Signatures of Fractional Exclusion Statistics in the Spectroscopy of Quantum Hall Droplets

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(Dated: February 3, 2015)

We show how spectroscopic experiments on a small Laughlin droplet of rotating bosons can directly demonstrate Haldane fractional exclusion statistics of quasihole excitations. The characteristic signatures appear in the single-particle excitation spectrum. We show that the transitions are governed by a "many-body selection rule" which allows one to relate the number of allowed transitions to the number of quasihole states on a finite geometry. We illustrate the theory with numerically exact simulations of small numbers of particles.

PACS numbers: 67.85.-d, 73.43.Cd, 05.30.Pr

One of the most dramatic features of strongly correlated phases is the emergence of quasiparticle excitations with unconventional quantum statistics. The archetypal example is the fractional, "anyonic", quantum statistics predicted for the quasiparticles of the fractional quantum Hall phases[1, 2]. While experiments on semiconductor devices have shown that these quasiparticles have fractional charges[3–5], a direct observation of the fractional statistics has remained lacking.

In this Letter we show how precision spectroscopy measurements of rotating droplets of ultracold atoms could be used to demonstrate the Haldane fractional exclusion statistics[6] of quasiholes in the Laughlin state of bosons. By involving only spectroscopic signatures of the rotating droplet, our proposal plays to the strengths of atomic physics experiments. We show that evidence of the fractional exclusion statistics appears in counting the numbers of lines in the radio-frequency (RF) absorption spectrum. In this sense, the method is conceptually similar to classic evidence of quantum statistics, as appearing in the rotational levels of homonuclear diatomic molecules (e.g. the Fermi statistics of the proton causing the rotational levels of H_2 to depend on whether the spins of the nuclei are in singlet or triplet state). Our method differs substantially from proposals to measure the fractional braiding statistics of quasiholes[7–9], notably by not requiring local time-dependent potentials for the adiabatic manipulation of the positions of the quasiholes.

We have in mind a fast rotating gas of identical bosonic atoms, initially in a single internal (hyperfine) state $\Uparrow$, and confined to a quasi-2D layer with oscillator length a_z. The gas is subjected to a tight circularly symmetric harmonic trap of frequency ω_0 and oscillator length a_0, with $\hbar\omega_0 \gg V_0$, in which $V_0 \equiv \sqrt{\frac{2}{\pi}}\frac{a_\mathrm{s}}{a_z}\hbar\omega_0$ is a characteristic interaction energy for atoms with scattering length a_s. Hence, the interactions leave the particles in the lowest Landau level (LLL)[10]. In addition, we shall consider a weak axisymmetric quartic potential $V_\mathrm{Q} = \eta_\mathrm{Q}(|\boldsymbol{r}|/a_0)^4$, with $\eta_\mathrm{Q} \ll \hbar\omega_0 \ll V_0$, for reasons to be described below. Specifically, we shall consider an initial state of N_i atoms which has been spun up to the angular momentum $L_\mathrm{i} = N_\mathrm{i}(N_\mathrm{i} - 1)$. Then, for the case of contact repulsive interactions relevant in typical cold gas experiments, the groundstate is the (exact) $\nu = 1/2$ Laughlin state. Furthermore, for the case of contact interactions, the quasihole excitations of these Laughlin droplets are non-interacting: this will allow us to find evidence of fractional exclusion statistics even in small systems of $N_\mathrm{i} \lesssim 10$ atoms[11]. Experimental protocols to generate this initial state for small numbers of atoms have been identified[12, 13], and experimental work on driven lattices[14] has investigated the properties of rapid rotation on multi-droplet systems. We shall focus on the properties of a single droplet, and the spectroscopic signatures we seek shall require single-atom imaging; such conditions are likely to require technologies developed in ultracold gas microscopes[15, 16].

Now consider making an RF excitation of a single atom from internal state $\Uparrow$ into an internal state $\Downarrow$ which does not interact with the initial $\Uparrow$ atoms. For hyperfine states, such situations can be found by tuning to the zero of a Feshbach resonance. The promoted atom can carry away an angular momentum, m_f, in the range $0 \leq m_\mathrm{f} \leq m_\mathrm{max}$, leaving the $N_\mathrm{f} = N_\mathrm{i} - 1$ atoms with angular momentum $L_\mathrm{f} = L_\mathrm{i} - m_\mathrm{f}$ in the range $L_\mathrm{i} - m_\mathrm{max} \leq L_\mathrm{f} \leq L_\mathrm{i}$. (The upper limit $m_\mathrm{max} = 2(N_\mathrm{i} - 1) = 2N_\mathrm{f}$ is the highest angular momentum carried by any one particle in the initial Laughlin state of N_i particles[17].) In the following, we shall imagine that the transition spectrum can be resolved into components labelled by this final angular momentum L_f. In principle this could be done by measuring the final angular momentum of the excited $\Downarrow$

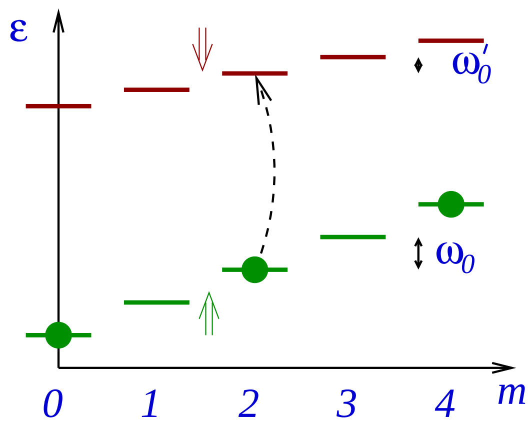

FIG. 1. Schematic illustration of RF excitation of a single particle from state ⇑ to ⇓. The energy of the transition reveals the angular momentum of the excited particle.

atom. However, note that, in general, the change in internal states of the atom will also change the confinement frequency. If the new confinement frequency ω_0' is such that $\hbar|\omega_0 - \omega_0'| \gg V_0$, then the final angular momentum of the excited particle can be found spectroscopically. This is illustrated in Fig. 1, which plots the single particle energies $\epsilon_{\Uparrow,m} = \epsilon_{\Uparrow,0} + m\hbar\omega_0$ and $\epsilon_{\Downarrow,m} = \epsilon_{\Downarrow,0} + m\hbar\omega_0'$, and indicates an RF transition, for which there is no change in orbital angular momentum $\delta m = 0$. Note that we also should assume $|\omega_0 - \omega_0'| \ll \omega_0$ so that the $|\Uparrow, m\rangle$ and the $|\Downarrow, m\rangle$ orbitals are roughly aligned spatially.

In considering the form of the spectrum of these RF transitions there are two questions of importance: what are the energies of the final states; and what are the matrix elements for transitions into these final states?

Among the possible final states, there is a set with zero interaction energy. These are of particular interest: they are the edge and quasihole excitations of the Laughlin state of N_f particles, and their properties provide a robust characterization of the Laughlin state. For $L_\mathrm{f} = N_\mathrm{f}(N_\mathrm{f} - 1)$ the only possible final state with zero interaction energy is the Laughlin state for N_f particles. As L_f increases, the number of zero interaction energy states increases, with well-defined counting rules, which for $N_\mathrm{f} = 5$ lead to the multiplicities shown in Table I. This integer sequence is highly indicative of the Laughlin state, giving information on the edge/quasihole excitations[18]. Note that an analogous experiment performed with a Laughlin state at a different filling factor $\nu = 1/p$ would show the same series of multiplicities (for the same N_f) although the final angular momentum will range from $pN_\mathrm{f}(N_\mathrm{f} - 1)/2$ to $pN_\mathrm{f}(N_\mathrm{f} + 1)/2$, covering the first $pN_\mathrm{f} + 1 = m_\mathrm{max} + 1$ values of this sequence.

The counting of the zero interaction energy states can be obtained from a simple picture based on the generalized clustering principle[19], which identifies the "root states" of the exact quantum states. These root states are single Fock states, with definite particle number n_m in orbitals $m = 0, 1, 2 \ldots$. For the $\nu = 1/2$ Laughlin state, the clustering principle is that no two particles can be in orbitals, m and m', with $|m - m'| < 2$. The resulting root configurations for $N_\mathrm{f} = 5$ are shown in Fig. 2 for small total angular momentum. Note that (nonorthogonal) basis vectors of the zero interaction energy space can be made from the root Fock state superposed with daughter Fock states that do not generally obey the clustering principle. The daughter states have the interesting feature that they are always "squeezed" from the root state[19], meaning that if we write out a string to represent the occupancies of single particle orbitals m, squeezing always numerically reduces the value of this string while preserving the total number of particles as well as the total angular momentum, L. For example if we write the root state string 1 0 1 0 0 1 to mean we have filled the orbitals $m = 5, m = 3, m = 0$ each once, a daughter state squeezed from this would be the string 1 0 0 1 1 0 which is numerically less than 1 0 1 0 0 1.

L_f	$\#_{N_\mathrm{f},L_\mathrm{f}}$	$\#^{\mathrm{allowed}}_{N_\mathrm{f},L_\mathrm{f}}$	$L_z^\mathrm{s} = L_\mathrm{f} - N_\mathrm{f}^2$
20	1	1	-5
21	1	1	-4
22	2	2	-3
23	3	2	-2
24	5	3	-1
25	7	3	0
26	10	3	1
27	13	2	2
28	18	2	3
29	23	1	4
30	30	1	5
31	37	0	
⋮	⋮	⋮	

TABLE I. The number of zero-energy final states of $N_\mathrm{f} = 5$ contact interacting bosons $\#_{N_\mathrm{f}=5,L_\mathrm{f}}$, and the number of these states with allowed transitions, $\#^{\mathrm{allowed}}_{N_\mathrm{f}=5,L_\mathrm{f}}$, following the many-body selection rule described in the text. The final column gives the z-component of the angular momentum of an analogous spherical system described in the text. No states are allowed for $L_\mathrm{f} > 30$ and here there is no meaningful value for L_z^s

If there were no quartic potential, all of the final states with zero interaction energy at a given L_f would be at exactly the same energy (since interaction energy is zero). The quartic potential splits these degenerate states, allowing separate transitions to be resolved. Fig. 3 shows the splitting for $N_\mathrm{f} = 5$. For weak quartic potential (compared to V_0) this splitting does not obscure the many-body gap separating these zero-interaction energy states from the high energy states. Therefore, by detecting the number of low-energy spectral lines (i.e. those below the

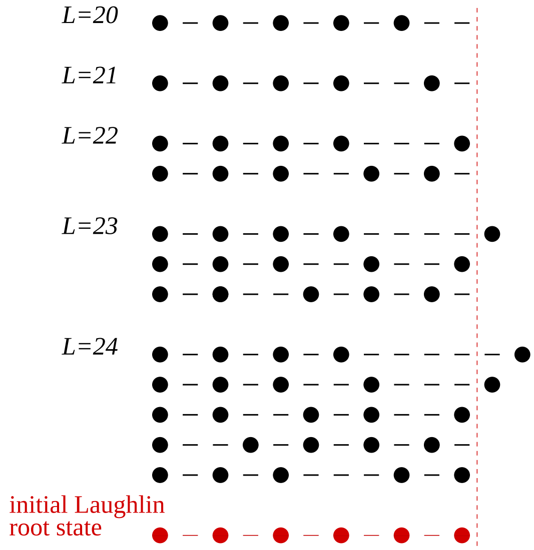

FIG. 2. Counting of the states for $N_\mathrm{f} = 5$ contact interacting bosons for different final angular momenta L. Shown in red is the root state for the initial Laughlin state of $N_\mathrm{i} = 6$ particles. This sets the cut-off (red dashed line) for the root configurations with sizeable matrix element: the allowed final states are those for which all particles are to the left of the red dashed line.

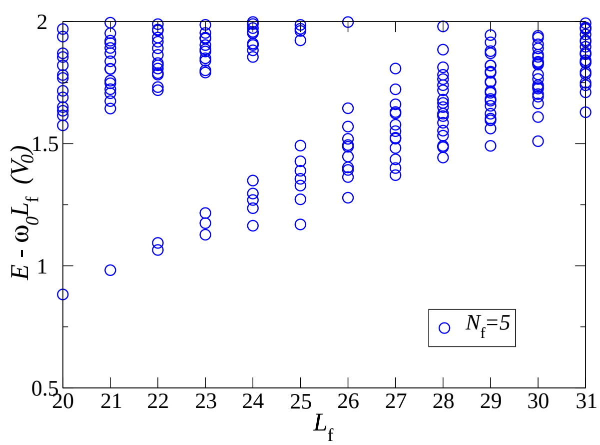

FIG. 3. Spectrum of final states of $N_\mathrm{f} = 5$ contact interacting bosons, showing the branch of zero interaction energy states below the (uninteresting) upper branch of high energy states. The zero interaction energy states (lower branch) are split in energy by a quartic potential with $\eta_\mathrm{Q} = 0.005V_0$. The x-axis is the final total angular momentum; the y-axis is energy in units of V_0.

gap, arising from many-body repulsion, in Fig. 3) as a function of L_f, if all transitions had non-zero matrix elements, one could measure the above counting sequence.

For a spectroscopic probe, it is important to determine the rate of transition into the possible final states. This rate is proportional to the squared matrix element

$$|\langle \mathrm{final}; N_\mathrm{f} = N_\mathrm{i} - 1, L_\mathrm{f} = L_\mathrm{i} - m_\mathrm{f}|\hat{b}_{m_\mathrm{f}}|\mathrm{Laughlin}; N_\mathrm{i}\rangle|^2$$

where $\hat{b}_m$ destroys a boson in state m. We find that there are very significant restrictions on such matrix elements. This leads to a strong "many-body selection rule" on the RF transitions in Laughlin clusters. Specifically, we find that strong transitions — which we shall refer to as "allowed" transitions — exist only to those states whose root configurations have all particles within the $m = 0 \to 2N_\mathrm{f}$ orbitals.

The origin of the selection rule lies in the fact that the initial Laughlin state, with $N_\mathrm{i} = N_\mathrm{f} + 1$ particles, is a state in which all particles are in these $m = 0 \ldots 2N_\mathrm{f}$ orbitals (see Fig. 2). Hence, removal of a particle from the Laughlin state cannot produce a Fock state with a particle in orbital $m > 2N_\mathrm{f}$. Since we find, from numerical diagonalizations, that the root configuration has a very large weight in the exact eigenstates, matrix elements are large only with those states whose root configurations have all particles within $0 \le m_\mathrm{f} \le 2N_\mathrm{f}$[20].

The allowed states are a well-defined subset of the zero-interaction energy states. This subset can be readily found from Fig. 2 by retaining only those root states for which all particles lie to the left of the red dashed line. For this case of $N_\mathrm{f} = 5$ particles, the number of states that have allowed transitions is shown in Table I.

The set of allowed states has a simple interpretation: it is the set of zero-energy states that can be formed for contact interacting bosons on a system of fixed area with 2 quasiholes. This follows from the fact that with m restricted to $m = 0 \ldots 2N_\mathrm{f}$, the number of orbitals over which the N_f particles can be distributed is $2N_\mathrm{f}+1$, while the Laughlin groundstate is formed when the number of states is $2N_\mathrm{f} - 1$. These $(2N_\mathrm{f}+1) - (2N_\mathrm{f} - 1) = 2$ excess orbitals can be viewed as two quasiholes in the Laughlin groundstate. Each can be placed in

$$d_\mathrm{qh} = N_\mathrm{f} + 1 \tag{1}$$

possible locations, with respect to the N_f particles. The total number of states is then simply given by the number of ways to put two identical quasiholes in $d_\mathrm{qh} = N_\mathrm{f} + 1$ states: $\#^\mathrm{allowed}_{N_\mathrm{f}} \equiv \sum_{L_\mathrm{f}} \#^\mathrm{allowed}_{N_\mathrm{f},L_\mathrm{f}} = \frac{1}{2}(N_\mathrm{f} + 1)(N_\mathrm{f} + 2)$. These states can be resolved into states of fixed angular momentum L_f. A compact way to state the result is provided by noting that the same counting problem appears for particles confined to the surface of a sphere with flux $N_\phi = 2N_\mathrm{f}$: our single particle orbitals $m = 0 \ldots 2N_\mathrm{f}$ on the disc just being viewed as states of z-projection of angular momentum $L^\mathrm{s}_z = -N_\mathrm{f} \ldots N_\mathrm{f}$ on the sphere, such that the z-projection of total angular momentum is $L^\mathrm{s}_z = L_\mathrm{f} - N_\mathrm{f}^2$. The flux $N_\phi = 2N_\mathrm{f}$ still corresponds to $n = 2$ quasiholes, so the known degeneracy[21] of zero energy states for n quasiholes on a sphere — the binomial coefficient $C(N_\mathrm{f} + n, n)$ (i.e., choose n from $N_\mathrm{f} + n$) — matches $\#^\mathrm{allowed}_{N_\mathrm{f}}$ derived above. The advantage of viewing the problem for the sphere is that these states can be indexed by a *total* angular momentum, L^s, with L^s_z taking values $-L^\mathrm{s}, \ldots L^\mathrm{s}$. The result for $n = 2$ is that the zero energy states have[21] $L^{\mathrm{s}} = 1, 3, 5, \ldots N_{\mathrm{f}}$ (N_{f} odd), or $L^{\mathrm{s}} = 0, 2, 4, \ldots N_{\mathrm{f}}$ (N_{f} even), which is consistent with the counting in Table I.

The relation (1), together with the fact that for a system of fixed area (here, the disc with $m = 0, \ldots 2N_{\mathrm{f}}$), the removal of a single particle ($\Delta N = -1$) corresponds to the creation of two quasiholes $\Delta n_{\mathrm{qh}} = -2\Delta N$ fixes the Haldane exclusion statistics of the quasiholes. These generalized exclusion statistics relate the change in dimension of the Hilbert space Δd available to a quasiparticle to the change in number of quasiparticles Δn via $\Delta d = -g\Delta n$ with g being the exclusion statistic parameter. For bosons or fermions, $g = 0$ or 1 respectively. Here we have $\Delta d = -\frac{1}{2}\Delta n$ showing that $g = \frac{1}{2}$, indicative of "semionic" statistics of the quasiholes. Thus, by counting the number of allowed transitions in the RF spectra, one obtains direct evidence for the counting formula (1). As described above, the dependence of the counting formulas on N_{f} lies at the heart of the fractional exclusion statistics for the quasholes: thus, detecting the multiplicities for different N_{f} (i.e. different initial $N_{\mathrm{i}} = N_{\mathrm{f}} + 1$), amounts to a direct detection of these exclusion statistics.

The preceding discussion is based on the existence of the many-body selection rule. How well does this many-body selection rule apply in practice? To test this, we have computed the many-body states and matrix elements numerically for $N_{\mathrm{i}} = 2 \ldots 10$ particles. In Fig. 4 we show the matrix elements (squared) for all excited states at each final momentum L, computed by exact diagonalization for the case of $N_{\mathrm{f}} = 5$. (Results for other values of N_{f} are consistent.) There is a clear separation between allowed states, with matrix element squared of order one, and forbidden states, with square matrix element smaller by at least two orders of magnitude, and imperceptible on the linear scale of Fig. 4. The counting of the allowed states (marked by red circles in Fig. 4) follows the pattern (1,1,2,2,3,3,3,2,2,1,1) expected from the counting rules described above (see Table I). The experimental goal will be to count the transitions with nonnegligible weights in each angular momentum sector.

One might wonder why the many-body selection rule works as well as it does (with the matrix elements for states that violate the selection rule being suppresed by factors of 100 or more). Our above argument that each eigenstate contains a large component of its root Fock state turns out not to be a sufficient explanation since there is substantial mixing with daughter states[19]. Let us examine a more general potential $V \sim r^{\gamma}$ instead of the quartic potential $\gamma = 4$ (while keeping the potential the smallest energy scale of the system). In terms of orbital occupations n_m, this yields potential energies $\sim \sum_m m^{\gamma/2} n_m$. In the limit $\gamma \to \infty$ this is dominated by the occupied orbitals with the largest m, and the potential energies of Fock states are then ordered with the same "squeezing" relationship as described above. Beginning with the basis of zero interaction energy states

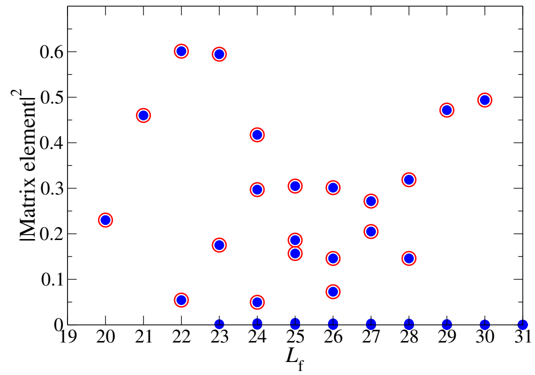

FIG. 4. Squared matrix element of transitions into the quasihole sector as a function of final angular momentum (blue dots) for $N_{\mathrm{f}} = 5$ particles in the final state. The red circles indicate those transitions which are allowed according to the many-body selection rule described in the text.

described by root states and their corresponding daughters, in the $\gamma \to \infty$ limit the *exact* energy eigenstates can then be found by successively orthogonalizing these wavefunctions, starting with the root state with lowest potential energy and continuing to successively higher root states. Since removing a particle from the Laughlin state must generate a superposition of wavefunctions in this zero-interaction energy space which also contains no occupied orbitals with $m > N_{\mathrm{f}}$ the selection rule becomes exact. We find numerically that this "orthogonalized root state basis" is an extremely accurate representation of the exact eigenstates even when $\gamma = 4$, providing some justification for why the selection rule works so well.

Although we have considered a rotating gas, similar features of fractional exclusion statistics should also appear for other forms of synthetic gauge field[22]. Our results amount to counting low-energy states of quasiholes, so can apply whenever quasihole-quashole interactions are weak compared to the Laughlin state energy gap. For gauge fields acting on small droplets with (near) axial symmetry[23, 24] our results can be directly applied, provided excitation to an undressed weakly-interacting state is possible. For lattice-based systems[25], the loss of continuous rotational symmetry precludes indexing the states by total angular momentum which can complicate the counting of the transitions.

In summary, the spectroscopic probe removing one atom creates two quasiholes in the Laughlin cluster. Measurements of the number of allowed transitions would determine the number of these quasihole states, testing theories of the properties (degeneracy and mutual statistics) of this pair of fractionalized particles. While experiments of this kind are certainly technically challenging — requiring control of small numbers of atoms with high fidelity, and the detection of single atoms in spectroscopic probes — these are within reach of new technologies of quantum gas microscopes. Our work provides a very appealing direct link between multiplicities in RF spectra, and counting formulas for fractionalized excitations in strongly correlated many-body systems.

We are grateful to Jean Dalibard for many enlightening comments. This work was supported by EPSRC Grants EP/J017639/1, EP/I032487/1, EP/I031014/1, and by the Royal Society.

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