up the bounds gives the claimed error. □

2.6 Proof of Proposition 2.1.4

Having proved the estimates for the full plane covariance decomposition, the torus analogue is not difficult to prove.

Proof of Proposition 2.1.4. By definition,

$$D_t^{\mathbb{Z}^d}(0,x;s,m^2) = \int_{[-\pi,\pi]^d} e^{ip\cdot x}\hat{D}_t(p;s,m^2)\,\frac{dp}{(2\pi)^d} \tag{2.88}$$

and we define

$$D_t^{\Lambda_N}(0,x;s,m^2) = \frac{1}{|\Lambda_N|}\sum_{p\in\Lambda_N^*} e^{ip\cdot x}\hat{D}_t(p;s,m^2), \tag{2.89}$$

where $\Lambda_N^* \subset (-\pi,\pi]^d$ is the dual torus. For $t < L^N$ the finite-range property and Poisson summation (2.21) imply that

$$D_t^{\mathbb{Z}^d}(0,x;s,m^2) = D_t^{\Lambda_N}(0,x;s,m^2). \tag{2.90}$$

So we are only left to prove (2.16) and the bound on $\tilde{D}_t^{\Lambda_N}$. Let $t_N = \int_{\frac{1}{4}L^{N-1}}^{\infty} \hat{D}_t(0;s,m^2)\,dt$ and

$$\tilde{D}_t^{\Lambda_N}(0,x;s,m^2) = \frac{1}{|\Lambda_N|}\sum_{p\in\Lambda_N^*\setminus\{0\}} e^{ip\cdot x}\hat{D}_t(p;s,m^2). \tag{2.91}$$

To see the bound for t_N, just notice that

$$t_N = \int_0^\infty \hat{D}_t(0;s,m^2)dt - \int_0^{\frac{1}{4}L^{N-1}} \hat{D}_t(0;s,m^2)dt \leqslant m^{-2} - C\rho_J^{-2}L^{2N-2} \tag{2.92}$$

by (2.41) and Proposition 2.4.3. The proof of the bound on $\tilde{D}_t^{\Lambda_N}$ is analogous to the argument below (2.62) using that all p that contribute satisfy $|p| > 2\pi L^{-N}$ and that $t > \frac{1}{2}L^N$. Indeed,

$$\frac{1}{|\Lambda_N|}\sum_{p\neq 0}\lambda^{|\alpha|/2}\hat{D}_t(p;s,m^2) \leqslant \frac{1}{|\Lambda_N|}\sum_{p\neq 0}\lambda^{|\alpha|/2}\rho_J^{-2}te^{-\tilde{c}(t\rho_J^{-1}\sqrt{\lambda_{J,m^2}}))^{1/4}}. \tag{2.93}$$

The contribution from $2\pi L^{-N} < |p| \leqslant \rho_J^{-1}$ is

$$\begin{aligned}
&\rho_J^{-2}t(\rho_J^{-1}v_Jt)^{-|\alpha|}\frac{1}{|\Lambda_N|}\sum_{0<|p|\leqslant\rho_J^{-1}}(\rho_J^{-1}v_Jt|p|)^{|\alpha|}e^{-c(\rho_J^{-1}v_Jt|p|)^{1/4}} \\
&\leqslant C_\alpha\rho_J^{-2}t(\rho_J^{-1}v_Jt)^{-|\alpha|}\frac{1}{|\Lambda_N|}\sum_{0<|p|\leqslant\rho_J^{-1}} e^{-\frac{1}{2}c(\rho_J^{-1}v_Jt|p|)^{1/4}},
\end{aligned} \tag{2.94}$$

but since $r \mapsto e^{-cr^{1/4}}$ is decreasing for $r \geqslant 0$ and $2\pi L^{-N} < \rho_J^{-1}$, we have the domination

$$\frac{1}{|\Lambda_N|} \sum_{0<|p|\leqslant\rho_J^{-1}} e^{-\frac{1}{2}c(\rho_J^{-1}v_J t|p|)^{1/4}} \leqslant 2d \int_{|p|\leqslant 2\rho_J^{-1}} e^{-\frac{1}{2}c(\rho_J^{-1}v_J|p|t)^{1/4}}\, dp. \tag{2.95}$$

Hence the contribution from $|p| \leqslant \rho_J^{-1}$ is bounded by $C''_\alpha \rho_J^{-2} t(\frac{\rho_J}{v_J t})^{|\alpha|+d}$. Finally, the contribution from $|p| \geqslant \rho_J^{-1}$ is bounded by

$$\frac{C}{|\Lambda_N|} \sum_{|p|\geqslant\rho_J^{-1}} |p|^{|\alpha|} \rho_J^{-2} t e^{-c(\rho_J^{-1}\theta_J^{1/2}|p|t)^{1/4}} \leqslant C_\alpha \rho_J^{-2} t(\rho_J^{-1}\theta_J^{-1/2}t)^{-|\alpha|} \sum_{|p|\geqslant\rho_J^{-1}} e^{-\frac{c}{2}(\rho_J^{-1}\theta_J^{1/2}|p|t)^{1/4}} \tag{2.96}$$

but again by the same domination, the estimate for $|\nabla^\alpha \tilde{D}_t^{\Lambda_N}|$ is the same as that for $|\nabla^\alpha D_t|$. The claim that $D_t^{\Lambda_N}(s,m^2)$ is continuous in m^2 and attains a limit as $m^2 \downarrow 0$ is deduced from the fact that the partial absolute sums $\sum_{|p|\leqslant R} |\hat{D}_t(p;s,m^2)|$ have a bound uniform in R and m^2. $\square$

Appendix 2.A Fourier transform of the standard bump function

In the proof of Lemma 2.3.1, the decay rate of the Fourier transform of the standard bump function κ was used. Since we were unable to locate a reference, we include the elementary proof here.

Proposition 2.A.1. *Define $\kappa(x) = e^{-\frac{1}{1-4x^2}} 1_{|x|<1/2}$ for $x \in \mathbb{R}$ and $\hat{\kappa}(p)$ to be its Fourier transform. Then $\hat{\kappa}(p) = O(e^{-|p|^{1/2}})$.*

Proof. Letting $\tau(x) = \kappa(x/2) = e^{-1/(1-x^2)} 1_{|x|<1}$, it is sufficient to prove $\hat{\tau}(p) = O(e^{-|2p|^{1/2}})$. One has $\hat{\tau}(p) = \int_{(-1,1)} e^{-ipx-\frac{1}{1-x^2}} dx$. Since τ is analytic and bounded on the rectangle $S = \{z \in \mathbb{C} : \mathrm{Re}(z) \in (-1,1),\ \mathrm{Im}(z) \in (-2,2)\}$, one may write alternatively

$$\hat{\tau}(p) = \int_{\Gamma_- \cup \Gamma_+} e^{-ipz-\frac{1}{1-z^2}} dz = 2\,\mathrm{Re}\left[\int_{\Gamma_+} e^{-ipz-\frac{1}{1-z^2}} dz\right] \tag{2.97}$$

where $\Gamma_\pm = \{\pm 1 + (\mp 1 + i)t \in \mathbb{C} : t \in (0,1]\}$ (with orientations as appropriate). Without loss of generality, take $p > 0$. Then change of parameter $v = (\frac{1+i}{\sqrt{2}})^{-1}\sqrt{2p}(1-z)$ gives

$$G(p) := \int_{\Gamma_+} e^{-ipz-\frac{1}{1-z^2}} dz = \frac{1}{\sqrt{2p}} e^{-ip-\frac{1}{4}} \int_0^{2\sqrt{p}} e^{-\sqrt{p}\frac{1-i}{\sqrt{2}}(v+v^{-1}) - g(\frac{1+i}{2\sqrt{p}}v)} dv \tag{2.98}$$

where $g(x) = \frac{x}{4(2-x)}$. Since $g(\frac{1+i}{2\sqrt{p}}v)$ is bounded uniformly on $v \in [0, 2\sqrt{p})$, there is $C > 0$ such that

$$|G(p)| \leqslant \frac{C}{2\sqrt{p}} \int_0^{2\sqrt{p}} e^{-\sqrt{\frac{p}{2}}(v+v^{-1})} dv \leqslant Ce^{-\sqrt{2p}} \tag{2.99}$$

utilising $v + v^{-1} \geqslant 2$.

□

Chapter 3

Polymer activities

In this chapter, we discuss the polymer activities, and the building blocks of the polymer expansions. The polymer activity depends on a polymer X and a field φ. It is allowed to be any real-analytic function in φ with a restricted growth rate as $\nabla\varphi \to \infty$ and decay as $|X| \to \infty$. They deal with the problem of large fields and the large set, respectively. The main results of this chapter are Propositions 3.3.5 and 3.3.7 which discuss how the large field bound is preserved under progressive integrals, i.e., integrating the fluctuation field at scale j.

3.1 Scales, polymers and polymer activities

In this chapter, Λ_N always denotes a discrete torus of side length L^N, for integers $L \in 2\mathbb{N}+3$, $N \geqslant 1$. Later we will further assume that $L = \ell^{N'}$ for integers $\ell \in 2\mathbb{N}+3$, $N' \geqslant 1$. We also let $\pi_N : \mathbb{Z}^d \to \Lambda_N$ be the canonical projection with $0 = \pi_N(0) \in \Lambda_N$.

3.1.1 Blocks and polymers

We follow the setup for the renormalisation group coordinates of [23]. For any scale $j = 0, 1, \dots, N$, we call *j-block* any set $B = x + ([-\frac{L^j-1}{2}, \frac{L^j-1}{2}] \cap \mathbb{Z})^d$ for $x \in L^j\mathbb{Z}^d$. The j-blocks in Λ_N are the projections $\pi_N(B)$. The set of j-blocks is denoted by $\mathscr{B}_j \equiv \mathscr{B}_j(\Lambda^*)$ for either $\Lambda^* = \mathbb{Z}^2$ or Λ_N. It induces a partition of Λ^* into j-blocks. A *j-polymer* is any set X which is obtained as a finite union of j-blocks, and we then denote by $\mathscr{B}_j(X) \subset \mathscr{B}_j$ the set of j-blocks contained in X. The set of j-polymers is denoted by $\mathscr{P}_j \equiv \mathscr{P}_j(\Lambda^*)$. Note that the family $\mathscr{P}_j$ is decreasing in j. For $X \in \mathscr{P}_j$, its closure $\overline{X} \in \mathscr{P}_{j+1}$ is the union of all $(j+1)$-blocks which intersect X, i.e., $\overline{X}$ is the 'smallest' $Y \in \mathscr{P}_{j+1}$ such that $X \subset Y$.

Next, a *connected polymer* is a polymer $X \neq \emptyset$ which forms a connected set in ℓ^∞-sense. Two connected polymers X_1, X_2 are called *connected* if $X_1 \cup X_2$ is a connected polymer; this is denoted by $X_1 \sim X_2$ and we write $X_1 \not\sim X_2$ if X_1 and X_2 are not connected. The set of connected j-polymers is denoted by $\mathscr{P}_j^c \equiv \mathscr{P}_j^c(\Lambda^*)$. It is worth highlighting that $\emptyset \notin \mathscr{P}_j^c$ by this definition. For $X \in \mathscr{P}_j$ we write $\mathrm{Comp}_j(X) \subset \mathscr{P}_j^c$ for the set of constituting connected polymers, i.e., each $Y \in \mathrm{Comp}_j(X)$ is a maximal connected polymer in X and the union over all such Y is X. A connected polymer $X \in \mathscr{P}_j^c$ is called a *small set* if $|X|_j \leqslant 2^d$, where $|X|_j = |\mathscr{B}_j(X)|$ denotes the number of j-blocks contained in X. We write $\mathscr{S}_j \equiv \mathscr{S}_j(\Lambda^*)$ for the set of small sets (at scale j). For any $X \in \mathscr{P}_j$, we define its *small-set neighbourhood* as $X^* = \bigcup S$ where the union ranges over all $S \in \mathscr{S}_j$ such that $S \cap X \neq \emptyset$. Finally, we call *polymer functions* for functions having polymers as their arguments.

For later reference, we note that the combinatorial results of Lemmas 6.15–6.19 from [23, Section 6.4] all hold in the present setup.

3.1.2 Massless finite-range decomposition

As usual, let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be a finite-range step distribution that is invariant under lattice symmetries, and recall the finite-range decomposition of the associated covariance matrix $C^\Lambda(s,m^2)$ from Propositions 2.1.2 and 2.1.4. To simplify the conditions, we will from now on always assume that $d = 2$ and that there is a constant $C > 0$ such that the parameters from (2.3)–(2.5) satisfy

$$|s| \leqslant \varepsilon_s \theta_J, \qquad \theta_J \geqslant C^{-1}, \qquad C^{-1}\rho_J \leqslant v_J \leqslant \rho_J/2. \tag{3.1}$$

All constants in the sequel are permitted to depend on this constant C but will be otherwise independent of J. In particular, this assumption holds for any fixed J as in the statement of Theorem 1.1.1, and it also holds uniformly in ρ for the standard range-ρ distribution J_ρ discussed in Example 2.1.1.

Since D_t is independent of Λ_N for scales $< \frac{1}{4}L^{N-1}$, setting $D_t = 0$ for $t < \rho_J$ (cf. (2.7)), we define for $j \geqslant 0$,

$$\begin{aligned} \Gamma_{j+1}(s,m^2) &= \int_{\frac{1}{4}L^j}^{\frac{1}{4}L^{j+1}} D_t(s,m^2)\,dt, \\ \Gamma_N^{\Lambda_N}(s,m^2) &= \int_{\frac{1}{4}L^{N-1}}^{\infty} \tilde{D}_t^{\Lambda_N}(s,m^2)\,dt, \end{aligned} \tag{3.2}$$

and set $\Gamma_{j,j'} = \sum_{k=j+1}^{j'} \Gamma_k$ so that, in view of (2.15), we obtain

$$\begin{aligned} C^{\Lambda_N}(s,m^2) &= \Gamma_1(s,m^2) + \cdots + \Gamma_{N-1}(s,m^2) + \Gamma_N^{\Lambda_N}(s,m^2) + t_N(s,m^2)Q_N \\ &= \Gamma_{0,N-1}(s,m^2) + \Gamma_N^{\Lambda_N}(s,m^2) + t_N(s,m^2)Q_N. \end{aligned} \tag{3.3}$$

The matrices Γ_j have range $\frac{1}{4}L^j$ by (2.8) and satisfy the following bounds, which are straightforward consequences of Propositions 2.1.2 and 2.1.4.

Corollary 3.1.1. *Let $d = 2$ and assume* (3.1)*. Then Γ_{j+1} is analytic in $|s| < \varepsilon_s \theta_J$,*

$$|\nabla^\alpha \Gamma_{j+1}(0,x;s)| \leqslant \begin{cases} C_\alpha \rho_J^{-2} L^{-j|\alpha|} & \text{if } |\alpha| \geqslant 1 \\ C_0 \rho_J^{-2} \log L & \text{if } \alpha = 0 \end{cases} \tag{3.4}$$

and

$$\Gamma_{j+1}(0,0;s) = \frac{\log L}{2\pi(v_J^2 + s)} + O(\rho_J^{-1} L^{-j}), \tag{3.5}$$

and the estimates (3.4) *also hold for $\Gamma_N^{\Lambda_N}$ and we have $t_N(s,m^2) = m^{-2} + O(\rho_J^{-2} L^{2N})$.*

We are ultimately interested in taking $m^2 \downarrow 0$. While the zero mode $t_N(s,m^2)$ diverges as m^{-2} as $m^2 \downarrow 0$ like the torus Green function, the covariances Γ_j and their discrete derivatives are continuous as $m^2 \downarrow 0$, and this allows to directly set $m^2 = 0$ in these. This is made precise by the following lemma. To simplify notation, we let $\mathbb{E}_\Gamma^\zeta$ be the expectation of a Gaussian field ζ with covariance Γ. We omit Γ or ζ whenever the choices are clear from the context, and we will abbreviate from now on $\Gamma_j = \Gamma_j(s) = \Gamma_j(s,0)$ and $\Gamma_N^{\Lambda_N} = \Gamma_N^{\Lambda_N}(s) = \Gamma_N^{\Lambda_N}(s,0)$, i.e., m^2 is set to 0 and the dependence on s is often made implicit.

Lemma 3.1.2. *Let s be as in* (3.1)*, let $\kappa < \theta_J + s$, and let $F : \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function satisfying $|F(\varphi)| \leqslant e^{\kappa(\nabla\varphi,\nabla\varphi)}$. Then as $m^2 \downarrow 0$,*

$$\mathbb{E}_{C^{\Lambda_N}(s,m^2)} F \sim \mathbb{E}^{\varphi'}_{t_N(s,m^2)Q_N} \mathbb{E}^{\zeta}_{\Gamma_{0,N-1}(s) + \Gamma_N^{\Lambda_N}(s)} F(\varphi' + \zeta), \tag{3.6}$$

where on the right-hand side φ' is (centered) Gaussian with covariance $t_N(s,m^2)Q_N$ and ζ is (centered) Gaussian with covariance $\Gamma_{0,N-1}(s) + \Gamma_N^{\Lambda_N}(s)$, and $a \sim b$ means $\lim a/b = 1$.

Proof. Provided sufficient integrability holds, by (3.3) and the fact that the sum of independent Gaussian vectors is Gaussian with covariance the sum of the convariances, we have the identity

$$\mathbb{E}_{C^{\Lambda_N}(s,m^2)} F = \mathbb{E}^{\varphi'}_{t_N(s,m^2)Q_N} \mathbb{E}^{\zeta}_{\Gamma_1(s,m^2) + \cdots + \Gamma_{N-1}(s,m^2) + \Gamma_N^{\Lambda_N}(s,m^2)} F(\varphi' + \zeta) \tag{3.7}$$

and thus

$$1 \sim \frac{\mathbb{E}^{\varphi'}_{t_N(s,m^2)Q_N}\mathbb{E}^{\zeta}_{\Gamma_1(s,0)+\cdots+\Gamma_{N-1}(s,0)+\Gamma_N^{\Lambda_N}(s,0)}F(\varphi'+\zeta)}{\mathbb{E}_{C^{\Lambda_N}(s,m^2)}F} \qquad (m^2 \downarrow 0), \tag{3.8}$$

where we used that $\Gamma_j(s,m^2)$ and $\Gamma_N^{\Lambda_N}(s,m^2)$ are continuous as $m^2 \downarrow 0$ which implies that the inner Gaussian expectation in the numerator is continuous as $m^2 \downarrow 0$ if F is integrable uniformly in m^2.

To see the integrability of the function $\varphi \mapsto e^{\kappa(\nabla\varphi,\nabla\varphi)} = e^{\kappa(\varphi,-\Delta\varphi)}$, it is enough to check that $\kappa(-\Delta) < C^{\Lambda_N}(s,m^2)^{-1}$ for each $m^2 > 0$ and sufficiently small κ. But by definition $C^{\Lambda_N}(s,m^2)^{-1} \geqslant -\Delta_J - s\Delta \geqslant -(\theta_J+s)\Delta$, so this holds as long as $\kappa < \theta_J + s$. □

Now suppose we have a function $Z_0^0(\,\cdot\,|\Lambda_N) : \mathbb{R}^{\Lambda_N} \to \mathbb{R}$ that describes the distribution of the field $\varphi \in \mathbb{R}^{\Lambda_N}$, i.e., the field theory of φ, whose precise form is given later in (6.40). Then functions $Z_j^0(\,\cdot\,|\Lambda_N) : \mathbb{R}^{\Lambda_N} \to \mathbb{R}$ are defined inductively by

$$Z_{j+1}^0(\varphi|\Lambda_N) = \mathbb{E}_{\Gamma_{j+1}}[Z_j^0(\varphi+\zeta|\Lambda_N)], \tag{3.9}$$

where the expectation is taken over $\zeta \sim \mathcal{N}(0,\Gamma_{j+1})$; here we emphasise again that $\Gamma_{j+1} = \Gamma_{j+1}(s,0)$, and we assume that Z_0 is such that the integrals exist. Then by the previous lemma, and again using that the sum of independent Gaussian vectors is Gaussian with covariance the sum of the covariances,

$$\begin{aligned}\mathbb{E}^{\zeta}_{C^{\Lambda_N}(s,m^2)}Z_0(\varphi+\zeta|\Lambda_N) &\sim \mathbb{E}_{\Gamma_N^{\Lambda_N}(s)+t_N(s,m^2)Q_N}Z_{N-1}^0(\varphi+\zeta_N+\zeta_{\hat N}|\Lambda_N) \quad (m^2\downarrow 0),\\ &=: \mathbb{E}_{t_N(s,m^2)Q_N}Z_N^0(\varphi+\zeta_{\hat N}|\Lambda_N)\end{aligned} \tag{3.10}$$

where as usual the expectations are over $\zeta \sim \mathcal{N}(0,C^{\Lambda_N}(s,m^2))$, $\zeta_N \sim \mathcal{N}(0,\Gamma_N^{\Lambda_N}(s))$ and $\zeta_{\hat N} \sim \mathcal{N}(0,t_N(s,m^2)Q_N)$. For the purpose of Theorem 1.1.1, we will see the integral over $\zeta_{\hat N} \sim \mathcal{N}(0,t_N(s,m^2)Q_N)$ as $N \to \infty$ only in Section 9.1. Therefore we can and will focus on the massless covariances $\Gamma_j(s)$ and $\Gamma_N^{\Lambda_N}(s)$ in Sections 3.2–Chapter 8.

We conclude this short section with the following factorisation property implied by the finite range property of the covariances.

Lemma 3.1.3. *Let $X,Y \subset \Lambda_N$ with $\min\{d_\infty(x,y) : x\in X, y\in Y\} \geqslant \frac{1}{4}L^{j+1}$. Then for all functions $F(X) : \mathbb{R}^X \to \mathbb{R}$ and $F(Y) : \mathbb{R}^Y \to \mathbb{R}$ such that the following integrals exist,*

$$\mathbb{E}^{\zeta}_{\Gamma_{j+1}}\Big(F(X,\varphi+\zeta)F(Y,\varphi+\zeta)\Big) = \mathbb{E}^{\zeta}_{\Gamma_{j+1}}(F(X,\varphi+\zeta))\mathbb{E}^{\zeta}_{\Gamma_{j+1}}(F(Y,\varphi+\zeta)). \tag{3.11}$$

In particular, assuming $L \geqslant 2^{d+2}$, this applies if X and Y are scale-$(j+1)$ polymers that do not touch, i.e., X and Y are distinct elements of $\mathrm{Comp}_{j+1}(X \cup Y)$*, and $F(X) : \mathbb{R}^{X^*} \to \mathbb{R}$ and $F(Y) : \mathbb{R}^{Y^*} \to \mathbb{R}$ where X^* and Y^* denote the small set neighbourhoods of X and Y at scale j.*

Proof. (3.11) is immediate from the finite-range property of the covariance Γ_{j+1} (recall that Γ_{j+1} has range at most $\frac{1}{4}L^{j+1}$, cf. (3.2) and (2.8)) and the fact that two jointly Gaussian random variables are independent if their covariance vanishes.

The claim below (3.11) then follows from the fact that if X and Y are scale-$(j+1)$ polymers that do not touch, their ℓ^∞-distance is at least L^{j+1} and their scale-j small set neighbourhoods X^* and Y^* then still have distance at least $L^{j+1} - 2^{d+1}L^j = L^{j+1}(1 - 2^{d+1}L^{-1}) \geqslant \frac{1}{2}L^{j+1} \geqslant \frac{1}{4}L^{j+1}$. □

In the rest of the thesis, we write $\mathbb{E}$ for $\mathbb{E}_{\Gamma_{j+1}}$ (respectively, $\mathbb{E}_{\Gamma_N^{\Lambda_N}}$) when we are working at scale j (respectively, $N-1$) , where $\Gamma_{j+1} = \Gamma_{j+1}(s,0)$ (respectively, $\Gamma_N^{\Lambda_N} = \Gamma_N^{\Lambda_N}(s,0)$) is the covariance introduced below in (3.2). Typically, j without further specification is allowed to take values $j = 1, \dots, N-1$, where $N \geqslant 1$ refers to the exponent of the underlying torus size L^N, for some $L > 1$.

3.1.3 Scale subdecomposition

In some places, it is necessary to subdecompose each Γ_{j+1} further to obtain better integrability and related better contractivity of the renormalisation group map. (For example, in the proof of Proposition 3.3.5 below this subdecomposition allows to choose κ_L of order $1/(\log L)$. Since $1/\kappa_L$ appears in various error terms, this integrability is especially important to get to the critical temperature or close to it). More precisely, we subdecompose each scale j further into fractional scales $j+s$ with

$$s \in I_{N'} := \{0, 1/N', \dots, 1 - 1/N'\} \tag{3.12}$$

where N' is an integer such that $L = \ell^{N'}$ for an integer ℓ. Corresponding to the fractional scales, we define covariances analogously to (3.2), i.e.,

$$\Gamma_{j+s,j+s'} = \int_{\frac{1}{4}L^{j+s}}^{\frac{1}{4}L^{j+s'}} D_t \, dt, \qquad s \in I_{N'},\ s' \in I_{N'} + \frac{1}{N'} \tag{3.13}$$

for $j < N-1$, and for $j = N-1$,

$$\Gamma^{\Lambda_N}_{j+s,j+s'} = \begin{cases} \int_{\frac{1}{4}L^{j+s}}^{\frac{1}{4}L^{j+s'}} \tilde{D}_t^{\Lambda_N}\, dt & \text{if } s' < 1 \\ \int_{\frac{1}{4}L^{j+s}}^{\infty} \tilde{D}_t^{\Lambda_N}\, dt & \text{if } s' = 1. \end{cases} \tag{3.14}$$

In particular $\Gamma_{j,j+1} = \Gamma_{j+1}$. These covariances admit estimates that are analogous to those for Γ_{j+1} in Corollary 3.1.1 and they are again corollaries of Proposition 2.1.2 and Proposition 2.1.4.

Lemma 3.1.4. *Let $d = 2$ and assume* (3.1). *Then for $s \in I_{N'}$ and $s' = s + (N')^{-1}$,*

$$|\nabla^\alpha \Gamma_{j+s,j+s'}| \leqslant \begin{cases} C_\alpha \rho_J^{-2} L^{-(j+s)|\alpha|} & \textit{if } |\alpha| \geqslant 1 \\ C_0 \rho_J^{-2} \log \ell & \textit{if } \alpha = 0. \end{cases} \tag{3.15}$$

and the estimates also hold for $\Gamma^{\Lambda_N}_{N-1+s,N-1+s'}$.

Finally, for each fractional scale $j+s$, we also introduce the corresponding division of the torus into blocks and polymers, exactly as in Section 3.1.1. Thus $\mathscr{P}_{j+s}$ is the set of polymers composed of blocks in $\mathscr{B}_{j+s}$ of (integer) side lengths $L^{j+s} = L^j \ell^k$ if $s = k/N'$. Given $X \in \mathscr{P}_{j+s}$, we define $X_{s'}$ ($s \leqslant s'$) to be the smallest $j+s'$-polymer that contains X, i.e., $X_{s'}$ consists of all blocks of side length $L^{j+s'}$ that intersect X. In particular, $(X_s)_{s'} = X_{s \vee s'}$ and $\overline{X} = X_1$.

3.1.4 Tilted expectations

When we consider the Gaussian integration with covariance Γ, we also consider tilted expectations $\mathbb{E}_{(\omega)}$ given by

$$\mathbb{E}_{(\omega),\Gamma}[F(\zeta)] = \frac{\mathbb{E}_\Gamma[e^{\omega(\tilde{\mathfrak{f}},\zeta)} F(\zeta)]}{\mathbb{E}_\Gamma[e^{\omega(\tilde{\mathfrak{f}},\zeta)}]}, \qquad \tilde{\mathfrak{f}} \in \mathbb{R}^\Lambda \tag{3.16}$$

for a (complex) parameter ω in order to facilitate the computation of the moment generating function, and we drop Γ whenever the choice is clear from the context. The tested function $\tilde{\mathfrak{f}}$ is called the *external field*. We restrict the amount of tilting by restricting $|\omega| < h_\omega$ for h_ω sufficiently small and imposing the following condition on $\tilde{\mathfrak{f}}$ for given $M, \rho > 0$ and $\mathfrak{n} \in \mathbb{Z}_{\geqslant 1}$ and confine the support in a range defined by the *observable scale* j_s.

(A'_f) $\tilde{\mathfrak{f}}$ is decomposed as $\tilde{\mathfrak{f}} = \sum_{i=1}^{\mathfrak{n}} T_{y_i}\tilde{\mathfrak{f}}_i$ where each $\tilde{\mathfrak{f}}_i \in \mathbb{R}^{\mathbb{Z}^2}$ is a function with compact support, $0 \in \mathrm{supp}(\tilde{\mathfrak{f}}_i)$. Also, given an observable scale $j_s \geqslant 0$,

$$\max_{i=1,\cdots,\mathfrak{n}} \{\mathrm{diam}(\mathrm{supp}(\tilde{\mathfrak{f}}_i))\} \leqslant \rho \leqslant \frac{1}{12} L^{j_s+1}, \qquad \max_{i=1,\cdots,\mathfrak{n}} \{\|\tilde{\mathfrak{f}}_i\|_{L^\infty}\} \leqslant M, \tag{3.17}$$

with $\mathfrak{n}M\rho^2 \leqslant C$ for some absolute constant C (that does not depend on any other parameter).

Note that we allow $\sum_{i=1}^{\mathfrak{n}} \sum_x \tilde{\mathfrak{f}}(x) \neq 0$ compared to (A_f) and take $\mathfrak{n}M\rho^2 \leqslant C$. In principle, the estimates coming below should be depending on this constant C, but as C does not depend on any other parameter, we never make this explicit. Thus the condition $\mathfrak{n}M\rho^2 \leqslant C$ makes the estimates uniform in the choice of M and ρ. $\tilde{\mathfrak{f}}$ will be defined in terms of f_ε or $\mathfrak{f}$ for Theorem 1.1.3 and Theorem 1.1.5, but we leave freedom of choice for now. We also mention that, since $\tilde{\mathfrak{f}}$ is a function on $\mathbb{Z}^2$ with compact support, $\tilde{\mathfrak{f}}$ can also be considered as a function on Λ_N as long as L^N is sufficiently large compared to $\max_i\{\|y_i\|_2\}$ and ρ. There are no restrictions on y_i's other than this (so in particular, $T_{y_i}\tilde{\mathfrak{f}}_i$'s are allowed to overlap).

In most situations, we will see that the tilted expectation can be treated by complex shift of variables, yielding

$$\mathbb{E}_{(\omega),\Gamma_j}[F(\zeta)] = \mathbb{E}[F(\zeta + \omega\Gamma_j\tilde{\mathfrak{f}})]. \tag{3.18}$$

Thus it is helpful to define

$$u_j = \sum_{i=1}^{\mathfrak{n}} u_{j,i}, \qquad u_{j,i} = \begin{cases} \Gamma_j\tilde{\mathfrak{f}}_i & (j > j_s) \\ 0 & (j \leqslant j_s) \end{cases} \tag{3.19}$$

for $i = 1, \cdots, \mathfrak{n}$, where each $\tilde{\mathfrak{f}}_i$ is as in (A'_f). Then we may simply note the following, according to the properties of Γ_j.

Lemma 3.1.5. *Let $\tilde{\mathfrak{f}}$ satisfy (A'_f) and $u_{j,\alpha}$ be defined by* (3.19)*. Then $u_{j,\alpha}$ is supported on B_0^j, the unique block in $\mathscr{B}_j$ that contains* 0 *and*

$$\max_{\mu \in \hat{e}^n} \|\nabla^\mu u_{j,\alpha}\|_{L^\infty} \leqslant \begin{cases} C_0 M\rho^2 \log L & (n = 0) \\ C_n M\rho^2 L^{-jn} & (n > 0). \end{cases} \tag{3.20}$$

Proof. Since $u_{j,\alpha} \equiv 0$ for $j \leqslant j_s$, we only consider $j > j_s$. If $x \notin B_j^0$, then $d_\infty(0,x) \geqslant \frac{L^j}{3}$ so for $y \in \Lambda$ such that $d_\infty(0,y) \leqslant \rho$, we have $d_\infty(x,y) \geqslant \frac{L^j}{3} - \rho \geqslant \frac{L^j}{4}$. Thus

$$u_{j,\alpha}(x) = \sum_{\|y\|_\infty \leqslant \rho} \Gamma_j(x-y)\tilde{\mathfrak{f}}(y) = 0 \tag{3.21}$$

since $\Gamma_j(x-y) = 0$ whenever $d_\infty(x,y) \geqslant L^j/4$. Thus $u_{j,\alpha}$ is supported on B_j^0.

To see the bound on $u_{j,\alpha}$, we may use the bound $\|\Gamma_j\|_{L^\infty} \leqslant C \log L$ (Corollary 3.1.1) to see

$$|u_{j,\alpha}(x)| = \Big| \sum_{\|y\|_\infty \leqslant \rho} \Gamma_j(x-y)\tilde{\mathfrak{f}}(y) \Big| \leqslant CM\rho^2 \log L, \tag{3.22}$$

while for the derivatives, we have even better bounds because $\|\nabla^n \Gamma_j\|_{L^\infty} \leqslant C_n L^{-nj}$, so

$$|\nabla u_{j,\alpha}(x)| = \Big| \sum_{\|y\|_\infty \leqslant \rho} \nabla\Gamma_j(x-y)\tilde{\mathfrak{f}}(y) \Big| \leqslant M\rho^2 C_n L^{-nj}. \tag{3.23}$$

□

Although $u_{j,\alpha}$ is supported inside B_0^j, its translation $T_{y_\alpha} u_{j,\alpha}$ is not necessarily contained in a j-block. Thus we let $Q_{y_\alpha}^j$ be the j-polymer of four blocks that necessarily contains $T_{y_\alpha} u_{j,\alpha}$. More precisely, let $y_\alpha = (y_{\alpha,1}, y_{\alpha,2}) \in \mathbb{Z}^2$ and

$$\begin{aligned} &B_{\alpha,1}^j = \text{unique block in } \mathscr{B}_j \text{ that contains } (y_1 - (L^j-1)/2, y_2 - (L^j-1)/2) \\ &B_{\alpha,2}^j = B_1^j + L^j e_1, \quad B_{\alpha,3}^j = B_1^j + L^j e_2, \quad B_{\alpha,4}^j = B_1^j + L^1(e_1+e_2) \end{aligned} \tag{3.24}$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{Z}^2$ and

$$P_{\vec{y}}^j = \cup_{\alpha=1}^{\mathfrak{n}} Q_{y_\alpha}^j, \qquad Q_{y_\alpha}^j = \begin{cases} \cup_{l=1}^4 B_{\alpha,1}^j & (j > j_s) \\ Q_{y_\alpha}^{j_s+1} & (j = j_s) \\ \emptyset & (j < j_s). \end{cases} \tag{3.25}$$

We have only seen the case $j > j_s$, but we have also defined $Q_{y_\alpha}^j$ for $j \leqslant j_s$ for later use. We summarise the list of properties of $u_{j,\alpha}$ as the following, where we use

$$\|u\|_{C_j^2} = \max_{n=0,1,2} \sup_{\mu \in \hat{e}^n} L^{nj} \|\nabla^\mu u\|_{L^\infty}. \tag{3.26}$$

(A'_u) The sequence $(u_j)_{j\geqslant 1}$ has decomposition $u_j = \sum_{i=1}^{\mathfrak{n}} T_{y_i} u_{j,i}$ such that $\|u_{j,i}\|_{C_j^2} \leqslant C\mathfrak{n}^{-1}\log L$ and $u_{j,i}$ is supported on B_0^j for each j,α. Thus $T_{y_i}u_{j,i}$ is supported on $Q_{y_i}^j \in \mathscr{P}_j$. Also, $u_{j,i} \equiv 0$ when $j \leqslant j_s$ for the observable scale j_s.

Again, C will be a constant independent of all the other parameters, so we do not mention the dependence on it.

3.2 Polymer activities and their norms

Our choices of norms are almost the same as that in [35, Section 5.1], which are closely related to those of [23, 30]. Compared to these references, we simplify the construction somewhat and make the estimates explicit to obtain uniform control in the range of the step distribution.

Recall that Λ_N denotes the discrete d-dimensional torus of side length L^N, for integers $N, L \geqslant 1$. Unless explicitly stated otherwise, all results in this section (implicitly) hold for any choice of N and L. In the sequel, we make frequent use of the notation and setup introduced in Sections 3.1.1-3.1.2. In particular, $\mathbb{E}$ is $\mathbb{E}^{\zeta}_{\Gamma_{j+1}}$ if $j+1 < N$ and $\mathbb{E}^{\zeta}_{\Gamma_N^\Lambda}$ if $j+1 = N$.

Basic definitions

Following Section 1.3.2, we suppose that a renormalised 'bulk' theories are described by $(Z_j^0)_{j\geqslant 0}$ inductively defined as

$$Z^0_{j+1}(\varphi|\Lambda) = \mathbb{E}[\theta_\zeta Z^0_j(\varphi|\Lambda)] \tag{3.27}$$

where we use notation $\theta_\zeta F(\cdot) = F(\cdot + \zeta)$ now, and each Z_j^0 admits expansion

$$Z^0_j(\varphi|\Lambda) = e^{-E_{j+1}|\Lambda|} \sum_{X\in\mathscr{P}_j(\Lambda)} e^{U_j(\Lambda\backslash X,\varphi)} K^0_j(X,\varphi), \tag{3.28}$$

where we recall the definition of polymers $\mathscr{P}_j$ from Section 3.1.1. Compared to Section 1.3.2, we added superscripts 0 to indicate that the external source field is set to 0, and to distinguish them from the 'observable' theories introduced later. In this representation, E_j is going to be a suitable scalar, and U_j and K_j^0 will be scale j polymer functions. U_j will be the leading term and K_j^0 will be a remainder coordinate, which are both functions of both the polymer $X \in \mathscr{P}_j$ and the field φ. Their main features will be characterised by factorisation properties:

$$e^{U_j(X,\varphi)} = (e^{U_j(\cdot,\varphi)})^X, \qquad K^0_j(X,\varphi) = (K^0_j)^{\otimes X}(\varphi) \tag{3.29}$$

where we have used *polymer powers*

$$F^X(\varphi) = \prod_{B \in \mathscr{B}_j(X)} F(B,\varphi), \qquad F^{\otimes X}(\varphi) = \prod_{Y \in \mathrm{Comp}_j(X)} F(Y,\varphi) \tag{3.30}$$

for any $F : \mathscr{P}_j \times \mathbb{R}^\Lambda \to \mathbb{C}$, with the convention that the product over the empty set equals 1. In particular, the tuple $(K_j^0(X))_{X \in \mathscr{P}_j}$ is determined by $(K_j^0(X))_{X \in \mathscr{P}_j^c}$. The latter is an example of a *polymer activity*. We formalise the space of polymer actitivities as follows, and then define norms on polymer activites in the remainder of this section. We also allow the dependence on a complex variable $\omega \in \mathbb{D}_{h_\omega} = \{z \in \mathbb{C} : |z| < h_\omega\}$.

Definition 3.2.1. *For $X \in \mathscr{P}_j$ and $h_\omega > 0$, we write $\mathscr{N}_j(X)$ for the space $C^\infty(\mathbb{R}^{X^*})$ and $\mathscr{N}_{j,h_\omega}(X)$ for $C^\infty(\mathbb{R}^{X^*} \times \mathbb{D}_{h_\omega})$, analytic in the second component. For $F \in \mathscr{N}_j(X)$ (or $\in \mathscr{N}_{j,h_\omega(X)}$) and $\varphi \in \mathbb{R}^\Lambda$, we make the identification $F(\varphi) = F(\varphi|_{X^*})$. In particular, $F(\varphi)$ only depends on $\varphi|_{X^*}$ and we have the natural inclusions $\mathscr{N}_j(X) \subset \mathscr{N}_j(Y)$ (and $\mathscr{N}_{j,h_\omega}(X) \subset \mathscr{N}_{j,h_\omega}(Y)$) if $X \subset Y$.*

A scale-j polymer activity *is a tuple $K = (K(X))_{X \in \mathscr{P}_j^c}$, where for each connected polymer $X \in \mathscr{P}_j^c$, the corresponding component is a function $K(X) \in \mathscr{N}_j(X)$. Any polymer activity K is identified with its extension $(K(X))_{X \in \mathscr{P}_j}$ to all (not necessarily connected) polymers by $K(X) = K^{\otimes X}$. Tuple $(K(X) \in \mathscr{N}_{j,h_\omega}(X))_{X \in \mathscr{P}_j^c}$ is called ω-polymer activity and the same remark applies.*

We denote the space of scale-j polymer activities by $\mathscr{N}_j$ and the scale-j ω-polymer activities by $\mathscr{N}_{j,h_\omega}$.

Note that (3.29) implies that $K(\emptyset) = 1$ for any polymer activity K according to Definition 3.2.1, and that, with the restriction to connected polymers, the scale-j polymer activities form a linear space with 0 element given by $K(X) = 1_{X=\emptyset}$.

3.2.1 Norms

To define norms on polymer activities, we first define norms of lattice functions which will enter the definition of norms on polymer activities. Recall that $\{e_1, e_2\}$ is the standard basis of unit vectors with nonnegative components spanning $\mathbb{Z}^2$ or the local coordinates of Λ_N, and $\hat{e} = \{\pm e_1, \pm e_2\}$ and for any multi-index $(\mu) \in \hat{e}^n$, $\nabla^{(\mu)} = \nabla^{\mu_1} \cdots \nabla^{\mu_n}$. For functions $f, g : \Lambda_N \to \mathbb{C}$ and $n \geqslant 0$, we denote $\nabla^n f$ for the collection $(\nabla^{\vec{\mu}} f : |\vec{\mu}| = n)$, define inner products

$$(f,g)_X = \sum_{x\in X} f(x)g(x),$$
$$(\nabla^n f, \nabla^n g)_X = 2^{-n} \sum_{(\mu)=(\mu_1,\cdots,\mu_n)\in \hat{e}^n} \sum_{x\in X} \nabla^{\vec{\mu}} f(x) \nabla^{\vec{\mu}} g(x) \tag{3.31}$$

(the factors 2^{-n} in (3.31) are natural because each coordinate direction appears with positive and negative sign in the sum) and

$$|\nabla^n f|_X^2 = (\nabla^n f, \nabla^n f)_X \tag{3.32}$$

so that $(f, -\Delta f)_{\Lambda_N} = |\nabla f|^2_{\Lambda_N}$ by summation by parts.

At scale j, it is further natural to consider the rescaled derivatives

$$\nabla_j^n f = L^{jn} \nabla^n f. \tag{3.33}$$

Definition 3.2.2. *Let $n \in \mathbb{N}$, $X \in \mathscr{P}_j$ and $f : \{x : d_1(x,X) \leqslant n\} \to \mathbb{C}$ where d_1 is the graph distance on Λ_N. With (μ) ranging over $\{\pm e_1, \pm e_2\}^n$ in the sequel, define*

$$\|\nabla_j^n f\|_{L^\infty(X)} = \max_{(\mu)} \max_{x\in X} |\nabla_j^{(\mu)} f(x)| \tag{3.34}$$

$$\|\nabla_j^n f\|^2_{L^2_j(X)} = L^{-2j} |\nabla_j^n f|^2_X \tag{3.35}$$

$$\|\nabla_j^n f\|^p_{L^2_j(\partial X)} = L^{-j} |\nabla_j^n f|^2_{\partial X} \tag{3.36}$$

$$\|f\|_{C_j^2(X)} = \max_{n=0,1,2} \|\nabla_j^n f\|_{L^\infty(X)}. \tag{3.37}$$

(In (3.36) *and elsewhere, ∂U refers to the inner vertex boundary of $U \subset \Lambda_N$ with respect to the graph distance d_1.)*

These norms on lattice functions provide the basis for the norms on polymer activities that we use and which we introduce next. The norm is scale-dependent and measures smoothness of polymer activities with respect to typical fields at scale j, which are lattice functions φ with bounded C_j^2 norm. The norm needs to permit growth when $\nabla\varphi$ is large and give small weight to large sizes of polymers X. These two aspects are accounted for by two weights: the *(large-field) regulator* G_j for growth in $\nabla\varphi$ (see Definition 3.2.3 and (3.44)) and the parameter $A > 1$ (the *large-set regulator*) for decay in the size of the polymer (see (3.45)).

We start by measuring the size of a polymer activity for fixed φ and X. For all $n \in \mathbb{N}$, given $K(X,\cdot) \in \mathscr{N}_j(X)$, its n-th order derivative $D^n K$ along the directions $f_1, \dots f_n \in \mathbb{R}^{X^*}$ is given by

$$D^n K(X,\varphi)(f_1,\cdots,f_n) = \sum_{x_1,\cdots,x_n \in X^*} \frac{\partial^n K(X,\varphi)}{\partial\varphi(x_1)\cdots\partial\varphi(x_n)} f(x_1)\cdots f(x_n), \tag{3.38}$$

with the convention $D^0 K = K$. For $X \in \mathscr{P}_j^c$, $K(X) \in \mathscr{N}_j(X)$ and $\varphi \in \mathbb{R}^{\Lambda_N}$, set

$$\|D^n K(X,\varphi)\|_{n,T_j(X,\varphi)} = \sup\left\{|D^n K(X,\varphi)(f_1,\cdots,f_n)| : \|f_k\|_{C_j^2(X^*)} \leqslant 1, k = 1,\ldots,n\right\}, \tag{3.39}$$

with the convention $\|D^0 K(X,\varphi)\|_{0,T_j(X,\varphi)} = |K(X,\varphi)|$. Then, for a parameter $\mathfrak{h} > 0$, define

$$\|K(X,\varphi)\|_{\mathfrak{h},T_j(X,\varphi)} = \sum_{n=0}^{\infty} \frac{\mathfrak{h}^n}{n!} \|D^n K(X,\varphi)\|_{n,T_j(X,\varphi)}. \tag{3.40}$$

In the subscripts, we used labels 'T' to indicate that the (semi-)norms are obtained by summing the absolute values of each term in the Taylor series. On the other hand, n, j, X, φ and $\mathfrak{h}$ are objects that actually appear in the definitions. Note that (3.38) only depends on the f_k in X^*, but that the norms $\|f_k\|_{C_j^2(X^*)}$ in (3.39) actually depend on f_k in a neighbourhood of X^*. The supremum in (3.39) is thus over all $f_k \in \mathbb{R}^\Lambda$ or equivalently over all extensions of $f_k \in \mathbb{R}^{X^*}$ to a suitable neighbourhood of X^*.

The $\|\cdot\|_{\mathfrak{h},T_j(X,\varphi)}$-norm measures the size of K in a manner depending on φ and X. The norms on functions of (X,φ) are defined by weighted supremum norms. The large-field regulator which is the φ-dependent weight is defined next.

Definition 3.2.3. *Given $c_2, \kappa \equiv \kappa_L > 0$, define the large-field regulator for $X \in \mathscr{P}_j$ and $\varphi \in \mathbb{R}^{\Lambda_N}$ by*

$$G_j(X,\varphi) = \exp\left\{\kappa\Big(\|\nabla_j\varphi\|^2_{L_j^2(X)} + c_2\|\nabla_j\varphi\|^2_{L_j^2(\partial X)} + W_j(X,\nabla_j^2\varphi)^2\Big)\right\} \tag{3.41}$$

where

$$W_j(X,\nabla_j^2\varphi)^2 = \sum_{B\in\mathscr{B}_j(X)} \|\nabla_j^2\varphi\|^2_{L^\infty(B^*)}. \tag{3.42}$$

The particular form of the regulator is motivated by its properties stated in Section 3.3 below. Finally, the definition of the norms on polymer activities is given by the following definition.

Definition 3.2.4. *For a j-scale polymer activity K and $X \in \mathscr{P}_j$, define*

$$\|D^n K(X)\|_{n,T_j(X)} = \sup_{\varphi \in \mathbb{R}^{X^*}} (G_j(X,\varphi))^{-1} \|D^n K(X,\varphi)\|_{n,T_j(X,\varphi)} \tag{3.43}$$

$$\|K(X)\|_{\mathfrak{h},T_j(X)} = \sup_{\varphi \in \mathbb{R}^{X^*}} (G_j(X,\varphi))^{-1} \|K(X,\varphi)\|_{\mathfrak{h},T_j(X,\varphi)} \tag{3.44}$$

$$\|K\|_{\mathfrak{h},T_j,A} = \sup_{X \in \mathscr{P}_j^c(\Lambda_N)} A^{|X|_j} \|K(X)\|_{\mathfrak{h},T_j(X)}. \tag{3.45}$$

We will sometimes abbreviate $\|K\|_j = \|K\|_{\mathfrak{h},T_j} = \|K\|_{\mathfrak{h},T_j,A}$ whenever the choice of $\mathfrak{h}$ and A are clear form the context.

Analogues on ω-polymer activities can also be defined. Derivative in variable $\omega \in \mathbb{D}_{h_\omega}$ will be denoted ∂_ω.

Definition 3.2.5. *Given $h_\omega > 0$ and a complex analytic function $f : \mathbb{D}_{h_\omega} \to \mathbb{C}$, let*

$$\|f\|_{h_\omega,T} = \sum_{m=0}^{\infty} \frac{h_\omega^m}{m!} |\partial_\omega^m f(0)|. \tag{3.46}$$

If $\vec{\mathfrak{h}} = (\mathfrak{h}, h_\omega)$, then for a j scale ω-polymer activity K, let

$$\|K(X,\varphi;\cdot)\|_{\vec{\mathfrak{h}},T_j(X)} = \sum_{m=0}^{\infty} \frac{h_\omega^m}{m!} \|\partial_\omega^m K(X,\varphi;\omega)|_{\omega=0}\|_{\mathfrak{h},T_j(X,\varphi)} \tag{3.47}$$

$$\|K(X)\|_{\vec{\mathfrak{h}},T_j(X)} = \sup_{\varphi \in \mathbb{R}^{X^*}} (G_j(X,\varphi))^{-1} \|K(X,\varphi)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \tag{3.48}$$

$$\|K\|_{\vec{\mathfrak{h}},T_j,A} = \sup_{X \in \mathscr{P}_j^c(\Lambda_N)} A^{|X|_j} \|K(X)\|_{\vec{\mathfrak{h}},T_j(X)}. \tag{3.49}$$

We will sometimes abbreviate $\|K\|_j = \|K\|_{\vec{\mathfrak{h}},T_j} = \|K\|_{\vec{\mathfrak{h}},T_j,A}$ whenever the choice of $\vec{\mathfrak{h}}$ and A are clear form the context.

There is a slight abuse of notation for $\|K\|_j$ since it could either mean $\|K\|_{\mathfrak{h},T_j,A}$ or $\|K\|_{\vec{\mathfrak{h}},T_j,A}$ depending on the context, but there should be no confusion since we almost always measure $\|K\|_{\vec{\mathfrak{h}},T_j,A}$ for K that depends on ω.

These norms and the associated spaces (of polymer activities of finite norm) implicitly depend on the choice of Λ_N. However, the definitions are essentially local and it is thus possible to define an infinite-volume analogue of the norm, see Section 7.2. The space of polymer activities in $\mathscr{N}_j(X)$ with finite $\|\cdot\|_{\mathfrak{h},T_j(X)}$ norm is complete, see Appendix 3.B, and as a consequence the space of polymer activities in $\mathscr{N}_j$ with finite $\|\cdot\|_{\mathfrak{h},T_j}$ norm is also complete.

3.2.2 Norm on U_j

In the expansion (3.28), we use a specific form of U_j given by

$$\begin{aligned} U_j(X,\varphi) &= \frac{1}{2}s_j|\nabla\varphi|_X^2 + W_j(X,\varphi) \\ W_j(X,\varphi) &= \sum_{x\in X}\sum_{q\geqslant 1} L^{-2j} z_j^{(q)} \cos(q\beta^{1/2}\varphi(x)) \end{aligned} \tag{3.50}$$

for some $s_j \in \mathbb{R}$ and $(z_j^{(q)})_{q\geqslant 1} \subset \mathbb{R}$, where we recall (3.32) for $|\nabla\varphi|^2$. We will see $z_j^{(q)}$ has an exponential decay as $q\to\infty$, so it will be natural to define the following norm on U_j.

Definition 3.2.6. *Let γ be as in* (2.1)*, A the large set regulator and $c_f = \frac{1}{4}\gamma$. Then given U_j of form* (3.50)*, let*

$$\|U_j\|_j \equiv \|U_j\|_{\Omega_j^U} = A \max\left\{ |s_j|, \sup_{q\geqslant 1} e^{c_f\beta q}|z_j^{(q)}| \right\}. \tag{3.51}$$

Denote Ω_j^U for the space of U_j with $\|U_j\|_{\Omega_j} < \infty$.

In particular, note for later purposes that $\|W_j\|_j \equiv \|W_j\|_{\Omega_j^U}$ is also defined by (3.51) and corresponds to $s_j = 0$.

We will see the legitimacy of $c_f = \frac{1}{4}\gamma$ in Chapter 6 once the decay rate of $|z_j^{(q)}|$ is known. We will now partially clarify the relationship between c_f and $\mathfrak{h}$ in Lemma 3.2.8, where we also establish the relation between $\|\cdot\|_{\Omega_j^U}$ and $\|\cdot\|_{\mathfrak{h},T_j}$. For this purpose, we state the following preparatory lemma.

Lemma 3.2.7. *For $B\in\mathscr{B}_j$, $\mu,\nu\in\hat{e}$ and $\mathfrak{h}>0$,*

$$\|(\nabla^\mu\varphi,\nabla^\nu\varphi)_B\|_{\mathfrak{h},T_j(B,\varphi)} \leqslant 4(\mathfrak{h}^2 + \|\nabla_j\varphi\|_{L^\infty(B)}^2) \tag{3.52}$$

and

$$\|e^{i\sqrt{\beta}q\varphi(x_0)}\|_{\mathfrak{h},T_j(B,\varphi)} = e^{\sqrt{\beta}|q|\mathfrak{h}}, \quad x_0\in B. \tag{3.53}$$

Proof. One has the following exact derivatives of $\frac{1}{2}|\nabla\varphi|_B^2$:

$$D_\varphi((\nabla^\mu\varphi,\nabla^\nu\varphi)_B)(f) = \sum_{y\in B} \partial^\mu f_y \partial^\nu\varphi(y) + \partial^\nu f_y \partial^\mu\varphi(y) \tag{3.54}$$

$$D_\varphi^2((\nabla^\mu\varphi,\nabla^\nu\varphi)_B)(f,g) = \sum_{y\in D} \partial^\mu f_y \partial^\nu g_y + \partial^\mu g_y \partial^\nu f_y \tag{3.55}$$

and hence $\|(\nabla^\mu\varphi,\nabla^\nu\varphi)_B\|_{\mathfrak{h},T_j(B,\varphi)} \leqslant 2(\mathfrak{h} + \|\nabla_j\varphi\|_{L^\infty(B)})^2$, from which (3.52) follows.

The identity (3.53) also follows easily from the definition of the norm since

$$D_\varphi^n e^{i\sqrt{\beta}q\varphi(x)}(f_1,\cdots,f_n) = (i\sqrt{\beta}q)^n e^{i\sqrt{\beta}q\varphi(x)} \prod_{k=1}^n f_k(x), \tag{3.56}$$

which gives the claimed bound when substituted in the definition of the norm. It can conceptually be understood from the fact that the right-hand side is the supremum of $|e^{i\sqrt{\beta}q\varphi}|$ for φ in a strip of width $\mathfrak{h}$ around the real axis. □

Lemma 3.2.8. *Assume $\beta \geqslant 2\mathfrak{h}^2 c_f^{-2}$ and $\mathfrak{h} \geqslant c_f^{1/2}$. Then for any $B \in \mathscr{B}_j$,*

$$\begin{aligned} &\|s_j|\nabla\varphi|_B^2\|_{\mathfrak{h},T_j(X)} \leqslant C|s_j|(\mathfrak{h}^2 + \|\nabla_j\varphi\|_{L^\infty(B)}^2), \\ &\|W_j(B,\varphi)\|_{\mathfrak{h},T_j(X)} \leqslant CA^{-1}\|W_j\|_{\Omega_j^U}. \end{aligned} \tag{3.57}$$

Proof. The first line of (3.57) is just (3.52). For the norm on W_j, we use the triangle inequality and the conditions on β, c_f, $\mathfrak{h}$ and (3.53).

$$\begin{aligned} \|W_j(B,\varphi)\|_{\mathfrak{h},T_j(B,\varphi)} &\leqslant \sum_{q\geqslant 1} \|e^{i\sqrt{\beta}q\varphi(x_0)}\|_{\mathfrak{h},T_j(B,\varphi)} |z_j^{(q)}| \\ &\leqslant A^{-1} \sum_{q\geqslant 1} e^{-q(c_f\beta - \sqrt{\beta}\mathfrak{h})} \|W_j\|_{\Omega_j^U} \leqslant CA^{-1}\|W_j\|_{\Omega_j^U} \end{aligned} \tag{3.58}$$

for any $B \in \mathscr{B}_j$. □

3.2.3 Choice of parameters

We now give an overview of how the parameters $\mathfrak{h}, A, c_2, \kappa_L$ will eventually be chosen; see also Definition 6.3.2 and Remark 6.3.3. The constants c_2, κ_L will be fixed below Proposition 3.3.5 (see Remark 3.3.8) as $c_2 > 0$ sufficiently small (independent of L), and κ_L of order $(\log L)^{-1}$. The large set weight A will be chosen large enough as a function of L in Theorem 6.1.3 (essentially in such a way that the conclusions of Proposition 4.1.5 below hold). This leaves $\mathfrak{h}$, which we allow its dependency on j by letting $\mathfrak{h} = h_j$ where

$$h_j = \begin{cases} 2h & (j \leqslant j_s) \\ h & (j > j_s) \end{cases} \tag{3.59}$$

where h will be picked large enough so that the conclusions of Lemma 5.4.7 hold. To be specific, if we are given $\beta \geqslant 2\max\{c_f^{-1}, c_f^{-2}\}$ and some parameter $r > 0$, we let

$$h = \max\{c_f^{1/2}, rc_h\rho_J^{-2}\sqrt{\beta}, \rho_J^{-1}\}. \tag{3.60}$$

c_h is a constant specified by Lemma 5.4.5 later (but is an absolute constant that only depends on the finite range decomposition of Chapter 2). If we further choose r sufficiently small so that $\rho_J^2 \geqslant \sqrt{2}rc_hc_f^{-1}$, then β and $\mathfrak{h} = h_j$ satisfy the assumptions of Lemma 3.2.8. But since only the order of magnitude of $\mathfrak{h}$ matters in most of the inequalities, we will persist with using $\mathfrak{h}$ instead of making the dependence on j explicit. We also try to make these choice of parameters explicit whenever it seems appropriate.

3.3 Properties of the norms and the regulator

We start by remarking that $\|\cdot\|_{\mathfrak{h},T_j(X,\varphi)}$ can be considered as $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}$ with $h_\omega = 0$. Thus, unless otherwise stated, all the properties on the $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}$ would also apply to $\|\cdot\|_{\mathfrak{h},T_j(X,\varphi)}$. Conversely, since $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}$ can be thought of as $\|\cdot\|_{\mathfrak{h},T_j(X,\varphi)}$ with an external block with field value ω, some results we cite from [23] should hold the same for $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}$.

3.3.1 Key inequalities

The most fundamental properties of the seminorm $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}$ is its submultiplicativity property, and its monotonicity in the base polymer X and in the scale j. These properties will be used heavily, but not always explicitly.

Lemma 3.3.1. *Suppose $X, Y \in \mathscr{P}_j$ with $Y \subset X$, and let $F(Y) \in \mathscr{N}_{j,h_\omega}(Y)$ (here recall the inclusion $\mathscr{N}_{j,h_\omega}(Y) \subset \mathscr{N}_{j,h_\omega}(X)$ from Definition 3.2.1). Then for each $n \geqslant 0$,*

$$\text{(i) } \|D^nF(Y,\varphi)\|_{n,T_{j+1}(Y,\varphi)} \leqslant \|D^nF(Y,\varphi)\|_{n,T_j(Y,\varphi)}, \quad \text{and} \tag{3.61}$$

$$\text{(ii) } \|D^nF(Y,\varphi)\|_{n,T_j(Y,\varphi)} \leqslant \|D^nF(Y,\varphi)\|_{n,T_j(X,\varphi)}. \tag{3.62}$$

Hence,

$$\|F(Y,\varphi)\|_{\vec{\mathfrak{h}},T_{j+1}(Y,\varphi)} \leqslant \|F(Y,\varphi)\|_{\vec{\mathfrak{h}},T_j(Y,\varphi)}, \quad \|F(Y,\varphi)\|_{\vec{\mathfrak{h}},T_j(Y,\varphi)} \leqslant \|F(Y,\varphi)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}. \tag{3.63}$$

Moreover, for $Y_1, Y_2, X \in \mathscr{P}_j$ with $Y_1, Y_2 \subset X$ (with Y_1 and Y_2 not necessarily disjoint), and $F(Y_i) \in \mathscr{N}_{j,h_\omega}(Y_i)$, the following submultiplicativity property holds:

$$\|F_1(Y_1,\varphi)F_2(Y_2,\varphi)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \leqslant \|F_1(Y_1,\varphi)\|_{\vec{\mathfrak{h}},T_j(Y_1,\varphi)}\|F_2(Y_2,\varphi)\|_{\vec{\mathfrak{h}},T_j(Y_2,\varphi)}. \tag{3.64}$$

Proof. To see (i), notice that for any $f \in \mathbb{R}^\Lambda$, we have $\|f\|_{C_j^2(Y^*)} \leqslant \|f\|_{C_{j+1}^2(Y^*)}$. Hence $\{f \in \mathbb{R}^\Lambda : \|f\|_{C_{j+1}^2(Y^*)} \leqslant 1\} \subset \{f \in \mathbb{R}^\Lambda : \|f\|_{C_j^2(Y^*)} \leqslant 1\}$ and (i) follows readily in view of (3.39). For (ii), we have for any $f : \Lambda \to \mathbb{R}$ that $\|f\|_{C_j^2(Y^*)} \leqslant \|f\|_{C_j^2(X^*)}$, and the result follows similarly. On account of (3.40), the inequalities in (3.63) are immediate consequences of (i) and (ii), respectively. For the submultiplicativity property, see [23, Lemma 6.7]. □

Corollary 3.3.2. *For each $k \geqslant -1$,*

$$\Big\|e^{F_1(X,\varphi;\omega)} - \sum_{m=0}^{k}\frac{1}{m!}(F_1(X,\varphi;\omega))^m\Big\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \leqslant \sum_{m=k+1}^{\infty}\frac{1}{m!}\|F_1(X,\varphi;\omega)\|^m_{\vec{\mathfrak{h}},T_j(X,\varphi)} \tag{3.65}$$

with convention $\sum_{m=0}^{-1}(\cdots) \equiv 0$.

Proof. Consider the sequence of polymer activities

$$H_l(X,\varphi;\omega) = \sum_{m=0}^{l}\frac{1}{m!}(F_1(X,\varphi))^m. \tag{3.66}$$

Then by the submultiplicativity, (3.64), we have

$$\|H_l(X,\varphi;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \leqslant \exp\left(\|F_1(X,\varphi;;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}\right). \tag{3.67}$$

Since e^{F_1} is a pointwise limit of H_l and $D^n\partial_\omega^m H_k(X,\varphi;\omega) \to D^n\partial_\omega^m e^{F_1}(X,\varphi;\omega)$ as $l\to\infty$ for each $n,m \geqslant 0$, we have

$$\sum_{n,m\geqslant 0}^{n+m\leqslant N'}\frac{h_\omega^m\mathfrak{h}^n}{m!n!}\left\|D^n\partial_\omega^m e^{F_1(X,\varphi;0)}\right\|_{n,T_j(X,\varphi)} \leqslant \limsup_{l\to\infty}\|H_l(X,\varphi;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \tag{3.68}$$

for any $N' > 0$ so we see in fact

$$\left\|e^{F_1(X,\varphi;\omega)}\right\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \leqslant \exp\left(\|F_1(X,\varphi;;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}\right) < \infty, \tag{3.69}$$

which is (3.65) with $k=-1$. Bounds for $k \geqslant 0$ are obtained similarly. □

The next property of the regulator is the following basic inequality that allows to absorb polynomial error bounds in the fields when changing from one scale to the next.

Lemma 3.3.3. *For all $c_2, \kappa_L > 0$, $X \in \mathscr{S}_j$ for some $0 \leqslant j < N$, all $x_0 \in X$ and $\varphi \in \mathbb{R}^{\Lambda_N}$, defining $\delta\varphi(x) = \varphi(x) - \varphi(x_0)$, one has*

$$\kappa_L^{k/2} \|\delta\varphi\|^k_{C^2_j(X^*)} \leqslant C(k) G_j(X,\varphi), \quad k \in \mathbb{N}. \tag{3.70}$$

Proof. See Section 3.7.2. □

The next property of the regulator involves what is called the *strong regulator* in [23]:

$$w_j(X,\varphi)^2 = \sum_{B \in \mathscr{B}_j(X)} \max_{n=1,2} \|\nabla_j^n \varphi\|^2_{L^\infty(B^*)}, \quad X \in \mathscr{P}_j. \tag{3.71}$$

(The term strong regulator refers to the left-hand side of (3.72) below.)

Lemma 3.3.4. *For all $\kappa_L > 0$ and sufficiently small c_2, c_w,*

$$e^{c_w \kappa_L w_j(X,\varphi)^2} \leqslant G_j(X,\varphi), \quad X \in \mathscr{P}_j. \tag{3.72}$$

Moreover, for all $X, Y \in \mathscr{P}_j$ satisfying $X \cap Y = \emptyset$,

$$e^{c_w \kappa_L w_j(X,\varphi)^2} G_j(Y,\varphi) \leqslant G_j(X \cup Y, \varphi). \tag{3.73}$$

Proof. See Section 3.7.2. □

3.3.2 Supermartingale property

We will need to analyse how the norm of the polymer activities is bounded after taking the Gaussian expectation with respect to the covariances Γ_{j+1} (recall (3.2)). For a scale-j polymer activity F, a common strategy to bound $\mathbb{E}[\theta_\zeta F(X,\varphi')]$, for fixed φ' and $X \in \mathscr{P}_j^c$, will be to first bound $\|\theta_\zeta F(X,\varphi')\|_{\mathfrak{h},T_j(X,\varphi')} \leqslant \|F(X)\|_{\mathfrak{h},T_j(X)} \theta_\zeta G_j(X,\varphi')$, which follows immediately from (3.44), so that the fluctuation integral acts effectively on the large field regulator G_j only. In this regard, the following Proposition 3.3.5 yields that the form of the large field regulator is stable under the fluctuation integral up to a factor $2^{|\overline{X}|_j}$, where $\overline{X}$ denotes the closure of X, cf. Section 3.2.

Recall that $\mathbb{E}$ means we are taking expectation over $\zeta \sim \mathscr{N}(0,\Gamma_{j+1})$ if $j+1 < N$ and $\zeta \sim \mathscr{N}(0,\Gamma_N^{\Lambda_N})$ if $j+1 = N$. The non-random part of the field is often denoted φ'.

Proposition 3.3.5. *Assume* (3.1) *and that $L = \ell^{N'}$ for integers $\ell, N' \geqslant 1$. For c_2 sufficiently small, there exists an integer $\ell \equiv \ell(c_2)$ and a constant $c = c(c_2) > 0$ such that with $c_\kappa =$*

$c(c_2)\ell^{-2} \in (0,1)$*, the following holds: for all* $0 \leqslant j \leqslant N-1$ *and all* $\kappa_L \leqslant c_\kappa \rho_J^2 (\log L)^{-1}$*,*

$$\mathbb{E}[\theta_\zeta G_j(X,\varphi')] \leqslant 2^{|X|_j} G_{j+1}(\overline{X},\varphi'), \qquad X \in \mathscr{P}_j^c,\ \varphi' \in \mathbb{R}^{\Lambda_N}. \tag{3.74}$$

Proof. See Section 3.7.4. □

The following lemma is a simple consequence of Proposition 3.3.5.

Lemma 3.3.6. *Suppose the assumptions of Proposition 3.3.5 hold. Let* $F \in \mathscr{N}_j(X)$ *with* $\|D^n F\|_{n,T_j(X)} < \infty$*, for* $X \in \mathscr{P}_j^c$ *and* $\varphi' \in \mathbb{R}^{\Lambda_N}$*. Then*

$$\|D^n_{\varphi'} \mathbb{E}[\theta_\zeta F(X,\varphi')]\|_{n,T_j(X,\varphi')} \leqslant 2^{|X|_j} \|D^n F(X)\|_{n,T_j(X)}\, G_{j+1}(\overline{X},\varphi'). \tag{3.75}$$

Proof. The derivative $D_{\varphi'}$ can be exchanged with the expectation $\mathbb{E}$, hence for all functions f_k with $\|f_k\|_{C_j^2(X^*)} \leqslant 1$, $k=1,\ldots,n$, by (3.39),

$$\begin{aligned} |D^n_{\varphi'} \mathbb{E}[F(X,\varphi'+\zeta)](f_1,\cdots,f_n)| &\leqslant \mathbb{E}[|D^n_{\varphi'} F(X,\varphi'+\zeta)(f_1,\cdots,f_n)|] \\ &\leqslant \mathbb{E}[\|D^n_{\varphi'} F(X,\varphi'+\zeta)\|_{n,T_j(X,\varphi'+\zeta)}] \end{aligned} \tag{3.76}$$

and so, taking suprema over the f_k's, recalling Definition 3.2.4 and applying the bound (3.74),

$$\begin{aligned} \|D^n_{\varphi'} \mathbb{E}[\theta_\zeta F(X,\varphi')]\|_{n,T_j(X)} &\leqslant \|D^n F(X)\|_{n,T_j(X)} \mathbb{E}[\theta_\zeta G_j(X,\varphi')] \\ &\leqslant 2^{|X|_j} \|D^n F(X)\|_{n,T_j(X)} G_{j+1}(\overline{X},\varphi'). \end{aligned} \tag{3.77}$$

□

As a particular application, by multiplying $\frac{\mathfrak{h}^n}{n!}$ and summing over $n \geqslant 0$, this implies

$$\|\mathbb{E}[\theta_\zeta F(X,\varphi')]\|_{\mathfrak{h},T_j(X,\varphi')} \leqslant 2^{|X|_j} \|F\|_{\mathfrak{h},T_j(X)}\, G_{j+1}(\overline{X},\varphi') \tag{3.78}$$

whenever $\|F\|_{\mathfrak{h},T_j(X)} < \infty$. In practice, we will also be needing similar estimate with $\mathbb{E}$ replaced by $\mathbb{E}_{(\omega)}$, as stated in the next proposition.

Proposition 3.3.7. *Let* $\tilde{\mathfrak{f}}$ *be as in* (A'_f)*,* u_{j+1} *be as in* (3.19)*,* $j \in [j_s, N-1]$ *and the parameters be chosen according to Proposition 3.3.5. Let* $F \in \mathscr{N}_{j,h_\omega}(X)$ *be such that* $\|F(X;\omega)\|_{\vec{\mathfrak{h}},T_j(X)} < \infty$ *and* $h_\omega \leqslant (C_1 \log L)^{-3/2}$ *for sufficiently large* $C_1 > 0$*. Then* $\mathbb{D}_{h_\omega} \ni \omega \mapsto \mathbb{E}_{(\omega)}[D^n F(X,\varphi'+\zeta;\omega)]$ *is analytic in* $\omega \in \mathbb{D}_{h_\omega}$ *for each* $n \geqslant 0$ *and satisfies*

$$\left\|\mathbb{E}_{(\omega)}[F(X,\varphi'+\zeta;\omega)]\right\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \leqslant C_2 2^{|X|_j} \|F(X;\omega)\|_{\vec{\mathfrak{h}},T_j(X)} G_{j+1}(\overline{X},\varphi') \tag{3.79}$$

for $X \in \mathscr{P}_j^c$ and some $C_2 > 0$.

Proof. See Section 3.8.2. □

We conclude this section by fixing the parameters c_2, κ_L appearing in (3.41).

Remark 3.3.8. We choose $c_2 > 0$ small enough such that both i) the estimate (3.73) in Lemma 3.3.4 holds whenever $c_w \leqslant c(c_2)$ and ii) Proposition 3.3.5 is in force. Having fixed c_2, we choose $\ell = C\rho_J$ according to Proposition 3.3.5 and set $\kappa_L = c_\kappa \rho_J^2 (\log L)^{-1}$ with $c_\kappa = c\ell^{-2}$, so that the conclusions of Proposition 3.3.5 (i.e. (3.74)) hold. We can thus freely apply the bounds derived in Lemmas 3.3.3 and 3.3.4 (in the latter case whenever $c_w \leqslant c(c_2)$) and Proposition 3.3.5 in the sequel. Throughout the rest of this thesis, we always implicitly assume that the base scale L is of the form $L = \ell^{N'}$ with ℓ as fixed above. Unless stated otherwise, all statements hold uniformly in $N' \geqslant 1$, and when we write $L \geqslant C$ in the sequel, we tacitly view this as a condition on N' being sufficiently large.

3.3.3 Subdecomposition of the regulator

The final property of the regulator is a technical property involving the scale subdecomposition from Section 3.1.3 and that is needed to obtain sharp integrability estimates. It is used as an ingredient of the proof of Proposition 3.3.5 above and also in the justification of complex translations in the proof of Lemma 5.4.6 below.

Throughout this section, assume $L = \ell^{N'}$ with integers ℓ and N'. For a parameter $c_4 > 0$ and $X \in \mathscr{P}_{j+s}$ for $s \in I_{N'} := \{0, 1/N', \dots, 1 - 1/N'\}$ (recall the notion of fractional scales from Section 3.1.3), let

$$g_{j+s}(X, \xi) = \exp\Big(c_4 \kappa_L \sum_{a=0,1,2} W_{j+s}(X, \nabla_{j+s}^a \xi)^2\Big), \tag{3.80}$$

with W_{j+s} defined analogously to (3.42). Then, with obvious notation, define $G_{j+s}(X, \cdot)$ for $X \in \mathscr{P}_{j+s}$ as in (3.41) but with $j+s$ in place of j everywhere. They will be stated explicitly in (3.107) again.

The following Lemmas 3.3.9 and 3.3.10 can be extracted from [35, Lemma 19] and its proof. For completeness, we have again included proofs in Section 3.7.3 and 3.7.1.

Lemma 3.3.9. *There exists $C > 0$ such that for any $X \in \mathscr{P}_{j+s}$ and $\zeta \in \mathbb{R}^{\Lambda_N}$,*

$$g_{j+s}(X, \zeta) \leqslant \exp(\frac{1}{2} Q_{j+s}(X, \zeta)) := \exp\Big(C c_4 \kappa_L \sum_{a=0}^{4} \sum_{(\mu)} \|\nabla_{j+s}^{(\mu)} \zeta\|_{L_{j+s}^2(X^*)}^2\Big), \tag{3.81}$$

where the sum ranges over multiindices $(\mu) = (\mu_1, \ldots, \mu_a) \in \{\pm e_1, \pm e_2\}^a$. *Moreover, for any* $c_4 > 0$, *any integer* ℓ, *there is* $c_\kappa = c_\kappa(c_4, \ell) > 0$ *such that if* $\kappa_L = c_\kappa \rho_J^2 (\log L)^{-1}$ *then*

$$\mathbb{E}_{\Gamma_{j+s,j+s'}}(e^{Q_{j+s}(X,\zeta)}) \leqslant 2^{(N')^{-1}|X|_{j+s}}. \tag{3.82}$$

Lemma 3.3.10. *For* $c_2 > 0$ *small enough, there exist* $c_4 = c_4(c_2) > 0$ *and an integer* $\ell_0 = \ell_0(c_2) > 1$ *(both large), such that for all* $\ell \geqslant \ell_0$, $N' \geqslant 1$, $0 \leqslant j < N$, $s \in I_{N'}$, $s' = s + (N')^{-1}$ *and* $\kappa_L > 0$, *for* $X \in \mathscr{P}^c_{j+s}$, $\varphi, \xi \in \mathbb{R}^{\Lambda_N}$,

$$G_{j+s}(X, \varphi + \xi) \leqslant g_{j+s}(X_{s'}, \xi) G_{j+s'}(X_{s'}, \varphi). \tag{3.83}$$

3.3.4 Continuity of the expectation

The next property shows that the expectation is continuous with respect to the parameter s of the covariances.

Lemma 3.3.11. *For any* $X \in \mathscr{P}^c_j$ *and* $F(X)$ *with* $\|F(X)\|_{\mathfrak{h},T_j(X)} < \infty$, *for* $|s|, |s'| < \theta_J \varepsilon_s$,

$$\lim_{s' \to s} \|\mathbb{E}_{\Gamma_{j+1}(s')}[F(X, \cdot + \zeta)] - \mathbb{E}_{\Gamma_{j+1}(s)}[F(X, \cdot + \zeta)]\|_{\mathfrak{h},T_{j+1}(\overline{X})} = 0. \tag{3.84}$$

More precisely, for any $C > 0$, *the convergence is uniform over all* F *with* $\|F(X)\|_{\mathfrak{h},T_j(X)} \leqslant C$. *The same conclusion holds when* $X \in \mathscr{P}^c_{j+1}$ *and we assume*

$$\sup_\varphi G_j(X, \varphi)^{-1} \|F(X, \varphi)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} < \infty, \tag{3.85}$$

i.e., the convergence is uniform in F *for which the left-hand side of* (3.85) *is bounded by a given* $C > 0$.

3.4 Analytic polymer activities

We use the observation made in [29] that the finiteness of the norm enforces analyticity of polymer activities in a strip. For open $U \subset \mathbb{C}^\Lambda$, the function $F : U \to \mathbb{C}$ is called complex analytic in U if it admits a local representation as a convergent power series around any point in U.

Proposition 3.4.1. *Let* $\vec{\mathfrak{h}} = (\mathfrak{h}, h_\omega)$, $X \in \mathscr{P}_j$ *and* $\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} < +\infty$. *Then* $F(X, \cdot; \omega)$ *can be extended to* $S_{\mathfrak{h}}(X) = \{\varphi + i\psi \in \mathbb{C}^{\Lambda_N} : \varphi(x), \psi(x) \in \mathbb{R}, \|\psi\|_{C^2_j(X^*)} < \mathfrak{h}\}$ *where each*

$D^n F(X,\cdot;\omega)$ is complex analytic and satisfies

$$|F(X,\varphi+\phi;\omega)| \leqslant \|F(X,\varphi;\omega)\|_{\mathfrak{h},T_j(X,\varphi)} \tag{3.86}$$

whenever $\varphi \in \mathbb{R}^{\Lambda_N}$, $\phi \in \mathbb{C}^{\Lambda_N}$, $\|\phi\|_{C_j^2(X^*)} < \mathfrak{h}$.

Proof. Let $D_{\mathfrak{h}}(0) = \{\psi \in \mathbb{C}^\Lambda : \|\psi\|_{C_j^2(X^*)} < \mathfrak{h}\}$. Note that $D_{\mathfrak{h}}(0) \subset \mathbb{C}^\Lambda$ is open because $\|\psi\|_{L^\infty(X^*)} < \frac{1}{4}L^{-2j}\mathfrak{h}$ implies $\psi \in D_{\mathfrak{h}}(0)$. For $\varphi \in \mathbb{R}^{X^*}$ and $\psi \in D_{\mathfrak{h}}(0)$, let

$$F_{[\varphi]}(X,\varphi+\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(X,\varphi)(\psi^{\otimes n}). \tag{3.87}$$

Since $\|F\|_{\mathfrak{h},T_j(X,\varphi)} \leqslant \|F\|_{\mathfrak{h},T_j(X)} G_j(X,\varphi) < +\infty$ and since $\|\psi\|_{C_j^2(X^*)} < \mathfrak{h}$, the series (3.87) converges absolutely. These considerations also imply that $\widetilde{F}(X,\cdot) : S_{\mathfrak{h}} \to \mathbb{C}$ given by

$$\widetilde{F}(X,z) \stackrel{\text{def.}}{=} F_{[\varphi]}(X,\varphi+\psi), \quad \text{for any } \varphi \in \mathbb{R}^\Lambda \text{ and } \psi \in D_{\mathfrak{h}}(0) \text{ s.t. } z = \varphi+\psi \tag{3.88}$$

is well-defined and extends F. Moreover, in view of (3.87), $\widetilde{F}(X,\cdot)$ is (plainly) given by a convergent power series in a neighbourhood of $z = \varphi$, for any $\varphi \in \mathbb{R}^\Lambda$.

Now consider an arbitrary point $\zeta \in S_{\mathfrak{h}}$. It remains to argue that $\widetilde{F}(X,\cdot)$ defined by (3.88) can be represented as convergent power series around ζ. Write $\zeta = \varphi + \psi$ where $\varphi = \mathrm{Re}(\zeta)$ componentwise. Now observe that for $\delta\zeta \in \mathbb{C}^\Lambda$ small enough (such that $\psi + \delta\zeta \in D_{\mathfrak{h}}(0)$), one has

$$\widetilde{F}(X,\zeta+\delta\zeta) \stackrel{(3.88)}{=} \sum_{n\geqslant 0} \frac{1}{n!} D^n F(X,\varphi)((\psi+\delta\zeta)^{\otimes n}) = \sum_{k\geqslant 0} \frac{1}{k!} A_k(\delta\zeta^{\otimes k}), \tag{3.89}$$

where

$$A_k(f_1,\cdots,f_k) = \sum_{l=0}^{\infty} \frac{1}{l!} D^{k+l} F(X,\varphi)(\psi^{\otimes l}, f_1,\cdots,f_k) \tag{3.90}$$

and the right-hand side of (3.89) is obtained by expanding $(\psi+\delta z)^{\otimes n}$, using multilinearity and re-arranging terms according to the number k that δz appears. Now use $\|F\|_{h,T_j} < +\infty$ once again to show the series in (3.89) converges. All in all, it follows that $\widetilde{F}$ is complex-analytic, as desired.

The bound (3.86) is a result of (3.87). □

By the proposition, we can make complex shift of variables in each Gaussian integrals using the Cauchy's integral theorem as long as the shift is not too large. This result is summarised in the next lemma, whose proof is presented in Section 3.8.1.

Lemma 3.4.2 (Gaussian complex shift of variable)**.** *Let $\mathfrak{f}$ be as in* (A'_f)*,* u_{j+1} *be as in* (3.19) *and* $j \geqslant j_s$*. Also, let* $F \in \mathscr{N}_j(X)$ *with* $\|F(X)\|_{\mathfrak{h},T_j(X)} < \infty$*. Then for* $h_\omega < (C\log L)^{-1}\mathfrak{h}$ *with* $C > 0$ *sufficiently large and* $\omega \in \mathbb{D}_{h_\omega}$*,*

$$\mathbb{E}_{(\omega)}\big[F(X,\varphi'+\zeta)\big] = \mathbb{E}\big[F\big(X,\varphi'+\zeta+\omega u_{j+1}\big)\big] \tag{3.91}$$

As the norm $\|F\|_{\vec{\mathfrak{h}},T_j}$ exploits the analyticity of F even further, we would have to study this a bit further.

Lemma 3.4.3. *Let* $\vec{\mathfrak{h}} = (\mathfrak{h}, h_\omega)$ *and* $F \in \mathscr{N}_{j,h_\omega}(X)$ *be such that* $\|F(X;\omega)\|_{\vec{\mathfrak{h}},T_j(X)} < +\infty$ *where* $X \in \mathscr{P}_j^c$*. Then* $\mathbb{E}[\partial_\omega^m D^n F(X,\varphi'+\zeta;\omega)] = \partial_\omega^m D^n \mathbb{E}[F(X,\varphi'+\zeta;\omega)]$ *for any* $n,m \geqslant 0$ *and* $\mathbb{D}_{h_\omega} \ni \omega \mapsto \mathbb{E}[D^n F(X,\varphi'+\zeta;\omega)]$ *is a complex analytic function.*

Proof. Let $\omega \in \mathbb{D}_{(1-2\delta)h_\omega}$ for some $\delta > 0$. Then by the Cauchy's integral formula, for any $n,m \geqslant 0$,

$$\begin{aligned} \|\partial_\omega^m D^n F(X,\varphi;\omega)\|_{n,T_j(X,\varphi)} &= \Big\| \frac{m!}{2\pi i} \int_{|z|=(1-\delta)h_\omega} \frac{D^n F(X,\varphi;z)}{(z-\omega)^{m+1}} dz \Big\|_{n,T_j(X,\varphi)} \\ &\leqslant \frac{(1-\delta')n!}{(\delta-\delta')^2 h_\omega^m h^n} \|F(X;\cdot)\|_{\vec{\mathfrak{h}},T_j(X)} G_j(X,\varphi) \end{aligned} \tag{3.92}$$

where we have used, for $z \in \mathbb{D}_{h_\omega}$,

$$\|D^n F(X,\varphi;z)\|_{n,T_j(X,\varphi)} \leqslant \sum_{k=0}^\infty \frac{|z|^k}{k!} \|\partial_\omega^k D^n F(X,\varphi;\omega)|_{\omega=0}\|_{n,T_j(X,\varphi)} \leqslant \frac{n!}{\mathfrak{h}^n} \|F(X,\varphi;\cdot)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)}. \tag{3.93}$$

By Proposition 3.3.5, $\mathbb{E}[G_j(X,\varphi'+\zeta)] \leqslant 2^{|X|_j} G_{j+1}(\overline{X},\varphi')$ for each $\varphi' \in \mathbb{R}^\Lambda$, so the Dominated convergence theorem guarantees that $\mathbb{E}[\partial_\omega^m D^n \theta_\zeta F(X,\varphi';\omega)] = \partial_\omega^m D^n \mathbb{E}[\theta_\zeta F(X,\varphi';\omega)]$, and they are continuous functions of ω.

Now let $\tilde{\gamma}$ be any piecewise C^1 curve in $\mathbb{D}_{h_\omega}$, and we consider

$$\int_{\tilde{\gamma}} \mathbb{E}[D^n F(X,\varphi'+\zeta;\omega)] d\omega. \tag{3.94}$$

Again by (3.93) and Proposition 3.3.5, we have

$$\mathbb{E}[|D^n F(X,\varphi'+\zeta;\omega)|] \leqslant \frac{n!}{\mathfrak{h}^n}\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} G_{j+1}(\overline{X},\varphi') \tag{3.95}$$

so $D^n F(X,\varphi'+\zeta;\omega)$ is integrable under the product measure $\mathbb{P}(d\zeta)\otimes d\omega$ of (3.94), and by the Fubini's theorem,

$$\int_{\tilde{\gamma}} \mathbb{E}[D^n F(X,\varphi'+\zeta;\omega)]d\omega = \mathbb{E}\Big[\int_{\tilde{\gamma}} D^n F(X,\varphi'+\zeta;\omega)d\omega\Big], \tag{3.96}$$

but by the Cauchy's integral theorem, $\int_{\tilde{\gamma}} D^n F(X,\varphi'+\zeta;\omega)d\omega = 0$, making the whole integral vanish. Hence by the Morera's theorem (recalling continuity in ω proved above), $\mathbb{E}[D^n F(X,\varphi'+\zeta;\omega)]$ is also complex analytic on $\mathbb{D}_{h_\omega}$.

□

Lemma 3.4.4. *Let $\tilde{\mathfrak{f}}$ satisfy (A'_f) and u_{j+1} be defined by* (3.19)*. Let $\mathfrak{h}, h_\omega > 0$ be such that $h_\omega < (C\log L)^{-1}\mathfrak{h}$ for sufficiently large C. Let $F \in \mathscr{N}_{j,h_\omega}(X)$ be such that $\|F(X;\omega)\|_{(\mathfrak{h}'',h_\omega),T_j(X)} < +\infty$ where $\mathfrak{h}'' = \mathfrak{h} + h_\omega \|u_{j+1}\|_{C_j^2}$ and $X \in \mathscr{P}_j^c$. If we define*

$$F'(X,\varphi;\cdot\,) : \mathbb{D}_{h_\omega} \to \mathbb{C}, \quad \omega \mapsto F(X,\varphi+\omega u_{j+1};\omega), \tag{3.97}$$

then $D^n F'(X,\varphi;\cdot)$ and $\mathbb{E}[D^n F'(X,\varphi'+\zeta;\cdot)]$ are complex analytic functions of $\omega \in \mathbb{D}_{h_\omega}$ for each $n \geqslant 0$, and satisfies

$$\|F'(X,\varphi;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \leqslant \|F(X,\varphi;\omega)\|_{(\mathfrak{h}'',h_\omega),T_j(X,\varphi)} \tag{3.98}$$

$$\|\mathbb{E}_{(\omega)}[F(X,\varphi'+\zeta;\omega)]\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \leqslant \|\mathbb{E}[F(X,\varphi'+\zeta;\omega)]\|_{(\mathfrak{h}'',h_\omega),T_j(X,\varphi')}. \tag{3.99}$$

Proof. By Proposition 3.4.1, $z \mapsto D^n F(X,\varphi+zu_{j+1};\omega)$ is analytic whenever $\|zu_{j+1}\|_{C_j^2(X^*)} < \mathfrak{h}$. However, since $\|u_{j+1}\|_{C_j^2(X^*)} \leqslant C\kappa^{-1}$, this condition is satisfied whenever $|z| \leqslant h_\omega < C^{-1}\kappa\mathfrak{h}$, making $F'(X,\varphi;\omega)$ analytic in $\omega \in \mathbb{D}_{h_\omega}$. Now by the Chain rule,

$$\frac{d^m}{d\omega^m} D^n F'(X,\varphi;\omega) = \sum_{k=0}^m \binom{m}{k} D^{n+k}\partial_\omega^{m-k} F(X,\varphi+\omega u_{j+1};\omega)(u_{j+1}^{\otimes k}) \tag{3.100}$$

(D and ∂_ω are partial derivatives) so

$$\begin{aligned}
\|F'(X,\varphi;\omega)\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} &\leqslant \sum_{n,m=0}^{\infty}\sum_{k=0}^{m} \frac{\mathfrak{h}^n h_\omega^m}{n!m!}\binom{m}{k} \|D^{n+k}\partial_\omega^{m-k}F(X,\varphi;0)\|_{n+k,T_j(X,\varphi)}\|u_{j+1}\|^k \\
&= \sum_{n,k=0}^{\infty}\sum_{m'=0}^{\infty} \frac{\mathfrak{h}^n h_\omega^{k+m'}}{n!k!m'!}\|u_{j+1}\|^k \|D^{n+k}\partial_\omega^{m'}F(X,\varphi;0)\|_{n+k,T_j(X,\varphi)} \\
&= \sum_{m'=0}^{\infty} \frac{h_\omega^{m'}}{m'!}\|\partial_\omega^{m'}F(X,\varphi;0)\|_{\mathfrak{h}+h_\omega\|u_{j+1}\|,T_j(X,\varphi)}, \qquad (3.101)
\end{aligned}$$

where the second line follows from change of variable $m' = m-k$ and $\|u_{j+1}\| = \|u_{j+1}\|_{C_j^2(X^*)}$. This yields (3.98), hence by Lemma 3.4.3, we also have that $\mathbb{E}[D^n F'(X,\varphi'+\zeta;\omega)]$ complex analytic in $\omega \in \mathbb{D}_{h_\omega}$ for each $n \geqslant 0$. Now (3.99) follows from Lemma 3.4.2, saying that $\mathbb{E}_{(\omega)}[F(X,\varphi'+\zeta;\omega)] = \mathbb{E}[F'(X,\varphi'+\zeta;\omega)]$ and applying the same type of argument on $\mathbb{E}[F'(X,\varphi'+\zeta;\omega)]$.

□

This lemma is usually used after setting $C_1 > 0$ sufficiently large so that $\mathfrak{h}'' \leqslant 2\mathfrak{h}$. Then if we bound the right-hand side of (3.99) using the definition of $\|\cdot\|_{\vec{\mathfrak{h}},T_j(X)}$-norm, we have

$$\|\mathbb{E}_{(\omega)}[F(X,\varphi'+\zeta;\omega)]\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \leqslant C2^{|X|_j}\|F(X,\cdot;\omega)]\|_{(2\mathfrak{h},h_\omega),T_j(X)}G_{j+1}(\overline{X},\varphi'), \qquad (3.102)$$

which is similar to the conclusion of Proposition 3.3.7, but weaker. However, (3.99) has use of its own as it makes it easy translate inequalities on $\|\cdot\|_{\mathfrak{h},T_j(X)}$.

3.5 Proofs of the technical claims: organisation

In the rest of the chapter, we prove the unproved claims from the previous sections. The proofs do not come in linear order, so it ill be helpful to give an overview of the proofs.

- In Section 3.6, we prove Lemma 3.3.11. This statement depends on Proposition 3.3.5, but it is not used to prove any other statements.
- In Section 3.7, we prove Lemma 3.3.10, Lemma 3.3.3, Lemma 3.3.4, Lemma 3.3.9 and Proposition 3.3.5 in order. They share the common theme of controlling the large-field regulators G_j and $e^{c_w \kappa_L w_j(B,\varphi)^2}$. We see that the subdecomposed regulator of Section 3.3.3 will be necessary to control the expectation of G_j, so Proposition 3.3.5 is proved the last.

- In Section 3.8, we prove Lemma 3.4.2 and Proposition 3.3.7. We prove them by exploiting the properties of the regulator and analyticity of the polymer activities. Subdecomposition of the regulator also plays an important role in the proof of Proposition 3.3.7.

3.6 Continuity of the expectation

Proof of Lemma 3.3.11. We start from the following elementary identity for the derivative of a Gaussian integral with respect to its covariance: abbreviating $\Gamma_{j+1}(x,y) \equiv \Gamma_{j+1}(x,y;s,m^2)$, considering first the centered Gaussian vector on Λ_N with covariance $\Gamma_{j+1,\varepsilon} = \Gamma_{j+1} + \varepsilon\mathrm{Id}$ with density f_ε, computing the derivatives $\partial f_\varepsilon / \partial \Gamma_{j+1,\varepsilon}(x,y)$ and letting $\varepsilon \to 0$ using dominated convergence, one finds that

$$\frac{\partial}{\partial s}\mathbb{E}_{\Gamma_{j+1}(s)}[F(X,\varphi+\zeta)] = \frac{1}{2}\sum_{x,y\in\Lambda_N} \frac{\partial \Gamma_{j+1}(x,y)}{\partial s} \mathbb{E}_{\Gamma_{j+1}(s)}\Big[\frac{\partial^2 F(X,\varphi+\zeta)}{\partial\varphi(x)\varphi(y)}\Big]. \tag{3.103}$$

Let $f^z(x) = \Gamma_{j+1}(z,x)$ and $g^z(x) = \delta(z,x)$. It then follows with the notation from (3.38) that

$$\begin{aligned}\mathbb{E}_{\Gamma_{j+1}(s)}&[F(X,\varphi+\zeta)] - \mathbb{E}_{\Gamma_{j+1}(s')}[F(X,\varphi+\zeta)] \\ &= \frac{1}{2}\sum_{z\in X^*}\int_s^{s'} ds''\,\mathbb{E}_{\Gamma_{j+1}(s'')}[D^2F(X,\varphi+\zeta;f^z,g^z)].\end{aligned} \tag{3.104}$$

By taking the $\|\cdot\|_{\mathfrak{h},T_{j+1}(\overline{X})}$ norm of this and using Proposition 3.3.5, it follows that the left-hand side of (3.84) is bounded by

$$|s-s'|\left(2^{|X|_j}\mathfrak{h}^{-2}\sum_{z\in X^*}\|f^z\|_{C_j^2(X^*)}\|g^z\|_{C_j^2(X^*)}\|F(X)\|_{h,T_j(X)}\right). \tag{3.105}$$

Since X^* is finite and $\|f^z\|_{C_j^2(X^*)}$ and $\|g^z\|_{C_j^2(X^*)}$ are bounded uniformly in $|s| \leqslant \varepsilon_s\theta_J$ (their dependence on j and X is not relevant), the claim follows. The case $X \in \mathscr{P}_{j+1}^c$ assuming (3.85) follows using the same proof, and we now obtain

$$|s-s'|\left(2^{L^2|X|_{j+1}}\mathfrak{h}^{-2}\sum_{z\in X^*}\|f^z\|_{C_{j+1}^2(X^*)}\|g^z\|_{C_{j+1}^2(X^*)}\sup_\varphi\frac{\|F(X)\|_{h,T_{j+1}(X,\varphi)}}{G_j(X,\varphi)}\right) \tag{3.106}$$

instead of (3.105). $\square$

3.7 Properties of the regulator

We prove Lemma 3.3.10, Lemma 3.3.3, Lemma 3.3.4, Lemma 3.3.9 and Proposition 3.3.5 in this section.

3.7.1 Proof of Lemma 3.3.10

In this section and the next, we focus on purely geometric results on the L^2 and L^∞ function spaces on the lattice. The proofs heavily depend on formulas derived from the lattice versions of the Sobolev estimates, see Appendix 3.A.

Proof of Lemma 3.3.10. We state the definition of G_{j+s} explicitly as

$$\begin{aligned} G_{j+s}(X,\varphi) &= \exp\Big\{\kappa_L\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(X)} + c_2\kappa_L\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} + \kappa_L W_{j+s}(X,\nabla^2_{j+s}\varphi)^2\Big\} \\ W_{j+s}(X,\nabla^a_j\varphi)^2 &= \sum_{B\in\mathscr{B}_{j+s}(X)} \|\nabla^a_{j+s}\varphi\|^2_{L^\infty(B^*)} \end{aligned} \tag{3.107}$$

For brevity, $s+M^{-1}$ will be denoted s' and $X_{s'}$ will be denoted X'. We will bound each term appearing in $\log G_{j+s}(X,\varphi+\xi)$. First, $\|\nabla\varphi\|^2_{L^2(X)}$ will be isolated from $\|\nabla(\varphi+\xi)\|^2_{L^2(X)}$. Let $B\in\mathscr{B}_{j+s}(X)$ and without loss of generality, let B, l_i ($i=1,2,3,4$) be as above but $B=[1,L^{j+s}]^2$. Then by discrete integration by parts,

$$\sum_{x\in B}\nabla^{e_1}\varphi(x)\nabla^{e_1}\xi(x) = -\sum_{x\in l_3}\xi(x)\nabla^{-e_1}\varphi(x) - \sum_{x\in l_1}\xi(x+e_1)\nabla^{e_1}\varphi(x) + \sum_{x\in B}\xi(x)\nabla^{e_1}\nabla^{-e_1}\varphi(x). \tag{3.108}$$

Hence in particular, summing this over each direction $\pm e_1,\pm e_2$, $B\in\mathscr{B}_{j+s}(X)$, and using the AM-GM inequality,

$$\begin{aligned} t(\nabla\varphi,\nabla\xi)_X &\leqslant \tau t\|\xi\|^2_{L^2_{j+s}(X)} + \tau^{-1}t\|\nabla^2_{j+s}\varphi\|^2_{L^2_{j+s}(X)} + \tau t\|\xi\|^2_{L^2_{j+s}(\partial X)} + \tau^{-1}t\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} \\ &\leqslant 2\tau W_{j+s}(X,\xi)^2 + \tau^{-1}\big(\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} + W_{j+s}(X,\nabla^2_{j+s}\varphi)^2\big) \end{aligned} \tag{3.109}$$

for any $\tau>0$, and hence

$$\begin{aligned} \|\nabla_{j+s}(\varphi+\xi)\|^2_{L^2_{j+s}(X)} \leqslant{}& \|\nabla_{j+s'}\varphi\|^2_{L^2_{j+s'}(X)} + \|\nabla_{j+s}\xi\|^2_{L^2_{j+s}(X)} \\ &+ 2\tau W_{j+s}(X,\xi)^2 + \tau^{-1}\big(\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} + W_{j+s}(X,\nabla^2_{j+s}\varphi)^2\big). \end{aligned} \tag{3.110}$$

Next, we will use rather trivial bound on the other two terms of $\log G_{j+s}$:

$$\|\nabla_{j+s}(\varphi+\xi)\|^2_{L^2_{j+s}(\partial X)} \leqslant 2\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} + 2W_{j+s}(X,\nabla_{j+s}\xi)^2 \tag{3.111}$$

$$\|\nabla_j^2(\varphi+\xi)\|^2_{L^\infty(B^*)} \leqslant 2\|\nabla_j^2\varphi\|^2_{L^\infty(B^*)} + 2\|\nabla_j^2\xi\|^2_{L^\infty(B^*)} \tag{3.112}$$

By (3.110), (3.111), (3.112) and setting $c_4 = \max\{2, 2\tau, 2c_2\}$,

$$\begin{aligned}\frac{1}{\kappa_L}\log G_{j+s}(X,\varphi+\xi) \leqslant \|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(X)} + (2c_2+\tau^{-1})\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} \\ +2(1+\tau^{-1})W_{j+s}(X,\nabla^2_{j+s}\varphi) + \frac{1}{\kappa_L}\log g_{j+s}(X,\xi).\end{aligned} \tag{3.113}$$

Now by repeated application of the discrete Sobolev trace theorem [12, (3.174)],

$$\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X)} \leqslant \|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X')} + 10\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(X'\backslash X)} + 10W_{j+s}(\nabla^2_{j+s}\varphi, X'\backslash X) \tag{3.114}$$

hence by choosing $\tau = c_2^{-1}$ and $30c_2 \leqslant 1$,

$$\begin{aligned}&\frac{\log(G_{j+s}(X,\varphi+\xi)/g_{j+s}(X,\xi))}{\kappa_L} \\ &\leqslant \|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(X')} + 3c_2\|\nabla_{j+s}\varphi\|^2_{L^2_{j+s}(\partial X')} + 2(1+\tau^{-1})W_{j+s}(\nabla^2_{j+s}\varphi, X') \\ &\leqslant \|\nabla_{j+s'}\varphi\|^2_{L^2_{j+s'}(X')} + 3\ell^{-1}c_2\|\nabla_{j+s'}\varphi\|^2_{L^2_{j+s'}(\partial X')} + 2\ell^{-2}(1+\tau^{-1})W_{j+s'}(\nabla^2_{j+s'}\varphi, X').\end{aligned} \tag{3.115}$$

Hence the conclusion follows upon taking ℓ large enough. □

3.7.2 Proof of Lemmas 3.3.3 and 3.3.4

Proof of Lemma 3.3.3. We first collect the following elementary but fundamental inequality. For any function $f : \Lambda_N \to \mathbb{R}$, any connected polymer $X \in \mathscr{P}_j^c$ (not necessarily small) and $x_0 \in X$,

$$\max_{x\in X}|f(x)-f(x_0)| \leqslant 2|X|_j L^j\|\nabla f\|_{L^\infty(X)} = 2|X|_j\|\nabla_j f\|_{L^\infty(X)}. \tag{3.116}$$

Observing that $2|X^*|_j \leqslant C_d$ for some $C_d > 0$ when $X \in \mathscr{S}_j$, this gives

$$\|\delta\varphi\|_{L^\infty(X^*)} \leqslant C_d\|\nabla_j\varphi\|_{L^\infty(X^*)}, \quad X \in \mathscr{S}_j. \tag{3.117}$$

Similarly, applying (3.116) with the choice $f = \nabla_j^\mu \varphi$ for $\mu \in \hat{e} = \{\pm e_1, \pm e_2\}$ to obtain that $|\nabla_j^\mu \varphi(x)| \leqslant |\nabla_j^\mu \varphi(y)| + C_d \|\nabla_j^2 \varphi\|_{L_\infty(X^*)}$ for any $x, y \in X^*$, averaging over y in X and μ in $\hat{e}$, taking squares and using that $(a+b)^2 \leqslant 2(a^2+b^2)$, one obtains for any $X \in \mathscr{C}_j$ that

$$\|\nabla_j \varphi\|_{L^\infty(X^*)}^2 \leqslant C\big(|X|_j^{-1} \|\nabla_j \varphi\|_{L_j^2(X)}^2 + W_j(X, \nabla_j^2 \varphi)^2\big), \tag{3.118}$$

where we also used that

$$\|\nabla_j^2 \varphi\|_{L^\infty(X^*)} \leqslant W_j(X, \nabla_j^2 \varphi), \tag{3.119}$$

which follows from (3.42). Recalling $\|\cdot\|_{C_j^2(X^*)}$ from (3.37), combining (3.117), (3.118) and (3.119) while noting that $\nabla_j \delta\varphi = \nabla_j \varphi$, one readily infers that

$$\|\delta\varphi\|_{C_j^2(X^*)}^2 \leqslant C\big(\|\nabla_j \varphi\|_{L_j^2(X)}^2 + W_j(X, \nabla_j^2 \varphi)^2\big), \quad X \in \mathscr{S}_j, \tag{3.120}$$

from which (3.70) follows in view of (3.41) by means of the elementary inequality $t^k \leqslant C(k)e^{t^2}$, valid for all $t \geqslant 0$. □

Proof of Lemma 3.3.4. The bound (3.72) is a direct consequence of the first estimate in (6.100) of [23, Lemma 6.21] upon taking a product over $B \in \mathscr{B}_j(X)$ (for the reader's orientation, the quantity $e^{c_w \kappa_L w_j(\varphi, B)}$ for $B \in \mathscr{B}_j$ corresponds to $G^2_{\text{strong},\varphi}(B)$ in the notation of [23]). In particular, the presence of the factors 2^{-n} in (3.35) and (3.36), absent in [23, (6.67)], is inconsequential for the validity of these results. The same applies to further references to [23] in the sequel.

Note also that, while the value of c_w is fixed in [23] as $c_w = 2$ and exponent inside the definition of G_j is multiplied by a factor (called c_1) chosen large enough, we take c_w small, which is equivalent. Finally, note that (3.72) does not rely on the presence of the $\|\cdot\|^2_{L_j^2(\partial X)}$-term in G_j, i.e., (3.72) holds with $c_2 = 0$ in (3.41).

The inequality (3.73) is the content of (6.103) in [23]. Here G_j and c_2 corresponds to $G_{\varphi'}$ and c_3, respectively, in the notation of [23]. Whereas c_1 (mentioned above) is fixed beforehand in [23], the asserted dependence of parameters above (3.73) on c_1 follows by inspection of the proof of [23, Lemma 6.22], see in particular [23, (6.105)]. □

3.7.3 Proof of Lemma 3.3.9

Proof of (3.81) *of Lemma 3.3.9.* By the Sobolev inequality, Lemma 3.A.3, for each $a = 0,1,2$, we have

$$W_{j+s}(X, \nabla^a_{j+s}\zeta) \leqslant C_{a,d} \|\nabla^a_{j+s}\zeta\|_{L^\infty(X^*)} \leqslant C'_{a,d} \sum_{b=0,1,2} \|\nabla^{a+b}_{j+s}\zeta\|_{L^2_{j+s}(X^*)}. \tag{3.121}$$

Plugging this into the definition of $g_{j+s}(X,\zeta)$ with scaled coefficients give the desired result. □

For the proof of (3.82), we will need the following general estimate for Gaussian fields; see [23, Lemma 6.28] for a proof.

Lemma 3.7.1. *Let ζ be a centered real-valued Gaussian field on a finite set X with covariance matrix C and suppose that the largest eigenvalue of C is smaller or equal to $\frac{1}{2}$. Then*

$$\mathbb{E}\big[\exp(\frac{1}{2}\sum_{x\in X}\zeta(x)^2)\big] \leqslant e^{\mathrm{Tr}C}. \tag{3.122}$$

Applying this lemma to gradients of the slices ξ_k (see Section 3.1.3) gives the following lemma.

Lemma 3.7.2. *For any $j \in \{1,\dots,N\}$, $k \in \{1,\dots,N'\}$, $s = \frac{k-1}{N'}$, $s' = \frac{k}{N'}$, let $Y \in \mathscr{P}^c_{j+s}$, and let ξ be a centered Gaussian field with covariance $\Gamma_{j+s,j+s'}$. For a multiindex $(\mu) = (\mu_1,\dots,\mu_a) \in \{\pm e_1, \pm e_2\}^a$ for $a \in \{0,1,2,3,4\}$, then let $\eta_{a,(\mu)}$ be the Gaussian field defined by*

$$\eta_{a,(\mu)}(x) = \rho_J^2(\log L)^{-1}L^{-(j+s)}\nabla^{\mu_1\cdots\mu_a}_{j+s}\xi(x). \tag{3.123}$$

Then there is a constant $C' > 0$ such that if $t \leqslant (2C'\ell^2)^{-1}$, then (recall $L = \ell^M$)

$$\mathbb{E}\big[e^{\frac{t}{2}\sum_{x\in Y}\eta(x)^2}\big] \leqslant e^{tC'M^{-1}|Y|_{j+s}} \tag{3.124}$$

where $|Y|_{j+s}$ denotes the number of L^{j+s}-blocks contained in Y.

Proof. By Lemma 3.1.4, for all $x,y \in \Lambda_N$ and (μ) as above, defining $\alpha = (\mu,-\mu)$ to be the concatenated multi-index (of length $2a$),

$$\big|\mathbb{E}^{\xi_k}\big[(\nabla^{(\mu)}_{j+s}\xi_k)(x)(\nabla^{(\mu)}_{j+s}\xi_k)(y)\big]\big| \leqslant C_\alpha \rho_J^{-2}\log\ell, \tag{3.125}$$

which follows by considering the (worst) case estimate $|\alpha| = 0$ in Lemma 3.1.4. Letting $H_{a,(\mu)}(x,y) = \mathrm{Cov}(\eta_{a,(\mu)}(x), \eta_{a,(\mu)}(y))$, it follows from (3.123) and (3.125) that there is a constant $C' > 0$ such that for all $x,y \in Y$ and $a = 0,1,\dots,4$,

$$|H_{a,(\mu)}(x,y)| \leqslant C'(\log L)^{-1} L^{-2(j+s)} \log \ell. \tag{3.126}$$

Since also $H_{a,(\mu)}(x,y) = 0$ for $|x-y|_\infty \geqslant \frac{1}{2} L^{j+s+(N')^{-1}} = \frac{1}{2}\ell L^{j+s}$ by the finite-range property (2.8), it follows from (3.126) that

$$t \sup_{a\in\{0,\dots,4\}} \|H_{a,(\mu)}\|_{\mathrm{op}} \leqslant tC'\ell^2(\log\ell)(\log L)^{-1} = \frac{C't}{N'}\ell^2 \leqslant \frac{1}{2N'} \leqslant \frac{1}{2}, \tag{3.127}$$

whenever $t \leqslant (2C'\ell^2)^{-1}$. Thus $\sqrt{t}\eta_{a,(\mu)}$ satisfies the assumption of Lemma 3.7.1 so that with (3.126),

$$\mathbb{E}[e^{\frac{t}{2}\sum_{x\in Y}\eta(x)^2}] \leqslant e^{tC'(N')^{-1}|Y|_{j+s}} \tag{3.128}$$

as claimed. □

Proof of (3.82) *of Lemma 3.3.9.* Let $\eta_{a,(\mu)}$ be as in Lemma 3.7.2 and write $\kappa_L = c_\kappa \rho_J^2 (\log L)^{-1}$ with $c_\kappa > 0$. Then by (3.81) there is a constant $C > 0$ such that

$$g_{j+s}(Y,\xi) \leqslant \prod_{a=0}^{4}\prod_{(\mu)} \exp\Big(\frac{Cc_4c_\kappa}{2}(5\cdot 2^a)^{-1}\|\eta_{a,(\mu)}\|^2_{L^2(Y)}\Big), \tag{3.129}$$

and hence by Hölder's inequality,

$$\mathbb{E}[g_{j+s}(Y,\xi)] \leqslant \prod_{a=0}^{4}\prod_{(\mu)} \mathbb{E}\Big[\exp\Big(\frac{Cc_4c_\kappa}{2}\|\eta_{a,(\mu)}\|^2_{L^2(Y)}\Big)\Big]^{1/(5\cdot 2^a)}. \tag{3.130}$$

Applying (3.124) with $t = Cc_4c_\kappa$, the right-hand side is bounded by $e^{Cc_4c_\kappa C'(N')^{-1}|Y|_{j+s}} \leqslant 2^{(N')^{-1}|Y|_{j+s}}$ when $c_\kappa \leqslant (2CC'c_4\ell^2)^{-1}$ is chosen small enough.

For the analogous conclusion for the last step with $\Gamma_N^{\Lambda_N}$ instead of Γ_{j+1}, we just need to use the decomposition (3.14) instead of (3.13) and recall from Lemma 3.1.4 that $\Gamma^{\Lambda_N}_{N-1+s,N-1+s'}$ satisfies the same estimates as $\Gamma_{N-1+s,N-1+s'}$. □

3.7.4 Proof of Proposition 3.3.5

Proof of Proposition 3.3.5. Assuming that c_2 is sufficiently small so that Lemma 3.3.10 applies, fix (with the right-hand sides as in the conclusion of Lemma 3.3.10)

$$\ell = \ell_0(c_2), \qquad c_4 = c_4(c_2). \tag{3.131}$$

By the subdecomposition $\Gamma_{j+1} = \Gamma_{j,j+1/N'} + \cdots + \Gamma_{j+1-1/N',j+1}$ and the corresponding decomposition of the field

$$\zeta = \sum_{k=1}^{N'} \xi_k \sim \mathscr{N}(0, \Gamma_{j+1}), \qquad \xi_k \sim \Gamma_{j+\frac{k-1}{N'}, j+\frac{k}{N'}}, \tag{3.132}$$

by repeated application of (3.83), for all $\varphi' \in \mathbb{R}^{\Lambda_N}$, $X \in \mathscr{P}_j^c$, it follows that

$$\mathbb{E}[G_j(X, \varphi' + \zeta)] \leqslant \prod_{k=1}^{N'} \mathbb{E}^{\xi_k}\big[g_{j+\frac{(k-1)}{N'}}(X_{\frac{k}{N'}}, \xi_k)\big] G_{j+1}(\overline{X}, \varphi'). \tag{3.133}$$

Now by Lemma 3.3.9, and recalling $|Y|_{j+s} = |X_{j+s}|_{j+s} \leqslant |X|_j$, we obtain the claim:

$$\mathbb{E}[G_j(X, \varphi' + \zeta)] \leqslant \big(2^{(N')^{-1}|X|_j}\big)^{N'} G_{j+1}(\overline{X}, \varphi') \leqslant 2^{|X|_j} G_{j+1}(\overline{X}, \varphi'). \tag{3.134}$$

The proof of the analogous conclusion for the last step with $\Gamma_N^{\Lambda_N}$ instead of Γ_{j+1} is analogous. □

3.8 Complex shift of variables

We prove Lemma 3.4.2 and Proposition 3.3.7 in this section.

3.8.1 Proof of Lemma 3.4.2

We prove Lemma 3.4.2 here. We first prove a lemma the justifies the complex shift of variables for subdecomposed fields.

Lemma 3.8.1. *Let $X \in \mathscr{P}_j$, F be a polymer activity such that $\|F(X)\|_{\mathfrak{h}, T_j(X)} < \infty$ and $\psi, \tilde{\psi} \in \mathbb{R}^{\Lambda_N}$ have $\|\psi\|_{C_j^2}, \|\tilde{\psi}\|_{C_j^2} < \mathfrak{h}/2$. Also, for $m^2 > 0$, let $C_s = \Gamma_{j+s,j+s'} + m^2)$ if $j+s' < N$ and $C_s = \Gamma^{\Lambda_N}_{j+s,j+s'} + m^2$ if $j+s' = N$. Then for $\varphi \in \mathbb{R}^{\Lambda_N}$,*

$$\int_{\mathbb{R}^{X^*}} d\xi_s e^{-\frac{1}{2}(\xi_s, C_s^{-1}\xi_s)} F(X, \varphi + \xi_s + \tilde{\psi}) = \int_{\mathbb{R}^{X^*}+i\psi} d\xi_s e^{-\frac{1}{2}(\xi_s, C_s^{-1}\xi_s)} F(X, \varphi + \xi_s + \tilde{\psi}) \tag{3.135}$$

Proof. Consider the orthonormal change of coordinates of $\mathbb{R}^{X^*}$ (with L^2-norm) from $(\delta_x : x \in X^*)$ to $(e_y : y \in A)$ (with $|A| = |X^*|$) such that $\psi = \alpha e_{y_0}$ for some $\alpha \in \mathbb{R}$ and $y_0 \in A$. Writing

$(\xi_{s,y})_{y\in A}$ for the coordinates for ξ_s in this basis (so $\xi_s = \sum_{y\in A}\xi_{s,y}e_y$), (3.135) is equivalent to

$$\int_{\mathbb{R}} d\xi_{s,y_0}\, e^{-\frac{1}{2}(\xi_s,C_s^{-1}\xi_s)}F(X,\varphi'+\xi_s+\tilde{\psi}) = \int_{\mathbb{R}+i\alpha} d\xi_{s,y_0}\, e^{-\frac{1}{2}(\xi_s,C_s^{-1}\xi_s)}F(X,\varphi'+\xi_s+\tilde{\psi}). \tag{3.136}$$

Thus it is enough to apply the Cauchy's integral formula and check

$$\left|e^{-\frac{1}{2}((\xi'+i\psi),C_s^{-1}(\xi'+i\psi)}F(X,\varphi'+\xi'+i(\tilde{\psi}+\psi))\right| \to 0 \quad \text{as } \xi'\to\infty \tag{3.137}$$

to take the limits $\xi_{s,y_0}\to\pm\infty$. But since $\|\tilde{\psi}+\psi\|_{C_j^2} < \mathfrak{h}$, Proposition 3.4.1 and Lemma 3.3.10 give

$$\begin{aligned}|F(X,\varphi'+\xi'+i(\psi+\tilde{\psi}))| &\leqslant \|F(X)\|_{\mathfrak{h},T_j(X)}G_j(X,\varphi'+\xi') \\ &\leqslant \|F(X)\|_{\mathfrak{h},T_j(X)}G_{j+s'}(X_{j+s'},\varphi')g_{j+s}(X_{j+s'},\xi')\end{aligned} \tag{3.138}$$

for $s = s+(N')^{-1}$. But also by Lemma 3.3.9,

$$\int_{\mathbb{R}^{X^*}} e^{-\frac{1}{2}(\xi',C_s^{-1}\xi')}g_{j+s}(X_{j+s'},\xi') < \infty, \tag{3.139}$$

thus in particular (3.137) holds. □

Lemma 3.8.2. *Let $X\in\mathscr{P}_j$, let F be a function such that $\|F\|_{\mathfrak{h},T_j(X)} < \infty$. Let $v_2\in\mathbb{R}^{\Lambda_N}$ be such that $\|\Gamma_{j+s,j+\bar{s}}v_2\|_{C_j^2} < \mathfrak{h}/2$ for each $s,\bar{s}\in\{0,(N')^{-1},\cdots,1\}$ and $s<\bar{s}$. If $\varphi'\in\mathbb{R}^{\Lambda_N}$, $v = v_1+iv_2$ for some $v_1\in\mathbb{R}^{\Lambda_N}$, then*

$$\mathbb{E}[F(X,\varphi'+\zeta+\Gamma_{j+1}v)] = e^{-\frac{1}{2}(v,\Gamma_{j+1}v)}\mathbb{E}\left[e^{(\zeta,v)}F(X,\varphi'+\zeta)\right]. \tag{3.140}$$

Proof. Since X does not play any role, we will drop it at most places. By change of variable $\zeta\to\zeta-\Gamma_{j+1}v_1$, we have

$$\mathbb{E}[F(\varphi+\zeta+\Gamma_{j+1}v_1+i\Gamma_{j+1}v_2)] = e^{-\frac{1}{2}(v_1,\Gamma_{j+1}v_1)}\mathbb{E}[e^{(\zeta,v_1)}F(\varphi+\zeta+i\Gamma_{j+1}v_2)]. \tag{3.141}$$

So we aim to prove, for any $m^2>0$ sufficiently small,

$$\mathbb{E}[e^{(\zeta^{(m^2)},v_1)}F(\varphi+\zeta^{(m^2)}+iCv_2)] = e^{-i(v_1,Cv_2)+\frac{1}{2}(v_2,Cv_2)}\mathbb{E}\left[e^{(\zeta^{(m^2)},v_1+iv_2)}F(\varphi+\zeta^{(m^2)})\right] \tag{3.142}$$

where now $C = \Gamma_{j+1} + N'm^2$ and $\zeta^{(m^2)} \sim \mathscr{N}(0, C|_{X^*})$. Once this is obtained, we can take the limit $m^2 \downarrow 0$ to conclude. But we have to be more careful than the real-valued shift of variables, as we only have restricted amount of analyticity for F.

Also having done the subscale decomposition $\zeta^{(m^2)} = \sum_{s=0}^{N'} \xi_s$ with $\xi_s \sim \mathscr{N}(0, \Gamma_{j+s,j+s'} + m^2)$ (where $s' = s + (N')^{-1}$), the proof of (3.142), upto an induction, reduces to proving

$$\begin{aligned} &\mathbb{E}\big[e^{(\xi_s, v_1)} F(\varphi' + \xi_s + i(C_s + C_{>s})v_2)\big] \\ &\qquad = e^{-i(v_1, C_s v_2) + \frac{1}{2}(v_2, C_s v_2)} \mathbb{E}\big[e^{(\xi_s, v_1 + iv_2)} F(\varphi' + \xi_s + iC_{>s} v_2)\big] \end{aligned} \tag{3.143}$$

where $C_s = \Gamma_{j+s,j+s'} + m^2$, $C_{>s} = \sum_{\bar{s}>s} C_{\bar{s}}$, $\xi_s \sim \mathscr{N}(0, C_s)$ and $\varphi' = \varphi + \sum_{\bar{s}\neq s} \xi_{\bar{s}}$. But this is just (3.135) with $\psi = C_s v_2$ and $\tilde{\psi} = C_{<s} v_2$, whose conditions can be checked by the assumptions on v_2.

□

Then we are only left to bound the size of $\Gamma_{j+s,j+s'}\tilde{\mathfrak{f}}$ in order to verify Lemma 3.4.2. In the next lemma, we recall $I_{N'} = \{0, \cdots, 1 - (N')^{-1}\}$.

Lemma 3.8.3. *For $\alpha \in \{1, \cdots, \mathfrak{n}\}$, let $\mathfrak{f}_\alpha$ be as in* (A'_f) *and let*

$$u_{j+s,j+\bar{s},\alpha} = \Gamma_{j+s,j+s'} \tilde{\mathfrak{f}}_\alpha, \quad \bar{s} > s,\ s \in I_{N'},\ \bar{s} \in (N')^{-1} + I_{N'} \tag{3.144}$$

for $j \geqslant j_s$. Then $u_{j+s,j+\bar{s},\alpha}$ is supported on $B^0_{j+\bar{s}}$, the unique $j+\bar{s}$-block containing 0 and for each $n \geqslant 0$ and

$$\|\nabla^n u_{j+s,j+\bar{s},\alpha}\|_{L^\infty} \leqslant \begin{cases} C_0 M \rho^2 \log L & \text{if } n = 0 \\ C_n M \rho^2 L^{-n(j+s)} & \text{if } n \geqslant 1 \end{cases} \tag{3.145}$$

Also, $u_{j+1,\alpha}$ defined by (3.19) *admits decomposition*

$$u_{j+1,\alpha} = \sum_{s \in I_{N'}} u_{j+s,j+s+(N')^{-1},\alpha}. \tag{3.146}$$

Proof. The proof is identical to Lemma 3.1.5.

□

Proof of Lemma 3.4.2. Recall, $u_{j+1} = \Gamma_{j+1}\tilde{\mathfrak{f}}$ if $j \geqslant 0$. By definition,

$$\mathbb{E}_{(\omega)}\big[F(X, \varphi' + \zeta)\big] = e^{-\frac{1}{2}\omega^2(\tilde{\mathfrak{f}}, \Gamma_{j+1}\tilde{\mathfrak{f}})} \mathbb{E}\Big[e^{(\zeta, \omega\tilde{\mathfrak{f}})} F(X, \varphi' + \zeta)\Big] \tag{3.147}$$

Also, Lemma 3.8.3 and the condition $|\omega| < h_\omega < (C\log L)^{-1}\mathfrak{h}$ implies

$$\|\omega\Gamma_{j+s,j+\bar{s}}\tilde{\mathfrak{f}}\|_{C_j^2} < h/2, \quad \bar{s} > s,\ s \in I_{N'},\ \bar{s} \in (N')^{-1} + I_{N'} \tag{3.148}$$

whenever $\omega < (C\log L)^{-1}h$ for sufficiently large C. This verifies the assumptions of Lemma 3.8.2, completing the proof. □

3.8.2 Proof of Proposition 3.3.7

Lemma 3.8.4 (Gaussian integration by parts)**.** *Given $\tilde{\mathfrak{f}}$ satisfying* (A'_f)*, let $u_{j+1} = \Gamma_{j+1}\tilde{\mathfrak{f}}$. Let F be a polymer activity such that $\|F\|_{\vec{\mathfrak{h}},T_j(X)} < \infty$. Then*

$$\mathbb{E}\Big[D^k F(X,\varphi'+\zeta)(u_{j+1}^{\otimes k})\Big] = \sum_{l=0}^{k}\binom{k}{l}(u_{j+1},\tilde{\mathfrak{f}})^{l/2}\,\mathrm{He}_l(0)\mathbb{E}^{\zeta}_{\Gamma_{j+1}}\big[(\zeta|_{X^*},\tilde{\mathfrak{f}})^{k-l}F(X,\varphi'+\zeta)\big] \tag{3.149}$$

when $\mathrm{He}_{2p+1}(0) = 0$ *and* $\mathrm{He}_{2p}(0) = (-1)^p\frac{(2p)!}{2^p p!}$ *for $p \in \mathbb{Z}_{\geqslant 0}$.*

Proof. We drop X so that $F(X,\varphi)$ is written as $F(\varphi)$. It is sufficient to prove this for $\zeta \sim \mathscr{N}(0,C|_{X^*})$ and $u_{j+1} = C\tilde{\mathfrak{f}}$, where $C = \Gamma_{j+1} + m^2$ and $m^2 > 0$. Also, by integration by parts and upto a normalisation factor, $\mathbb{E}_C[D^k F(X,\varphi'+\zeta)(u_{j+1}^{\otimes k})]$ can be written in terms of the Lebesgue integral

$$\int_{\mathbb{R}^{X^*}} d\zeta\, e^{-\frac{1}{2}(\zeta,C^{-1}\zeta)}D^k F(\varphi'+\zeta)(u_{j+1}^{\otimes k}) = \int_{\mathbb{R}^{X^*}} d\zeta\,(-D)^k e^{-\frac{1}{2}(\zeta,C^{-1}\zeta)}(u_{j+1}^{\otimes k})F(\varphi'+\zeta). \tag{3.150}$$

The derivative on the right-hand side can be written as

$$(-1)^k\frac{d^k}{ds^k}\Big|_{s=0} e^{-\frac{1}{2}(\zeta+su_{j+1},C^{-1}(\zeta+su_{j+1}))} = (-1)^k\frac{d^k}{ds^k}\Big|_{s=0} e^{-\frac{1}{2}As^2-Bs-E} \tag{3.151}$$

where $A = (u_{j+1},C^{-1}u_{j+1}) = (u_{j+1},\tilde{\mathfrak{f}})$, $B = (\zeta,C^{-1}u_{j+1}) = (\zeta,\tilde{\mathfrak{f}})$ and $E = \frac{1}{2}(\zeta,C^{-1}\zeta)$. By the Leibniz formula,

$$(-1)^k\frac{d^k}{ds^k}e^{-\frac{1}{2}As^2-Bs-E} = \sum_{l=0}^{k}\binom{k}{l}\mathrm{He}_l(\sqrt{A}s)A^{l/2}B^{k-l}e^{-\frac{1}{2}As^2-Bs-E}, \tag{3.152}$$

where $\mathrm{He}_l(x)$ is the Hermite polynomial of degree l. Therefore we obtain

$$\begin{aligned}
&\mathbb{E}_C\Big[D^k F(X,\varphi'+\zeta)(u_{j+1}^{\otimes k})\Big] \\
&\quad \propto \sum_{l=0}^{k}\binom{k}{l}(u_{j+1},\tilde{\mathfrak{f}})^{l/2}\,\mathrm{He}_l(0)\int_{\mathbb{R}^{X^*}} d\zeta\, e^{-\frac{1}{2}(\zeta,C^{-1}\zeta)}(\zeta,\tilde{\mathfrak{f}})^{k-l}F(X,\varphi'+\zeta)\big]. \quad (3.153)
\end{aligned}$$

To conclude, take limit $m^2\to 0$. □

Lemma 3.8.5. *Let* $\tilde{\mathfrak{f}}$ *be as in* (A'_f) *and* u_{j+1} *be defined by* (3.19) *and* $j \geqslant j_s$. *If* $\|D^n F(X)\|_{n,T_j(X)} < \infty$*, there is some constant* $C>0$ *such that*

$$\begin{aligned}
&\big\|\mathbb{E}^{\zeta}_{\Gamma_{j+1}}\big[(\zeta|_{X^*},\tilde{\mathfrak{f}})^k D^n F(X,\varphi'+\zeta)\big]\big\|_{n,T_j(X,\varphi)} \\
&\qquad \leqslant C2^{|X|_j}(C\log L)^{\frac{3}{2}k}\Big(\Big\lceil\frac{k}{2}\Big\rceil !\Big)\,G_{j+1}(\overline{X},\varphi')\|D^nF(X)\|_{n,T_j(X)}.
\end{aligned} \quad (3.154)$$

Proof. Again, we drop X. Since $|D^nF(\varphi'+\zeta)| \leqslant \|D^nF\|_{n,T_j(X)}G_j(X,\varphi'+\zeta)$ we just need to bound $\mathbb{E}[|(\zeta|_{X^*},\tilde{\mathfrak{f}})|^k G_j(X,\varphi'+\zeta)]$. Firstly, after the subdecomposition $\zeta|_{X^*} = \sum_{s=0}^{N'-1}\xi_s|_{X^*}$ (as in (3.132)) and using the Jensen's inequality, we have

$$\begin{aligned}
|(\zeta|_{X^*},\tilde{\mathfrak{f}})|^k = \Big|\textstyle\sum_{s=0}^{N'-1}(\xi_s|_{X^*},\tilde{\mathfrak{f}})\Big|^k &\leqslant (N')^{k-1}\textstyle\sum_{s=0}^{N'-1}|(\xi_s|_{X^*},\tilde{\mathfrak{f}})|^k \\
&\leqslant (N')^{k-1}\textstyle\sum_{s=0}^{N'-1}\|\tilde{\mathfrak{f}}\|_{L^1}^k\|\xi_s\|^k_{L^\infty(X^*)}
\end{aligned} \quad (3.155)$$

But since

$$\|\xi_s\|^k_{L^\infty(X^*)} \leqslant (c_4\kappa)^{-\frac{k}{2}}\Gamma\Big(\frac{k+2}{2}\Big)\exp\big(c_4\kappa\|\xi_s\|^2_{L^\infty(X^*)}\big) \quad (3.156)$$

(Γ is the gamma function) and $\|\tilde{\mathfrak{f}}\|_{L^1} \leqslant \mathfrak{n}M\rho^2 \leqslant 1$, we see for some $C>0$ that

$$|(\zeta|_{X^*},\tilde{\mathfrak{f}})|^k \leqslant (N')^{k-1}\sum_{s=0}^{N'-1}C^k\kappa^{-\frac{k}{2}}\Big(\Big\lceil\frac{k}{2}\Big\rceil !\Big)e^{c_4\kappa\|\xi_s\|^2_{L^\infty(X^*)}}. \quad (3.157)$$

Secondly, by repeated application of Lemma 3.3.10,

$$G_j(X,\varphi'+\zeta) \leqslant \prod_{s\in I_{N'}} g_{j+(N')^{-1}s}(X_{j+(N')^{-1}s},\xi_s)G_{j+1}(\overline{X},\varphi') \quad (3.158)$$

(as in (3.133)). But since $e^{c_4\kappa\|\xi_s\|^2_{L^\infty(X^*)}} \leqslant g_{j+(N')^{-1}s}(X,\xi_s)$, combining this with (3.157), we have

$$\left|(\zeta|_{X^*},\tilde{\mathfrak{f}})\right|^k G_j(X,\varphi'+\zeta) \leqslant (N')^k C^k \kappa^{-\frac{k}{2}} \left\lceil\frac{k}{2}\right\rceil! \, G_{j+1}(\overline{X},\varphi') \prod_{s=0}^{N'-1} g_{j+(N')^{-1}s}(X_{j+(N')^{-1}s},\xi_s)^2. \tag{3.159}$$

Also Lemma 3.3.9 says

$$\mathbb{E}\Big[\prod_{s=0}^{N'-1} g_{j+(N')^{-1}s}(X_{j+(N')^{-1}s},\xi_s)^2\Big] \leqslant 2^{|X|_j}, \tag{3.160}$$

so the conclusion follows from recalling that $N' = \frac{\log L}{\log \ell}$ and $\kappa = c_\kappa \rho_J^2 (\log L)^{-1}$.

□

Lemma 3.8.6. *Let $\tilde{\mathfrak{f}}$ and u_{j+1} be as in the previous Lemma. Let $h_\omega \leqslant (C\log L)^{-3/2}$ for sufficiently large C and $\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} < \infty$. Then*

$$\|\mathbb{E}[F(X,\varphi'+\zeta+\omega u_{j+1};\omega)]\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \leqslant C_2 2^{|X|_j} \|F(X)\|_{\vec{k}h,T_j(X)} G_{j+1}(\overline{X},\varphi') \tag{3.161}$$

for some $C_2 > 0$.

Proof. Again, we drop X. We prove this by making the bound

$$\begin{aligned}\sum_{l=0}^{\infty} \frac{h_\omega^l}{l!} \Big\| \frac{d^l}{d\omega^l}\Big|_{\omega=0} D^n \mathbb{E}[F(\varphi'+\zeta+\omega u_{j+1};\omega)\Big\|_{n,T_j(X,\varphi)} \\ \leqslant C 2^{|X|_j} \sum_{l=0}^{\infty} \frac{h_\omega^l}{l!} \|\partial_\omega^l D^n F(;\omega)\|_{n,T_j(X)} G_{j+1}(\overline{X},\varphi')\end{aligned} \tag{3.162}$$

for each $n \geqslant 0$ and C independent of n. Also, since n does not play any role in the proof, we will just prove this for the case $n = 0$. To show this, first make expansion

$$\frac{d^l}{d\omega^l}\Big|_{\omega=0} F(\varphi'+\zeta+\omega u_{j+1};\omega) = \sum_{k=0}^{l} \binom{l}{k} D^{l-k}\partial_\omega^k F(\varphi'+\zeta;0)(u_{j+1}^{\otimes l-k}) \tag{3.163}$$

and by Lemma 3.8.4,

$$\begin{aligned}&\mathbb{E}\big[D^{l-k}\partial_\omega^k F(\varphi'+\zeta;0)(u_{j+1}^{\otimes l-k})\big] \\ &= \sum_{m=0}^{l-k} \binom{l-k}{m} (u_{j+1},\tilde{\mathfrak{f}})^{\frac{m}{2}} \,\mathrm{He}_m(0) \mathbb{E}\big[(\zeta|_{X^*},\tilde{\mathfrak{f}})^{l-k-m} \partial_\omega^k F(\varphi'+\zeta;0)\big],\end{aligned} \tag{3.164}$$

while by Lemma 3.8.5,

$$\begin{aligned} &\left|\mathbb{E}\left[(\zeta|_{X^*},\tilde{\mathfrak{f}})^{l-k-m}\partial_\omega^k F(\varphi'+\zeta;0)\right]\right| \\ &\leqslant C_2 2^{|X|_j}(C_1\log L)^{\frac{3}{2}(l-k-m)}\left(\left\lceil\frac{l-k-m}{2}\right\rceil!\right)G_{j+1}(\overline{X},\varphi')\|\partial_\omega^k F\|_{0,T_j(X)} \end{aligned} \tag{3.165}$$

Combining these bounds, using $(u_{j+1},\tilde{\mathfrak{f}}) \leqslant C(M,\rho)\log L$ and $\mathrm{He}_{2p-1}(0)=0$, $\mathrm{He}_{2p}(0) = (-1)^p\frac{(2p)!}{2^p p!}$ for $p\in\mathbb{Z}_{\geqslant 0}$, we have

$$\begin{aligned} &\sum_{l=0}^{\infty}\frac{h_\omega^l}{l!}\left\|\frac{d^l}{d\omega^l}\right|_{\omega=0}\mathbb{E}[F(\varphi'+\zeta+\omega;\omega)]\Big\|_{0,,T_j(X,\varphi)} \\ &\leqslant C_2 2^{|X|_j}G_{j+1}(\overline{X},\varphi')\sum_{k=0}^{\infty}\sum_{l'=0}^{\infty}\frac{h_\omega^{l'+k}}{k!}\sum_{p=0}^{\lfloor\frac{l'}{2}\rfloor}C_3^{2p}\frac{(C_1\log L)^{\frac{3}{2}l'-2p}}{2^p p!(l'-2p)!}\left(\left\lceil\frac{l'-2p}{2}\right\rceil!\right)\|\partial_\omega^k F\|_{0,T_j(X)} \end{aligned} \tag{3.166}$$

after reparametrising $l'=l-k$, $m=2p$ and $C_3=(C_1/C)^{1/2}$. This is bounded by

$$C_2 2^{|X|_j}\sum_{k=0}^{\infty}\frac{h_\omega^k}{k!}\|\partial_\omega^k F(\cdot;\omega)\|_{0,T_j(X)}G_{j+1}(\overline{X},\varphi')\times\sum_{l'=0}^{\infty}\Big(\cdots\Big) \tag{3.167}$$

where

$$\sum_{l'=0}^{\infty}\Big(\cdots\Big) = \sum_{l'=0}^{\infty}\left((C_1\log L)^{\frac{3}{2}}h_\omega\right)^{l'}\sum_{p=0}^{\lfloor\frac{l'}{2}\rfloor}C_3^p\frac{(C_1\log L)^{-2p}}{2^p p!(l'-2p)!}\left\lceil\frac{l'-2p}{2}\right\rceil!. \tag{3.168}$$

But after using the trivial bound $\sum_{p=0}^{\lfloor\frac{l'}{2}\rfloor}\frac{1}{p!(l'-2p)!}\left\lceil\frac{l'-2p}{2}\right\rceil!\leqslant e$ and setting h_ω and L so that $\frac{1}{2}C_3(C_1\log L)^{-2}\leqslant 1$ and $h_\omega\leqslant\frac{1}{2}(C_1\log L)^{-3/2}$, we see that (3.167) is bounded by a constant that is independent of L.

□

Proof of Proposition 3.3.7. By Lemma 3.4.2,

$$\begin{aligned} \mathbb{E}_{(\omega)}[F(X,\varphi';\omega)] &= \frac{\mathbb{E}\left[e^{(\zeta,\tilde{\mathfrak{f}})}F(X,\varphi'+\zeta;\omega)\right]}{\exp\left(\frac{1}{2}\omega^2(\tilde{\mathfrak{f}},\Gamma_{j+1}\tilde{\mathfrak{f}})\right)} \\ &= \mathbb{E}[F(X,\varphi'+\zeta+\omega u_{j+1};\omega)]. \end{aligned} \tag{3.169}$$

Hence in fact

$$\|\mathbb{E}_{(\omega)}[F(X,\varphi'+\zeta;\omega)]\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} = \|\mathbb{E}[F(X,\varphi'+\zeta+\omega u_{j+1};\omega)]\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \tag{3.170}$$

and we obtain the bound (3.79) by applying Lemma 3.8.6. Also the analyticity of $\omega \mapsto \mathbb{E}_{(\omega)}[D^n F(X, \varphi' + \zeta; \omega)]$ is already proved in Lemma 3.4.4. $\square$

Appendix 3.A Lattice Sobolev estimates

The proof of Lemma 3.3.10 heavily depends on formulas derived on lattice versions of the Sobolev inequality and the trace theorem for Sobolev spaces. We include the versions we need here. To simplify notation, from now on we fix $d = 2$.

Trace theorem Consider a block $B \in \{1,\dots,R\}^2$ with $l_1 = \{0\} \times [1,R]$, $l_2 = [1,R] \times \{R+1\}$, $l_3 = \{R+1\} \times [1,R]$, $l_4 = [1,R] \times \{0\}$. Note that l_i's are the outer boundary, which are different from ∂B, which is the inner boundary.

Lemma 3.A.1. *For any $u : \Lambda \to \mathbb{R}$, if $(k,\mu_k) \in \{(1,-e_1),(2,e_2),(3,e_1),(4,-e_2)\}$,*

$$R^{-1} \sum_{x \in l_k} u(x)^2 \leqslant R^{-2} \sum_{x \in B} \Big(u(x)^2 + R|\nabla^{\mu_k} u(x)^2| \Big). \tag{3.171}$$

Proof. Without loss of generality, fix $k = 1$. Define a function $\xi : B \cup (\cup_{k=1}^4 l_k) \to \mathbb{R}$ by

$$\xi(ae_1 + be_2) = (R-a)/R. \tag{3.172}$$

Then

$$\begin{aligned} R^{-1} \sum_{x \in l_1} u(x)^2 &= R^{-1} \sum_{x \in l_1} \xi(x) u(x)^2 \\ &= R^{-1} \sum_{k=1}^{L^j} \nabla^{-e_1} \Big[\sum_{b=1}^{L^j} \xi(ke_1 + be_2) u(ke_1 + be_2)^2 \Big] \\ &= R^{-1} \sum_{x \in B} \nabla^{-e_1} (\xi(x) u(x)^2) \\ &= R^{-2} \sum_{x \in B} \Big((R\nabla^{-e_1} \xi(x)) u(x)^2 + \xi(x - e_1) R \nabla^{-e_1} (u(x)^2) \Big). \end{aligned} \tag{3.173}$$

But $|\xi(x)|, R|\nabla^{-e_1}\xi(x)| \leqslant 1$ for $x \in B$ and hence the result follows. $\square$

In particular, this lemma can be applied to the control the field on the boundary of the box by the field inside the box.

Corollary 3.A.2. *Let B be a box with outer boundary $\cup_k l_k$ as above and diameter $R \geqslant 10$. Then for $\varphi : \Lambda \to \mathbb{R}$,*

$$R^{-1} \sum_{x \in \cup_k l_k} |\nabla \varphi(x)|^2 \leqslant 10 \Big(R^{-2} \sum_{x \in B} |\nabla \varphi(x)|^2 + \|\nabla^2 \varphi\|_{L^\infty(B)} \Big). \tag{3.174}$$

Proof. In Lemma 3.A.1, set

$$u(x)^2 = |\partial^\mu \varphi(x)|^2. \tag{3.175}$$

Then for $\nu \in \hat{e} = \{\pm e_1, \pm e_2\}$,

$$\begin{aligned} R|u(x+\nu)^2 - u(x)^2| &= R|\nabla^\mu \varphi(x+\nu) + \nabla^\mu \varphi(x)|\,|\nabla^\nu \nabla^\mu \varphi(x)| \\ &\leqslant \frac{1}{2}|\nabla^\mu \varphi(x+\nu) + \nabla^\mu \varphi(x)|^2 + R^2 \|\nabla^2 \varphi\|_{L^\infty(B)}. \end{aligned} \tag{3.176}$$

so summation over $x \in B$ gives

$$R\sum_{x\in B} |\nabla^\nu u(x)^2| \leqslant \sum_{x\in B\cup l_k} |\nabla^\mu \varphi(x)|^2 + R^4 \|\nabla^2 \varphi\|_{L^\infty(B)}. \tag{3.177}$$

Therefore the lemma gives

$$R^{-1}\sum_{x\in l_k} |\nabla \varphi(x)|^2 \leqslant 2R^{-2} \sum_{x\in B\cup l_k} |\nabla \varphi(x)|^2 + R^2 \|\nabla^2 \varphi\|_{L^\infty(B)}. \tag{3.178}$$

If $R \geqslant 10$, we may send the l_k part in the sum $\sum_{x\in B\cup l_k}$ to the left-hand side to obtain

$$R^{-1}\sum_{x\in l_k} |\nabla \varphi(x)|^2 \leqslant \frac{10}{4}\Big(R^{-2} \sum_{x\in B} |\nabla \varphi(x)|^2 + \|\nabla^2 \varphi\|_{L^\infty(B)}\Big). \tag{3.179}$$

□

Sobolev inequality While the large field regulator G_j contains $\exp(\|\nabla^2 \varphi\|^2_{L^\infty(B^*)})$ for $B \in \mathscr{B}_j$, we have a nice estimate of Gaussian integration only for exponentials of quadratic forms. Hence it is desirable to bound $\|\nabla^2 \varphi\|^2_{L^\infty(B^*)}$ in terms of $\|\nabla^a \varphi\|^2_{L^2(B^*)}$, $a \geqslant 2$. This follows from the following Sobolev inequality. Here, we are using the convention $\|f\|_{L^2(X)} = \sum_{x\in X} |f(x)|^2$.

Lemma 3.A.3. *For B be square of diameter R as above. There exists constant $C > 0$ uniform in R such that for all $f : \{x \in \Lambda : d_1(x,B) \leqslant 2\} \to \mathbb{R}$,*

$$\|f\|^2_{L^\infty(B)} \leqslant C \sum_{a=0}^{2} R^{2a-2} \|\nabla^a f\|^2_{L^2(B)}. \tag{3.180}$$

Proof. Take $x \in [1, \frac{R+1}{2}]^2 \cap B$. By symmetry, the conclusion follows if we bound $f(x)^2$ in terms of $\|\nabla^a f\|^2_{L^2(B)}$, $a = 0,1,2$. Take

$$\xi_x(x + ae_1 + be_2) = \begin{cases} (1 - 3R^{-1}a)(1 - 3R^{-1}b) & \text{if } 0 \leqslant a, b \leqslant \frac{1}{3}R \\ 0 & \text{otherwise} \end{cases}. \tag{3.181}$$

Also let $D_x = \{x + ae_1 + be_2 : 0 \leqslant a, b \leqslant \frac{1}{3}R + 2\}$. Then

$$\begin{aligned} f(x)^2 = f(x)^2 \xi_x(x) &= \sum_{a,b=0}^{\lfloor R/3 \rfloor} \nabla^{e_1} \nabla^{e_2} (f(x + ae_1 + be_2)^2 \xi_x(x + ae_1 + be_2)) \\ &= \sum_{a,b=0}^{\lfloor R/3 \rfloor} \big(\nabla^{e_1}\nabla^{e_2} f(x + ae_1 + be_2)^2\big) \xi_x(x + (a+1)e_1 + (b+1)e_2) \\ &\qquad + \big(\nabla^{e_1} f(x + ae_1 + be_2)^2\big)\big(\nabla^{e_1}\xi_x(x + ae_1 + (b+1)e_2)\big) \\ &\qquad + \big(\nabla^{e_1} f(x + ae_1 + be_2)^2\big)\big(\nabla^{e_2}\xi_x(x + ae_1 + (b+1)e_2)\big) \\ &\qquad\qquad + f(x + ae_1 + be_2)^2 \nabla^{e_2}\nabla^{e_1} \xi_x(x + ae_1 + be_2)\Big). \end{aligned} \tag{3.182}$$

Noticing that $\|\nabla_j^a \xi_x\|_{L^\infty(B)} \leqslant 3^a$ for $a = 0,1,2$,

$$f(x)^2 \leqslant 9 \sum_{a=0}^{2} R^{a-2} \|\nabla^a f^2\|_{L^1(D_x)} \tag{3.183}$$

but also the AM-GM inequality implies

$$\|\nabla^2 f^2\|_{L^1(D_x)} \leqslant \frac{1}{2}\|\nabla f\|^2_{L^2(B)} + R^{-1}\|f\|^2_{L^2(B)} + R\|\nabla^2 f\|^2_{L^2(B)} \tag{3.184}$$

$$\|\nabla f^2\|_{L^1(D_x)} \leqslant R^{-1}\|f\|_{L^2(B)} + R\|\nabla f\|^2_{L^2(B)} \tag{3.185}$$

which completes the inequality. □

Appendix 3.B Completeness of the space of polymer activities

We prove the following proposition in this appendix.

Proposition 3.B.1. *For any $\mathfrak{h} > 0$, the space $\{F \in \mathcal{N}_j(X) : \|F\|_{\mathfrak{h}, T_j(X)} < \infty\}$ is a Banach space.*

Proof. Suppose $(F_k)_{k\geqslant 1}$ is a Cauchy sequence in the norm $\|\cdot\|_{\mathfrak{h},T_j(X)}$. Without loss of generality, we will assume $\|F_k - F_{k+1}\|_{\mathfrak{h},T_j(X)} \leqslant 2^{-k}$. In particular,

$$\frac{\mathfrak{h}^n}{n!}\|D^n(F_k - F_{k+1})(\varphi)\|_{n,T_j(X,\varphi)} \leqslant 2^{-k}G_j(X,\varphi) \tag{3.186}$$

for each $n \geqslant 0$. Therefore the pointwise limit exists for $(F_k)_{k\geqslant 1}$, say F. From the completeness of the spaces $C^k(\mathbb{R}^{X^*})$, it is also clear that F is smooth. In fact, if we define another normed space

$$\mathcal{N}_j' = \{K \text{ polymer activity } : \|K\|_{\mathfrak{h},T_j(X)}' < \infty\}, \tag{3.187}$$

$$\|K\|_{\mathfrak{h},T_j(X)}' = \sup_{\varphi\in\mathbb{R}^{X^*}} G_j(X,\varphi)^{-1}\Big(\sup_{n\geqslant 0}\frac{\mathfrak{h}^n}{n!}\|D^nK(\varphi)\|_{n,T_j(X,\varphi)}\Big) \tag{3.188}$$

then the pointwise limit satisfies $F \in \mathcal{N}_j'$. Now suppose $\|F\|_{\mathfrak{h},T_j(X)} = +\infty$. Then for each $M > 0$, there exists $\varphi_M \in \mathbb{R}^{X^*}$ and $N_M \in \mathbb{Z}$ such that

$$\sum_{n=0}^{N_M}\frac{\mathfrak{h}^n}{n!}\|D^nF(\varphi_M)\|_{n,T_j(X,\varphi_M)} \geqslant MG_j(X,\varphi_M). \tag{3.189}$$

But $\sum_{n=0}^{N_M}\frac{\mathfrak{h}^n}{n!}\|D^n(F_k - F_{k+1})(\varphi_M)\|_{n,T_j(X,\varphi_M)} \leqslant 2^{-k}G_j(X,\varphi_M)$ for each k so if we set $M > 1 + \|F_1\|_{\mathfrak{h},T_j(X)}$, this gives a contradiction. This proves $\|F\|_{\mathfrak{h},T_j(X)} < \infty$.

Finally, we have to prove $F_k \to F$ as $k \to \infty$ in the $\|\cdot\|_{\mathfrak{h},T_j(X)}$ norm. To see this, let $\overline{F}_k = F_k - F$, and notice that $(\overline{F}_k)_k$ is still Cauchy in the $\|\cdot\|_{\mathfrak{h},T_j(X)}$ norm. Suppose $\overline{F}_k$ does not converge to 0 as $k \to \infty$. By scaling and taking a subsequence if necessary, this means there is $\varphi \in \mathbb{R}^{X^*}$ such that

$$\sum_{n=0}^{\infty}\frac{\mathfrak{h}^n}{n!}\|D^n\overline{F}_k(\varphi_k)\|_{n,T_j(X,\varphi_k)} \geqslant G_j(X,\varphi_k). \tag{3.190}$$

But also since $\|D^n\overline{F}_k(\varphi,k)\|_{n,T_j(X,\varphi)} \to 0$ as $k \to \infty$, up to a subsequence, there exist sequences $(N_k)_{k\geqslant 0}$, $(M_k)_{k\geqslant 0}$ such that $N_k < M_k < N_{k+1}$ and

$$\sum_{n=N_k}^{M_k}\frac{\mathfrak{h}^n}{n!}\|D^n\overline{F}_k(\varphi_k)\|_{n,T_j(X,\varphi_k)} \geqslant \frac{2}{3}G_j(X,\varphi_k) \tag{3.191}$$

$$\sum_{n\in\mathbb{N}\setminus[N_k,M_k]}\frac{\mathfrak{h}^n}{n!}\|D^n\overline{F}_k(\varphi)\|_{n,T_j(X,\varphi)} \leqslant \frac{1}{3}G_j(X,\varphi) \quad \text{for all } \varphi \in \mathbb{R}^{X^*}. \tag{3.192}$$

But this implies $\|(\overline{F}_k - \overline{F}_{k+1})(\varphi_{k+1})\|_{\mathfrak{h},T_j(X,\varphi_{k+1})} \geqslant \frac{1}{3}G_j(X,\varphi_{k+1})$ which contradicts that $\overline{F}_k$ is a Cauchy sequence. Therefore $F_k \to F$ as $k \to \infty$. $\square$

Chapter 4

Polymer expansions

This chapter has two purposes. First, we discuss how the expansion (3.28)

$$Z_j^0(\varphi|\Lambda) = e^{-E_j|\Lambda|} \sum_{X \in \mathscr{P}_j(\Lambda)} e^{U_j(\Lambda \backslash X, \varphi)} K_j^0(X, \varphi), \tag{4.1}$$

can be propagated under the recursive fluctuation integrals $\mathbb{E}_{\Gamma_{j+1}}$ in (3.27). Due to the flexibility of the polymer expansion, we will see that it is not difficult to devise operations on polymer expansion to find triple $(E_{j+1}, U_{j+1}, K_{j+1}^0)$ satisfying the definition. However, the difficult part is to obtain E_{j+1} and U_{j+1} that approximate $\log Z_j^0$ relatively well and the remainder K_j^0 is suppressed small. This point is related to the second purpose of this chapter. By identifying the first-order terms of the reblocking operation and proving bounds on the higher-order terms, we reduce the problem of determining (E_{j+1}, U_{j+1}) to a problem of solving a linear equation, although the actual choice of (E_{j+1}, U_{j+1}) will be deferred to Chapter 6. This chapter has applicability in general renormalisation group analysis, as it does not require a specific form of the field theory. This is in contrast with what comes in the following chapters, where we rely on the symmetries of the polymer activities.

We outline the organisation of this chapter. In Section 4.1, we introduce a number of different ways to re-expand the polymer expansion after the fluctuation integral, exploiting the high degree of freedom of the polymer activities. In Section 4.2, we explain a clever expansion technique that cancels unwanted contributions from the artificial grid structure we have implemented. The polymer activity K_{j+1}^0 obtained from this operation exhibits a simple linear approximation, which is used to determine (E_{j+1}, U_{j+1}) later. In Section 4.3, we insert external fields $\tilde{\mathfrak{f}}$ and v inside the polymer expansions. These external fields are required to bring the mesoscopic and the macroscopic observable of Theorem 1.1.3 and Theorem 1.1.5 into the picture. Combining with polymer operation techniques in Section 4.1, we see that this only adds a slight amount of complexity to the polymer expansions. In Section 4.4, we prove that the non-linear term of K_{j+1} is differentiable in its arguments and the derivative vanishes at 0. Thus the problem of how to keep K_j suppressed small reduces to the problem of how to control the linear approximation K_j.

4.1 Polymer operations

We start the discussion with basic operations on polymer activities. The ideas of this section is based on the principles developed in [23, Section 5].

4.1.1 Polymer powers

We introduce convenient notations

$$(F \circ_j G)(X) = \sum_{Y \in \mathscr{P}_j(X)} F(X \backslash Y) G(Y), \qquad (F \circledast_j G)(X) = \sum_{Y \in \mathscr{P}_j(X)}^{Y \not\sim X \backslash Y} F(X \backslash Y) G(Y), \tag{4.2}$$

for any $X \in \mathscr{P}_j$ and scale j polymer functions F, G. Then (3.28) is equivalent to

$$Z_j^0(\cdot|\Lambda) = e^{-E_j|\Lambda|}(e^{U_j} \circ_j K_j^0)(\Lambda, \cdot). \tag{4.3}$$

Each $\circ_j$ and $\circledast_j$ is a commuting binary operator. These expansions have natural connections with our polymer functions of interest, in view of the next lemma.

Lemma 4.1.1. *Let F_1, F_2 be multiplicative polymer functions and G_1, G_2 be polymer activities at scale j, i.e., $F_\alpha(X) = F_\alpha^X$ and $G_\alpha(X) = G_\alpha^{\otimes X}$ for $X \in \mathscr{P}_j$ and $\alpha = 1, 2$. Then*

$$(F_1 + F_2)^X = (F_1 \circ_j F_2)(X), \qquad (G_1 + G_2)^{\otimes X} = (G_1 \circledast_j G_2)(X). \tag{4.4}$$

Proof. Recall that the polymer powers F^X and $F^{\otimes X}$ are defined by (3.30). The proof is direct from the definitions. $\square$

4.1.2 Reblocking the perturbed field

We will encounter polymer expansions with two different types of field perturbation. The first type is perturbing the whole field, so we consider

$$F^{(1)}(\varphi) = (e^{U_j} \circ_j K_j)(\Lambda, \varphi + \omega u) \tag{4.5}$$

and the second type is perturbing only the field in U_j, so we consider

$$F^{(2)}(\varphi) = (e^{U_j(\cdot+\omega u)} \circ_j K_j)(\Lambda, \varphi), \tag{4.6}$$

cf. the shift of variable of Lemma 3.4.2. It is not difficult to put them in the standard form (4.3) using the polymer powers.

Definition 4.1.2. *Given $u \in \mathbb{C}^\Lambda$ and (U_j, K_j) be functions of (X, φ). Define*

$$\mathscr{R}_j^{(1)}[u, U_j, K_j](X) = \Big(\big(e^{U_j(\cdot+u)} - e^{U_j}\big) \circ_j K_j(\cdot + u)\Big)(X) \tag{4.7}$$

$$\mathscr{R}_j^{(2)}[u, U_j, K_j](X) = \Big(\big(e^{U_j(\cdot+u)} - e^{U_j}\big) \circ_j K_j\Big)(X). \tag{4.8}$$

Lemma 4.1.3. *Suppose U_j and K_j satisfy $e^{U_j(X)} = (e^{U_j})^X$ and $K_j(X) = K_j^{\otimes X}$, respectively. Then*

$$F^{(\alpha)}(\varphi) = \Big(e^{U_j} \circ_j \mathscr{R}_j^{(\alpha)}[\omega u, U_j, K_j]\Big)(\Lambda, \varphi) \tag{4.9}$$

for both $\alpha = 1, 2$.

If we assume in addition that U_j has form (3.50) *and K_j is a polymer activity, and fi u is supported on some $Y_u \in \mathscr{P}_j$, then $\mathscr{R}_j^{(2)}[u, U_j, K_j](X) = K_j(X)$ whenever $X \cap Y_u = \emptyset$ and $\mathscr{R}_j^{(1)}[u, U_j, K_j](X) = K_j(X)$ whenever $X \cap (Y_u)^* = \emptyset$ (where we recall that $(\cdot)^*$ is the small set neighbourhood).*

Proof. For brevity, denote $U_j(X, \varphi + \omega u) = U_j'(X, \varphi)$, $K_j(X, \varphi + \omega u) = K_j'(X, \varphi)$ and drop φ. Then

$$\begin{aligned} F^{(1)} = (e^{U_j'} \circ K_j')(\Lambda) &= \big((e^{U_j} + e^{U_j'} - e^{U_j}) \circ_j K_j')\big)(\Lambda) \\ &= \Big(e^{U_j'} \circ_j \big((e^{U_j'} - e^{U_j}) \circ_j K_j'\big)\Big)(\Lambda) \end{aligned} \tag{4.10}$$

where we used Lemma 4.1.1 in the second line. This gives (4.9) for $\alpha = 1$. The proof is the same for $\alpha = 2$, just replacing K_j' by K_j.

To see the second point, just observe that $U_j(X, \varphi) = U_j(X, \varphi + u)$ if $X \cap Y_u = \emptyset$ and $K_j(Z, \varphi) = K_j(Z, \varphi + u)$ if $Z \cap Y_u^* = \emptyset$ (see Definition 3.2.1 for the definition of polymer activity).

□

The following lemma also suggests that the estimates on these operations are well-behaved.

Lemma 4.1.4. *Suppose $\beta \geqslant 8\mathfrak{h}^2 c_f^{-2}$, $2h_\omega \|u\|_{C_j^2} \leqslant \mathfrak{h}$, U_j is given in form* (3.50) *and $K_j \in \mathscr{N}_{j,h_\omega}$. Denoting $\vec{\mathfrak{h}} = (\mathfrak{h}, h_\omega)$, there exists $\varepsilon_r \equiv \varepsilon_r(\beta, L) > 0$ such that whenever $\max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\vec{\mathfrak{h}},T_j,A}\} \leqslant \varepsilon_r$ and $\omega \in \mathbb{D}_{h_\omega}$,*

$$\big\|\mathscr{R}_j^{(1)}[\omega u, U_j, K_j]\big\|_{(\mathfrak{h}/2,h_\omega),T_j,\frac{1}{2}A} \leqslant C \max\big\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\vec{\mathfrak{h}},T_j,A}\big\},, \tag{4.11}$$

$$\big\|\mathscr{R}_j^{(2)}[\omega u, U_j, K_j]\big\|_{\vec{\mathfrak{h}},T_j,\frac{1}{2}A} \leqslant C \max\big\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\vec{\mathfrak{h}},T_j,A}\big\} \tag{4.12}$$

for some absolute constant C (that does not depend on any other parameters).

Proof. For brevity, denote $\mathscr{R}^{(\alpha)} = \mathscr{R}_j^{(\alpha)}[\omega u, U_j, K_j]$ for $\alpha = 1,2$. By Lemma 3.4.4 and Lemma 3.2.8 with the assumption $\mathfrak{h} + h_\omega \|u\|_{C_j^2} < 2\mathfrak{h}$, one may write

$$\begin{aligned}\|U_j(B,\varphi+\omega u) - U_j(B,\varphi)\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} &\leqslant 2\|U_j(B,\varphi)\|_{\mathfrak{h}+h_\omega\|u\|,T_j(B,\varphi)} \\ &\leqslant CA^{-1}\|U_j\|_{\Omega_j^U}(\mathfrak{h}^2 + w_j(B,\varphi)^2).\end{aligned} \tag{4.13}$$

Then by (3.64), for $\|U_j\|_{\Omega_j^U} \leqslant \min\{1, \beta^{-1}\}$,

$$\begin{aligned}\|e^{U_j(B,\varphi+\omega u)} - e^{U_j(B,\varphi)}\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} &\leqslant \|e^{U_j(B,\varphi)}\|_{\vec{h},T_j(B,\varphi)} \|e^{U_j(B,\varphi+\omega u)-U_j(B,\varphi)} - 1\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \\ &\leqslant CA^{-1}\|U_j\|_{\Omega_j^U} e^{CA^{-1}\|U_j\|_{\Omega_j^U} w_j(B,\varphi)^2}.\end{aligned} \tag{4.14}$$

Hence, if $x := \max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\vec{\mathfrak{h}},T_j,A}\}$ is sufficiently smaller than $\min\{1, \beta^{-1}, c_w\kappa\}$, then by (3.64) and Lemma 3.3.4,

$$\begin{aligned}&\Big\|\Big(e^{U_j(\cdot,\varphi+\omega u)} - e^{U_j(\cdot,\varphi)}\Big)^{X\setminus Y} K_j(Y,\varphi)\Big\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} \\ &\qquad \leqslant A^{-|X|_j}(Cx)^{|X\setminus Y|_j} e^{c_w\kappa w_j(X\setminus Y,\varphi)^2} x^{|\mathrm{Comp}_j(Y)|} G_j(Y,\varphi) \\ &\qquad \leqslant A^{-|X|_j}(Cx)^{|X\setminus Y|_j} x^{|\mathrm{Comp}_j(Y)|} G_j(X,\varphi)\end{aligned} \tag{4.15}$$

for $X, Y \in \mathscr{P}_j$, $Y \subset X$. By the definition of $\mathscr{R}^{(2)}$, now with x sufficiently smaller than A,

$$\|\mathscr{R}^{(2)}(X)\|_{\vec{\mathfrak{h}},T_j(X)} \leqslant A^{-|X|_j} \sum_{Y \in \mathscr{P}_j(X)} (Cx)^{|X\setminus Y|_j} x^{|\mathrm{Comp}_j(Y)|} \leqslant C(A/2)^{-|X|_j} x, \tag{4.16}$$

which is exactly (4.11).

For the bound on $\mathscr{R}^{(1)}$, we just have to replace the norm on K_j using Lemma 3.4.4 with the assumption $\frac{\mathfrak{h}}{2} + h_\omega \|u\| < \mathfrak{h}$,

$$\|K_j(Y, \varphi + \omega u)\|_{(\mathfrak{h}/2, h_\omega), T_j(Y,\varphi)} \leqslant \|K_j(Y,\varphi)\|_{\vec{\mathfrak{h}}, T_j(X,\varphi)}. \tag{4.17}$$

Thus we obtain (4.12). □

4.1.3 Reblocking to next scale

Changing the scale of the polymer expansion is based on the *coarse-graining-reblocking* operations defined as

$$\overline{K}^0_j(X,\varphi) = \sum_{Y \in \mathscr{P}_j(X)}^{\overline{Y}=X} e^{U_j(X \setminus Y, \varphi)} K^0_j(Y,\varphi), \qquad X \in \mathscr{P}_{j+1}. \tag{4.18}$$

Then it is basic to check that

$$Z^0_j(\cdot|\Lambda) = e^{-E_j|\Lambda|}(e^{U_j} \circ_j K^0_j)(\Lambda,\cdot) = e^{-E_j|\Lambda|}(e^{U_j} \circ_{j+1} \overline{K}^0_j)(\Lambda,\cdot). \tag{4.19}$$

The linearised coarse-graining-reblocking is also of importance, defined as

$$\mathbb{S}K^0_j(X,\varphi) = \sum_{Y \in \mathscr{P}^c_j(X)}^{\overline{Y}=X} K^0_j(Y,\varphi), \qquad X \in \mathscr{P}^c_{j+1}. \tag{4.20}$$

As the name indicates, $\mathbb{S}$ is a linear operator mapping a j-scale polymer activity to a $j+1$-scale polymer activity. We see from the following proposition that only a limited number of terms actually contribute in $\mathbb{S}K^0_j$, since the norm of the polymer activities supported on large sets contract upon $\mathbb{S}$.

Proposition 4.1.5. *There exists a geometric constant $\eta > 0$ such that the following holds when $L \geqslant 2^d + 1 = 5$. For $h_\omega \leqslant (C \log L)^{-\frac{3}{2}}$ for sufficiently large C, $A^{\frac{\eta}{2}} \geqslant 2^{\frac{2+\eta}{2}} eL^3$, $X \in \mathscr{P}^c_{j+1}$, and any scale-$j$ polymer activity F with $\|F\|_{h,T_j} < \infty$,*

$$\|\mathbb{S}(\mathbb{E}_{(\omega)}[F 1_{Y \notin \mathscr{S}_j}])(X)\|_{\vec{\mathfrak{h}}, T_{j+1}(X)} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}} \|F\|_{\vec{\mathfrak{h}}, T_j}. \tag{4.21}$$

The factor L^{-1} will compensate the loss of the factor of 2 in the $A/2$ factor in Lemma 3.3.6. The proof is a consequence of the following combinatorial lemma.

Lemma 4.1.6 (Lemmas 6.14–15 of [23])**.** *There exists a geometric constant $\eta > 0$ such that the following holds when $L \geqslant 2^d + 1 = 5$. For every $X \in \mathscr{P}_j$,*

$$(1+\eta)|\overline{X}|_{j+1} \leqslant |X|_j + 8(1+\eta)|\mathrm{Comp}_j(X)|. \tag{4.22}$$

Moreover, if X is connected but not a small set, then

$$(1+\eta)|\overline{X}|_{j+1} \leqslant |X|_j. \tag{4.23}$$

Proof of Proposition 4.1.5. By (4.23), we have $|Y|_j \geqslant (1+\eta)|X|_{j+1}$ if $\overline{Y} = X$ and $Y \in \mathscr{P}_j^c \backslash \mathscr{S}_j$, so applying successively (4.20), (3.44)-(3.63) and Proposition 3.3.7, one obtains

$$\begin{aligned} &\|\mathbb{SE}_{(\omega)}[F1_{Y \notin \mathscr{S}_j}](X,\varphi')\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')} \\ &\leqslant C \sum_{Y \in \mathscr{P}_j^c \backslash \mathscr{S}_j}^{\overline{Y}=X} (A/2)^{-|Y|_j} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j} \\ &\leqslant C \sum_{Y \in \mathscr{P}_j^c \backslash \mathscr{S}_j}^{\overline{Y}=X} (A/2)^{-|Y|_j/2} (A/2)^{-(1+\eta)|X|_{j+1}/2} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j}. \end{aligned} \tag{4.24}$$

Next, observe that for any $z > 0$, decomposing a polymer $Y \in \mathscr{P}_j$ with $\overline{Y} = X$ over $(j+1)-$blocks constituting X, one can rewrite

$$\sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} z^{|Y|_j} = \prod_{B \in \mathscr{B}_{j+1}(X)} \sum_{Y' \in \mathscr{P}_j}^{\overline{Y'}=B} z^{|Y'|_j} = \left((1+z)^{L^2} - 1\right)^{|X|_{j+1}}. \tag{4.25}$$

Returning to (4.24), using (4.25) with the choice $z = (A/2)^{-1/2}$, one obtains that the quantity $\|\mathbb{SE}[F1_{Y \notin \mathscr{S}_j}](X,\varphi')\|_{\mathfrak{h},T_{j+1}(X,\varphi')}$ is bounded by

$$\begin{aligned} &(A/2)^{-\frac{1+\eta}{2}|X|_{j+1}} \left((1+(A/2)^{-\frac{1}{2}})^{L^2} - 1\right)^{|X|_{j+1}} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j} \\ &\qquad \leqslant (A/2)^{-\frac{1+\eta}{2}|X|_{j+1}} \left(e(A/2)^{-\frac{1}{2}} L^2\right)^{|X|_{j+1}} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j} \end{aligned} \tag{4.26}$$

under the assumption $\sqrt{2}A^{-\frac{1}{2}}L^2 \leqslant 1$ where we use $(1+b)^c - 1 \leqslant \exp(bc) - 1 \leqslant ebc$ for any $b,c \geqslant 0$, $bc \leqslant 1$ to obtain the last inequality. If we assume further that A is large enough so that $e(A/2)^{-(2+\eta)/2}L^2 \leqslant L^{-1}A^{-1}$, then this is bounded by

$$(e(A/2)^{-\frac{2+\eta}{2}} L^2)^{|X|_{j+1}} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}} G_{j+1}(X,\varphi') \|F\|_{\vec{\mathfrak{h}},T_j}, \tag{4.27}$$

giving the desired bound. □

We also have the following lemma which is of a slightly different flavour, but has its use in various places related to large sets.

Lemma 4.1.7. *Let* $X \in \mathscr{P}_{j+1}$, $0 \leqslant x \leqslant \varepsilon_{rb} = A^{-16}$, η *be as in Lemma 4.1.6 and* $L^2 A^{-\eta/(1+\eta)} \leqslant 1$. *Then*

$$\sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} 1_{|\mathrm{Comp}_j(Y)|=1} x A^{-|Y|_j} \leqslant \left(eL^2 A^{-(1+2\eta)/(1+\eta)}\right)^{|X|_{j+1}} x \tag{4.28}$$

and

$$\sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} 1_{|\mathrm{Comp}_j(Y)|\geqslant 2} x^{|\mathrm{Comp}_j(Y)|} A^{-|Y|_j} \leqslant A^{16}\left(eL^2 A^{-(1+2\eta)/(1+\eta)}\right)^{|X|_{j+1}} x^2. \tag{4.29}$$

Proof. For the first estimate, for $x \leqslant 1$, (4.23) implies $|Y|_j = \frac{1}{1+\eta}|Y|_j + \frac{\eta}{1+\eta}|Y|_j \geqslant |X|_{j+1} + \frac{\eta}{1+\eta}|Y|_j$ so that

$$\sum_{Y \in \mathscr{P}_j^c}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} x^{|\mathrm{Comp}_j(Y)|} A^{-|Y|_j} \leqslant A^{-|X|_{j+1}} \sum_{Y \in \mathscr{P}_j^c}^{\overline{Y}=X} A^{-\frac{\eta}{1+\eta}|Y|_j} x \tag{4.30}$$

and we get the desired bound after applying (4.25).

For the second estimate, observe that (4.22) implies $|Y|_j = \frac{1}{1+\eta}|Y|_j + \frac{\eta}{1+\eta}|Y|_j \geqslant |X|_{j+1} - 8|\mathrm{Comp}_j(Y)| + \frac{\eta}{1+\eta}|Y|_j$ so that

$$\begin{aligned}\sum_{|\mathrm{Comp}_j(Y)|\geqslant 2}^{\overline{Y}=X} x^{|\mathrm{Comp}_j(Y)|} A^{-|Y|_j} &\leqslant \sum_{|\mathrm{Comp}_j(Y)|\geqslant 2}^{\overline{Y}=X} A^{-\frac{\eta}{1+\eta}|Y|_j} A^{-|X|_{j+1}+8|\mathrm{Comp}_j(Y)|} x^{|\mathrm{Comp}_j(Y)|} \\ &\leqslant A^{16} A^{-|X|_{j+1}} \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} A^{-\frac{\eta}{1+\eta}|Y|_j} x^2. \end{aligned} \tag{4.31}$$

where the final line follows under the assumption $x \leqslant \varepsilon_{rb} = A^{-16}$. Now (4.25) implies

$$\begin{aligned} A^{-|X|_{j+1}} \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} A^{-\frac{\eta}{1+\eta}|Y|_j} &= A^{-|X|_{j+1}} \left[(1 + A^{-\eta/(1+\eta)})^{L^2} - 1\right]^{|X|_{j+1}} \\ &\leqslant A^{-|X|_{j+1}} \left(e^{A^{-\eta/(1+\eta)}L^2} - 1\right)^{|X|_{j+1}}, \end{aligned} \tag{4.32}$$

If A is chosen so that $A^{-\eta/(1+\eta)}L^2 \leqslant 1$, then this can be bounded by

$$(eL^2 A^{-(2+\eta)/(1+\eta)})^{|X|_{j+1}}, \tag{4.33}$$

completing the proof of the second estimate. □

4.2 Reexpansion in the succeeding scale

In this section, we assume Z_j^0 of form (4.1) and we have good candidates for $E_{j+1} - E_j$ and U_{j+1}, the leading terms for the expansion in scale $j+1$. To emphasize that they will eventually given as functions of (U_j, K_j^0), we denote them $\mathscr{E}_{j+1}$ and $\mathscr{U}_{j+1}$. Then we may now expand Z_{j+1}^0 in terms of polymer activities in scale $j+1$ and in form (4.1). Note that this section is a modification of [23, Section 5], with the small difference that the order of expectation and reblocking is reversed, following the set-up of [35], and the operations are purely algebraic. The justification of convergence of expectation $\mathbb{E}$ and estimates are subjects of Section 4.4 and later chapters.

Recall that the change of scale of the expansion was already presented in (4.18)–(4.19) using the coarse-graining-reblocking. So we are left to find a way to integrate the fluctuation integral $\mathbb{E}$ with the polymer expansion. The first step is to replace U_j by U_{j+1} using

$$\overline{\mathscr{U}}_{j+1}^0(X,\varphi) := -\mathscr{E}_{j+1}|X| + \mathscr{U}_{j+1}(X,\varphi) \tag{4.34}$$

and Lemma 4.1.1:

$$e^{\theta_\zeta U_j(X)} = (e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}^0} + e^{\overline{\mathscr{U}}_{j+1}^0})^X = \Big((e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}^0}) \circ_{j+1} e^{\overline{\mathscr{U}}_{j+1}^0}\Big)(X) \tag{4.35}$$

for $X \in \mathscr{P}_{j+1}$, where we recall the notation $\theta_\zeta F(X,\varphi') = F(X,\varphi'+\zeta)$. Thus by (3.27) and (4.19),

$$\begin{aligned} Z_{j+1}^0(\cdot|\Lambda) &= e^{-E_j|\Lambda|}\mathbb{E}^\zeta\left[e^{\theta_\zeta U_j} \circ_{j+1} \theta_\zeta \overline{K}_j^0\right](\Lambda) \\ &= e^{-E_j|\Lambda|}\Big(e^{\overline{\mathscr{U}}_{j+1}^0} \circ_{j+1} \mathbb{E}^\zeta\left[(e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}^0}) \circ_{j+1} \theta_\zeta \overline{K}_j^0\right]\Big)(\Lambda). \end{aligned} \tag{4.36}$$

where we have inserted (4.35) for the second inequality. Naively, this should complete the reexpansion if $\overline{\mathscr{U}}_{j+1}^0$ is chosen correctly, but we will further manipulate this expansion to obtain a linear approximation of the polymer activity at scale $j+1$ more suitable for the analysis.

4.2.1 Redistribution of local terms

An additional expansion step can be performed in order to replace $\overline{K}_j$ with a polymer activity with delicate cancellations. Suppose we are given a polymer function $Q_j^0 : \mathscr{B}_j \times \mathscr{S}_j \times \mathbb{R}^\Lambda \to \mathbb{R}$ that $\sum_{B\in\mathscr{B}_j} Q_j^0(B,X,\varphi')$ approximates $\mathbb{E}\theta_\zeta K_j(X,\varphi')$. Such functions are called *localisations*. The localisation depends on the model of interest, and for the DG model, we define it in Section 5.2 which are used to define Q_j^0 in (6.63), but the choice of Q_j can be arbitrary at this point. We also define *redistribution operators* on the localisation by

$$\mathscr{D}Q_j^0(B,X) = 1_{X\in\mathscr{S}_j} 1_{B\in\mathscr{B}_{j+1}(X)} \sum_{D\in\mathscr{B}_j(B)} \sum_{Y\in\mathscr{S}_j}^{D\in\mathscr{B}_j(Y)} Q_j^0(D,Y)(1_{\overline{Y}=X} - 1_{B=X}) \tag{4.37}$$

$$\overline{\mathscr{D}}Q_j^0(X) = \sum_{B\in\mathscr{B}_{j+1}(X)} \mathscr{D}Q_j^0(B,X). \tag{4.38}$$

Restrictions $X\in\mathscr{S}_j$ and $B\in\mathscr{B}_{j+1}(X)$ on $\mathscr{D}Q_j^0(B,X)$ are redundant, but we have included them for clarity. The role of $\mathscr{D}Q_j^0$ is to reorder the summation. If X was given, the sum with $1_{\overline{Y}=X}$ asks to first seek $Y\in\mathscr{S}_j$ such that $\overline{Y}=X$ and then sum over $D\in\mathscr{P}_j(B\cap Y)$, but the sum with $1_{B=X}$ asks to first find $D\in\mathscr{B}_j(B)$ and sum over $Y\in\mathscr{S}_j$ such that $D\in\mathscr{B}_j(Y)$. See Remark 4.2.3 to see the advantage of these operations.

Then we use Lemma 4.1.1 to obtain

$$\theta_\zeta\overline{K}_j^0(X) = ((\theta_\zeta\overline{K}_j^0 - \overline{\mathscr{D}}Q_j^0) \circledast_{j+1} \overline{\mathscr{D}}Q_j^0)(X) \tag{4.39}$$

and (4.36) becomes

$$\begin{aligned} Z_{j+1}^0(\cdot|\Lambda) &= e^{-E_j|\Lambda|}\Big(e^{\overline{\mathscr{U}}_{j+1}^0} \circ_{j+1} \mathbb{E}^\zeta\left[(e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}^0}) \circ_{j+1} \left((\theta_\zeta\overline{K}_j - \overline{\mathscr{D}}K_j^0) \circledast_{j+1} \overline{\mathscr{D}}K_j^0\right)\right]\Big)(\Lambda) \\ &= e^{-E_j|\Lambda|} \sum_{X',X_0,X_1,Z\in\mathscr{P}_{j+1}} e^{\overline{\mathscr{U}}_{j+1}^0(X')}\mathbb{E}^\zeta\left[(e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}^0})^{X_0}(\theta_\zeta\overline{K}_j^0 - \overline{\mathscr{D}}Q_j^0)^{[X_1]}\right] \\ &\quad \times \prod_{Z'\in\mathrm{Comp}_{j+1}(Z)} \sum_{B_{Z'}\in\mathscr{B}_{j+1}(Z')} \mathscr{D}Q_j^0(B_{Z'},Z') \end{aligned} \tag{4.40}$$

where $X', X_0, X_1, Z \in \mathscr{P}_{j+1}$ are disjoint, $X' \cup X_0 \cup X_1 \cup Z = \Lambda$ and $X_1 \not\sim Z$. We let $X = X_0 \cup X_1 \cup (\cup_{Z' \in \mathrm{Comp}_{j+1}(Z)} B^*_{Z'})$ and let $\mathscr{K}^0_{j+1}$ be what is present on X:

$$\mathscr{K}^0_{j+1}(U_j, K_j; X) := \sum^*_{X_0, X_1, Z, (B_{Z'})} e^{-\overline{\mathscr{U}}^0_{j+1}(T) + \mathscr{U}_{j+1}(X)} \\ \times \mathbb{E}^\zeta \Big[(e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}^0_{j+1}})^{X_0} (\theta_\zeta \overline{K}^0_j - \overline{\mathscr{D}} Q^0_j)^{[X_1]} \Big] \prod_{Z' \in \mathrm{Comp}_{j+1}(Z)} \mathscr{D} Q^0_j (B_{Z'}, Z') \tag{4.41}$$

where the sum $\sum^*_{X_0, X_1, Z, (B'_Z))}$ indicates $T = X_0 \cup X_1 \cup Z$, $X_1 \not\sim Z$ and $B_{Z'} \in \mathscr{B}_{j+1}(Z')$ for each $Z' \in \mathrm{Comp}_{j+1}(Z)$. Then it is natural to see that

$$Z^0_{j+1}(\cdot|\Lambda) = e^{-E_{j+1}|\Lambda|} \big(e^{U_{j+1}} \circ_{j+1} \mathscr{K}^0_{j+1} \big)(\Lambda). \tag{4.42}$$

It will be useful to summarise these results as the following.

Definition 4.2.1. *Given U_j, K_j, U_+, $\overline{U}_+$ and Q, define*

$$\mathbb{K}_{j+1}(U_j, K_j, U_+, \overline{U}_+, Q; X) := \sum^*_{X_0, X_1, Z, (B_{Z'})} e^{-\overline{U}_+(T) + U_+(X)} \\ \times \mathbb{E} \Big[(e^{\theta_\zeta U_j} - e^{\overline{U}_+})^{X_0} (\theta_\zeta \overline{K}_j - \overline{\mathscr{D}} Q)^{[X_1]} \Big] \prod_{Z' \in \mathrm{Comp}_{j+1}(Z)} \mathscr{D} Q (B_{Z'}, Z') \tag{4.43}$$

for $X \in \mathscr{P}^c_{j+1}$, where the sum $\sum^*_{X_0, X_1, Z, (B'_Z))}$ indicates $X_0, X_1, Z \in \mathscr{P}_{j+1}$, $(B_{Z'})$ is a collection such that $Z' \in \mathrm{Comp}_{j+1}(Z)$ and $B_{Z'} \in \mathscr{B}_{j+1}(Z')$, $T = X_0 \cup X_1 \cup Z$, $X_1 \not\sim Z$.

Proposition 4.2.2. *For any choice of $U_+, \overline{U}_+$ and Q such that $(\overline{U}_+ - U_+)(X)$ is scalar-valued (i.e., independent of the field φ), let*

$$K_{j+1} = \mathbb{K}_{j+1}(U_j, K_j, U_+, \overline{U}_+, Q; X). \tag{4.44}$$

Then it satisfies

$$\mathbb{E}^\zeta [\theta_\zeta (e^{U_j} \circ_j K_j)(\Lambda)] = e^{(\overline{U}_+ - U_+)(\Lambda)} (e^{U_+} \circ_{j+1} K_{j+1})(\Lambda), \tag{4.45}$$

if the convergence of the expectation $\mathbb{E}$ is guaranteed.

4.2.2 Linear expansion of $\mathscr{K}^0_{j+1}$

At the first sight, it is not so clear what is the purpose of the redistribution operator. This point will get clear once we restrict ourselves to the linear order. The first order analysis does not only help the conceptual understanding of the map $\mathscr{K}^0_{j+1}$, but it will also be a crucial component of our analysis.

The first order terms of $\mathscr{K}^0_{j+1}$ can be identified by the following steps on (4.41). (1) Only consider $X_0, X_1, Z, (B_{Z'})$ in the sum $\sum^*$ such that

$$\#(X_0, X_1, Z) := |X_0|_j + |\mathrm{Comp}_j(X_1)| + |\mathrm{Comp}_j(Z)| = 1. \tag{4.46}$$

(2) Replace $e^{\mathscr{E}_{j+1}|T| + \mathscr{U}_{j+1}(X\setminus T)}$ by 1. (3) Replace $e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}}$ by $\theta_\zeta U_j - \overline{\mathscr{U}}_{j+1}$. (4) Replace $\overline{K}^0_j$ by $\mathbb{S}K^0_j(X)$. These will leave us with the linearised version of $\mathscr{K}^0_{j+1}$ as follows: for $X \in \mathscr{P}^c_{j+1}$,

$$\begin{aligned} \mathscr{L}^0_{j+1}(U_j, K^0_j; X, \varphi') := &\sum_{Y : \overline{Y} = X} \Big(1_{Y \in \mathscr{P}^c_j} \mathbb{E}^\zeta \theta_\zeta K^0_j(Y, \varphi') - 1_{Y \in \mathscr{S}_j} \sum_{D \in \mathscr{B}_j(Y)} Q^0_j(D, Y, \varphi') \Big) \\ &+ \sum_{D \in \mathscr{B}_j}^{\overline{D} = X} \Big(\mathbb{E}^\zeta [\theta_\zeta U_j(D, \varphi')] + \mathscr{E}_{j+1}|D| - \mathscr{U}_{j+1}(D, \varphi') + \sum_{Y \in \mathscr{S}_j}^{D \in \mathscr{B}_j(Y)} Q^0_j(D, Y, \varphi') \Big). \end{aligned} \tag{4.47}$$

In more detail, the terms in the first line above and the Q^0_j-terms in the second line come from $X = T = X_1$ (replacing $e^{U_j(X\setminus Y)}$ by 1 in $\overline{K}_j$, which corresponds to replacing $\overline{K}_j$ by $\mathbb{S}K_j$, and keeping only connected polymers Y), the remaining terms in the second line are due to $X = T = X_0$ (and linearising the exponentials), and finally the terms with $T = Z$ (and thus $X = B^*_Z$) actually vanish by the construction of $\overline{\mathscr{D}}Q^0_j$ (after the replacement of the exponential outside the expectation, i.e., in the first line of (4.41), by 1). Indeed, to see that the contribution from $X = B^*_Z$ cancels, note that for any $B \in \mathscr{B}_{j+1}$,

$$\begin{aligned} \sum_{Z \in \mathscr{S}_{j+1}}^{B \in \mathscr{B}_{j+1}(Z)} \overline{\mathscr{D}}Q^0_j(B, Z) &= \sum_{Z \in \mathscr{S}_{j+1}}^{B \in \mathscr{B}_{j+1}(Z)} \sum_{D \in \mathscr{B}_j(B)} \sum_{Y \in \mathscr{S}_j}^{D \in \mathscr{B}_j(Y)} Q_j(D, Y)(1_{\overline{Y} = Z} - 1_{B = Z}) \\ &= \sum_{D \in \mathscr{B}_j(B)} \sum_{Y \in \mathscr{S}_j}^{D \in \mathscr{B}_j(Y)} Q_j(D, Y) \sum_{Z \in \mathscr{S}_{j+1}}^{B \in \mathscr{B}_{j+1}(Z)} (1_{\overline{Y} = Z} - 1_{B = Z}) = 0. \end{aligned} \tag{4.48}$$

Now, in light of Proposition 4.1.5, we see that the very first term of (4.47) has meaningful contribution only when $Y \in \mathscr{S}_j$. We thus see that it is enough to choose Q^0_j that cancels $\mathbb{E}\theta_\zeta K_j$ on small sets.

We may also define for $X \in \mathscr{P}^c_{j+1}$

$$\mathscr{M}^0_{j+1}(U_j, K_j) = \mathscr{K}^0_{j+1}(U_j, K_j) - \mathscr{L}^0_{j+1}(U_j, K_j), \tag{4.49}$$

the remaining part of $\mathscr{K}^0_{j+1}$. We will later see in Lemma 4.4.2 that $\mathscr{M}^0_{j+1}$ is of order $\geqslant 2$.

Remark 4.2.3. As mentioned, the redistribution operator was introduced in order to facilitate the first order analysis. Suppose we have not introduced Q_j and rather have defined

$$K'_{j+1}(X) = e^{\mathscr{E}_{j+1}|X|}\mathbb{E}\big[(e^{\theta_\zeta U_j} - e^{\overline{\mathscr{U}}_{j+1}}) \circ_{j+1} \theta_\zeta \overline{K}^0_j\big](X) \tag{4.50}$$

Then by (4.36), it also satisfies

$$Z_{j+1}(\cdot|\Lambda) = e^{-\mathscr{E}_{j+1}|\Lambda|}(e^{\mathscr{U}_{j+1}} \circ_{j+1} K'_{j+1})(X) \tag{4.51}$$

so K'_{j+1} is also a valid candidate for the remainder coordinate in scale $j+1$. However, if we expand out K'_{j+1} in first order, we get

$$\mathscr{L}'_{j+1}(X) = \sum_{Y:\overline{Y}=X} 1_{Y\in\mathscr{P}^c_j}\mathbb{E}\theta_\zeta K^0_j(Y) + \sum_{D\in\mathscr{B}_j}^{\overline{D}=X} \mathbb{E}[\theta_\zeta U_j(D)] + \mathscr{E}_{j+1}|D| - \mathscr{U}_{j+1}(D). \tag{4.52}$$

In above definition of $\mathscr{L}^0_{j+1}$, we see that we can have cancellations in both of the sum by choosing Q^0_j, $\mathscr{E}_{j+1}$ and $\mathscr{U}_{j+1}$ appropriately. But for $\mathscr{L}'_{j+1}K_j$, since the form of the two sums are incompatible, there is no obvious way to make a cancellation between the two terms. The role of the redistribution operator is to add and subtract terms in both $\sum_{Y:\overline{Y}=X}$ and $\sum_{D\in\mathscr{B}_j}^{\overline{D}=X}$ in a compatible way so there there are natural cancellations.

4.3 Polymer expansion with external field

In general, an expansion of form (3.28) is used when we are interested in macroscopic observables, such as the free energy or the torus scaling limit of Theorem 1.1.1. However, if we are interested in observables at lower scales, it is desirable to consider the Laplace transformations, or the tilted expectation of (3.16),

$$\mathbb{E}_{(\omega)}[F(\zeta)] = \mathbb{E}\big[e^{\omega(\tilde{\mathfrak{f}},\zeta)}F(\zeta)\big]/\mathbb{E}[e^{\omega(\tilde{\mathfrak{f}},\zeta)}] \tag{4.53}$$

for the external field $\tilde{\mathfrak{f}}$.

If we are further given the observable scale $j_s > 0$ (also present in (A'_f)) and field $v \in \mathbb{R}^\Lambda$, we may consider a flow of Z_j defined inductively by

$$Z_0(\varphi;\omega|\Lambda) = Z^0_0(\varphi + \omega v|\Lambda) \tag{4.54}$$

and

$$Z_{j+1}(\varphi;\omega|\Lambda) = \begin{cases} \mathbb{E}[\theta_\zeta Z_j(\varphi;\omega|\Lambda)] & (j < j_s) \\ \mathbb{E}_{(\omega)}\big[\theta_\zeta Z_j(\varphi;\omega|\Lambda)\big] & (j \geqslant j_s). \end{cases} \tag{4.55}$$

In the case when ω is real, then we can use a change of variables to write

$$Z_{j+1}(\varphi;\omega|\Lambda) = \mathbb{E}\big[\theta_\zeta Z_j(\varphi + \omega\Gamma_{j+1}\tilde{\mathfrak{f}};\omega|\Lambda)\big] \tag{4.56}$$

but for general complex ω, this is not always possible, as we haven't checked the complex analyticity of Z_j. We also impose a restriction on v similar to that of (A'_u).

(A_v) Given the observable scale $j_s \geqslant 0$, the field v has decomposition $v = \sum_{i=1}^{\mathfrak{n}} T_{y_i} v_\alpha$ such that $\|v_i\|_{C^2_{j_s}} \leqslant C\mathfrak{n}^{-1}\log L$. Also, $\mathrm{supp}(v_i) \subset \mathscr{B}_0^{j_s+1}$ for each i.

The norm of v is bounded in scale j_s, but it is allowed to be supported on a larger region $B_0^{j_s+1}$. This makes a problem if it happens at all scales, but v only lives at a single scale j_s, so this is acceptable. Again, C will be a constant independent of all the other parameters, so we do not mention the dependence on it.

Nevertheless, we will be tracing the complex analyticity when required, and seek for a representation for $j > j_s$

Now, we seek for polymer representations of Z_j,

$$Z_j(\varphi;\omega|\Lambda) = \begin{cases} e^{-E_j|\Lambda|}(e^{U_j} \circ_j K_j^0)(\Lambda, \varphi + \omega v) & (j \leqslant j_s) \\ e^{-E_j|\Lambda| + e_j(\omega)}(e^{U_j(\cdot,\varphi+\omega u_j)} \circ_j K_j(\cdot,\varphi))(\Lambda) & (j > j_s) \end{cases} \tag{4.57}$$

when ω is sufficiently small and $u_j = 1_{j>j_s}\Gamma_j\tilde{\mathfrak{f}}$ (see (3.19)). (Note that case $j \leqslant j_s$ in (4.57) is not the same as what is given in (4.55). This point will be justified in Section 4.3.1.) To this end, we define the renormalisation group step as a slight modification of (4.43), whose justifications will be given in Section 4.3.1, 4.3.2.

Definition 4.3.1. *Let v and $(u_j)_j$ satisfy* (A_v) *and* (A'_u)*, respectively. Suppose $\mathscr{E}_{j+1}$, U_j, $\mathscr{U}_{j+1}$ and K_j are given. Then define K_{j+1} as the following.*

- *When $j < j_s$, suppose localisation Q_j^0 is given. Then consider K_{j+1}^0 defined by* (4.43)*, i.e.,*

$$K_{j+1}^0 = \mathscr{K}_{j+1}^0(U_j, K_j^0) := \mathbb{K}_{j+1}(U_j, K_j^0, \mathscr{U}_{j+1}, \overline{\mathscr{U}}_{j+1}^0, Q_j^0; X), \tag{4.58}$$

and let $K_{j+1} = K_{j+1}^0$.

- *When $j \geqslant j_s$, let*

$$K_j^{\dagger}(\cdot;\omega) = \begin{cases} \mathscr{R}_j^{(1)}[\omega v, U_j, K_j(\cdot;\omega)] & (j = j_s) \\ \mathscr{R}_j^{(2)}[\omega u_j, U_j, K_j(\cdot;\omega)] & (j > j_s) \end{cases} \tag{4.59}$$

with $\mathscr{R}^{(1)}, \mathscr{R}^{(2)}$ defined by (4.7),(4.8). *If we are given a localisation Q_j and $\mathfrak{g}_{j+1}(X;\omega)$ is some polymer function of $X \in \mathscr{P}_{j+1}$ such that $\mathfrak{g}_{j+1}(X \cup Y) = \mathfrak{g}_{j+1}(X) + \mathfrak{g}_{j+1}(Y)$ for any $X \cap Y = \emptyset$, then let*

$$K_{j+1} = \mathscr{K}_{j+1}(U_j, K_j^{\dagger}) := \mathbb{K}_{j+1}(U_j, K_j^{\dagger}, \mathscr{U}_{j+1}, \overline{\mathscr{U}}_{j+1}, Q_j; X) \tag{4.60}$$

where now

$$\overline{\mathscr{U}}_{j+1}(X,\varphi;\omega) = \mathfrak{g}_{j+1}(X;\omega) - \mathscr{E}_{j+1}|X|_{j+1} + \mathscr{U}_{j+1}(X,\varphi + \omega u_{j+1}), \tag{4.61}$$

and recall $\mathbb{K}_{j+1}$ from Definition 4.2.1.

Although it is conceptually better to think of $\mathscr{K}_{j+1}$ as a function of (U_j, K_j), it is often more convenient to think of it as a function of $(U_j, K_j^{\dagger})$ for its estimates. K_j does not depend on ω for $j \leqslant j_s$, but we are denoting $K_j(\cdot;\omega) \equiv K_j^0$ to minimize notational changes. Then we have the following analogue of Proposition 4.2.2.

Proposition 4.3.2. *Suppose Z_j satisfies* (4.57). *Let Z_{j+1} be defined by* (4.55) *and K_{j+1} be defined by Definition 4.3.1 with integrability in $\mathbb{E}$ being assumed. Then Z_{j+1} also satisfies* (4.57) *with $E_{j+1} = E_j + \mathscr{E}_{j+1}$ and $e_{j+1}(\Lambda) = e_j(\Lambda) + \mathfrak{g}_{j+1}(\Lambda)$ (with convention $e_j \equiv 0$ for $j \leqslant j_s$).*

The proof will be given in the following sections.

4.3.1 Regime $j < j_s$

The $j < j_s$ case of Definition 4.3.1 is valid only when we have equivalence of (4.55) and (4.57). Indeed, we verify that these two definitions match in the following lemma, where we drop Λ for brevity.

Lemma 4.3.3. *Let v satisfy (A_v) and $h_\omega \leqslant (C \log L)^{-1} \mathfrak{h}$ for large C. Let Z_j be defined iteratively by $Z_{j+1}(\varphi;\omega) = \mathbb{E}^{\zeta}_{\Gamma_{j+1}}[\theta_\zeta Z_j(\varphi;\omega)]$ for $\varphi \in \mathbb{R}^\Lambda$ and $Z_0(\varphi;\omega) = Z_0^0(\varphi + \omega v)$. Suppose there is a sequence $(E_j, U_j, K_j)_{j \leqslant j_s}$ such that $Z_j^0(\varphi)$ admits expansion* (4.1), *$\|K_j^0\|_{\mathfrak{h},T_j} < +\infty$*

and $\|U_j\|_{\Omega_j^U}$ are sufficiently small (polynomially in β, L) for each $j \leqslant j_s$. Then

$$Z_j(\varphi;\omega) = Z_j^0(\varphi+\omega v) \qquad \text{for all } j \leqslant j_s,\ \omega \in \mathbb{D}_{h_\omega}. \tag{4.62}$$

Proof. When $\omega \in \mathbb{R} \cap \mathbb{D}_{h_\omega}$, then the identity is trivial by definition. Thus it is enough to verify that Z_j and $Z_j^0(\varphi+\omega v)$ are complex analytic functions of $\omega \in \mathbb{D}_{h_\omega}$. First, by Proposition 3.4.1, we see that $\omega \mapsto K_j^0(X, \varphi+\omega v)$ is analytic for each $X \in \mathscr{P}_j$ with

$$|K_j^0(X,\varphi+\omega v)| \leqslant \|K_j^0(X,\varphi)\|_{\mathfrak{h},T_j(X,\varphi)} \leqslant \|K_j^0(X)\|_{\mathfrak{h},T_j(X)} G_j(X,\varphi) \tag{4.63}$$

Also, by the explicit form of U_j, we see that $\omega \mapsto Z_j^0(\varphi+\omega v)$ is analytic. To prove analyticity of $\omega \mapsto Z_{j+1}(\varphi;\omega)$, we use induction, so assume that $Z_j(\cdot;\omega) = Z_j^0(\cdot+\omega v)$. Observe that

$$\sup_{\omega\in\mathbb{D}_{h_\omega}} |U_j(X,\varphi+\omega v)| \leqslant O(\|U_j\|_{\Omega_j^U})(\mathfrak{h}^2 + w_j(X,\varphi)^2) \leqslant c_w \kappa_L (1+w_j(X,\varphi)^2) \tag{4.64}$$

for $\|U_j\|_{\Omega_j^U}$ sufficiently small. Thus combined with (4.63),

$$\sup_{\omega\in\mathbb{D}_{h_\omega}} \left| e^{U_j(\Lambda\backslash X,\varphi+\omega v)} K_j^0(X,\varphi+\omega v) \right| \leqslant O(\|K_j^0(X)\|_{\mathfrak{h},T_j(X)} G_j(\Lambda,\varphi) \tag{4.65}$$

where we used Lemma 3.3.4 to bound $e^{c_w \kappa_L w_j(\Lambda\backslash X,\varphi)^2} G_j(X,\varphi)$. But by Proposition 3.3.5, we see $\mathbb{E}_{\Gamma_{j+1}} \theta_\zeta G_j(X,\varphi) < \infty$, so by the Dominated Convergence Theorem,

$$\omega \mapsto Z_{j+1}(\cdot;\omega) = e^{-E_j|\Lambda|} \mathbb{E}^\zeta_{\Gamma_{j+1}} \Big[\theta_\zeta \sum_{X\in\mathscr{P}_j} e^{U_j(\Lambda\backslash X,\cdot+\omega v)} K_j(X,\cdot+\omega v) \Big] \tag{4.66}$$

is continuous in ω. The same bound also allows to use the Fubini's theorem, so

$$\int_\gamma d\omega Z_{j+1}(\varphi;\omega) = e^{-E_j|\Lambda|} \mathbb{E}^\zeta_{\Gamma_{j+1}} \Big[\theta_\zeta \sum_{X\in\mathscr{P}_j} \int_\gamma d\omega e^{U_j(\Lambda\backslash X,\varphi+\omega v)} K_j^0(X,\varphi+\omega v) \Big] = 0 \tag{4.67}$$

for any C^1 curve γ in $\mathbb{D}_{h_\omega}$, where the second equality follows from the Cauchy's integral theorem. We conclude that $\omega \mapsto Z_{j+1}(\varphi;\omega)$ is analytic, completing the induction. $\square$

4.3.2 Regime $j \geqslant j_s$

We now justify (4.60) in the proof of Proposition 4.3.2.

Proof of Proposition 4.3.2. The case $j < j_s$ is already checked by Lemma 4.3.3. For $j \geqslant j_s$, (4.57) and Lemma 4.9 show

$$Z_{j_s}(\cdot|\Lambda) = e^{-E_j|\Lambda|+e_j(\Lambda)}(e^{U_j} \circ_j K_j^\dagger)(\Lambda). \tag{4.68}$$

Thus Proposition 4.2.2 guarantees (4.57) for Z_{j+1} and K_{j+1}.

□

4.4 Estimate on the non-linear part

In this section, we define the linearisation $\mathscr{L}_{j+1}$ of $\mathscr{K}_{j+1}$ and prove estimates on the remaining terms. The results of this section also applies to $\mathscr{K}_{j+1}^0$, since it can simply be considered as the case with vanishing external fields, so we do not discuss it separately.

Exactly following the procedure of Section 4.2.2, we see that the linear part of $\mathscr{K}_{j+1}$ is given by

$$\begin{aligned}\mathscr{L}_{j+1}(U_j, K_j^\dagger; X, \varphi') := \sum_{Y:\overline{Y}=X} \Big(1_{Y\in\mathscr{P}_j^c} \mathbb{E}_{(\omega)} \theta_\zeta K_j^\dagger(Y,\varphi') - 1_{Y\in\mathscr{S}_j} \sum_{D\in\mathscr{B}_j(Y)} Q_j(D,Y,\varphi') \Big) \\ + \sum_{\substack{\overline{D}=X \\ D\in\mathscr{B}_j}} \Big(\mathbb{E}_{(\omega)} \theta_\zeta U_j(D,\varphi') - \overline{\mathscr{U}}_{j+1}(D,\varphi') + \sum_{\substack{D\in\mathscr{B}_j(Y) \\ Y\in\mathscr{S}_j}} Q_j(D,Y,\varphi') \Big)\end{aligned} \tag{4.69}$$

where $\zeta \sim \mathscr{N}(0,\Gamma_{j+1})$ if $j < N-1$ and $\zeta \sim \mathscr{N}(0,\Gamma_N^{\Lambda_N})$ if $j = N-1$. Then again, the remaining part will be defined by

$$\mathscr{M}_{j+1}(U_j, K_j^\dagger) := \mathscr{K}_{j+1}(U_j, K_j^\dagger) - \mathscr{L}_{j+1}(U_j, K_j^\dagger) \tag{4.70}$$

The aim of this section is to expand out $\mathscr{M}_{j+1}$ and prove that $\mathscr{M}_{j+1}$ is of order $O((\|U_j\|_{\Omega_j^U} + \|K_j^\dagger\|_{\vec{h},T_j})^2)$ in Lemma 4.4.2. At this stage, we are not fixing $\mathscr{E}_{j+1}, \mathscr{U}_{j+1}$ and Q_j, so the estimate will only be proved modulo some estimates on them.

Below we write $U_{j+1}, \overline{U}_{j+1}$ in place of $\mathscr{U}_{j+1}, \overline{\mathscr{U}}_{j+1}$ for simplicity of notation and recall

$$\overline{U}_{j+1}(X,\varphi';\omega) = -\mathscr{E}_{j+1}^\dagger(X;\omega) + U_{j+1}(X,\varphi' + \omega u_{j+1}) \tag{4.71}$$

where we also let

$$\mathscr{E}_{j+1}^\dagger(X;\omega) = \mathscr{E}_{j+1}|X|_{j+1} - \mathfrak{g}_{j+1}(X;\omega). \tag{4.72}$$

Accordingly, we introduce the map

$$\omega_j = (U_j, K_j) \mapsto \overline{\mathfrak{K}}_j(\omega_j) \equiv (\mathscr{E}^\dagger_{j+1}|X|, U_j, \overline{U}_{j+1}, K^\dagger_j, \overline{K}_j, Q_j)(\omega_j). \tag{4.73}$$

In terms of ω_j, one may decompose $\mathscr{M}_{j+1}$ into four terms as follows: for $X \in \mathscr{P}^c_{j+1}$, using the notation (4.73),

$$\mathscr{M}_{j+1}(U_j, K^\dagger_j, X, \varphi') = \sum_{k=1}^{4} \mathfrak{M}^{(k)}_{j+1}(\overline{\mathfrak{K}}_j(\omega_j), X, \varphi'), \tag{4.74}$$

where the $\mathfrak{M}^{(k)}_{j+1}$ are given as follows:

$$\begin{aligned}\mathfrak{M}^{(1)}_{j+1}(\overline{\mathfrak{K}}_j(\omega_j), X) = &\sum^{*}_{X_0, X_1, Z, (B_{Z''})} 1_{\#(X_0,X_1,Z)\geqslant 2} e^{\mathscr{E}^\dagger_{j+1}(X)} e^{\overline{U}_{j+1}(X\setminus T)} \\ &\times \mathbb{E}^{\zeta}_{(\omega)}\Big[(e^{\theta_\zeta U_j} - e^{\overline{U}_{j+1}})^{X_0}(\theta_\zeta \overline{K}_j - \overline{\mathscr{D}Q}_j)^{[X_1]}\Big] \prod_{Z'' \in \mathrm{Comp}_{j+1}(Z)} \mathscr{D}Q_j(B_{Z''}, Z'')\end{aligned} \tag{4.75}$$

$$\begin{aligned}\mathfrak{M}^{(2)}_{j+1}(\overline{\mathfrak{K}}_j(\omega_j), X) = &\sum^{*}_{X_0, X_1, Z, (B_{Z''})} 1_{\#(X_0,X_1,Z)\leqslant 1} (e^{\mathscr{E}^\dagger_{j+1}(X)} e^{\overline{U}_{j+1}(X\setminus T)} - 1) \\ &\times \mathbb{E}^{\zeta}_{(\omega)}\Big[(e^{\theta_\zeta U_j} - e^{\overline{U}_{j+1}})^{X_0}(\theta_\zeta \overline{K}_j - \overline{\mathscr{D}Q}_j)^{[X_1]}\Big] \prod_{Z'' \in \mathrm{Comp}_{j+1}(Z)} \mathscr{D}Q_j(B_{Z''}, Z'')\end{aligned} \tag{4.76}$$

$$\mathfrak{M}^{(3)}_{j+1}(\overline{\mathfrak{K}}_j(\omega_j), X) = \sum_{|X_0|_{j+1}=1}^{X_0=X} \mathbb{E}^{\zeta}_{(\omega)}\Big[\Big(e^{\theta_\zeta U_j} - e^{\overline{U}_{j+1}} - \theta_\zeta U_j + \overline{U}_{j+1}\Big)^{X_0}\Big] \tag{4.77}$$

$$\mathfrak{M}^{(4)}_{j+1}(\overline{\mathfrak{K}}_j(\omega_j), X) = \mathbb{E}^{\zeta}_{(\omega)}\Big[\theta_\zeta \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} e^{U_j(Y)} K^\dagger_j(X\setminus Y) - \mathbb{S}[K^\dagger_j](X)\Big] \tag{4.78}$$

where $\sum^*$ is as in Definition 4.2.1, thus in particular $T = X_0 \cup X_1 \cup Z$. Each $\mathfrak{M}^{(b)}_{j+1}$ ($b = 1,2,3,4$) arise as the remainder of the linearisation process (1)–(4) described above (4.47).

The bound of $\mathscr{M}_{j+1}(U_j, K^\dagger_j)$ will follow by bounding each $\mathfrak{M}^{(k)}_{j+1}(\overline{\mathfrak{K}}_j)$ separately. Although the above is not the most efficient way to express $\mathscr{M}_{j+1}$, writing it in this way will make it easier to generalise the estimates to different settings. Indeed, one may deduce a bound on each $\mathfrak{M}^{(b)}_{j+1}$ that only depends on the estimates on $\overline{\mathfrak{K}}_j$. The next definition collects a list of bounds on various terms that appear in the above formulas. These estimates are sufficient to imply the desired bounds on the $\mathfrak{M}^{(b)}_{j+1}$, as asserted in Lemma 4.4.2.

These "building block estimates" require a suitable notion of derivative. Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, and let $F : \mathbb{X} \to \mathbb{Y}$. The directional derivative of F at a point $x \in \mathbb{X}$, in direction $\dot{x}$ is denoted by $DF(x,\dot{x})$, i.e., when the limit exists,

$$DF(x,\dot{x}) = \lim_{t\to 0} \frac{1}{t}(F(x+t\dot{x}) - F(x)), \tag{4.79}$$

and if F is Fréchet-differentiable the norm of the derivative is the operator norm

$$\|DF(x,\cdot)\| := \sup\{\|DF(x,\dot{x})\|_{\mathbb{Y}} : \|\dot{x}\|_{\mathbb{X}} \leqslant 1\}. \tag{4.80}$$

Definition 4.4.1. *Let $(\mathbb{X}, |\cdot|)$ be a closed subset of a normed vector space. Given $\delta, \eta > 0$, define $\mathscr{X}_j^{\mathfrak{K}}(\mathbb{Y})$, to be the set of functions $\mathfrak{K}_j : x(\in \mathbb{X}) \mapsto \mathfrak{K}_j(x) = (\mathscr{E}^{\dagger}_{j+1}(X), U_j, \overline{U}_{j+1}, K^{\dagger}_j, \overline{K}_j, Q_j)(x)$ where each component takes polymer activity value as in the right-hand side of (4.73), such that $\mathfrak{K}_j(0) = 0$ and $x \mapsto (K^{\dagger}_j(x), Q_j(x))$ is linear and $K^{\dagger}_j(x)$ is bounded as a function $\mathbb{X} \to \mathscr{N}_{j,h_\omega}$ and satisfies the following estimates for all $B \in \mathscr{B}_{j+1}$, $Z \in \mathscr{P}_{j+1}$ and $\varphi \in \mathbb{R}^{\Lambda_N}$: for $k \in \{0,1,2\}$,*

$$\|\mathfrak{U}(B,\varphi)\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \leqslant C(\delta,L)(1+\delta c_w \kappa_L w_j(B,\varphi)^2)|x| \tag{4.81}$$

$$\|e^{\mathfrak{U}(B,\varphi)} - \sum_{m=0}^{k} \frac{1}{m!}(\mathfrak{U}(B,\varphi))^m\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \leqslant C(\delta,L)e^{\delta c_w \kappa_L w_j(B,\varphi)^2}|x|^{k+1}, \tag{4.82}$$

for $\mathfrak{U} \in \{U_j, \overline{U}_{j+1}, \mathscr{E}^{\dagger}_{j+1}\}$ and some $C(L)$, and when $\mathfrak{U} = \mathscr{E}^{\dagger}_{j+1}$, the inequality holds with $C(\delta,L)$ replaced by $C(L)$ and δ set to 0. Moreover,

$$\|De^{\mathfrak{U}'(B,\varphi)}\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \leqslant C(L)e^{c_w \kappa_L w_j(B,\varphi)^2}, \tag{4.83}$$

$$\|D^2 e^{\mathfrak{U}'(B,\varphi)}\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \leqslant C(L)e^{c_w \kappa_L w_j(B,\varphi)^2}, \tag{4.84}$$

$$\|D\overline{K}_j(Z,\varphi)\|_{\vec{\mathfrak{h}},T_j(Z,\varphi)} \leqslant C(A,L)A^{-(1+\eta)|Z|_{j+1}} G_j(Z,\varphi), \tag{4.85}$$

$$\|DQ_j(D,Y,\varphi)\|_{\vec{\mathfrak{h}},T_j(Y,\varphi)} \leqslant C(L)A^{-1}e^{c_w \kappa_L w_j(D,\varphi)^2}, \tag{4.86}$$

for any $Y \in \mathscr{S}_j$, $D \in \mathscr{B}_j(Y)$, and $\mathfrak{U}' \in \{U_j, \overline{U}_{j+1}, \mathscr{E}^{\dagger}_{j+1}\}$, and in the case of $\mathscr{E}^{\dagger}_{j+1}$, the factor $e^{c_w \kappa_L w_j(B,\varphi)}$ can be omitted. The derivatives exist in the space of polymer activities with finite $\|\cdot\|_{\vec{\mathfrak{h}},T_j(B)}$-norm for $e^{\mathfrak{U}'}$, $De^{\mathfrak{U}'}$, finite $\|\cdot\|_{\vec{\mathfrak{h}},T_j(Z)}$-norm for $\overline{K}_j$ and finite $\|\cdot\|_{\vec{\mathfrak{h}},T_j(Y)}$-norm for Q_j.

Having these assumptions at hand, we can prove bounds on each $\mathfrak{M}^{(k)}_{j+1}$ as in the following lemma.

Lemma 4.4.2. *Then there exist $L_0, A_0(L)$ such that the following holds. Let $(\mathbb{X}, |\cdot|)$ be a closed subset of a normed space. If $L \geqslant L_0$, $A \geqslant A_0(L)$ and if $\mathfrak{K}_j$ is in $\mathscr{X}_j^{\mathfrak{K}}(\mathbb{X})$, there exist $\eta_0 \equiv \eta_0(\eta) > 0$ and $\varepsilon_{nl} > 0$ such that each $\mathfrak{M}_{j+1}^{(k)}(\mathfrak{K}_j(x))$ is continuously differentiable on $\{x \in \mathbb{X} : |x| \leqslant \varepsilon_{nl}\}$ for $k \in \{1,2,3,4\}$ and satisfies*

$$\left\| D\mathfrak{M}_{j+1}^{(k)}(\mathfrak{K}_j(x), X) \right\|_{\vec{\mathfrak{h}}, T_{j+1}(X)} \leqslant C_2(A,L) A^{-(1+\eta_0(\eta))|X|_{j+1}} |x| \tag{4.87}$$

for some $C_2(A,L) > 0$.

In practice, we set $\mathbb{X}$ to be a set of $(U_j, K_j^\dagger)$, thus the derivative in (4.87) is a derivative in $(U_j, K_j^\dagger)$. In this case, the assumption $\mathfrak{K}_j \in \mathscr{X}_j^{\mathfrak{K}}(\mathbb{X})$ is verified later in Lemma 6.4.1 for the bulk RG and in Lemma 8.3.1 for the observable RG, after making the choice of $\mathscr{E}_{j+1}$, $\mathfrak{g}_{j+1}$, $\mathscr{U}_{j+1}$ and Q_j. The lemma has two important implications in this setting. First is that $\mathscr{M}_{j+1}$ is continuously differentiable function of order 2 in $(U_j, K_j^\dagger)$. Second is that the large set regulator of $\mathscr{M}_{j+1}$ is improved by a power of η_0, thus we have relatively large amount of freedom on modifying the large set regulator in the reblocking steps. For example, we see that the large set regulator gets worse in Lemma 4.1.4, but this can be compensated by taking A sufficiently large so that $2A^{-\eta_0} \leqslant 1$. Note that η_0 is a function that solely depends on η in Definition 4.4.1 (that defines $\mathscr{X}_j^{\mathfrak{K}}(\mathbb{X})$). We will later see that η is a purely geometric constant (i.e., only depends on the graph, $\mathbb{Z}^2$ for our case), thus η_0 is also a purely geometric constant.

We again emphasize that we can also apply Lemma 4.4.2 to the setting without the external fields v and $\tilde{\mathfrak{f}}$, as it would be equivalent to setting $v \equiv \tilde{\mathfrak{f}} \equiv 0$. The rest of the chapter will be devoted to proving this lemma. The principles used to prove the lemma does not depend heavily on the system, so they have potential to be applied to much general settings.

4.4.1 Product rule for polymer activities

In preparation of the proof of Lemma 4.4.2, we first prove a product rule for polymer activities defined as in (4.75)–(4.76). For general polymer activities K, K' with $\|K\|_{\Omega_j^K}, \|K'\|_{\Omega_j^K} < \infty$, the polymer activity defined by $K''(X) = \sum_{Y \in \mathscr{P}_j(X)} K_j(Y) K_j'(X \backslash Y)$ is not necessarily differentiable. There are obstacles related both to the large field and the large set regulators. The first obstable is that it is not true that $G_j(X)G_j(Y) = G_j(X \cup Y)$ for general disjoint $X, Y \in \mathscr{P}_j$. The second obstacle is that summing over all $Y \in \mathscr{P}_j(X)$ would create a combinatorial factor $2^{|X|_j}$ in the end, so taking the supremum over $X \in \mathscr{P}_j^c$ would make $\|K''\|_j$ diverge. Fortunately, we can circumvent these problems in (4.75)–(4.76) due to the the specific form of the polymers involved. Sufficient conditions for the former operations are implied by the following conditions:

(Q) Let $(\mathbb{X}, |\cdot|)$ be a normed space and $B_\varepsilon^{\mathbb{X}}$ be the open ball with radius $\varepsilon > 0$. Let φ', ζ be fields taking value in $\mathbb{R}^\Lambda$, and $\zeta \sim \mathscr{N}(0, \Gamma_{j+1})$. Let $\mathscr{Q} = \cup_{\alpha=1}^4 \mathscr{Q}_\alpha$ be a partition of $\mathscr{B}_{j+1}$ and let $F_x(\cdot, \varphi', \zeta)$ and $f_x^y(\cdot, \varphi', \zeta)$ be polymer activities supported on $\mathscr{Q}$ and labelled by $x \in B_\varepsilon^{\mathbb{X}}$, $y \in \mathbb{X}$. Assume that $F_0(Q, \varphi', \zeta) \equiv 1_{Q=\emptyset}$, that f_x^y is linear in y, and that there are $C > 0$, $\vec{\eta}(Q) \geqslant 0$ and a function $\psi : \mathbb{X} \to \mathbb{R}$ with $\psi(z) = o(1)$ as $z \to 0$ such that

(i) (Boundedness)
$\|f_x^y(Q, \varphi', \zeta)\|_{\vec{\mathfrak{h}}, T_j(Q,\varphi')} \leqslant C A^{-(1+\vec{\eta}(Q))|Q|_{j+1}} \mathscr{G}(Q, \varphi', \zeta)|y|$;

(ii) (Continuity)
$\|(f_{x+z}^y - f_x^y)(Q, \varphi', \zeta)\|_{\vec{\mathfrak{h}}, T_j(Q,\varphi')} \leqslant C A^{-(1+\vec{\eta}(Q))|Q|_{j+1}} \mathscr{G}(Q, \varphi', \zeta)|y|\psi(z)$;

(iii) (Derivative)
$\|(F_{x+y} - F_x - f_x^y)(Q, \varphi', \zeta)\|_{\vec{\mathfrak{h}}, T_j(Q,\varphi')} \leqslant C A^{-(1+\vec{\eta}(Q))|Q|_{j+1}} \mathscr{G}(Q, \varphi', \zeta)|y|\psi(y)$

where $\mathscr{G}(Q, \varphi', \zeta)$ is 1 if $Q \in \mathscr{Q}_1$, is $e^{c_w \kappa w_j(Q,\varphi')}$ if $Q \in \mathscr{Q}_2$, is $e^{c_w \kappa w_j(Q,\varphi'+\zeta)}$ if $Q \in \mathscr{Q}_3$, and is $G_j(Q, \varphi' + \zeta)$ if $Q \in \mathscr{Q}_4$, and $Y, Z \in \mathscr{Q}_4$ implies $Y \not\sim Z$. Also, assume that F_x does not depend on ζ if $Q \in \mathscr{Q}_1 \cup \mathscr{Q}_2$, and $\vec{\eta}(Q)$ takes value 0 or $\eta_0 > 0$ and if $\vec{\eta}(Q) = 0$, then $Q \in \mathscr{S}_{j+1}$.

The flexibility of $\mathscr{G}_j$ will save us from the problem of regulators and the extra decay due to $\vec{\eta}(Q)$ will save us from the problem of combinatorial factor $2^{|X|_j}$. The choices of $\mathscr{G}_j$'s will have the following consequence, where we denote $Q \in \mathscr{Q}(X)$ if $Q \in \mathscr{Q}$ and $Q \subset X$.

Lemma 4.4.3. *Let $\mathscr{Q} = \cup_{\alpha=1}^4 \mathscr{Q}_\alpha$ be as in* (Q) *for $X \in \mathscr{P}_{j+1}$ and suppose $f(Q, \varphi', \zeta)$ is a polymer function such that*

$$\|f(Q, \varphi, \zeta)\|_{\vec{\mathfrak{h}}, T_j(Q,\varphi)} \leqslant a(Q)\mathscr{G}_j(Q, \varphi', \zeta). \tag{4.88}$$

Also, let $f(Q, \varphi', \zeta)$ be independent of ζ when $Q \in \mathscr{Q}_1 \cup \mathscr{Q}_2$. Then

$$\Big\| \mathbb{E}_{(\omega)} \prod_{Q \in \mathscr{Q}(X)} f(Q, \varphi', \zeta) \Big\|_{\vec{\mathfrak{h}}, T_{j+1}(X,\varphi')} \leqslant C 2^{|X|_j} G_{j+1}(X, \varphi') \prod_{Q \in \mathscr{Q}(X)} a(Q). \tag{4.89}$$

for some constant $C > 0$.

Proof. Since $f(Q)$ does not depend on ζ when $Q \in \mathscr{Q}_1 \cup \mathscr{Q}_2$, we first consider $Q \in \mathscr{Q}_3 \cup \mathscr{Q}_4$ to see that

$$\Big\| \prod_{Q \in \mathscr{Q}_3(X) \cup \mathscr{Q}_4(X)} f(Q, \varphi', \zeta) \Big\|_{\vec{\mathfrak{h}}, T_j(X_{34})} \leqslant G_j(X_{34}, \varphi' + \zeta) \prod_{Q \in \mathscr{Q}_3(X) \cup \mathscr{Q}_4(X)} a(Q) \tag{4.90}$$

where $X_{34} = \cup_{Q\in\mathcal{Q}_3(X)\cup\mathcal{Q}_4(X)} Q$ and we have used Lemma 3.3.4 to bound the products of $\mathcal{G}$'s. Thus by Proposition 3.3.7,

$$\begin{aligned} \Big\| \mathbb{E}_{(\omega)} \prod_{Q\in\mathcal{Q}_3(X)\cup\mathcal{Q}_4(X)} f(Q,\varphi',\zeta) \Big\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')} \\ \leqslant C2^{|X_{34}|_j} G_{j+1}(X_{34},\varphi') \prod_{Q\in\mathcal{Q}_3(X)\cup\mathcal{Q}_4(X)} a(Q). \end{aligned} \tag{4.91}$$

Now we have the desired inequality if we just multiply the norm on $f(Q,\varphi',\zeta)$ for $Q \in \mathcal{Q}_3 \cup \mathcal{Q}_4$. $\square$

Then we have the product rule for the case of our interest.

Proposition 4.4.4 (Product rule). *Let $\mathbb{X}$, $\mathcal{Q}$, $\mathcal{Q}(X)$, f, F, ψ and $\vec{\eta}$ be as in* (Q) *for $X \in \mathcal{P}_{j+1}$. Given collection of parameters $\vec{x} = \{x(Q) \in \mathbb{X} : Q \in \mathcal{Q}\}$, define for $X \in \mathcal{P}_{j+1}$,*

$$\mathcal{L}_{\vec{x}}(X,\varphi') = \begin{cases} \mathbb{E}_{(\omega)}\Big[\prod_{Q\in\mathcal{Q}(X)} F_{x(Q)}(Q,\varphi',\zeta)\Big] & \text{if } |\mathcal{Q}(X)| \geqslant 2,\ X = \cup_{Q\in\mathcal{Q}(X)} Q \\ 0 & \text{otherwise.} \end{cases} \tag{4.92}$$

Then for A sufficiently large and ε sufficiently small (polynomially in L, A, C), the partial derivatives of $\vec{x} \mapsto \mathcal{L}_{\vec{x}}$ exist (as a map from $\mathbb{X}$ to the space of polymer activities of finite $\|\cdot\|_{h,T_{j+1}(X)}$-norm), the partial derivatives in directions $P \in \mathcal{Q}$ are given by

$$d_P^y := \partial_{y,P}\mathcal{L}_{\vec{x}} := \mathbb{E}_{(\omega)}\Big[f^y_{x(P)}(P,\varphi',\zeta) \prod_{Q\in\mathcal{Q}(X\setminus P)} F_{x(Q)}(\varphi')\Big] \quad \text{if } |X|_{j+1} \geqslant 2, \tag{4.93}$$

and they are continuous in the domain $\{\vec{x} : |x(Q)| < \varepsilon,\ \forall Q \in \mathcal{Q}\}$. Moreover, in the case $x(Q) \equiv x$, if we let $\mathcal{L}_x = \mathcal{L}_{\vec{x}}$, then $\mathcal{L}_x$ is differentiable in $\{|x| < \varepsilon\}$, the derivative satisfies the bounds

$$\|D\mathcal{L}_x(X)\|_{\vec{\mathfrak{h}},T_{j+1}(X)} \leqslant CA^{-(1+\eta_0/2)|X|_{j+1}} |x|^{\max\{1,\frac{|\mathcal{Q}(X)|-1}{2}\}}, \tag{4.94}$$

and $D\mathcal{L}_x(X)$ is continuous in x.

Proof. All X used below are assumed to satisfy $|\mathcal{Q}(X)| \geqslant 2$ and $\cup_{Q\in\mathcal{Q}(X)} Q = X$ which is sufficient by the definition of $\mathcal{L}_{\vec{x}}$. We first show that d_P^y has finite norm. Indeed, Lemma 4.4.3

gives

$$
\begin{aligned}
&\|d_P^y(\varphi',X)\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')}\\
&\quad\leqslant C^{|\mathscr{Q}(X)|}A^{-\sum_{Q\in\mathscr{Q}(X)}(1+\vec{\eta}(Q))|Q|_{j+1}}2^{|X|_j}G_{j+1}(X,\varphi')\,|x|^{|\mathscr{Q}(X)|-1}|y|\\
&\quad\leqslant C^{|\mathscr{Q}(X)|}A^{\eta_0\sum_{Q\in\mathscr{Q}(X)}^{\vec{\eta}(Q)=0}|Q|_{j+1}}(2^{L^2}A^{-(1+\eta_0)})^{|X|_{j+1}}G_{j+1}(X,\varphi')|x|^{|\mathscr{Q}(X)|-1}|y|\\
&\quad\leqslant C^{|\mathscr{Q}(X)|}A^{4\eta_0|\mathscr{Q}(X)|}(2^{L^2}A^{-(1+\eta_0)})^{|X|_{j+1}}G_{j+1}(X,\varphi')|x|^{|\mathscr{Q}(X)|-1}|y|. \qquad (4.95)
\end{aligned}
$$

Thus $(y\mapsto d_P^y)$ is a bounded linear map from $\mathbb{X}$ to the polymer activities of finite $\|\cdot\|_{\vec{\mathfrak{h}},T_{j+1}(X)}$-norm.

To show that d_P^y is the derivative of $\mathscr{L}_x$, let $\delta\mathscr{L}_{\vec{x},y}:=\mathscr{L}_{\vec{x}+y\delta_{P,Q}}-\mathscr{L}_{\vec{x}}-d_P^y$. Then by essentially the same computation as above,

$$
\begin{aligned}
&\|\delta\mathscr{L}_{\vec{x},y}(X,\varphi')\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')}\\
&\quad=\Big\|\mathbb{E}_{(\omega)}\Big[(F_{x(P)+y}-F_{x(P)}-f^y_{x(P)})(P)\prod_{Q\in\mathscr{Q}(X\setminus P)}F_{x(Q)}(Q)\Big]\Big\|_{\mathfrak{h},T_{j+1}(X,\varphi')}\\
&\quad\leqslant(CA^{4\eta_0})^{|\mathscr{Q}(X)|}|x|^{|\mathscr{Q}(X)|-1}A^{-(1+\eta_0)|X|_{j+1}}|y|2^{|X|_j}G_{j+1}(X,\varphi')\psi(y)\\
&\quad\leqslant(CA^{4\eta_0})^{|\mathscr{Q}(X)|}|x|^{|\mathscr{Q}(X)|-1}(2^{L^2}A^{-(1+\eta_0)})^{|X|_{j+1}}G_{j+1}(X,\varphi')|y|\psi(y), \qquad (4.96)
\end{aligned}
$$

proving the existence of the partial derivatives of $\mathscr{L}_{\vec{x}}$ as a function from $\mathbb{X}$ to the space of polymer activities of finite $\|\cdot\|_{\vec{\mathfrak{h}},T_{j+1}(X)}$-norm. To see the differentiability of $\mathscr{L}_x$, let

$$
d^y(X,\varphi'):=D\mathscr{L}_x(X,\varphi'):=\sum_{P\in\mathscr{Q}(X)}\partial_{y,P}\mathscr{L}_{\vec{x}}|_{x(Q)\equiv x}. \qquad (4.97)
$$

Then

$$
\|d^y(X,\cdot)\|_{\vec{\mathfrak{h}},T_{j+1}(X)}\leqslant|\mathscr{Q}(X)|(CA^{4\eta_0})^{|\mathscr{Q}(X)|}|x|^{|\mathscr{Q}(X)|-1}(2^{L^2}A^{-(1+\eta_0)})^{|X|_{j+1}}G_{j+1}(X,\varphi')|y| \qquad (4.98)
$$

and hence $(y\mapsto d^y)$ is bounded linear from $\mathbb{X}$ to the space of polymer activities with finite $\|\cdot\|_{\vec{\mathfrak{h}},T_{j+1}(X)}$-norm. Also applying (4.96) multiple times shows that

$$
\begin{aligned}
&\|(\mathscr{L}_{x+y}-\mathscr{L}_x-d^y)(X,\cdot)\|_{\vec{\mathfrak{h}},T_{j+1}(X)}\\
&\qquad\leqslant(CA^{4\eta_0})^{|\mathscr{Q}(X)|}|\mathscr{Q}(X)|\,|x|^{|\mathscr{Q}(X)|-1}(2^{L^2}A^{-(1+\eta_0)})^{|X|_{j+1}}|y|\psi(y) \qquad (4.99)
\end{aligned}
$$

proving differentiability of $\mathscr{L}_x$. The bound for the derivative is obtained once we choose ε small and A large so that

$$A^{|X|_{j+1}}\|D\mathscr{L}_x(X)\|_{\vec{\mathfrak{h}},T_{j+1}(X)} \leqslant (CA^{4\eta_0})^{|\mathscr{Q}(X)|}|\mathscr{Q}(X)|A^{-\frac{\eta_0}{2}|X|_{j+1}}|x|^{|\mathscr{Q}(X)|-1}$$
$$\leqslant C'A^{-\frac{\eta_0}{2}|X|_{j+1}}|x|^{\frac{|\mathscr{Q}(X)|-1}{2}}. \tag{4.100}$$

But for the case $|\mathscr{Q}(X)|=2$, one could just have bounded the left-hand by $C'A^{-\frac{\eta_0}{2}|X|_{j+1}}|x|$ instead.

The continuity of the derivative follows from the assumption on the continuity of f. □

4.4.2 Proof of Lemma 4.4.2

In this subsection, $\mathscr{E}_{j+1}|X|, U_j, \overline{U}_{j+1}, K_j$ and Q_j will always be a function of x implicitly. Also, since Q_j contributes only through $\mathscr{D}Q_j$ and $\overline{\mathscr{D}}Q_j$, it is convenient to also think $\mathfrak{K}_j$ is a collection $(\mathscr{E}_{j+1}|X|, U_j, \overline{U}_{j+1}, K_j, \mathscr{D}Q_j, \overline{\mathscr{D}}Q_j)$ and use the following estimates.

Lemma 4.4.5. *Under the assumptions of Lemma 4.4.2,*

$$\|D\mathscr{D}Q_j(B,Z,\varphi)\|_{\vec{\mathfrak{h}},T_j(B,\varphi)} \leqslant C(L)A^{-1}e^{c_w\kappa_L w_j(B,\varphi)^2}, \tag{4.101}$$
$$\|D\overline{\mathscr{D}}Q_j(Z,\varphi)\|_{\vec{\mathfrak{h}},T_j(Z,\varphi)} \leqslant C(A,L)A^{-(1+\eta)|Z|_{j+1}}e^{c_w\kappa_L w_j(Z,\varphi)^2}, \tag{4.102}$$

for some constants $C(L), C(A,L) > 0$, being differentiable in the indicated space.

Proof. The proof is direct recalling $x \mapsto Q_j(x)$ is linear, the bound (4.86) and (4.37),(4.38) for the definition of $\mathscr{D}Q_j$ and $\overline{\mathscr{D}}Q_j$.

□

Now for brevity, we define the following expressions which appear as part of the definitions of the $\mathfrak{M}^{(k)}_{j+1}$:

$$F(\mathfrak{K}_j,T,X) = e^{\mathscr{E}^{\dagger}_{j+1}|X|+\overline{U}_{j+1}(X\backslash T)} \tag{4.103}$$
$$H(\mathfrak{K}_j,X_0,X_1,Z,(B_{Z''})) = \mathbb{E}\Big[(e^{U_j}-e^{\overline{U}_{j+1}})^{X_0}(\overline{K}_j-\overline{\mathscr{D}}Q_j)^{[X_1]}\Big]$$
$$\times \prod_{Z''\in\mathrm{Comp}_{j+1}(Z)} \mathscr{D}Q_j(B,Z''). \tag{4.104}$$

Lemma 4.4.6. *Under the same assumptions as in Lemma 4.4.2,*

$$\mathscr{A}_1(\mathfrak{K}_j(x),X) = 1_{|X|_{j+1}=1}\mathbb{E}[e^{U_j(X)} - U_j(X) - e^{\overline{U}_{j+1}(X)} + \overline{U}_{j+1}(X)] \tag{4.105}$$

$$\mathscr{A}_2(\mathfrak{K}_j(x),T) = \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} H(x,X_0,X_1,Z,(B_{Z''})) \tag{4.106}$$

with $T = X_0 \cup X_1 \cup (\cup_{Z''\in \mathrm{Comp}_{j+1}(Z)} B^*_{Z''})$ *are differentiable in x with*

$$\|DH(\mathfrak{K}_j(x),X_0,X_1,Z,(B_{Z''}))\|_{\vec{\mathfrak{h}},T_{j+1}(T)} \leqslant C_A A^{-(1+\frac{\eta}{4})|X|_{j+1}}|x|^{\max\{1,\frac{\#(X_0,X_1,Z)-1}{2}\}} \tag{4.107}$$

$$\|D\mathscr{A}_l(\mathfrak{K}_j(x),X)\|_{\vec{\mathfrak{h}},T_{j+1}(T)} \leqslant CA^{-(1+\eta)|X|_{j+1}}|x|, \quad l = 1,2 \tag{4.108}$$

for some $\eta > 0$*,* $C_A \equiv C_A(A,L)$*,* $C \equiv C(L)$ *and H defined by* (4.104)*. Moreover, each derivative is continuous in x.*

Proof. The differentiability of $\mathscr{A}_1$ follows from (4.82) and (4.83). To see its bound, let $X = B \in \mathscr{B}_{j+1}$. We have $\mathbb{E}[e^{U_j(B)} - U_j(B) - e^{\overline{U}_{j+1}(B)} + \overline{U}_{j+1}(B)] = \mathbb{E}\big[\big((e^{U_j} - 1 - U_j) + (e^{\overline{U}_{j+1}} - 1 - \overline{U}_{j+1})\big)(B)\big]$, and (4.84) implies

$$\|D\mathbb{E}[(e^{U_j} - 1 - U_j)](B,\varphi')\|_{\vec{\mathfrak{h}},T_{j+1}(B,\varphi')} \leqslant C\mathbb{E}[e^{c_w\kappa_L w_j(B,\varphi'+\zeta)^2}]|x| \leqslant C'G_{j+1}(B,\varphi')|x| \tag{4.109}$$

where the second inequality follows from $\mathbb{E}[e^{c_w\kappa_L w_j(B,\varphi'+\zeta)^2}] \leqslant \mathbb{E}[G_j(B,\varphi'+\zeta)] \leqslant 2^{L^2}G_{j+1}(B,\varphi')$, see Lemma 3.3.4 and Proposition 3.3.5. The same estimate applies to $e^{\overline{U}_{j+1}} - 1 - \overline{U}_{j+1}$ and hence

$$\|D\mathscr{A}_1(\mathfrak{K}_j(x),B)\|_{\vec{\mathfrak{h}},T_{j+1}(B)} \leqslant C|x|. \tag{4.110}$$

To show the differentiability of $\mathscr{A}_2$, we can apply Proposition 4.4.4. To see this, expand

$$\begin{aligned} H(\mathfrak{K}_j(x),X_0,X_1,Z,(B_{Z''})) &= \sum_{Y_0,Y_1}(-1)^{|Y_0|_{j+1}+|\mathrm{Comp}_{j+1}(Y_1)|} \\ &\times \mathbb{E}\Big[(e^{U_j}-1)^{X_0\setminus Y_0}(e^{\overline{U}_{j+1}}-1)^{Y_0}(\overline{K_j})^{[X_1\setminus Y_1]}(\overline{\mathscr{D}}Q_j)^{[Y_1]}\Big]\prod_{Z''\in\mathrm{Comp}_{j+1}(Z)}\mathscr{D}Q_j(B,Z'') \end{aligned} \tag{4.111}$$

where the sum runs over $Y_0 \in \mathscr{P}_{j+1}(X_0)$ and $\mathrm{Comp}_{j+1}(Y_1) \subset \mathrm{Comp}_{j+1}(X_1)$. For fixed X_0, X_1, Z, $Y_0 \subset X_0$ and $Y_1 \subset X_1$, let $\mathscr{Q} = \mathscr{B}_{j+1}(X_0) \cup \mathrm{Comp}_{j+1}(X_1) \cup \mathrm{Comp}_{j+1}(Z) \cup \mathscr{B}_{j+1}(T^c)$

where $T = X_0 \cup X_1 \cup Z$ and define

$$F_x(Q,\varphi',\zeta) = \begin{cases} e^{U_j(Q,\varphi'+\zeta)} - 1 & \text{if } Q \in \mathscr{B}_{j+1}(X_0 \backslash Y_0) \\ e^{\overline{U}_{j+1}(Q,\varphi')} - 1 & \text{if } Q \in \mathscr{B}_{j+1}(Y_0) \\ \overline{K}_j(Q,\varphi'+\zeta) & \text{if } Q \in \mathrm{Comp}_{j+1}(X_1 \backslash Y_1) \\ \overline{\mathscr{D}}Q_j(Q,\varphi') & \text{if } Q \in \mathrm{Comp}_{j+1}(Y_1) \\ \mathscr{D}Q_j(B_Q,Q,\varphi') & \text{if } Q \in \mathrm{Comp}_{j+1}(Z) \\ 1 & \text{if } Q \in \mathscr{B}_{j+1}(T^c), \end{cases} \tag{4.112}$$

and

$$\mathscr{G}(Q,\varphi',\zeta) = \begin{cases} e^{c_w \kappa_L w_j(Q,\varphi'+\zeta)} & \text{if } Q \in \mathscr{B}_{j+1}(X_0 \backslash Y_0) \\ e^{c_w \kappa_L w_j(Q,\varphi')} & \text{if } Q \in \mathscr{B}_{j+1}(Y_0) \cup \mathrm{Comp}_{j+1}(Y_1) \cup \mathrm{Comp}_{j+1}(Z) \\ G_j(Q,\varphi'+\zeta) & \text{if } Q \in \mathrm{Comp}_{j+1}(X_1 \backslash Y_1) \\ 1 & \text{if } Q \in \mathscr{B}_{j+1}(T^c). \end{cases} \tag{4.113}$$

Then Proposition 4.4.4 with the assumption that $\mathfrak{K}_j \in \mathscr{X}_j^{\mathfrak{K}}(\mathbb{X})$ (i.e., it satisfies the bounds (4.81)–(4.102)) shows $\mathscr{A}_2$ is differentiable and

$$\|D\mathscr{A}_2(\mathfrak{K}_j(x),X)\|_{\vec{\mathfrak{h}},T_{j+1}(X)} \leqslant \sum_{X_0,X_1,Y_0,Y_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} E(X_0,X_1,Y_0,Y_1,Z,(B_{Z''})) \tag{4.114}$$

$$E(X_0,X_1,Y_0,Y_1,Z,(B_{Z''})) = C_A A^{-(1+\eta/2)|T|_{j+1}} |x|^{\max\{1,\frac{\#(X_0,X_1,Z)-1}{2}\}} \tag{4.115}$$

First consider the cases $\#(X_0,X_1,Z) \geqslant 4$ and note that $\max\{1,\frac{\#(X_0,X_1,Z)-1}{2}\} = \frac{\#(X_0,X_1,Z)-1}{2}$. Since $X_0,X_1,Y_0,Y_1,(B_{Z''}),Z\backslash(\cup_{Z''}B_{Z''})$ and $X\backslash T$ partition X, one may bound the sum by a sum running over partitions of X partitioned into 7 subsets. This gives a crude combinatorial bound

$$\sum_{X_0,X_1,Y_0,Y_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 4} E(X_0,X_1,Y_0,Y_1,Z,(B_{Z''})) \leqslant C_A 7^{|X|_{j+1}} \sup A^{-(1+\frac{\eta}{2})|T|_{j+1}} |x|^{\frac{\#(X_0,X_1,Z)-1}{2}} \tag{4.116}$$

where the supremum also runs over the choices of $X_0,X_1,Y_0,Y_1,Z,(B_{Z''})$. Also with the assumption $7A^{-\eta/4} \leqslant 1$, this can also be bounded by

$$C_A A^{-(1+\frac{\eta}{4})|X|_{j+1}} \sup A^{(1+\frac{\eta}{2})|X\backslash T|_{j+1}} |x|^{\frac{\#(X_0,X_1,Z)-1}{2}}. \tag{4.117}$$

But $|X\backslash T|_{j+1} = |\cup_{Z''}(B^*_{Z''}\backslash Z'')|_{j+1} \leqslant 48|Z|_{j+1}$, so $A^{|X\backslash T|_{j+1}} \leqslant A^{48|Z|_{j+1}}$. Since each connected component of Z is a small set, it follows that $|Z|_{j+1} \leqslant 4|\mathrm{Comp}_{j+1}(Z)|$, and hence the condition $A^{192(1+\eta/4)}|x|^{1/8} \leqslant 1$ gives

$$A^{(1+\frac{\eta}{4})|X|_{j+1}} \sum_{X_0,X_1,Y_1,Y_2,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 4} E(X_0,X_1,Y_0,Y_1,Z,(B_{Z''})) \leqslant C|x|. \tag{4.118}$$

For the cases $\#(X_0,X_1,Z) \in \{2,3\}$, we have $|X_0|_{j+1} \leqslant 3$ and $|\cup_{Z''} B^*_{Z''}|_{j+1} \leqslant 3 \times 49$ so

$$A^{(1+\frac{\eta}{4})|X|_{j+1}} \sum_{X_0,X_1,Y_1,Y_2,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\in\{2,3\}} E(X_0,X_1,Y_0,Y_1,Z,(B_{Z''})) \leqslant C_A|x| \tag{4.119}$$

by just setting C_A sufficiently large depending on A.

The continuity of DG and $D\mathscr{A}_l$ is a result of the continuity of the derivative in Proposition 4.4.4. □

Proof of Lemma 4.4.2, case $k \in \{1,2,3\}$. Consider the function

$$M_{j+1}(x,x') = \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} F(\mathfrak{K}_j(x),T,X)H(\mathfrak{K}_j(x'),X_0,X_1,Z,(B_{Z''})) \tag{4.120}$$

and recall that F and H are defined by (4.103) and (4.104), and we emphasise that abpve F uses x to define E_{j+1} and U_{j+1} while H uses x', so that $\mathfrak{M}^{(1)}_{j+1}(\mathfrak{K}_j(x)) = M_{j+1}(x,x)$. By Lemmas 6.4.4 and 4.4.6, $M_{j+1}(x,x')$, $\mathfrak{M}^{(2)}_{j+1}(\mathfrak{K}_j(x'))$ and $\mathfrak{M}^{(3)}_{j+1}(\mathfrak{K}_j(x'))$ are differentiable in x' with the desired bounds. For the x derivative of $M_{j+1}(x,x')$, we justify the differentiability more carefully: let

$$f^{\dot{x}}_x(T,X,\varphi') = (D\mathscr{E}^{\dagger}_{j+1}(\dot{x})|X| + D\overline{U}_{j+1}(X\backslash T)(\dot{x}))F(\mathfrak{K}_j(x),T,X) \tag{4.121}$$

$$m^{\dot{x}}_x(X,\varphi') = \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} f^{\dot{x}}_x(T,X)H(\mathfrak{K}_j(x'),X_0,X_1,Z,(B_{Z''})). \tag{4.122}$$

Letting $\delta_{\dot{x}} M_{j+1}(x,x') = M_{j+1}(x+\dot{x},x') - M_{j+1}(x,x') - m_x^{\dot{x}}$, the bounds (4.82) and (4.107) give

$$
\begin{aligned}
&\|\delta_{\dot{x}} M_{j+1}(x,x')(X,\varphi')\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')} \\
&\leqslant C_A \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} C^{|X|_{j+1}} (A^{-1-\eta})^{|X|_{j+1}} e^{c_w \kappa_L w_j(X\setminus T,\varphi')^2} G_{j+1}(T,\varphi')|\dot{x}|^2 |x'|^{\max\{2,\frac{\#(X_0,X_1,Z)+1}{2}\}} \\
&\leqslant C_A \sup (5C)^{|X|_{j+1}} (A^{-1-\eta})^{|X|_{j+1}} G_{j+1}(X,\varphi')|\dot{x}|^2 |x'|^{\max\{2,\frac{\#(X_0,X_1,Z)+1}{2}\}}
\end{aligned} \tag{4.123}
$$

where the supremum ranges over $X_0, X_1, Z, (B_{Z''})$ with $\#(X_0,X_1,Z) \geqslant 2$. Choosing $5CA^{-\frac{\eta}{2}} \leqslant 1$,

$$
\|\delta_{\dot{x}} M_{j+1}(x,x')(X,\varphi')\|_{\vec{\mathfrak{h}},T_{j+1}(X,\varphi')} \leqslant C_A A^{-(1+\frac{\eta}{2})|X|_{j+1}} G_{j+1}(X,\varphi')|\dot{x}|^2|x'|^2. \tag{4.124}
$$

Therefore $M_{j+1}(x,x')(X)$ is differentiable in x and the same computation gives the bound

$$
\|\partial_x M_{j+1}(x,x')(X)\|_{\vec{\mathfrak{h}},T_{j+1}(X)} \leqslant C_A A^{-(1+\frac{\eta}{2})|X|_{j+1}} |x'|^2. \tag{4.125}
$$

The continuity of the derivatives are results of continuity of derivatives in Lemma 4.4.6. □

Proof of Lemma 4.4.2, case $k=4$. We may alternatively write $\mathfrak{M}_{j+1}^{(4)} = \mathbb{E} M_-^{(4)} := \mathbb{E}[M_-^{(4,1)} + M_-^{(4,2)}]$ where

$$
M_-^{(4,1)}(\mathfrak{K}_j(x),X,\varphi',\zeta) = \sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} 1_{Y\in\mathscr{P}_j^c}(e^{U_j(X\setminus Y,\varphi'+\zeta)}-1)K_j^\dagger(Y,\varphi'+\zeta) \tag{4.126}
$$

$$
M_-^{(4,2)}(\mathfrak{K}_j(x),X,\varphi',\zeta) = \sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} 1_{Y\notin\mathscr{S}_j} 1_{Y\notin\mathscr{P}_j^c} e^{U_j(X\setminus Y,\varphi'+\zeta)} \prod_{Z\in\mathrm{Comp}_j(Y)} K_j^\dagger(Z,\varphi'+\zeta) \tag{4.127}
$$

as $1_{Y\notin\mathscr{S}_j}1_{Y\notin\mathscr{P}_j^c} = 1_{Y\notin\mathscr{P}_j^c}$. By (4.82),

$$
\begin{aligned}
\|De^{U_j(X\setminus Y,\varphi)}(\dot{x})\|_{\vec{\mathfrak{h}},T_j(X,\varphi)} &\leqslant C(\delta,L)e^{(1+C(\delta,L)|x|)(|X\setminus Y|_j+\delta c_w\kappa_L w_j(X\setminus Y,\varphi)^2)}|\dot{x}| \\
&\leqslant C(L)e^{2|X\setminus Y|_j}e^{c_w\kappa_L w_j(X\setminus Y,\varphi)}|\dot{x}|
\end{aligned} \tag{4.128}
$$

for $\delta < 1/2$ and $|x| \leqslant \frac{1}{C(\delta,L)}$ and then the mean value theorem gives

$$
\|e^{U_j(X\setminus Y,\varphi)} - 1\|_{h,T_j(X,\varphi)} \leqslant C(L)e^{2|X\setminus Y|_j}e^{c_w\kappa_L w_j(X\setminus Y,\varphi)}|x|. \tag{4.129}
$$

So using (4.128) to bound $\partial_{U_j} M_-^{(4)}$ and (4.129) to bound $\partial_{K_j} M_-^{(4)}$, and since $x \mapsto K_j$ is linear and bounded, we see that

$$\begin{aligned}\|DM_-^{(4,1)}(\mathfrak{K}_j(x), X, \varphi', \zeta)\|_{\vec{\mathfrak{h}}, T_j(X,\varphi')} \\ \leqslant C(L) \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} 1_{Y \in \mathscr{P}_j^c} e^{2|X\setminus Y|_{j+1}} e^{c_w \kappa_L w_j (X\setminus Y)^2} |x| A^{-|Y|_j} G_j(Y, \varphi' + \zeta).\end{aligned} \quad (4.130)$$

If $Y \in \mathscr{S}_j$, then $X \in \mathscr{S}_{j+1}$ and $|X|_{j+1} \leqslant |Y|_j$ so the summand on the right-hand side is bounded by $C'(A,L)A^{-(1+\eta)|X|_{j+1}}|x|$ where $C'(A,L) = C(L)A^{4\eta}$. If $Y \notin \mathscr{S}_j$, then Lemma 4.1.6 implies $|Y|_j \geqslant \frac{\eta}{2(1+\eta)}|Y|_j + \frac{2+\eta}{2}|X|_{j+1}$ so that

$$C(L) \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} (2e^2)^{|X|_{j+1}} A^{-|Y|_j} |x| \leqslant C(L)(2e^2 A^{-\frac{2+\eta}{2}})^{|X|_{j+1}} |x| \sum_{Y \in \mathscr{P}_j}^{\overline{Y}=X} A^{-\frac{\eta}{2(1+\eta)}|Y|_j}. \quad (4.131)$$

But for $L^2 \leqslant A^{\frac{\eta}{2(1+\eta)}}$,

$$\sum_{Y : \overline{Y}=X} A^{-\frac{\eta}{2(1+\eta)}|Y|_j} \leqslant (1 + A^{-\frac{\eta}{2(1+\eta)}})^{|X|_j} \leqslant e^{L^2 A^{-\frac{\eta}{2(1+\eta)}} |X|_{j+1}} \leqslant e^{|X|_{j+1}} \quad (4.132)$$

so we may conclude

$$\|DM_-^{(4,1)}(\mathfrak{K}_j(x), X, \varphi')\|_{\vec{\mathfrak{h}}, T_j(X,\varphi')} \leqslant C(A,L)(e^3 A^{-(1+\frac{\eta}{2})})^{|X|_{j+1}} |x| G_j(X, \varphi' + \zeta) \quad (4.133)$$

For $M_-^{(4,2)}$, we have

$$\begin{aligned}&\|DM_-^{(4,2)}(\mathfrak{K}_j(x), X, \varphi', \zeta)\|_{\vec{\mathfrak{h}}, T_{j+1}(X,\varphi')} \\ &\leqslant C(L) \sum_{Y \in \mathscr{P}_j \setminus \mathscr{P}_j^c}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} |\mathrm{Comp}_j(Y)| A^{-|Y|_j} e^{2|X\setminus Y|_j} e^{c_w \kappa_L w_j (X\setminus Y, \varphi)} G_j(Y, \varphi' + \zeta) |x|^{|\mathrm{Comp}_j(Y)|-1} \\ &\leqslant C(L) G_j(X, \varphi' + \zeta) e^{2L^2 |X|_{j+1}} \sum_{Y \in \mathscr{P}_j \setminus \mathscr{P}_j^c}^{\overline{Y}=X} 1_{Y \notin \mathscr{S}_j} (e^2 A/2)^{-|Y|_j} |x|^{|\mathrm{Comp}_j(Y)|-1}.\end{aligned} \quad (4.134)$$

But by Lemma 4.1.7, this is bounded by

$$C(L) G_j(X, \varphi' + \zeta) e^{2L^2 |X|_{j+1}} (2e^{-1} L^2 A^{-1-\eta})^{|X|_{j+1}} |x| \quad (4.135)$$

for some $\eta > 0$. Hence for sufficiently large A, we have

$$\|DM_{-}^{(4,2)}(\mathfrak{K}_j(x), X, \varphi')\|_{\vec{\mathfrak{h}}, T_j(X,\varphi')} \leqslant C(L) A^{-(1+\eta/2)|X|_{j+1}} |x| G_j(X, \varphi' + \zeta) \tag{4.136}$$

and the same bounds also imply the differentiability of $D\mathfrak{M}_{j+1}^{(4)}$ with bound

$$\|D\mathfrak{M}_{j+1}^{(4)}(\mathfrak{K}_j(x))\|_{\vec{\mathfrak{h}}, T_{j+1}(X,\varphi')} \leqslant C(A,L) A^{-(1+\frac{\eta}{3})|X|_{j+1}} |x| G_{j+1}(X, \varphi'). \tag{4.137}$$

The continuity of the derivative is a consequence of continuity of derivatives in Lemma 4.4.6.

□

Chapter 5

Localisation of periodic polymer activities

This chapter is dedicated to identifying the localisation and the mechanisms that are used to prove the contraction of the linearised renormalisation group map. In Chapter 4, we have already seen that terms in polymer expansions that are algebraically of degree $\geqslant 2$ are also analytically of degree $\geqslant 2$ in appropriate norms. Thus the results of this chapter combined with the previous chapter would complete the picture for proving the analytic properties of the polymer expansion introduced in Definition 4.2.1. Ultimately, the contraction estimates will become the key component for showing the stability of the renormalisation group flow, when the localisation and the coupling constants are chosen correctly.

There are essentially three sources of contraction in our setup, one relying on the periodicity (which is inherited from the original potential), one from terms only involving the gradients $\nabla^n \varphi$ of order $n > 2$, and one coming from large polymers $X \notin \mathscr{S}_j$. The final one is already proved in Proposition 4.1.5, so we focus on the first two mechanisms.

The main results of this chapter are Propositions 5.2.2 and 5.2.3, which concern small polymers. Most of the remainder of this section consists of supporting arguments that are used only for the proof of these propositions and will not be applied directly in the rest of this paper.

5.1 Periodicity, charge decomposition, and lattice symmetries

For a field $\varphi = (\varphi_x)$ and scalar $t \in \mathbb{R}$ we often write $\varphi + t = (\varphi_x + t)$ in the sequel. Our starting point is the following *charge decomposition* of a globally periodic field functional, introduced in [29]. The period would depend on the parameter β, but we would not make the dependence always explicit.

Definition 5.1.1. *Let $\beta > 0$. Then a polymer activity F is $2\pi/\sqrt{\beta}$-*periodic *if $t \in \mathbb{R} \mapsto F(X,\varphi+t)$ is $2\pi/\sqrt{\beta}$-periodic. Its Fourier expansion in the constant field is denoted by*

$$F(X,\varphi+t) = \sum_{q\in\mathbb{Z}} e^{i\sqrt{\beta}qt}\hat{F}_q(X,\varphi), \quad t \in \mathbb{R} \tag{5.1}$$

where

$$\hat{F}_q(X,\varphi) = \int_0^1 dc\, e^{-2\pi iqc}F(X,\varphi+2\pi\beta^{-1/2}c), \quad q \in \mathbb{Z}. \tag{5.2}$$

The polymer activity $\hat{F}_q$ is called the charge-q part of F (and the neutral part when $q = 0$). Moreover, a polymer activity F is said to have charge q if $F = \hat{F}_q$ and neutral if has charge 0.

We simply refer to a $2\pi/\sqrt{\beta}$-periodic polymer activity for some $\beta > 0$ as *periodic* in the sequel. In doing so, we always assume that statements hold for any value of β, unless explicitly stated otherwise.

Notice that the smoothness assumption on F guarantees the existence and absolute convergence of the Fourier series (5.1). Moreover, F having charge q is equivalent to the condition that

$$F(X,\varphi+t) = e^{i\sqrt{\beta}qt}F(X,\varphi), \quad \text{for all } t \in \mathbb{R} \tag{5.3}$$

(the direct implication follows plainly from (5.2) and the converse by comparing (5.3) and (5.1)).

For later use, we record the following instance of the above setup. For any polymer activity $F(X,\varphi)$ as appearing in Definition 5.1.1, fixing a point $x_0 \in X$ and denoting $\delta\varphi(x) = \varphi(x) - \varphi(x_0)$, using that $F(X,\varphi) = F(X,\varphi(x_0)+\delta\varphi)$, one sees that

$$F(X,\varphi) = \sum_{q\in\mathbb{Z}} e^{i\sqrt{\beta}q\varphi(x_0)}\hat{F}_q(X,\delta\varphi). \tag{5.4}$$

The following elementary lemma states that the charge-q part $\hat{F}_q$ of a polymer activity is bounded in terms of the norm of the polymer activity (defined in Definition 3.2.4).

Lemma 5.1.2. *Let F be a periodic polymer activity. For all $\varphi \in \mathbb{R}^{\Lambda_N}$ and $X \in \mathscr{P}_j^c$,*

$$\|\hat{F}_q(X,\varphi)\|_{\mathfrak{h},T_j(X,\varphi)} \leqslant \|F(X)\|_{\mathfrak{h},T_j(X)}G_j(X,\varphi). \tag{5.5}$$

Proof. The inequality (5.5) is obtained by starting from (5.2) and then using the definition of the norm: for $(f_k)_{k=1}^n$ with $\|f_k\|_{C_j^2(X^*)} \leqslant 1$ for each k,

$$\begin{aligned} \left|D^n \hat{F}_q(X,\varphi)(f_1,\cdots,f_n)\right| &\leqslant \int_0^1 ds\,|D^n F(X,\varphi+2\pi\beta^{-1/2}s)(f_1,\cdots,f_n)| \\ &\leqslant \int_0^1 ds\|D^n F(X,\varphi+2\pi\beta^{-1/2}s)\|_{n,T_j(X,\varphi)} \end{aligned} \tag{5.6}$$

hence

$$\|\hat{F}_q(X,\varphi)\|_{\mathfrak{h},T_j(X,\varphi)} \leqslant \int_0^1 ds\|F(X,\cdot)\|_{\mathfrak{h},T_j(X)} G_j(X,\varphi+2\pi\beta^{-1/2}s) \tag{5.7}$$

$$\stackrel{(3.41)}{=} \|F(X,\cdot)\|_{\mathfrak{h},T_j(X)} G_j(X,\varphi). \tag{5.8}$$

□

The localisation operators which will be used to exact the relevant and marginal part from the remainder coordinates rely on the charge decomposition as well as on lattice symmetries, so we define these first.

Definition 5.1.3. *A scale-j polymer activity $F = (F(X))_{X\in\mathscr{P}_j^c}$ is* even *if $F(X,\varphi) = F(X,-\varphi)$ for each (X,φ). F is* invariant under lattice symmetries *if for every graph automorphism A of the torus Λ_N that maps any block in $\mathscr{B}_j$ to a block in $\mathscr{B}_j$ one has $F(AX,A\varphi) = F(X,\varphi)$ where $(A\varphi)(x) = \varphi(A^{-1}x)$. The meaning is analogues for polymer activities on $\mathbb{Z}^2$.*

5.2 Localisation operator

The main result of Section 5 are the following localisation operators $\mathrm{Loc}_{X,B}$ which will be used to exact the relevant and marginal part from the remainder coordinates. Our notation $\mathrm{Loc}_{X,B}$ is inspired by that of [26], but compared to this reference, the contraction mechanisms in this section rely on oscillations under the Gaussian expectation for the charged terms in addition. These operators are given in the next definition, but the explicit definition does not play a direct role in the remainder of the paper: all that we will require in the following sections are its main properties which are stated in Propositions 5.2.2 and 5.2.3 below.

The intuition that motivates the definition of Loc (and is substantiated by its properties stated in the next propositions) is related to which terms of a given periodic polymer F are *relevant* or *marginal*: all the higher order Fourier coefficients $\hat{F}_q$, $q \geqslant 1$, contract at large β (i.e., they become irrelevant along the renormalisation group flow), cf. Lemma 5.4.7 below, and so does the neutral part $\hat{F}_0$ after removal of its Taylor expansion in $\nabla\varphi$ up to terms of second order, cf. Lemma 5.4.12 below. The combination of these mechanisms culminates in Proposition 5.2.3. These considerations motivates the localisation to be an approximation of the Taylor expansion of the neutral part of the polymer activities. Moreover, it is sufficient to exhibit these cancellations for small polymers $X \in \mathscr{S}_j$. Large polymers contract automatically (due to their size), as was seen in Section 4.1.3, in particular Proposition 4.1.5.

Definition 5.2.1. *Let F be a periodic scale-j polymer activity, and let $\hat{F}_0$ be its neutral part. For $X \in \mathscr{S}_j$ and $B \in \mathscr{B}_j(X)$, define*

$$\mathrm{Loc}^{(0)}_{X,B} F(X,\varphi) = \hat{F}_0(X,0). \tag{5.9}$$

If H is an even periodic scale-j polymer activity and respects lattice symmetries, define

$$\begin{aligned}\mathrm{Loc}^{(2)}_{X,B} H(X,\varphi) = \frac{1}{8|X||B|} &\sum_{x_0,y_0 \in B} \sum_{x_1,x_2 \in X^*} \partial_{\varphi(x_1)} \partial_{\varphi(x_2)} \hat{H}_0(X,0) \\ &\times \sum_{\mu,\nu \in \hat{e}} (1+\delta_{\mu,\nu} - \delta_{\mu,-\nu}) \nabla^\mu \varphi(y_0)(\delta x_1)^\mu \, \nabla^\nu \varphi(y_0)(\delta x_2)^\nu\end{aligned} \tag{5.10}$$

where $\delta x_i = x_i - x_0$ for $i \in \{1,2\}$ and y^μ is the μ-component of y with the convention $y^{-\mu} = -y^\mu$, and let

$$\mathrm{Loc}_{X,B} H(X) = \mathrm{Loc}^{(0)}_{X,B} H(X) + \mathrm{Loc}^{(2)}_{X,B} H(X). \tag{5.11}$$

Then define

$$\begin{aligned}&\mathrm{Loc}^{(0)}_X F(X) = \sum_{B \in \mathscr{B}_j(X)} \mathrm{Loc}^{(0)}_{X,B} F(X) \\ &\mathrm{Loc}_X H(X) = \sum_{B \in \mathscr{B}_j(X)} \mathrm{Loc}_{X,B} H(X), \qquad \mathrm{Loc}^{(2)}_X H(X) = \sum_{B \in \mathscr{B}_j(X)} \mathrm{Loc}^{(2)}_{X,B} H(X)\end{aligned} \tag{5.12}$$

In the definition of $\mathrm{Loc}^{(2)}$, although the points x_0, x_1, x_2 live in Λ_N, since they are restricted to a small polymer $X \in \mathscr{S}_j$ (that doe not 'wrap' around the torus), we can define the subtraction $\delta x_i = x_i - x_0$ in the local coordinates of X.

As one can see from the definition, we use $\mathrm{Loc}^{(0)}$ for localising periodic polymer activities without any symmetry and use Loc for localising even periodic polymer activities with lattice symmetries. Without the presence of the external field, we can define the (bulk) polymer activities that always preserve the symmetries, so only Loc appears in the bulk renormalisation group flow.

5.2.1 Algebraic property

Following our convention, recall that j is tacitly allowed to take values $j = 1,\dots,N-1$ for a given torus of side length L^N and the following statements hold uniformly in N (and L unless stated otherwise). We emphasise that $\mathrm{Loc}_X F$ is meant to be the Taylor expansion of $\hat{F}_0$ in the second order, so it is a second degree polynomial of $\nabla\varphi$. But by the specific definition of of it, we see that the localisation only contains terms of specific form.

In the following proposition and the rest of the chapter, again recall that $\mathbb{E}$ means we are taking expectation over $\zeta \sim \mathscr{N}(0,\Gamma_{j+1})$ if $j+1 < N$ and $\zeta \sim \mathscr{N}(0,\Gamma_N^{\Lambda_N})$ if $j+1 = N$. The non-random part of the field is often denoted φ'.

Proposition 5.2.2. *Let F,H be periodic scale-j polymer activity such that H is even and invariant under lattice symmetries. Then there are scalars $\overline{E} = \overline{E}(F)$, $\overline{s} = \overline{s}(H)$ satisfying (with purely geometric implicit constants)*

$$\overline{E} = O(A^{-1}L^{-2j}\|F\|_{\mathfrak{h},T_j}), \quad \overline{s} = O(A^{-1}\mathfrak{h}^{-2}\|H\|_{\mathfrak{h},T_j}) \tag{5.13}$$

such that for any $B \in \mathscr{B}_j$,

$$\sum_{X\in\mathscr{S}_j : X\supset B} \mathrm{Loc}^{(0)}_{X,B}\mathbb{E}\theta_\zeta F(X,\varphi') = \overline{E}|B| \tag{5.14}$$

$$\sum_{X\in\mathscr{S}_j : X\supset B} \mathrm{Loc}^{(2)}_{X,B}\mathbb{E}\theta_\zeta H(X,\varphi') = \frac{1}{2}\overline{s}|\nabla\varphi'|^2_B \tag{5.15}$$

where the localisation operators are applied on the variable φ'. Moreover, whenever $\|F\|_{\mathfrak{h},T_j}, \|H\|_{\mathfrak{h},T_j} < \infty$, both $\overline{E} = \overline{E}(F)$ and $\overline{s} = \overline{s}(H)$ are continuous functions of the implicit parameter $s \in [-\varepsilon_s\theta_J, \varepsilon_s\theta_J]$ (inherent to $\mathbb{E}$).

5.2.2 Analytic properties

Since localisations approximate the polymer activities, we can also write estimates on the error of the approximation, stated in terms of

$$\alpha_{\mathrm{Loc}} = L^{-3}(\log L)^{3/2} + \min\left\{1, \sum_{q\geqslant 1} e^{4\sqrt{\beta}qh}e^{-(q-1/2)r\beta\Gamma_{j+1}(0)}\right\} \tag{5.16}$$

$$\alpha^{(0)}_{\mathrm{Loc}} = L^{-1}(\log L)^{1/2} + \min\left\{1, \sum_{q\geqslant 1} e^{4\sqrt{\beta}qh}e^{-(q-1/2)r\beta\Gamma_{j+1}(0)}\right\}. \tag{5.17}$$

As mentioned in Section 3.2.3, h will be a constant in the end, while $\mathfrak{h}$ is allowed to depend on the scale. We distinguish the role of h and $\mathfrak{h}$ because of this reason.

Proposition 5.2.3. *There exists a constant $c_h > 0$ such that the following holds. Let $r \in (0,1]$ and assume that $2h \geqslant \mathfrak{h} \geqslant \max\{rc_h\rho_J^{-2}\sqrt{\beta}, \rho_J^{-1}\}$, that $\kappa_L = c_\kappa \rho_J^2(\log L)^{-1}$ as in Proposition 3.3.5, and that $L \geqslant C$ and $A \geqslant 1$. Then for all $\varphi' \in \mathbb{R}^{\Lambda_N}$, $h_{h_\omega} < (C_1 \log L)^{-1}\mathfrak{h}$ for sufficiently large C_1 and some $C_2 > 0$,*

(i) if F is a periodic ω-polymer activity at scale j and $X \in \mathscr{S}_j$,

$$\left\|(\mathrm{Loc}_X^{(0)} - 1)\mathbb{E}_{(\omega)}\theta_\zeta F(X,\cdot)\right\|_{\vec{\mathfrak{h}},T_{j+1}(\overline{X})} \leqslant C_2 \alpha_{\mathrm{Loc}}^{(0)} A^{-|X|_j} \|F\|_{\vec{\mathfrak{h}},T_j}\,; \tag{5.18}$$

(i) if F is an even periodic ω-polymer activity at scale j and $X \in \mathscr{S}_j$,

$$\left\|(\mathrm{Loc}_X - 1)\mathbb{E}_{(\omega)}\theta_\zeta F(X,\cdot)\right\|_{\vec{\mathfrak{h}},T_{j+1}(\overline{X})} \leqslant C_2 \alpha_{\mathrm{Loc}} A^{-|X|_j} \|F\|_{\vec{\mathfrak{h}},T_j}. \tag{5.19}$$

Proposition 5.2.4. *Let F be a periodic ω-polymer activity and $h_\omega < (C_1 \log L)^{-1}\mathfrak{h}$ for sufficiently large C_1. Then $\mathrm{Loc}_{X,B}^{(0)}$ and $\mathrm{Loc}_{X,B}$ are bounded in the sense (note the T_j instead of T_{j+1} norm on the left-hand side)*

$$\|\mathrm{Loc}_X^{(0)}\,\mathbb{E}_{(\omega)}\theta_\zeta F(X,\varphi')\|_{h_\omega,T} \leqslant C\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} \tag{5.20}$$

$$\|\mathrm{Loc}_{X,B}\,\mathbb{E}_{(\omega)}\theta_\zeta F(X,\varphi')\|_{\vec{\mathfrak{h}},T_j(X,\varphi')} \leqslant C(\log L)\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} e^{c_w \kappa_L w_j(B,\varphi')^2}, \tag{5.21}$$

and $\mathrm{Loc}_{X,B}^{(0)}\,\mathbb{E}_{(\omega)}\theta_\zeta F(X)$ and $\mathrm{Loc}_{X,B}\,\mathbb{E}_{(\omega)}\theta_\zeta F(X)$ are continuous in the implicit parameter $s \in [-\varepsilon_s\theta_J, \varepsilon_s\theta_J]$ (inherent to $\mathbb{E}_{(\omega)}$) with respect to the same norms.

In our application (carried out precisely in Section 7.3), we will choose $h \leqslant rc_h\rho_J^{-2}\sqrt{\beta}$. The expression for α_{Loc} can then be simplified as follows: since then

$$e^{4\sqrt{\beta}h} \leqslant C' e^{4rc_h\rho_J^{-2}\beta} \tag{5.22}$$

and for $j+1 < N$, the maximum in (5.16) is bounded by

$$e^{-\frac{1}{2}r\beta\Gamma_{j+1}(0)} e^{4\sqrt{\beta}h} \sum_{q\geqslant 0} e^{4\sqrt{\beta}qh} L^{-qr\beta\Gamma_{j+1}(0)} \leqslant \left(Ce^{-\frac{1}{2}\Gamma_{j+1}(0)}\right)^{r\beta} \sum_{q\geqslant 0}\left(Ce^{-\Gamma_{j+1}(0)}\right)^{qr\beta} \tag{5.23}$$

for $C = C'e^{4c_h}$. By Corollary 3.1.1, the covariances satisfy $\Gamma_j(0) \sim (4/\beta_{\mathrm{free}} + O(s/\beta_{\mathrm{free}}^2))\log L$ with $\beta_{\mathrm{free}} = 8\pi(v_J^2 + s)$. For any $\theta \in (0,1/2]$ and β is sufficiently large so that $r\beta >$

$\beta_{\text{free}}(1+2\theta)$, it follows that if L is sufficiently large depending on C, θ, and v_J (to ensure that $C \leqslant e^{\frac{1}{4}\theta\Gamma_{j+1}(0)}$), and s is sufficiently small,

$$(Ce^{-\frac{1}{2}\Gamma_{j+1}(0)})^{r\beta} \leqslant L^{-2r\beta(1-\theta/2)\left((1/\beta_{\text{free}}+O(s/\beta_{\text{free}}^2)\right)} \leqslant L^{-2(1+2\theta)(1-\theta/2)} \leqslant L^{-2(1+\theta)}, \quad (5.24)$$

and hence (5.23) is bounded by $CL^{-2-2\theta}$. In particular,

$$\alpha_{\text{Loc}} \leqslant C(L^{-3}(\log L)^{3/2} + L^{-2-2\theta}) \leqslant L^{-2-\theta}. \quad (5.25)$$

The contractivity of the (bulk) renormalisation group map will later be ensured by $CL^2\alpha_{\text{Loc}} < CL^{-\theta} < 1$.

Much of the remainder of this section is concerned with the proof of these propositions. Proposition 5.2.2 is a relatively straightforward consequence of the definitions. Proposition 5.2.3 is more involved and combines different contraction mechanisms for neutral and charged terms. We thus discuss these mechanisms separately.

5.3 Symmetries of Loc–proof of Proposition 5.2.2

Proof of Proposition 5.2.2. (5.14) is relatively easy to see, as

$$\overline{E} = \frac{1}{|B|}\sum_{X\in\mathscr{S}_j:X\supset B} \text{Loc}^{(0)}_{X,B}\mathbb{E}\theta_\zeta F(X,\varphi') = \sum_{X\in\mathscr{S}_j:X\supset B}\frac{1}{|X|}\mathbb{E}\hat{F}_0(X,\zeta). \quad (5.26)$$

Also, by Definition 5.2.1, the left-hand side of (5.15) equals

$$\sum_{X\in\mathscr{S}_j:X\supset B}\frac{1}{|X||B|}\sum_{x_0,y_0\in B}\sum_{x_1,x_2\in X^*}\frac{1}{2}\partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}\hat{H}_0(X,\zeta)\langle\nabla\varphi'(y_0),x_1-x_0,\nabla\varphi'(y_0),x_2-x_0\rangle, \quad (5.27)$$

where

$$\langle\nabla\varphi'(y_0),y_1,\nabla\varphi'(y_0),y_2\rangle = \frac{1}{4}\sum_{\mu,\nu\in\hat{e}}(1+\delta_{\mu,\nu}-\delta_{\mu,-\nu})\nabla^\mu\varphi'(y_0)y_1^\mu\nabla^\nu\varphi'(y_0)y_2^\nu. \quad (5.28)$$

As we now explain, by invariance under lattice rotations, only the diagonal terms in the inner product contribute and we see that this expression equals the right-hand side of (5.15) with

$$\overline{s} = \sum_{X\in\mathscr{S}_j:X\supset B}\frac{1}{|X||B|}\sum_{x_0\in B,x_1,x_2\in X^*}\partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}\hat{H}_0(X,\zeta)(x_1-x_0,x_2-x_0), \quad (5.29)$$

where $(\cdot,\cdot)$ is the standard ℓ^2 inner product. To see this in detail, expand (5.27) using the definition (5.28) and let $I_{\mu\nu}$ be the (scaled) coefficient of $\nabla^\mu\varphi'(y_0)\nabla^\nu\varphi'(y_0)$ written explicitly as

$$I_{\mu\nu} = i_{\mu\nu} \sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X|} \sum_{x_0\in B} \sum_{x_1,x_2\in X^*} \partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}\hat{F}_0(X,\zeta)(x_1-x_0)^\mu(x_2-x_0)^\nu \tag{5.30}$$

where $i_{\mu\mu}=2$, $i_{(-\mu)\mu}=0$ and $i_{\mu\nu}=1$ if $\mu\perp\nu$. But by rotational invariance, we have $I_{\mu\mu}=I_{\nu\nu}$ for any $\mu,\nu\in\hat{e}$, so

$$I_{\mu\mu} = \frac{1}{4}\sum_{\nu\in\hat{e}} I_{\nu\nu} = \sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X|} \sum_{x_0\in B} \sum_{x_1,x_2\in X^*} \partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}\hat{F}_0(X,\zeta)(x_1-x_0,x_2-x_0) \tag{5.31}$$

Therefore summing over $\mu=\pm\nu$ and $y_0\in B$ simply gives $\frac{1}{2}\bar{s}|\nabla\varphi'|_B^2$. Now for the case $\mu\perp\nu$, it is direct from the expression that $I_{(-\mu)\nu}=-I_{\mu\nu}$ and $I_{\mu\nu}=I_{\nu\mu}$. But since $\mu\perp\nu$, by rotation invariance, $I_{\nu(-\mu)}=I_{\mu\nu}$ and it follows that $I_{\mu\nu}=0$.

To bound $\bar{s}$, let $f_\nu^{x_0}(x_1)=(x_1-x_0)^\nu$ for $x_1\in X^*$ and a fixed $x_0\in X$. Then $\|f_\nu^{x_0}\|_{C_j^2(X^*)}\leqslant CL^j$ and with (3.78) it follows that

$$\begin{aligned}
|\bar{s}| &= \Big|\sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X||B|} \sum_{x_0\in B,\,\nu=1,2} D^2\mathbb{E}\hat{F}_0(X,\zeta)(f_\nu^{x_0},f_\nu^{x_0})]\Big| \\
&\leqslant C\mathfrak{h}^{-2} \sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X||B|} \sum_{x_0\in B,\,\nu=1,2} L^{2j}\|\mathbb{E}F(X,\zeta)\|_{\mathfrak{h},T_j(X,0)} \\
&\leqslant 2^4C\mathfrak{h}^{-2}A^{-1}\|F\|_{h,T_j} \sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X|_j} \leqslant C'\mathfrak{h}^{-2}A^{-1}\|F\|_{\mathfrak{h},T_j}.
\end{aligned} \tag{5.32}$$

The bound for E is proved similarly:

$$|\bar{E}| = \Big|\sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X|}\mathbb{E}\hat{F}_0(X,0)\Big| \leqslant \sum_{X\in\mathscr{S}_j:X\supset B} \frac{1}{|X|}(2A)^{-|X|_j}\|F\|_{\mathfrak{h},T_j} \leqslant CL^{-2j}A^{-1}\|F\|_{\mathfrak{h},T_j}. \tag{5.33}$$

The asserted continuity in the implicit parameter s follows from the expressions in the first line of (5.32) and (5.33) in combination with Lemma 3.3.11. This completes the proof. □

5.4 Proof of Proposition 5.2.3

5.4.1 Preliminaries

As a preliminary to the proof of Proposition 5.2.3, we state how the norm of a polymer activity changes when measured in terms of T_{j+1} compared to the T_j-norm. We will use the following elementary inequality.

Lemma 5.4.1. *Let $X \in \mathscr{S}_j$. Fix $x_0 \in X$ and for $f : X^* \to \mathbb{C}$, define $\delta f(x) = f(x) - f(x_0)$. Then*

$$\|\delta f\|_{C_j^2(X^*)} \leqslant C_d L^{-1} \max_{m=1,2} \|\nabla_{j+1}^m f\|_{L^\infty(X^*)}. \tag{5.34}$$

for some geometric constant $C_d > 0$.

Proof. Since $X \in \mathscr{S}_j$, its small set neighbourhood X^* contains at most $2^d b$ blocks, where $b = |B^*|_j$ for any $B \in \mathscr{B}_j$. Thus the ℓ^∞-diameter of X^* is at most $C_d L^j$ and thus

$$\|\delta f\|_{L^\infty(X^*)} = \max_{x \in X^*} |f(x) - f(x_0)| \leqslant C_d L^j \max_{x \in X^*, \mu \in \hat{e}} |\nabla^\mu f(x)| \leqslant C_d L^{-1} \|\nabla_{j+1} f\|_{L^\infty(X^*)}. \tag{5.35}$$

Also, for $m \geqslant 1$, $\nabla^m \delta f = \nabla^m f$, and the result follows. □

This lemma has the following important consequence for neutral polymer activities.

Lemma 5.4.2. *Let F be a neutral scale-j polymer activity. Then for $X \in \mathscr{S}_j$, $\varphi' \in \mathbb{R}^{\Lambda_N}$ and $n \geqslant 0$,*

$$\|D^n F(X,\varphi)\|_{n,T_{j+1}(X,\varphi)} \leqslant (C_d^{-1} L)^{-n} \|D^n F(X,\varphi)\|_{n,T_j(X,\varphi)}. \tag{5.36}$$

In particular, if F is in addition supported on $X \in \mathscr{S}_j$ (i.e., $F(X) = 0$ when $X \notin \mathscr{S}_j$),

$$\|F(X,\varphi)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant \|F(X,\varphi)\|_{C_d L^{-1}\mathfrak{h},T_j(X,\varphi)}. \tag{5.37}$$

Proof. Since F is neutral, i.e. has charge $q = 0$, cf. (5.3), for f_1 constant-valued one has

$$DF(X,\varphi)(f_1) = f_1 \sum_{x_0 \in X} \frac{\partial}{\partial \varphi_{x_0}} F(X,\varphi) = f_1 \frac{d}{dc} F(X,\varphi + c)\big|_{c=0} = 0 \tag{5.38}$$

and the same reasoning implies that $D^n F(X,\varphi)(f_1, \cdots, f_n) = 0$ whenever any of $f_1, \dots, f_n$ is constant-valued. Therefore, having fixed $x_0 \in X$, for any $f_i \in \mathbb{R}^{\Lambda_N}$, by multilinearity,

$$D^n F(X,\varphi)(f_1, \dots, f_n) = D^n F(X,\varphi)(\delta f_1, \dots, \delta f_n),$$

where $(\delta f_k)(x) = f_k(x) - f_k(x_0)$. Therefore if $\|f_k\|_{C^2_{j+1}(X^*)} \leqslant 1$ for $k = 1, \dots, n$,

$$|D^n F(X,\varphi)(f_1,\dots,f_n)| \leqslant (C_d^{-1}L)^{-n} \|D^n F(X,\varphi)\|_{n,T_j(X,\varphi)} \tag{5.39}$$

by Lemma 5.4.1. In view of (3.40), the claim follows. □

The following similar but weaker bound holds for charged polymer activities.

Lemma 5.4.3. *Let F be a scale-j polymer activity of charge q that is supported on $X \in \mathscr{S}_j$. Then*

$$\|F(X,\varphi)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant e^{\sqrt{\beta}|q|\mathfrak{h}} \|F(X,\varphi)\|_{C_dL^{-1}\mathfrak{h},T_j(X,\varphi)}. \tag{5.40}$$

Proof. One may decompose $F(X,\varphi) = e^{i\sqrt{\beta}q\varphi(x_0)}F(X,\delta\varphi)$ where $\delta\varphi(x) = \varphi(x) - \varphi(x_0)$. Define $\overline{F}(X,\varphi) := F(X,\delta\varphi)$, then $\overline{F}$ is now neutral. The estimate of Lemma 5.4.2 applies to $\overline{F}$, giving

$$\|\overline{F}(X,\varphi)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant \|F\|_{C_dL^{-1}\mathfrak{h},T_j(X,\varphi)}. \tag{5.41}$$

The conclusion now follows from (3.53) and the submultiplicativity property of the norm (3.64). □

5.4.2 Proof of Proposition 5.2.3: charged part

We will prove Proposition 5.2.3 by decomposing F into its neutral and charged part and considering both contributions separately, starting with the latter. The estimate (5.18) and (5.19) for charged F relies crucially on how the expectation acts on the charged components. The contraction mechanism for charged polymer activities is a generalisation of the elementary identity

$$\mathbb{E}[e^{i\sqrt{\beta}q\zeta_{x_0}}] = e^{-\frac{1}{2}\beta q^2\Gamma_{j+1}(0)}, \tag{5.42}$$

valid for all integers q and $\beta > 0$, where here and in the sequel, $\Gamma_{j+1}(x) = (\delta_0, \Gamma_{j+1}\delta_x)$. The generalisation uses the analyticity of polymer activities with finite $\|\cdot\|_{h,T_j}$-norm, see Proposition 3.4.1, which justifies the following complex translation.

Lemma 5.4.4. *Let $\mathfrak{h} > 0$, and let F be a charge-q polymer activity with $\|F(X)\|_{\mathfrak{h},T_j(X)} < \infty$, $q \in \mathbb{Z}$. Then for any constant $c \in \mathbb{R}$ with $|c| < \mathfrak{h}$,*

$$F(X,\varphi + ic) = e^{-\sqrt{\beta}qc}F(X,\varphi). \tag{5.43}$$

Proof. First recall that by Proposition 3.4.1, $F(\varphi+z)$ is well-defined and complex analytic for $z \in \{w \in \mathbb{C} : |w| < \mathfrak{h}\}$. Hence $f : \{z \in \mathbb{C} : |z| < \mathfrak{h}\} \to \mathbb{C}$, $z \mapsto F(X,\varphi+z) - e^{iq\sqrt{\beta}z} F(X,\varphi)$ is a complex analytic function that takes value 0 on the real line by (5.3). Therefore $f \equiv 0$. $\square$

Before we jump into the main result, we first discuss a technical point, which defines the constant c_h appearing in the statement of Proposition 5.2.3. Ultimately we are interested in the covariance Γ_{j+1}, but to obtain the optimal estimates we must work with its subdecomposition into fractional scales introduced in Section 3.1.3. Thus for ℓ, N' as in Section 3.3.3, let $I_{N'} = \{0, (N')^{-1}, 2(N')^{-1}, \dots 1-(N')^{-1}\}$ be the set of fractional scales. Then for $s = k(N')^{-1} \in I_{N'}$ and $s' = s + (N')^{-1}$ (cf. Remark 3.3.8 regarding N'), set

$$\xi_s(x) = \sqrt{\beta}(\Gamma_{j+s,j+s'}(x-x_0) - \Gamma_{j+s,j+s'}(0)), \qquad \xi_{<s} = \sum_{t \in I_{N'}, t<s} \xi_t. \tag{5.44}$$

Lemma 5.4.5 (Choice of c_h)**.** *There exists $c_h > 0$ such that for any $X \in \mathscr{S}_j$, $s \in I_{N'}$ and $\beta > 0$,*

$$\|\xi_s\|_{C_j^2(X^*)} \vee \|\xi_{<s}\|_{C_j^2(X^*)} < \frac{1}{2} c_h \rho_J^{-2} \sqrt{\beta}. \tag{5.45}$$

Proof. By Lemma 3.1.4,

$$\|\sum_{s \in J} \nabla_j^\alpha \xi_s\|_{L^\infty(X^*)} \leqslant C_\alpha \rho_J^{-2} \sqrt{\beta} \tag{5.46}$$

for any $J \subset I_{N'}$ and $|\alpha| \in \{1,2\}$. Also for $\alpha = 0$ and $X \in \mathscr{S}_j$, with the same constant C_d as in (5.35),

$$\sup_{x \in X^*} |\sum_{s \in J} \xi_s(x)| \leqslant C_d L^j \|\sum_{s \in J} \nabla \xi_s\|_{L^\infty(X^*)} \leqslant C_d \|\sum_{s \in J} \nabla_j \xi_s\|_{L^\infty(X^*)} \leqslant C_d C_\alpha \sqrt{\beta} \rho_J^{-2}. \tag{5.47}$$

Combining both inequalities gives the claim with $c_h = 3C_\alpha(C_d \vee 1)$. $\square$

Henceforth, we fix c_h so that the conclusions of Lemma 5.4.5 hold. The formula (5.42) can now be generalised to the following identity.

Lemma 5.4.6. *Let $r \in (0,1]$, $\mathfrak{h} \geqslant r c_h \rho_J^{-2} \sqrt{\beta}$, and let F be a charge-q ω-polymer activity with $\|F(X)\|_{\vec{\mathfrak{h}}, T_j(X)} < \infty$. Then for $X \in \mathscr{S}_j$, $q \in \mathbb{Z}$, $x_0 \in X$ and $\xi(x) = \sqrt{\beta}(\Gamma_{j+1}(x-x_0) - \Gamma_{j+1}(0))$, for all $\varphi' \in \mathbb{R}^{\Lambda_N}$,*

$$\mathbb{E}[F(X,\varphi'+\zeta)] = e^{-\frac{1}{2}\beta\Gamma_{j+1}(0)(2r|q|-r^2)} \mathbb{E}\big[e^{-i\sqrt{\beta} r \sigma_q \zeta_{x_0}} F(X, \varphi' + \zeta + i r \sigma_q \xi(x))\big], \tag{5.48}$$

where $\sigma_q = \mathrm{sign}(q)$.

Proof. Recall that $\zeta \sim \mathscr{N}(0,\Gamma_{j+1})$ under $\mathbb{E}$. (or $\zeta \sim \mathscr{N}(0,\Gamma_N^{\Lambda_N})$ when $j+1=N$, but we will drop label Λ_N for brevity). We will need to work with the subdecomposition of the covariance Γ_{j+1} discussed above the lemma. Since $\xi = \sum_{s\in I_{N'}} \xi_s$, it is sufficient to show the lemma for $\|F\|_{\mathfrak{h},T_{j+s}} < +\infty$ and Γ_{j+1} replaced by $\Gamma_{j+s,j+s'}$ where $s = s' - M^{-1} \in I_{N'}$ and ζ replaced by $\zeta_s + ir\xi_{<s}$ where $\zeta_s \sim \mathscr{N}(0,\Gamma_{j+s,j+s'})$. It is convenient to work with invertible covariance matrices, so we will work with $C = \Gamma_{j+s,j+s'} + \delta$ for $\delta > 0$ so that C is strictly positive definite, and then take the limit $\delta \downarrow 0$ to conclude. All in all, it thus suffices to show that

$$\begin{aligned}&\int e^{-\frac{1}{2}(\zeta_s,C^{-1}\zeta_s)}F(X,\varphi'+\zeta_s+ir\sigma_q\xi_{<s})\,d\zeta_s \\ &\quad= e^{-\frac{1}{2}\beta C(0)(2r|q|-r^2)}\int e^{-\frac{1}{2}(\zeta_s,C^{-1}\zeta_s)-i\sqrt{\beta}r\sigma_q\zeta_s(x_0)}F(X,\varphi'+\zeta_s+ir\sigma_q(\xi_{<s}+\xi_s))\,d\zeta_s,\end{aligned} \tag{5.49}$$

from which (5.48) readily follows by integrating successively over ζ_s, $s\in I_{N'}$ and letting $\delta\downarrow 0$. Here, with a slight abuse of notation, we define $\xi_s(x) = \sqrt{\beta}(C(x-x_0)-C(0))$, from which ξ_s as introduced in (5.44) is obtained in the limit $\delta \to 0$. Then the bound of Lemma 5.4.5 holds the same for this modified ξ_s when δ is sufficiently small, which we henceforth tacitly assume.

We now show (5.49). Let $X' = \{x\in\Lambda : d_1(x,X^*)\leqslant 2\}$ so that $\|\psi\|_{C_j^2(X^*)}$ only depends on $\psi|_{X'}$. Performing a change of variable from ζ_s to $\zeta_s - ir\sigma_q\xi_s$, the integral on the right-hand side of (5.49) can be recast as

$$\int_{\mathbb{R}^{X'}} e^{-\frac{1}{2}(\zeta_s,C^{-1}\zeta_s)-i\sqrt{\beta}r\sigma_q\zeta_s(x_0)}F(X,\varphi'+\zeta_s+ir\sigma_q\xi_{<s}+ir\sigma_q\xi_s)\,d\zeta_s = e^{-\frac{1}{2}\beta r^2 C(0)}R_{r\sigma_q\xi_s}(r) \tag{5.50}$$

where

$$R_\psi(r') = \int_{\mathbb{R}^{X'}+i\psi} e^{-\frac{1}{2}(\zeta+ir'z,C^{-1}(\zeta+ir'z))}F(X,\varphi'+\zeta+ir\sigma_q\xi_{<s})\,d\zeta. \tag{5.51}$$

and $z = \sigma_q\sqrt{\beta}C(0)$. By Lemma 5.4.5 (which is just the Cauchy's integral formula) and assumption on δ, the condition $\mathfrak{h}\geqslant c_h r\rho^{-2}\sqrt{\beta}$ guarantees $\|r\sigma_q\xi_s\|_{C_j^2(X^*)}$, $\|r\sigma_q\xi_{<s}\|_{C_j^2(X^*)} < \frac{1}{2}\mathfrak{h}$, we can apply Lemma 3.8.1 to see that

$$R_{r\sigma_q\xi_s}(r) = R_0(r) \tag{5.52}$$

and thus it will be enough to compute $R_0(r)$.

To compute $R_0(r)$, consider $R_0(r'+\delta r')$ for sufficiently small $\delta r'$. Another application of Lemma 3.8.1 shows that

$$R_0(r'+\delta r') = \int_{\mathbb{R}^{X'}} e^{-\frac{1}{2}(\zeta+ir'z, C^{-1}(\zeta+ir'z))} F(X, \varphi'+\zeta+ir\sigma_q\xi_{<s} - i(\delta r')z)\, d\zeta. \tag{5.53}$$

But by (5.43), $F(X,\varphi'+\zeta+ir\sigma_q\xi_{<s}-i(\delta r')z) = e^{\sqrt{\beta}qz\delta r'}F(X,\varphi'+\zeta+ir\sigma_q\xi_{<s})$, so $R_0(r')$ satisfies

$$\frac{d}{dr'}R_0(r') = \sqrt{\beta}qzR_0(r') = \beta|q|C(0)R_0(r'). \tag{5.54}$$

Solving this differential equation yields

$$R_0(r) = e^{|q|r\beta C(0)} \int_{\mathbb{R}^{X'}} e^{-\frac{1}{2}(\zeta, C^{-1}\zeta)} F(X,\varphi'+\zeta+ir\sigma_q\xi_{<s}) d\zeta, \tag{5.55}$$

and thus (5.49) is obtained once we recall (5.50) (5.52).

□

This identity leads to the following contraction mechanism for charge-q polymer activities.

Lemma 5.4.7. *Let $r\in(0,1]$, $\mathfrak{h} \geqslant rc_h\rho_J^{-2}\sqrt{\beta}$, $L\geqslant 4C_d$ and $h_\omega \leqslant (C_1\log L)^{-1}\mathfrak{h}$ for sufficiently large C_1. Then there exists $C>0$ such that for $X\in\mathscr{S}_j$, and any charge-q polymer activity F with $|q|\geqslant 1$ and $\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)}<\infty$, and all $\varphi'\in\mathbb{R}^{\Lambda_N}$,*

$$\|\mathbb{E}[\theta_\zeta F(X)]\|_{(2\mathfrak{h},h_\omega),T_{j+1}(X)} \leqslant Ce^{2\sqrt{\beta}|q|\mathfrak{h}}e^{-(|q|-1/2)r\beta\Gamma_{j+1}(0)}\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)}. \tag{5.56}$$

Proof. We will hide the dependence of $F(X,\varphi)$ on X for brevity. Let us start from (5.48) with $r\in(0,1]$. Then

$$D^n\theta_\zeta\mathbb{E}[F(\varphi')] = e^{-\frac{2|q|r-r^2}{2}\beta\Gamma_{j+1}(0)}\mathbb{E}[e^{-i\sqrt{\beta}r\sigma_q\zeta_{x_0}}D^n\theta_\zeta F(\varphi'+ir\sigma_q\xi)] \tag{5.57}$$

where $\xi(x)=\sqrt{\beta}(\Gamma_{j+1}(x-x_0)-\Gamma_{j+1}(0))$. By our assumptions and Lemma 5.4.5 (with the choice $M=1$), we have $2C_dL^{-1}\mathfrak{h}+r\|\xi\|<\mathfrak{h}$, where $\|\cdot\|=\|\cdot\|_{C_j^2}$. Thus by Proposition 3.4.1, F is analytic in the strip $S_{2C_dL^{-1}\mathfrak{h}+r\|\xi\|}$, and hence the Taylor expansion

$$D^n\theta_\zeta F(\varphi'+ir\sigma_q\xi) = \sum_{k=0}^{\infty}\frac{1}{k!}D^k_{\varphi'}(D^n\theta_\zeta F)(\varphi')((ir\sigma_q\xi)^{\otimes k}) \tag{5.58}$$

is convergent and so, combining with (5.57), and since $|\sigma_q| = 1$,

$$\|D^n_{\varphi'}\mathbb{E}[F(\varphi'+\zeta)]\|_{n,T_j(X,\varphi')} \leqslant e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}\mathbb{E}\Big[\sum_{k=0}^{\infty}\frac{\|r\xi\|^k}{k!}\|D^{n+k}F(\varphi'+\zeta)\|_{n+k,T_j(X,\varphi')}\Big]. \tag{5.59}$$

Therefore, for $\mathfrak{h}' > 0$ left to be chosen,

$$\begin{aligned}\|\mathbb{E}\theta_\zeta[F(\varphi')]\|_{\mathfrak{h}',T_j(X,\varphi')} &\leqslant e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}\mathbb{E}\Big[\sum_{n,k}\frac{\mathfrak{h}'^n\|r\xi\|^k}{n!k!}\|D^{n+k}\theta_\zeta F(\varphi')\|_{n+k,T_j(X,\varphi')}\Big]\\ &\leqslant e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}\mathbb{E}\Big[\sum_{n=0}^{\infty}\frac{(\mathfrak{h}'+\|r\xi\|)^n}{n!}\|D^n\theta_\zeta F(\varphi')\|_{n,T_j(X,\varphi')}\Big]\\ &= e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}\mathbb{E}[\|\theta_\zeta F(\varphi')\|_{\mathfrak{h}'+r\|\xi\|,T_j(X,\varphi')}]. \end{aligned} \tag{5.60}$$

To complete the lemma, one is just left to compare $\|\theta_\zeta F(\varphi')\|_{\mathfrak{h},T_{j+1}(\varphi',X)}$ with a quantity in a lower scale. This is where Lemma 5.4.3 comes in, yielding the bound

$$\|\mathbb{E}\theta_\zeta F(\varphi')\|_{2\mathfrak{h},T_{j+1}(\varphi',X)} \leqslant e^{2\sqrt{\beta}|q|\mathfrak{h}}\|\mathbb{E}[F(\varphi'+\zeta)]\|_{2C_dL^{-1}\mathfrak{h},T_j(\varphi',X)} \tag{5.61}$$

and we see that the choice $\mathfrak{h}' = 2C_dL^{-1}\mathfrak{h}$ gives

$$\|\mathbb{E}[\theta_\zeta F(\varphi')]\|_{2\mathfrak{h},T_{j+1}(\varphi'+\zeta)} \leqslant e^{2\sqrt{\beta}|q|\mathfrak{h}}e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}\mathbb{E}[\|F(\varphi'+\zeta)\|_{2C_dL^{-1}\mathfrak{h}+r\|\xi\|,T_j(\varphi',X)}]. \tag{5.62}$$

Now invoking Proposition 3.3.5,

$$\begin{aligned}\|\mathbb{E}[\theta_\zeta F(\varphi')]\|_{2\mathfrak{h},T_{j+1}(\varphi',X)} &\leqslant e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}e^{2\sqrt{\beta}|q|\mathfrak{h}}\mathbb{E}[\theta_\zeta G_j(X,\varphi')]\|F\|_{2C_dL^{-1}\mathfrak{h}+r\|\xi\|,T_j(X)}\\ &\leqslant e^{-\frac{2|q|-1}{2}r\beta\Gamma_{j+1}(0)}e^{2\sqrt{\beta}|q|\mathfrak{h}}2^{|X|_j}G_{j+1}(\overline{X},\varphi')\|F\|_{2C_dL^{-1}\mathfrak{h}+r\|\xi\|,T_j(X)}.\end{aligned} \tag{5.63}$$

Since $\|F\|_{2C_dL^{-1}\mathfrak{h}+r\|\xi\|,T_j(X)} \leqslant \|F\|_{\mathfrak{h},T_j(X)}$ and X is a small set, we obtain

$$\|\mathbb{E}[\theta_\zeta F(X,\varphi')]\|_{2\mathfrak{h},T_{j+1}(X,\varphi')} \leqslant Ce^{2\sqrt{\beta}|q|\mathfrak{h}}e^{-(|q|-1/2)r\beta\Gamma_{j+1}(0)}\|F(X)\|_{\mathfrak{h},T_j(X)}G_{j+1}(\overline{X},\varphi'). \tag{5.64}$$

The same estimate holds for each $\|D^m_\omega\mathbb{E}[\theta_\zeta F(X,\varphi')]\|_{2\mathfrak{h},T_{j+1}(X,\varphi')}$ ($m \geqslant 0$), so the desired conclusion holds. □

Finally, we conclude the Proposition 5.2.3 for charged F.

Lemma 5.4.8. *Under the setting of Proposition 5.2.3, let F be a periodic scale-j polymer activity and $X \in \mathscr{S}_j$. Also, assume that the neutral part of $F(X)$ vanishes, i.e., $\hat{F}_0(X) = 0$. Then*

$$\|\mathbb{E}_{(\omega)} \theta_\zeta F(X, \cdot)\|_{\vec{\mathfrak{h}}, T_{j+1}(\overline{X})} \leqslant C \alpha_{\mathrm{Loc}} \|F(X)\|_{\vec{\mathfrak{h}}, T_j(X)} \tag{5.65}$$

for some $C > 0$ independent of all the other parameters.

Proof. Since $\hat{F}_0(X) = 0$, by (5.1),

$$F = \sum_{q \neq 0} \hat{F}_q. \tag{5.66}$$

The triangle inequality, (3.63), Lemma 5.4.7 and the assumption $\mathfrak{h} \leqslant 2h$ give

$$\begin{aligned} &\|\mathbb{E}\theta_\zeta F(X, \varphi')\|_{(2\mathfrak{h}, h_\omega), T_{j+1}(\overline{X}, \varphi')} \\ &\quad \leqslant C \left[\sum_{q \geqslant 1} e^{4\sqrt{\beta}|q|h} e^{-(|q|-1/2) r \beta \Gamma_{j+1}(0)} \right] \|F(X)\|_{\vec{\mathfrak{h}}, T_j(X)} G_{j+1}(\overline{X}, \varphi'), \end{aligned} \tag{5.67}$$

But by Lemma 3.4.4,

$$\|\mathbb{E}_{(\omega)} \theta_\zeta F(X, \varphi')\|_{\vec{\mathfrak{h}}, T_{j+1}(\overline{X}, \varphi')} \leqslant \|\mathbb{E}\theta_\zeta F(X, \varphi')\|_{(2\mathfrak{h}, h_\omega), T_{j+1}(\overline{X}, \varphi')} \tag{5.68}$$

whenever $2\mathfrak{h} \geqslant \mathfrak{h}'' = \mathfrak{h} + h_\omega \|u_{j+1}\|_{C_j^2}$, thus we have the desired bound.

□

5.4.3 Proof of Proposition 5.2.3: neutral part

For neutral F, the contractions in Proposition 5.2.3 does not rely on the fluctuation integral, but instead uses that gradients contract under change of norm. In all of the following lemmas, we assume that $\mathfrak{h} \geqslant \rho_J^{-1}$ as appearing in the assumptions of Proposition 5.2.3, and we also suppose that all remaining assumptions of Proposition 5.2.3 are in force. We will also frequently abbreviate $\mathbb{E}F(X) = \mathbb{E}\theta_\zeta F(X)$.

$$\begin{aligned} \mathrm{Loc}_X \mathbb{E}F(X) - \mathbb{E}F(X) &= (\mathrm{Loc}_X \mathbb{E}F(X) - \overline{\mathrm{Tay}}_2 \mathbb{E}F(X)) + (\overline{\mathrm{Tay}}_2 \mathbb{E}F(X) - \mathbb{E}F(X)) \\ &= (\mathrm{Loc}_X \mathbb{E}F(X) - \overline{\mathrm{Tay}}_2 \mathbb{E}F(X)) - \overline{\mathrm{Rem}}_2 \mathbb{E}F(X), \end{aligned} \tag{5.69}$$

where the Taylor approximation and its remainder are defined as follows: For $F(X) \in \mathscr{N}_{j,h_\omega}(X)$ with $\|F\|_{\vec{\mathfrak{h}},T_j(X)} < \infty$, define the Taylor approximation and remainder of degree n in the variable φ by

$$\mathrm{Tay}_n^{\varphi} F(X,\varphi+\psi) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{x_1,\cdots,x_k \in X^*} \frac{\partial^k F(X,\psi)}{\partial\varphi(x_1)\cdots\partial\varphi(x_k)} \varphi(x_1)\cdots\varphi(x_k) \tag{5.70}$$

$$\mathrm{Rem}_n^{\varphi} F(X,\varphi+\psi) = F(X,\varphi+\psi) - \mathrm{Tay}_n^{\varphi} F(X,\varphi+\psi). \tag{5.71}$$

If F is neutral, then we can also define the Taylor expansion in the gradient of φ,

$$\overline{\mathrm{Tay}}_n^{\varphi} F(X,\varphi+\psi) = \frac{1}{|X|} \sum_{x_0 \in X} \mathrm{Tay}_n^{\delta\varphi} F(X,\delta\varphi+\psi) \tag{5.72}$$

$$\overline{\mathrm{Rem}}_n^{\varphi} F(X,\varphi+\psi) = F(X,\varphi+\psi) - \overline{\mathrm{Tay}}_n^{\varphi} F(X,\varphi+\psi) \tag{5.73}$$

where $\delta\varphi(x) := \varphi(x) - \varphi(x_0)$ is dependent of the choice of $x_0 \in X$.

We first collect two auxiliary results that will be used to bound the first term in (5.69).

Lemma 5.4.9. *For $\varphi \in \mathbb{R}^{\Lambda_N}$, $X \in \mathscr{S}_j$ and $x_0, y_0 \in X$,*

$$\begin{aligned} &\|\nabla^{e_1}\varphi(x_0)\nabla^{e_2}\varphi(x_0) - \nabla^{e_1}\varphi(y_0)\nabla^{e_2}\varphi(y_0)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \\ &\qquad \leqslant CL^{-2j-3}(\mathfrak{h} + \|\nabla_{j+1}\varphi\|_{L^\infty(X)} + \|\nabla_{j+1}^2\varphi\|_{L^\infty(X)})^2 \end{aligned} \tag{5.74}$$

and for any $\mu \in \hat{e}$ and $x \in X$ (see below (3.31) *for notation),*

$$\|\nabla^{\mu}\varphi(x)\nabla^{(\mu,-\mu)}\varphi(x)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant CL^{-3j-3}(\mathfrak{h} + \|\nabla_{j+1}\varphi\|_{L^\infty(X)} + \|\nabla_{j+1}^2\varphi\|_{L^\infty(X)})^2. \tag{5.75}$$

Proof. To see the first inequality, observe that

$$|\nabla^{\mu}\varphi(x_0) - \nabla^{\mu}\varphi(y_0)| \leqslant CL^{-j-2}\|\nabla_{j+1}^2\varphi\|_{L^\infty(X)}, \tag{5.76}$$

hence

$$|D_\varphi(\nabla^{\mu}\varphi(x_0) - \nabla^{\mu}\varphi(y_0))(f)| = |\nabla^{\mu}f(x_0) - \nabla^{\mu}f(y_0)| \leqslant CL^{-j-2}\|f\|_{C_{j+1}^2(X^*)}. \tag{5.77}$$

Since $\nabla^{\mu}\varphi(x_0) - \nabla^{\mu}\varphi(y_0)$ is linear in φ, all but the first two terms in the series expansion (3.40) of $\|\nabla^{\mu}\varphi(x_0) - \nabla^{\mu}\varphi(y_0)\|_{\mathfrak{h},T_{j+1}(\varphi,X)}$ vanish and therefore, using (5.76) and (5.77),

$$\|\nabla^{\mu}\varphi(x_0) - \nabla^{\mu}\varphi(y_0)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant CL^{-j-2}(\mathfrak{h} + \|\nabla_{j+1}^2\varphi\|_{L^\infty(X)}). \tag{5.78}$$

Analogously,

$$\|\nabla^\mu \varphi(x_0)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} + \|\nabla^\mu \varphi(y_0)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant CL^{-j-1}(\mathfrak{h} + \|\nabla_{j+1}\varphi\|_{L^\infty(X)}), \tag{5.79}$$

and (5.74) follows using the submultiplicativity of the norm. The second inequality (5.75) follows from similar direct computations:

$$\|\nabla^\mu \varphi(x)\nabla^{(\mu,-\mu)}\varphi(x)\|_{0,T_j(X,\varphi)} \leqslant CL^{-3j-3}\|\nabla_{j+1}\varphi\|_{L^\infty(X)}\|\nabla^2_{j+1}\varphi\|_{L^\infty(X)} \tag{5.80}$$

$$\|D\nabla^\mu \varphi(x)\nabla^{(\mu,-\mu)}\varphi(x)\|_{1,T_j(X,\varphi)} \leqslant CL^{-3j-3}(\|\nabla_{j+1}\varphi\|_{L^\infty(X)} + \|\nabla^2_{j+1}\varphi\|_{L^\infty(X)}) \tag{5.81}$$

$$\|D^2\nabla^\mu \varphi(x)\nabla^{(\mu,-\mu)}\varphi(x)\|_{2,T_j(X,\varphi)} \leqslant CL^{-3j-3}, \tag{5.82}$$

and higher-order derivatives vanish. □

Lemma 5.4.10. *Let $F \in \mathscr{N}_{j,h_\omega}(X)$ with $\|F\|_{\mathfrak{h},T_j(X)} < \infty$, and let $X \in \mathscr{S}_j$. Choose any $x_0 \in X$ and denote $\delta x_1 = x_1 - x_0$, $\delta x_2 = x_2 - x_0$. Then*

$$\Big|\sum_{x_1,x_2\in X^*} \partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}F(X,\zeta)\delta x_1^\mu \delta x_2^\nu\Big| \leqslant C\mathfrak{h}^{-2}L^{2j}\|F(X)\|_{\mathfrak{h},T_j(X)}. \tag{5.83}$$

Proof. By definition of $\|\cdot\|_{\mathfrak{h},T_j(X)}$ followed by (3.78) with $G(\overline{X},0) = 1$,

$$\begin{aligned}\mathfrak{h}^2\Big|\sum_{x_1,x_2} \partial_{\varphi(x_1)}\partial_{\varphi(x_2)}\mathbb{E}F(X,\zeta)\delta x_1^\mu \delta x_2^\nu\Big| &\leqslant \|\mathbb{E}F(X,\zeta)\|_{\mathfrak{h},T_j(X,0)}\|\delta x_1^\mu\|_{C_j^2(X^*)}\|\delta x_2^\nu\|_{C_j^2(X^*)} \\ &\leqslant CL^{2j}\|F(X)\|_{\mathfrak{h},T_j(X)}\end{aligned} \tag{5.84}$$

where we used $\|\delta x_1^\mu\|_{C_j^2(X^*)}, \|\delta x_2^\nu\|_{C_j^2(X^*)} = O(L^j)$. □

Lemma 5.4.11. *Let $\mathfrak{h} \geqslant \rho_J^{-1}$ and $\kappa_L = c_\kappa \rho_J^2(\log L)^{-1}$. Then for all $X \in \mathscr{S}_j$ and neutral and even $F(X) \in \mathscr{N}_{j,h_\omega}(X)$, we have*

$$\big\|\big(\mathrm{Loc}_X - \overline{\mathrm{Tay}}_2\big)\mathbb{E}\theta_\zeta F(X,\cdot)\big\|_{(2\mathfrak{h},h_\omega),T_{j+1}(\overline{X})} \leqslant CL^{-3}(\log L)A^{-|X|_j}\|F\|_{\vec{\mathfrak{h}},T_j} \tag{5.85}$$

for $\vec{\mathfrak{h}} = (\mathfrak{h}, h_\omega)$.

Proof. We will prove the statement without the ω-derivatives, i.e., we aim to prove

$$\|\mathrm{Loc}_X\,\mathbb{E}\theta_\zeta F(X,\cdot) - \overline{\mathrm{Tay}}_2\mathbb{E}\theta_\zeta F(X,\cdot)\|_{2\mathfrak{h},T_{j+1}(\overline{X})} \leqslant CL^{-3}(\log L)A^{-|X|_j}\|F\|_{\mathfrak{h},T_j}. \tag{5.86}$$

Then the same estimates on $D_\omega^m F$ are essentially the same.

By definition (see (5.11) and (5.12)), denoting by $B_j(x_0)$ the block $B \in \mathscr{B}_j$ such that $x_0 \in B$,

$$\begin{aligned}
\mathrm{Loc}_X \mathbb{E}F(X) = \sum_{B \in \mathscr{B}_j(X)} \mathrm{Loc}_{X,B} \mathbb{E}F(X) = \mathbb{E}\hat{F}_0(X,\zeta) \\
+ \frac{1}{|X|} \sum_{x_0 \in X} \frac{1}{|B|} \sum_{y_0 \in B_j(x_0)} \sum_{x_1,x_2 \in X^*} \frac{1}{2} \partial_{\varphi_{x_1}} \partial_{\varphi_{x_2}} \mathbb{E}\hat{F}_0(X,\zeta) \langle \nabla\varphi(y_0), \delta x_1, \nabla\varphi(y_0), \delta x_2 \rangle, \quad (5.87)
\end{aligned}$$

where $\delta x_1 = x_1 - x_0, \delta x_2 = x_2 - x_0$ and, following the notation of Lemma 5.4.10 (cf. (5.28)),

$$\langle \nabla\varphi(y_0), \delta x_1, \nabla\varphi(y_0), \delta x_2 \rangle := \frac{1}{4} \sum_{\mu,\nu \in \hat{e}} (1 + \delta_{\mu,\nu} - \delta_{\mu,-\nu}) \nabla^\mu \varphi(y_0) \delta x_1^\mu \nabla^\nu \varphi(y_0) \delta x_2^\nu. \tag{5.88}$$

We firstly replace $\langle \nabla\varphi(y_0), \delta x_1, \nabla\varphi(y_0), \delta x_2 \rangle$ by $(\nabla\varphi(y_0), \delta x_1)(\nabla\varphi(y_0), \delta x_2)$ and secondly replace y_0 by x_0 in (5.87) where $(\nabla\varphi(x), y) = \frac{1}{2}\sum_{\mu \in \hat{e}} \nabla^\mu \varphi(x) y^\mu$. This gives

$$\frac{1}{|X|} \sum_{x_0 \in X} \sum_{x_1,x_2 \in X^*} \frac{1}{2} \partial_{\varphi_{x_1}} \partial_{\varphi_{x_2}} \mathbb{E}\hat{F}_0(X,\zeta) (\nabla\varphi(x_0), \delta x_1)(\nabla\varphi(x_0), \delta x_2) \tag{5.89}$$

and, as we now explain, an error term bounded in the $\|\cdot\|_{2\mathfrak{h},T_{j+1}(X,\varphi)}$-norm by

$$CL^{-3} 2^{|X|_j} \|F(X)\|_{\mathfrak{h},T_j(X)} \mathfrak{h}^{-2} (\mathfrak{h} + \|\nabla_{j+1}\varphi\|_{L^\infty(X^*)} + \|\nabla_{j+1}^2 \varphi\|_{L^\infty(X^*)})^2. \tag{5.90}$$

Indeed, to obtain this error bound, we proceed as follows: observing that

$$\begin{aligned}
(\nabla\varphi(y_0), \delta x_1)(\nabla\varphi(x_0), \delta x_2) - \langle \nabla\varphi(y_0), \delta x_1, \nabla\varphi(x_0), \delta x_2 \rangle \\
= \frac{1}{4} \sum_{\mu \in \hat{e}} \nabla^\mu \varphi(y_0) \nabla^{(\mu,-\mu)} \varphi(y_0) \delta x_1^\mu \, \delta x_2^\mu, \quad (5.91)
\end{aligned}$$

the claimed bound for the first replacement is justified by (5.75) and (5.83), whereas the claimed bound for the second replacement follows from (5.74) and (5.83). The factor $2^{|X|_j}$ appearing in (5.90) follows hereby from an application of Proposition 3.3.5. Rather than including full details here, we refer to (5.96)-(5.97) below, which estimate a similar but slightly more involved error term, yielding the bound (5.95). The bound (5.90) is readily obtained by adapting these arguments.

Next we replace $\langle \nabla\varphi(x_0), \delta x_i\rangle$ in (5.89) by $\delta\varphi(x_i) = \varphi(x_i) - \varphi(x_0)$. For $X \in \mathscr{S}_j$, one has

$$\|\delta\varphi(x)\|_{C^2_{j+1}(X^*)},\ \|\langle\nabla\varphi(x_0), \delta x\rangle\|_{C^2_{j+1}(X^*)} \leqslant CL^{-1}\|\nabla_{j+1}\varphi\|_{L^\infty(X^*)} \tag{5.92}$$

$$\|\delta\varphi(x) - \langle\nabla\varphi(x_0), \delta x\rangle\|_{C^2_{j+1}(X^*)} \leqslant CL^{-2}\|\nabla^2_{j+1}\varphi\|_{L^\infty(X^*)}. \tag{5.93}$$

Using again the definition of the norm (3.40), we may thus replace (5.89) by

$$\frac{1}{2|X|}\sum_{x_0\in X}\sum_{x_1,x_2\in X^*}\partial_{\delta\varphi(x_1)}\partial_{\delta\varphi(x_2)}\mathbb{E}\hat{F}_0(X,\zeta)(\varphi(x_1)-\varphi(x_0))(\varphi(x_2)-\varphi(x_0)) \tag{5.94}$$

with an error in the $\|\cdot\|_{2\mathfrak{h},T_{j+1}(X,\varphi)}$-norm bounded by

$$CL^{-3}2^{|X|_j}\|F(X)\|_{\mathfrak{h},T_j(X)}\mathfrak{h}^{-2}\big(\mathfrak{h}+\|\nabla_{j+1}\varphi\|_{L^\infty(X^*)}+\|\nabla^2_{j+1}\varphi\|_{L^\infty(X^*)}\big)^2. \tag{5.95}$$

Indeed,

$$\begin{aligned}
&\Big|\sum_{x_1,x_2\in X^*}\partial_{\delta\varphi(x_1)}\partial_{\delta\varphi(x_2)}\mathbb{E}\hat{F}_0(X,\zeta)(\delta\varphi(x_1)-(\nabla\varphi(x_0),\delta x_1))\delta\varphi(x_1)\Big| \\
&\leqslant \mathbb{E}\|D^2\hat{F}_0(X,\zeta)\|_{n,T_{j+1}(X,\zeta)}\|\delta\varphi(x_1)-(\nabla\varphi(x_0),\delta x_1)\|_{C^2_{j+1}(X^*)}\|\varphi(x_2)-\varphi(x_0)\|_{C^2_{j+1}(X^*)} \\
&\leqslant C\mathfrak{h}^{-2}\|F(X)\|_{h,T_j(X)}\mathbb{E}[G_j(X,\zeta)]L^{-3}\|\nabla_{j+1}\varphi\|_{L^\infty}\|\nabla^2_{j+2}\varphi\|_{L^\infty(X^*)} \\
&\leqslant CL^{-3}2^{|X|_j}\|F(X)\|_{h,T_j(X)}\mathfrak{h}^{-2}\|\nabla_{j+1}\varphi\|_{L^\infty(X^*)}\|\nabla^2_{j+1}\varphi\|_{L^\infty(X^*)}
\end{aligned} \tag{5.96}$$

using (5.5) for the second inequality and Proposition 3.3.5 in the last step, and since each $\delta\varphi(x_1)-(\nabla\varphi(x_0),\delta x_1)$ and $\delta\varphi(x_1)$ are linear in φ, we immediately see (see around (5.77) for a similar reasoning) that

$$\begin{aligned}
&\|\sum_{x_1,x_2\in X^*}\partial_{\delta\varphi(x_1)}\partial_{\delta\varphi(x_2)}\mathbb{E}\hat{F}_0(X,\zeta)(\delta\varphi(x_1)-(\nabla\varphi(x_0),\delta x_1))\delta\varphi(x_1)\|_{2\mathfrak{h},T_{j+1}(X,\varphi)} \\
&\leqslant CL^{-3}2^{|X|_j}\|F(X)\|_{\mathfrak{h},T_j(X)}\mathfrak{h}^{-2}(2\mathfrak{h}+\|\nabla_{j+1}\varphi\|_{L^\infty(X^*)})(2\mathfrak{h}+\|\nabla^2_{j+1}\varphi\|_{L^\infty(X^*)})
\end{aligned} \tag{5.97}$$

A similar bound holds for

$$\sum_{x_1,x_2\in X^*}\frac{\partial^2\mathbb{E}\hat{F}_0(X,\zeta)}{\partial(\delta\varphi(x_1))\partial(\delta\varphi(x_2))}(\delta\varphi(x_1)-(\nabla\varphi(x_0),\delta x_1))(\nabla\varphi(x_0),\delta x_2) \tag{5.98}$$

and hence the claim follows.

Recognizing $\mathbb{E}\hat{F}_0(X,\zeta)$ in (5.87) together with (5.94) as $\overline{\mathrm{Tay}}_2\mathbb{E}F(X)$ and collecting the errors, we have thus overall shown

$$\|\operatorname{Loc}_X \mathbb{E}F(X) - \overline{\mathrm{Tay}}_2\mathbb{E}F(X)\|_{2\mathfrak{h},T_{j+1}(\overline{X},\varphi')} \leqslant CL^{-3}\|F(X)\|_{h,T_j(X)}(1+\mathfrak{h}^{-1}\max_{n=1,2}\|\nabla^n_{j+1}\varphi\|_{L^\infty(X*)})^2. \tag{5.99}$$

The claim now follows from Lemma 3.3.3, along with (3.63), using that $\mathfrak{h}^{-2}\kappa_L^{-1} = O(\log L)$ which holds since $\mathfrak{h}^{-2} = O(\rho_J^2)$ by our assumption $\mathfrak{h} \geqslant \rho_J^{-1}$ and since $\kappa_L^{-1} = O(\rho_J^{-2}\log L)$. □

Lemma 5.4.12. *Under the setting of Lemma 5.4.11, for all $X \in \mathscr{S}_j$, neutral $F(X) \in \mathscr{N}_{j,h_\omega}(X)$ and $\alpha \geqslant 0$,*

$$\|\overline{\mathrm{Rem}}^{\varphi'}_\alpha \mathbb{E}\theta_\zeta F(X,\cdot)\|_{(2\mathfrak{h},h_\omega),T_{j+1}(\overline{X})} \leqslant C_\alpha\big(L^{-1}(\log L)^{1/2}\big)^{\alpha+1}A^{-|X|_j}\|F\|_{\vec{\mathfrak{h}},T_j} \tag{5.100}$$

with $\vec{\mathfrak{h}} = (\mathfrak{h},h_\omega)$.

Proof. Again, we will only prove the bound without the ω-derivative.

Recall that $F(X,\zeta+\varphi') = F(X,\zeta+\delta\varphi')$ with $\delta\varphi'(x) = \varphi'(x) - \varphi'(x_0)$ for neutral F and any x_0 by (5.4). Thus, $\overline{\mathrm{Rem}}^{\varphi'}_\alpha \mathbb{E}F = \frac{1}{|X|}\sum_{x_0\in X}\mathrm{Rem}^{\delta\varphi'}_\alpha \mathbb{E}F$ with $\delta\varphi'(x)$ defined for varying x_0's, we just need to prove the statement for a fixed $x_0 \in X$ and $\overline{\mathrm{Rem}}^{\varphi'}_\alpha$ replaced by $\mathrm{Rem}_\alpha \equiv \mathrm{Rem}^{\delta\varphi'}_\alpha$.

We need to estimate $\|D^n\mathrm{Rem}_\alpha\mathbb{E}F(X)\|_{n,T_{j+1}(X,\varphi')}$. We will divide the cases $n \geqslant \alpha+1$ and $n \leqslant \alpha$. Using that $\mathrm{Rem}_\alpha\mathbb{E}\theta_\zeta F(X,\varphi')$ is neutral, the estimate for $n \geqslant \alpha+1$ follows simply from Lemma 5.4.2 and the fact that $D^n\mathrm{Rem}_\alpha = D^n$ for $n \geqslant \alpha+1$:

$$\|D^n\mathrm{Rem}_\alpha\mathbb{E}\theta_\zeta F(X,\varphi')\|_{n,T_{j+1}(X,\varphi')} \leqslant (C_d^{-1}L)^{-n}\|D^n\mathbb{E}\theta_\zeta F(X,\varphi')\|_{n,T_j(X,\varphi')}. \tag{5.101}$$

Multiplying by $\mathfrak{h}^n/n!$, summing over n, and combining with Lemma 3.3.6, noting that $2^{|X|_j} \leqslant C$ since X is small, this readily yields

$$\sum_{n\geqslant\alpha+1}\frac{(2\mathfrak{h})^n}{n!}\|D^n\mathrm{Rem}_2\mathbb{E}\theta_\zeta F(X,\varphi')\|_{n,T_{j+1}(X,\varphi')} \leqslant CL^{-(\alpha+1)}A^{-|X|_j}\|F\|_{\mathfrak{h},T_j}G_{j+1}(\overline{X},\varphi'). \tag{5.102}$$

The cases $n = 0, \cdots, \alpha$ require a bit of effort and represent in fact the dominant contributions. We use Taylor's theorem and neutrality of F to write

$$D^n \mathrm{Rem}_\alpha \mathbb{E}\theta_\zeta F(X,\varphi')(f_1,\cdots,f_n) = \sum_{k=n}^{\alpha} \frac{1}{(k-n)!} D^k \mathrm{Rem}_\alpha \mathbb{E}F(X,\zeta)(f_1,\cdots,f_n,(\delta\varphi')^{\otimes k-n})$$
$$+ \int_0^1 dt \, \frac{(1-t)^{\alpha+1-n}}{(\alpha-n)!} D^{\alpha+1} \mathrm{Rem}_\alpha \mathbb{E}F(X,\zeta+t\varphi')(f_1,\cdots,f_n,(\delta\varphi')^{\otimes \alpha+1-n}). \tag{5.103}$$

But since $D^k \mathrm{Rem}_\alpha \mathbb{E}F(X,\cdot) = D^k \mathbb{E}F(X,\cdot)$ for $k \geqslant \alpha+1$, applying successively (3.39), (5.36) and (3.75), one sees that

$$\begin{aligned}
&|D^{\alpha+1} \mathrm{Rem}_\alpha \mathbb{E}F(X,\zeta+t\varphi')(f_1,\cdots,f_n,(\delta\varphi')^{\otimes\alpha+1-n})| \\
&\quad \leqslant \|D^{\alpha+1}\mathbb{E}F(X,\zeta+t\varphi')\|_{\alpha+1,T_{j+1}(X,t\varphi')} \|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)} \prod_{1\leqslant l\leqslant n} \|f_l\|_{C^2_{j+1}(X^*)} \\
&\quad \leqslant (C_d^{-1}L)^{-(\alpha+1)} \|D^{\alpha+1}\mathbb{E}F(X,\zeta+t\varphi')\|_{\alpha+1,T_j(X,t\varphi')} \|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)} \prod_{1\leqslant l\leqslant n} \|f_l\|_{C^2_{j+1}(X^*)} \\
&\quad \leqslant C'(hL)^{-(\alpha+1)} \|F(X)\|_{\alpha+1,T_j(X)} G_{j+1}(\overline{X},t\varphi') \|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)} \prod_{1\leqslant l\leqslant n} \|f_l\|_{C^2_{j+1}(X^*)}.
\end{aligned} \tag{5.104}$$

Moreover, since $D^k \mathrm{Rem}_2 \mathbb{E}F(X,\zeta) = 0$ for $k \in \{0,\cdots,\alpha\}$, whenever $\|f_l\|_{C^2_{j+1}(X^*)} \leqslant 1$ for each $l \in \{1,\ldots,n\}$, one obtains that the left-hand side of (5.103) is bounded in absolute value by

$$(C_d^{-1}L\mathfrak{h})^{-(\alpha+1)} \int_0^1 dt \, \frac{(1-t)^{\alpha+1-n}}{(\alpha-n)!} \|F(X)\|_{\alpha+1,T_j(X)} G_{j+1}(\overline{X},t\varphi') \|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)}. \tag{5.105}$$

Now since $G_{j+1}(\overline{X},\varphi') = G_{j+1}(\overline{X},t\varphi')G_{j+1}(\overline{X},\sqrt{1-t^2}\varphi')$ by definition of G_{j+1} in (3.41), and then using Lemma 3.3.3 applied with $\varphi = \sqrt{1-t^2}\varphi'$, we obtain, for $n \leqslant \alpha$

$$\sup_{\varphi'} \frac{G_{j+1}(\overline{X},t\varphi')\|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)}}{G_{j+1}(\overline{X},\varphi')} = \sup_{\varphi'} \frac{\|\delta\varphi'\|^{\alpha+1-n}_{C^2_{j+1}(X^*)}}{G_{j+1}(\overline{X},\sqrt{1-t^2}\varphi')} \leqslant O(\kappa_L^{-(\alpha+1-n)/2})(1-t^2)^{-(\alpha+1-n)/2}. \tag{5.106}$$

All in all, since

$$\int_0^1 (1-t)^{\alpha+1-n}(1-t^2)^{-(\alpha+1-n)/2}\, dt \leqslant C_n \int_0^1 t^{(\alpha+1-n)/2} < \infty, \quad n \leqslant \alpha, \tag{5.107}$$

this implies, for $n \leqslant \alpha$,

$$\|D^n \mathrm{Rem}_\alpha \mathbb{E}F(X,\varphi)\|_{n,T_{j+1}(\overline{X},\varphi')} \\ \leqslant O(L^{-(\alpha+1)}\kappa_L^{-(\alpha+1-n)/2})\mathfrak{h}^{-(\alpha+1)}\|F(X)\|_{\alpha+1,T_j(X)}G_{j+1}(\overline{X},\varphi'). \quad (5.108)$$

Multiplying by $\mathfrak{h}^n/n!$, summing over n, using that

$$\sum_{0\leqslant n\leqslant\alpha} \frac{\mathfrak{h}^{n-\alpha-1}}{n!}\kappa_L^{-(\alpha+1-n)/2} \leqslant C(\log L)^{(\alpha+1)/2}\mathfrak{h}^{-(\alpha+1)} \quad (5.109)$$

by assumption on κ_L and $\mathfrak{h}$ (the dominant term being $n=0$), it follows that

$$\sum_{n\leqslant\alpha}\frac{(2\mathfrak{h})^n}{n!}\|D^n\mathrm{Rem}_\alpha\mathbb{E}F(X,\varphi)\|_{n,T_{j+1}(\overline{X},\varphi')} \leqslant C\big(L^{-1}(\log L)^{1/2}\big)^{\alpha+1}A^{-|X|_j}\|F\|_{\mathfrak{h},T_j}G_{j+1}(\overline{X},\varphi'). \quad (5.110)$$

The claim follows immediately by combining the estimates (5.102) (with (3.63)) and (5.110).

□

Proof of Proposition 5.2.3. The Fourier decomposition (5.1.1) yields, since $\mathrm{Loc}_X^{(\alpha)}\mathbb{E}\theta_\zeta\hat{F}_q(X) = 0$ whenever $q \neq 0$ (cf. (5.11)),

$$\mathrm{Loc}_X^{(\alpha)}\mathbb{E}_{(\omega)}\theta_\zeta F(X) - \mathbb{E}_{(\omega)}\theta_\zeta F(X) = \big(\mathrm{Loc}_X^{(\alpha)} - 1\big)\mathbb{E}_{(\omega)}\theta_\zeta\hat{F}_0(X) - \sum_{q\neq 0}\mathbb{E}_{(\omega)}\theta_\zeta\hat{F}_q(X), \quad (5.111)$$

which allows to bound them by bounding the terms of different charge separately. The last sum is the charged part of F and was already bounded by Lemma 5.4.8.
(i) Since $\overline{\mathrm{Tay}}_0\mathbb{E}\theta_\zeta\hat{F}_0(X) = \mathrm{Loc}_X^{(0)}\mathbb{E}\theta_\zeta\hat{F}_0(X)$, Lemma 5.4.12 (with the choice $F = \hat{F}_0$) is enough to obtain

$$\|(\mathrm{Loc}_X^{(0)} - 1)\mathbb{E}\theta_\zeta\hat{F}_0(X)\|_{(2\mathfrak{h},h_\omega),T_{j+1}(X)} \leqslant C\alpha_{\mathrm{Loc}}^{(0)}A^{-|X|_j}\|F\|_{\vec{\mathfrak{h}},T_j} \quad (5.112)$$

and also by Lemma 3.4.4,

$$\|(\mathrm{Loc}_X^{(0)} - 1)\mathbb{E}_{(\omega)}\theta_\zeta\hat{F}_0(X)\|_{\vec{\mathfrak{h}},T_{j+1}(X)} \leqslant \|(\mathrm{Loc}_X^{(0)} - 1)\mathbb{E}\theta_\zeta\hat{F}_0(X)\|_{(2\mathfrak{h},h_\omega),T_{j+1}(X)}. \quad (5.113)$$

These prove (5.18).
(ii) Assume F is even and periodic. Then Lemmas 5.4.11 and 5.4.12 (with the choice $F = \hat{F}_0$) yield bounds on $(\mathrm{Loc}_X - \overline{\mathrm{Tay}}_2)\mathbb{E}\theta_\zeta\hat{F}_0(X,\varphi')$ and $(1 - \overline{\mathrm{Tay}}_2^{\varphi'})\mathbb{E}\theta_\zeta\hat{F}_0(X,\varphi')$, respectively, and

thus

$$\|(\mathrm{Loc}_X - 1)\mathbb{E}\theta_\zeta \hat{F}_0(X)\|_{(2\mathfrak{h},h_\omega),T_{j+1}(X)} \leqslant C\alpha_{\mathrm{Loc}} A^{-|X|_j} \|F\|_{\vec{\mathfrak{h}},T_j}. \tag{5.114}$$

As above, this proves (5.19) once we recall Lemma 3.4.4. □

Boundedness of localisation

The following will be used to deduce (5.21).

Lemma 5.4.13. *For F a neutral j scale ω-polynomial activity $B \in \mathscr{B}_j$ and $X \in \mathscr{S}_j$*

$$\left|\mathrm{Loc}^{(0)}_{X,B}\mathbb{E}\theta_\zeta F(X,\varphi')\right| \leqslant C\|F(X)\|_{\mathfrak{h},T_j(X)} \tag{5.115}$$

$$\|\mathrm{Loc}^{(2)}_{X,B}\mathbb{E}\theta_\zeta F(X,\varphi')\|_{(2\mathfrak{h},h_\omega),T_j(X,\varphi')} \leqslant C(\log L)\|F(X)\|_{\vec{\mathfrak{h}},T_j(X)} e^{c_w\kappa_L w_j(B,\varphi')^2}. \tag{5.116}$$

Proof. $\mathrm{Loc}^{(0)}_X \mathbb{E}F(X)$ is bounded using (3.75) with $\varphi' \equiv 0$ by

$$|\mathbb{E}\hat{F}_0(X,\zeta)| \leqslant 2^{|X|_j}\|F(X)\|_{\mathfrak{h},T_j(X)} \leqslant C\|F(X)\|_{\mathfrak{h},T_j(X)}, \tag{5.117}$$

since X is small. Now consider

$$\mathrm{Loc}^{(2)}_{X,B}\mathbb{E}F(X) = \frac{1}{|X|}\sum_{x_0,y_0\in B}\frac{1}{|B|}\sum_{x_1,x_2\in X^*}\frac{1}{2}\partial_{\varphi_{x_1}}\partial_{\varphi_{x_2}}\mathbb{E}\hat{F}_0(X,\zeta)\langle\nabla\varphi'(y_0),\delta x_1,\nabla\varphi'(y_0),\delta x_2\rangle. \tag{5.118}$$

(see (5.88) for the notation). For $\mu,\nu \in \hat{e}$, $\sum_{x_1,x_2\in X^*}\partial_{\varphi_{x_1}}\partial_{\varphi_{x_2}}\mathbb{E}\hat{F}_0(X,\zeta)(\delta x_1)^\mu(\delta x_2)^\nu$ can be bounded using Lemma 5.4.10. Moreover since $y_0 \in B$,

$$L^j\|\nabla^\mu\varphi'(y_0)1_{y_0\in B}\|_{\mathfrak{h},T_j(X,\varphi')} \leqslant \mathfrak{h} + \|\nabla_j\varphi'\|_{L^\infty(B)}. \tag{5.119}$$

Putting these together, using the submultiplicativity of the norm and recalling the definition of w_j from (3.71), the $\|\cdot\|_{\mathfrak{h},T_j(\varphi,X)}$-norm of the second term of (5.118) is readily seen to be bounded by

$$C\mathfrak{h}^{-2}(\mathfrak{h}+\|\nabla_j\varphi'\|^2_{L^\infty(B)})^2\|F\|_{\mathfrak{h},T_j(X)} \leqslant C'\mathfrak{h}^{-2}\kappa_L^{-1}\|F(X)\|_{\mathfrak{h},T_j(X)}e^{c_w\kappa_L w_j(B,\varphi')^2}. \tag{5.120}$$

The claim again follows from the fact that $\mathfrak{h}^{-2}\kappa_L{}^{-1} = O(\log L)$. □

Proof of Proposition 5.2.4. The bounds are direct results of Lemma 5.4.13 and Lemma 3.4.4.

The continuity in s again follows from Lemma 3.3.11, similarly as in the proof of Proposition 5.2.2. $\square$

Chapter 6

Renormalisation group on the Discrete Gaussian model at high temperature

This chapter reformulates the Discrete Gaussian model in terms of a statistical physics model with continuous spin values and applies the renormalisation group analysis without the external field. The RG analysis is composed of two parts. We construct the (bulk) RG map Φ^0_{j+1}, a function of (U_j, K_j^0) that constructs the coordinates in the next scale in Section 6.3.3. The (bulk) RG map is then used to construct the (bulk) RG flow, whose convergence is subject to the choice of the initial condition, and the set of converging initial values form a stable manifold. The stable manifold and the (bulk) RG flow is constructed in Chapter 7. The (bulk) RG flow becomes the basis of proving the torus scaling limit theorem in Chapter 9, and observable RG flows are discussed in Chapter 8.

The bulk RG map of this chapter only considers scales $j+1 < N$, which we tacitly assume.

6.1 Main results

6.1.1 Reformulation into a continuous spin model

We have two inputs in the reformulated J-DG model: the regularising mass m^2 and the stiffness renormalisation factor s. These give covariances $\Gamma_j(s)$, $\Gamma_N^{\Lambda_N}(s)$ and $t_N(s,m^2)Q_N$ introduced in Section 3.1 and

$$\tilde{C}(s) \equiv \tilde{C}^{\Lambda_N}(s) = \sum_{j=1}^{N-1} \Gamma_j(s) + \Gamma_N^{\Lambda_N} \tag{6.1}$$

(see Lemma 3.1.2), the tilted expectation

$$\mathbb{E}_{(\omega),C}[F(\varphi)] = \frac{\mathbb{E}_C\left[e^{\omega(\tilde{\mathfrak{f}},\varphi)}F(\zeta)\right]}{\mathbb{E}_C[F(\zeta)]}, \qquad \tilde{\mathfrak{f}} = (1+s\gamma\Delta)\mathfrak{f} \tag{6.2}$$

for given $\mathfrak{f}$, and the following reformulation.

Proposition 6.1.1. *Suppose $\sum_x \mathfrak{f}(x) = 0$ and let $\tilde{\mathfrak{f}} = (1+s\gamma\Delta)\mathfrak{f}$. Then for $|s|$ sufficiently small, $\beta > 0$ and $\omega \in \mathbb{C}$,*

$$\langle e^{\beta^{-1/2}\omega(\mathfrak{f},\sigma)}\rangle_{J,\beta}^{\Lambda_N} = e^{\frac{1}{2}\omega^2(\tilde{\mathfrak{f}},\tilde{C}(s)\tilde{\mathfrak{f}})+\frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}})} \lim_{m^2\downarrow 0} F_{N,m^2}[\mathfrak{f}](\omega) \tag{6.3}$$

$$F_{N,m^2}[\mathfrak{f}](\omega) = \frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta}_{(\omega)}[Z_0(\varphi'+\zeta+\omega\gamma\mathfrak{f})]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta}[Z_0(\varphi'+\zeta)]} \tag{6.4}$$

with $\varphi' \sim \mathcal{N}(0, t_N(m^2)Q_N)$ and $\zeta \sim \mathcal{N}(0, \tilde{C}(s))$ and

$$Z_0^0(\varphi|\Lambda_N) = \exp\Big[\frac{1}{2}s_0|\nabla\varphi|^2_{\Lambda_N} + \sum_{x\in\Lambda_N,\, q\geqslant 1} z_0^{(q)}\cos(q\sqrt{\beta}\varphi(x))\Big], \tag{6.5}$$

for $s_0 = s$ and some $z_0^{(q)} = O(e^{-c_f\beta(1+q)})$.

The proposition simply states that the J-DG model is equivalently described by the continuous-valued spin model with potential $U_0 \in \Omega_0^U$ (of form given by Definition 3.2.6). The proof is given in Section 6.2.

6.1.2 Renormalisation group map

As in Chapter 4, we study $\mathbb{E}^{\zeta}[Z_0^0(\varphi'+\zeta)]$ by inductively defining $Z_{j+1}^0(\varphi'|\Lambda_N) = \mathbb{E}^{\zeta}_{\Gamma_{j+1}}[\theta_\zeta Z_j^0(\varphi'|\Lambda_N)]$ when $Z_j^0 = e^{-E_j|\Lambda_N|}(e^{U_j}\circ_j K_j^0)(\Lambda_N)$ for some $U_j \in \Omega_j^U$ and some polymer activity K_j^0. We construct a map

$$\Phi_{j+1}^0 : (E_j, s_j, z_j, K_j^0) \mapsto (E_j + \mathscr{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathscr{K}_{j+1}^0), \tag{6.6}$$

so that if $(E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}^0) = (E_j + \mathscr{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathscr{K}_{j+1}^0)$, then

$$Z_{j+1}^0 = e^{-E_{j+1}|\Lambda_N|}(e^{U_{j+1}}\circ_j K_{j+1}^0)(\Lambda_N) \tag{6.7}$$

(see Theorem 6.3.5). We also make use of the symmetries and periodicity under constant summations, so only even periodic K_j^0 respecting lattice symmetry at scale j will be considered.

We denote this by letting $\Omega^K_{j,0}$ be the space of such K^0_j and also

$$\|K^0_j\|_{\Omega^K_{j,0}} = \|K^0_j\|_{2h,T_j,A}, \qquad \|(U_j,K^0_j)\|_{\Omega_{j,0}} = \max\{\|U_j\|_{\Omega^U_j}, \|K^0_j\|_{\Omega^K_{j,0}}\} \tag{6.8}$$

We choose $(\mathscr{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1})$ that satisfy the following analytic results. The first theorem is about the coupling constants (s_j, z_j) (equivalently, U_j), which form the leading part of the RG map.

Theorem 6.1.2 (Estimates for coupling constants)**.** *Let $U_j = (s_j, z_j) \in \Omega^U_j$ and $K^0_j \in \Omega^K_{j,0}$. For any choice of $L > 1$, $A > 1$, and $h > 0$, we have $\mathfrak{z}^{(q)}_{j+1}(z_j) = L^2 e^{-\frac{1}{2}\beta q^2 \Gamma_{j+1}(0)} z^{(q)}_j$ for all $q \geqslant 1$ and $\mathfrak{s}_{j+1}$, $\mathscr{E}_{j+1}$ only depend on (s_j, K^0_j) with the following estimates:*

$$|\mathfrak{s}_{j+1}(s_j, K^0_j) - s_j| \leqslant Ch^{-2}A^{-1}\|K^0_j\|_{\Omega^K_{j,0}} \tag{6.9}$$

$$|\mathscr{E}_{j+1}(s_j, K^0_j) + s_j \nabla^{(e_1,-e_1)}\Gamma_{j+1}(0)| \leqslant CL^{-2j}A^{-1}\|K^0_j\|_{\Omega^K_{j,0}}. \tag{6.10}$$

Moreover, all maps above are continuous in the implicit parameter s for fixed (s_j, z_j, K^0_j).

The next theorem is the estimate on the remaining coordinate K^0_j, saying that the norm always contracts if U_j and K^0_j are small enough. This contraction is necessary, as it makes K^0_j 'irrelevant'.

Theorem 6.1.3 (Estimate for remainder coordinate)**.** *Let $j+1 < N$, $U_j = (s_j, z_j) \in \Omega^U_j$ and $K^0_j \in \Omega^K_{j,0}$. Then $\mathscr{K}^0_{j+1}(U_j, K_j) \in \Omega^K_{j+1,0}$ and admits a decomposition*

$$\mathscr{K}^0_{j+1}(U_j, K^0_j) = \mathscr{L}^0_{j+1}(K^0_j) + \mathscr{M}^0_{j+1}(U_j, K^0_j) \tag{6.11}$$

into polymer activities at scale $j+1$ such that the following holds for any β large and r small (satisfying (6.56)*), h given by* (3.60) *provided $L \geqslant L_0$, $A \geqslant A_0(L)$, for $K^0_j \in \Omega^K_{j,0}$.*

(i) The map $\mathscr{L}^0_j$ is linear in K^0_j and independent of U_j. There is a constants $C_1 > 0$ independent of all the parameters such that, with α_{Loc} as in (5.16)*,*

$$\|\mathscr{L}^0_{j+1}(K^0_j)\|_{\Omega^K_{j+1,0}} \leqslant C_1 L^2 \alpha_{\mathrm{Loc}} \|K^0_j\|_{\Omega^K_{j,0}}. \tag{6.12}$$

(ii) The remainder maps $\mathscr{M}^0_{j+1}$ satisfy $\mathscr{M}^0_{j+1} = O(U_j, K_j)^2$ in the sense that there exist $\varepsilon_{nl} \equiv \varepsilon_{nl}(\beta, A) > 0$ (only polynomially small in β) and $C_3 = C_3(\beta, A, L) > 0$ (only polynomially large in β) such that $\mathscr{M}^0_{j+1}(U_j, K^0_j)$ is continuously Fréchet-differentiable

and, for $\|(U_j, K_j^0)\|_{\Omega_{j,0}} \leqslant \varepsilon_{nl}$,

$$\|D\mathscr{M}_{j+1}^0(U_j, K_j^0)\|_{\Omega_{j+1,0}^K} \leqslant C_3 \|(U_j, K_j^0)\|_{\Omega_{j,0}} \tag{6.13}$$

with $\mathscr{M}_{j+1}^0(0,0) = 0$.

The proofs are given in Section 6.3 and 6.4.

6.2 Reformulating the Discrete Gaussian model

In this section, we prepare for the renormalisation group analysis by performing several initial manipulations of the Discrete Gaussian model as defined in (1.12). In this section, we will mostly keep the interaction J and the underlying torus $\Lambda = \Lambda_N$ (of side length L^N) implicit throughout this section. The corresponding statements are then simply understood to hold for any choice of J satisfying the conditions above (1.8), all $N \geqslant 0$ and $L \geqslant 1$. In fact the choice of side length for Λ will play no role in the present section.

6.2.1 Mass regularisation

In the first regularisation step, we replace the Discrete Gaussian model $\langle\cdot\rangle_{J,\beta}$ supported on $\sigma \in \Omega^{\Lambda_N}$ by a mass-regularised version without fixing σ_0 to be 0, thus restoring translation invariance. To this end, given $m^2 > 0$, let

$$\mathbb{Z}_\beta = 2\pi\beta^{-1/2}\mathbb{Z} \tag{6.14}$$

and for any bounded function $F : \mathbb{Z}_\beta^{\Lambda_N} \to \mathbb{R}$, let

$$\langle F \rangle_{\beta,m^2} = \frac{1}{Z_{\beta,m^2}} \sum_{\sigma \in \mathbb{Z}_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,(-\Delta_J + m^2)\sigma)} F(\sigma), \tag{6.15}$$

where Z_{β,m^2} is the corresponding normalisation constant. The following lemma shows that we recover $\langle\cdot\rangle_{J,\beta}$ in the limit $m^2 \downarrow 0$. In the sequel, for $t \in \mathbb{Z}_\beta$ and $\sigma \in \mathbb{Z}_\beta^{\Lambda_N}$ we write $\sigma + t$ for the shifted configuration with entries $(\sigma+t)_x = \sigma_x + t$, $x \in \Lambda_N$.

Lemma 6.2.1. *Let* $F : \mathbb{Z}_\beta^\Lambda \to \mathbb{R}$ *be such that* $F(\sigma) = F(\sigma + t)$ *for any constant (field)* $t \in \mathbb{Z}_\beta$, *and assume that* $F|_{\Omega_\beta^\Lambda}$ *is integrable with respect to* $\langle\cdot\rangle_{J,\beta}$. *Then* F *is integrable under* $\langle\cdot\rangle_{\beta,m^2}$ *for all* $m^2 > 0$ *and*

$$\langle F(\beta^{-1/2}\sigma)\rangle_{J,\beta} = \lim_{m^2 \downarrow 0} \langle F(\sigma)\rangle_{\beta,m^2}. \tag{6.16}$$

Proof. Recall (1.12) for $\langle\cdot\rangle_{J,\beta}$. By rescaling the spins by $\beta^{-1/2}$,

$$\langle F(\beta^{-1/2}\sigma)\rangle_{J,\beta} = \sum_{\sigma\in\Omega_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,-\Delta_J\sigma)}F(\sigma)/Z_{J,\beta}, \qquad Z_{J,\beta} = \sum_{\sigma\in\Omega_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,-\Delta_J\sigma)} \tag{6.17}$$

where $\Omega_\beta^\Lambda = \{\sigma \in \mathbb{Z}_\beta^\Lambda : \sigma_0 = 0\}$. Thus it will be sufficient to prove

$$\frac{1}{Z_{J,\beta}} \sum_{\sigma\in\Omega_\beta^\Lambda e^{-\frac{1}{2}(\sigma,-\Delta_J\sigma)}} F(\sigma) \sim \langle F(\sigma)\rangle_{\beta,m^2} \tag{6.18}$$

as $m^2 \downarrow 0$.

For F having the above properties, writing any element of $\mathbb{Z}_\beta^\Lambda$ as $\sigma+t$ with $t\in\mathbb{Z}_\beta$ and $\sigma\in\Omega_\beta^\Lambda$, one has that

$$\begin{aligned}\sum_{\sigma\in\mathbb{Z}_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,(-\Delta_J+m^2)\sigma)}|F(\sigma)| &= \sum_{t\in\mathbb{Z}_\beta}\sum_{\sigma\in\Omega_\beta^\Lambda} e^{-\frac{1}{2}(\sigma+t,(-\Delta_J+m^2)(\sigma+t))}|F(\sigma+t)| \\ &= \sum_{\sigma\in\Omega_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,-\Delta_J\sigma)}|F(\sigma)| \sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2(\sigma+t,\sigma+t)},\end{aligned} \tag{6.19}$$

where the second line is obtained using that $F(\sigma)=F(\sigma+t)$ and expanding the quadratic form (note that $\Delta_J t = 0$ when t is constant-valued). Since, uniformly in $\sigma\in\Omega_\beta^\Lambda$,

$$\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2(\sigma+t,\sigma+t)} = \sum_{t\in\mathbb{Z}_\beta}\prod_{x\in\Lambda} e^{-\frac{1}{2}m^2(\sigma_x+t)^2} \leqslant \sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}(mt)^2}, \tag{6.20}$$

where the inequality follows by retaining only $x=0$ with $\sigma_x=0$, and combining with the integrability of $F|_{\Omega_\beta^\Lambda}$, it follows that the left-hand side of (6.19) is finite; hence F is in $L^1(\langle\cdot\rangle_{\beta,m^2})$. Moreover (6.19) continues to hold without absolute values, as follows readily by the Dominated convergence theorem. Now, as $m^2\downarrow 0$, for any fixed $\sigma\in\Omega_\beta^\Lambda$, we claim that

$$\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2(\sigma+t,\sigma+t)} \sim \sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2t^2|\Lambda|}. \tag{6.21}$$

Indeed, since $|(\sigma,1)t| \leqslant \frac{1}{2}\varepsilon t^2 + \frac{1}{2\varepsilon}(\sigma,1)^2$ for any $\varepsilon>0$ by Young's inequality, the left-hand side is

$$e^{-\frac{1}{2}m^2(\sigma,\sigma)}\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2t^2}e^{-m^2(\sigma,1)t} \leqslant e^{-\frac{1}{2}m^2(\sigma,\sigma)}e^{\frac{1}{2\varepsilon}m^2(\sigma,1)^2}\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}(1-\varepsilon)m^2t^2}. \tag{6.22}$$

Therefore, for all $\varepsilon > 0$,

$$\limsup_{m^2 \downarrow 0} \frac{\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2(\sigma+t,\sigma+t)}}{\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}(1-\varepsilon)m^2t^2}} \leqslant 1, \tag{6.23}$$

and analogously

$$\liminf_{m^2 \downarrow 0} \frac{\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2(\sigma+t,\sigma+t)}}{\sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}(1+\varepsilon)m^2t^2}} \geqslant 1. \tag{6.24}$$

From this, (6.21) follows by taking $\varepsilon \to 0$. By (6.19) and the Dominated convergence theorem, thus

$$\sum_{\sigma\in\mathbb{Z}_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,(-\Delta_J+m^2)\sigma)}F(\sigma) \sim \sum_{t\in\mathbb{Z}_\beta} e^{-\frac{1}{2}m^2t^2} \sum_{\sigma\in\Omega_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,-\Delta_J\sigma)}F(\sigma) \tag{6.25}$$

and the claim follows by taking a ratio of this and the expression with F replaced by 1. $\square$

6.2.2 Smoothing the discrete model

In the next step, we replace the Discrete Gaussian model with mass $m^2 \in (0,1]$ given by (6.15) with a continuous-valued spin model. For this we write

$$(-\Delta_J + m^2)^{-1} = \gamma \mathrm{id} + C(m^2) \tag{6.26}$$

where $\gamma > 0$ is a positive constant chosen such that $C(m^2)$ is positive definite. Assuming $m^2 \in (0,1]$, we have $0 < -\Delta_J + m^2 \leqslant 3\,\mathrm{id}$ as quadratic forms, and one can choose any $\gamma \in (0,1/3)$. Note that $C(m^2)$ inherits symmetry from $-\Delta_J + m^2$ by (6.26). In fact, the parameter γ is fixed by Proposition 2.1.2, but the rest of the section holds true for any choice of $\gamma \in (0,1/3)$. We often omit the superscript Λ_N for the remainder of Section 6.2 to avoid unnecessary clutter.

As we now explain, the decomposition (6.26) implies that for any $\sigma \in \mathbb{R}^\Lambda$, one can rewrite

$$e^{-\frac{1}{2}(\sigma,(-\Delta_J+m^2)\sigma)} = c(\gamma,m^2)\int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2\gamma}\sum_x(\varphi_x-\sigma_x)^2} e^{-\frac{1}{2}(\varphi,C(m^2)^{-1}\varphi)}\,d\varphi, \tag{6.27}$$

for a suitable constant $c(\gamma,m^2) \in (0,\infty)$. The identity (6.27) is of central importance. It is in fact equivalent to the well-known property that the sum of two independent Gaussian vectors is Gaussian with covariance the sum of the two covariances. To wit, observe that for any symmetric matrices A and B acting on $\mathbb{R}^\Lambda$ such that all of A, B and $A+B$ are invertible and all $\sigma, \varphi \in \mathbb{R}^\Lambda$, letting $\sigma' = (A+B)^{-1}A\sigma$, one has the identity

$$\big(\varphi-\sigma, A(\varphi-\sigma)\big) + (\varphi, B\varphi) = \big(\varphi-\sigma', (A+B)(\varphi-\sigma')\big) + \big(\sigma, (A^{-1}+B^{-1})^{-1}\sigma\big), \quad (6.28)$$

as can be verified immediately upon rewriting the last term as $(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$. Choosing $A = \frac{1}{\gamma}\mathrm{Id}$, $B = C(m^2)^{-1}$, the left-hand side of (6.28) corresponds to (twice) the exponential appearing on the right of (6.27). With these choices, $(A^{-1}+B^{-1})^{-1} = -\Delta_J + m^2$ on account of (6.26). Thus, integrating over φ and completing the square using (6.28) readily gives (6.27).

Inserting the identity (6.27) into the partition function of the (mass regularised) Discrete Gaussian model (6.15), one obtains, for all $\beta, m^2 > 0$,

$$Z_{\beta,m^2} = \sum_{\sigma \in \mathbb{Z}_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,(-\Delta_J+m^2)\sigma)} = c(\gamma, m^2, \beta) \int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2}(\varphi, C(m^2)^{-1}\varphi)} e^{\sum_x \tilde{U}(\varphi_x)}\, d\varphi, \quad (6.29)$$

where for $\varphi \in \mathbb{R}$ and all $\gamma > 0$ we define

$$\tilde{F}(\varphi) = c(\gamma, \beta) \sum_{\sigma \in \mathbb{Z}_\beta} e^{-\frac{1}{2\gamma}(\sigma-\varphi)^2}, \qquad \tilde{U}(\varphi) = \log \tilde{F}(\varphi). \quad (6.30)$$

Here $c(\gamma, \beta) > 0$ is a constant that is chosen for later convenience such that

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \tilde{F}(\varphi/\sqrt{\beta})\, d\varphi = \frac{c(\gamma,\beta)}{2\pi} \int_{\mathbb{R}} e^{-\frac{1}{2\gamma\beta}\varphi^2}\, d\varphi. \quad (6.31)$$

Both $\tilde{F}$ and $\tilde{U}$ are smooth periodic functions of the single real variable $\varphi \in \mathbb{R}$. For later application, we record the following properties of their Fourier representations.

Lemma 6.2.2. *For any $\gamma > 0$ and $\beta > 0$, the Fourier representation of $\tilde{F}$ is given by*

$$\tilde{F}(\varphi) = 1 + \sum_{q=1}^{\infty} 2e^{-\frac{\gamma\beta}{2}q^2} \cos(q\sqrt{\beta}\varphi), \quad \varphi \in \mathbb{R}. \quad (6.32)$$

Moreover, there exists $C \in (0,\infty)$ such that for any $\gamma\beta \geqslant C$, the function $\varphi \mapsto \tilde{U}(\varphi)$ has the Fourier representation

$$\tilde{U}(\varphi) = \sum_{q=1}^{\infty} \tilde{z}^{(q)}(\beta) \cos(q\sqrt{\beta}\varphi), \quad \varphi \in \mathbb{R}, \quad (6.33)$$

with coefficients satisfying

$$|\tilde{z}^{(q)}(\beta)| \leqslant 16 e^{-\frac{1}{4}\gamma\beta(1+q)}. \quad (6.34)$$

Proof. Let $F(\varphi) = \tilde{F}(\varphi/\sqrt{\beta}) = e^{\tilde{U}(\varphi/\sqrt{\beta})}$. Then F is 2π even and periodic, see (6.30) and recall $\Omega_\beta^\Lambda = \{\sigma \in \mathbb{Z}_\beta^\Lambda : \sigma_0 = 0\}$. Its Fourier coefficients are given by

$$\hat{F}(q) = \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)e^{iq\varphi}\,d\varphi = \frac{c(\gamma,\beta)}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2\gamma\beta}\varphi^2} e^{iq\varphi}\,d\varphi \overset{(6.31)}{=} e^{-\frac{\gamma\beta}{2}q^2}, \quad q \in \mathbb{Z}. \tag{6.35}$$

Thus (6.32) follows. To prove (6.33), (6.34), consider the following norm on 2π-periodic functions f (for which the norm is finite): for $c = \frac{1}{4}\gamma\beta$, denoting by $\hat{f}(q) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{iq\varphi}\,d\varphi$ the corresponding Fourier coefficients, one sets

$$\|f\|_{\ell^1(c)} = \sum_{q\in\mathbb{Z}} e^{c|q|}|\hat{f}(q)|. \tag{6.36}$$

Using the fact that $\widehat{fg}(q) = \sum_{q'\in\mathbb{Z}} \hat{f}(q)\hat{g}(q'-q)$ for periodic f and g, one readily deduces that $\|\cdot\|_{\ell^1(c)}$ is submultiplicative, i.e. that $\|fg\|_{\ell^1(c)} \leqslant \|f\|_{\ell^1(c)}\|g\|_{\ell^1(c)}$, making the space of 2π-periodic functions with finite norm a unital Banach algebra with unit $f \equiv 1$. Moreover, for $\beta\gamma \geqslant 4$,

$$\|F-1\|_{\ell^1(c)} = 2\sum_{q\geqslant 1} e^{-\frac{\gamma\beta}{2}q^2+cq} \leqslant 4e^{-\frac{1}{4}\gamma\beta}, \tag{6.37}$$

where the second inequality follows for instance by completing the square, comparing with a Gaussian integral and applying a standard Gaussian tail estimate. Since $\tilde{U}(\varphi/\sqrt{\beta}) = \log F(\varphi)$, we have

$$\|\tilde{U}(\cdot/\sqrt{\beta})\|_{\ell^1(c)} \leqslant 2\|F-1\|_{\ell^1(c)} \leqslant 8e^{-\frac{1}{4}\gamma\beta}, \tag{6.38}$$

where we have used the estimate $\|\log F\| \leqslant 2\|F-1\|$ which is valid in any (unital) Banach algebra with norm $\|\cdot\|$ if $\|F-1\|$ is small, as follows e.g. by bounding the relevant Taylor remainder. In view of (6.36), this yields that $|\tilde{z}^{(q)}(\beta)| \leqslant 16e^{-\frac{1}{4}\gamma\beta - c|q|}$ for all $q \geqslant 1$ with $\tilde{z}^{(q)}(\beta)$ as defined by (6.33). □

6.2.3 Stiffness renormalisation

The identity (6.29) for the partition function and its extension to the moment generating function in Lemma 6.2.3 below reformulates the analysis of the Discrete Gaussian model in terms of a smooth periodic potential integrated against a Gaussian field. Ideas of this flavour have been used in various contexts in the past. However, to achieve sufficient precision to control the scaling limit, it is crucial for our work to allow for the parameter $s \neq 0$ below, which will correspond to the stiffness renormalisation of the limiting Gaussian free field, or equivalently, the exact identification of the effective temperature β_{eff} in Theorem 1.1.1.

To set up this stiffness renormalisation, first recall that Δ_J is the normalised Laplacian, see (1.8), with step distribution J. For convenience, we will denote by Δ without subscript the standard *unnormalised* nearest-neighbour Laplacian; the irrelevant omission of the normalisation for Δ simplifies some formulas later. For $|s|$ sufficiently small, $C(m^2)^{-1} - s\Delta$ is positive definite, as shown in Proposition 2.1.4 below (see in particular (2.15), where $C \equiv C^{\Lambda_N}$), hence

$$C(s,m^2) = (C(m^2)^{-1} - s\Delta)^{-1}, \tag{6.39}$$

is well-defined and positive-definite. We then introduce, for $s_0 \in \mathbb{R}$,

$$Z_0^0(\varphi) \equiv Z_0^0(\varphi|\Lambda) = e^{U_0(\varphi)} \stackrel{\text{def.}}{=} e^{\frac{s_0}{2}(\varphi,-\Delta\varphi)+\sum_x \tilde{U}(\varphi_x)} \tag{6.40}$$

with $\tilde{U}$ given by (6.30). We return to discuss the interpretation of Z_0^0 below the proof of Proposition 6.1.1.

The following lemma generalises the partition function identity (6.29), both by allowing a test function and by allowing the parameter $s \neq 0$ that will later correspond to the stiffness renormalisation.

Lemma 6.2.3. *For all $\beta > 0$, $\gamma \in (0,1/3)$, $m^2 \in (0,1]$, $\omega \in \mathbb{C}$, s sufficiently small and $\mathfrak{f} \in \mathbb{C}^\Lambda$,*

$$\left\langle e^{\omega(\mathfrak{f},\sigma)} \right\rangle_{\beta,m^2}^{\Lambda_N} = e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}})} \frac{\mathbb{E}_{C(s,m^2)}\left[e^{\omega(\tilde{\mathfrak{f}},\varphi)} Z_0^0(\varphi + \gamma\omega\mathfrak{f})\right]}{\mathbb{E}_{C(s,m^2)}\left[Z_0^0(\varphi)\right]} \tag{6.41}$$

where $\tilde{\mathfrak{f}} = (1 + s\gamma\Delta)\mathfrak{f}$ and

$$Z_0^0(\varphi) = \exp\Big(\frac{1}{2}s_0|\nabla\varphi|_\Lambda^2 + \sum_{x\in\Lambda}\sum_{q\geqslant 1} z_0^{(q)} \cos\big(q\beta^{1/2}\varphi(x)\big)\Big) \tag{6.42}$$

with $s_0 = s$ and $z_0^{(q)} = \tilde{z}^{(q)}$ (which are defined by (6.33)*).*

Proof. Completing the square and recalling (6.30), one sees that for any $\varphi \in \mathbb{R}$ and $a \in \mathbb{C}$,

$$\sum_{\sigma\in\mathbb{Z}_\beta} e^{a\sigma} e^{-\frac{1}{2\gamma}(\sigma-\varphi)^2} = e^{\frac{\gamma}{2}a^2} e^{a\varphi} \sum_{\sigma\in\mathbb{Z}_\beta} e^{-\frac{1}{2\gamma}(\sigma-\varphi-\gamma a)^2} \propto e^{\frac{\gamma}{2}a^2} e^{a\varphi} e^{\tilde{U}(\varphi+\gamma a)}. \tag{6.43}$$

Using the convolution identity (6.27), for any $f \in \mathbb{C}^\Lambda$, one therefore obtains that

$$\sum_{\sigma\in\mathbb{Z}_\beta^\Lambda} e^{-\frac{1}{2}(\sigma,(-\Delta_J+m^2)\sigma)} e^{\omega(\mathfrak{f},\sigma)} \propto e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\mathfrak{f})} \mathbb{E}_{C(m^2)}[e^{\omega(\mathfrak{f},\varphi)} e^{\tilde{U}(\varphi+\gamma\omega\mathfrak{f})}]. \tag{6.44}$$

By definition of $C(s,m^2)$, see (6.39), the right-hand side is proportional to

$$\begin{aligned}
&e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\mathfrak{f})}\mathbb{E}_{C(s,m^2)}[e^{\omega(\mathfrak{f},\varphi)}e^{\frac{s}{2}(\varphi,-\Delta\varphi)}e^{\tilde{U}(\varphi+\gamma\omega\mathfrak{f})}] \\
&= e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\mathfrak{f})-\frac{s}{2}\gamma^2\omega^2(\mathfrak{f},-\Delta\mathfrak{f})}\mathbb{E}_{C(s,m^2)}[e^{\omega(\tilde{\mathfrak{f}},\varphi)}e^{\frac{s}{2}(\varphi+\gamma\omega\mathfrak{f},-\Delta(\varphi+\gamma\omega\mathfrak{f}))}e^{\tilde{U}(\varphi+\gamma\omega\mathfrak{f})}] \\
&= e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}})}\mathbb{E}_{C(s,m^2)}[e^{\omega(\tilde{\mathfrak{f}},\varphi)}Z_0^0(\varphi+\gamma\omega\mathfrak{f})] \qquad (6.45)
\end{aligned}$$

where in the second line we again completed the square and used that $s=s_0$ along with (6.42).

□

Finally, we take limit $m^2\downarrow 0$ using Lemma 3.1.2 and notation

$$\tilde{C}(s)=\sum_{j=1}^{N-1}\Gamma_j(s)+\Gamma_N^{\Lambda_N}(s)=\lim_{m^2\downarrow 0}\sum_{j=1}^{N-1}\Gamma_j(s,m^2)+\Gamma_N^{\Lambda_N}(s,m^2) \qquad (6.46)$$

Proof of Proposition 6.1.1. By Lemma 6.2.1 and Lemma 6.2.3,

$$\langle e^{\beta^{-1/2}\omega(\mathfrak{f},\sigma)}\rangle_{J,\beta}^{\Lambda_N}=\lim_{m^2\downarrow 0}\langle e^{\omega(\mathfrak{f},\sigma)}\rangle_{\beta,m^2,\Lambda_N}=\lim_{m^2\downarrow 0}e^{\frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}})}\frac{\mathbb{E}_{C(s,m^2)}\left[e^{\omega(\varphi,\tilde{\mathfrak{f}})}Z_0^0(\varphi+\gamma\omega\mathfrak{f})\right]}{\mathbb{E}_{C(s,m^2)}\left[Z_0^0(\varphi)\right]} \qquad (6.47)$$

Since we have decomposition $\varphi=\varphi'+\zeta^{(m^2)}$ for some independent Gaussian random variables $\varphi'\sim\mathcal{N}(0,t_NQ_N)$, $\zeta^{(m^2)}\sim\mathcal{N}(0,\sum_{j=1}^{N-1}\Gamma_j(s,m^2)+\Gamma_N^{\Lambda_N}(s,m^2))$, and since $t_NQ_N\tilde{\mathfrak{f}}\equiv 0$, Lemma 3.1.2 implies

$$\frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta^{(m^2)}}[e^{\omega(\tilde{\mathfrak{f}},\zeta^{(m^2)})}Z_0^0(\varphi'+\zeta^{(m^2)}+\gamma\omega\mathfrak{f})]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta^{(m^2)}}[Z_0^0(\varphi'+\zeta^{(m^2)})]}\sim\frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta}[e^{\omega(\tilde{\mathfrak{f}},\zeta)}Z_0^0(\varphi'+\zeta+\gamma\omega\mathfrak{f})]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta}[Z_0^0(\varphi'+\zeta)]} \qquad (6.48)$$

as $m^2\downarrow 0$ (for $\zeta\sim\mathcal{N}(0,\tilde{C}(s))$).

□

To conclude this section, we briefly discuss the role of U_0 and $\tilde{C}(s)$ introduced in (6.40),(6.46). Compared to $\tilde{U}$ the potential U_0 includes an additional Dirichlet energy term $(\varphi,-\Delta\varphi)=(\nabla\varphi,\nabla\varphi)$ with prefactor s_0. This parameter s_0 can be essentially arbitrary for the moment and is compensated by the s-dependence of $\tilde{C}(s)$ on F_{N,m^2} of (6.4) when $s=s_0$ as in the assumption of the last lemma. Thus the parameter $s=s_0$ corresponds to a division of the Gaussian free field into a part that serves as reference measure, i.e., the Gaussian measure with covariance $\tilde{C}(s)+t_NQ_N$, and a part that is interpreted as a perturbation of it. A careful choice of this division will be made at the end of the analysis. This choice will be such that the covariance $\tilde{C}(s)$ is that of a limiting Gaussian field that approximates the Discrete Gaussian model on large scales and that converges to the multiple of the Gaussian free field in Theorem 1.1.1. Namely, if $f \in C^\infty(\mathbb{T}^2)$ and f_N is as in the statement of Theorem 1.1.1, then

$$\lim_{N\to\infty} (f_N, \tilde{C}(s) f_N) = \frac{1}{s+v_J^2} (f, (-\Delta_{\mathbb{T}^2})^{-1} f)_{\mathbb{T}^2}, \tag{6.49}$$

(see Lemma 9.1.3, where we look at a slightly modified covariance) and our choice of s depending J and β will be precisely such that $\beta_{\mathrm{eff}}(J,\beta) = \beta(1+sv_J^{-2})$, where $\beta_{\mathrm{eff}}(J,\beta)$ is the effective temperature in Theorem 1.1.1.

6.3 Bulk renormalisation group map

The present section is at the heart of the argument. We define a suitable renormalisation group map Φ^0_{j+1} from scale j to scale $j+1$, which corresponds to integrating out the covariance Γ_{j+1}, and exhibits Theorem 6.3.5 (algebraic property) and Theorems 6.1.2, 6.1.3 (analytic properties). These are the only features which will be needed in the sequel and, roughly speaking, will allow to perform a suitable fixed-point argument in the next chapter. The map Φ^0_{j+1} has two components, one describing the evolution of coupling constants, and one describing that of the remainder coordinate. The latter is an evolution on polymer activities, whose growth will be controlled in terms of the norms introduced in Section 3.2. The estimates corresponding to these two components appear separately in Theorems 6.1.2 and 6.1.3. The actual definition of the remainder coordinate (Definition 6.3.7) involves the localisation operator introduced in Section 5.2, which is used to extract the relevant terms. The most involved part, which occupies most of this section, is to obtain the relevant bounds for the resulting remainder coordinate, and in particular for its non-linear part, cf. (6.13), but we have done most of the work in Section 4.4.

The parameter s was arbitrary in Proposition 6.1.1 (provided $|s| \leqslant \varepsilon_s \theta_J$). A careful choice will be necessary in the analysis of the stable manifold of the renormalisation group map (in Chapter 7), but in the present section the parameter does not play an important role. We will therefore usually leave the s-dependence implicit in our notation. Thus all definitions in this section do implicitly depend on s, but all estimates will be uniform in $|s| \leqslant \varepsilon_s \theta_J$. Thoughout this section, the distribution J is allowed to be any finite-range step distribution that is invariant under lattice symmetries (cf. above (1.8)) and we assume (3.1), which is no loss of generality.

Each RG map for $j+1<N$ only depends on the massless covariance $\Gamma_j(s)$, so the RG flow does not feel the mass regularisation effect in these intermediate scales.

6.3.1 Coordinates for the renormalisation group map

The initial condition for the renormalisation group map is the interaction function $Z_0^0(\varphi|\Lambda)$. This function will eventually be chosen as in (6.40) with $s_0 = s$ and s chosen carefully, but we allow $s_0 \neq s$ for the RG map. Given such a function $Z_0^0(\varphi|\Lambda)$, we define Z_j^0 (with $\Lambda = \Lambda_N$) as in Chapter 4:

$$Z_{j+1}^0(\varphi';\omega|\Lambda) = \mathbb{E}\theta_\zeta Z_j^0(\varphi'|\Lambda), \qquad (j \leqslant N-1,\, \varphi \in \mathbb{R}^\Lambda), \tag{6.50}$$

(recall that $\mathbb{E}$ integrates the Gaussian field ζ with covariance $\Gamma_{j+1} = \Gamma_{j+1}(s)$). Then the renormalisation group map is defined to parametrise the polymer expansions

$$Z_{j'}^0(\varphi|\Lambda) = e^{-E_{j'}|\Lambda|}(e^{U_{j'}} \circ_j K_{j'}^0)(\Lambda,\varphi), \qquad j' \in \{j, j+1\}. \tag{6.51}$$

A careful inductive choice of E_j, U_j and K_j^0 for the representation (6.51) will later constitute the bulk (or unperturbed) renormalisation group flow.

For the remainder of this section, we merely specify general conditions that we impose on the form of U_j and K_j and how to measure their size. The coordinate U_j is an explicit leading part that is defined in terms of coupling constants (s_j, z_j) as follows.

Definition 6.3.1. *The coordinate U_j given by* (3.50) *is parametrised in terms of the coupling constants (s_j, z_j) where $s \in \mathbb{R}$ and $z_j = (z_j^{(q)})_{q\geqslant 1}$ is itself a sequence of real coupling constants:*

$$\begin{aligned} U_j(X,\varphi) &= \frac{1}{2}s_j|\nabla\varphi|_X^2 + W_j(X,\varphi) \\ W_j(X,\varphi) &= \sum_{x\in X}\sum_{q\geqslant 1} L^{-2j} z_j^{(q)} \cos(\beta^{1/2} q\varphi(x)), \end{aligned} \tag{6.52}$$

We will always identify U_j with the coupling constants (s_j, z_j) and use the norm

$$\|U_j\|_j \equiv \|U_j\|_{\Omega_j^U} = A \max\left\{|s_j|, \sup_{q\geqslant 1} e^{c_f \beta q}|z_j^{(q)}|\right\} \tag{6.53}$$

(recall Definition 3.2.6 for $\|U_j\|_{\Omega_j^U}$) for $c_f = \frac{1}{4}\gamma$, where the constant γ is the one from Proposition 2.1.2. Also recall that Ω_j^U is the Banach space of such U_j with finite $\|\cdot\|_{\Omega_j^U}$-norm.

The quantity K_j^0 is a remainder coordinate on whose form we only impose the following generic conditions. Note that this includes in particular the important component factorisation property (3.29) which is implied by Definition 3.2.1.

Definition 6.3.2. *The coordinate K_j^0 is a polymer activity (see Definition 3.2.1) satisfying the following.*

- *The periodicity condition $K_j^0(X,\varphi) = K_j^0(X,\varphi + 2\pi\beta^{-1/2}n)$ for any $n \in \mathbb{Z}$.*
- *Even and invariant under the lattice symmetries (see Definition 5.1.3) and $K_j^0(\cdot,\varphi) = K_j^0(\cdot,-\varphi)$.*

For such polymer activitives K_j^0 we use the norm (3.45)*, i.e.,*

$$\|K_j^0\|_j \equiv \|K_j^0\|_{\Omega_{j,0}^K} = \|K_j^0\|_{2h,T_j,A}, \tag{6.54}$$

with

$$h = \max\{c_f^{1/2}, rc_h\rho_J^{-2}\sqrt{\beta}, \rho_J^{-1}\}, \tag{6.55}$$

where $r \in (0,1]$, c_f is as in (3.51) *and c_h is chosen by* (5.45)*. Let $\Omega_{j,0}^K$ be the Banach space of polymer activies K_j^0 with finite $\|\cdot\|_{\Omega_{j,0}^K}$-norm.*

Remark 6.3.3. We will take β and r such that

$$\beta \geqslant 32\max\{c_f^{-1}, c_f^{-2}\}, \qquad \rho_J^2 \geqslant \sqrt{32}rc_hc_f^{-1}. \tag{6.56}$$

Then the choice of $\mathfrak{h} = 4h$ satisfies the assumptions of Lemma 3.2.8. Thus

$$\|W_j(B,\varphi)\|_{4h,T_j(B,\varphi)} \leqslant CA^{-1}\|W_j\|_{\Omega_j^U}. \tag{6.57}$$

Indeed, if $\mathfrak{h} = 4c_f^{1/2}$, then $2\mathfrak{h}^2c_f^{-2} = 32c_f^{-1} \leqslant \beta$, if $\mathfrak{h} = 4rc_h\rho_J^{-2}\sqrt{\beta}$, then $2\mathfrak{h}^2c_f^{-2} \leqslant 32r^2c_h^2\rho_J^{-4}\beta \leqslant \beta$ and if $\mathfrak{h} = 4\rho_J^{-1}$, then $2\mathfrak{h}^2c_f^{-2} \leqslant 32c_f^{-2} \leqslant \beta$.

Since c_h and c_f are absolute constants, the conditions (6.56) can be achieved either by taking r small enough with ρ_J fixed or ρ_J large enough with $r = 1$. Note that by Proposition 5.2.3 (observe that all of its assumptions hold) and the discussion below its statement, in particular the second term in the definition of α_{Loc} in (5.16) indicates that the price to pay for having r small is to take β sufficiently large so that $e^{-\frac{1}{2}r\beta\Gamma_{j+1}(0)} < L^{-2}$ (which we will later need). We will eventually impose one of these choices of parameters; this choice occurs in the proof of Corollary 7.3.3.

Finally, we define the norm on the product space of (U_j, K_j) as follows.

Definition 6.3.4. *Let $\Omega_{j,0} = \Omega_j^U \times \Omega_{j,0}^K$ with norm*

$$\|\omega_j\|_{j,0} \equiv \|\omega_j\|_{\Omega_{j,0}} = \max\{\|U_j\|_{\Omega_j^U}, \|K_j^0\|_{\Omega_{j,0}^K}\}. \tag{6.58}$$

Ultimately we will choose $W_0(X,\varphi) = \sum_{x\in X} \tilde{U}(\varphi_x)$ with $\tilde{U}$ as in (6.30), i.e., with $z_0^{(q)} = \tilde{z}^{(q)}$ as in Lemma 6.2.2. Then Lemma 6.2.2 implies

$$\|W_0\|_{\Omega_0^U} \leqslant 16Ae^{-c_f\beta}, \qquad c_f = \frac{1}{4}\gamma. \tag{6.59}$$

6.3.2 Introduction to the renormalisation group map

Due to the flexibility of the polymer expansion, there are many choices of maps that act on the renormalisation group coordinates $(E_j, s_j, z_j, K_j^0) \mapsto (E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}^0)$ such that (6.50)–(6.52) hold. The renormalisation group map corresponds to a careful choice in which the remainder coordinates K_j^0 contract from scale to scale in an appropriate sense, while the evolution of the coordinates U_j can be analysed explicitly. Such a choice of the renormalisation group map

$$\Phi_{j+1}^0 : (E_j, s_j, z_j, K_j^0) \mapsto (E_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}^0) \tag{6.60}$$

is explicitly given in Definitions 6.3.6–6.3.7 below. Note that, throughout the chapter, Φ_j^0 depends implicitly on Λ_N and $0 \leqslant j < N-1$ (but see Section 7.2 for its infinite-volume extension). The precise choice of the definition of Φ_{j+1}^0 is not significant for later sections, however, save for certain key properties that follow from this definition, which was stated in Theorem 6.1.2, 6.1.3 and will be stated below in Theorem 6.3.5. Any definition that implies these properties would have been equally good.

We briefly set up some convenient notation. In what follows, we either denote the components of the map Φ_{j+1}^0 by $(E_j + \mathscr{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathscr{K}_{j+1}^0)$ or by $(E_j + \mathscr{E}_{j+1}, \mathscr{U}_{j+1}, \mathscr{K}_{j+1}^0)$ where $\mathscr{U}_{j+1} = (\mathfrak{s}_{j+1}, \mathfrak{z}_{j+1})$. Note that the coupling constant E_j contributes to (6.51) only by a φ-independent factor, and therefore its influence on (6.50) is trivial. As indicated above, we will thus assume that $E_j = 0$ in the definition of Φ_{j+1}^0, and that the definition is then extended to general E_j by setting $\mathscr{E}_{j+1}(E_j, s_j, z_j, K_j^0) = \mathscr{E}_{j+1}(0, s_j, z_j, K_j^0)$, $\mathfrak{s}_{j+1}(E_j, s_j, z_j, K_j^0) = \mathfrak{s}_{j+1}(0, s_j, z_j, K_j^0)$, and analogously for the other components. To emphasise the dependence on Λ_N, we will sometimes write Φ_{j+1}^{0,Λ_N} and $\mathscr{K}_{j+1}^{0,\Lambda_N}$ instead of Φ_{j+1}^0 and $\mathscr{K}_{j+1}^0$. Whenever we write only a subset of the arguments (E_j, s_j, z_j, K_j^0) below, we implicitly mean that the given map is a function of these arguments alone. For instance, $\mathfrak{s}_{j+1}(s_j, K_j^0)$ means that $\mathfrak{s}_{j+1}$ is a function of (s_j, K_j^0).

The following theorem, along with Theorem 6.1.2, 6.1.3 refers to the map Φ^0_{j+1} introduced below in Definitions 6.3.6–6.3.7 and exhibit its salient features.

Theorem 6.3.5 (Algebraic properties)**.** *The renormalisation group map Φ^0_{j+1} is consistent with* (6.50)–(6.51)*, i.e., if Z^0_j has the form* (6.51) *at scale j with parameters (E_j, s_j, z_j, K^0_j) then Z^0_{j+1} defined by* (6.50) *has this form at scale $j+1$ with $(E_{j+1}, s_{j+1}, z_{j+1}, K^0_{j+1}) = \Phi^0_{j+1}(E_j, s_j, z_j, K^0_j)$. Moreover, if K^0_j is a scale-j polymer activity (see Definition 3.2.1) that is even and invariant under lattice symmetries (see Definition 5.1.3) and is ($2\pi\beta^{-1/2}$-)periodic, then K^0_{j+1} is a scale-($j+1$) polymer activity with the same properties.*

The remainder of this section is concerned with the definition of the renormalisation group map and the proof of the above theorems.

6.3.3 Definition of the renormalisation group map

The first definition concerns the coupling constants (E_j, s_j, z_j). These are given by first order perturbation theory, plus a correction from the remainder coordinate K^0_j, which involves its localisation as introduced in Section 5.2.

Definition 6.3.6. *For U_j of the form* (6.52)*, define the map $(\mathscr{E}_{j+1}, \mathscr{U}_{j+1}) : (U_j, K^0_j) \mapsto (E_{j+1}, U_{j+1})$ to be the unique solution of*

$$\begin{aligned} -\mathscr{E}_{j+1}(U_j, K^0_j)|B| + \mathscr{U}_{j+1}(U_j, K^0_j, B, \varphi') \\ = \mathbb{E}U_j(B, \varphi' + \zeta) + \sum_{X \in \mathscr{S}_j : X \supset B} \mathrm{Loc}_{X,B}\, \mathbb{E}K^0_j(X, \varphi' + \zeta), \end{aligned} \tag{6.61}$$

where $B \in \mathscr{B}_j$ is any scale-j block, $\mathscr{E}_{j+1}(U_j, K^0_j) \in \mathbb{R}$ and $\mathscr{U}_{j+1}(U_j, K^0_j) \in \Omega^U_j$. For general $X \in \mathscr{P}_j$, the definition extends by setting

$$\mathscr{U}_{j+1}(X) = \sum_{B \in \mathscr{B}_j(X)} \mathscr{U}_{j+1}(B). \tag{6.62}$$

That $(\mathscr{E}_{j+1}, \mathscr{U}_{j+1})$ is well-defined via (6.61), i.e., that the right-hand side of (6.61) can be uniquely written in the form of the left-hand side, follows by explicitly evaluating the Gaussian expectation in the first term and by Proposition 5.2.2 for the sum over $\mathrm{Loc}_{X,B}$, as will become apparent in the proof of Theorem 6.1.2 below.

The following definition gives the evolution of the remainder coordinate K^0_j, where we recall $\mathbb{K}_{j+1}$ from Definition 4.2.1.

Definition 6.3.7. *Given* $(U_j, K_j^0) \in \Omega_{j,0}$*,* $(\mathscr{E}_{j+1}, \mathscr{U}_{j+1})$ *given by Definition 6.3.6 and* $\overline{\mathscr{U}}^0_{j+1}(X) = -\mathscr{E}_{j+1}|X| + \mathscr{U}_{j+1}(X)$*, we define*

$$Q_j^0(D,Y,\varphi') = 1_{Y \in \mathscr{S}_j} \operatorname{Loc}^{(2)}_{Y,D} \mathbb{E}_{(\omega)}[K_j^0(Y, \varphi' + \zeta)] \tag{6.63}$$

and $\mathscr{K}^0_{j+1} : (U_j, K_j^0) \mapsto K^0_{j+1}$ *by*

$$\mathscr{K}^0_{j+1}(U_j, K_j^0; X, \varphi) = \mathbb{K}_{j+1}(U_j, K_j^0, \mathscr{U}_{j+1}, \overline{\mathscr{U}}_{j+1}, Q_j^0; X, \varphi). \tag{6.64}$$

We prove Theorem 6.3.5 and Theorem 6.1.2 first, which can be seen from the definitions without much difficulty.

Proof of Theorem 6.3.5. That Z^0_{j+1} satisfies

$$Z^0_{j+1}(\cdot|\Lambda_N) = e^{-(E_j + \mathscr{E}_{j+1})|\Lambda_N|}\big(e^{\mathscr{U}_{j+1}} \circ_{j+1} K^0_{j+1}\big)(\Lambda_N) \tag{6.65}$$

follows from Proposition 4.2.2. The remark on the symmetries of K^0_{j+1} follows from the definition of $\mathscr{K}^0_{j+1}$, as the reexpansion map $\mathbb{K}_{j+1}$ does not affect any of the indicated symmetries. □

Proof of Theorem 6.1.2. By evaluating the expectation $\mathbb{E}U_j$ on the right-hand side of (6.61) explicitly using (6.52), the fact that ζ is centred and invariant under lattice rotations and (5.42),

$$\begin{aligned} \mathbb{E}U_j(B, \varphi' + \zeta) &= \frac{1}{2}s_j\big(|\nabla\varphi'|^2_B + \mathbb{E}|\nabla\zeta|^2_B\big) + \sum_{x\in B}\sum_{q\geqslant 1} L^{-2j} z_j^{(q)} \cos(\beta^{1/2} q\varphi'(x))\mathbb{E}[e^{i\sqrt{\beta}q\zeta(x)}] \\ &= \frac{1}{2}s_j\Big(|\nabla\varphi'|^2_B + |B|\sum_{\sigma=\pm}\mathbb{E}\big[(\zeta(x_0+\sigma e_1) - \zeta(x_0))^2\big]\Big) + W_{j+1}(B,\varphi') \end{aligned} \tag{6.66}$$

with $z^{(q)}_{j+1} = \mathfrak{z}^{(q)}_{j+1}(z_j)$ implicit in W_{j+1} given by

$$\mathfrak{z}^{(q)}_{j+1}(z_j^{(q)}) = L^2 e^{-\frac{1}{2}\beta q^2 \Gamma_{j+1}(0)} z_j^{(q)} \tag{6.67}$$

as declared in Theorem 6.1.2. Hence, combining (6.66) with (5.13) and (5.15), it follows that the right-hand side of (6.61) corresponds to the change of coupling constants

$(\mathscr{E}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}) : (s_j, z_j, K_j^0) \mapsto (E_{j+1}s_{j+1}, z_{j+1})$ given by (6.67) and

$$\mathfrak{s}_{j+1}(s_j, K_j^0) = s_j + O\big(A^{-1}h^{-2}\|K_j^0\|_{\Omega^K_{j,0}}\big) \tag{6.68}$$

$$\mathscr{E}_{j+1}(s_j, K_j^0) = -s_j\nabla^{(e_1,-e_1)}\Gamma_{j+1}(0) + O(L^{-2j}A^{-1}\|K_j^0\|_{\Omega^K_{j,0}}) \tag{6.69}$$

where

$$\nabla^{(e_1,-e_1)}\Gamma_{j+1}(0) = \frac{1}{2}\sum_{\sigma}\mathbb{E}[(\zeta(x_0+\sigma e_1) - \zeta(x_0))^2] = -\Gamma_{j+1}(e_1) - \Gamma_{j+1}(-e_1) + 2\Gamma_{j+1}(0). \tag{6.70}$$

The conclusions (6.9) and (6.10) are different ways to state (6.68) and (6.69).

We now argue that the asserted continuity properties in the implicit parameter s hold. With regards to $\mathfrak{z}^{(q)}_{j+1}$, this is immediate by (6.67) and the continuity of $s \mapsto \Gamma_{j+1}(s)$, cf. Proposition 2.1.2,(ii). Next, referring to Proposition 5.2.2, we have $\mathfrak{s}_{j+1}(s_j, K_j^0) = s_j + \mathfrak{s}_{j+1}(0, K_j^0)$ and $\mathscr{E}_{j+1}(s_j, K_j^0) = -s_j\nabla^{(e_1,-e_1)}\Gamma_{j+1}(0) + \mathscr{E}_{j+1}(0, K_j^0)$, whereby $\mathscr{E}_{j+1}(0, K_j^0) = -\overline{E}(K_j^0)$ and $\mathfrak{s}_{j+1} = \overline{s}(K_j^0)$. Thus, Proposition 5.2.2 immediately yields that $\mathscr{E}_{j+1}(0, K_j^0)$ and $\mathfrak{s}_{j+1}(0, K_j^0)$ are both continuous in the implicit parameter s whenever $\|K_j^0\|_{\Omega^K_{j,0}} < \infty$. The claim follows. □

The proof of Theorem 6.1.3 occupies the remainder of the chapter. More precisely, in Section 6.3.4 we find the explicit expression of $\mathscr{L}^0_{j+1}$ and prove its bound, and in Sections 6.4, we bound the nonlinear part $\mathscr{M}^0_{j+1}$.

6.3.4 Proof of Theorem 6.1.3: bound on the linear part

We have already observed in Section 4.2 that we can extract out the linear part of $\mathscr{K}^0_{j+1}$ (also see Section 4.4) and it has form

$$\begin{aligned}\mathscr{L}^0_{j+1}(U_j, K_j^0; X, \varphi') := \sum_{Y:\overline{Y}=X}&\Big(1_{Y\in\mathscr{P}_j^c}\mathbb{E}K_j^0(Y, \varphi'+\zeta) - 1_{Y\in\mathscr{S}_j}\sum_{D\in\mathscr{B}_j(Y)} Q_j^0(D, Y, \varphi')\Big)\\ &+ \sum_{D\in\mathscr{B}_j}^{\overline{D}=X}\Big(\mathbb{E}[U_j(D, \varphi'+\zeta)] + \mathscr{E}_{j+1}|D|\\ &\qquad - \mathscr{U}_{j+1}(D, \varphi') + \sum_{Y\in\mathscr{S}_j}^{D\in\mathscr{B}_j(Y)} Q_j^0(D, Y, \varphi')\Big).\end{aligned} \tag{6.71}$$

By the choice of $(\mathscr{E}_{j+1}, \mathscr{U}_{j+1}, Q_j^0)$ from Definition 6.3.6–6.3.7, we see that the second summation vanishes and the first line becomes

$$\mathscr{L}_{j+1}^0(U_j, K_j^0; X, \varphi') = \sum_{b=1,3} \mathscr{L}_{j+1}^{0,(b)}(K_j^0; X, \varphi') \tag{6.72}$$

where

$$\mathscr{L}_{j+1}^{0,(1)} = \sum_{Y:\overline{Y}=X} 1_{Y \in \mathscr{S}_j}(1 - \mathrm{Loc}_Y^{(2)})\mathbb{E}\theta_\zeta K_j^0(Y, \varphi') \tag{6.73}$$

$$\mathscr{L}_{j+1}^{0,(3)} = \mathbb{S}\big[1_{Y \in \mathscr{P}_j^c \setminus \mathscr{S}_j}\mathbb{E}[\theta_\zeta K_j^0(Y, \varphi')]\big], \tag{6.74}$$

and we recall $\mathbb{S}$ from Section 4.1.3. We have omitted $b = 2$ to make the notation consistent with what comes in Chapter 8. The bound on $\mathscr{L}_{j+1}^0$ is obtained by bounding each $\mathscr{L}_{j+1}^{0,(b)}$.

Proof of Theorem 6.1.3,(i). By Proposition 5.2.3 (ii),

$$\begin{aligned}
\|\mathscr{L}_{j+1}^{0,(1)}(K_j^0; X)\|_{2h, T_{j+1}(X)} &\leqslant C \sum_{Y:\overline{Y}=X} 1_{Y \in \mathscr{S}_j} \alpha_{\mathrm{Loc}} A^{-|Y|_j} \|K_j^0\|_{\Omega_{j,0}^K} \\
&\leqslant C' A^{-|X|_{j+1}} L^2 \alpha_{\mathrm{Loc}} \|K_j^0\|_{\Omega_{j,0}^K}
\end{aligned} \tag{6.75}$$

where in the second inequality, we have used that $|Y|_j \geqslant |X|_{j+1}$ and that there are at most $O(L^2)$ number of small polymers $Y \in \mathscr{S}_j$ such that $\overline{Y} = X$.

For $\mathscr{L}_{j+1}^{0,(3)}$, we use Proposition 4.1.5 to obtain

$$\|\mathscr{L}_{j+1}^{0,(3)}(K_j^0; X)\|_{2h, T_{j+1}(X)} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}} \|K_j^0\|_{\Omega_{j,0}^K}. \tag{6.76}$$

□

6.4 Proof of Theorem 6.1.3: bound on the non-linear part

Bound on $\mathscr{M}_{j+1}^0 = \mathscr{K}_{j+1}^0 - \mathscr{L}_{j+1}^0$ is not of triviality, but thanks to Lemma 4.4.2, the bound reduces to the following lemma, where we recall $\overline{\mathfrak{K}}_j$ from (4.73)–for the bulk RG flow, it becomes

$$\overline{\mathfrak{K}}_j^0(\omega_j^0) = (\mathscr{E}_{j+1}|X|, U_j, \overline{U}_{j+1}^0, K_j^0, \overline{K}_j^0, Q_j^0)(\omega_j^0) \tag{6.77}$$

for $\omega_j^0 = (U_j, K_j^0)$ and

$$\overline{U}_{j+1}^0(X,\varphi') = -\mathscr{E}_{j+1}|X| + U_{j+1}(X,\varphi') \tag{6.78}$$

and we also recall $\mathscr{X}_j^{\mathfrak{K}}$ from Definition 4.4.1. For notational sake, we write $s_{j+1}, z_{j+1}, U_{j+1}$ for $\mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathscr{U}_{j+1}$.

Lemma 6.4.1. *Under the assumptions of Theorem 6.1.3, for any $\delta > 0$ and parameters satisfying* (6.56)*, there exists $\varepsilon(L) > 0$ only polynomially small in L, and constants $C(\delta,L) \equiv C(\delta,\beta,L)$, $C(L) \equiv C(\beta,L)$, $C(A,L)$, $\varepsilon(\delta,L) \equiv \varepsilon(\delta,\beta,L)$ and $\eta > 0$ such that if $\mathscr{X}_j^{\mathfrak{K}}(\cdot)$ is defined with these δ, η, $C(\delta,L)$, $C(L)$, $C(A,L)$ then $\overline{\mathfrak{K}}_j^0$ is in $\mathscr{X}_j^{\mathfrak{K}}(\{\omega_j^0 \in \Omega_{j,0} : \|\omega_j^0\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,L)\})$.*

We defer the proof of the lemma to Section 6.4.1 and first complete the proof of Theorem 6.1.3 (ii).

Proof of Theorem 6.1.3 (ii). The continuous differentiability of $\mathscr{M}_{j+1}^0$ is a direct consequence of Lemma 4.4.2 applied with $\delta > 0$ sufficiently small, $\mathbb{X} = \{\omega_j^0 \in \Omega_{j,0} : \|\omega_j^0\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,\beta,L)\}$, $\mathfrak{K}_j = \overline{\mathfrak{K}}_j^0$, and the decomposition $\mathscr{M}_{j+1}^0 = \sum_{k=1}^4 \mathfrak{M}_{j+1}^{0,(k)}$ from (4.74), with the assumptions of Lemma 4.4.2 being verified by Lemma 6.4.1. The bound (6.13) is obtained by summing (4.87) for $k = 1,2,3,4$, so

$$\|\mathscr{M}_{j+1}^0(U_j, K_j^0)(X)\|_{2h,T_{j+1}(X)} \leqslant CA^{-(1+\eta_0)|X|_{j+1}} \|(U_j, K_j^0)\|_{\Omega_{j,0}}, \tag{6.79}$$

as desired.

□

6.4.1 Proof of Lemma 6.4.1

In this section we prove Lemma 6.4.1, i.e., that $\overline{\mathfrak{K}}_j(\omega_j^0)$ defined above satisfies $\overline{\mathfrak{K}}_j \in \mathscr{X}_j^{\mathfrak{K}}(\Omega_{j,0})$ whenever $\omega_j^0 = (U_j, K_j^0)$ is sufficiently small. Indeed, in Lemma 6.4.2 we verify that (4.81) and (4.82) hold, and in Lemmas 6.4.3–6.4.4 we verify (4.83)–(4.86). We would have to note that $\|\cdot\|_{\vec{\mathfrak{h}},T_j}$ in Definition 4.4.1 should be interpreted as $\|\cdot\|_{2h,T_j}$, since the bulk RG coordinates do not have ω-dependence.

To control the term $\frac{1}{2}|\nabla\varphi|_B^2$ that appears in the expressions to be bounded (cf. for instance (6.52)), the expression (3.71) will appear repeatedly, i.e.,

$$w_j(X,\varphi)^2 = \sum_{D \in \mathscr{B}_j(X)} \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(D^*)}^2, \tag{6.80}$$

see (3.52) for example. We recall that w_j is related to the large field regulator G_j by the inequalities (3.72) and (3.73).

Lemma 6.4.2. *Under the assumptions of Theorem 6.1.3, there exists* $\varepsilon(\delta,\beta,L)>0$ *only polynomially small in* L *and* β *such that the following holds: for any* $\delta>0$, *suppose* $\|\omega_j^0\|_{j,0} := \|(U_j,K_j^0)\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,\beta,L)$. *Then* (4.81), (4.82) *hold with* $\mathfrak{h}=2h$, *i.e.,*

$$\|\mathfrak{U}(B,\varphi)\|_{2h,T_j(B,\varphi)} \leqslant C(\delta,L)(1+\delta c_w \kappa_L w_j(B,\varphi)^2)\|\omega_j^0\|_{j,0} \tag{6.81}$$

$$\|e^{\mathfrak{U}(B,\varphi)} - \sum_{m=0}^{k}\frac{1}{m!}(\mathfrak{U}(B,\varphi))^m\|_{2h,T_j(B,\varphi)} \leqslant C(\delta,L)e^{\delta c_w\kappa_L w_j(B,\varphi)^2}\|\omega_j^0\|_{j,0}^{k+1} \tag{6.82}$$

for $\mathfrak{U}\in\{U_j,\overline{U}_{j+1}^0\}$ *and the same holds when* $\mathfrak{U}=\mathscr{E}_{j+1}$ *but with* $\delta=0$.

Proof. By Remark 6.3.3, Theorem 6.1.2, (3.4) (estimate on $\nabla^{(e_1,-e_1)}\Gamma_{j+1}(0)$) and the choice of W_{j+1} for $j^*\in\{j,j+1\}$ and $B\in\mathscr{B}_{j+1}$,

$$|\mathscr{E}_{j+1}||B|,\ \ \|W_{j^*}(B,\varphi)\|_{2h,T_j(B,\varphi)} \leqslant CA^{-1}L^2\|\omega_j^0\|_{j,0}, \tag{6.83}$$

and

$$\begin{aligned}\|\frac{1}{2}s_{j^*}|\nabla\varphi|_B^2\|_{2h,T_j(B,\varphi')} &\leqslant CA^{-1}\sum_{D\in\mathscr{B}_j(B)}(h^2+w_j(D,\varphi)^2)\|\omega_j^0\|_{j,0}\\ &\leqslant CA^{-1}(L^2h^2+w_j(B,\varphi)^2)\|\omega_j^0\|_{j,0}.\end{aligned} \tag{6.84}$$

Since $h=\max\{c_f^{1/2}, rc_h\rho_J^{-2}\sqrt{\beta},\rho_J^{-1}\}$, by taking $L\geqslant c_f^{-1/2}$ we have $L^2h^2\geqslant 1$. Then for $\mathfrak{U}\in\{U_j,\overline{U}_{j+1}^0\}$

$$\|\mathfrak{U}(B,\varphi)\|_{2h,T_j(B,\varphi)} \leqslant C(\delta)\kappa_L^{-1}L^2h^2\big(1+\delta c_w\kappa_L w_j(B,\varphi)^2\big)\|\omega_j^0\|_{j,0}. \tag{6.85}$$

Now, since $\beta\geqslant 32c_f^{1/2}$, there exists $C>0$ such that $h\leqslant C\sqrt{\beta}$, so for some $C(\delta,\beta,L)$ only polynomially large in β and L,

$$\|\mathfrak{U}(B,\varphi)\|_{2h,T_j(B,\varphi)} \leqslant C(\delta,\beta,L)\big(1+\delta c_w\kappa_L w_j(B,\varphi)^2\big)\|\omega_j^0\|_{j,0}. \tag{6.86}$$

Also using the trivial fact that $1+x\leqslant e^x$ for $x\geqslant 0$,

$$\|\mathfrak{U}(B,\varphi)\|_{2h,T_j(B,\varphi)} \leqslant C(\delta,\beta,L)e^{\delta c_w\kappa_L w_j(B,\varphi)^2}\|\omega_j^0\|_{j,0}. \tag{6.87}$$

This shows (6.81). To deduce (6.82), assume $\|\omega_j^0\|_{j,0} \leqslant \varepsilon(\delta,\beta,L) = \frac{1}{C(\delta,\beta,L)}$. Together with the submultiplicativity (3.64) of the norm, (6.86) then implies

$$\|e^{\mathfrak{U}}\|_{2h,T_j(B,\varphi)} \leqslant e^{\|\mathfrak{U}\|_{2h,T_j(B,\varphi)}} \leqslant e^{1+\delta c_w \kappa_L w_j(B,\varphi)^2} \leqslant C e^{\delta c_w \kappa_L w_j(B,\varphi)^2} \tag{6.88}$$

and furthermore, using (6.87) to bound $(\mathfrak{U})^{k+1}$ for $k \in \{0,1,2\}$,

$$\begin{aligned}\|e^{\mathfrak{U}} - \sum_{m=0}^{k} \frac{1}{m!}(\mathfrak{U})^m\|_{2h,T_j(B,\varphi)} &\leqslant \frac{1}{(k+1)!}\|\mathfrak{U}\|^{k+1}_{2h,T_j(B,\varphi)} e^{\|\mathfrak{U}\|_{2h,T_j(B,\varphi)}} \\ &\leqslant C(\delta,\beta,L)\|\omega_j^0\|^{k+1}_{j,0} \exp\big(4\delta c_w \kappa_L w_j(B,\varphi)^2\big), \end{aligned} \tag{6.89}$$

which is equivalent to the claim, by replacing 4δ by δ. The remark about $\mathscr{E}_{j+1}$ follows from the same computations starting just from (6.83). □

Lemma 6.4.3. *Under the assumptions of Theorem 6.1.3, there exist $c_w > 0$, $\varepsilon \equiv \varepsilon(\beta,L) > 0$ (only polynomially small in β), $C \equiv C(c_w,\beta,L)$, and $C_A \equiv C_A(c_w,\beta,L,A)$ such that the bounds* (4.83), (4.84), (4.86) *hold whenever $\|\omega_j^0\|_{j,0} \leqslant \varepsilon$, i.e.,*

$$\|De^{\mathfrak{U}'(B,\varphi)}\|_{2h,T_j(B,\varphi)} \leqslant C(L) e^{c_w \kappa_L w_j(B,\varphi)^2}, \tag{6.90}$$

$$\|D^2 e^{\mathfrak{U}'(B,\varphi)}\|_{2h,T_j(B,\varphi)} \leqslant C(L) e^{c_w \kappa_L w_j(B,\varphi)^2}, \tag{6.91}$$

$$\|DQ_j^0(D,Y,\varphi)\|_{2h,T_j(Y,\varphi)} \leqslant C(L) e^{c_w \kappa_L w_j(D,\varphi)^2}, \tag{6.92}$$

for any $Y \in \mathscr{S}_j$, $D \in \mathscr{B}_j(Y)$, and $\mathfrak{U}' \in \{U_j, \overline{U}^0_{j+1}, \mathscr{E}_{j+1}\}$, and in the case of $\mathscr{E}_{j+1}$, the factor $e^{c_w \kappa_L w_j(B,\varphi)}$ can be omitted. The derivatives exist in the asserted spaces of polymer activities.

Proof. The twice differentiability of $e^{\mathfrak{U}'}$ is a consequence of Lemma 6.4.2, as we will show in detail below. To start with, we will fix $c_w > 0$ small enough so that Lemma 3.3.4 holds. Let $\|\dot\omega_{j,0}\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,\beta,L)$ for small $\delta > 0$, where $\varepsilon(\delta,\beta,L)$ is as in Lemma 6.4.2. By Lemma 6.4.2 and (6.88),

$$\begin{aligned}\|e^{\mathfrak{U}'(\omega_j+\dot\omega_j,B,\varphi)} - (1+\mathfrak{U}'(\dot\omega_j,B,\varphi))e^{\mathfrak{U}'(\omega_j,B,\varphi)}\|_{2h,T_j(\varphi,B)} \\ \leqslant C(\delta,\beta,L)\|\dot\omega_j\|^2_{j,0}\, e^{2\delta c_w \kappa_L w_j(B,\varphi)^2}\end{aligned} \tag{6.93}$$

so $De^{\mathfrak{U}'(\omega_j,B,\varphi)}(\dot\omega_j) = e^{\mathfrak{U}'(\omega_j,B,\varphi)}\mathfrak{U}'(\dot\omega_j,B,\varphi)$. Moreover, as asserted, the differentiability is uniform in φ after dividing by $G_j(B,\varphi)$ by Lemma 3.3.4, i.e., the derivatives exist in the space of polymer activities. Similarly, for $\|\ddot{\omega}_j\|_{j,0} \leqslant \varepsilon(\delta,\beta,L)$,

$$\|De^{\mathfrak{U}'(\omega_j+\ddot{\omega}_j,B,\varphi)}(\dot{\omega}_j) - (1+\mathfrak{U}'(\ddot{\omega}_j,B,\varphi))De^{\mathfrak{U}'(\omega_j,B,\varphi)}(\dot{\omega}_j)\|_{2h,T_j(\varphi,B)} \\ \leqslant C(\delta,\beta,L)\|\dot{\omega}_j\|_{j,0}\|\ddot{\omega}_j\|_{j,0}^2 e^{2\delta c_w \kappa_L w_j(B,\varphi)^2} \quad (6.94)$$

so $D^2 e^{\mathfrak{U}'(\omega_j,B,\varphi)}(\dot{\omega}_j,\ddot{\omega}_j) = e^{\mathfrak{U}'(\omega_j,B,\varphi)}\mathfrak{U}'(\dot{\omega}_j,B,\varphi)\mathfrak{U}'(\ddot{\omega}_j,B,\varphi)$. It follows that $e^{\mathfrak{U}'}$, $De^{\mathfrak{U}'}$ are differentiable and $De^{\mathfrak{U}'(\omega_j,B,\varphi)}, D^2e^{\mathfrak{U}'(\omega_j,B,\varphi)}$ satisfy the desired bounds again using Lemma 6.4.2 and (6.88). Claims on $\mathscr{E}_{j+1}$ follow from the same principles, but it does not have dependence on φ.

Finally, because of (5.21),

$$\|\mathrm{Loc}^{(2)}_{Y,D}\mathbb{E}_{(\omega)}\theta_\zeta K_j^0(Y,\varphi')\|_{2h,T_j(Y,\varphi')} \leqslant C(\log L)\|K_j^0(Y)\|_{2h,T_j(Y)}e^{c_w\kappa_L w_j(D,\varphi')^2}. \quad (6.95)$$

Since Q_j^0 is a linear function of K_j^0, its differentiability follows from boundedness, and the derivative satisfies (6.92).

$\square$

Lemma 6.4.4. *Under the assumptions of Theorem 6.1.3, there exist $\varepsilon \equiv \varepsilon(\beta,L) > 0$ (only polynomially small in β) and $C(A,L)$ such that* (4.85) *holds whenever $\|\omega_j^0\|_{j,0} \leqslant \varepsilon$, i.e.,*

$$\|D\overline{K}_j^0(Z,\varphi)\|_{2h,T_j(Z,\varphi)} \leqslant C(A,L)A^{-(1+\eta)|Z|_{j+1}}G_j(Z,\varphi) \quad (6.96)$$

for some purely geometric constant $\eta > 0$.

Proof. Recall (4.18) for the definition of $\overline{K}_j^0$. We may rewrite, for $X \in \mathscr{P}_{j+1}^c$,

$$\overline{K}_j^0(X,\varphi) = \sum_{Y\in\mathscr{P}_j^c}^{\overline{Y}=X} e^{U_j(X\setminus Y,\varphi)}K_j^0(Y,\varphi) + \sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} e^{U_j(X\setminus Y,\varphi)}K_j^{(n)}(Y,\varphi) \quad (6.97)$$

where

$$K_j^{(n)}(Y,\varphi) = 1_{Y\in\mathscr{P}_j\setminus\mathscr{P}_j^c}K_j^0(Y,\varphi). \quad (6.98)$$

We will bound the two terms in (6.97) separately. Observe that, for $Y \in \mathscr{P}_j^c$, $\overline{Y} = X$ and any $\delta > 0$, applying submultiplicativity, Lemma 6.4.2 (also see (6.88)) implies

$$\|e^{U_j(X\setminus Y)}\|_{2h,T_j(X,\varphi)} \leqslant e^{|X\setminus Y|_j + \delta c_w\kappa_L w_j(X\setminus Y,\varphi)^2\|\omega_j^0\|_{\Omega_{j,0}}} \quad (6.99)$$

whenever $\|\omega_j^0\|_{j,0} \leqslant \varepsilon(\delta,\beta,L)$ for suitable $\varepsilon(\delta,\beta,L)$. Using this bound, together with (3.73), Lemma 4.1.6, and estimating $|X\backslash Y|_j \leqslant L^2|X|_{j+1}$, one obtains for $Y \in \mathscr{P}_j^c$

$$\begin{aligned} \|e^{U_j(X\backslash Y,\varphi)}K_j^0(Y,\varphi)\|_{2h,T_j(X,\varphi)} &\leqslant e^{|X\backslash Y|_j+\delta c_w\kappa_L\varepsilon w_j(X\backslash Y,\varphi)^2}G_j(Y,\varphi)A^{-|Y|_j}\|K_j^0\|_{\Omega_{j,0}^K} \\ &\leqslant A^{8(1+\eta)}e^{L^2|X|_{j+1}}G_j(X,\varphi)A^{-(1+\eta)|X|_{j+1}}\|K_j^0\|_{\Omega_{j,0}^K} \end{aligned} \tag{6.100}$$

for some $\eta > 0$ and $\|\omega_j^0\|_{j,0} \leqslant \varepsilon(\delta,L)$. Hence for the first term of (6.97),

$$\|\sum_{Y\in\mathscr{P}_j^c}^{\overline{Y}=X} e^{U_j(X\backslash Y,\varphi)}K_j^0(Y,\varphi)\|_{2h,T_j(X,\varphi)} \leqslant C(A)e^{L^2|X|_{j+1}}G_j(X,\varphi)\sum_{Y:\overline{Y}=X}A^{-(1+\eta)|X|_{j+1}}\|K_j^0\|_{\Omega_{j,0}^K} \tag{6.101}$$

but $\sum_{Y:\overline{Y}=X}1 \leqslant 2^{|X|_j} \leqslant 2^{L^2|X|_{j+1}}$ so this is bounded by $C(A)A^{-(1+\frac{\eta}{2})|X|_{j+1}}G_j(X,\varphi)\|K_j^0\|_{\Omega_{j,0}^K}$ for $A \geqslant C(L)$ sufficiently large. Now by the linearity of the map $K_j^0 \mapsto \sum_{Y\in\mathscr{P}_j^c}^{\overline{Y}=X} e^{U_j(X\backslash Y)}K_j^0(Y)$, we immediately have, for $\eta' = \eta/2$,

$$\Big\|\partial_{K_j^0}\Big[\sum_{Y\in\mathscr{P}_j^c}^{\overline{Y}=X} e^{U_j(X\backslash Y,\varphi)}K_j^0(Y,\varphi)\Big](\dot{K}_j)\Big\|_{2h,T_j(X,\varphi)} \leqslant C(A)A^{-(1+\eta')|X|_{j+1}}\|\dot{K}_j\|_{\Omega_{j,0}^K}G_j(X,\varphi). \tag{6.102}$$

Next, for $Y \in \mathscr{P}_j\backslash\mathscr{P}_j^c$ and $\overline{Y} = X$, we have by (3.29) that

$$(K_j^0+\dot{K}_j)(Y) - K_j^0(Y) = \prod_{Z\in\mathrm{Comp}_j(Y)}(K_j(Z)+\dot{K}_j(Z)) - \prod_{Z\in\mathrm{Comp}_j(Y)}K_j^0(Z) \tag{6.103}$$

so, denoting by $\overline{K_j^{(n)}}$ the object defined by (4.18) with $K_j^{(n)}$ from (6.98) in place of K_j^0, we obtain

$$\begin{aligned} &\Big\|\overline{(K_j^0+\dot{K}_j)^{(n)}} - \overline{K_j^{(n)}} - \sum_{Y\notin\mathscr{P}_j^c}^{\overline{Y}=X}\sum_{Z\in\mathrm{Comp}_j(Y)} e^{U_j(X\backslash Y)}\dot{K}_j(Z)\prod_{Z'\in\mathrm{Comp}_j(Y\backslash Z)}K_j^0(Z')\Big\|_{2h,T_j(X)} \\ &\leqslant \sum_{Y\notin\mathscr{P}_j^c}^{\overline{Y}=X} e^{|X\backslash Y|_j}A^{-|Y|_j}\Big(\big(\varepsilon+\|\dot{K}_j\|_{\Omega_{j,0}^K}\big)^{|\mathrm{Comp}_j(Y)|} - \|\dot{K}_j\|_{\Omega_{j,0}^K}^{|\mathrm{Comp}_j(Y)|} - |\mathrm{Comp}_j(Y)|\|\dot{K}_j\|_{\Omega_{j,0}^K}^{|\mathrm{Comp}_j(Y)|-1}\varepsilon\Big) \\ &\leqslant C\sum_{Y\notin\mathscr{P}_j^c}^{\overline{Y}=X} e^{|X\backslash Y|_j}A^{-|Y|_j}|\mathrm{Comp}_j(Y)|^2\|\dot{K}_j\|_{\Omega_{j,0}^K}^2\,\varepsilon^{|\mathrm{Comp}_j(Y)|-2} \\ &\leqslant C'e^{L^2|X|_{j+1}}\sum_{Y\notin\mathscr{P}_j^c}^{\overline{Y}=X} e^{-\frac{1}{2}|Y|_j}A^{-|Y|_j}\|\dot{K}_j\|_{\Omega_{j,0}^K}^2\,\varepsilon^{|\mathrm{Comp}_j(Y)|-2} \end{aligned} \tag{6.104}$$

where the second inequality holds under the assumption $\|\dot{K}_j\|_{\Omega^K_{j,0}} \leqslant \frac{1}{2}\varepsilon$. By Lemma 4.1.7, this is bounded by $C(A)(eL^2e^{L^2}A^{-(1+\eta'')})^{|X|_{j+1}}\|\dot{K}_j\|^2_{\Omega^K_{j,0}}$ for some $\eta''>0$, and hence $\overline{K_j^{(n)}}$ is differentiable in K_j^0. The derivative satisfies a similar bound:

$$\Big\|\sum_{Y\notin\mathscr{P}_j}^{\overline{Y}=X}\sum_{Z\in\mathrm{Comp}_j(Y)} e^{U_j(X\backslash Y)}\dot{K}_j(Z)K_j^0(Y\backslash Z)\Big\|_{2h,T_j(X)} \leqslant C(A)A^{-(1+\eta''/2)|X|_{j+1}}\|\dot{K}_j\|_{\Omega^K_{j,0}}\varepsilon \tag{6.105}$$

when A is chosen sufficiently large. So only the derivative in U_j is left to be studied. But

$$\begin{aligned}
&\Big\|\partial_{U_j}\Big[\sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} e^{U_j(X\backslash Y,\varphi)}K_j^0(Y,\varphi))\Big](\dot{U}_j)\Big\|_{2h,T_j(X,\varphi)}\\
&\leqslant \sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X}\|\dot{U}_j(X\backslash Y,\varphi)\|_{2h,T_j(X\backslash Y,\varphi)}\|e^{U_j(X\backslash Y)}K_j^0(Y,\varphi)\|_{2h,T_j(X,\varphi)}\\
&\leqslant C(\beta,L)\sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} e^{L^2|X|_{j+1}}e^{-|Y|_j}G_j(X,\varphi)A^{-|Y|_j}\|K_j^0\|^{|\mathrm{Comp}_j(Y)|}_{\Omega^K_{j,0}}\|\dot{U}_j\|_{\Omega^U_j}\\
&\leqslant C(\beta,L)e^{L^2|X|_{j+1}}G_j(X,\varphi)\|\dot{U}_j\|_{\Omega^U_j}(eL^2A^{-(1+2\eta)/(1+\eta)})^{|X|_{j+1}}\|K_j^0\|_{\Omega^K_{j,0}}
\end{aligned} \tag{6.106}$$

where the final inequality follows again by Lemma 4.1.7 assuming $\|K_j^0\|_{\Omega^K_{j,0}} \leqslant \varepsilon_{rb}$. Also, since $C(\beta,L)$ is a constant only polynomially large in β, we obtain

$$\Big\|\partial_{U_j}\Big[\sum_{Y\in\mathscr{P}_j}^{\overline{Y}=X} e^{U_j(X\backslash Y,\varphi)}K_j^0(Y,\varphi))\Big](\dot{U}_j)\Big\|_{2h,T_j(X,\varphi)} \leqslant C(L)A^{-(1+\eta''')|X|_{j+1}}G_j(X,\varphi)\|\dot{U}_j\|_{\Omega^U_j} \tag{6.107}$$

after choosing A large in L and $\|K_j^0\|_{\Omega^K_{j,0}}$ polynomially small in β. Hence we have the bound for $\partial_{U_j}\overline{K}_j^0$ when A is sufficiently large and together, (6.102), (6.105) and (6.107) yield (6.96). □

Chapter 7

Stable manifold

In Chapter 6, we defined a renormalisation group map Φ_{j+1}^{0,Λ_N} for $j+1<N$. Later in this chapter, we also construct the RG map on the final scale, Φ_N^{N,Λ_N} whose properties will be given in Proposition 7.4.4. These RG maps construct a renormalisation group flow $(s_j,z_j,K_j^0)_{j\leqslant N}$, defined iteratively by

$$(s_{j+1},z_{j+1},K_{j+1}^0)=\Phi_{j+1}^{0,\Lambda_N}(s_j,z_j,K_j^0), \qquad j\leqslant N-1 \tag{7.1}$$

provided that (s_j,z_j,K_j^0) remains in the domain of the renormalisation group maps. Compared to the definition in Chapter 6, we have dropped the E-coordinate from the renormalisation group map as it does not influence its dynamics and thus does not play a role in this section.

Our goal is now to show that for appropriate initial conditions (s_0,z_0,K_0), independent of Λ_N, the renormalisation group flow exists indefinitely (in the sense explained below). Moreover, we will address the point that our renormalisation group map actually depends on a parameter s (mostly suppressed in our notation so far), which we ultimately need to set equal to s_0 (see Proposition 6.1.1), but which has been arbitrary so far. Thus a renormalisation group flow depends both on the parameter s and the initial condition (s_0,z_0,K_0^0), but we will show that it is possible to choose $s=s_0$.

7.1 Statement of result

Recall the definition of the reference temperature $\beta_{\mathrm{free}}(J)$ from (1.23) for a given finite-range step distribution J

$$\beta_{\mathrm{free}}(J)=8\pi v_J^2, \tag{7.2}$$

c_f given by Definition 3.2.6 and $\tilde{z}(\beta)$ given by Lemma 6.2.2. In the sequel, we frequently write $K_0 = 0$ to denote the zero element in the linear space of polymer activities, i.e., the polymer activity given by $K_0(X) = 0$, $X \in \mathscr{P}_j^c$, whence $K_0(X) = 1_\emptyset(X)$, cf. below Definition 3.2.1.

Proposition 7.1.1. *(i) For any finite-range step distribution J (as always invariant under lattice symmetries and satisfying* (3.1)*) there exist $r \in (0,1]$ and $\beta_0(J) \in (0,\infty)$ such that the following holds for $\beta \geqslant \beta_0(J)$. There exist $s_0^c(J,\beta) = O(e^{-c_f\beta})$ and $\alpha = \alpha(J,\beta) > 0$, and positive integers $L = L(J)$ and $A = A(J)$ such that there exists a solution $(U_j, K_j^0)_{0\leqslant j\leqslant N}$ to* (7.1) *with parameter $s = s_0^c(J,\beta)$ and initial conditions $s_0 = s_0^c(J,\beta)$, $z_0 = \tilde{z}(\beta)$ and $K_0^0 = 0$. Moreover, they satisfy*

$$\|U_j\|_{\Omega_j^U} = O(e^{-c_f\beta}L^{-\alpha j}), \qquad \|K_j^0\|_{\Omega_{j,0}^K} = O(e^{-c_f\gamma\beta}L^{-\alpha j}), \tag{7.3}$$

for any $0 \leqslant j \leqslant N$ where the norms are as in Definitions 6.3.1–6.3.2 (and thus depend on A, L, r, β, ρ_J) and α is such that $CL^2\alpha_{\mathrm{Loc}} < L^{-\alpha}$ for sufficiently large C.

(ii) If $\mathscr{J}$ is a family of finite-range step distributions and (3.1) *holds with the same constant for all $J \in \mathscr{J}$, then there exists $C(\mathscr{J}) > 0$ such that for any $\delta > 0$ and $J \in \mathscr{J}$ with $v_J^2 \geqslant C(\mathscr{J})|\log\delta|$, one may take $\beta_0(J) = (1+\delta)\beta_{\mathrm{free}}(J)$ in (i).*

We remark that in terms of the function $s_0^c(J,\beta)$ of the proposition, the effective temperature in Theorem 1.1.1 will be defined by (cf. the discussion around (6.49))

$$\beta_{\mathrm{eff}}(J,\beta) = (1 + s_0^c(J,\beta)v_J^{-2})^{-1}\beta. \tag{7.4}$$

The initial conditions of Proposition 7.1.1 will be repeated multiple number of times, so we summarise them as the following.

(Φ_{IC}) Let $\beta_0(J)$, $s_0^c(J,\beta)$, $\alpha(J,\beta)$, L and A be as in Proposition 7.1.1. Let $\beta \geqslant \beta_0(J)$, the parameter s be set to be $s_0^c(\beta)$, the initial coupling constants $U_0(X,\varphi) = \frac{1}{2}s_0|\nabla\varphi|_X^2 + \sum_{x\in X}\sum_{q\geqslant 1} z_0^{(q)}\cos(q\beta^{1/2}\varphi(x))$ are given by $s_0 = s_0^c(\beta)$, $z_0^{(q)} = \tilde{z}^{(q)}(\beta)$, and the initial remainder coordinate is $K_0(X) = 1_{X=\emptyset}$.

Proposition 7.1.1 will be proved in the rest of the chapter.

7.2 Infinite-volume RG flow

In Chapter 6, we considered Λ_N fixed and corresponding scales $j < N$. In particular the renormalisation group map (7.1) also depends on Λ_N. However, in order to talk about the convergence of the flow (s_j, z_j, K_j^0) as $j \to \infty$, we now introduce notions of polymer activities and renormalisation flow that is free of this dependence by being defined in infinite volume. To distinguish polymer activities that depend on the torus from those defined in $\mathbb{Z}^2$, we now write $K(\cdot|\Lambda_N)$ or $K^{\Lambda_N}(\cdot)$ for the former and $K(\cdot)$ without index for the latter.

We first have to define an infinite-volume analogues of the polymers and polymer activities. We do not attempt to write everything explicitly when the extensions are clear.

Definition 7.2.1. *Let $\Omega^K_{j,0}(\mathbb{Z}^2)$ be the set of even periodic j-polymer activities $(K_j^0(X,\varphi) : X \in \mathscr{P}_j)$ such that $K_j^0(X,\varphi)$ only depends on $\varphi|_{X^*}$, $K_j^0(X) = (K_j^0)^{[X]}$ for any $X \in \mathscr{P}_j(\mathbb{Z}^2)$, respecting lattice symmetries and*

$$\|K_j^0\|_{\Omega^K_{j,0}(\mathbb{Z}^2)} := \|K_j^0\|_{2h,T_j} = \sup_{X \in \mathscr{P}^c_{j,0}(\mathbb{Z}^d)} A^{|X|_j} \|K_j^0(X)\|_{2h_j,T_j(X)} < +\infty. \tag{7.5}$$

where $\mathscr{P}^c_{j,0}(\mathbb{Z}^2)$ is the set of connected j-scale polymers $X \subset \mathbb{Z}^2$ such that $0 \in X$. Also, for $\omega_j^0 = (U_j, K_j^0)$, let

$$\|\omega_j^0\|_{j,0} \equiv \|\omega_j^0\|_{\Omega_{j,0}(\mathbb{Z}^2)} = \max\{\|U_j\|_{\Omega_j^U}, \|K_j^0\|_{\Omega^K_{j,0}(\mathbb{Z}^2)}\}. \tag{7.6}$$

It follows from Appendix 3.B that $\Omega^K_{j,0}$ is complete. Also, for $K_j^0 \in \Omega^K_{j,0}(\mathbb{Z}^2)$, we can think of the the infinite volume RG map

$$\Phi^{0,\mathbb{Z}^2}_{j+1} : (s_j, z_j, K_j^0) \mapsto (\mathfrak{s}_{j+1}(s_j, K_j^0), \mathfrak{z}_{j+1}(z_j), \mathscr{K}^{0,\mathbb{Z}^2}_{j+1}(s_j, z_j, K_j^0)) \tag{7.7}$$

defined exactly according to the procedure described in Section 6.3. Note that these quantities are well-defined because of the local dependence of the polymer activities and the covariance Γ_{j+1} has finite range. The dependence of $\mathfrak{s}_{j+1}$ and $\mathfrak{z}_{j+1}$ on $\mathbb{Z}^2$ are not made explicit because they will turn out to be essentially the same as those on Λ_N, see Proposition 7.4.2. Also, we see that the infinite volume RG map satisfies all the properties proved in Chapter 6.

Proposition 7.2.2. *Let $U_j \in \Omega_j^U$ and $K_j^0 \in \Omega^K_{j,0}(\mathbb{Z}^2)$. Then $\Phi^{0,\mathbb{Z}^2}_{j+1}(E_j, U_j, K_j^0)$ satisfies the estimates of of Theorem 6.1.2 and Theorem 6.1.3 also hold when K_j^0 is measured in norm $\|\cdot\|_{\Omega^K_{j,0}(\mathbb{Z}^2)}$.*

Proof. The proof of Theorem 6.1.2 and Theorem 6.1.3 did not use finiteness of the system Λ_N, so the proofs applies exactly the same.

□

We also need the continuity in s, whose proof will be deferred to Section 7.5.

Lemma 7.2.3. *Let β, r, A, L be as in Theorem 6.1.3. Then there exists $\varepsilon_c \equiv \varepsilon_c(\beta, A, L)$ (only polynomially small in β) such that the family $(\mathscr{K}_{j+1}^{0,\mathbb{Z}^2})_N$ with $\mathscr{K}_{j+1}^{0,\mathbb{Z}^2} : D_j \times [-\varepsilon_s\theta_J, \varepsilon_s\theta_J] \to \Omega_{j+1,0}^K(\mathbb{Z}^2)$ is continuous as a function of the implicit parameter $s \in [-\varepsilon_s\theta_J, \varepsilon_s\theta_J]$ when $D_j = \{\|(U_j, K_j^0)\|_{j,0} \leqslant \varepsilon_c\} \subset \Omega_{j,0}(\mathbb{Z}^2)$.*

7.3 Stable manifold for the infinite volume RG flow

In this section, we prove an analogue of Proposition 7.1.1 for the infinite volume RG flow.

It is somewhat more convenient to represent $z_j = (z_j^{(q)})$ and its evolution in terms of W_j as defined in Definition 6.3.1. This is mainly so that so that we can use the notation $\|W_j\|_{\Omega_j^U}$ from that definition (and do not need to introduce further notation). Thus given the map $\mathscr{U}_{j+1} = (\mathfrak{s}_{j+1}, \mathfrak{z}_{j+1})$, we define

$$\mathscr{W}_{j+1}(\omega_j^0)(B,\varphi) = \sum_{q\geqslant 1}\sum_{x\in B} L^{-2(j+1)}\mathfrak{z}_{j+1}^{(q)}(z_j)\cos(\sqrt{\beta}q\varphi(x)). \tag{7.8}$$

Then by Proposition 7.2.2 and Theorems 6.1.2–6.1.3, the infinite-volume renormalisation flow is given by

$$s_{j+1} = \mathfrak{s}_{j+1}(s_j, K_j^0) = s_j + \mathscr{H}_{j+1}(K_j^0) \tag{7.9}$$

$$W_{j+1}(B,\varphi') = \mathscr{W}_{j+1}(W_j)(B,\varphi') = \mathbb{E}_{\Gamma_{j+1}}[W_j(B,\varphi'+\zeta)] \tag{7.10}$$

$$K_{j+1}^0 = \mathscr{K}_{j+1}^{0,\mathbb{Z}^2}(s_j, W_j, K_j^0) = \mathscr{L}_{j+1}^{0,\mathbb{Z}^2}(K_j^0) + \mathscr{M}_{j+1}^{0,\mathbb{Z}^2}(s_j, W_j, K_j^0) \tag{7.11}$$

where $\mathscr{H}_{j+1}(K_j^0)$ is given by Definition 6.3.6 (whose extension to $\mathbb{Z}^2$ is clear, as it only uses small polymers) and $\mathscr{L}_{j+1}^{0,\mathbb{Z}^2}$, $\mathscr{M}_{j+1}^{0,\mathbb{Z}^2}$ are given by Theorem 6.1.3, extended to $\mathbb{Z}^2$ by Proposition 7.2.2. Our goal is to apply the stable manifold theorem in the form stated in [23, Theorem 2.16] to show the existence of s_0^c explained earlier. For this it is essential that the maps $\mathscr{K}_{j+1}^{0,\mathbb{Z}^2}$ contract. According to (6.12) and the definition of α_{Loc} in (5.16), this requires control of the lower bound on $\Gamma_{j+1}(0)$. The covariance estimate (3.5) implies a good lower bound on $\Gamma_{j+1}(0)/\log L$ once j is larger than a *critical scale* j_0, defined precisely by the next lemma. In the following we will write (note the extra argument s compared to (7.4)):

$$\beta_{\mathrm{eff}}(J,\beta,s) = (1+sv_J^{-2})^{-1}\beta. \tag{7.12}$$

Proposition 7.3.1. *For given $r \in (0,1]$ and $\delta > 0$, assume β is such that $r\beta_{\text{eff}}(J,\beta,s) \geqslant (1+\delta)\beta_{\text{free}}(J)$. Then there exists $j_0 \equiv j_0(\rho_J, L, \delta)$ such that*

$$L^{j_0} = O\Big(L\rho_J(1+\delta^{-1})\Big) \tag{7.13}$$

and that, for $j \geqslant j_0$,

$$L^2 e^{-\frac{1}{2}r\beta\Gamma_{j+1}(0;s)} \leqslant L^{-\delta}. \tag{7.14}$$

Proof. By (2.11) and (3.1), there exists $c_0 \geqslant 1$ such that $|2\pi t(v_J^2+s)\dot{D}_t(0,0|s)-1| \leqslant c_0\rho_J/t$ for all $t \geqslant \rho_J$. Hence define

$$t_0 := c_0\Big(\frac{1}{4} - \frac{1}{4(1+\delta)}\Big)^{-1}\rho_J \geqslant c_0\Big(\frac{1}{4} - \frac{\beta_{\text{free}}(J)}{4r\beta_{\text{eff}}(J,\beta,s)}\Big)^{-1}\rho_J \tag{7.15}$$

and $j_0 := \lceil \log_L(2t_0) \rceil$. Then for $j \geqslant j_0$,

$$\frac{r\beta}{2\log L}\Gamma_{j+1}(0;s) - 2 \geqslant \frac{2r\beta_{\text{eff}}(J,\beta,s)}{\beta_{\text{free}}(J)}\Big(1 - \frac{c_0 r(J)}{2t_0}\Big) - 2 \geqslant \frac{7}{4}\delta \tag{7.16}$$

so the claim holds. □

We explain some terminologies for the following theorem. We assume that $r \in (0,1]$, $\beta > 0$, $\rho_J \geqslant 1$ satisfy the assumptions of Remark 6.3.3 and that $r\beta > \beta_{\text{free}}(J)$. Let L and A be at least those given in Theorem 6.1.3, $j_0(\rho_J, L, \delta)$ be as in Proposition 7.3.1, and recall (2.5), the definition of θ_J. There are various ε's turning up. Given $\delta > 0$, we let $\varepsilon_\delta > 0$ be such that $r\beta_{\text{eff}}(s,J) \geqslant (1+\delta)\beta_{\text{free}}(J)$ for $|s| \leqslant \varepsilon_\delta$. Let $\varepsilon_{nl} \equiv \varepsilon_{nl}(\beta, A, L)$, a rational function of its arguments, be as in Theorem 6.1.3, ε_s be as in Lemma 7.5.1, ε_c be as in Lemma 7.2.3 and let

$$\varepsilon'_\delta = \min\{\varepsilon_\delta, \theta_J\varepsilon_s, \tfrac{1}{4}\}, \qquad \varepsilon'_{nl} = \min\{\varepsilon_{nl}, (2L)^{-1}C_3(\beta,A,L)^{-1}, \varepsilon_c\}. \tag{7.17}$$

with C_3 as in (6.13)). Thus ε'_δ is a bound for parameter s and ε'_{nl} is a bound for various polymer activities. Also, let $\varepsilon_0 = L^{-3j_0(\rho_J,L,\delta)}\varepsilon'_{nl}(\beta,A,L)$ and $\theta_0 = \frac{1}{8}\min\{1,\delta\} > 0$.

Theorem 7.3.2. *Let ℓ be sufficiently large and $r, \delta > 0$. Then for $L \geqslant L_0(\theta_0)$ of form $L = \ell^{N'}$, $A \geqslant A_0(L)$, $|s| \leqslant \varepsilon'_\delta$ and $\|W_0\|_{\Omega_0^U} \leqslant \varepsilon_0$ there exists $\mathfrak{s}_0^c(\beta,s) = O(\|W_0\|_{\Omega_0^U})$ such that $(s_j, W_j, K_j^0) \to 0$ exponentially in j, satisfying the flow equations* (7.9)–(7.11) *with initial conditions $s_0 = \mathfrak{s}_0^c(\beta,s)$, W_0 given as above, and $K_0^0 = 0$. Moreover, $\mathfrak{s}_0^c$ is continuous in s and*

$$|s_j|,\ \|W_j\|_{\Omega_j^U},\ \|K_j^0\|_{\Omega_{j,0}^K} \leqslant O(\|W_0\|_{\Omega_0^U})L^{-\alpha j} \tag{7.18}$$

for some $\alpha > 0$ satisfying $CL^2\alpha_{\mathrm{Loc}} \leqslant L^{-\alpha}$ for sufficiently large C.

Proof. We drop $\mathbb{Z}^2$ in the proof. The proof is an application of the stable manifold theorem in the form of [23, Theorem 2.16], only with smoothness replaced by continuous differentiability in its assumption and conclusion. To obtain the continuity in s we will work with spaces of continuous functions in s. For this application, we begin by defining Banach spaces $(I_j)_j$, $(F_j)_j$ for $j \in \mathbb{N}_{\geqslant 0}$ by

$$I_j = \big\{ s_j(s) \in C([-\varepsilon'_\delta, \varepsilon'_\delta], \mathbb{R}) : \|s_j\|_{I_j} < +\infty \big\}, \tag{7.19}$$
$$F_j = \big\{ (W_j, K_j)(s) \in C\big([-\varepsilon'_\delta, \varepsilon'_\delta], \Omega_j^W \times \Omega_{j,0}^K\big) : \|(W_j, K_j^0)\|_{F_j} < +\infty \big\}, \tag{7.20}$$

where $\Omega_j^W \subset \Omega_j^U$ is the (closed) subspace of elements with s-component equal to 0,

$$\begin{aligned} \|s_j\|_{I_j} &= \tau(j) \sup_{s\in[-\varepsilon'_\delta,\varepsilon'_\delta]} |s_j(s)|, \\ \|(W_j, K_j^0)\|_{F_j} &= \tau(j) \sup_{s\in[-\varepsilon'_\delta,\varepsilon'_\delta]} \max\{\|W_j(s)\|_{\Omega_j^U}, \|K_j^0(s)\|_{\Omega_{j,0}^K}\}, \end{aligned} \tag{7.21}$$

and

$$\tau(j) = L^{3(j_0-j)_+} = L^{3\max\{j_0-j,0\}}. \tag{7.22}$$

The weight $\tau(j)$ will ensure contractiveness of the map for scales $j \leqslant j_0$ where it is not guaranteed that $\Gamma_{j+1}(0)$ is not bounded below. Since Ω_j^U and $\Omega_{j,0}^K(\mathbb{Z}^2)$ are Banach spaces, I_j and F_j are Banach spaces. Also let $B_a^{\mathbb{X}}$ be the open ball in normed space $\mathbb{X}$ centred at 0 with radius $a > 0$. Define

$$\begin{aligned} T_{j+1} :& B_{\varepsilon_0}^{I_j} \times B_{\varepsilon_0}^{F_j} \to I_{j+1} \times F_{j+1}, \\ & (s_j, W_j, K_j^0) \mapsto (\mathfrak{s}_{j+1}(s_j, K_j^0), \mathscr{W}_{j+1}(W_j), \mathscr{K}_{j+1}^0(s_j, W_j, K_j^0)). \end{aligned} \tag{7.23}$$

Since $\mathscr{H}_{j+1}, \mathscr{W}_{j+1}, \mathscr{L}_{j+1}^0$ are bounded linear functions and $\mathscr{M}_{j+1}^0$ is a continuously differentiable function, T_{j+1} is also continuously differentiable. Also, the operators T_{j+1} are uniformly invertible in a neighbourhood of $(0,0)$ in the following sense: by Theorem 7.3.2 (and using estimates of Theorems 6.1.2, 6.1.3), there are constants C_1, C_2 independent of j

such that

(C1) $\sup_j\{|\mathscr{H}_{j+1}(K_j^0)| : \|K^0\|_{\Omega^K_{j,0}} \leqslant 1\} < +\infty$;
(C2) $\sup_j\{\|\mathscr{L}^0_{j+1}(K_j^0)\|_{\Omega^K_{j+1,0}} : \|K\|_{\Omega^K_{j,0}} \leqslant 1\} \leqslant C_1 L^2 \alpha_{\mathrm{Loc}}$;
(C3) $\sup_j\{\|\mathscr{W}_{j+1}(W_j)\|_{\Omega^U_{j+1}} : \|W_j\|_{\Omega^U_j} \leqslant 1\} \leqslant L^2 e^{-\frac{1}{2}\beta\Gamma_{j+1}(0)}$;
(C4) $(s_j, W_j, K_j^0) \mapsto \mathscr{M}^0_{j+1}$ is continuously differentiable ;
(C5) $\|D\mathscr{M}^0_{j+1}(s_j,W_j,K_j^0)\|_{\Omega^K_{j+1,0}} \leqslant C_2\|(s_j,W_j,K_j^0)\|_{\Omega_{j,0}}$ for $(s_j,W_j,K_j^0) \in B^{I_j}_{\varepsilon_0} \times B^{F_j}_{\varepsilon_0}$, and $\mathscr{M}^0_{j+1}(0,0,0) = 0$.

Note that Proposition 7.3.1 implies, for $e^{2\sqrt{\beta}h} \leqslant (e^{\frac{1}{2}r\beta\Gamma_{j+1}(0;s)})^{\theta_0}$ (which is implied always possible by choosing $L \geqslant L_0(\theta)$ sufficiently large)

$$L^2\alpha_{\mathrm{Loc}} \leqslant L^{-1}(\log L)^{3/2} + L^2 \sum_{q\geqslant 1} L^{-(2+\delta)(2q-1)(1-\theta_0)} \leqslant C'(L^{-1}(\log L)^{3/2} + L^{-\delta/2}). \quad (7.24)$$

Together with (C2), (C3), and (7.14), this implies

$$\sup_j \|(\mathscr{W}_{j+1}, \mathscr{L}^0_{j+1})\|_{F_j \to F_{j+1}} < 2C_1 L^2 \alpha_{\mathrm{Loc}} \leqslant L^{-\alpha} < 1 \quad (7.25)$$

when L is chosen sufficiently large. Then, by (C1), (C4), (C5), and (7.25), T_{j+1} is as required for the proof of [23, Theorem 2.16] (with smoothness of $\mathscr{M}^0_{j+1}$ replaced by continuous differentiability) to apply, thus yielding the existence of a continuously differentiable function $S^{(s)} : B^{F_0}_{\varepsilon_0} \to I_0$ such that the initial condition $(S^{(s)}(W_0,K_0^0), W_0, K_0^0)$ solves the flow equations (7.9)–(7.11) with the final condition $(s_j, W_j, K_j^0) \to (0,0,0)$ exponentially. The rate of the exponential decay also follows from the proof.

Then $\mathfrak{s}_0^c = S^{(s)}(W_0, 0)$ is as desired: Indeed,

$$|\mathfrak{s}_0^c(\beta,s)| \leqslant \sup_{(W_0',0)\in B^{F_0}_{\varepsilon_0}} \|D_{(W_0,K_0)}S^{(s)}(W_0',0)\|_{\mathrm{op}} \|W_0\|_{\Omega^U_0} = O(\|W_0\|_{\Omega^U_0}), \quad (7.26)$$

and continuity in s follows because all elements are by construction continuous functions in s by Lemma 7.2.3. □

Corollary 7.3.3. *Let $\mathfrak{s}_0^c(\beta,s)$, L_0, and A_0 be as in Theorem 7.3.2 applied with $W_0 = \tilde{U}$ as in* (6.30)*, and set $N_0' = \lceil \log_\ell L_0 \rceil$. The following hold for $L = \ell^{N_0'}$ and $A = A_0(L)$.*

(i) If J is fixed and β is sufficiently large, there exists $s_0^c(J,\beta)$ such that $\mathfrak{s}_0^c(\beta, s_0^c(J,\beta)) = s_0^c(J,\beta)$.

(ii) *Let $\mathscr{J}$ be a family of finite-range step distributions and suppose that* (3.1) *holds with the same constants for all $J \in \mathscr{J}$. Then for any $\delta > 0$, there exists $C > 0$ such that whenever $J \in \mathscr{J}$, $v_J^2 \geqslant C|\log\delta|$ and $\beta \geqslant (1+\delta)\beta_{\text{free}}(J)$, there exists $s_0^c(J,\beta)$ such that $\mathfrak{s}_0^c(\beta, s_0^c(J,\beta)) = s_0^c(J,\beta)$.*

The proof of the corollary is an application of the intermediate value theorem.

Proof. To see *(i)*, first choose $r > 0$ small enough and $\beta > 0$ large enough so that the assumption of Lemma 3.2.8 is satisfied and $r\beta \geqslant 2\beta_{\text{free}}$. Also choose $\varepsilon_0 > 0$ as in Theorem 7.3.2 and fix $\delta = 1/2$. Then $\varepsilon_\delta > 0$ is chosen to be less than $1/10$.

Now note that Lemma 6.2.2 implies that $\|W_0\|_{\Omega_0^U} \leqslant Ce^{-\frac{1}{4}\gamma\beta}$. By Theorem 6.1.3 and Proposition 7.3.1, $\varepsilon_0 = L^{-3j_0(\rho_J,L,\delta)}\varepsilon_{nl}(\beta,A,L)$ is only polynomially decaying in β. Therefore $\|W_0\|_{\Omega_0^U} \leqslant Ae^{-\frac{1}{4}\gamma\beta} < \varepsilon_0$ for sufficiently large β, and the assumption concerning W_0 of Theorem 7.3.2 is satisfied with $W_0 = \tilde{U}$. Also by the choice of $|s| \leqslant \varepsilon'_\delta = \min\{\varepsilon_\delta, \theta_J\varepsilon_s, \frac{1}{4}\}$ above and because $v_J^2 \geqslant 1/2$, it is also true that $r\beta_{\text{eff}}(s,J) = r(1+sv_J^{-2})^{-1}\beta \geqslant \frac{10}{12}r\beta \geqslant \frac{20}{12}\beta_{\text{free}}$, verifying the other assumption of Theorem 7.3.2. Hence by the theorem, there is $\mathfrak{s}_0^c(\beta,s) = O(e^{-\frac{1}{4}\gamma\beta})$ so taking β sufficiently large so that $|\mathfrak{s}_0^c(\beta,s)| \leqslant \varepsilon'_\delta/2$ for all $|s| \leqslant \varepsilon'_\delta$ then *(i)* follows from continuity: if $f(s) = s - \mathfrak{s}_0^c(\beta,s)$ then $f(+\varepsilon'_\delta) \geqslant \varepsilon'_\delta/2$ and $f(-\varepsilon'_\delta) < -\varepsilon'_\delta/2$. By the intermediate value theorem there is s such that $f(s) = 0$ which is the claim.

To see *(ii)*, first fix $r = 1$ and ρ_J large enough to satisfy the assumptions of Lemma 3.2.8. Having v_J^2 sufficiently large and $\beta \geqslant (1+\delta)\beta_{\text{free}}(J) = 8\pi(1+\delta)v_J^2$ is again sufficient to obtain $\|W_0\|_{\Omega_0^U} \leqslant CAe^{-\frac{1}{4}\gamma\beta} \leqslant \varepsilon_0$. Then we choose $\varepsilon''_\delta = \min\{\varepsilon_\delta, \theta_{\mathscr{J}}\varepsilon_s, \frac{1}{4}\}$ (in place of ε'_δ) so we have a common domain $[-\varepsilon''_\delta, \varepsilon''_\delta]$ of s on which Theorem 7.3.2 is satisfied for all $J \in \mathscr{J}$. Moreover, whenever $|s| \leqslant \varepsilon''_\delta \leqslant \frac{\delta}{4}$,

$$\beta_{\text{eff}}(s,J) = (1+sv_J^{-2})^{-1}\beta \geqslant (1+\delta/2)^{-1}\beta \geqslant (1+\delta)(1+\delta/2)^{-1}\beta_{\text{free}}(J) \tag{7.27}$$

hence one has uniform lower bound of $\beta_{\text{eff}}(s,J)/\beta_{\text{free}}(J)$ greater than 1. Since $\mathfrak{s}_0^c(\beta,c) = O(e^{-\frac{1}{4}\gamma\beta}) = O(e^{-2\pi\gamma v_J^2})$ by Theorem 7.3.2, taking v_J^2 large enough gives $|\mathfrak{s}_0^c| \leqslant \varepsilon''_\delta/2$. The same continuity argument as in *(i)* then applies to give the conclusion. □

7.4 Infinite volume RG as a limit of finite volume RG

We have seen in the previous section that the analogue of Proposition 7.1.1 holds for the infinite volume RG flow $(U_j, K_j^{0,\mathbb{Z}^2})_{j\geqslant 0}$. In order to convert it to a statement about the finite volume RG flow, we would like to consider $K_j^{0,\mathbb{Z}^2} \in \Omega_{j,0}^K(\mathbb{Z}^2)$ as a limit of polymer activities in $\Omega_{j,0}^K(\Lambda_N)$. We introduce some notations for this purpose.

Definition 7.4.1. *For each Λ_N, fix an origin $0 \in \Lambda_N$ and recall that $\pi_N : \mathbb{Z}^2 \to \Lambda_N$ is the quotient map such that $\pi_N(0) = 0$. Define $R_N = \mathbb{Z}^2 \cap [-\frac{L^N-1}{2}, \frac{L^N-1}{2}]^2 \subset \mathbb{Z}^2$, so $\pi_N|_{R_N} : R_N \to \Lambda_N$ is a bijection with inverse $\iota_N : \Lambda_N \to R_N$. For $\varphi \in \mathbb{R}^{\Lambda_N}$, the push-forward $(\iota_N)_\# \varphi \in \mathbb{R}^{\mathbb{Z}^2}$ is well-defined.*

Proposition 7.4.2. *Given $N > k > 0$, let $(U_j^{\mathbb{Z}^2}, K_j^{0,\mathbb{Z}^2})_{0 \leqslant j \leqslant k}$ and $(U_j^{\Lambda_N}, K_j^{0,\Lambda_N})_{0 \leqslant j \leqslant k}$ satisfy*

$$(U_{j+1}^{\mathbb{Z}^2}, K_{j+1}^{0,\mathbb{Z}^2}) = \Phi_{j+1}^{0,\mathbb{Z}^2}(U_j^{\mathbb{Z}^2}, K_j^{0,\mathbb{Z}^2}), \qquad (U_{j+1}^{\Lambda_N}, K_{j+1}^{0,\Lambda_N}) = \Phi_{j+1}^{0,\Lambda_N}(U_j^{\Lambda_N}, K_j^{0,\Lambda_N}) \tag{7.28}$$

for each $j \leqslant k-1$ with initial condition $U_j = U_0$ and $K_0(X) = 1_{X=\emptyset}$. Then for any $X \in \mathscr{P}_j(\pi_N R_{N-1}) \subset \mathscr{P}_j(\Lambda_N)$, we have

$$K_j^{0,\Lambda_N}(X, \varphi) = K_j^{0,\mathbb{Z}^2}(\iota_N(X), (\iota_N)_\# \varphi), \tag{7.29}$$

$$(E_j^{\Lambda_N}, s_j^{\Lambda_N}, z_j^{\Lambda_N}) = (E_j^{\mathbb{Z}^2}, s_j^{\mathbb{Z}^2}, z_j^{\mathbb{Z}^2}) \tag{7.30}$$

for any $\varphi \in \mathbb{R}^{\Lambda_N}$ and $j \leqslant k$.

Proof. We see that (7.29) follows from an induction concerning the definition of the RG map, see (4.43) and Definition 6.3.7. Then (7.30) follows because $(E_j^{\Lambda_N}, U_j^{\Lambda_N})$ only depend on $(K_{j-1}(Y) : Y \in \mathscr{S}_{j-1})$ by definition. □

Thus we immediately prove Proposition 7.1.1 for $j \leqslant N-1$.

Corollary 7.4.3. *The conclusion of Proposition 7.1.1 holds, up to $j \leqslant N-1$.*

Proof. The proof is a consequence of Corollary 7.3.3 and Proposition 7.4.2: Indeed, the flow $(s_j, W_j, K_j^{0,\Lambda_N})$ determined by Φ_j^{0,Λ_N} has the same coupling constants s_j and W_j as the analogously defined flow of $\Phi^{0,\mathbb{Z}^2}$ with the same initial conditions, thus in particular $\|U_j\|_j \leqslant O(\|W_0\|_0) L^{-\alpha j}$. Now by (6.12) and (6.13), since $2C_1 L^2 \alpha_{\mathrm{Loc}} \leqslant L^{-\alpha}$,

$$\|K_{j+1}^{0,\Lambda_N}\|_{j+1,0} \leqslant \frac{1}{2} C_1 L^{-\alpha} \|K_j^0\|_{j,0} + C_2(\|K_j^0\|_{j,0} + \|U_j\|_j)^2, \tag{7.31}$$

for $j \leqslant N-2$. The flow of $(K_j^{0,\Lambda_N}))_{0 \leqslant j \leqslant N-1}$ is thus dominated by an exponentially converging sequence uniformly in N, i.e., if $(k_j)_{j \in \mathbb{N}}$ solves $k_0 = 0$ and

$$k_{j+1} = \frac{1}{2} C_1 L^{-\alpha} k_j + C_2(k_j + \|U_j\|_j)^2, \tag{7.32}$$

then $\|K_j^{0,\Lambda_N}\|_{j,0} \leqslant k_j \leqslant O(\|W_0\|_0 L^{-\alpha j})$ for any $j \leqslant N-1$.

□

Now we are only left to study the final integration by performing the integral over covariance $\Gamma_N^{\Lambda_N}$. The case $j+1=N$ is not within the scope of Theorem 6.1.3, so additional analysis is requires. The result is almost the same, but the contraction of (6.12) is not present.

Proposition 7.4.4 (Integration with respect to the bounded covariance)**.** *Let*

$$\Phi_{0,N}^{\Lambda_N} : (E_{N-1}, s_{N-1}, W_{N-1}, K_{N-1}) \mapsto (E_N^{\Lambda_N}, s_N^{\Lambda_N}, W_N^{\Lambda_N}, K_N^{0,\Lambda_N}) \tag{7.33}$$

be defined according to Definition 6.3.6 and Definition 6.3.7 but with Γ_{j+1} replaced by $\Gamma_N^{\Lambda_N}$. Then

$$Z_N^0(\varphi'|\Lambda_N) := e^{-E_N^{\Lambda_N}|\Lambda_N|}(e^{U_N^{\Lambda_N}(\Lambda_N,\varphi')} + K_N^{0,\Lambda_N})(\Lambda_N,\varphi')) = \mathbb{E}^{\zeta}_{\Gamma_N^{\Lambda_N}}[Z_{N-1}^0(\varphi'+\zeta)] \tag{7.34}$$

where $U_N^{\Lambda_N}(\Lambda_N,\varphi') = \frac{1}{2}s_N^{\Lambda_N}|\nabla\varphi'|^2_{\Lambda_N} + W_N^{\Lambda_N}(\Lambda_N,\varphi')$ and $E_N^{\Lambda_N}, s_N^{\Lambda_N}, W_N^{\Lambda_N}$ satisfy the estimates of Theorem 6.1.2 and

$$\|(U_N^{\Lambda_N}, K_N^{0,\Lambda_N})\|_{\Omega_{N,0}} \leqslant CL^2 \|(U_{N-1,0}, K_{N-1}^0)\|_{\Omega_{N-1,0}} \tag{7.35}$$

for some $C>0$ whenever $\|(U_{N-1}, K_{N-1}^0)\|_{\Omega_{N-1,0}} \leqslant \varepsilon \equiv \varepsilon(\beta, A, L)$ is sufficiently small (only polynomially small in β).

Proof. The identity (7.34) is true by construction since $\mathscr{B}_N(\Lambda_N)$ only consists of the empty polymer and Λ_N itself. Also the estimates of Theorem 6.1.2 hold because $\Gamma_N^{\Lambda_N}$ satisfies the same upper bounds as Γ_N, cf. Corollary 3.1.1 and Lemma 3.1.4 for the covariance estimates and Proposition 3.3.5 for the corresponding regulators.

To see the final remark, notice that

$$W_N^{\Lambda_N}(\Lambda_N,\varphi) = L^{-2N}\sum_{q=1}^{\infty} L^2 e^{-\frac{1}{2}\beta q^2 \Gamma_N^{\Lambda_N}(0)} z_{N-1}^{(q)} \sum_{x\in B}\cos(q\sqrt{\beta}\varphi(x)) \tag{7.36}$$

and $\Gamma_N^{\Lambda_N}(0) \geqslant 0$, we have $\|U_N^{\Lambda_N}\|_N \leqslant CL^2\|U_{N-1}\|_{N-1}$. Also, we see that analogues of (6.12) and (6.13) bound $(K_N^0)'$, but now α_{Loc} replaced by

$$\alpha_{\mathrm{Loc}}^{\Lambda_N} = CL^{-3}(\log L)^{3/2} + C\min\left\{1, \sum_{q\geqslant 1} e^{4\sqrt{\beta}qh} e^{-(q-1/2)r\beta\Gamma_N^{\Lambda_N}(0)}\right\} \leqslant 2C. \tag{7.37}$$

This does not provide contraction because we do not have a lower bound on $\Gamma_N^{\Lambda_N}(0)$, but we have $\|K_N^{0,\Lambda_N}\|_{\Omega_{N,0}^K} \leqslant 2CL^2\|(U_{N-1}, K_{N-1}^0)\|_{\Omega_{N-1,0}}$. □

Proof of Proposition 7.1.1. The proposition follows by combining Corollary 7.4.3 and Proposition 7.4.4. □

7.5 Continuity in s: proof of Lemma 7.2.3

In the proof, we will always be considering the RG map on $\mathbb{Z}^2$, so we omit them in the notation. The proof of continuity in s of the RG map uses the following lemma which extends Lemma 3.3.6.

Lemma 7.5.1. *For any $C > 0$ and any scale-j polymer activity F that is invariant under translations and satisfies $\|F\|_{\mathfrak{h},T_j} \leqslant C$, for $|s|, |s'| < \theta_J \varepsilon_s$,*

$$\lim_{s'\to s} \sup_{X\in\mathscr{P}_j^c} \left(\frac{A}{3}\right)^{|X|_j} \|\mathbb{E}_{\Gamma_{j+1}(s')}[F(X,\cdot+\zeta)] - \mathbb{E}_{\Gamma_{j+1}(s)}[F(X,\cdot+\zeta)]\|_{\mathfrak{h},T_{j+1}(\overline{X})} = 0 \tag{7.38}$$

and the limit is uniform in F satisfying $\|F\|_{\mathfrak{h},T_j} \leqslant C$. An analogous statement holds if we assume

$$\sup_{X\in\mathscr{P}_{j+1}^c} A^{|X|_{j+1}} \sup_{\varphi} G_j(X,\varphi)^{-1} \|F(X,\varphi)\|_{\mathfrak{h},T_{j+1}(X,\varphi)} \leqslant C \tag{7.39}$$

with the conclusion now being

$$\lim_{s'\to s} \sup_{X\in\mathscr{P}_{j+1}^c} \left(\frac{2A}{3\cdot 2^{L^2}}\right)^{|X|_{j+1}} \|\mathbb{E}_{\Gamma_{j+1}(s')}[F(X,\cdot+\zeta)] - \mathbb{E}_{\Gamma_{j+1}(s)}[F(X,\cdot+\zeta)]\|_{\mathfrak{h},T_{j+1}(X)} = 0. \tag{7.40}$$

Proof. We first claim that any scale-j polymer activity F with $\|F\|_{\mathfrak{h},T_j} \leqslant C$ can be approximated by polymer activities that are supported on polymers consisting of a bounded number of blocks. Indeed, $\|F\|_{\mathfrak{h},T_j} = \sup_{X\in\mathscr{P}_j^c} A^{|X|_j} \|F(X)\|_{\mathfrak{h},T_j(X)} \leqslant C$ implies that $(2A/3)^{|X|_j}\|F(X)\|_{\mathfrak{h},T_j(X)} \to 0$ as $|X|_j \to \infty$. More precisely, for any $\delta > 0$, there exist $M > 0$ only depending on C such that

$$\sup_{X\in\mathscr{P}_j^c} (2A/3)^{|X|_j} \|F(X)1_{|X|_j\leqslant M} - F(X)\|_{\mathfrak{h},T_j(X)} \leqslant \delta. \tag{7.41}$$

By Lemma 3.3.6, then also

$$\sup_{X\in\mathscr{P}_j^c} (A/3)^{|X|_j} \|\mathbb{E}_{\Gamma_{j+1}(s)}[F(X,\cdot+\zeta)1_{|X|_j\leqslant M} - F(X,\cdot+\zeta)]\|_{\mathfrak{h},T_{j+1}(\overline{X})} \leqslant \delta. \tag{7.42}$$

Since $\mathbb{E}_{\Gamma_{j+1}(s)} F(X,\cdot+\zeta)1_{|X|_j\leqslant M}$ is continuous in s by Lemma 3.3.11 uniformly in $X \in \mathscr{P}_j^c$ and F with $\|F(X)\|_{h,T_j(X)} \leqslant C$ (by translation invariance there are only a bounded number of polymers X with $|X|_j \leqslant M$ to consider), the claim follows. For the case (7.39), the conclusion follows from the same argument and (7.42) replaced by

$$\sup_{X \in \mathscr{P}^c_{j+1}} (3^{-1} 2^{-L^2+1} A)^{|X|_{j+1}} \|\mathbb{E}_{\Gamma_{j+1}(s)}[F(X, \cdot + \zeta) 1_{|X|_j \leqslant M} - F(X, \cdot + \zeta)]\|_{\mathfrak{h}, T_{j+1}(X)} \leqslant \delta \tag{7.43}$$

because $\mathbb{E}[G_j(X,\zeta)] \leqslant 2^{|X|_j} = 2^{L^2 |X|_{j+1}}$. $\square$

We begin with the continuity of the maps $\mathscr{L}^0_{j+1}$. To make their s-dependence explicit we write $\mathscr{L}^{0,s}_{j+1}$ for $\mathscr{L}^0_{j+1}$ defined with $\mathbb{E} = \mathbb{E}_{\Gamma_{j+1}(s)}$.

Lemma 7.5.2. *Under the assumptions of Theorem 6.1.3 and $s, s' \in [-\varepsilon_s \theta_J, \varepsilon_s \theta_J]$, we have*

$$\lim_{s' \to s} \|\mathscr{L}^{0,s}_{j+1}(K_j) - \mathscr{L}^{0,s'}_{j+1}(K_j)\|_{\Omega^K_{j+1,0}} = 0. \tag{7.44}$$

Proof. By (6.72), for $X \in \mathscr{P}^c_{j+1}$,

$$\mathscr{L}^{0,s}_{j+1}(K^0_j)(X, \varphi') = \mathscr{L}^{0,s}_{j+1}(K^0_j 1_{Y \in \mathscr{S}_j})(X, \varphi') + \mathbb{S}\big[\mathbb{E}_{\Gamma_{j+1}(s)}[K^0_j 1_{Y \notin \mathscr{S}_j}]\big](X, \varphi') \tag{7.45}$$

where $\mathscr{L}^{0,s}_{j+1}(K^0_j 1_{Y \in \mathscr{S}_j})$ is generated by $K^0_j(Y)$ on $Y \in \mathscr{S}_j$ and we recall the reblocking operator $\mathbb{S}$ from (4.20). Since by translation invariance the norm effectively only uses bounded number of $Y \in \mathscr{S}_j$, Lemma 3.3.11 and the continuity statement of Proposition 5.2.3 directly imply the continuity of $\mathscr{L}^{0,s}_{j+1}(K^0_j 1_{Y \in \mathscr{S}_j})$ in s. Concerning the continuity of the second term, (7.38) shows that

$$y(s,s') := \sup_{Y \in \mathscr{P}^c_j} \left(\frac{A}{3}\right)^{|Y|_j} \big\|\big(\mathbb{E}_{\Gamma_{j+1}(s)} - \mathbb{E}_{\Gamma_{j+1}(s')}\big)[\theta_\zeta K^0_j 1_{Y \notin \mathscr{S}_j}(Y)]\big\|_{2h, T_{j+1}(X)} \tag{7.46}$$

tends to 0 as $s' \to s$ and

$$\|\mathbb{S}\big(\mathbb{E}_{\Gamma_{j+1}(s)} - \mathbb{E}_{\Gamma_{j+1}(s')}\big)[\theta_\zeta K^0_j 1_{Y \notin \mathscr{S}_j}](X)\|_{2h, T_{j+1}(X)} \leqslant \sum_{Y \in \mathscr{P}^c_j \setminus \mathscr{S}_j}^{\overline{Y} = X} \left(\frac{A}{3}\right)^{-|Y|_j} y(s,s')^{|\mathrm{Comp}_j(Y)|}. \tag{7.47}$$

But then Lemma 4.1.7 directly implies, whenever $y(s,s') \leqslant (A/3)^{-8}$,

$$\sum_{Y \in \mathscr{P}^c_j \setminus \mathscr{S}_j}^{\overline{Y} = X} (A/3)^{-|Y|_j} y(s,s')^{|\mathrm{Comp}_j(Y)|} \leqslant (eL^2 (A/3)^{-(1+2\eta)/(1+\eta)})^{|X|_{j+1}} y(s,s'). \tag{7.48}$$

By setting $eL^2(A/3)^{-(1+2\eta)/(1+\eta)} \leqslant A^{-1}$, we have that $\mathbb{S}\big[\mathbb{E}_{\Gamma_{j+1}(s)}[\theta_\zeta K_j^0 1_{Y\notin\mathscr{S}_j}]\big]$ is continuous in s. $\square$

In the definition of the maps $\mathscr{M}_{j+1}^0$, there are two sources of dependence on s, the first one coming from $\mathscr{E}_{j+1}$, $\overline{U}_{j+1}^0$, $\mathscr{D}Q_j$ and $\overline{\mathscr{D}}Q_j$, and the second one coming from the expectation $\mathbb{E} = \mathbb{E}_{\Gamma_{j+1}(s)}$ written explicitly in (4.43). Concerning the first dependence, by the continuity statement of Proposition 5.2.3 and Theorem 6.1.2, we have that

$$\overline{\mathfrak{K}}_j(U_j,K_j^0) = (\mathscr{E}_{j+1}|X|,U_j,\overline{U}_{j+1}^0,K_j,\overline{K}_j^0,Q_j^0)(U_j,K_j^0) \tag{7.49}$$

is continuous in the implicit parameter s, so if we can show that $\mathfrak{M}_{j+1}^{(k)}(\mathfrak{K}_j^0)$ depends 'continuously' on $\mathfrak{K}_j^0$, then the dependence on s coming from the first source is continuous. Indeed, this will be shown in the following corollary. For given $\eta > 0$, define $\Omega_{j,\eta}^{\mathfrak{K}}$ to be the linear space of coordinates $(\mathscr{E}_{j+1}|X|,U_j,\overline{U}_{j+1}^0,K_j,\overline{K}_j^0,Q_j^0)$ where the following norm takes finite value:

$$\begin{aligned}
&\|(\mathscr{E}_{j+1}|X|,U_j,\overline{U}_{j+1}^0,K_j,\overline{K}_j^0,Q_j^0)\|_{j,\eta,\mathfrak{K}}\\
&= \max\Big\{L^{2j}|\mathscr{E}_{j+1}|, \|U_j\|_{\Omega_j^U}, \|\overline{U}_{j+1}+\mathscr{E}_{j+1}|X|\|_{\Omega_j^U}, \|K_j\|_{\Omega_{j,0}^K},\\
&\qquad \sup_{X\in\mathscr{P}_{j+1}^c,\ \varphi\in\mathbb{R}^{X^*}} A^{(1+\eta)|X|_{j+1}} G_j(X,\varphi)^{-1}\|\overline{K}_j(X,\varphi)\|_{h,T_j(X,\varphi)},\\
&\qquad \sup_{Y\in\mathscr{S}_{j+1},\ D\in\mathscr{B}_j(Y)}\ \sup_{\varphi\in\mathbb{R}^{X^*}} Ae^{-c_w\kappa_L w_j(B,\varphi)^2}\|Q_j^0(D,Y,\varphi)\|_{2h,T_j(D,\varphi)}\Big\}.
\end{aligned} \tag{7.50}$$

Then $(\Omega_{j,\eta}^{\mathfrak{K}}, \|\cdot\|_{j,\eta,\mathfrak{K}})$ forms a normed space. Note that this norm is essentially defined by the conditions in Definition 4.4.1.

Corollary 7.5.3. *Let $\eta,\delta > 0$ and $B_a^{\mathfrak{K}} = \{x\in\Omega_{j,\eta}^{\mathfrak{K}} : \|x\|_{j,\eta,\mathfrak{K}} \leqslant a\}$. Then there exists $a \equiv a(\delta,\beta,L) > 0$ (independent of j and N) such that the identity map $\mathrm{id}|_{B_a^{\mathfrak{K}}}$ is in $\mathscr{X}_j^{\mathfrak{K}}(B_a^{\mathfrak{K}})$. In particular, if we set $\mathfrak{K}_j(x) = x$ for $x\in B_a^{\mathfrak{K}}$, then each $\mathfrak{M}_{j+1}^{(k)}(\mathfrak{K}_j(x))$ ($k = 1,2,3,4$) is differentiable in $x\in B_a^{\mathfrak{K}}$ with the derivative uniformly bounded in j and N.*

Proof. The first statement is obvious because $\mathrm{id} : \Omega_{j,\eta}^{\mathfrak{K}} \to \Omega_{j,\eta}^{\mathfrak{K}}$ is a linear function with norm 1. For the second statement, we just need to apply Lemma 4.4.2 with $(\mathbb{Y},|\cdot|) = (B_a^{\mathfrak{K}},\|\cdot\|_{j,\eta,\mathfrak{K}})$. $\square$

Note that by Lemma 6.4.1, there exist $\varepsilon(\delta,\beta,L)$ and $C(\delta,\beta,L)$ such that $\|(U_j,K_j^0)\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,\beta,L)$ gives $\|\overline{\mathfrak{K}}_j(U_j,K_j^0)\|_{j,\eta,\mathfrak{K}} \leqslant C(\delta,\beta,L)\varepsilon(\delta,\beta,L)$. So if we set $\|(U_j,K_j)\|_{\Omega_{j,0}} \leqslant \varepsilon(\delta,\beta,L) \leqslant a(\delta,\beta,L)/C(\delta,\beta,L)$, then this corollary implies that each $\mathfrak{M}_{j+1}^{(k)}(\overline{\mathfrak{K}}_j(U_j,K_j))$ is continuous in s coming from the first source described above.

For the second source of s-dependence of $\mathscr{M}^0_{j+1}$, we will make the dependence due to $\mathbb{E} = \mathbb{E}_{\Gamma_{j+1}(s)}$ visible in (4.74) and (4.75)–(4.78) by writing $\mathscr{M}^{0,s}_{j+1}$ and $\mathfrak{M}^{(k),s}_{j+1}$ for $\mathscr{M}_{j+1}$ and $\mathfrak{M}^{(k)}_{j+1}$ evaluated by taking the expectation over $\zeta \sim \mathscr{N}(0, \Gamma_{j+1}(s))$. This dependence will be studied in the next lemma.

Lemma 7.5.4. *Under the assumptions of Theorem 6.1.3 and $s, s' \in [-\varepsilon_s \theta_J, \varepsilon_s \theta_J]$, we have*

$$\lim_{s' \to s} \|\mathscr{M}^{0,s}_{j+1}(U_j, K^0_j) - \mathscr{M}^{0,s'}_{j+1}(U_j, K^0_j)\|_{\Omega^K_{j+1,0}} = 0. \tag{7.51}$$

Proof. Since we have (4.74) and Lemma 6.4.1, we only have to verify

$$\lim_{s' \to s} \|\mathfrak{M}^{(k),s}_{j+1}(\overline{\mathfrak{K}}_j(\omega^0_j)) - \mathfrak{M}^{(k),s'}_{j+1}(\overline{\mathfrak{K}}_j(\omega^0_j))\|_{j+1} = 0 \tag{7.52}$$

for $\omega^0_j = (U_j, K^0_j)$ and each $k \in \{1,2,3,4\}$. Define

$$H_-(\mathfrak{K}_j, X_0, X_1, Z, (B_{Z''}), \varphi', \zeta) = (e^{U_j} - e^{\overline{U}_{j+1}})^{X_0} (\overline{K}_j - \mathscr{E} K_j)^{[X_1]} \prod_{Z'' \in \mathrm{Comp}_{j+1}(Z)} J_j(B_{Z''}, Z'') \tag{7.53}$$

and, as in (4.104),

$$H^s(\overline{\mathfrak{K}}_j, X_0, X_1, Z, (B_{Z''}), \varphi') = \mathbb{E}_{\Gamma_{j+1}(s)} H_-(\overline{\mathfrak{K}}_j, X_0, X_1, Z, (B_{Z''}), \varphi', \zeta). \tag{7.54}$$

Expanding (4.111), i.e.,

$$(e^{U_j} - e^{\overline{U}_{j+1}})^{X_0} (\overline{K}_j - \mathscr{E} K_j)^{[X_1]} = \sum_{Y_0, Y_1} (e^{U_j} - 1)^{Y_0} (-e^{\overline{U}_{j+1}} + 1)^{X_0 \setminus Y_0} (\overline{K}_j)^{[Y_1]} (-\mathscr{E} K_j)^{[X_1 \setminus Y_1]} \tag{7.55}$$

where Y_0, Y_1 run over $Y_0 \in \mathscr{P}_{j+1}(X_0)$, $Y_1 \in \mathscr{P}_{j+1}(Y_1)$, $Y_1 \not\sim X_1 \setminus Y_1$, the bounds (4.83)–(4.102) imply

$$\begin{aligned} &\|H_-(\overline{\mathfrak{K}}_j(\omega^0_j), X_0, X_1, Z, (B_{Z''}), \varphi', \zeta)\|_{2h, T_j(T, \varphi')} \\ &\qquad \leqslant \sum_{Y_0, Y_1} \big(C(A, L) \|\omega^0_j\|_{\Omega_j}\big)^{\#(X_0, X_1, Z)} A^{-(1+\eta)|X_0 \cup X_1|_{j+1}} G(X_0, Y_0, X_1, Y_1, Z, \varphi', \zeta) \end{aligned} \tag{7.56}$$

for some $\eta > 0$ where

$$G(X_0,Y_0,X_1,Y_1,Z,\varphi',\zeta) = e^{c_w\kappa_L\left(w_j((X_0\setminus Y_0)\cup(X_1\setminus Y_1)\cup Z,\varphi')+w_j(Y_0,\varphi'+\zeta)\right)}G_j(Y_1,\varphi'+\zeta). \tag{7.57}$$

Choosing $C(A,L)\|\omega_j^0\|_{\Omega_{j,0}}^{1/4} \leqslant 1$ and $(C(A,L)\|\omega_j^0\|_{\Omega_{j,0}})^{1/196} \leqslant A^{-(1+\eta)}$, since $49|\operatorname{Comp}_{j+1}(Z)| \leqslant |\cup_{Z''} B^*_{Z''}|_{j+1}$, we have

$$\left(4C(A,L)\|\omega_j^0\|_{\Omega_{j,0}}\right)^{\#(X_0,X_1,Z)} A^{-(1+\eta)|X_0\cup X_1|_{j+1}} \leqslant 4^{-\#(X_0,X_1,Z)}\|\omega_j^0\|_{\Omega_{j,0}}^{\frac{\#(X_0,X_1,Z)}{2}} A^{-(1+\eta)|X|_{j+1}} \tag{7.58}$$

where $X = X_0 \cup X_1 \cup (\cup_{Z''}B^*_{Z''})$. Therefore

$$\begin{aligned}\|H_-(\overline{\mathfrak{K}}_j(\omega_j^0),X_0,X_1,Z,(B_{Z''}),\varphi',\zeta)\|_{2h,T_j(T,\varphi')}&\\ \leqslant \|\omega_j^0\|_{\Omega_{j,0}}^{\frac{\#(X_0,X_1,Z)}{2}} A^{-(1+\eta)|X|_{j+1}} &\sup_{Y_0,Y_1} G(X_0,Y_0,X_1,Y_1,Z,\varphi',\zeta).\end{aligned} \tag{7.59}$$

since $H_-(\cdot,\varphi',\zeta)$ is a function of two field variables, Lemma 7.5.1 does not apply directly. Nevertheless, since G serves the role of the regulator satisfying

$$\mathbb{E}[G(X_0,Y_0,X_1,Y_1,Z,\varphi',\zeta)] \leqslant 2^{|X|_j} G_{j+1}(X,\varphi'),$$

the proof of (7.40) shows that, defining

$$\begin{aligned}&H^{s,s'}(\overline{\mathfrak{K}}_j(\omega_j^0),X_0,X_1,Z,(B_{Z''}),\varphi')\\ &\quad= \left(\mathbb{E}_{\Gamma_{j+1}(s)} - \mathbb{E}_{\Gamma_{j+1}(s')}\right)\left[H_-(\overline{\mathfrak{K}}_j(\omega_j^0),X_0,X_1,Z,(B_{Z''}),\varphi',\zeta)\right],\end{aligned} \tag{7.60}$$

in the limit $s' \to s$, one has

$$|H^{s,s'}|_{j+1} := \sup_{T\in\mathscr{P}_{j+1}} \left(\frac{2A^{1+\eta}}{3\cdot 2^{L^2}}\right)^{|T|_{j+1}} \|H^{s,s'}(\overline{\mathfrak{K}}_j(\omega_j^0),X_0,X_1,Z,(B_{Z''}))\|_{h,T_{j+1}(T)} \to 0. \tag{7.61}$$

In particular each $\|H^{s,s'}(\overline{\mathfrak{K}}_j(\omega_j^0), X_0, X_1, Z, (B_{Z''}))\|_{2h,T_{j+1}(T)}$ is finite. Hence

$$\begin{aligned}
&\|(\mathfrak{M}_{j+1}^{(1),s} - \mathfrak{M}_{j+1}^{(1),s'})(\overline{\mathfrak{K}}_j(\omega_j^0), X, \varphi')\|_{2h,T_{j+1}(X,\varphi')} \\
&= \Big\| \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} F(\overline{\mathfrak{K}}_j, T, X, \varphi') H^{s,s'}(\overline{\mathfrak{K}}_j(\omega_j^0), X_0, X_1, Z, (B_{Z''}), \varphi') \Big\|_{2h,T_{j+1}(X,\varphi')} \\
&\leqslant \sum_{X_0,X_1,Z,(B_{Z''})}^{\#(X_0,X_1,Z)\geqslant 2} C^{|X|_{j+1}} e^{c_w \kappa_L w_j(X\setminus T,\varphi')} |H^{s,s'}|_{j+1} \Big(\frac{2}{3\cdot 2^{L^2}} A^{(1+\eta)}\Big)^{-|X|_{j+1}} G_{j+1}(T,\varphi') \\
&\leqslant C^{|X|_{j+1}} |H^{s,s'}|_{j+1} 5^{|X|_{j+1}} \Big(\frac{2}{3\cdot 2^{L^2}} A^{(1+\eta)}\Big)^{-|X|_{j+1}} G_{j+1}(X,\varphi'),
\end{aligned} \tag{7.62}$$

where 5 in the last line is a combinatorial factor arising from choices of X_0, X_1, Z and $(B_{Z''})$. Taking $A^\eta \geqslant 15C2^{L^2-1}$, we see

$$\|(\mathfrak{M}_{j+1}^{(1),s} - \mathfrak{M}_{j+1}^{(1),s'})(\overline{\mathfrak{K}}_j(\omega_j^0))\|_{\Omega_{j+1}^K} \leqslant C|H^{s,s'}|_{j+1} \to 0 \text{ as } s' \to s. \tag{7.63}$$

A similar but simpler computations shows the same for $\mathfrak{M}_{j+1}^{(2),s}$. The continuity of $\mathfrak{M}_{j+1}^{(3),s}$ in s is implied directly by Lemma 3.3.11 because it only allows the case $|X|_{j+1} = 1$.

To see the same for $\mathfrak{M}_{j+1}^{(4),s}$, recall from (4.133) and (4.136) that

$$\|M_-^{(4)}(\overline{\mathfrak{K}}_j(\omega_j^0), X, \varphi)\|_{2h,T_j(X,\varphi')} \leqslant C(A,L) A^{-(1+\eta)|X|_{j+1}} G_j(X, \varphi' + \zeta) \|\omega_j^0\|_{\Omega_j}^2 \tag{7.64}$$

for some $\eta > 0$. Since $\mathfrak{M}_{j+1}^{(4),s,s'} = (\mathbb{E}_{\Gamma_{j+1}(s)} - \mathbb{E}_{\Gamma_{j+1}(s')}) M_-^{(4)}$, (7.40) implies

$$\lim_{s'\to s} \sup_{X\in\mathscr{P}_j} \Big(\frac{2}{3\cdot 2^{L^2}} A^{1+\eta}\Big)^{-|X|_{j+1}} \|\mathfrak{M}_{j+1}^{(4),s,s'}(\overline{\mathfrak{K}}_j(\omega_j^0), X)\|_{2h,T_{j+1}(X)} = 0. \tag{7.65}$$

Just taking $A^\eta \geqslant 3\cdot 2^{L^2-1}$, this implies continuity of $\mathfrak{M}_{j+1}^{(4),s}$ in s. □

Proof of Lemma 7.2.3. The continuity statement of $\mathscr{K}_{j+1}^0 = \mathscr{L}_{j+1}^0 + \mathscr{M}_{j+1}^0$ is now a direct consequence of Lemma 7.5.2, Corollary 7.5.3, and Lemma 7.5.4. The continuity of $\mathscr{U}_{j+1}$ follows from the continuity statement in Theorem 6.1.2.

□

Chapter 8

Observable renormalisation group flow

In this chapter, we reproduce the results of Chapter 6, 7 for the RG flow with observables. However, we do not have to tune the value of s_0 as in the previous chapter, as we can consider the observable RG flow as a perturbation of the bulk RG flow. Indeed, we will prove the stability of observable RG flow assuming the initial condition (Φ_{IC}) from Proposition 7.1.1.

8.1 Main results

The results are about the observable RG flow

$$\Phi_{j+1} : (E_j, e_j, s_j, z_j, K_j) \mapsto (E_j + \mathscr{E}_{j+1}, e_j + \mathfrak{g}_{j+1}, \mathfrak{s}_{j+1}, \mathfrak{z}_{j+1}, \mathscr{K}_{j+1}) \tag{8.1}$$

where now K_j is a periodic ω-polymer activity and $e_j(X;\omega)$ is a polymer function analytic in $\omega \in \mathbb{D}_{h_\omega}$. The flow depends on the external fields v and $\tilde{\mathfrak{f}}$ satisfying (A_v) and (A'_f), respectively, and we will assume this throughout the chapter. The coordinates are defined so that, if $(Z_j)_{j \geqslant 0}$ are defined by (4.55) and

$$(E_{j+1}, e_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}) = \Phi_{j+1}(E_j, e_j, s_j, z_j, K_j), \tag{8.2}$$

then

$$Z_j(\varphi;\omega|\Lambda) = \begin{cases} e^{-E_j|\Lambda|}(e^{U_j} \circ_j K_j^0)(\Lambda, \varphi + \omega v) & (j \leqslant j_s) \\ e^{-E_j|\Lambda| + e_j(\Lambda;\omega)}(e^{U_j(\cdot, \varphi + \omega u_j)} \circ_j K_j(\cdot, \varphi))(\Lambda) & (j > j_s) \end{cases} \tag{8.3}$$

(recall j_s is the observable scale). When $\omega = 0$, it means that the system is unperturbed, so $K_j(\omega = 0)$ corresponds to the bulk polymer activity K_j^0. The goal of this chapter is to obtain the stability of the observable RG flow when we tune the initial condition to (Φ_{IC}). In the following, $(\Omega_j^K, \|\cdot\|_{\Omega_j^K})$ is the set of ω-polymer activities specific later in Section 8.2.1.

Proposition 8.1.1. *Suppose v and $\tilde{\mathfrak{f}}$ satisfy* (A_v) *and* (A'_f)*, respectively. Let $r \in (0,1]$, $\beta_0(J) \in (0,\infty)$, $\alpha = \alpha(J,\beta) > 0$, $s_0^c(J,\beta)$, $L = L(J)$ and $A = A(J)$ be as in* (Φ_{IC})*. Then whenever $\beta \geqslant \beta_0(J)$, there is a solution $(U_j, K_j)_{0 \leqslant j \leqslant N}$ to* (8.2) *exists with initial conditions $s_0 = s = s_0^c(J,\beta)$, $z_0 = \tilde{z}(\beta)$ and $K_0 = 0$. Moreover,*

$$\|\mathfrak{g}_j(\Lambda_N)\|_{h_\omega,T} \leqslant O(e^{-c_f\beta}L^{-\alpha j}), \qquad \|K_j\|_{\Omega_j^K} = O(e^{-c_f\beta}L^{-\alpha j}). \tag{8.4}$$

for each $0 \leqslant j \leqslant N$ (with U_j identical to that of Proposition 7.1.1) and $\beta_0(J)$ satisfies the same remark as that of Proposition 7.1.1 (ii).

Remark 8.1.2. In the proof, we will see that we have to choose $L = L(J)$ and $A = A(J)$ sufficiently large for the observable flow on top of (Φ_{IC}). However, we could have taken $L(J)$ and $A(J)$ in (Φ_{IC}) that covers the case of Proposition 8.1.1, so we would not be emphasising this point.

To prove the stability, we will need analogues of Theorem 6.1.2, 6.1.3, with U_j chosen the same as for the bulk RG flow.

Theorem 8.1.3. *Suppose v and $\tilde{\mathfrak{f}}$ satisfy* (A_v) *and* (A'_f)*, respectively, with observable scale j_s. If $h_\omega < (C\log L)^{-1}\mathfrak{h}$ for C sufficiently large, there exists $\varepsilon_r \equiv \varepsilon_r(A,L) > 0$ such that whenever* $\max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\Omega_j^K}\} \leqslant \varepsilon_r$ *and $\omega \in \mathbb{D}_{h_\omega}$,*

$$\|\mathfrak{g}_{j+1}(D)\|_{h_\omega,T} \leqslant 1_{D \subset P_j^*} CA^{-1} \max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\Omega_j^K}\} \tag{8.5}$$

for any $D \in \mathscr{B}_j$.

To prove bounds on $\mathscr{K}_{j+1}$, it is more convenient to think of it as a function of $(U_j, K_j^\dagger)$. We use $(\Omega_{j,\dagger}^K, \|\cdot\|_{\Omega_{j,\dagger}^K})$ defined later in Section 8.2.1.

Theorem 8.1.4 (Estimate for remainder coordinate)**.** *Suppose v and $\tilde{\mathfrak{f}}$ satisfy* (A_v) *and* (A'_f)*, respectively. Let $j+1 < N$ and assume* (Φ_{IC})*. Then the map $\mathscr{K}_{j+1}$ admits a decomposition*

$$\mathscr{K}_{j+1}(U_j, K_j^\dagger) = \mathscr{L}_{j+1}(K_j^\dagger) + \mathscr{M}_{j+1}(U_j, K_j^\dagger) \tag{8.6}$$

into polymer activities at scale $j+1$ such that the following holds for any $\beta \geqslant \beta_0(J)$:

(i) *The map $\mathscr{L}_j$ is linear in $K_j^\dagger$ and independent of U_j. There is a constants $C_1, C_2 > 0$ independent of all the parameters such that, with α_{Loc} and $\alpha_{\mathrm{Loc}}^{(0)}$ as in* (5.16) *and* (5.17)

$$\|\mathscr{L}_{j+1}(K_j)\|_{\Omega_{j+1}^K} \leqslant C_1 L^2 \alpha_{\mathrm{Loc}} \|K_j^0\|_{\Omega_{j,0}^K} + C_2 \alpha_{\mathrm{Loc}}^{(0)} \|K_j^\dagger - K_j^0\|_{\Omega_{j,\dagger}^K} \times \begin{cases} 1 & (j \neq j_s) \\ L^2 & (j = j_s) \end{cases}. \tag{8.7}$$

(ii) *The remainder maps $\mathscr{M}_{j+1}$ satisfy $\mathscr{M}_{j+1} = O(U_j, K_j^\dagger)^2$ in the sense that there exist $\varepsilon_{nl} \equiv \varepsilon_{nl}(\beta, A, L) > 0$ (only polynomially small in β) and $C_3 = C_3(\beta, A, L) > 0$ (only polynomially large in β) such that $\mathscr{M}_{j+1}(U_j, K_j^\dagger)$ is continuously Fréchet-differentiable and, for $\|(U_j, K_j^\dagger)\|_{\Omega_{j,\dagger}} \leqslant \varepsilon_{nl}$,*

$$\|D\mathscr{M}_{j+1}(U_j, K_j^\dagger)\|_{\Omega_{j+1}^K} \leqslant C_3 \max\left\{\|U_j\|_{\Omega_j^U}, \|K_j^\dagger\|_{\Omega_{j,\dagger}^K}\right\} \tag{8.8}$$

with $\mathscr{M}_{j+1}(0,0) = 0$.

We prove Theorem 8.2.2 in Section 8.2.2 directly after defining $\mathfrak{g}_{j+1}$, while the proof of Theorem 8.1.4 is given in Section 8.2.3, 8.3.

8.2 Observable renormalisation group map

In this section, we construct the observable RG map (8.1) based on the bulk RG map (6.6). Given initial $Z_0^0(\varphi|\Lambda)$ by (6.40) with $s_0 = s = s_0^c(J, \beta)$, we define Z_j (with $\Lambda = \Lambda_N$) as in Chapter 4: in the presence of external field with observable scale j_s, we have defined $Z_0(\varphi; \omega|\Lambda) = Z_0^0(\varphi + \omega v|\Lambda)$ and

$$Z_{j+1}(\varphi; \omega|\Lambda) = \begin{cases} \mathbb{E}[\theta_\zeta Z_j(\varphi; \omega|\Lambda)] & (j < j_s) \\ \mathbb{E}_{(\omega)}\left[\theta_\zeta Z_j(\varphi; \omega|\Lambda)\right] & (j \geqslant j_s). \end{cases} \tag{8.9}$$

and the renormalisation group flow map is defined to parametrise

$$Z_j(\varphi; \omega|\Lambda) = \begin{cases} e^{-E_j|\Lambda|}(e^{U_j} \circ_j K_j^0)(\Lambda, \varphi + \omega v) & (j \leqslant j_s) \\ e^{-E_j|\Lambda| + e_j(\omega)}(e^{U_j(\cdot, \varphi + \omega u_j)} \circ_j K_j(\cdot, \varphi))(\Lambda) & (j > j_s). \end{cases} \tag{8.10}$$

8.2.1 Coordinates for the RG map

As promised, we now specify the space of remainder coordinate K_j. Since we would like to have K_j as a perturbation of K_j^0, the condition $K_j(\omega = 0) = K_j^0$ is crucial. Since we are assuming (Φ_{IC}), we are always assuming that $(U_j, K_j^0)_{0\leqslant j\leqslant N}$ can be constructed as in Proposition 7.1.1.

Definition 8.2.1. *The coordinate K_j is a ω-polymer activity (see Definition 3.2.1) satisfying*

- *the periodicity condition $K_j(X,\varphi) = K_j(X,\varphi + 2\pi\beta^{-1/2}n)$ for any $n \in \mathbb{Z}$,*
- *$K_j(\cdot;\omega = 0) = K_j^0$,*
- *$K_j(X;\omega) = K_j^0(X)$ whenever $X \cap (P_{\vec{y}}^j)^* = \emptyset$ when $j > j_s$ and $K_j \equiv K_j^0$ when $j \leqslant j_s$.*

For such polymer activitives K_j we use the norm (3.45)*, i.e.,*

$$\|K_j\|_j \equiv \|K_j\|_{\Omega_j^K} = \|K_j\|_{\vec{h}_j,T_j,A}, \tag{8.11}$$

with $\vec{h}_j = (h_j, h_\omega)$,

$$h_j = \begin{cases} 2h & (j \leqslant j_s) \\ h & (j > j_s), \end{cases} \qquad h_\omega = (C_w \log L)^{-\frac{3}{2}} \tag{8.12}$$

where we recall the choice of h from (6.55)*. Let Ω_j^K be the Banach space of polymer activies K_j with finite $\|\cdot\|_{\Omega_j^K}$-norm.*

As we have already seen, it is more convenient to use $K_j^\dagger$ in place of K_j, so we also define a space for them (see Lemma 4.1.3,4.1.4 for the motivation).

Definition 8.2.2. *The coordinate $K_j^\dagger$ is a ω-polymer activity satisfying all of Definition 8.2.1 but has $K_j(X;\omega) = K_j^0(X)$ whenever $X \cap (P_{\vec{y}}^j)^* = \emptyset$ for any j and*

$$\|K_j\|_{j,\dagger} \equiv \|K_j\|_{\Omega_{j,\dagger}^K} = \|K_j\|_{\vec{h}_{j+1},T_j,A/2}. \tag{8.13}$$

We also let $\Omega_{j,\dagger}^K$ be the space of such $K_j^\dagger$ with finite $\|\cdot\|_{\Omega_{j,\dagger}^K}$-norm.

By the definition (3.25) of $P_{\vec{y}^j}$, the condition $K_j(X;\omega) = K_j^0(X)$ whenever $X \cap (P_{\vec{y}}^j)^* = \emptyset$ for $j < j_s$ just means that $K_j \equiv K_j^0$.

Finally, we define the norm on the product space of (U_j, K_j) as follows.

Definition 8.2.3. *Let $\Omega_j = \Omega_j^U \times \Omega_j^K$ with norm*

$$\|\omega_j\|_j \equiv \|\omega_j\|_{\Omega_j} = \max\{\|U_j\|_{\Omega_j^U}, \|K_j\|_{\Omega_j^K}\}. \tag{8.14}$$

Also let $\Omega_{j,\dagger} = \Omega_j^U \times \Omega_{j,\dagger}^K$ *with norm*

$$\|\omega_j^\dagger\|_{j,\dagger} \equiv \|\omega_j^\dagger\|_{\Omega_{j,\dagger}} = \max\{\|U_j\|_{\Omega_j^U},\ \|K_j^\dagger\|_{\Omega_{j,\dagger}}\}. \tag{8.15}$$

8.2.2 Definition of the renormalisation group map

We first define the perturbed free energy $\mathfrak{g}_{j+1}$ and remind the definition of $K_j^\dagger$.

Definition 8.2.4. *Given* $K_j \in \Omega_j^K$, *define*

$$\mathfrak{g}_{j+1}(D, K_j; \omega) = 1_{D\subset(P_{\vec{y}}^j)^*} \sum_{Y\in\mathscr{S}_j}^{Y\supset D} \frac{1}{|Y\cap(P_{\vec{y}}^j)^*|_j} \mathrm{Loc}_Y^{(0)}\, \mathbb{E}_{(\omega)} \theta_\zeta [K_j^\dagger(Y,\varphi';\omega) - K_j^0(Y,\varphi')] \tag{8.16}$$

for $D \in \mathscr{B}_j$, *where*

$$K_j^\dagger(\cdot;\omega) = \begin{cases} \mathscr{R}_j^{(1)}[\omega u_j, U_j, K_j(\cdot;\omega)] & (j = j_s) \\ \mathscr{R}_j^{(2)}[\omega u_j, U_j, K_j(\cdot;\omega)] & (j > j_s). \end{cases} \tag{8.17}$$

For $X \in \mathscr{P}_j$ *(or* $\in \mathscr{P}_{j+1}$*), let*

$$\mathfrak{g}_{j+1}(X, K_j; \omega) = \sum_{D\in\mathscr{B}_j(X)} \mathfrak{g}_{j+1}(D, K_j; \omega). \tag{8.18}$$

The following definition gives the evolution of the remainder coordinate K_j. It does not distinguish the case $j < j_s$ from $j \geqslant j_s$ because $K_j \equiv K_j^0$ when $j < j_s$.

Definition 8.2.5. *Given* $(\mathscr{E}_{j+1}, \mathfrak{g}_{j+1}, \mathscr{U}_{j+1}, K_j)$, *we define* $\mathscr{K}_{j+1} : (U_j, K_j) \mapsto K_{j+1}$ *as in Definition 4.3.1 with*

$$\begin{aligned} Q_j(D,Y,\varphi') = 1_{Y\in\mathscr{S}_j}\Big(&\mathrm{Loc}_{Y,D}\, \mathbb{E}_{(\omega)}[K_j^0(Y,\varphi'+\zeta)] \\ &+ \frac{1_{D\in(Y\cap P_j)^*}}{|Y\cap(P_j)^*|_j} \mathrm{Loc}_Y^{(0)}\, \mathbb{E}_{(\omega)}[K_j^\dagger(Y,\varphi'+\zeta;\omega) - K_j^0(Y,\varphi'+\zeta)]\Big). \end{aligned} \tag{8.19}$$

Theorem 8.2.6 (Algebraic properties)**.** *The renormalisation group map* Φ_{j+1} *is consistent with* (6.50)–(6.51)*, i.e., if* Z_j *has the form* (6.51) *at scale* j *with parameters* (E_j, s_j, z_j, K_j) *then* Z_{j+1} *defined by* (6.50) *has this form at scale* $j+1$ *with* $(E_{j+1}, e_{j+1}, s_{j+1}, z_{j+1}, K_{j+1}) = \Phi_{j+1}(E_j, e_j, s_j, z_j, K_j)$. *Moreover, if* K_j *is a periodic* j*-scale* ω*-polymer activity (see Definition 3.2.1) then* K_{j+1} *is also a periodic* $j+1$*-scale* ω*-polymer activity with the same* *properties (perhaps except smoothness in the field) and if $K_j(\omega = 0) = K_j^0$, then $K_{j+1}(\omega = 0) = K_{j+1}^0$.*

Proof. The first part follows from Proposition 4.3.2. The second part also follows from the definition, as the map $\mathbb{K}_{j+1}$ preserves the periodicity of the inputs. We also have the final part because if $K_j(\omega = 0) = K_j^0$, then $K_j^\dagger(\omega = 0) = K_j^0$, thus $\mathfrak{g}_{j+1} = 0$ and $Q_j \equiv Q_j^0$. □

Estimate on $\mathfrak{g}_{j+1}$ is direct from its definition.

Proof of Theorem 8.1.3. By definition of $\mathrm{Loc}^{(0)}$, (8.16) can be restated as

$$\mathfrak{g}_{j+1}(D, K_j; \omega) = 1_{D \subset (P_j)^*} \sum_{Y \in \mathscr{S}_j}^{Y \supset D} \frac{1}{|Y \cap P_j^*|_j} \mathbb{E}_{(\omega)}[\hat{K}_{j,0}^\dagger(Y, \zeta; \omega) - \hat{K}_{j,0}^0(Y, \zeta)] \tag{8.20}$$

By Lemma 3.4.4, $\mathbb{E}_{(\omega)}[\hat{K}_{j,0}^\dagger(Y, \zeta; \omega)]$ and $\mathbb{E}_{(\omega)}[\hat{K}_{j,0}^0(Y, \zeta)]$ are analytic in $\omega \in \mathbb{D}_{h_\omega}$ with

$$\|\mathbb{E}_{(\omega)}[\hat{K}_{j,0}^\dagger(Y, \zeta; \omega)]\|_{h_\omega, T}, \ \|\mathbb{E}_{(\omega)}[\hat{K}_{j,0}^0(Y, \zeta)]\|_{h_\omega, T} \leqslant CA^{-|Y|_j} \|K_j^\dagger\|_{\Omega_{j,\dagger}^K}. \tag{8.21}$$

But also by Lemma 4.1.4, norm on $K_j^\dagger$ can also be bounded by $\|(U_j, K_j)\|_j$. Summing these estimates gives the desired bound.

□

8.2.3 Proof of Theorem 8.1.4: bound on the linear part

We use the observation in Section 4.4 to extract out the linear approximation

$$\begin{aligned} \mathscr{L}_{j+1}(U_j, K_j^\dagger; X, \varphi') := \sum_{Y : \overline{Y} = X} \Big(& 1_{Y \in \mathscr{P}_j^c} \mathbb{E}_{(\omega)} K_j^\dagger(Y, \varphi' + \zeta) - 1_{Y \in \mathscr{S}_j} \sum_{D \in \mathscr{B}_j(Y)} Q_j(D, Y, \varphi') \Big) \\ + \sum_{D \in \mathscr{B}_j}^{\overline{D} = X} \Big(& \mathbb{E}_{(\omega)}[U_j(D, \varphi' + \zeta)] + \mathscr{E}_{j+1}|D| - \mathfrak{g}_{j+1}(D; \omega) \\ & - \mathscr{U}_{j+1}(D, \varphi' + \omega u_j) + \sum_{Y \in \mathscr{S}_j}^{D \in \mathscr{B}_j(Y)} Q_j(D, Y, \varphi') \Big). \end{aligned} \tag{8.22}$$

By the choice of $(\mathscr{E}_{j+1}, \mathscr{U}_{j+1}, \mathfrak{g}_{j+1}, Q_j)$ from Definition 6.3.6, 8.2.4, 8.2.5, we see that the second summation vanishes and the first line becomes

$$\mathscr{L}_{j+1}(U_j, K_j^\dagger; X, \varphi') = \sum_{b=1,2,3} \mathscr{L}_{j+1}^{(b)}(K_j^\dagger; X, \varphi') \tag{8.23}$$

where

$$\mathscr{L}^{(1)}_{j+1} = \sum_{Y:\overline{Y}=X} 1_{Y\in\mathscr{S}_j}(1-\mathrm{Loc}^{(2)}_Y)\mathbb{E}_{(\omega)}\theta_\zeta K^0_j(Y,\varphi') \tag{8.24}$$

$$\mathscr{L}^{(2)}_{j+1} = \sum_{Y:\overline{Y}=X} 1_{Y\in\mathscr{S}_j}(1-\mathrm{Loc}^{(0)}_Y)\mathbb{E}_{(\omega)}\theta_\zeta D_j(Y,\varphi';\omega) \tag{8.25}$$

$$\mathscr{L}^{(3)}_{j+1} = \mathbb{S}\big[1_{Y\in\mathscr{P}^c_j\setminus\mathscr{S}_j}\mathbb{E}_{(\omega)}[\theta_\zeta K^\dagger_j(Y,\varphi')]\big], \tag{8.26}$$

where we recall $\mathbb{S}$ from Section 4.1.3and let

$$D_j(Y,\varphi';\omega) = K^\dagger_j(Y,\varphi';\omega) - K^0_j(Y,\varphi'). \tag{8.27}$$

Then the bound on $\mathscr{L}_{j+1}$ is obtained by bounding each $\mathscr{L}^{(b)}_{j+1}$.

Proof of Theorem 8.1.4,(i). By Proposition 5.2.3 (2),

$$\|\mathscr{L}^{(1)}_{j+1}(K^\dagger_j;X,\omega)\|_{\Omega^K_{j+1}} \leqslant \sum_{Y:\overline{Y}=X} 1_{Y\in\mathscr{S}_j}\alpha_{\mathrm{Loc}}(A/2)^{-|Y|_j}\|K^0_j\|_{\Omega^K_{j,0}} \leqslant CA^{-|X|_{j+1}}L^2\alpha_{\mathrm{Loc}}\|K^0_j\|_{\Omega^K_{j,0}} \tag{8.28}$$

where in the second inequality, we have used that $|Y|_j \geqslant |X|_{j+1}$ and that there are at most $O(L^2)$ number of small polymers $Y\in\mathscr{S}_j$ such that $\overline{Y}=X$.

For $\mathscr{L}^{(2)}_{j+1}$, notice that $D_j(Y)\neq 0$ only if $Y\cap(P^j_{\vec{y}})^*\neq\emptyset$ (see Definition 8.2.2). Thus by Proposition 5.2.3 (1),

$$\|\mathscr{L}^{(2)}_{j+1}(K^\dagger_j;X,\omega)\|_{\Omega^K_{j+1}} \leqslant \sum_{Y:\overline{Y}=X} 1_{Y\in\mathscr{S}_j}1_{Y\cap P^*_j\neq\emptyset}\alpha^{(0)}_{\mathrm{Loc}}(A/2)^{-|Y|_j}\|D_j\|_{\Omega^K_{j,\dagger}} \tag{8.29}$$

When $j\neq j_s$, then there are at most $O(1)$ number of $\mathscr{S}_j$-polymers such that $Y\cap(P^j_{\vec{y}})^*\neq\emptyset$, thus this is bounded by

$$\leqslant CA^{-|X|_{j+1}}\alpha^{(0)}_{\mathrm{Loc}}\|K^\dagger_j - K^0_j\|_{\Omega^K_{j,\dagger}}. \tag{8.30}$$

However, if $j=j_s$, then $(P^j_{\vec{y}})^*$ contains $O(L^2)$ number of j-blocks, thus

$$\|\mathscr{L}^{(2)}_{j_s+1}(K^\dagger_{j_s};\omega)\|_{\Omega^K_{j_s+1}} \leqslant CA^{-|X|_{j+1}}\alpha^{(0)}_{\mathrm{Loc}}\|K^\dagger_j - K^0_j\|_{\Omega^K_{j,\dagger}}\times L^2 \tag{8.31}$$

For $\mathscr{L}^{(3)}_{j+1}$, we use Proposition 4.1.5 to obtain

$$\|\mathscr{L}^{(3)}_{j+1}(K_j^\dagger;X,\omega)\|_{\Omega^K_{j+1}} \leqslant (L^{-1}A^{-1})^{|X|_{j+1}}\|K_j^\dagger\|_{\Omega^K_{j,\dagger}} \tag{8.32}$$

□

8.3 Proof of Theorem 6.1.3: bound on the non-linear part

The strategy for bounding $\mathscr{M}_{j+1} = \mathscr{K}_{j+1} - \mathscr{L}_{j+1}$ is the same as that of Section 6.4. Using Lemma 4.4.2, the bound reduces to the following lemma, where we recall $\overline{\mathfrak{K}}$ from (4.73),

$$\overline{\mathfrak{K}}_j(\omega_j^\dagger) = (\mathscr{E}^\dagger_{j+1}|X|, U_j, \overline{U}_{j+1}, K_j^\dagger, \overline{K}_j, Q_j)(\omega_j^\dagger) \tag{8.33}$$

and $\mathscr{X}_j^{\mathfrak{K}}$ from Definition 4.4.1.

Lemma 8.3.1. *Under the assumptions of Theorem 8.1.4, for any $\delta > 0$ and parameters satisfying* (6.56)*, there exists $\varepsilon(L) > 0$ only polynomially small in L, and constants $C(\delta,L) \equiv C(\delta,\beta,L)$, $C(L) \equiv C(\beta,L)$, $C(A,L)$, $\varepsilon(\delta,L) \equiv \varepsilon(\delta,\beta,L)$ and $\eta > 0$ such that if $\mathscr{X}_j^{\mathfrak{K}}(\cdot)$ is defined with these δ, η, $C(\delta,L)$, $C(L)$, $C(A,L)$ then $\overline{\mathfrak{K}}_j$ is in $\mathscr{X}_j^{\mathfrak{K}}(\{\omega_j^\dagger \in \Omega_{j,\dagger} : \|\omega_j^\dagger\|_{\Omega_{j,\dagger}} \leqslant \varepsilon(\delta,L)\})$.*

We defer the proof of the lemma to Section 8.3.1 and first complete the proof of Theorem 6.1.3 (ii).

Proof of Theorem 6.1.3 (ii). The continuous differentiability of $\mathscr{M}_{j+1}$ is a direct consequence of Lemma 4.4.2 applied with $\delta > 0$ sufficiently small, $\mathbb{X} = \{\omega_j^\dagger \in \Omega_{j,\dagger} : \|\omega_j^\dagger\|_{\Omega_{j,\dagger}} \leqslant \varepsilon(\delta,\beta,L)\}$, $\mathfrak{K}_j = \overline{\mathfrak{K}}_j$, and the decomposition $\mathscr{M}_{j+1} = \sum_{k=1}^4 \mathfrak{M}^{(k)}_{j+1}$ from (4.74), with the assumptions of Lemma 4.4.2 being verified by Lemma 8.3.1. The bound (6.13) is obtained by summing (4.87) for $k = 1,2,3,4$, so

$$\|\mathscr{M}_{j+1}(U_j,K_j^\dagger)(X)\|_{\vec{h}_j,T_{j+1}(X)} \leqslant C(A/2)^{-(1+\eta_0)|X|_{j+1}}\|(U_j,K_j^\dagger)\|_{\Omega_{j,\dagger}}, \tag{8.34}$$

with $A/2$ coming from the definition of the norm on $\|\cdot\|_{\Omega^K_{j,\dagger}}$. But by choosing A sufficiently large so that $2^{1+\eta_0}A^{-\eta_0} \leqslant 1$, this is bounded by

$$\leqslant CA^{-|X|_{j+1}}\|(U_j,K_j^\dagger)\|_{\Omega_{j,\dagger}}, \tag{8.35}$$

thus we have the desired conclusion.

□

8.3.1 Proof of Lemma 8.3.1

In this section we prove Lemma 8.3.1, i.e., that $\mathfrak{K}_j(\omega_j^\dagger)$ defined above (4.71) satisfies $\mathfrak{K}_j \in \mathscr{X}_j^{\mathfrak{K}}(\Omega_j)$ whenever $\omega_j^\dagger = (U_j, K_j^\dagger)$ is sufficiently small. In Lemma 8.3.2 we verify that (4.81) and (4.82) hold, and in Lemmas 8.3.3–8.3.4 we verify (4.83)–(4.86).

Lemma 8.3.2. *Under the assumptions of Theorem 8.1.4, there exists $\varepsilon(\delta,\beta,L) > 0$ only polynomially small in L and β such that the following holds: for any $\delta > 0$, suppose $\|\omega_j^\dagger\|_{j,\dagger} := \|(U_j,K_j^\dagger)\|_{\Omega_{j,\dagger}} \leqslant \varepsilon(\delta,\beta,L)$. Then* (4.81)*,* (4.82) *hold with $\vec{\mathfrak{h}} = \vec{h}_{j+1}$, i.e.,*

$$\|\mathfrak{U}(B,\varphi)\|_{\vec{h}_{j+1},T_j(B,\varphi)} \leqslant C(\delta,L)(1+\delta c_w \kappa_L w_j(B,\varphi)^2)\|\omega_j^\dagger\|_{j,\dagger} \tag{8.36}$$

$$\|e^{\mathfrak{U}(B,\varphi)} - \sum_{m=0}^{k} \frac{1}{m!}(\mathfrak{U}(B,\varphi))^m\|_{\vec{h}_{j+1},T_j(B,\varphi)} \leqslant C(\delta,L)e^{\delta c_w \kappa_L w_j(B,\varphi)^2}\|\omega_j^\dagger\|_{j,\dagger}^{k+1} \tag{8.37}$$

for $\mathfrak{U} \in \{U_j, \overline{U}_{j+1}\}$ and the same holds when $\mathfrak{U} = \mathscr{E}_{j+1}^\dagger$ but with $\delta = 0$.

Proof. The case $\mathfrak{U} = U_j$ was already proved in Lemma 6.4.2, so we focus on $\mathfrak{U} \in \{\overline{U}_{j+1}, \mathscr{E}_{j+1}^\dagger\}$. By (6.83) and Theorem 8.1.3, for $j^* \in \{j, j+1\}$, for $B \in \mathscr{B}_{j+1}$,

$$|\mathscr{E}_{j+1}||B|,\ \|\mathfrak{g}_{j+1}(B)\|_{h_\omega,T},\ \|W_{j^*}(B,\varphi)\|_{2h_{j+1},T_j(B,\varphi)} \leqslant CA^{-1}L^2\|\omega_j^\dagger\|_{j,\dagger}, \tag{8.38}$$

and by (6.84) and $L^2h^2 \geqslant 1$,

$$\|\frac{1}{2}s_{j^*}|\nabla\varphi|_B^2\|_{2h_{j+1},T_j(B,\varphi')} \leqslant CA^{-1}L^2h^2(1+w_j(B,\varphi)^2)\|\omega_j^\dagger\|_{j,\dagger}. \tag{8.39}$$

Hence if we let $\tilde{U}_{j+1}(B) = -\mathscr{E}_{j+1}|B| + \mathfrak{g}_{j+1}(B) + U_{j+1}(B)$,

$$\|\tilde{U}_{j+1}(B,\varphi)\|_{2h_{j+1},T_j(B,\varphi)} \leqslant C(\delta)\kappa_L^{-1}L^2h^2\big(1+\delta c_w\kappa_L w_j(B,\varphi)^2\big)\|\omega_j^\dagger\|_{j,\dagger}. \tag{8.40}$$

Also, since $\overline{U}_{j+1}(B,\varphi') = \tilde{U}_{j+1}(B,\varphi'+\omega u_{j+1})$ (see (4.71)), by Lemma 3.4.4,

$$\|\overline{U}_{j+1}(B,\varphi)\|_{h_{j+1},T_j(B,\varphi)} \leqslant C(\delta)\kappa_L^{-1}L^2h^2\big(1+\delta c_w\kappa_L w_j(B,\varphi)^2\big)\|\omega_j^\dagger\|_{j,\dagger}. \tag{8.41}$$

Then (8.37) is purely a consequence of (8.36), see the proof of Lemma 8.3.2.

The remark about $\mathscr{E}_{j+1}^\dagger$ follows from the same computations starting just from (8.38).

□

Lemma 8.3.3. *Under the assumptions of Theorem 8.1.4, there exist $c_w > 0$, $\varepsilon \equiv \varepsilon(\beta,L) > 0$ (only polynomially small in β), $C \equiv C(c_w,\beta,L)$, and $C_A \equiv C_A(c_w,\beta,L,A)$ such that the* *bounds* (4.83)*,* (8.43)*,* (4.86) *hold whenever* $\|\omega_j^\dagger\|_{j,\dagger} \leqslant \varepsilon$*, i.e.,*

$$\|De^{\mathfrak{U}'(B,\varphi)}\|_{\vec{h}_{j+1},T_j(B,\varphi)} \leqslant C(L)e^{c_w\kappa_L w_j(B,\varphi)^2}, \tag{8.42}$$

$$\|D^2e^{\mathfrak{U}'(B,\varphi)}\|_{\vec{h}_{j+1},T_j(B,\varphi)} \leqslant C(L)e^{c_w\kappa_L w_j(B,\varphi)^2}, \tag{8.43}$$

$$\|DQ_j(D,Y,\varphi)\|_{\vec{h}_{j+1},T_j(Y,\varphi)} \leqslant C(L)e^{c_w\kappa_L w_j(D,\varphi)^2}, \tag{8.44}$$

for any $Y \in \mathscr{S}_j$*,* $D \in \mathscr{B}_j(Y)$*, and* $\mathfrak{U}' \in \{U_j, \overline{U}_{j+1}, \mathscr{E}_{j+1}^\dagger\}$*, and in the case of* $\mathscr{E}_{j+1}^\dagger$*, the factor* $e^{c_w\kappa_L w_j(B,\varphi)}$ *can be omitted. The derivatives exist in the asserted spaces of polymer activities.*

Proof. The twice differentiability of $e^{\mathfrak{U}'}$ is a consequence of Lemma 8.3.2, as it was seen in the proof of Lemma 6.4.3.

Finally, because of Proposition 5.2.4,

$$\|\mathrm{Loc}_{Y,D}^{(2)}\mathbb{E}_{(\omega)}\theta\zeta K_j^0(Y,\varphi')\|_{\vec{h}_{j+1},T_j(Y,\varphi')} \leqslant C(\log L)\|K_j^0(Y)\|_{h_j,T_j(Y)}e^{c_w\kappa_L w_j(D,\varphi')^2}, \tag{8.45}$$

$$\|\mathrm{Loc}_Y^{(0)}\mathbb{E}\theta\zeta(K_j^\dagger - K_j^0)(Y,\varphi';\omega)\|_{\vec{h}_{j+1},T_j(Y,\varphi')} \leqslant C\|K_j^\dagger(Y)\|_{\vec{h}_{j+1},T_j(Y)}, \tag{8.46}$$

but since Q_j is a linear function of K_j, its differentiability follows from boundedness and the derivative satisfies (8.44).

□

Lemma 8.3.4. *Under the assumptions of Theorem 8.1.4, there exist* $\varepsilon \equiv \varepsilon(\beta, L) > 0$ *(only polynomially small in* β*) and* $C(A,L)$ *such that* (4.85) *holds whenever* $\|\omega_j^\dagger\|_{j,\dagger} \leqslant \varepsilon$*, i.e.,*

$$\|D\overline{K}_j(Z,\varphi)\|_{\vec{h}_{j+1},T_j(Z,\varphi)} \leqslant C(A,L)(A/2)^{-(1+\eta)|Z|_{j+1}}G_j(Z,\varphi) \tag{8.47}$$

for some purely geometric constant $\eta > 0$.

Proof. The proof is exactly the same as that of Lemma 6.4.4, but we have $A/2$ on the right-hand side because $K_j^\dagger$ is measured in $\|\cdot\|_{\vec{h}_{j+1},T_j(X),A/2}$-norm, the large set regulator halved.

□

8.4 Stability of the observable RG flow

We prove Proposition 8.1.1 considering the observable RG flow as the perturbation of the bulk RG flow, thus we need Proposition 7.1.1 as a reference point. We first need an analogue of Proposition 7.2.2. As in Definition 7.4.1, but for $N < N'$, let $\pi_{N,N'} : \Lambda_{N'} \to \Lambda_N$ be the canonical projection with $\pi_{N,N'}(0)=0$ and for $R_N=[-\frac{L^N-1}{2},\frac{L^N-1}{2}]\subset\Lambda_{N'}$, let $\iota_{N,N'}:\Lambda_N\to R_N$ be the inverse of $\pi_{N,N'}|_{R_N}$. The push-forward $(\iota_{N,N'})_\#$ is well-defined.

Lemma 8.4.1. *Assume* (Φ_{IC}) *and let* L^N *be sufficiently large compared to* L^{j_s}*,* $\max_{i=1,\cdots,\mathfrak{n}}\{\|y_i\|_2\}$ *and* ρ*. Suppose there are solutions* $(U_j,K_j^{\Lambda_N})_{0\leqslant j\leqslant k}$ *and* $(U_j,K_j^{\Lambda_{N'}})_{0\leqslant j\leqslant k}$ *of* (8.2) *for some* $N<N'$*. Then* $\mathfrak{g}_j^{\Lambda_N}=\mathfrak{g}_j^{\Lambda_{N'}}$ *whenever* $j<N$ *and* $X\in\mathscr{P}_j(\pi_{N,N'}R_{N-1})\subset\mathscr{P}_j(\Lambda_N)$*,*

$$K_j^{\Lambda_N}(X,\varphi)=K_j^{\Lambda_{N'}}(\iota_{N,N'}X,(\iota_{N,N'})_\#\varphi). \tag{8.48}$$

Proof. The proof is identical to that of Proposition 7.4.2. □

Then Proposition 8.1.1 is almost direct.

Proof of Proposition 8.1.1. Recall from Proposition 7.1.1 that

$$\|U_j\|_{\Omega_j^U}\leqslant Ce^{-\frac{1}{4}\gamma\beta}L^{-\alpha N},\qquad \|K_j^0\|_{\Omega_{j,0}^K}\leqslant Ce^{-\frac{1}{4}\gamma\beta}L^{-\alpha N} \tag{8.49}$$

for any $0\leqslant j\leqslant N$. We now prove, using induction, that

$$\|K_j\|_{\vec{h}_j,T_j}\leqslant CL^2e^{-c_f\beta}L^{-\alpha j}. \tag{8.50}$$

By the choice of $\beta_0(J)$, this is smaller than ε_{nl} whenever $\beta\geqslant\beta_0(J)$, thus inside the domain of Theorem 8.1.4. Since $K_j\equiv K_j^0$ for $j\leqslant j_s$, we only have to verify the bound for $j>j_s$. Since $K_{j+1}-K_{j+1}^0=\mathscr{K}_{j+1}(\omega_j^\dagger)-\mathscr{K}_{j+1}(U_j,K_j^0)$, by (8.7) and (8.8),

$$\|K_{j+1}-K_{j+1}^0\|_{\Omega_{j+1}^K}\leqslant 2C_1\alpha_{\mathrm{Loc}}^{(0)}\|K_j^\dagger-K_j^0\|_{\Omega_{j,\dagger}^K}\times\begin{cases}1 & (j\neq j_s)\\ L^2 & (j=j_s)\end{cases} \tag{8.51}$$

whenever $\|\omega_j^\dagger\|_{\Omega_{j,\dagger}}=\|(U_j,K_j^\dagger)\|_{\Omega_{j,\dagger}}$ is sufficiently small compared to $\alpha_{\mathrm{Loc}}^{(0)}$–this again holds due to Lemma 4.1.4 and the choice $\beta\geqslant\beta_0(J)$. Also by Lemma 4.1.4,

$$\|K_j^\dagger\|_{\Omega_{j,\dagger}^K},\ \ \|K_j^0\|_{\Omega_{j,\dagger}^K}\leqslant\|\omega_j\|_{\Omega_j}. \tag{8.52}$$

Thus when $j=j_s$, since $K_{j_s}=K_{j_s}^0$, combining with (8.49) gives

$$\|K_{j+1}\|_{\Omega_{j+1}^K}\leqslant\left(C+2C_1CL^{2+\alpha}\alpha_{\mathrm{Loc}}^{(0)}\right)e^{-c_f\beta}L^{-\alpha(j+1)}. \tag{8.53}$$

But if we use the fact that L^α is chosen in such a way that $L^2\alpha_{\text{Loc}} \leqslant L^{-\alpha}$ (see Proposition 7.1.1) and $\alpha_{\text{Loc}}^{(0)} \leqslant L^2(\log L)^{-1}\alpha_{\text{Loc}}$, we may take L sufficiently large to obtain

$$C + 2C_1CL^{2+\alpha}\alpha_{\text{Loc}}^{(0)} \leqslant CL^2, \tag{8.54}$$

thus we have (8.50) for $j = j_s$. When $j > j_s$, combining (8.51), (8.52) and (8.50) gives

$$\|K_{j+1}\|_{\Omega_{j+1}^K} \leqslant \big(C + 2C_1CL^{2+\alpha}\alpha_{\text{Loc}}^{(0)}\big)e^{-c_f\beta}L^{-\alpha(j+1)}. \tag{8.55}$$

Again, since $\alpha_{\text{Loc}}^{(0)} \leqslant L^2(\log L)^{-1}\alpha_{\text{Loc}}$, we may take L sufficiently large to obtain

$$C + 2C_1CL^{2+\alpha}\alpha_{\text{Loc}}^{(0)} \leqslant CL^2, \tag{8.56}$$

completing the induction. Then the bound on $\mathfrak{g}_j(\Lambda)$ follows from Theorem 8.1.3. $\square$

Chapter 9

Scaling limits

In this chapter, we prove the main results of Chapter 1. These results are more or less implied by the renormalisation group flows constructed in the previous chapters.

9.1 Torus scaling limit

The first object of this chapter is to prove the torus scaling limit, Theorem 1.1.1, restated as the following in terms of

$$f_N(x) = \frac{1}{|\Lambda_N|}\left(f(L^{-N}x) - \frac{1}{|\Lambda_N|}\sum_{y\in\Lambda_N} f(L^{-N}y)\right), \qquad x \in \Lambda_N,. \tag{9.1}$$

Theorem 9.1.1. *Let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. Then there exists $\beta_0(J) > 0$ and an integer $L = L(J)$ such that for the J-DG model on the torus Λ_N of side length L^N at temperature $\beta \geqslant \beta_0(J)$, there is $\beta_{\mathrm{eff}}(J,\beta) > 0$ such that for any $f \in C^\infty(\mathbb{T}^2)$ with $\int f\,dx = 0$, as $N \to \infty$,*

$$\log\left\langle e^{(f_N,\sigma)_{\Lambda_N}}\right\rangle^{\Lambda_N}_{J,\beta} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}. \tag{9.2}$$

Moreover, $\beta_{\mathrm{eff}}(J,\beta) = \beta + O_J(e^{-c\beta})$ for some $c > 0$ (independent of J).

In order to apply Proposition 7.1.1, we will always fix $s = s_0 = s_0^c(J,\beta)$ in the proof.

9.1.1 Final integral

We start from the conclusion of Proposition 7.1.1. The proof of Theorem 9.1.1 only requires K_N^0, thus the choice of j_s does not matter, so we only consider $j_s = \infty$. In particular, by the choice $s = s_0 = s_0^c(J,\beta)$, the renormalisation group flow (E_j, U_j, K_j) satisfies (7.3) for $j \leqslant N$. By Proposition 6.1.1, understanding

$$\tilde{Z}_N^0(\varphi; m^2) := \mathbb{E}^{\zeta}_{\tilde{C}(s)+t_N Q_N} Z_0^0(\varphi+\zeta) = \mathbb{E}^{\zeta}_{t_N(s,m^2)Q_N} Z_N^0(\varphi+\zeta) \tag{9.3}$$

would be the key component of the proof. Thus we now consider the final renormalisation group step corresponding to the covariance $t_N(s,m^2)Q_N$.

By Proposition 7.1.1 and Proposition 7.4.4, we have

$$Z_N^0(\varphi'|\Lambda_N) := e^{-E_N^{\Lambda_N}|\Lambda_N|}\big(e^{U_N^{\Lambda_N}(\Lambda_N,\varphi')} + K_N^{0,\Lambda_N}\big)(\Lambda_N,\varphi') \tag{9.4}$$

with estimates

$$\|U_N^{\Lambda_N}\|_N = O(e^{-\frac{1}{4}\gamma\beta}L^{-\alpha j}), \qquad \|K_N^{0,\Lambda_N}\|_{N,0} = O(e^{-\frac{1}{4}\gamma\beta}L^{-\alpha j}). \tag{9.5}$$

The following bounds (9.3).

Lemma 9.1.2. *$\tilde{Z}_N^0(\varphi;m^2)$ satisfies*

$$\tilde{Z}_N^0(\varphi;m^2) = e^{-E_N^{\Lambda_N}|\Lambda_N|}\Big(e^{\frac{1}{2}s_N^{\Lambda_N}|\nabla\varphi|^2_{\Lambda_N}}\big(1+O(\|W_N^{\Lambda_N}\|_N)\big) + O(\|K_N^{0,\Lambda_N}\|_{N,0})G_N(\Lambda_N,\varphi)\Big) \tag{9.6}$$

uniformly in $m^2 > 0$.

Proof. Since Q_N is the orthogonal projection onto the constant vectors in $\mathbb{R}^\Lambda$,

$$\mathbb{E}_{t_N(s,m^2)Q_N} Z_N^0(\varphi+\zeta) = \sqrt{\frac{|\Lambda_N|}{2\pi t_N}}\int_{\mathbb{R}} e^{-\frac{|\Lambda_N|}{2t_N}\zeta^2} Z_N^0(\varphi+\zeta)\,d\zeta. \tag{9.7}$$

For constant field ζ, using $G_N(\varphi+\zeta,\Lambda) = G_N(\varphi,\Lambda)$ for such ζ (see (3.41)),

$$\begin{aligned} e^{E_N^{\Lambda_N}|\Lambda_N|} Z_N^0(\varphi+\zeta) &= e^{U_N^{\Lambda_N}(\Lambda_N,\varphi+\zeta)} + K_N^{0,\Lambda_N}(\Lambda_N,\varphi+\zeta) \\ &= e^{\frac{1}{2}s_N^{\Lambda_N}|\nabla\varphi|^2_{\Lambda_N}}\big(1+O(\|W_N^{\Lambda_N}\|_N)\big) + O(\|K_N^{0,\Lambda_N}\|_{N,0})G_N(\Lambda_N,\varphi). \end{aligned} \tag{9.8}$$

whenever $\|W_N^{\Lambda_N}\|_{\Omega_N^U} \leqslant 1$. The last right-hand side is independent of ζ, so we have the desired conclusion. $\square$

9.1.2 Proof of Theorem 9.1.1

To prove the theorem, we will apply Lemma 9.1.2 with

$$\overline{C}^{\Lambda_N}(s) = \gamma(1+s\gamma\Delta) + (1+s\gamma\Delta)\tilde{C}(s)(1+s\gamma\Delta) \tag{9.9}$$

$$\varphi = \phi_N := \overline{C}^{\Lambda_N}(s)(1+s\gamma\Delta)^{-1} f_N \tag{9.10}$$

where f_N is as in Theorem 9.1.1. The next lemma shows that the exponential term and the regulator above are bounded for this choice.

Lemma 9.1.3. *Let $J \subset \mathbb{Z}^2 \setminus 0$ be any finite-range step distribution as in Section 2, and assume that θ_J is bounded below (see* (2.5)*). Let $f \in C^\infty(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} f\, dx = 0$, let f_N be given by* (9.1)*, and define ϕ_N by* (9.10)*. Then there are constants $C, c > 0$ uniform in $m^2 \geqslant 0$ and $N \in \mathbb{N}$ such that for $|s| \leqslant c$,*

$$|\nabla\phi_N|^2_{\Lambda_N} \leqslant C, \qquad G_N(\Lambda_N, \phi_N) \leqslant C. \tag{9.11}$$

Further,

$$\lim_{N\to\infty} (f_N, \overline{C}^{\Lambda_N}(s) f_N) = \frac{1}{s+v_J^2}(f, (-\Delta_{\mathbb{T}^2})^{-1} f)_{\mathbb{T}^2}. \tag{9.12}$$

The proof of the lemma is given after concluding the proof of our main theorem.

Proof of Theorem 9.1.1. By assumption, the conditions of Proposition 7.1.1(i) hold and Z_N and $\tilde{Z}_N$ are then defined as above. By Proposition 6.1.1 (applied with $\omega = 1$),

$$\langle e^{(f_N,\varphi)} \rangle_{J,\beta}^{\Lambda_N} = e^{\frac{\beta}{2}(f_N, \overline{C}(s) f_N)} \lim_{m^2 \downarrow 0} F_{N,m^2}[\beta^{1/2} f_N](1), \tag{9.13}$$

but for $x \in \mathbb{R}$ and $\tilde{\mathfrak{f}} = (1+s\gamma\Delta) f_N$,

$$F_{N,m^2}[\beta^{1/2} f_N](x) = \frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\overline{\zeta}}[e^{x\beta^{1/2}(\tilde{\mathfrak{f}},\overline{\zeta})} Z_0^0(\varphi' + \overline{\zeta} + x\beta^{1/2}\gamma f_N)]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\overline{\zeta}}[Z_0^0(\varphi'+\overline{\zeta})]} = \frac{\mathbb{E}^{\varphi'} Z_N^0(\varphi' + x\beta^{1/2}\phi_N)]}{\mathbb{E}^{\varphi'} Z_N^0(\varphi')]} \tag{9.14}$$

where $\varphi' \sim \mathscr{N}(0, t_N(m^2)Q_N)$ and $\overline{\zeta} \sim \mathscr{N}(0, \tilde{C}(s))$ and change of variable $\overline{\zeta} \mapsto \overline{\zeta} + x\beta^{1/2}\tilde{C}(s)\tilde{\mathfrak{f}}$ and the definition of Z_N were used for the second equality. But by Lemma 9.1.2 and (9.11),

$$\begin{aligned}
&\frac{\mathbb{E}_{t_N Q_N}[Z_N^0(\varphi' + \beta^{1/2}\phi_N)]}{\mathbb{E}_{t_N Q_N}[Z_N^0(\varphi')]} \\
&= e^{\frac{\beta}{2} s_N^{\Lambda_N}(\phi_N, -\Delta\phi_N)}\big(1 + O(\|W_N^{\Lambda_N}\|_N)\big) + O(\|K_N^{0,\Lambda_N}\|_{K,0}) G_N(\Lambda_N, \beta^{1/2}\phi_N) \\
&= e^{O(s_N^{\Lambda_N})}\big(1 + O(\|W_N^{\Lambda_N}\|_N)\big) + O(\|K_N^{0,\Lambda_N}\|_{N,0})
\end{aligned} \tag{9.15}$$

while Proposition 7.1.1(i) implies that $|s_N^{\Lambda_N}| + \|W_N^{\Lambda_N}\|_N + \|K_N^{0,\Lambda_N}\|_{N,0} \to 0$ as $N \to \infty$, provided that s_0 and s are tuned to the correct initial value $s_0^c(J,\beta)$. Therefore the limit in $N \to \infty$ converges to 1, uniformly in $m^2 > 0$, hence in particular

$$\lim_{N\to\infty} \lim_{m^2 \downarrow 0} F_{N,m^2}[\beta^{1/2} f_N] = 1. \tag{9.16}$$

Also by (9.12),

$$\lim_{N\to\infty} e^{\frac{\beta}{2}(f_N, \overline{C}(s) f_N)} = \exp\left(\frac{\beta}{2(v_J^2 + s_0^c(J,\beta))} (f, (-\Delta_{\mathbb{T}^2})^{-1} f) \right). \tag{9.17}$$

This proves the main conclusion with $\beta_{\text{eff}}(J,\beta) = \beta v_J^2/(v_J^2 + s_0^c(J,\beta))$. □

Proof of Theorem 1.2.1. Let $\mathscr{J} = \{J_\rho : \rho \in \mathbb{N}\}$ be the family of range-ρ step distribution. Then by Lemma 2.2.2, if we let $\theta_{\mathscr{J}} = \frac{1}{3^2} = \frac{1}{9}$, then $\theta_J \geqslant \theta_{\mathscr{J}}$ for each $J \in \mathscr{J}$ and $v_{J_\rho}^2 \sim \frac{1}{6}\rho^2$. Hence $\mathscr{J}$ satisfies the assumptions of Proposition 7.1.1(ii), so there exists $C > 0$ such that for any $\delta > 0$, $\|U_j\|_j$ and $\|K_j\|_{j,0}$ both decay exponentially in j and uniformly in Λ_N whenever $\rho^2 \geqslant C|\log\delta|$ and $\beta \geqslant \beta_0(J_\rho) = (1+\delta)\beta_{\text{free}}(J_\rho) \sim \frac{4\pi(1+\delta)}{3}\rho^2$ as $\rho \to \infty$. Therefore we may follow exactly the same proof as that of Theorem 1.1.1, but in the temperature range $\beta \geqslant (1+\delta)\beta_{\text{free}}(J_\rho)$. □

9.1.3 Proof of Lemma 9.1.3

The proof of Lemma 9.1.3 uses the following estimates for the Fourier coefficients of the functions f_N.

Lemma 9.1.4. *For $f \in C^\infty(\mathbb{T}^2)$, let f_N be given by* (9.1) *Then there exist constants $C_a = C_a(f)$ for $a \geqslant 0$ independent of Λ_N such that, for any $p \in \Lambda_N^* \subset [-\pi,\pi]^2$,*

$$|\hat{f}_N(p)| \leqslant C_a \|\nabla^{2a} f\|_{L^\infty(\mathbb{T}^2)} L^{-2Na} |p|^{-2a}. \tag{9.18}$$

Proof. Define two components of the Fourier multiplier

$$\lambda_1(p) = 2 - 2\cos(p_1), \quad \lambda_2(p) = 2 - 2\cos(p_2) \tag{9.19}$$

for $p = (p_1, p_2)$ so that $\lambda(p) = \lambda_1(p) + \lambda_2(p)$. One has

$$\hat{f}_N(p) = \begin{cases} \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} f(L^{-N}x) e^{-ip\cdot x} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases} \tag{9.20}$$

hence for $k \in \{1, 2\}$,

$$\lambda_k^a(p) |\hat{f}_N(p)| = \Big| \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} e^{-ip\cdot x} (\partial_{\Lambda_N}^{(e_k, -e_k)})^a f(L^{-N}x) \Big| \leqslant \sup_{x \in \Lambda_N} |(\partial_{\Lambda_N}^{(e_k,-e_k)})^a f(L^{-N}x)| \tag{9.21}$$

where $\partial_{\Lambda_N}^{(e_k,-e_k)} f(x/L^N) = -f((x+e_k)/L^N) - f((x-e_k)/L^N) + 2f(x/L^N)$ for $x \in \Lambda_N$. But since $\lambda_k(p) \geqslant \frac{4}{\pi^2} p_k^2$ by Lemma 2.2.1, we are just left to bound $|(\partial^{(e_k,-e_k)})^a f(x/L^N)|$. We now claim that

$$(\partial_{\Lambda_N}^{(e_k,-e_k)})^a f(z) = (-1)^a \int_{[0,L^{-N}]^{2a}} \prod_{l=1}^{a} ds_l\, dt_l\, \partial_{x_k}^{2a} f\big(z + \sum_{l=1}^{a} (s_l + t_l - L^{-N}) e_l\big). \tag{9.22}$$

To see this, start from the elementary observation

$$\begin{aligned} \partial_{\Lambda_N}^{(e_k,-e_k)} f(z) &= 2f(z) - f(z + L^{-N}e_k) - f(z - L^{-N}e_k) \\ &= - \int_{[0,L^{-N}]^2} ds\, dt\, \partial_{x_k}^2 f\big(z + (s + t - L^{-N}) e_k\big) \end{aligned} \tag{9.23}$$

and proceed by induction. Now by (9.21) and (9.22),

$$|\hat{f}_N(p)| \leqslant C_a |p_k|^{-2a} L^{-2Na} \|\nabla^{2a} f\|_{L^\infty(\mathbb{T}^2)} \tag{9.24}$$

for $k = 1, 2$, which concludes the proof. □

Proof of Lemma 9.1.3. For $p \in 2\pi\mathbb{T}^2$, let $\lambda(p)$ and $\lambda_J(p)$ be the Fourier multiplier of $-\Delta$ and $-\Delta_J$ as defined by (2.22). We also define

$$\phi_N^{(m^2)} = (1 + s\gamma\Delta)^{-1} \overline{C}(s, m^2) f_N \tag{9.25}$$

$$\overline{C}(s, m^2) = \gamma(1 + s\gamma\Delta) + (1 + s\gamma\Delta)\tilde{C}(s, m^2)(1 + s\gamma\Delta) \tag{9.26}$$

so ϕ_N and $\overline{C}(s)$ can be considered as $m^2 \downarrow 0$ limits and the desired conclusions follow upon the limit $m^2 \downarrow 0$. We claim a bit stronger statement than the first inequality in (9.11): for any $f \in C^\infty(\mathbb{T}^2)$ and all $a \in \{1,2,3,4\}$, the norm $\|\nabla_N^a \phi_N^{(m^2)}\|_{L_N^2(\Lambda_N)}$ is bounded uniformly in N. Indeed, in Fourier space, and recalling that λ (resp. λ_J) denote the Fourier multipliers of $-\Delta$ (resp. $-\Delta_J$) and $\hat{\cdot}$ is the Fourier transform,

$$\hat{\phi}_N^{(m^2)}(p) = \frac{(\lambda_J(p)+m^2)^{-1}}{1+s\lambda(p)\big(\lambda_J(p)+m^2)^{-1}-\gamma\big)}\hat{f}_N(p). \tag{9.27}$$

Since $\lambda(p) \in [0,2]$ and $(\lambda_J(p)+m^2)^{-1}\lambda(p) \leqslant \theta_J^{-1}$, for $|s|$ small,

$$|\hat{\phi}_N^{(m^2)}(p)| \leqslant C\theta_J^{-1}\lambda(p)^{-1}|\hat{f}_N(p)| \tag{9.28}$$

and for $\Lambda_N^* = 2\pi L^{-N}\Lambda_N$,

$$\begin{aligned}
\|\nabla_N^a \phi_N^{(m^2)}\|_{L_N^2(\Lambda_N)} &:= L^{2Na-2N}\|\nabla^a \phi_N^{(m^2)}\|_{L^2(\Lambda_N)}^2 \\
&\leqslant \frac{C\theta_J^{-2}|\Lambda_N|^{a-2}}{4\pi^2}\sum_{p\in\Lambda_N^*}\lambda(p)^{a-2}|\hat{f}_N(p)|^2 \\
&= \frac{C\theta_J^{-2}}{4\pi^2}\sum_{k\in\Lambda_N}\big(|\Lambda_N|\lambda(2\pi L^{-N}k)\big)^{a-2}|\hat{f}_N(2\pi L^{-N}k)|^2.
\end{aligned}$$

By Lemma 2.2.1 and the lower bound $\cos(x) \geqslant 1-x^2/2$,

$$16|k|^2 \leqslant |\Lambda_N|\lambda(2\pi L^{-N}k) = 2L^{2N}(2-\cos(2\pi L^{-N}k_1)-\cos(2\pi L^{-N}k_2)) \leqslant 4\pi^2|k|^2 \tag{9.29}$$

and together with Lemma 9.1.4, we have

$$\sum_{k\in\Lambda_N}\big(|\Lambda_N|\lambda(2\pi L^{-N}k)\big)^{a-2}|\hat{f}_N(2\pi L^{-N}k)|^2 \leqslant C_a(1+\sum_{k\in\Lambda_N\setminus\{0\}}|k|^{2(a-2)}|k|^{-2a}) < \infty \tag{9.30}$$

for $a \in \{1,2,3,4\}$. The case $a=1$ concludes the proof of the first inequality (9.11). Moreover, by the lattice Sobolev inequality (Lemma 3.A.3) there exists $c' > 0$ such that

$$\log G_N(\Lambda_N,\phi_N) \leqslant c'\kappa_L\sum_{a=1}^4\|\nabla_N^a\phi_N^{(m^2)}\|_{L_N^2(\Lambda_N)}, \tag{9.31}$$

also giving the second inequality in (9.11).

For the final claim (9.12), recalling $\hat{f}_N(0) = 0$, one has

$$\begin{aligned}
\lim_{m^2\to 0}(f_N, \overline{C}(s,m^2) f_N) &= \lim_{m^2\to 0}(\phi_N^{(m^2)}, (1+s\gamma\Delta) f_N) \\
&= \frac{1}{4\pi^2|\Lambda_N|}\sum_{k\in\Lambda_N\setminus\{0\}} \frac{\lambda_J(2\pi L^{-N}k)^{-1}\big(1 - s\gamma\lambda(2\pi L^{-N}k)\big)}{1+s\lambda(2\pi L^{-N}k)\big(\lambda_J(2\pi L^{-N}k)^{-1}-\gamma\big)}|\hat{f}_N(2\pi L^{-N}k)|^2 \\
&= \frac{1}{4\pi^2}\sum_{k\in 2\pi\Lambda_N\setminus\{0\}} \frac{\big(|\Lambda_N|\lambda_J(L^{-N}k)\big)^{-1}\big(1 - s\gamma\lambda(L^{-N}k)\big)}{1+s|\Lambda_N|\lambda(L^{-N}k)|\Lambda_N|^{-1}\big(\lambda_J(L^{-N}k)^{-1}-\gamma\big)}|\hat{f}_N(L^{-N}k)|^2. \qquad (9.32)
\end{aligned}$$

Since $f \in C^\infty(\mathbb{T}^2)$, we have $\hat{f}_N(L^{-N}k) \to \hat{f}(k)$ as $N \to \infty$ for each $k \in (2\pi\mathbb{Z})^2\setminus\{0\}$. By Lemma 2.2.1,

$$\lim_{N\to\infty} L^{2N}\lambda(L^{-N}k) = |k|^2, \qquad \lim_{N\to\infty} L^{2N}\lambda_J(L^{-N}k) = v_J^2|k|^2, \qquad (9.33)$$

where v_J^2 is defined by (2.4). Also by Lemma 9.1.4, the sum is dominated by $C\sum_{k\in(2\pi\mathbb{Z})^2\setminus\{0\}}|k|^{-4}$ for some $C > 0$, and therefore the Dominated convergence theorem implies

$$\lim_{N\to\infty}\lim_{m^2\to 0}(f_N, \overline{C}(s,m^2) f_N) = \frac{1}{4\pi^2}\sum_{k\in(2\pi\mathbb{Z})^2\setminus\{0\}} \frac{1}{v_J^2+s}|k|^{-2}|\hat{f}(k)|^2 = \frac{1}{v_J^2+s}(f, (-\Delta_{\mathbb{T}^2})^{-1} f) \qquad (9.34)$$

as needed. $\square$

9.2 Scaling limit on $\mathbb{R}^2$

The second aim of this chapter is to prove the scaling limit on $\mathbb{R}^2$ of Theorems 1.1.3, 1.1.4, restated as the following. Given $f \in C_c^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} f(x)\,dx = 0$, we recall $f_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ satisfy $\sum_{x\in\mathbb{Z}^2} f_\varepsilon(x) = 0$ and

$$\begin{aligned}
&\max_{0\leqslant k\leqslant 2}\max_{x\in\mathbb{Z}^d}|(\varepsilon^{-1}\nabla)^k f_\varepsilon(x)| \leqslant C_f\varepsilon^2, \qquad \operatorname{supp} f_\varepsilon \subset [-R_f\varepsilon^{-1}, R_f\varepsilon^{-1}]^2, \\
&\max_{x\in\mathbb{Z}^d}\big|\varepsilon^{-2} f_\varepsilon(x) - f(\varepsilon x)\big| \to 0,
\end{aligned} \qquad (9.35)$$

for some constants $C_f, R_f > 0$.

Theorem 9.2.1. *Let $J \subset \mathbb{Z}^2 \setminus \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. Then there exists $\beta_0(J) > 0$ such that the following holds for $\beta \geqslant \beta_0(J)$ and $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$*

such that there exists discretisation f_ε as in (9.35)*. As $\varepsilon \to 0$,*

$$\log\left\langle e^{(f_\varepsilon,\sigma)_{\mathbb{Z}^2}}\right\rangle_{J,\beta}^{\mathbb{Z}^2} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \tag{9.36}$$

where β_{eff} is the same as Theorem 1.1.1 and $\langle\cdot\rangle_{J,\beta}^{\mathbb{Z}^2}$ is the infinite volume measure defined by Proposition 1.1.2.

Theorem 9.2.2. *Under the setting of Theorem 9.2.1, there exists $L \equiv L(J)$ such that, whenever $(\varepsilon_N)_{N\geqslant 0}$ is a sequence such that $\varepsilon_N \to 0$ and $L^N\varepsilon_N \to \infty$,*

$$\log\left\langle e^{(f_{\varepsilon_N},\sigma)_{\Lambda_N}}\right\rangle_{J,\beta}^{\Lambda_N} \to \frac{\beta_{\mathrm{eff}}(J,\beta)}{2v_J^2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2} \tag{9.37}$$

as $N \to \infty$.

In the proofs, we always tune $s = s_0 = s_0^c(J,\beta)$ for $s_0^c(J,\beta)$ as in Theorem 9.1.1.

9.2.1 External field

We first have to choose the observable scale j_s and external fields v and $\tilde{\mathfrak{f}}$ where we can apply Proposition 8.1.1. By Proposition 6.1.1,

$$F_{N,m^2}[f_\varepsilon](\omega) = \frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\overline{\zeta}}_{(\omega)}[Z_0(\varphi'+\overline{\zeta}+\omega\gamma\mathfrak{f})]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\overline{\zeta}}[Z_0(\varphi'+\overline{\zeta})]} \tag{9.38}$$

for $\varphi' \sim \mathscr{N}(0,t_NQ_N)$ and $\overline{\zeta} \sim \mathscr{N}(0,\tilde{C}(s))$. Also after letting

$$j_s = \min\{j \geqslant 0 \;:\; L^{j+1} \geqslant 24R_f\varepsilon^{-1}\}, \tag{9.39}$$

where R_f is given by (9.35), we can subdecompose $\tilde{C}(s)$ as

$$\tilde{C}(s) = \Gamma_{\leqslant j_s} + \Gamma_{>j_s} := \sum_{j=1}^{j_s}\Gamma_j + \Big(\sum_{j=j_s+1}^{N-1}\Gamma_j + \Gamma_N^{\Lambda_N}\Big), \tag{9.40}$$

thus if we let $\zeta_1 \sim \mathscr{N}(0,\Gamma_{\leqslant j_s})$ and $\zeta_2 \sim \mathscr{N}(0,\Gamma_{>j_s})$ be independent Gaussian variables and take $\omega \in \mathbb{R}$,

$$F_{N,m^2}[f_\varepsilon](1) = \frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta_1}_{(\omega)}\mathbb{E}^{\zeta_2}_{(\omega)}[Z_0(\varphi'+\zeta_1+\zeta_2+\gamma f_\varepsilon)]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta_1}\mathbb{E}^{\zeta_2}[Z_0(\varphi'+\overline{\zeta})]}$$
$$= \frac{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta_1}_{(\omega)}\mathbb{E}^{\zeta_2}[Z_0(\varphi'+\zeta_1+\zeta_2+\Gamma_{\leqslant j_s}\tilde{\mathfrak{f}}+\gamma f_\varepsilon)]}{\mathbb{E}^{\varphi'}\mathbb{E}^{\zeta_1}\mathbb{E}^{\zeta_2}[Z_0(\varphi'+\overline{\zeta})]} \tag{9.41}$$

where we used change of variable $\zeta_2 \to \zeta_2 + \omega\tilde{C}(s)\tilde{\mathfrak{f}}$ when $\omega \in \mathbb{R}$ and

$$\mathbb{E}_{(\omega)}[F(\zeta)] = \frac{\mathbb{E}[e^{(\tilde{\mathfrak{f}},\zeta)}F(\zeta)]}{\mathbb{E}[F(\zeta)]}, \qquad \tilde{\mathfrak{f}} \equiv \tilde{\mathfrak{f}}_\varepsilon = (1+s\gamma\Delta)f_\varepsilon. \tag{9.42}$$

In this setting, the natural choice of v is

$$v \equiv v_\varepsilon = \Gamma_{\leqslant j_s}\tilde{\mathfrak{f}}_\varepsilon + \gamma f_\varepsilon = \big(\Gamma_{\leqslant j_s}(1+s\gamma\Delta)+\gamma\big)f_\varepsilon. \tag{9.43}$$

We see that they satisfy the requirements.

Lemma 9.2.3. *Let f_ε satisfy* (9.35)*, and let v and $\tilde{\mathfrak{f}} = (1+s\gamma\Delta)f_\varepsilon$ be defined by* (9.43) *and* (9.42)*, respectively. If $R_f^2C_f$ is sufficiently small, then then they satisfy* (A_v) *and* (A'_f)*, respectively, with $\mathfrak{n}=1$ and j_s given by* (9.39)*.*

Proof. We have from the assumption that f_ε has support on a block of side length $\frac{1}{24}L^{j_s+1}$ centred at 0, thus v is supported on $B_0^{j_s+1}$ and $\tilde{\mathfrak{f}}$ is supported on a block of side length $\frac{1}{12}L^{j_s+1}$, centred at 0. Also,

$$\mathrm{diam}(\mathrm{supp}\,\tilde{\mathfrak{f}})^2 \times \|\tilde{\mathfrak{f}}\|_{L^\infty(\mathbb{Z}^2)} \leqslant 4CR_f^2C_f \leqslant 1 \tag{9.44}$$

is implied by assuming $R_f^2C_f$ sufficiently small.

We are only left to bound v. By definition of $\tilde{\mathfrak{f}}$ and (9.35), we have $\|\tilde{\mathfrak{f}}\|_{C_0^2} \leqslant CC_f\varepsilon^2$, thus $\|\tilde{\mathfrak{f}}\|_{C_{j_s}^2} \leqslant C'$. To bound each $\Gamma_j\tilde{\mathfrak{f}}$ for $j \leqslant j_s$, we can identify Γ_j with its convolution kernel throughout the proof, i.e., $\Gamma_j\tilde{\mathfrak{f}} = \Gamma_j * \tilde{\mathfrak{f}}$. Then Γ_j is supported in a block of side length $\frac{1}{4}L^j$ and satisfies $\|\nabla_j^\alpha\Gamma_j\|_{L^\infty} \leqslant C_\alpha \log L$ for $|\alpha| \geqslant 0$ where $\nabla_j^\alpha = L^{j|\alpha|}\nabla^\alpha$, see Corollary 3.1.1. Thus $|\alpha| \geqslant 0$, using that $\nabla^\alpha(\Gamma_j * \tilde{\mathfrak{f}}) = \Gamma_j * (\nabla^\alpha\tilde{\mathfrak{f}})$ we obtain

$$\|\nabla_j^\alpha\Gamma_j\tilde{\mathfrak{f}}\|_{L^\infty} \leqslant \|\tilde{\mathfrak{f}}\|_{L^\infty}\sum_x |\nabla_j\Gamma_j(x)| \leqslant CC_fL^{2j}\varepsilon^2\log L. \tag{9.45}$$

Summing up,

$$\|v\|_{C^2_{j_s}} \leqslant CC_f L^{2j_s}\varepsilon^2 \log L \leqslant C' C_f R_f^2 \log L \tag{9.46}$$

where the second inequality follows from the choice of j_s.

□

9.2.2 Proof of Theorem 9.2.1

Lemma 9.2.4. *Let $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\,dx = 0$ and f_ε be as in* (9.35) *and recall*

$$\overline{C}(s) = \gamma(1+s\gamma\Delta) + (1+s\gamma\Delta)\tilde{C}(s)(1+s\gamma\Delta). \tag{9.47}$$

Then

$$\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}(f_\varepsilon, \overline{C}^{\Lambda_N}(s) f_\varepsilon) = \frac{1}{v_J^2+s}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}, \tag{9.48}$$

and the statement also holds if the two leftmost limits are replaced by $N\to\infty$ with $\varepsilon = \varepsilon_N \to 0$ while $\varepsilon_N L^N \to \infty$.

Proof. In what follows, given $f_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$, we denote by $\hat{f}_\varepsilon$ its Fourier transform, defined as in (2.19). Since $\hat{f}_\varepsilon(0) = 0$,

$$\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}(f_\varepsilon, \tilde{C}(s) f_\varepsilon) = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\lim_{m^2\downarrow 0}(f_\varepsilon, \overline{C}(s,m^2) f_\varepsilon) \tag{9.49}$$

where

$$\overline{C}(s,m^2) = \gamma(1+s\gamma\Delta) + (1+s\gamma\Delta)C(s,m^2)(1+s\gamma\Delta), \tag{9.50}$$

so we will study $\overline{C}(s)$ as a limit of $\overline{C}(s,m^2)$. Again, since $\hat{f}_\varepsilon(0) = 0$,

$$\begin{aligned}\lim_{N\to\infty}\lim_{m^2\to 0}(f_\varepsilon, \overline{C}(s,m^2) f_\varepsilon) &= \frac{1}{4\pi^2}\int_{[-\pi,\pi]^2} \frac{\lambda_J(p)^{-1}(1-s\gamma\lambda(p))}{1+s\lambda(p)(\lambda_J(p)^{-1}-\gamma)}|\hat{f}_\varepsilon(p)|^2\,dp \\ &= \frac{1}{4\pi^2}\int_{[-\pi/\varepsilon,\pi/\varepsilon]^2} \frac{\varepsilon^2\lambda_J(\varepsilon p)^{-1}(1-s\gamma\lambda(\varepsilon p))}{1+s\lambda(\varepsilon p)(\lambda_J(\varepsilon p)^{-1}-\gamma)}|\hat{f}_\varepsilon(\varepsilon p)|^2\,dp,\end{aligned} \tag{9.51}$$

where $\lambda(p)$ is the Fourier multiplier of the (unnormalised) discrete Laplacian $-\Delta$ and $\lambda_J(p)$ that of the (normalised) range-J Laplacian $-\Delta_J$, see Chapter 2. Then by (2.2.1),

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-2}\lambda(\varepsilon p) = |p|^2, \qquad \lim_{\varepsilon\downarrow 0}\varepsilon^{-2}\lambda_J(\varepsilon p) = v_J^2|p|^2, \tag{9.52}$$

and the fraction in the integrand in (9.51) is bounded by $C|p|^{-2}$ uniformly in ε and $p \in [-\pi/\varepsilon, \pi/\varepsilon]^2$. Moreover, as we now argue, (9.35) implies that $\hat{f}_\varepsilon(\varepsilon p) \to \hat{f}(p)$ as $\varepsilon \downarrow 0$ for each $p \in \mathbb{R}^2$ and that $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|(1+|p|)^{-3}$. To see this in detail, we start from

$$\hat{f}_\varepsilon(\varepsilon p) = \sum_{y \in \varepsilon\mathbb{Z}^2} f_\varepsilon(y/\varepsilon) e^{-iy\cdot p}. \tag{9.53}$$

For $|\hat{f}(p) - \hat{f}_\varepsilon(\varepsilon p)| \to 0$ pointwise, use $f \in C_c^\infty(\mathbb{R}^2)$ and the last condition in (9.35) to see that, with $[\cdot]$ denoting the integer part,

$$|\hat{f}(p) - \hat{f}_\varepsilon(\varepsilon p)| \leqslant \int_{\mathbb{R}^2} |f(y)(e^{-iy\cdot p} - e^{-i\varepsilon[y/\varepsilon]\cdot p})|\,dy + \int_{\mathbb{R}^2} |f(y) - \varepsilon^{-2} f_\varepsilon([y/\varepsilon])|\,dy \to 0. \tag{9.54}$$

To see the bound on $\hat{f}_\varepsilon(\varepsilon p)$, use summation by parts to write

$$\lambda(p)|\hat{f}_\varepsilon(p)| = |\widehat{\Delta f}_\varepsilon(p)| = |\sum_{x\in\mathbb{Z}^2} e^{-ip\cdot x}\Delta f_\varepsilon(x)| \leqslant \|\Delta f_\varepsilon\|_{L^1(\mathbb{Z}^2)}. \tag{9.55}$$

By (9.35),

$$\|\Delta f_\varepsilon\|_{L^1(\mathbb{Z}^2)} \leqslant R_f^2(\varepsilon^{-1}+1)^2\|\Delta f_\varepsilon\|_{L^\infty(\mathbb{Z}^2)} \leqslant 2R_f^2\|(\varepsilon^{-1}\nabla)^2 f_\varepsilon\|_{L^\infty(\mathbb{Z}^2)} \leqslant 2C_f R_f^2 \varepsilon^2, \tag{9.56}$$

and by [12, Lemma 2.2.1], we have that $\frac{1}{\varepsilon^2|p|^2}\lambda(\varepsilon p) \geqslant \frac{4}{\pi^2}$. Thus it follows that $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|^{-2}$. On the other hand, since $\sum f_\varepsilon = 0$ and $\|f_\varepsilon\|_{L^\infty} \leqslant C_f\varepsilon^2$, also

$$|\hat{f}_\varepsilon(\varepsilon p)| = \Big|\sum_{y\in\varepsilon\mathbb{Z}^2} f_\varepsilon(y/\varepsilon)(e^{-iy\cdot p}-1)\Big| \leqslant \|f_\varepsilon\|_{L^\infty} \sum_{y\in\varepsilon\mathbb{Z}^2:|y|\leqslant R_f} |y\cdot p| \leqslant CC_f(R_f/\varepsilon)^3|p|, \tag{9.57}$$

and therefore $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|(1+|p|)^{-3}$ when combined with $|\hat{f}_\varepsilon(\varepsilon p)| \leqslant C|p|^{-2}$.

Finally, using the convergence in Fourier space and that the integrand is dominated by $C|p|^{-2} \times (|p|(1+|p|)^{-3})^2 \leqslant C(1+|p|)^{-6}$ which is integrable over $\mathbb{R}^2$, the dominated convergence theorem implies

$$\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\lim_{m^2\to 0}(f_\varepsilon, \overline{C}(s,m^2)f_\varepsilon) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{1}{v_J^2+s}|p|^{-2}|\hat{f}(p)|^2\,dp = \frac{1}{v_J^2+s}(f,(-\Delta_{\mathbb{R}^2})^{-1}f) \tag{9.58}$$

as claimed. ☐

We are ready to prove the second main theorem.

Proof of Theorem 9.2.1. We first prove the theorem with the assumption that $R_f^2 C_f \leqslant \bar{c}$ is sufficiently small. By (9.41), we have

$$F_{N,m^2}[f_\varepsilon](1) = \frac{\mathbb{E}^{\varphi'} Z_N(\varphi', \omega = 1|\Lambda_N)}{\mathbb{E}^{\varphi'} Z_N(\varphi', \omega = 0|\Lambda_N)}, \tag{9.59}$$

so we only have to study Z_N. By Lemma 9.2.3 (using the assumption that $R_f^2 C_f$ is sufficiently small), we see that we are in place to apply Proposition 8.1.1. Since $\mathfrak{g}_j$ is non-zero only for $j > j_s$, when $\omega = 1$,

$$\begin{aligned} Z_N(\varphi', \omega = 1|\Lambda_N) &= e^{-E_N|\Lambda_N| + \sum_{j>j_s} \mathfrak{g}_j(\Lambda;1)} \big(e^{U_N^{\Lambda_N}(\Lambda_N, \varphi' + u_N)} + K_N^{\Lambda_N}(\Lambda_N, \varphi'; 1) \big) \\ &= e^{-E_N|\Lambda_N| + \sum_{j>j_s} \mathfrak{g}_j(\Lambda;1)} \big(e^{U_N^{\Lambda_N}(\Lambda_N, \varphi')} + K_N^{\Lambda_N,\dagger}(\Lambda_N, \varphi'; 1) \big), \end{aligned} \tag{9.60}$$

but by (7.3), (8.4) and Lemma 4.1.4,

$$= e^{-E_N|\Lambda_N| + O(L^{-\alpha j_s})} \big(e^{\frac{1}{2} s_N^{\Lambda_N} |\nabla \varphi'|^2} (1 + O(L^{-\alpha N})) + O(L^{-\alpha N}) G_N(\Lambda_N, \varphi') \big). \tag{9.61}$$

But since $\nabla \varphi' = 0$ almost surely when $\varphi' \sim \mathscr{N}(0, t_N Q_N)$, we have

$$F_{N,m^2}[f_\varepsilon](1) = \frac{\mathbb{E}^{\varphi'} Z_N(\varphi', \omega = 1|\Lambda_N)}{\mathbb{E}^{\varphi'} Z_N(\varphi', \omega = 0|\Lambda_N)} = 1 + O(L^{-\alpha j_s}). \tag{9.62}$$

Also, since $\langle e^{\beta^{-1/2}(f_\varepsilon, \sigma)} \rangle_{\beta,J}^{\Lambda_N}$ and $(f_\varepsilon, \overline{C}^{\Lambda_N}(s) f_\varepsilon)$ have well-defined limits as $N \to \infty$ by Proposition 1.1.2 and Lemma 9.2.4, respectively, the limit

$$\begin{aligned} \lim_{N\to\infty} \lim_{m^2 \downarrow 0} F_{N,m^2}[f_\varepsilon](1) &= \lim_{N\to\infty} e^{-\frac{1}{2}(\tilde{\mathfrak{f}}_\varepsilon, \tilde{C}(s) \tilde{\mathfrak{f}}_\varepsilon) - \frac{\gamma}{2}(f_\varepsilon, \tilde{\mathfrak{f}}_\varepsilon)} \langle e^{\beta^{-1/2}(f_\varepsilon, \sigma)} \rangle_{\beta,J}^{\Lambda_N} && (9.63) \\ &= \lim_{N\to\infty} e^{-\frac{1}{2}(f_\varepsilon, \overline{C}^{\Lambda_N}(s) f_\varepsilon)} \langle e^{\beta^{-1/2}(f_\varepsilon, \sigma)} \rangle_{\beta,J}^{\Lambda_N} \\ &= 1 + O(L^{-\alpha j_s}). && (9.64) \end{aligned}$$

is also well-defined. But since $L^{-\alpha j_s} \to 0$ as $\varepsilon \downarrow 0$, by Proposition 6.1.1,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \langle e^{\beta^{-1/2}(f_\varepsilon, \sigma)} \rangle_{\beta,J}^{\mathbb{Z}^2} &= \lim_{\varepsilon \downarrow 0} \lim_{N\to\infty} e^{\frac{1}{2}(f_\varepsilon, \overline{C}^{\Lambda_N}(s) f_\varepsilon)} \lim_{m^2 \downarrow 0} F_{N,m^2}[f_\varepsilon](1) \\ &= \exp\Big(\frac{1}{2(v_J^2 + s)} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2} \Big) && (9.65) \end{aligned}$$

is as desired due to Lemma 9.2.4.

Now we extend this result to general function f, without the restriction on $R_f^2 C_f$ using Gaussian domination inequality. Indeed, there exists $\tau > 0$ such that, if we consider τf instead of f, then $R_{\tau f}^2 C_{\tau f} = \tau R_f^2 C_f \leqslant \bar{c}$ is small enough to apply the arguments above. Then by (9.65) applied on τf_ε,

$$\begin{aligned} &\langle (f_\varepsilon, \sigma)_{\mathbb{Z}^2}^{2n} \rangle_{J,\beta}^{\mathbb{Z}^2} \to \frac{(2n)!}{2^n n!} \frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2} (f, (-\Delta)^{-1} f)_{\mathbb{R}^2}^n \\ &\langle (f_\varepsilon, \sigma)_{\mathbb{Z}^2}^{2n+1} \rangle_{J,\beta}^{\mathbb{Z}^2} = 0 \end{aligned} \tag{9.66}$$

as $\varepsilon \to 0$, for each $n \in \mathbb{N}$. Also,

$$\sum_{n>k} \frac{1}{n!} |(f_\varepsilon, \sigma)|^n \leqslant \frac{e^{(f_\varepsilon,\sigma)} + e^{-(f_\varepsilon,\sigma)}}{(k+1)!}, \tag{9.67}$$

but by the upper bound of (1.6), we see

$$\Big| \Big\langle \sum_{n>k} \frac{1}{n!} (f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N} \Big| \leqslant \frac{2}{(k+1)!} e^{\frac{\beta}{2}(f_\varepsilon, (-\Delta)^{-1} f_\varepsilon)}. \tag{9.68}$$

In other words, $\langle \sum_{n=0}^k \frac{1}{n!} (f_\varepsilon, \sigma)^n \rangle_{J,\beta}^{\Lambda_N}$ converges to $\langle e^{(f_\varepsilon,\sigma)} \rangle_{J,\beta}^{\Lambda_N}$ as $k \to \infty$ uniformly in ε and N, proving

$$\lim_{\varepsilon \to 0, N \to \infty} \lim_{k \to \infty} \Big\langle \sum_{n=0}^k \frac{1}{n!} (f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N} = \lim_{k \to \infty} \lim_{\varepsilon \to 0, N \to \infty} \Big\langle \sum_{n=0}^k \frac{1}{n!} (f_\varepsilon, \sigma)^n \Big\rangle_{J,\beta}^{\Lambda_N}. \tag{9.69}$$

But by (9.66), the right-hand side is $\frac{\beta_{\text{eff}}(J,\beta)}{2v_J^2} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2}$, completing the proof.

□

The proof of Theorem 9.2.2 is a by-product.

Proof of Theorem 9.2.2. Following the same extension procedure as in the proof of Theorem 1.1.4, it is enough to prove the statement with the assumption that $R_f^2 C_f$ is sufficiently small.

Let $j_{s,N}$ be the observable scale of f_{ε_N}. Then $\varepsilon_N \to 0$ as $N \to \infty$ implies $j_{s,N} \to \infty$. Thus by combining Proposition 6.1.1 and (9.62),

$$\langle e^{(f_{\varepsilon_N}, \sigma)_{\Lambda_N}} \rangle_{J,\beta}^{\Lambda_N} \sim e^{\frac{1}{2}(f_{\varepsilon_N}, \overline{C}^{\Lambda_N}(s) f_{\varepsilon_N})} (1 + O(L^{-j_{s,N}})) \sim e^{\frac{1}{2}(f_{\varepsilon_N}, \overline{C}^{\Lambda_N}(s) f_{\varepsilon_N})}, \tag{9.70}$$

as $N \to \infty$. An argument similar to that of Lemma 9.2.4 shows

$$\lim_{N\to\infty}(f_{\varepsilon_N}, \overline{C}^{\Lambda_N}(s) f_{\varepsilon_N}) = \frac{1}{v_J^2 + s}(f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2} \tag{9.71}$$

as long as $L^N \varepsilon_N \to \infty$ as $N \to \infty$, giving the desired conclusion.

□

9.3 Multi-point functions

We now prove Theorem 1.1.5, restated as the following, where we recall that (A_f) is a condition on $\mathfrak{f} = \sum_{i=1}^{\mathfrak{n}} T_{y_i}\mathfrak{f}_i$ restricting to take $\sum_{i=1}^{\mathfrak{n}} \mathfrak{f}_i = 0$ and the size $\mathfrak{n} M \rho^2 \leqslant 1$ where M is the upper bound of each component $\mathfrak{f}_i$ and ρ is the diameter.

Theorem 9.3.1. *Let $J \subset \mathbb{Z}^2 \backslash \{0\}$ be any finite-range step distribution that is invariant under lattice rotations and reflections and includes the nearest-neighbour vertices of 0. There exists a translation invariant covariance matrix $\mathfrak{C}_\beta \equiv \mathfrak{C}_{J,\beta}$, $\beta_0 \equiv \beta_0(J)$ and $h_\omega \equiv h_\omega(J)$ such that the following holds. Let $\omega \in \mathbb{D}_{h_\omega}$ with $h_\omega > 0$, $\beta \geqslant \beta_0$ and $\sigma \sim \mathbb{P}_{J,\beta}^{\mathbb{Z}^2}$. Then for $\mathfrak{f} = \sum_{i=1}^{\mathfrak{n}} T_{y_i}\mathfrak{f}_i$ satisfying (A_f),*

$$\log\langle e^{\beta^{-1/2}\omega(\mathfrak{f},\sigma)}\rangle_{J,\beta}^{\mathbb{Z}^2} = \frac{1}{2}\omega^2(\mathfrak{f}, \mathfrak{C}_\beta \mathfrak{f}) + \sum_{i=1}^{\mathfrak{n}} h_\beta^{(1)}[\mathfrak{f}_i](\omega) + h_\beta^{(2)}[\vec{y}, \vec{\mathfrak{f}}](\omega) \tag{9.72}$$

where $h_\beta^{(a)}$ ($a \in \{1,2\}$) are analytic functions in $\mathbb{D}_{h_\omega} \ni \omega$ satisfying the following.

- $|h_\beta^{(2)}[\vec{y}, \vec{\mathfrak{f}}](\omega)| = O_\beta(d_{\vec{y}}^{-\alpha})$ *uniformly in $\omega \in \mathbb{D}_{h_\omega}$ and some $\alpha > 0$.*
- $h_\beta^{(1)}[\mathfrak{f}_1] = h_\beta^{(1)}[-\mathfrak{f}_1]$, $h_\beta^{(1)}[0] = 0$ *and* $h_\beta^{(2)}[\vec{y}, \mathfrak{f}_1, 0, \cdots, 0] = 0$.

Again, in the proof, we always tune $s = s_0 = s_0^c(J, \beta)$ for $s_0^c(J, \beta)$ as in Theorem 9.1.1.

9.3.1 External field

Based on Proposition 6.1.1, the choice of the external fields are easier than the previous section. Let $\mathfrak{f} = \sum_{i=1}^{\mathfrak{n}} T_{y_i}\mathfrak{f}_i$ satisfy (A_f). It is enough to take $j_s = 0$ and

$$v_i = \gamma \mathfrak{f}_i, \qquad \tilde{\mathfrak{f}}_i = (1 + s\gamma\Delta)\mathfrak{f}_i, \tag{9.73}$$

$v = \sum_{i=1}^{\mathfrak{n}} T_{y_i} v_i$ and $\tilde{\mathfrak{f}} = \sum_{i=1}^{\mathfrak{n}} T_{y_i} \tilde{\mathfrak{f}}_i$. Then

$$F_{N,m^2}[\mathfrak{f}](\omega) = \frac{\mathbb{E}^{\varphi'} \mathbb{E}^{\overline{\zeta}}_{(\omega)} [Z_0^0(\varphi' + \overline{\zeta} + \omega v)]}{\mathbb{E}^{\varphi'} \mathbb{E}^{\overline{\zeta}} [Z_0^0(\varphi' + \overline{\zeta})]}. \tag{9.74}$$

It is not difficult to see that v and $\tilde{\mathfrak{f}}$ satisfy the desired assumptions.

Lemma 9.3.2. *For* $\mathfrak{f}$ *satisfying* (A_f)*, let* $L \geqslant 12\rho$ *and* $v, \tilde{\mathfrak{f}}$ *be given by* (9.73)*. Then they satisfy* (A_v) *and* (A'_f)*, respectively, with and* $j_s = 0$.

Proof. The condition on the support of v follows from $L \geqslant 12\rho$. The bound on norm of v_i follows because $\|v\|_{C_0^2} = \gamma \|\mathfrak{f}\|_{C_0^2} \leqslant \gamma M$. This verifies (A_v) for v. Also, since $\mathrm{diam}(\mathrm{supp}\, \tilde{\mathfrak{f}}) \leqslant \mathrm{diam}(\mathrm{supp}\, \mathfrak{f}) + 2$ and $\|\tilde{\mathfrak{f}}\|_{L^\infty} \leqslant (1 + 2|s|\gamma) \|\mathfrak{f}\|_{L^\infty}$, (A'_f) is readily verified for $\tilde{\mathfrak{f}}$. □

9.3.2 Reduction of the proof

In view of Proposition 6.1.1, control of the moment generating function is just due to the control of the ratio F_{N,m^2}. Thus it is the objective of Proposition 9.3.3 to show how F_{N,m^2} is controlled. In the statement, the *coalescence scale* $j_{\vec{y}}$ is used: when $\vec{y} = \{y_1, \cdots, y_{\mathfrak{n}}\}$,

$$j_{\vec{y}} = \min \Big\{ j \geqslant 0 \,:\, (Q^j_{y_a})^{***} \cap Q^j_{y_b} \neq \emptyset \text{ for some } a \neq b \Big\}. \tag{9.75}$$

(Recall Q^j_y is given by (3.25) and $(Q^j_{y_a})^{***}$ is taking the small set neighbourhood three times.) The coalescence scale is formally just $\log_L d_{\vec{y}}$, but stated in the language of blocks. If there is only one $a \in \{1, \cdots, \mathfrak{n}\}$ such that $\mathfrak{f}_a \neq 0$, we use convention $j_{\vec{y}} = \infty$. Since $Q^j_{y_a} \supset \mathrm{supp}(T_{y_a} u_{j,a})$, this definition implies $\mathrm{supp}(T_{y_a} u_{j,a}) \cap \mathrm{supp}(T_{y_b} u_{j,b}) = \emptyset$ for $j < j_{\vec{y}}$ and any $a \neq b$, and there exists $C > 0$ such that

$$C^{-1} d_{\vec{y}} \leqslant L^{j_{\vec{y}}} \leqslant C d_{\vec{y}} \quad (= C \min\{\|y_a - y_b\|_2 : a \neq b \in \{1, \cdots, \mathfrak{n}\}\}). \tag{9.76}$$

Proposition 9.3.3. *Let* $\mathfrak{f}$ *be as in* (A'_f)*. Then under the assumptions of Theorem 9.3.1,*

$$\lim_{m^2 \downarrow 0} F_{N,m^2}[\mathfrak{f}](\omega) = e^{\sum_{a=1}^{\mathfrak{n}} \tilde{g}_\infty[\mathfrak{f}_a](\omega) + \tilde{g}^{(2)}_\infty[\vec{y}, \vec{\mathfrak{f}}](\omega)} \big(1 + \tilde{\psi}^{\Lambda_N}(\omega)\big) \tag{9.77}$$

where $\tilde{g}_\infty[\mathfrak{f}_i](\omega)$, $g^{(2)}_\infty[\vec{y}, \vec{\mathfrak{f}}](\omega)$ *and* $\tilde{\psi}^{\Lambda_N}(\omega)$ *are analytic functions in* $\omega \in \mathbb{D}_{h_\omega}$ *and satisfy*

$$\|\tilde{g}^{(2)}_\infty[\vec{y}, \vec{\mathfrak{f}}](\omega)\|_{L^\infty(\mathbb{D}_{h_\omega})} \leqslant O(d_{\vec{y}}^{-\alpha}), \tag{9.78}$$

$$\|\tilde{\psi}^{\Lambda_N}\|_{L^\infty(\mathbb{D}_{h_\omega})} \leqslant O(L^{-\alpha N}) \tag{9.79}$$

for some $\alpha > 0$.

Proof of Theorem 9.3.1. By Proposition 6.1.1 and Proposition 9.3.3,

$$\begin{aligned}\log\langle e^{\beta^{-1/2}\omega(\sigma,\mathfrak{f})}\rangle_{\beta,J}^{\Lambda_N} = \frac{1}{2}\omega^2(\tilde{\mathfrak{f}},\tilde{C}(s)\tilde{\mathfrak{f}}) + \frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}}) + \sum_{a=1}^{\mathfrak{n}} \tilde{g}_\infty(\omega)[\mathfrak{f}_a] \\ + \tilde{g}_\infty^{(2)}[\vec{y},\vec{\mathfrak{f}}](\omega) + \log\big(1+\tilde{\psi}_{\Lambda_N}^{\mathfrak{r}}(\tau,y)\big).\end{aligned} \tag{9.80}$$

By (9.78),(9.79) and (9.76), they satisfy

$$\left|\tilde{g}_\infty^{(2)}[\vec{y},\vec{\mathfrak{f}}]\right| \leqslant O(d_{\vec{y}}^{-\alpha}), \qquad |\tilde{\psi}_{\Lambda_N}| \leqslant O(L^{-\alpha N}) \leqslant O(|\Lambda_N|^{-\alpha}). \tag{9.81}$$

If we let

$$\begin{aligned}H_N(\omega,\vec{y}) &= \log\langle e^{\beta^{-1/2}\omega(\sigma,\mathfrak{f})}\rangle_{\beta,J}^{\Lambda_N} - \frac{1}{2}\omega^2(\tilde{\mathfrak{f}},\tilde{C}(s)\tilde{\mathfrak{f}}) - \frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}}) \\ &= \sum_{a=1}^{\mathfrak{n}} \tilde{g}_\infty[\mathfrak{f}_a](\omega) + \tilde{g}_\infty^{(2)}[\vec{y},\vec{\mathfrak{f}}] + \log(1+\tilde{\psi}^{\Lambda_N}(\omega,y)),\end{aligned} \tag{9.82}$$

then it is an analytic function of $\omega \in D_{h_\omega}$), and

$$h_\beta^{(1)}[\mathfrak{f}_a](\omega) = \frac{\gamma}{2}\omega^2(\mathfrak{f}_a,\tilde{\mathfrak{f}}_a) + \tilde{g}_\infty[\mathfrak{f}_a](\omega) \tag{9.83}$$

$$h_\beta^{(2)}[\vec{y},\vec{\mathfrak{f}}](\omega) = \frac{\gamma}{2}\omega^2(\mathfrak{f},\tilde{\mathfrak{f}}) - \frac{\gamma}{2}\omega^2\sum_{a=1}^{\mathfrak{n}}(\mathfrak{f}_a,\tilde{\mathfrak{f}}_a) + \tilde{g}_\infty^{(2)}[\vec{y},\vec{\mathfrak{f}}](\omega) \tag{9.84}$$

$$\psi_{\beta,N}(\omega) = \log(1+\tilde{\psi}^{\Lambda_N}(\omega)) \tag{9.85}$$

are also analytic functions of ω. Thus with the above choices of $h_\beta^{(1)}, h_\beta^{(2)}, \psi_{\beta,N}$,

$$\mathfrak{C}_{\beta,\Lambda_N} = (1+s\gamma\Delta)\tilde{C}(s)(1+s(\beta)\gamma\Delta), \tag{9.86}$$

we have

$$\langle e^{\beta^{-1/2}\omega(\mathfrak{f},\sigma)}\rangle_{\beta,J}^{\Lambda_N} = \frac{1}{2}(\mathfrak{f},\mathfrak{C}_{\beta,\Lambda_N}\mathfrak{f}) + \sum_{a=1}^{\mathfrak{n}} h_\beta^{(1)}[\mathfrak{f}_a](\omega) + h_\beta^{(2)}[\vec{y},\vec{\mathfrak{f}}](\omega) + \psi_{\beta,N}(\omega,\vec{y}) \tag{9.87}$$

and we have the desired conclusion upon taking limit $N\to\infty$, with the required estimates following from (9.81). (To see that the limit $\lim_{N\to\infty}(\mathfrak{f},\mathfrak{C}_{\beta,\Lambda_N}\mathfrak{f})$ is well-defined, take any $g = g_1 + T_y g_2$ that satisfies the assumptions of (A_f), then we have $|(g,\Gamma_j g)| = O\big(L^{-j}|y|\big)$ and $|(g,\tilde{\Gamma}_N^{\Lambda_N} g)| = O\big(L^{-N}|y|\big)$ (see Corollary 3.1.1), so $\lim_{N\to\infty}(g,\mathfrak{C}_{\beta,\Lambda_N} g)$ absolutely converges.

Hence, if we let

$$\mathfrak{C}_\beta := \sum_{j=1}^{\infty}(1+s\gamma\Delta)\Gamma_j(s)(1+s\gamma\Delta), \tag{9.88}$$

then $(\mathfrak{f}, \mathfrak{C}_\beta \mathfrak{f})$ is well-defined an equals $\lim_{N\to\infty}(\mathfrak{f}, \mathfrak{C}_{\beta,\Lambda_N})\mathfrak{f})$.) □

We also prove Lemma 1.2.3 and Lemma 1.2.8 here.

Proof of Lemma 1.2.3. With $f_y = \delta_0 - \delta_y$, we will compute

$$(f_y, \mathfrak{C}_\beta f_y) := \lim_{N\to\infty}(\mathfrak{f}, \mathfrak{C}_{\beta,\Lambda_N})\mathfrak{f}). \tag{9.89}$$

We will always fix $s = s_0^c$. Since $\tilde{C}^{\Lambda_N}(s) = \lim_{m^2\downarrow 0} C(s,m^2) - t_N(m^2)Q_N$ and $\mathfrak{C}_{\Lambda_N,\beta} = (1+s\gamma\Delta)^2\tilde{C}(s)$, we have in the Fourier space

$$\begin{aligned}(f_y, \mathfrak{C}_{\beta,\Lambda_N} f_y) = \frac{1}{4\pi^2|\Lambda_N|}&\sum_{p\in\Lambda_N^*}|1-e^{-iy\cdot p}|^2|1-s_0^c\gamma\lambda(p)|^2 \\ &\times \lim_{m^2\downarrow 0}\Big(\frac{(\lambda_J(p)+m^2)^{-1}-\gamma}{1+s\lambda(p)((\lambda_J(p)+m^2)^{-1}-\gamma)} - t_N\delta_0(p)\Big)\end{aligned} \tag{9.90}$$

where $\lambda(p) = 4-2\cos(p_1)-2\cos(p_2)$ and $\lambda_J(p) = \frac{1}{|J|}\sum_{x\in J}(1-\cos(x\cdot p))$ (see (2.22)) are the Fourier multipliers of $-\Delta$ and $-\Delta_J$, respectively, and Λ_N^* is the Fourier dual lattice of Λ_N. Since $1-e^{-iy\cdot p} = 0$ when $p=0$, we may ignore $t_N\delta_0(p)$ term. In the limit $N\to\infty$, this discrete sum converges to the integral

$$\frac{1}{4\pi^2}\int_{[-\pi,\pi]^2} dp\,|1-e^{-iy\cdot p}|^2|1-s_0^c\gamma\lambda(p)|^2 \lim_{m^2\downarrow 0}\Big(\frac{(\lambda_J(p)+m^2)^{-1}-\gamma}{1+s_0^c\lambda(p)((\lambda_J(p)+m^2)^{-1}-\gamma)}\Big). \tag{9.91}$$

From this representation, as $\|y\|_2\to\infty$,

$$\begin{aligned}(f_y, \mathfrak{C}_{\mathbb{Z}^2,\beta} f_y) &= C_1(\beta) + \frac{1}{1+v_J^{-2}s}\big(f_y,(-\Delta)^{-1}f_y\big) + O(\|y\|_2^{-2}) \\ &= C_2(J,\beta) + \frac{\beta_{\mathrm{eff}}(J,\beta)/\beta}{\pi}\log\|y\|_2 + O(\|y\|_2^{-2})\end{aligned} \tag{9.92}$$

for some $C_1(\beta), C_2(J,\beta)\in\mathbb{R}$, and we have used that $\beta_{\mathrm{eff}}(J,\beta)/\beta = \frac{1}{1+v_J^{-2}s_0^c}$ (recall (7.4)). □

Proof of Lemma 1.2.8. Suppose $\mathrm{supp}(\mathfrak{f}_1) \subset [-R,R]^2 \cap \mathbb{Z}^2$ and $\sum_x \mathfrak{f}_1(x) = 0$, then there exist $a : [-R,R]^2 \cap \mathbb{Z}^2 \to \mathbb{R}$ such that $\mathfrak{f}_1 = \nabla^{e_1} a = \sum_x a(x)(\delta_{x+e_1} - \delta_x)$—$a(x) = -\sum_{n\geqslant 0} \mathfrak{f}_1(x - ne_1)$ would suffice. The same remark applies for $\mathfrak{f}_2$, thus proving Lemma 1.2.8 is actually equivalent to showing

$$(\delta_{y+e_1} - \delta_y, \mathfrak{C}_\beta(\delta_{e_1} - \delta_0)) = \nabla^{-e_1}\nabla^{e_1}\mathfrak{C}_\beta(0,y) = O(\|y\|_2^{-2}). \tag{9.93}$$

As in the proof of Lemma 1.2.3 above, $(\delta_{y+e_1} - \delta_y, \mathfrak{C}_{\beta,\Lambda_N}(\delta_{e_1} - \delta_0))$ converges as $N \to \infty$ to

$$\frac{1}{4\pi^2}\int_{[-\pi,\pi]^2} dp\, e^{-iy\cdot p}|1-e^{-ip_1}|^2|1-s_0^c\gamma\lambda(p)|^2 \lim_{m^2\downarrow 0}\Big(\frac{(\lambda(p)/4+m^2)^{-1}-\gamma}{1+s_0^c\lambda(p)((\lambda(p)/4+m^2)^{-1}-\gamma)}\Big). \tag{9.94}$$

(λ_J is replace by $\lambda/4$ because $J = J_{\mathrm{nn}}$, and $1/4$ accounts for the normalising factor.) Since the integrand is smooth away from the singularity, as $\|y\|_2 \to \infty$, the integral is asymptotically equivalent to the integral of the singular part,

$$\begin{aligned} &\sim \frac{1}{4\pi^2(1+s_0^c/4)}\int_{[-\pi,\pi]^2} dp\, e^{-iy\cdot p}|1-e^{-ip_1}|^2\lambda(p)^{-1} + O(\|y\|_2^{-10}) \\ &= \frac{1}{1+s_0^c/4}\nabla^{(e_1,-e_1)}(-\Delta_{\mathbb{Z}^2})^{-1}(0,y) + O(\|y\|_2^{-10}) \end{aligned} \tag{9.95}$$

where $(-\Delta_{\mathbb{Z}^2})^{-1}$ is the lattice Green's function of the usual Laplacian and 10 is an arbitrary large number. Thus it is bounded by $O(\|y\|_2^{-2})$.

□

9.3.3 Infinite volume limit

Based on the RG analysis, we can show that

$$F_{N,m^2}[\mathfrak{f}](\omega) = \frac{\mathbb{E}_{(\omega)}\big[Z_0^0(\phi^{(m^2)} + \gamma\omega\mathfrak{f})\big]}{\mathbb{E}\big[Z_0^0(\phi^{(m^2)})\big]} \tag{9.96}$$

exhibits a well-defined limit as $m^2 \downarrow 0$ and $N \to \infty$ described in terms of the RG coordinates, where $\phi^{(m^2)} \sim \mathcal{N}(0, \tilde{C}(s) + t_N(m^2)Q_N)$ (see Proposition 6.1.1).

Proposition 9.3.4. *Let $\mathfrak{f}$ be as in* (A'_f). *Under the assumptions of Proposition 8.1.1, if* $\omega \in \mathbb{D}_{h_\omega}$, *then*

$$F_{N,m^2}[\mathfrak{f}] \xrightarrow{N\to\infty} \exp\Big(\sum_{j=1}^{\infty} \sum_{B\in\mathscr{B}_j((P^j_{\vec{y}})^*)} \mathfrak{g}_j(B;\omega)[\mathfrak{f}]\Big) \quad \textit{uniformly in } m^2 \in (0,1]. \tag{9.97}$$

Proof. By Proposition 8.1.1,

$$Z_N(\varphi';\omega) = e^{-E_N^{\Lambda_N}|\Lambda_N|} \exp\Big(\sum_{j=1}^{N} \sum_{B\in\mathscr{B}_j((P^j_{\vec{y}})^*)} \mathfrak{g}_j^{\Lambda_N}[\mathfrak{f}](B;\omega)\Big) \times \big(e^{U_N^{\Lambda_N}(\Lambda_N,\varphi'+\omega u_N)} + K_N^{\Lambda_N}(\Lambda_N,\varphi';\omega)\big) \tag{9.98}$$

with $u_N = \Gamma_N^{\Lambda_N}\mathfrak{f}$, and satisfy estimates

$$\|\mathfrak{g}_N^{\Lambda_N}\|_{h_\omega,T},\ \ \|U_N^{\Lambda_N}\|_N,\ \ \|K_N^{\Lambda_N}\|_N \leqslant Ce^{-\frac{1}{4}\gamma\beta}L^{-\alpha N}. \tag{9.99}$$

Also by Lemma 4.1.4,

$$e^{U_N^{\Lambda_N}(\Lambda_N,\varphi'+\omega u_N)} + K_N^{\Lambda_N}(\Lambda_N,\varphi';\omega) = e^{U_N^{\Lambda_N}(\Lambda_N,\varphi')} + (K_N^{\Lambda_N})^\dagger(\Lambda_N,\varphi';\omega) \tag{9.100}$$

with

$$\big\|(K_N^{\Lambda_N})^\dagger(\Lambda_N,\varphi')\big\|_{\Omega^K_{N,\dagger}} \leqslant C'e^{-\frac{1}{4}\gamma\beta}L^{-\alpha N}G_N(\Lambda_N,\varphi'). \tag{9.101}$$

Finally, we commence the integral in $\varphi' \sim \mathscr{N}(0,t_NQ_N)$. Observe that, if we take $Y \sim \mathscr{N}(0,t_NL^{-N})$, then $Y1$ has the same distribution as φ' (where 1 is the constant field taking value 1) so $G_N(\Lambda_N,\varphi') \equiv 1$ almost surely. Also $|\nabla\varphi'|^2 = 0$ almost surely, so $|U_N^{\Lambda_N}(\Lambda_N,\varphi')| \leqslant \|U_N^{\Lambda_N}\|_{\Omega^U_N}$. Therefore

$$\mathbb{E}^{\varphi'}\big[Z_N(\varphi';\omega)\big] = e^{-E_N^{\Lambda_N}|\Lambda_N|} \exp\Big(\sum_{j=1}^{N} \sum_{B\in\mathscr{B}_j((P^j_{y})^*)} \mathfrak{g}_j^{\Lambda_N}[\mathfrak{f}](B;\omega)\Big)\big(1+O(L^{-\alpha N})\big). \tag{9.102}$$

Since $\mathfrak{g}_j^{\Lambda_N}$ is independent of Λ_N for $N > \max\{j, j_{0y}+2\}$, the convergence (9.97) holds by (9.99) and estimates

$$\|\mathfrak{g}_j(B)\|_{h_\omega,T} \leqslant Ce^{-\frac{1}{4}\gamma\beta}L^{-\alpha j}, \qquad B\in\mathscr{B}_j. \tag{9.103}$$

The convergence is uniform in m^2 because the convergence rate only depends on $O(L^{-\alpha N})$ and the bound on $\mathfrak{g}_N^{\Lambda_N}[\mathfrak{f}](B;\omega)$, which are independent of $m^2 > 0$. $\square$

In this proposition, the finite volume function was computed only to justify that the infinite volume limit exists, but we can also record the finite volume result as a by-product. In what follows, we use the notation

$$\mathfrak{g}_j[\mathfrak{f}](\mathbb{Z}^2;\omega) := \sum_{B \in \mathscr{B}_j((P_{\vec{y}}^j)^*)} \mathfrak{g}_j[\mathfrak{f}](B;\omega) \tag{9.104}$$

although $\mathfrak{g}_j(B;\omega)[\mathfrak{f}]$ is actually defined on Λ_N.

Corollary 9.3.5. *Fix $R > 0$ and let $\omega \in \mathbb{D}_{h_\omega}$. Then for sufficiently large L (depending on R) and N, and under the same assumptions as in Proposition 9.3.4 but $\mathfrak{f}$ satisfying (A_f),*

$$\lim_{m^2 \downarrow 0} F_{N,m^2}[\mathfrak{f}] = e^{\sum_{j=1}^{\infty} \mathfrak{g}_j[\mathfrak{f}](\mathbb{Z}^2;\omega)} \big(1 + \tilde{\psi}_N[\mathfrak{f}](\omega)\big) \tag{9.105}$$

where $\tilde{\psi}_N(\omega)$ is an analytic function of $\omega \in \mathbb{D}_{h_\omega}$ and $\|\tilde{\psi}_N(\omega)\|_{h_\omega,T} = O(L^{-\alpha N})$.

Proof. By Proposition 6.1.1, F_{N,m^2} admits a limit as $m^2 \downarrow 0$ when $\mathfrak{f}$ satisfies (A_f) and the limit is an analytic function. It follows from the uniformity of (9.102) in m^2 and definition of F_{N,m^2} that

$$1 + \tilde{\psi}_N(\omega) := e^{-\sum_{j=1}^{\infty} \mathfrak{g}_j(\mathbb{Z}^2;\omega)} \lim_{m^2 \downarrow 0} F_{N,m^2}[\mathfrak{f}] \tag{9.106}$$

satisfies

$$1 + \tilde{\psi}_N(\omega) = e^{-\sum_{j>N} \mathfrak{g}_j(\mathbb{Z}^2;\omega)} \lim_{m^2 \downarrow 0} \mathbb{E}\Big[e^{U_N^{\Lambda_N}(\Lambda_N, Y1)} + (K_N^{\Lambda_N})^\dagger(\Lambda_N, Y1;\omega) \Big] = 1 + O(L^{-\alpha N}) \tag{9.107}$$

with $Y \sim \mathscr{N}(0, t_N(m^2)L^{-N})$. It also satisfies $\|\tilde{\psi}_N(\omega)\|_{h_\omega,T} = O(L^{-\alpha N})$ as

$$\|\sum_{j>N} \mathfrak{g}_j(\mathbb{Z}^2;\omega)\|_{h_\omega,T} \leqslant O(L^{-\alpha N}) \tag{9.108}$$

$$\|\mathbb{E}[(K_N^{\Lambda_N})^\dagger(\Lambda_N, Y1;\omega)]\|_{h_\omega,T} \leqslant C\|(K_N^{\Lambda_N})^\dagger(\Lambda_N)\|_{\Omega_{N,\dagger}^K} \tag{9.109}$$

for any $n \geqslant 0$. Thus we have (9.105).

$\square$

9.3.4 One and multi-point energies

Due to Corollary 9.3.5, the proof of Proposition 9.3.3 is complete once we show that the sum $\sum_j \mathfrak{g}_j[\mathfrak{f}](\mathbb{Z}^2;\omega)$ is not subject to a bias due to the multi-scale grid structure. In this section, we will see that this is the case, using translation invariance of $F_{N,m^2}[\mathfrak{f}]$.

One-point energy

For any $j \geqslant 1$ and $\alpha \in \{1,\cdots,\mathfrak{n}\}$, we have

$$\sum_{B\in\mathscr{B}_j((P^j_{\vec{y}})^*)} \mathfrak{g}_j(B;\omega)[T_{y_\alpha}\mathfrak{f}_\alpha] = \sum_{B\in\mathscr{B}_j((Q^j_{y_\alpha})^*)} \mathfrak{g}_j(B;\omega)[T_{y_\alpha}\mathfrak{f}_\alpha]. \tag{9.110}$$

Hence, by (9.97), this implies

$$\exp\Big(\sum_{j=1}^{\infty}\sum_{B\in\mathscr{B}_j((Q^j_{y_\alpha})^*)} \mathfrak{g}_j(B;\omega)[T_{y_\alpha}\mathfrak{f}_\alpha]\Big) = \lim_{N\to\infty} F_{N,m^2}[T_{y_\alpha}\mathfrak{f}_\alpha] \quad \text{uniformly in } m^2>0. \tag{9.111}$$

Also, if $\mathfrak{f}=\mathfrak{f}_\alpha$ with $y_\alpha=0$, the same principles give

$$\exp\Big(\sum_{j\geqslant 1}\sum_{B\in\mathscr{B}_j((Q^j_0)^*)} \mathfrak{g}_j(B;\omega)[\mathfrak{f}_\alpha]\Big) = \lim_{N\to\infty} F_{N,m^2}[\mathfrak{f}_\alpha] \quad \text{uniformly in } m^2>0. \tag{9.112}$$

But since $F_{N,m^2}[T_y\mathfrak{f}_\alpha]$ is independent of y by definition, we also see that the expression on the left side of (9.111),

$$\exp\Big(\sum_{j=1}^{\infty}\sum_{B\in\mathscr{B}_j((Q^j_y)^*)} \mathfrak{g}_j(B;\omega)[T_y\mathfrak{f}_\alpha]\Big) = \exp\Big(\sum_{j\geqslant 1}\mathfrak{g}_j(\mathbb{Z}^2;\omega)[T_y\mathfrak{f}_2]\Big) \tag{9.113}$$

should also be independent of y, i.e.,

$$\sum_{j\geqslant 1}\mathfrak{g}_j(\mathbb{Z}^2;\omega)[T_y\mathfrak{f}_2] = \sum_{j\geqslant 1}\sum_{B\in\mathscr{B}_j((Q^j_0)^*)} \mathfrak{g}_j(B;\omega)[\mathfrak{f}_2]. \tag{9.114}$$

In summary, we have the following.

Proposition 9.3.6. *Let $\omega \in \mathbb{D}_{h_\omega}$ and $\beta > 0$ be sufficiently large and $\mathfrak{f} = \sum_{a=1}^{\mathfrak{n}} \mathfrak{f}_a$ be as in* (A_f). *Define the infinite-volume one-point energy of $\mathfrak{f}_\alpha$ by*

$$\tilde{g}_\infty[\mathfrak{f}_\alpha](\omega) = \sum_{j\geqslant 1} \mathfrak{g}_j[\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega). \tag{9.115}$$

Then $\tilde{g}_\infty[\mathfrak{f}_\alpha](\omega) = \sum_{j\geqslant 1} \mathfrak{g}_j[T_y\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega)$ for any y. This series converges absolutely with rate

$$\left\| \tilde{g}_\infty[\mathfrak{f}_\alpha](\omega) - \sum_{j=1}^{m} \mathfrak{g}_j[T_y\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega) \right\|_{h_\omega,T} \leqslant CL^{-\alpha m} \tag{9.116}$$

with rate uniform on $y \in \mathbb{Z}^2$, and $\tilde{g}_\infty[\mathfrak{f}_\alpha](\omega)$ is analytic on $\mathbb{D}_{h_\omega}$.

Proof. The first part follows from the discussion above. Also the second part is a consequence of the estimate of (8.4) and recalling that any uniform limit of analytic functions is also an analytic function. □

Multi-point energy

The multi-point energy is defined to be the free energy in scales after $j_{\vec{y}}$. But for doing so, we will have to make sure that the free energy before the scale $j_{\vec{y}}$ can be expressed as sum of the one-point energies.

Lemma 9.3.7. *Let $j \leqslant j_{\vec{y}} - 1$ and $\mathfrak{f} = \sum_{\alpha=1}^{\mathfrak{n}} T_{y_\alpha}\mathfrak{f}_\alpha$ be as in* (A_f). *Then $\mathfrak{g}_j[\mathfrak{f}](B) = \mathfrak{g}_j[\mathfrak{f}_\alpha](B)$ for $B \in \mathscr{B}_{j-1}((Q_{y_\alpha}^{j-1})^*)$.*

Proof. These follow from the 'local dependence' of the RG flow as in the proof of Proposition 7.4.2. For the proof, we make the dependence on $\mathfrak{f}$ explicit by putting them inside $[\cdot]$. We assume, as an induction hypothesis, that

$$\begin{aligned} &K_{j'}(X)[\mathfrak{f}] = K_{j'}(X)[\mathfrak{f}_\alpha] \\ &\qquad\qquad \text{for any } X \in \mathscr{P}_{j'},\ X \cap (Q_{y_a}^{j_{\vec{y}}-1})^* = \emptyset,\ a \neq \alpha,\ j' < j_{0y} - 1. \end{aligned} \tag{9.117}$$

We have $X^* \cap \operatorname{supp}(T_{y_a}u_{j,a}) = \emptyset$ for any such X, so this would imply $K_{j'}^\dagger(X)[\mathfrak{f}] = K_{j'}^\dagger(X)[\mathfrak{f}_\alpha]$. Also, by definition of $j_{\vec{y}}$, saying $j' < j_{\vec{y}} - 1$ would mean $(Q_{y_\alpha}^{j'+1})^{**} \cap (Q_{y_a}^{j_{\vec{y}}-1})^* = \emptyset$ for each $a \neq \alpha$, so $\mathfrak{g}_{j'+1}[\mathfrak{f}](B) = \mathfrak{g}_{j'+1}[\mathfrak{f}_\alpha](B)$ for any $B \in \mathscr{B}_{j'+1}((Q_{y_\alpha}^{j'+1})^{**})$ by the definition of $\mathfrak{g}_{j'+1}$. If we also had $j' < j_{0y} - 1$, $Y \in \mathscr{P}_{j'+1}$, and $Y \cap (Q_{y_a}^{j'+1})^* = \emptyset$ for each $a \neq \alpha$, since $K_{j'+1}(Y)$ only depends on $\mathfrak{f}$ via $u_{j+1}|_{Y^*}$, $(\mathfrak{g}_{j'+1}(B))_{B\in(B_{y_\alpha}^{j'+1})^*}$ and $(K_{j'}^\dagger(X') : X' \in \mathscr{P}_{j'}(X^*), X \in \mathscr{P}_{j'}(Y))$, the induction proceeds. The conclusion was also obtained in the course of the induction argument. □

As we have claimed, this lemma implies that the free energy before scale $j_{\vec{y}}$ is just the sum of two one-point energies.

Corollary 9.3.8. *If $j < j_{\vec{y}}$, then*

$$\mathfrak{g}_j[\sum_{\alpha=1}^{\mathfrak{n}} T_{y_\alpha}\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega) = \sum_{\alpha=1}^{\mathfrak{n}} \mathfrak{g}_j[T_{y_\alpha}\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega) \tag{9.118}$$

Thus we can see that the multi-point energy, defined in the following lemma, has diminishing contribution in the limit $d_{\vec{y}} \to \infty$.

Lemma 9.3.9. *Let $\omega \in \mathbb{D}_{h_\omega}$ and $\beta > 0$ be sufficiently large. Define*

$$\tilde{g}_\infty^{(2)}[\vec{y},\mathfrak{f}_1,\cdots,\mathfrak{f}_\mathfrak{n}](\omega) = \sum_{j=1}^{\infty} \mathfrak{g}_j[\mathfrak{f}](\mathbb{Z}^2;\omega) - \sum_{\alpha=1}^{\mathfrak{n}} \tilde{g}_\infty[\mathfrak{f}_\alpha](\omega). \tag{9.119}$$

Then $\tilde{g}_\infty^{(2)}[\vec{y},\mathfrak{f}_1,\cdots,\mathfrak{f}_\mathfrak{n}](\omega)$ is analytic in $\mathbb{D}_{h_\omega} \ni \omega$ and satisfies the bound

$$\big\|\tilde{g}_\infty^{(2)}[\vec{y},\mathfrak{f}_1,\cdots,\mathfrak{f}_\mathfrak{n}](\cdot)\big\|_{h_\omega,T} = O\big(d_{\vec{y}}^{-\alpha}\big). \tag{9.120}$$

Proof. By Corollary 9.3.8 and Proposition 9.3.6 we have

$$\tilde{g}_\infty^{(2)}[\vec{y},\mathfrak{f}_1,\cdots,\mathfrak{f}_\mathfrak{n}](\omega) = \sum_{j=j_{\vec{y}}}^{\infty} \Big(\mathfrak{g}_j[\mathfrak{f}](\mathbb{Z}^2;\omega) - \textstyle\sum_{\alpha=1}^{\mathfrak{n}} \mathfrak{g}_j[T_{y_\alpha}\mathfrak{f}_\alpha](\mathbb{Z}^2;\omega)\Big). \tag{9.121}$$

Hence by (8.4), the norm on $\tilde{g}_\infty^{(2)}$ is bounded by $O(L^{-\alpha j_{\vec{y}}})$. But since $L^{-j_{\vec{y}}} = O(d_{\vec{y}}^{-1})$, we have the desired conclusion. □

Thus we are equipped with all components required to prove Proposition 9.3.3.

Proof of Proposition 9.3.3. Our aim is to rewrite the limit $\lim_{m^2\downarrow 0} F_{N,m^2}[\mathfrak{f}]$. By Corollary 9.3.5, Proposition 9.3.6 and Lemma 9.3.9,

$$\lim_{m^2\downarrow 0} F_{N,m^2}[\mathfrak{f}] = e^{\sum_{\alpha=1}^{\mathfrak{n}} \tilde{g}_\infty[\mathfrak{f}_\alpha] + \tilde{g}_\infty^{(2)}[\vec{y},\mathfrak{f}_1,\cdots,\mathfrak{f}_\mathfrak{n}])}(1+\tilde{\psi}_N). \tag{9.122}$$

and the required properties also follow from the same references. □

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