

THE IMPACT OF TECHNICAL ANALYSIS ON ASSET PRICE DYNAMICS

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Abstract

This paper formalises feedback trading arising from the popularity of technical analysis. It provides a systematic study on the effect of feedback trading on price dynamics. The analysis shows under different conditions how prices asymptotically approach the fundamental equilibrium and how a significant feedback effect drives them off the equilibrium path: prices are observed to exhibit patterns such as momentous overshooting and prolonged cycles. Fluctuations off the fundamental equilibrium can be systematically and endogenously induced by feedback trading. The results suggest that non-fundamental multiple equilibria are possible even though socially undesirable.

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1. Introduction

In the economics and finance literature, the applications of technical analysis have long remained controversial. Technical analysis employs historical price information to predict future market activity; it includes techniques from simple visual pattern recognition, such as head-and-shoulders, to sophisticated neural networks. The ability of technical analysis in generating systematic trading profits has been subject to a high degree of scepticism by the implications of one important cornerstone in the field of financial economics, the efficient market and random walk hypotheses. Previous efforts have been made to investigate the performance of technical analysis, but conclusions differ¹. Although the issue remains unsolved, it is undeniable that technical analysis has been widely adopted among institutional and individual traders (Frankel and Froot 1990; Taylor and Allen 1992). From our perspective, the popularity of technical analysis, regardless of its true profitability, will have an impact on markets.

While much of the existing literature seeks to provide analytical solutions or empirical evidence concerning the validity of technical analysis as an investment tool, it is only until recent years that studies concerning its market impact have started to emerge. On the empirical side, the investigation on its market impact has been largely impeded by a lack of appropriate data, and perhaps more importantly, by the difficulty in correctly identifying the underlying attributes of market phenomena.

On the theoretical side, several efforts have been made. Levy, Levy and Solomon (1994, 1995) and Levy and Levy (1996) contrasted the market behaviour before and after the introduction of heterogeneous expectations in terms of varying spans of price memory, and found the latter leads to more realistic price dynamics. Based on this model, Levy and Solomon (1996) demonstrate a convergence of the agents' wealth distribution to a power law. In a partial equilibrium setting², Farmer and Joshi (2000) showed that some commonly used trading strategies induce excess volatility.

¹ In favour of technical analysis, see Brock, Lakonishok and LeBaron (1992), Lo and MacKinlay (1988, 1999), Neftci (1991); against its predictive value, see Dempster and Jones (1999a, 1999b, 2001), Goodhart and O'Hara (1997), Sullivan, Timmerman and White (1999).

² In the authors' terminology, a 'market maker' is included in the model to absorb excess demand or supply; unlike a general equilibrium setting, there is no market clearing.

Other related work studies the market dynamics arising from the interaction and adaptation of heterogeneous agents; these are often referred to as agent-based modelling³ where agents with varying risk attitudes and beliefs switch between investment strategies according to trading performance (Brock and Hommes 1997a, 1997b, 1998; Chiarella 1992; Chiarella and He 2001, 2002; Day and Huang 1990; Gaunersdorfer 2000; Hommes 2001, 2002; Lux 1997, 1998; Lux and Marchesi 1999, 2000). Models with artificial intelligence agents⁴ also look into their resulting market behaviour (Arthur et al. 1997; LeBaron et al. 1999). Research in this area expands partly owing to the unsatisfactory assumptions of homogeneous agents and complete rationality in conventional models. Many of these studies have successfully connected their results with some empirical stylised facts.

In the present paper, we develop independently a simple framework to study the market impact of technical analysis. Technical analysis, involving studies of historical data, has a feedback effect on prices through trading. We study the feedback effect and how it induces systematic price deviations from the fundamental equilibrium, using both bifurcation analysis and numerical simulation. A related empirical study is by Sentana and Wadhwani (1992), who examine the links between volatility and serial correlation via the role of feedback trading.

The concern of the present study is related to the work by Farmer and Joshi (2000) but with different approaches. One major difference is that, instead of having a market maker to absorb excess demand or supply, we conduct the investigation in a general equilibrium setting where the market clears in each trading period. Besides, the focuses are somehow different in that their analysis on price behaviour is centred at volatility issues while we ask whether price dynamics exhibits certain recognisable patterns under the impact of technical analysis.

When compared with most computational agent-based models, our model is simple yet unrealistically free from agents' adaptation and interaction. This enables us to

³ See Hommes (2002) for a recent review.

⁴ See LeBaron (2000) for a recent review.

investigate the feedback effect on the market in isolation from the complication of strategic behavioural issues. The aim here is not to reflect real markets but to answer the more fundamental question that arises with the increasing popularity of technical analysis.

This paper is organised as follows. Section Two provides a simple asset pricing model with myopic mean-variance optimising investors. Section Three discusses the roles of heterogeneous conditional expectations. Section Four defines a feedback function that has some desirable properties of the average technical forecast. Section Five studies price dynamics using bifurcation analysis and generates numerical simulation results. Section Six discusses the implications of our finding and concludes.

2. A Simple Asset Pricing Model

Consider the following classic portfolio choice problem. Two assets are available to an investor at time t . One is risk free and one is risky. The risk free asset pays a fixed rate of return r_f for each time period, thus the gross rate of risk free return is $R_f = 1 + r_f$. The risky asset has a gross rate of return R_{t+1} from time t to time $t + 1$. Let $E_t[R_{t+1}]$ and $V_t[R_{t+1}]$ denote the conditional mean and conditional variance; they are the mean and variance of R_{t+1} , conditional on the investor's information at time t . The investor places a portfolio weight w_t on the risky asset at time t . The portfolio return is given by

$$R_{t+1}^P = R_f(1 - w_t) + R_{t+1}w_t. \quad (1)$$

Investors are assumed to be myopic mean-variance maximisers. That is, investors trade off mean and variance in a linear fashion:

$$\text{Max}_{w_t} \quad E_t[R_{t+1}^P] - \frac{a}{2} V_t[R_{t+1}^P], \quad (2)$$

where a is the risk aversion parameter. The solution to this maximisation problem is

$$w_t = \frac{E_t[R_{t+1}] - R_f}{a V_t[R_{t+1}]} \quad (3)$$

This is the well-known result of mean-variance analysis: the optimal portfolio share in the risky asset is given by the expected excess rate of return divided by the conditional variance times the risk aversion coefficient.

Denote by P_t the price per share of the risky asset at time t . We assume the risky asset pays periodic dividends and denote by d_t the stochastic dividend process of the risky asset. d_t is assumed to be an IID process, $d_t \sim IID(\bar{d}, \mathbf{s}_d^2)$. The gross risky payoff between time t and time $t+1$ is given by $P_{t+1} + d_{t+1}$. The gross rate of return on the risky asset, R_{t+1} , is defined as

$$R_{t+1} = \frac{P_{t+1} + d_{t+1}}{P_t}, \quad (4)$$

and the net rate of return is simply $R_{t+1} - 1$. Substituting in the definition of R_{t+1} , the portfolio weight on the risky asset at time t , given by (3), can be rewritten as

$$w_t = \frac{P_t E_t[P_{t+1} + d_{t+1}] - P_t^2 R_f}{a V_t[P_{t+1} + d_{t+1}]} . \quad (5)$$

We assume heterogeneous investors and add superscript i for investor type i . Investors differ in their forecasting strategies (or beliefs) on the risky payoff. Let \mathbf{q}_t^i denote the fraction of investor type i at time t , representing the popularity of strategy i at time t and satisfying $\sum_{i=1}^N \mathbf{q}_t^i = 1$, where N is the number of different investor (or strategy) types.

Let K denote the total size of capital in this two-asset market, and let x^s denote the supply of risky shares; both are assumed to be constant⁵. $K \mathbf{q}_t^i$ measures the capital size invested in strategy i at time t . Market equilibrium requires

$$K \sum_{i=1}^N \mathbf{q}_t^i w_t^i = P_t x^s . \quad (6)$$

Substituting in the portfolio weight (5) with superscript i , and assuming the conditional

⁵ These are simplifying assumptions. In a short time horizon, the total size of market capital and the supply of risky shares may be regarded as approximately constant.

variance of the risky payoff to be constant⁶ and equal for all types⁷, i.e. $V_t^i[P_{t+1} + d_{t+1}] = \mathbf{s}^2$, the market equilibrium equation can be rewritten as

$$R_f P_t = \sum_{i=1}^N q_t^i E_t^i[P_{t+1} + d_{t+1}] - \frac{a \mathbf{s}^2 x^s}{K}. \quad (7)$$

Market equilibrium yields the equilibrium price dependent on economic fundamentals and also the conditional forecasts influenced by investor psychology and emotion. In the next section, we will discuss the formation of heterogeneous investors' conditional forecasts. We now consider a conventional economic world of homogeneous and perfectly rational investors. First, equation (7) can be written as

$$E_t[P_{t+1} + d_{t+1}] - R_f P_t = \frac{a \mathbf{s}^2 x^s}{K}. \quad (8)$$

The term $\frac{a \mathbf{s}^2 x^s}{K}$ measures the expected excess amount of risky payoff and therefore may be interpreted as a risk premium.

Solving equation (8) by repeatedly substituting out next period's prices and assuming the transversality (no-bubble) condition $\lim_{h \rightarrow \infty} \frac{E_t[P_{t+h}]}{(R_f)^h} = 0$ holds⁸, we then obtain

$$P_t = \sum_{h=1}^{\infty} \frac{1}{(R_f)^h} \left(E_t[d_{t+h}] - \frac{a \mathbf{s}^2 x^s}{K} \right). \quad (9)$$

The equilibrium price is the discounted sum of future dividends minus the risk premium. Equation (9) gives the expression known as the fundamental value of the risky asset. Since the dividend process is assumed to be an IID process, we know that $E_t[d_{t+h}] = \bar{d}$. The fundamental price can be written as

$$P^F = \frac{1}{r_f} \left(\bar{d} - \frac{a \mathbf{s}^2 x^s}{K} \right). \quad (10)$$

⁶ A detailed derivation that solves the conditional variance to a constant value under some distributional assumptions can be found in Hoel (1962).

⁷ This is an approximation in a world where volatility forecasts are well established and agreed but mean forecasts are not; such a situation arises when there is a dominant risk management system or a implied volatility methodology that is universally accepted, see Merton (1980) who agrees that means are much harder to forecast than variances.

⁸ Relaxing this assumption leads to "rational bubbles". There are however theoretical and empirical arguments that can be used to rule out the existence of rational bubbles (see Campbell, Lo and MacKinlay 1997 for a brief discussion).

In the world of homogeneous and perfectly rational investors, the price of the risky asset should equal its fundamental value, independent of the trading and price history of the asset. Changes in the fundamental price can only be caused by exogenous shocks on economic fundamentals.

3. Heterogeneous Forecasts

In the asset pricing model with heterogeneous investors, market equilibrium (7) states that the price of the risky asset equals the discounted weighted average of heterogeneous forecasts on the risky payoff minus the risk premium, with the weights being the popularity of different forecasting strategies. In this section, we will discuss the risky payoff forecasts by heterogeneous, boundedly rational investors.

The attributes to investors heterogeneity can go beyond the conventional paradigm of asymmetric information to include diversity in prior beliefs. Kurz (1997) argues that the centre of individuals' disagreement lies in their diverse prior beliefs instead of information asymmetry; diverse beliefs explain why different interpretations arise given the same information. On the other hand, prior beliefs also influence information selection. Investors with different beliefs are likely to pick up dissimilar sources for their forecasts.

We will mainly focus on two classes of investors: fundamentalists and technical traders. Fundamentalists believe that the price of the asset should reveal its fundamental value. For fundamentalists, the stock price reflects the underlying value of the company and its potential for growth. Fundamental analysis thus involves studying the overall economy condition as well as the financial condition and management of the company, but dismisses market activity as the behaviour of unreliable, emotional herd. In contrast, technical analysis involves analysing statistics generated by market activity. Technical traders focus on the price and trading histories to seek to identify patterns in price movement and to forecast future activity.

Denote by $P_t^{TA,i}$ investor i 's technical forecast on the risky asset price, made at time t , and denote by \mathbf{e}_t some random noise at time t . We make the following assumptions about heterogeneous investors' conditional forecasts on future dividends and prices:

$$E_t^i[d_{t+1}] = E_t[d_{t+1}] = \bar{d}. \quad (11)$$

$$E_t^i[P_{t+1}] = (1 - \mathbf{b}^i)P^F + \mathbf{b}^i P_t^{TA,i} + \mathbf{e}_t. \quad (12)$$

(11) is a simplified assumption about the conditional mean dividend. Investors are assumed to share the same information and beliefs about the dividend payments so that they have a common conditional expectation on future dividends. For a stochastic IID dividend process, this implies that the conditional mean is the unconditional one.

We assume the fundamental value P^F of the risky asset is common knowledge. Technical traders however believe that short term prices will deviate from the fundamental value. (12) expresses the conditional expectation on the risky asset price as a weighted sum of the fundamental price and technical forecast, plus some random noise assumed to be common for all investors. The noise \mathbf{e}_t is to capture the effect of all other sources that may influence the price forecast. The weight \mathbf{b}^i reveals the investor type. $1 - \mathbf{b}^i$ and \mathbf{b}^i are investor i 's forecasting weights on the fundamental price and technical forecast respectively.

We will consider fundamentalists to be using only fundamental analysis ($\mathbf{b}^i = 0$) and technical traders to be using only technical analysis ($\mathbf{b}^i = 1$), although the mixture of both analyses is possible.

Assumptions (11) and (12) are consistent with the asset pricing model discussed in the previous section. The consistency can be seen in the following illustration: if all investors are fundamentalists, assumptions (11) and (12) should lead to the equilibrium price being the fundamental price.

When all investors are fundamentalists, the asset pricing (7), with assumptions (11) and (12), can be rewritten as

$$R_f P_t = P^F + \bar{d} - \frac{a \mathbf{S}^2 x^s}{K} + \mathbf{e}_t. \quad (13)$$

Substituting out P^F in (13) using the formula given by (10), we obtain the equilibrium price as

$$P_t = \frac{1}{r_f} \left(\bar{d} - \frac{a \mathbf{S}^2 x^s}{K} \right) + \frac{\mathbf{e}_t}{R_f}. \quad (14)$$

The equilibrium price (14) is in fact the fundamental price (10) with some noise.

Now we turn to the case when both investor types are present. One aim of this study is to investigate the market impact of the popularity of technical analysis. This investigation is made possible in a controlled experiment that looks into what outcome arises, given *exogenously* different ratios of technical traders in the market. Therefore, the ratio of investors will not be modelled as an time-varying endogenous variable. As shown later, the investor ratio is in fact part of the control variable in the price dynamic system. We thus drop the time subscript, and denote by \mathbf{q} and $1 - \mathbf{q}$ the fractions of technical traders and fundamentalists in the market. With assumptions (11) and (12), the asset pricing (7) now becomes

$$R_f P_t = (1 - \mathbf{q}) P^F + \mathbf{q} \left(\frac{\sum_{i=TA}^{N_1} \mathbf{q}^i P_t^{TA,i}}{\mathbf{q}} \right) + \left(\bar{d} - \frac{a \mathbf{S}^2 x^s}{K} \right) + \mathbf{e}_t, \quad (15)$$

where N_1 is the number of technical traders. Define $P_t^{TA} = \frac{1}{\mathbf{q}} \sum_{i=TA}^{N_1} \mathbf{q}^i P_t^{TA,i}$ as the *average* technical forecast among technical traders, made at time t , on the price of the risky asset. Substituting out the third term on the right hand side of (15) using the formula given by (10), we rearrange (15) to obtain

$$P_t = \left(1 - \frac{\mathbf{q}}{R_f} \right) P^F + \frac{\mathbf{q}}{R_f} P_t^{TA} + \frac{\mathbf{e}_t}{R_f}. \quad (16)$$

Therefore, in the presence of both fundamentalists and technical traders, the equilibrium price is a weighted combination of the fundamental price and the average technical forecast, plus some random noise. The weights depend on the fractions of different investors, which indeed represent the popularity of different forecasting strategies. In the next section, we will discuss the formation of the average technical forecast on the price of the risky asset.

4. The Feedback Function

Although technical analysis can take many different forms, all of them share a common feature: the use of price history. Investors take positions in the market based on their forecasts. The forecasts by technical traders however come from the analysis of price history. Thus, through trading, price history is partially incorporated into the new price to give a “feedback”. We shall refer to P_t^{TA} , the average technical forecast on the price of the risky asset, as the feedback function. A nonlinear feedback function gives rise to nonlinear price dynamics.

We propose that the feedback function satisfies the following properties:

- (i) It is a function of past prices. More precisely, it is a function of a trend indicator, which is a function of past prices.
- (ii) In order to be self-consistent, the feedback function is considered to be either monotonically increasing or decreasing in its trend indicator.
- (iii) Since the feedback function represents the average of collective technical forecasts, we assumed that it is bounded between two real numbers.
- (iv) When the trend indicator is neutral, the feedback function is assumed to become the fundamental price. That is, when past prices provide no information on future price movement, the average predicted asset value by technical traders coincides with the asset’s fundamental value.

There is virtually no information available to empirically estimate the feedback function. Properties (ii) and (iii) make any cumulative distribution function⁹ (CDF) a good choice for the feedback function without loss of generality. First, let $\mathbf{t}_t^P = f(P_{t-1}, P_{t-2}, \dots, P_{t-M})$ denote the trend indicator at time t ; it is a function of past prices of M lags. We define the feedback function by

$$P_t^{TA} = P^F \frac{CDF(\mathbf{h}\mathbf{t}_t^P)}{CDF(0)}, \quad (17)$$

⁹ Alternatively, see Sentana and Wadhwani (1992) who assume linear feedback to estimate the demand functions of technical traders.

where \mathbf{h} is the sensitivity parameter to the trend indicator. A large size of \mathbf{h} means, in average, technical traders being more responsive to the trend signal. A positive (negative) \mathbf{h} leads to the feedback function monotonically increasing (decreasing) in its trend indicator. When $\mathbf{h} > 0$, the trend chasing strategy dominates, and when $\mathbf{h} < 0$, the contrarian strategy that “buys low and sells high” prevails. In either case, the feedback function is bounded between $\left(0, \frac{P^F}{CDF(0)}\right)$. If the probability density function (PDF) is symmetric, the bound is simply $(0, 2P^F)$.

The definition of the feedback function given by (17) is consistent with the asset pricing model. When historical prices provide no information (for example, when prices are in steady state) and the trend indicator is neutral, i.e. $\mathbf{t}_t^P = 0$, it is clear that the feedback function (17) becomes only the fundamental price. In this case, it is straightforward to show that the equilibrium price (16) will coincide with the one when all investors are fundamentalists.

We now rewrite the equilibrium price (16) using the feedback function given by (17) as

$$P_t = b_0 + b_1 CDF(\mathbf{h}\mathbf{t}_t^P) + b_2 \mathbf{e}_t, \quad (18)$$

$$\text{where } b_0 = \left(1 - \frac{\mathbf{q}}{R_f}\right)P^F, \quad b_1 = \frac{\mathbf{q}}{R_f} \frac{P^F}{CDF(0)}, \text{ and } b_2 = \frac{1}{R_f}.$$

We will mainly focus on the deterministic dynamics¹⁰ free from the stochastic noise \mathbf{e}_t .

For convenience, let

$$q_t = \frac{P_t - b_0}{b_1} = CDF(\mathbf{h}\mathbf{t}_t^P). \quad (19)$$

q_t is bounded between $(0, 1)$. We shall call q_t the *normalised* price at time t .

Let $\mathbf{t}_t^q = f(q_{t-1}, q_{t-2}, \dots, q_{t-M})$ denote the trend indicator at time t of past normalised

¹⁰ The stochastic noise can disguise the detection of a particular structure in a dynamic system. With the stochastic noise above a certain level, any dynamic system would behave like a purely stochastic process no matter what the deterministic component is.

prices. In order to express q_t in terms of \mathbf{t}_t^q , we make an additional assumption on \mathbf{t}_t^P that it is a linear function of price differences. Based on the relationship between P_t and q_t given by (19), this assumption implies $\mathbf{t}_t^P = b_1 \mathbf{t}_t^q$. Now we can rewrite the normalised price as

$$q_t = CDF(\mathbf{m} \mathbf{t}_t^q), \quad (20)$$

where $\mathbf{m} = \mathbf{h} b_1 = \mathbf{h} \frac{\mathbf{q}}{R_f} \frac{P^F}{CDF(0)}$. The parameter \mathbf{m} essentially reflects the influence of feedback trading on prices. The size of \mathbf{m} decreases with R_f : an increase in the riskfree rate of return will divert risky investments into bonds and reduce feedback trading on the risky stock. P^F as a factor in \mathbf{m} is simply due to the design of the feedback function (17).

The two crucial terms, \mathbf{h} and \mathbf{q} , represent different aspects of feedback trading. As discussed before, \mathbf{h} measures the average responsiveness among technical traders to the trend signal, and the sign of \mathbf{h} indicates whether the trend chasing strategy or the contrarian strategy prevails; \mathbf{q} is the popularity of technical analysis. Therefore, the size of \mathbf{m} increases with the size of the feedback effect, holding all other variables in \mathbf{m} constant. Besides, the sign of \mathbf{m} indicates a positive or negative feedback. As shown later, \mathbf{m} is in fact the control variable that determines the price dynamics.

5. Price Dynamics

The use of the normalised price simplifies the analysis of price dynamics. The normalised price given by (20) is indeed a difference equation of order M . The dynamics has a simple steady-state solution. In steady state, because the (normalised) prices provide no information, i.e. $\mathbf{t}_t^q = 0$, we will have $q^* = CDF(0)$. For a symmetric PDF, the steady-state normalised price is simply $q^* = \frac{1}{2}$. The steady-state price is the fundamental price as discussed before.

For the dynamic analysis, the M -th order difference equation (20) is transformed into a first order difference equation using vectors:

$$Q_t = f(Q_{t-1}), \text{ where } Q_t = (q_t, q_{t-1}, \dots, q_{t-M+1})'. \quad (21)$$

We provide in Appendix 1 some necessary background knowledge for the analysis of a dynamic system. It is given on three subject matters relating to first, the definitions of Lyapunov (weak) and asymptotic (strong) stability, second, the stability conditions of a non-linear dynamic system in terms of eigenvalues, and third, the types of bifurcation and bifurcation analysis.

We now calculate the Jacobian matrix for the difference equation (21). The Jacobian matrix is an $M \times M$ matrix given by

$$J = \frac{\partial f(Q_{t-1})}{\partial Q_{t-1}} \bigg|_{q^*} = \begin{bmatrix} \frac{\partial q_t}{\partial q_{t-1}} \big|_{q^*} & \frac{\partial q_t}{\partial q_{t-2}} \big|_{q^*} & \dots & \frac{\partial q_t}{\partial q_{t-M+1}} \big|_{q^*} & \frac{\partial q_t}{\partial q_{t-M}} \big|_{q^*} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (22)$$

Let \mathbf{I} denote the eigenvalues. The characteristic equation is given in the following proposition.

Proposition: The characteristic equation for the Jacobian matrix (22), using the normalised price defined by (20), is given by

$$\mathbf{I} \mathbf{g} = 0, \quad (23)$$

$$\text{where } \mathbf{I} = (\mathbf{I}^M, \mathbf{I}^{M-1}, \dots, \mathbf{I}, 1)_{1 \times (M+1)}, \text{ and } \mathbf{g} = \begin{pmatrix} 1 \\ -mPDF(0) \frac{\partial t^q}{\partial Q_{t-1}} \big|_{q^*} \end{pmatrix}_{(M+1) \times 1}.$$

Proof: See Appendix 2.

To solve the characteristic equation, we proceed with an arbitrary choice of the CDF and the trend indicator function defined by the common moving average trading rule.

Corollary: Choose the CDF to be the logistic function, and define the trend indicator function by¹¹ $\mathbf{t}_t^q = q_{t-1} - \frac{1}{M} \sum_{j=1}^M q_{t-j}$. Then the coefficient vector in the characteristic equation (23) becomes

$$\mathbf{g} = (1, \mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_M)', \text{ where } \mathbf{g}_k = \begin{cases} -\frac{\mathbf{m} M - 1}{4 M}, & \text{for } k = 1; \\ \frac{\mathbf{m}}{4 M}, & \text{for } k = 2, \dots, M. \end{cases} \quad (24)$$

Proof: See Appendix 3.

We start our analysis with the simple case $M = 2$. The characteristic equation is now

$$\mathbf{l}^2 - \frac{\mathbf{m}}{8} \mathbf{l} + \frac{\mathbf{m}}{8} = 0, \text{ and the eigenvalues are } \mathbf{l} = \frac{\mathbf{m}}{16} \pm \frac{1}{16} \sqrt{\mathbf{m}(\mathbf{m} - 32)}.$$

Price dynamics depends on the parameter \mathbf{m} . The difference equation (21) can be viewed as $Q_t = f(Q_{t-1}, \mathbf{m})$, and \mathbf{m} is the control parameter. That is, price dynamics changes with \mathbf{m} , and \mathbf{m} defines stability. As discussed before, an increase in the size of \mathbf{m} can be caused by either a higher ratio of technical traders, or traders being more responsive to trend signals. The size of \mathbf{m} reflects the size of the feedback effect, and the sign of \mathbf{m} indicates a positive or negative feedback. Figure 1 plots the eigenvalue trajectory in the complex plane with respect to \mathbf{m} , in the case of $M = 2$. This plot is useful for the analysis of stability and bifurcation.

¹¹ This is equivalent to defining $\mathbf{t}_t^p = P_{t-1} - \frac{1}{M} \sum_{j=1}^M P_{t-j}$. As discussed before, based on (19), the relationship between \mathbf{t}_t^p and \mathbf{t}_t^q is given by $\mathbf{t}_t^p = b_1 \mathbf{t}_t^q$. From here, it is not difficult to see the two definitions are equivalent.

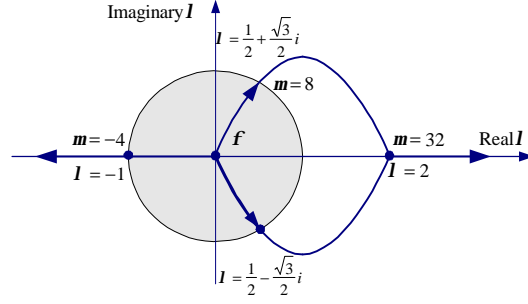


Figure 1. The eigenvalue trajectory with respect to \mathbf{m} in the case of $M = 2$.

In addition to the theoretical analysis, we simulate price time series for different values of \mathbf{m} to see how prices reach their fundamental value and how they behave when out of equilibrium. The simulation is carried out using (20), with the logistic CDF and the trend indicator defined by the moving average rule, as discussed in Corollary. Based on the simulation results, phase diagrams are drawn on two-dimensional q_t against q_{t-1} . The sets of Figure 2 to 5 show the results for varying lengths of M . We simulate 6000 trading periods in total. In order to see how prices evolve, the first 2000 points are shown in red, the next 2000 in green, and the last 2000 in blue.

Asymptotic stability, as discussed in Appendix 1, implies an attracting fixed point. It requires all eigenvalues lie strictly inside the unit circle. In the case of $M = 2$, if the feedback effect is sufficiently small, i.e. $-4 < \mathbf{m} < 8$, then the asymptotic stability condition holds, and q_t converges to the steady-state equilibrium q^* as shown by Figure 2b, or equivalently, prices P_t asymptotically approach the asset's fundamental value P^F .

Even in the simplest case $M = 2$, different bifurcation routes occur depending on \mathbf{m} . Suppose \mathbf{m} is initially in the region of $(-4, 8)$, where prices asymptotically converge to P^F . Consider now there is a significant negative feedback effect. When \mathbf{m} reaches the critical bifurcation point $\mathbf{m}^* = -4$, one eigenvalue will just cross the boundary of stability at -1 . At this point, the fundamental equilibrium is destabilised by a periodic doubling bifurcation, and a period-two orbit is created. Thus, in the presence of a large

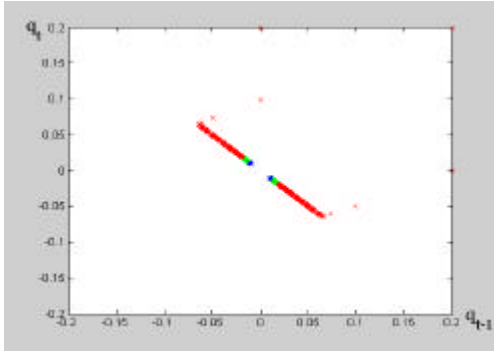
negative feedback, prices start to hop around between two non-fundamental points, as shown by Figure 2a. As m further decreases $m < -4$, the system has one eigenvalue outside and one inside the unit circle. Pulled by an unstable and a stable manifold, the fundamental equilibrium becomes an unstable saddle point coexisting with the period-two orbit.

What happens when there exists a large positive feedback? When m reaches the bifurcation point $m_+^* = 8$, the pair of complex conjugate eigenvalues pierces the unit circle. The fundamental equilibrium undergoes a Hopf bifurcation and loses its stability. The bifurcation yields a quasiperiodic orbit that behaves like an invariant curve, points on which hop around in circles. The angular frequency is determined by the angle at which the eigenvalue passes through the unit circle. In this case, the eigenvalues are $1 = \frac{1 \pm i\sqrt{3}}{2}$, so the angular frequency is $\frac{P}{3}$. The dynamics is described by a period-six cyclic motion, $A \cos(\frac{P}{3}t) + B \sin(\frac{P}{3}t)$, as shown by Figure 2c. As the feedback effect amplifies $m > 8$, centred at the repelling fundamental equilibrium, the size of cyclic motion expands and prices wonder further way. However, the phase diagram does not pursue an unlimited expansion as the feedback function is assumed to be bounded.

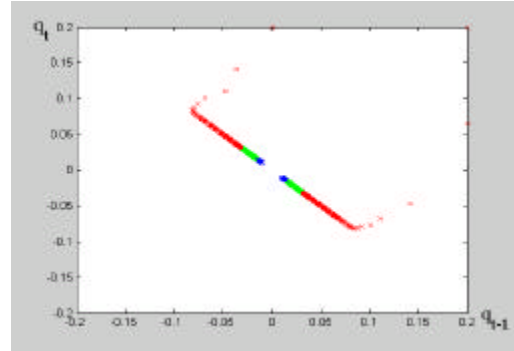
So far the analysis has focused on the simple case $M = 2$. For a longer M , direct computation of the stability condition however requires tedious work. We thus plots in Appendix 4 the absolute eigenvalues against m to obtain the stability condition and the critical bifurcation points for different lengths of M . For instance, when $M = 7$, the fundamental equilibrium is asymptotically stable if $-5.8 < m < 3.5$.

One might expect a wealth of distinct phase portraits as M gets longer. The simple logic is that a longer M increases the order of the characteristic equation and hence the number of eigenvalues. Since bifurcation occurs when an eigenvalue crosses the unit circle, the system with a longer M is likely to undergo a secondary or even higher-level bifurcation. The plots shown in Appendix 4 are however counter-intuitive. The system always has only one positive and one negative bifurcation point despite the length of M . This implies that the dynamic system with a longer price memory, as shown by the sets

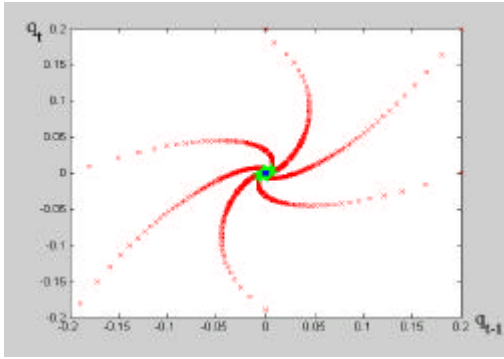
of Figure 3 to 5, will have a similar pattern as the system with $M = 2$. The only difference is that a significant positive feedback will now lead the prices to trace out a periodic orbit of a higher order than six.



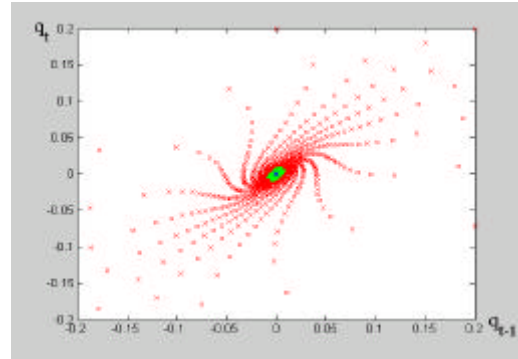
$M = 2, m = -4.2$
Figure 2.a



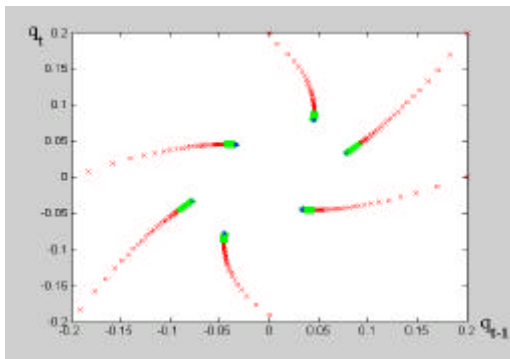
$M = 4, m = -4$
Figure 3.a



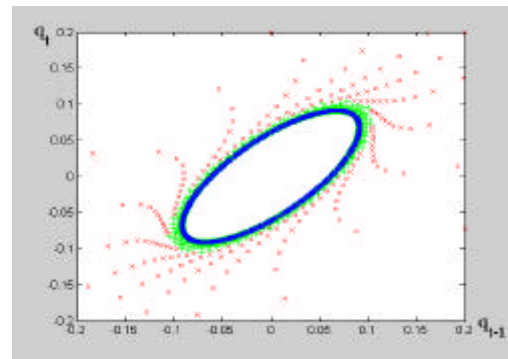
$M = 2, m = 7.9$
Figure 2.b



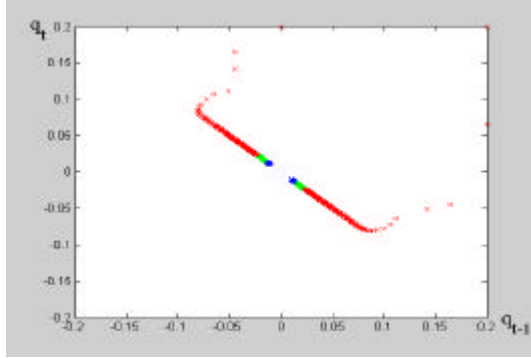
$M = 4, m = 4.3$
Figure 3.b



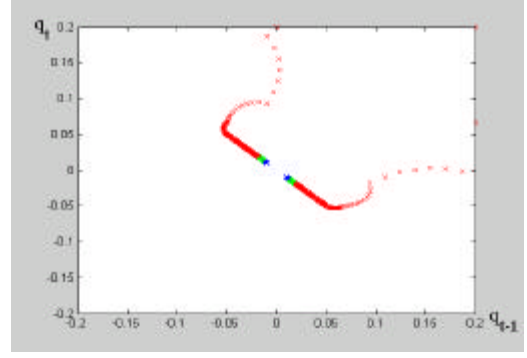
$M = 2, m = 8.2$
Figure 2.c



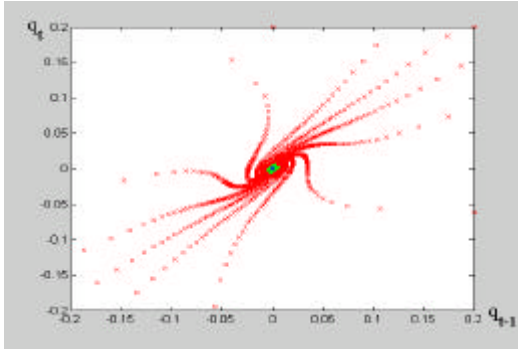
$M = 4, m = 4.5$
Figure 3.c



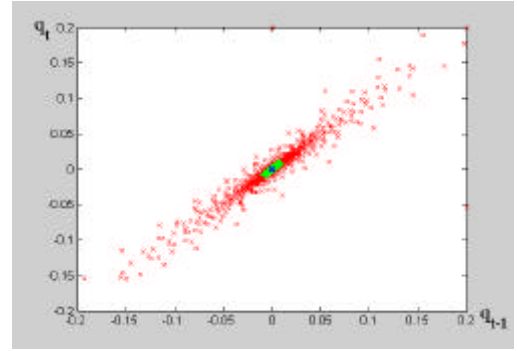
$M = 6, m = -4$
Figure 4.a



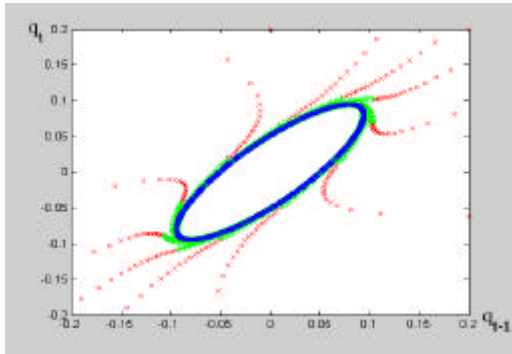
$M = 20, m = -4$
Figure 5.a



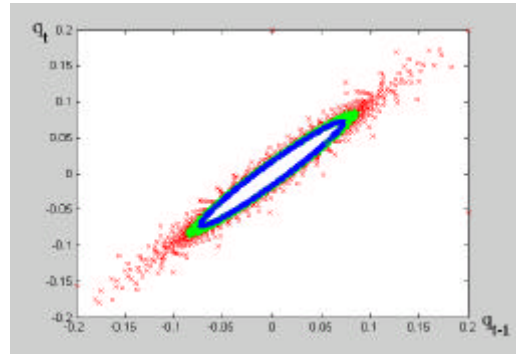
$M = 6, m = 3.6$
Figure 4.b



$M = 20, m = 3.2$
Figure 5.b



$M = 6, m = 3.8$
Figure 4.c



$M = 20, m = 3.3$
Figure 5.c

6. Discussions and Concluding Remarks

In this paper, we study the impact of technical analysis on asset pricing dynamics in a simple setting of myopic mean-variance optimising investors. There are two classes of investors: fundamentalists, who believe the asset price should reflect its fundamental value, and technical traders, who forecast using price history and result in price feedback. We discuss the properties of the average technical forecast (the feedback function) and formalise it with a plausible mathematical expression.

Our analysis has shown how prices reach the fundamental equilibrium and how a significant feedback effect drives prices off the equilibrium path. When the feedback effect is sufficiently small, the fundamental equilibrium is asymptotically stable (e.g. Figure 4b). When the feedback effect is intensified, two situations arise. On one hand, if the contrarian trading strategy prevails, the fundamental equilibrium is destabilised by a large negative feedback effect (e.g. Figure 4a). Contrarians trade against trend signals. Their long positions often coincide with the neighbourhood of falling prices. As a consequence, high demand causes a sharp rebound that diverts the falling prices to overshoot the equilibrium path. Similarly, the selling pressure near the mounting prices triggers a forceful pullback that again overshoots the fundamental value. The fundamental equilibrium is in fact an unstable saddle path. Prices oscillate up and down and do not settle on the fundamental equilibrium path.

On the other hand, if the trend chasing strategy prevails, the fundamental equilibrium is destabilised by a large positive feedback effect (e.g. Figure 4c). Investors buy into a rising market and sell into a falling one. As a consequence, fluctuations off the fundamental equilibrium are reinforced by the positive feedback effect, which can even lead to bear or bull markets. Instead of frequently overshooting the fundamental equilibrium, prices now exhibit a prolonged cycle. This observation may be in common with the real world where our economy can sometimes spend long periods away from equilibrium and where strategies based on buying past winners are commonplace.

The benchmark market in the absence of technical traders would reach the fundamental equilibrium with fluctuations only due to exogenous shocks. In the presence of

technical traders, fluctuations off the fundamental equilibrium can be systematically and endogenously induced by the feedback effect brought about by technical analysis. Our results suggest that non-fundamental multiple equilibria are possible in asset pricing. They arise as a result of different degrees of speculation by technical traders. Speculation allows possible allocations other than the fundamental value to be reached in equilibrium, even though they may not be socially desirable. The finding of multiple equilibria is in common with several studies, e.g. Pagano¹² (1989).

There is a certain lack of realism about these dynamics. The focus of our model has been placed on the impact of technical analysis, but the strategic interaction among investors has been ignored. In reality, investors are interdependent and are unlikely to remain unchanged. Interaction and adaptation will lead to a change in the ratio of different types of investors. The “ecology” of market participants can transform over a long horizon through market selection, or it can even change abruptly due to emotional herding. Exogenous factors, such as institutional changes, can also have a significant impulse on market ecology. Modelling the population ratio as a system variable helps answer how the feedback effect varies over time in the present study, and is certainly an interesting topic that deserves further exploration.

¹² Pagano (1989) finds the existence of multiple equilibria as a result of an increasing trade volume; the idea is in line with that speculation enlarges the trading space and allows multiple equilibria. However, his study, with a different focus from the present study, identifies that a large trade volume implies a less volatile market which is socially desirable.

Appendix 1

There are two types of stability, Lyapunov (weak) stability and asymptotic (strong) stability. Their definitions are given as follows.

Definition A1: The fixed point (equilibrium point) Q^* is said to be *Lyapunov stable* if for any $\epsilon > 0$ there exists a neighbourhood $\Omega(Q^*)$ of Q^* such that for all $Q \in \Omega(Q^*)$, the iterates of Q satisfy

$$\left| \mathbf{f}^t(Q) - Q^* \right| < \epsilon \text{ for iterations } t \geq 0.$$

Or alternatively, the fixed point Q^* is *Lyapunov stable*, if for any $\epsilon > 0$ there exists a $d = d(s, \epsilon) > 0$ such that

$$\text{if } \left| Q_s - Q^* \right| < d, \text{ then } \left| Q_t - Q^* \right| < \epsilon \text{ for all } t \geq s.$$

Definition A2: The fixed point Q^* is *asymptotically stable* if

- (i) it is Lyapunov stable and
- (ii) $\lim_{t \rightarrow \infty} \mathbf{f}^t(Q) = Q^*$ for all $Q \in \Omega(Q^*)$.

The difference between Lyapunov stability and asymptotic stability is that Lyapunov stability only requires a trajectory that starts in a neighbourhood of the fixed point to remain close to the fixed point. Asymptotic stability further requires a convergence of the trajectory to the fixed point. A fixed point that is only Lyapunov stable but not asymptotic stable is called *marginally stable*. In the present study, we only consider asymptotic stability which implies an attracting fixed point.

In applications, it is useful to characterise stability conditions in terms of eigenvalues.

Proposition A1: Consider a discrete *linear* map $Q_t = A Q_{t-1}$, where A is non-singular. The fixed point Q^* is asymptotically stable if and only if all the eigenvalues of A lie strictly inside the complex unit circle. For completeness, Q^* is asymptotically unstable

if it is not asymptotically stable. If one eigenvalue lies outside and one inside the unit circle, Q^* is a saddle point. The proof can be found in Hale and Kocak (1991, p.73) for example.

For a nonlinear dynamic system, the theorem developed independently by Hartman in 1964 and Grobman in 1965 is helpful.

Theorem A1: The behaviour of a discrete dynamic system $Q_t = \mathbf{f}(Q_{t-1})$ in the vicinity of an equilibrium point Q^* is topologically equivalent to the behaviour of the linear system $Q_t = \mathbf{f}'(Q^*)Q_{t-1}$, provided $|\mathbf{f}'(Q^*)| \neq 1$. The proof of the Hartman- Grobman linearisation theorem for discrete maps can be found in Robinson (1999, Theorem 6.2).

This theorem basically states that, in discrete nonlinear dynamics, the behaviour of its linearisation mimics the true behaviour, provided that the eigenvalues do not lie on the unit circle. Proposition A2 hence follows.

Proposition A2: For a discrete nonlinear dynamic system $Q_t = \mathbf{f}(Q_{t-1})$, the behaviour of its fixed point Q^* is determined by the eigenvalues \mathbf{I} of its linearisation (Jacobian) matrix $\mathbf{f}'(Q^*)$ as follows

- (i) If all $|\mathbf{I}| < 1$, then Q^* is an attracting equilibrium, or asymptotically stable.
- (ii) If all $|\mathbf{I}| > 1$, then Q^* is a repelling equilibrium, or asymptotically unstable.
- (iii) If one $|\mathbf{I}| > 1$ and one $|\mathbf{I}| < 1$, then Q^* is an unstable saddle.
- (iv) If any $|\mathbf{I}| = 1$, then a bifurcation takes place.

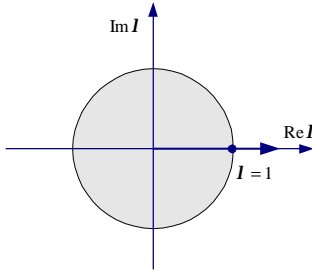
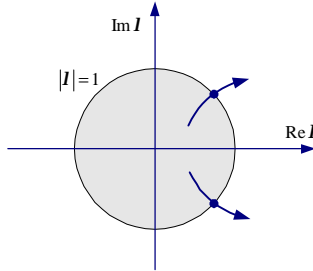
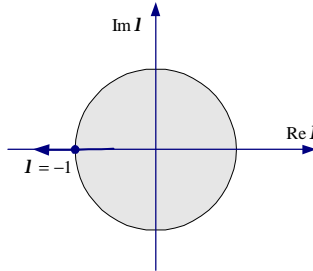
A change in stability is called *bifurcation*. There are different types of bifurcation. Lemma A1, A2, and A3 give three typical types of bifurcation. The details and the proofs can be found in Robinson (1999, Theorem 1.1, 3.1, and 5.1). Alternatively, see Kuznetsov (1995) for a comprehensive illustration.

Lemma A1: A *transcritical bifurcation* takes place when an eigenvalue crosses the boundary of stability at +1, i.e. $\mathbf{I} = 1$. A transcritical bifurcation causes a swap of the

stability of two equilibria. Suppose there exists two fixed points, one stable and one unstable, before a bifurcation takes place. When the dynamic system undergoes a transcritical bifurcation, the two equilibria exchange their stability properties.

Lemma A2: A *periodic doubling bifurcation* occurs when an eigenvalue crosses the boundary of stability at -1 , i.e. $\lambda = -1$. The stable equilibrium Q^* loses its stability and a period-two orbit emerges. Note that this type of bifurcation is absent in a continuous-time system.

Lemma A3: When a pair of complex conjugate eigenvalues crosses the boundary of stability, the dynamic system undergoes a *Hopf bifurcation*. The stable equilibrium Q^* loses its stability and a limit cycle (or a quasi-periodic orbit) bifurcates. The angular frequency is given by the angle at which the eigenvalue crosses the unit circle in the complex plane.

		
Transcritical bifurcation	Hopf bifurcation	Periodic doubling bifurcation

Appendix 2

Proof of Proposition

The characteristic equation for the Jacobian matrix (22) is given by

$$\mathbf{I}^M + \mathbf{g}_1 \mathbf{I}^{M-1} + \mathbf{g}_2 \mathbf{I}^{M-2} + \dots + \mathbf{g}_{M-1} \mathbf{I} + \mathbf{g}_M = 0,$$

where $\mathbf{g}_1 = -\frac{\partial q_t}{\partial q_{t-1}} \Big|_{q^*}$, $\mathbf{g}_2 = -\frac{\partial q_t}{\partial q_{t-2}} \Big|_{q^*}$, \dots , $\mathbf{g}_M = -\frac{\partial q_t}{\partial q_{t-M}} \Big|_{q^*}$.

The proof of a similar problem can be found in Hamilton (1994, p. 21).

Now, we rewrite the characteristic equation using vectors:

$$\mathbf{I} \mathbf{g} = 0,$$

where $\mathbf{I} = (\mathbf{I}^M, \mathbf{I}^{M-1}, \dots, \mathbf{I}, 1)_{1 \times (M+1)}$, and $\mathbf{g} = \begin{pmatrix} 1 \\ -\frac{\partial q_t}{\partial Q_{t-1}} \Big|_{Q^*} \end{pmatrix}_{(M+1) \times 1}$.

Using the normalised price given by (20), the $M \times 1$ sub-vector in \mathbf{g} becomes

$$-\frac{\partial q_t}{\partial Q_{t-1}} \Big|_{Q^*} = -\frac{\partial CDF(\mathbf{m} \mathbf{t}_t^q)}{\partial Q_{t-1}} \Big|_{Q^*} = -\mathbf{m} PDF(\mathbf{m} \mathbf{t}_t^q) \frac{\partial \mathbf{t}_t^q}{\partial Q_{t-1}} \Big|_{Q^*} = -\mathbf{m} PDF(0) \frac{\partial \mathbf{t}_t^q}{\partial Q_{t-1}} \Big|_{Q^*}.$$

Appendix 3

Proof of Corollary

Following the proposition, we calculate the coefficient vector in the characteristic equation, using the CDF as the logistic function and the trend indicator function defined

$$\text{by } \mathbf{t}_t^q = q_{t-1} - \frac{1}{M} \sum_{j=1}^M q_{t-j}.$$

It is easy to obtain the following derivatives.

$$\left. \frac{\partial \mathbf{t}_t^q}{\partial q_{t-1}} \right|_{q^*} = \frac{M-1}{M}, \text{ and } \left. \frac{\partial \mathbf{t}_t^q}{\partial q_{t-k}} \right|_{q^*} = \frac{-1}{M}, \text{ for } k = 2, \dots, M.$$

The logistic CDF and the PDF are given by

$$CDF(\mathbf{m}x) = \frac{\exp(\mathbf{m}x)}{1 + \exp(\mathbf{m}x)}.$$

$$PDF(\mathbf{m}x) = CDF(\mathbf{m}x)[1 - CDF(\mathbf{m}x)] = \frac{\exp(\mathbf{m}x)}{(1 + \exp(\mathbf{m}x))^2}.$$

$$\text{Thus, } PDF(0) = \frac{1}{4}.$$

Therefore, we can obtain the coefficients of the characteristic equation (23) as

$$\mathbf{g}_1 = -\mathbf{m}PDF(0) \left. \frac{\partial \mathbf{t}_t^q}{\partial q_{t-1}} \right|_{q^*} = -\frac{\mathbf{m}}{4} \frac{M-1}{M}, \text{ and}$$

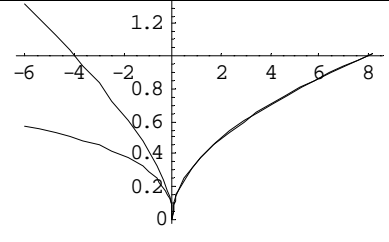
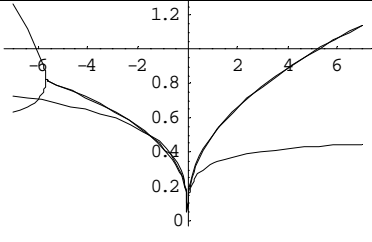
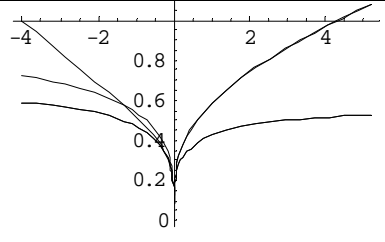
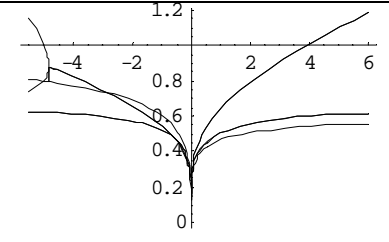
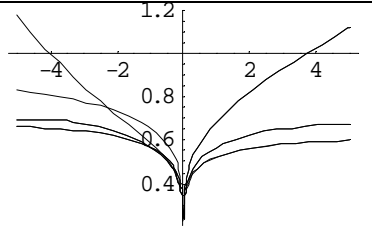
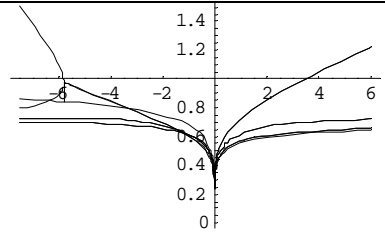
$$\mathbf{g}_k = -\mathbf{m}PDF(0) \left. \frac{\partial \mathbf{t}_t^q}{\partial q_{t-k}} \right|_{q^*} = \frac{\mathbf{m}}{4} \frac{1}{M}, \text{ for } k = 2, \dots, M.$$

Appendix 4

x-axis: \mathbf{m} , the control parameter of the dynamic system.

y-axis: $|\mathbf{I}|$, the absolute eigenvalue.

Since bifurcation takes place when $|\mathbf{I}| = 1$, the x-axis is shifted up to the point $|\mathbf{I}| = 1$ for convenience. The bifurcation points \mathbf{m}^* and also the stability condition of $|\mathbf{I}| < 1$ can then be easily found. The results are produced using Mathematica.

		
$M = 2, \mathbf{m}^* = -4, 8$	$M = 3, \mathbf{m}^* = -6, 5.21$	$M = 4, \mathbf{m}^* = -4, 4.33$
		
$M = 5, \mathbf{m}^* = -5, 4$	$M = 6, \mathbf{m}^* = -4, 3.7$	$M = 7, \mathbf{m}^* = -5.8, 3.5$

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