UNIVERSITY OF CAMBRIDGE

Gross-Siebert Mirror Ring for Smooth log Calabi-Yau Pairs

Yu Wang

Darwin College

This dissertation is submitted for the degree of Doctor of Philosophy

July 26, 2023

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or am concurrently submitting, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. This dissertation does not exceed the prescribed limit of 60 000 words.

Yu Wang
July, 2023

Abstract

In this paper, we exhibit a formula relating punctured Gromov-Witten invariants used by Gross and Siebert in [GS2] to 2-point relative/logarithmic Gromov-Witten invariants with one point-constraint for any smooth log Calabi-Yau pair (W, D). Denote by $N_{a,b}$ the number of rational curves in W meeting D in two points, one with contact order a and one with contact order b with a point constraint. (Such numbers are defined within relative or logarithmic Gromov-Witten theory). We then apply a modified version of deformation to the normal cone technique and the degeneration formula developed in [KLR] and [ACGS1] to give a full understanding of $N_{e-1,1}$ with D nef where e is the intersection number of D and a chosen curve class. Later, by means of punctured invariants as auxiliary invariants, we prove, for the projective plane with an elliptic curve $(\mathbb{P}^2, D)$, that all standard 2-pointed, degree d, relative invariants with a point condition, for each d, can be determined by exactly one of these degree d invariants, namely $N_{3d-1,1}$, plus those lower degree invariants. In the last section, we give full calculations of 2-pointed, degree 2, one-point-constrained relative Gromov-Witten invariants for $(\mathbb{P}^2, D)$.

Acknowledgements

My deepest thanks go to all the people who have supported me through my continuing mathematical education. First, to my family, my parents and my wife Yiyun Lin, who always come first in everything and a particular thanks goes to my wife for her encouragement and recommendations regarding my research career. My parents and my wife have never asked me what mathematics was good for and how much I could make out of math. I am grateful. Even though the relatives and friends of my mother put efforts in convincing both my parents and me for how much better studying practical sciences such as computer science, machine learning, statistics could be compared to pure mathematics, my mother would always put it aside and neglect it. She used to encourage me to pursue math about which I am passionate by saying to me that as a life, I am just the same as everybody else in the world, not better, not worse; as an individual, I could be extraordinary if I could do something that can be used to benefit others, and solving math problems is something that could be used to benefit human beings, so I am already extraordinary in this sense.

Mathematically, I must start by expressing my gratitude to my PhD supervisor Professor Mark Gross, who taught me everything I have used in this thesis, for his continuous advice, both mathematical and grammatical, which helped mould this thesis from my research paper. His patience and kindness have encouraged me to pursue solving these mathematical problems showing up in this thesis. He has often introduced novel insights in my own ideas. Secondly, I must thank Dr. Dhruv Ranganathan who is my second advisor at University of Cambridge for sharing different perspectives and applications of log geometry through his seminar talks, lectures and personal conversations. He is always willing to teach us some new viewpoints in log geometry and log Gromov-Witten theory as soon as they appeared. In the meanwhile, he suggested me and other PhD students several interesting research projects and directions. I am grateful for these.

I also want to thank Helge Ruddat for his support and several useful conversations about degeneration formula and his work with Tom Graber and Michel van Garrel showing that local Gromov-Witten invariants are log invariants, Michel van Garrel for carefully reading this thesis, pointing out several minors and constantly showing his appreciation for my work and suggesting some related problems to me, Bernd Siebert for allowing me to visit the University of Texas at Austin during the spring semester, 2023. The author also benefits from conversations with F. You and T. Grafnitz. In addition, I would also like to thank several Professors at University of Colorado-Boulder such as Agnes Beaudry, Sebastian (Yano) Casalaina-Martin and Jonathan Wise for their support when I was a student at CU-Boulder. I greatly appreciate that Jonathan and Yano were organizing a learning seminar on classifying spaces of degenerating polarized Hodge structure, which literally guides me into studying log geometry and Hodge theory.

Finally, I do want to thank Jonathan Quartin for his constant support as my close friend and sharing his PhD work, and all my friends for their warm consideration and boundless love.

The Author is funded by the ERC grant MSAG awarded by the European Research Council

Contents

Chapter 1

Introduction

1.1 Mirror Symmetry

Mirror symmetry is a concept that originated in theoretical physics and has since been developed further in the fields of algebraic geometry and symplectic geometry. It suggests that there is a duality between two seemingly different spaces or objects, where the symplectic and algebro-geometric properties of one are mirrored in the other. Mirror Symmetry proposes an equivalence between A-model and B-model where, in its original formulation in mathematics, A-model corresponds to Gromov-Witten theory which deals with so called Gromov-Witten invariants reflecting enumerative information and B-model consists of many period integrals encoding some complex geometry information. Usually, Gromov-Witten invariants coming out of A-model are hard to calculate, whilst period integrals are relatively easier to figure out and can be reduced to classical integrals which can be used to understand Hodge theory.

In symplectic geometry, mirror symmetry is often studied in the context of Calabi-Yau manifolds, which are complex manifolds with special holonomy properties. The theory of mirror symmetry in symplectic geometry was enhanced by physicists in the language of D-branes and also by Strominger-Yau-Zaslow in the geometric set-up of (special) Lagrangian torus fibrations [1]. Because the Gross-Siebert approach is an algebro-geometric approach inspired by Strominger-Yau-Zaslow(SYZ)'s mirror conjecture, let us recall some fundamental concepts in symplectic geometry in order to state the main picture of SYZ version of mirror symmetry.

Definition 1.1.1. A *symplectic manifold* is a $2n$-dimensional smooth manifold M together with a choice ω of closed, non-degenerate 2-form on M. A *Lagrangian submanifold* of a symplectic manifold M is maximally isotropic with respect to the symplectic form, meaning that the dimension of the submanifold is half the dimension of M and the restriction of the symplectic form to the submanifold vanishes.

The next object is presumably the most important object in the 20th century in both symplectic geometry and algebraic geometry, especially studying mirror symmetry. It is called a *Calabi-Yau* manifold.

Definition 1.1.2. A *Calabi-Yau manifold* is a complex manifold that has a Ricci-flat metric meaning that their Ricci curvature vanishes.

Note that an n-dimensional Calabi-Yau manifold has a natural symplectic structure coming from its Ricci-flat metric, and it also has a holomorphic $(n, 0)$-form.

Definition 1.1.3. A submanifold of a Calabi-Yau manifold is then said to be a *special Lagrangian submanifold* if it satisfies two conditions:

- it is a Lagrangian submanifold with respect to the symplectic structure.
- the restriction of the real part of the holomorphic $(n, 0)$-form to the submanifold vanishes.

Work of Greene and Plesser [GP] and Candelas, Lynker and Schimmrigk [CLR] gave the first hint that there should be a concrete mathematical framework for this idea that Calabi-Yau three-folds should come in pairs $(X, \check{X})$ with the property that $\chi(X) = -\chi(\check{X})$. More precisely, the Hodge numbers of these pairs obey the relation

$$h^{1,1}(X) = h^{1,2}(\check{X}), h^{1,2}(X) = h^{1,1}(\check{X}).$$

In 1991, Candelas, de la Ossa, Green and Parkes used string theory as a guideline to carry out calculations of period integrals for the mirror of the quintic three-fold in $\mathbb{P}^4$ and find the generating function for the genus 0 Gromov-Witten invariants of degree d. This immediately attracted attention from mathematicians and the first idea giving a geometric interpretation of mirror symmetry was due to Strominger-Yau-Zaslow [SYZ].

We are in position to state the core conjecture named SYZ mirror symmetry conjecture.

Conjecture 1.1.4. *Let $X, \check{X}$ be a mirror pair of Calabi-Yau manifolds. Then there is an integral affine manifold with singularities B and maps $\phi : X \to B, \check{\phi} : \check{X} \to B$ which are dual special Lagrangian torus fibrations which means that a non-singular point in B corresponds to a special Lagrangian torus in X and a dual torus in $\check{X}$.*

In the conjecture, finding the base B is very technical part and inspired Gross and Siebert to study toric degenerations of Calabi-Yau manifolds as described in [GS0]. The idea is to deform to a degenerate central fiber and construct the base B using tropical geometry which gives rise to a mirror algebra that can be thought of as a ring of functions on its mirror space. Then one attempts to deform the mirror algebra to incorporate the corrections to the complex structure determined by singular fibers. The Gross-Siebert program is an algebro-geometric framework developed by Mark Gross and Bernd Siebert for understanding SYZ mirror symmetry picture within algebraic geometry. In the Gross-Siebert program, the goal is to construct these mirror pairs directly and understand the correspondence between them. The central objects in their approach are the so-called "tropical" and "affine" structures on the base of a special degeneration of X, which is a real n-dimensional manifold called the "base" of the SYZ fibration. Starting with a log Calabi-Yau pair or a maximally unipotent degeneration of a Calabi-Yau manifold, here is a rough sketch of the procedures in the Gross-Siebert program to construct a mirror family:

- The base B of the SYZ fibration carries a piecewise-linear structure called a "tropical" structure. The combinatorics of this structure encode the degenerations of the torus fibers of the SYZ fibration. The tropical structure is related to a certain "real" or "tropical" limit of the complex structure on X.

- In addition to the tropical structure, B also carries an "affine" structure, which is a flat connection with possible monodromy around the discriminant locus (where the fibers of the SYZ fibration degenerate). This affine structure is related to the complexified symplectic form on X.

- The tropical and affine structures on B can be encoded in an object called a "scattering diagram", which is a certain collection of walls in B, together with data assigned to each wall. This is one of the main technical tools used in the Gross-Siebert program in order to corrects monodromy around those singularities.

- From the data of the scattering diagram, one can construct a mirror space $\check{X}$ as a certain toric degeneration. This involves a wall crossing phenomenon in the scattering diagram motivated from above..

In algebraic geometry, mirror symmetry is often studied in the context of complex algebraic varieties, which are solutions to polynomial equations. The Homological Mirror Symmetry (HMS) conjecture proposes a correspondence between the derived category of coherent sheaves on a Calabi-Yau manifold and the Fukaya category of its mirror. The HMS conjecture provides a way to translate questions about geometry into questions about algebra, and vice versa [Kon1]. Let M be a smooth projective Calabi-Yau manifold. The Lagrangian submanifolds naturally arise in the construction of the A-model as the boundary conditions of strings in the string theory, called D-branes. Mathematically, these boundary conditions should form a category, in this case the *Fukaya category.* In the following, we will depict a broad overview of the Fukaya category:

- **Objects**: The objects of the Fukaya category are Lagrangian submanifolds of the symplectic manifold, possibly equipped with additional data (like a flat line bundle).

- **Morphisms**: The morphisms (i.e., the arrows between objects) in the Fukaya category are defined using Floer homology. Given two Lagrangian submanifolds, the morphism space between them is the Floer homology group of the pair. This is generated by intersection points of the two submanifolds and relations come from counting certain pseudoholomorphic (J-holomorphic) disks with boundary on the submanifolds.

- **Compositions of morphisms**: The composition of morphisms is defined by counting pseudoholomorphic polygons with vertices at intersection points. The counts of these polygons are used to define the higher A_∞ structure on the Fukaya category.

The Fukaya category encodes a wealth of information about the symplectic geometry of the manifold, especially the behavior of Lagrangian submanifolds. In the Homological Mirror Symmetry conjecture, the Fukaya category of a Calabi-Yau manifold is conjectured to be equivalent to the bounded derived category of coherent sheaves on the mirror Calabi-Yau manifold, thus providing a deep link between symplectic and algebraic geometry.

Studying mirror symmetry via Gross-Siebert has applications in a variety of fields, including theoretical physics, algebraic geometry, algebra and topology. Understanding mirror symmetry in a variety of situations becomes more and more instructive. Recent work by Gross, Hacking, Keel and Kontsevich have applied Gross-Siebert mirror construction to cluster algebras to succeed in finding a canonical bases. Bousseau and Argüz have explored a relationship between Foch-Goncharov dual and Gross-Siebert mirror for cluster varieties. Many leading experts in the field try to compactify moduli of K3 surfaces using Gross-Siebert mirror as well.

1.2 Gromov-Witten theory

Gromov-Witten theory was developed in the late 20th century, inspired by ideas from quantum field theory and string theory. The theory provides tools to count the number of pseudo-holomorphic curves/algebraic curves in a given homology class in a symplectic manifold/algebraic variety. Gromov-Witten invariants, named after Mikhail Gromov and Edward Witten, are central objects of study in symplectic and algebraic geometry, and have provided deep insights into the geometry of symplectic manifolds and algebraic varieties. They have played a critical role in the development of mirror symmetry.

To define Gromov-Witten invariants, fixing integers g, n, we consider the moduli space $\mathscr{M}_{g,n}(X, \boldsymbol{\beta})$ of stable maps to X with domain curves of genus g and n smooth marked points representing a homology class $\boldsymbol{\beta}$ in the second homology group $H_2(X, \mathbb{Z})$. A stable map is a map from a nodal curve (a curve that might have nodal singularities but no other singularities) to the target space X(e.g., a symplectic manifold or an algebraic variety), where the map has only finitely many automorphisms. This moduli space is in general not represented by a scheme because of the existence of automorphisms. It is so-called Deligne-Mumford stack admitting a coarse moduli space. For concrete discussions of algebraic stacks, readers may refer to [LMB]. The *expected dimension* of the space is given by the following formula

$$\text{vir.dim}(\mathscr{M}_{g,n}(X,\boldsymbol{\beta})) = \int_X \beta \cdot c_1(T_X) + \dim(X)(1-g) + 3g - 3 + n.$$

where T_X is the tangent bundle of X and $c_1(T_X)$ is the first Chern class. Due to degenerations of stable maps, the expected dimension may not be equal to the actual dimension of the moduli space and it is actually the dimension of a special homology class of the moduli space that represents a nice part in which this special class is often called virtual fundamental class.

The construction of the virtual fundamental class is due to Li-Tian in [LT] and Behrend-Fantechi in [BF], both of which require some understanding of language of Deligne-Mumford stacks and deformation theory. We will not go over the theory of Deligne-Mumford stacks in details, whilst, roughly speaking, étale locally, it is isomorphic to the quotient of a scheme by a finite group, so is our moduli space. In the following, we will go over some crucial concepts and facts of obstruction theory developed by K. Behrend and B. Fantechi in the paper [BF] for the construction of virtual fundamental class.

Formally, let X be a Deligne-Mumford stack, which we can think of as a "space" parameterizing some sort of geometric objects, possibly with some extra data.

Definition 1.2.1. Let X be a Deligne-Mumford stack and $E^\bullet$ be a two-term complex as an object in the derived category of coherent sheaves on X. A *perfect obstruction theory* on X is a morphism $\phi : E^\bullet \to L^\bullet_X$ in the derived category of coherent sheaves to the cotangent complex $L^\bullet_X$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is an epimorphism.

Note that the general construction of cotangent complex requires a background of simplicial objects. However, if a morphism $f : U \to M$ is a local immersion with M smooth, then we can take the cotangent complex $L^\bullet_U$ to be the two-term complex $[I/I^2 \to f^*\Omega_M]$. Moreover, if a stack or scheme X admits an open cover $\{U_i\}$ such that each U_i embeds into a smooth scheme M_i, then all local cotangent complexes defined using these $U_i's$ will glue to a global object $L^\bullet_X$ in the derived category of coherent sheaves. In fact, many schemes or DM stacks admit an open cover in which each open subscheme/DM substack can be embedded into a smooth scheme or stack.

For any algebraic stack X, the cotangent complex $L^\bullet_X$ gives rise to so called *intrinsic normal sheaf* $\mathfrak{N}_X$ and *intrinsic normal cone* $\mathfrak{C}_X$. With the local description of the cotangent complex mentioned above, étale locally, if we have a local immersion $f : U \to M$, one has $\mathfrak{N}_X|_U = [N_{U/M}/f^*T_M]$ where $N_{U/M}$ is the normal sheaf associated to f. Similarly for $\mathfrak{C}_X$, étale locally, for the local immersion f, we have $\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]$ where $C_{U/M}$ is the cone associated to f. *Theorem* 4.5 in [BF] says that $\mathfrak{N}_X$ embeds into the quotient stack $h^1/h^0((E^\bullet)^\vee)$ when $E^\bullet$ is an obstruction theory. Also, we have a natural embedding of stacks $\mathfrak{C}_X \hookrightarrow h^1/h^0((E^\bullet)^\vee)$ and the image of $\mathfrak{C}_X$ under this embedding is called the *obstruction cone*. By the time K. Behrend and B. Fantechi developed the theory of intrinsic normal cone, they did not have intersection theory of Artin stacks at disposal. So, in [BF], they always assumed that their obstruction theory admits a global resolution while constructing the virtual fundamental class. After A. Kresch developed Chow groups for Artin stacks, we can just simply define the *virtual fundamental class* associated to a perfect obstruction theory $E^\bullet$ is the intersection of the obstruction cone ϕ_X with the vertex (zero section) of $h^1/h^0((E^\bullet)^\vee)$.

Furthermore, the whole theory can easily be generalized to relative cases for Deligne-Mumford type morphisms. We will just omit the details here and readers can refer to the paper [BF] for the details of the generalization.

Now, back to our original construction problem of the virtual fundamental class of the moduli space $\mathscr{M}_{g,n}(X, \boldsymbol{\beta})$ of genus g stable maps with n markings,

$$E^\bullet := (\Re^\bullet \pi_{X*} \mathrm{ev}^* T_X)^\vee$$

defines a perfect obstruction theory on $\mathscr{M}_{g,n}(X, \boldsymbol{\beta})$ relative to the moduli space $\mathfrak{M}_{g,n}$ of genus g pre-stable curves with n markings where π_X is the map from the universal stable maps $\mathscr{M}_{g,n+1}(X, \boldsymbol{\beta})$ forgetting the last marked point and stabilizing and ev is the evaluation map at the last marked point. Then it yields the virtual fundamental class $[\mathscr{M}_{g,n}(X, \boldsymbol{\beta})]^{\mathrm{vir}}$. Now we are in position to define these Gromov-Witten invariants as follows:

Definition 1.2.2. Let $\gamma_i \in H^{k_i}(X)$ be a collection of cohomology classes for $i = 1, 2, \ldots, n$. The *Gromov-Witten invariants* associated to these classes are defined to be

$$\int_{[\mathscr{M}_{g,n}(X,\boldsymbol{\beta})]^{\mathrm{vir}}} \mathrm{ev}_i^* \gamma_i$$

The idea of stable maps and Gromov-Witten invariants was first used by Maxim Kontsevich to give solutions for counting rational curves in various varieties [Kon] . For example, let N_d be the number of degree d rational curves through $3d-1$ points in $\mathbb{P}^2$. Then we have

$$N_d = \int_{[\mathscr{M}(\mathbb{P}^2,d)]} \prod_{i=1}^{3d-1} \mathrm{ev}_i^*[pt],$$

in this case, the virtual fundamental class is just the fundamental class. Kontsevich in [Kon] gave a recursion formula for these $N_d's$. In general, Gromov-Witten invariants may only be virtual counts with no any actual enumerative meaning.

The next milestone in the field is development of the relative Gromov-Witten theory which was first successfully investigated by Gathmann in the genus 0 case in [Ga] and was completely developed together with the degeneration formula by Jun Li in [Li1] and [Li2] in algebraic geometry, and was developed by An-Ming Li and Yongbin Ruan in [LR] and by Eleny-Nicoleta Ionel and Thomas Parker in [IP] with the symplectic geometry. The main difference between absolute Gromov-Witten invariansts and relative invariants is the appearance of tangency condition imposed relative to a smooth divisor. For instance, we can consider such an enumerative problem that how many conics are there in $\mathbb{P}^2$ which intersect a fixed elliptic curve at two points in which one of the two intersection has been fixed. Relative Gromov-Witten invariants are crucial as well in the mirror construction of smooth log Calabi-Yau pairs that is the main topic in this thesis.

Formally, define a topological type Γ be to a tuple $(g, n, \beta, \rho, \overrightarrow{\mu})$ where g, n are non-negative integers, $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ is a curve class and $\overrightarrow{\mu} = (\mu_1, \mu_2, \ldots, \mu_\rho) \in \mathbb{N}^\rho$ is a partition of the intersection number $\beta \cdot [D]$. By [Li1] and [Li2], there exists a moduli space $\mathscr{M}_\Gamma(X, D)$ of relative stable maps to the pair (X, D) with the prescribed topological type Γ. We will describe this moduli space by only describing its $\mathbb{C}$-points and readers may refer to the paper [Li1] for the general construction of a family of relative stable maps. Before giving a concrete description, people might guess the moduli space of relative stable maps were just a straightforward imitation of the moduli space of stable maps by just manually adding the piece of the constraints given by the tangency conditions into the moduli space of (absolute) stable maps. For the time being, we call this space the "naive moduli space". The big issue in this naive construction is that the naive moduli space is not compact, which basically means we are not able to define invariants over it. It would just take a moment for many people to figure out the reason that the space is not compact. In a word, one may have a family of maps to (X, D) satisfying the prescribed topological type generally but with some components mapped completely into the divisor D under the limiting map. It took roughly ten years for mathematicians to have relative Gromov-Witten theory perfectly defined for smooth pairs (X, D) from absolute Gromov-Witten theory being developed. Since we are mainly focusing on mirror symmetry within algebraic geometry, we are just going to introduce the technique called "expanded target" used by Jun Li in [Li1] and [Li2] to compactify the naive moduli space of relative stable maps.

Let (X, D) be a smooth pair meaning X is a smooth projective variety and D is a smooth divisor. Let $N_{D/X}$ be the normal bundle of D inside X. Define $Y := \mathbb{P}(N_{D/X} \oplus \mathscr{O}_D)$ the projective completion of the normal bundle which is obviously a $\mathbb{P}^1$-bundle over D. Y has two natural disjoint sections, one with the normal bundle $N_{D/X}^\vee$ and the other with the normal bundle $N_{D/X}$. We call these zero section and infinity section of Y. For any non-negative integer l, define Y_l by gluing l copies of Y, where the infinity section of the i^{th} component is glued to the zero section of the $(i+1)^{\text{th}}$ where $1 \leq i \leq l-1$. Denoting the zero section of the i^{th} component by D_{i-1} and the infinity section by D_i, the singular locus of Y is $\bigcup_{i=1}^{l-1} D_i$. We will also denote D_l by D_∞. Then define X_l by gluing X along D to Y_l along D_0. (For example, $X_0 = X$.) Let $\mathrm{Aut}_D(Y_l) \cong (\mathbb{C}^*)^l$ be the obvious group of automorphisms of Y_l preserving D_0, D_∞ and morphisms to D, then let $\mathrm{Aut}_D X_l$ be the group of automorphisms of X_l preserving X with restriction to Y_l contained in $\mathrm{Aut}_D Y_l$. Notice that $\mathrm{Aut}_D X_l \cong \mathrm{Aut}_D Y_l \cong (\mathbb{C}^*)^l$. Now, we are in position to describe $\mathbb{C}$-points of the moduli space $\mathscr{M}_\Gamma(X, D)$ of relative stable maps of a given topological type Γ.

The $\mathbb{C}$-points of this moduli space correspond to morphisms $C \xrightarrow{f} X_l \to X$ where C is a nodal curve of arithmetic genus g equipped with a collection of smooth marked points $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_\rho$. The morphism f is required to have the property that $f^{-1}(D_\infty) = \sum_{i=1}^{\rho} \mu_i q_i$ and satisfy the *predeformability condition* above the singular locus of X_l, meaning that the pre-image of the singular locus is a union of nodes of C, and if p is one such node, then the two branches of C at p map into different two consecutive irreducible components of X_l and their contact orders with the divisor D_i are the same. The morphism f is also required to satisfy the *stability condition* that there are no infinitesimal automorphisms of the sequence of maps $(C, p_1, \dots, q_\rho) \xrightarrow{f} X_l \to X$ where the allowed automorphisms of the map from X_l to X are $\mathrm{Aut}_D(X_l)$.

Furthermore, the paper [Li1] defines a good notion of a family of relative stable maps, i.e. a moduli functor or groupoid. A family of relative stable maps over a base scheme S is a chain of morphisms $C \xrightarrow{f} \overline{X} \to X \times S$ where for a $\mathbb{C}$-point s of S, the fiber $C_s \xrightarrow{f_s} \overline{X}_s \to X$ is a relative stable map. There is also a predeformability condition, that is, in a neighborhood of a node of C_s mapping to the singular locus of $\overline{X}_s$, one can choose an étale local coordinates on S, C, and $\overline{X}$ with the charts of the form $\mathrm{Spec}(R), \mathrm{Spec}(R[u,v]/uv = b)$ and $\mathrm{Spec}(R[x, y, z_1, z_2, \dots, z_m]/xy = a)$ respectively, with the map of the form $x \mapsto \alpha u^m, y \mapsto \beta v^m$ with α, β units and no restrictions on $z_i' s$.

The above theory admits a perfect relative obstruction theory and thus possesses a virtual fundamental class against which we can define invariants by integrating cohomology classes. This is the sketch of the definition of relative Gromov-Witten invariants. However, it was still a difficult problem to generalize the whole theory to pairs (X, D) with D a normal crossings divisor until several new kinds of Gromov-Witten theory were defined in which the logarithmic Gromov-Witten theory is one of the successful generalizations and was developed independently by Abramovich-Chen in [AC0], Gross-Siebert in [GS1]. In general, logarithmic Gromov-Witten invariants can be defined for any log smooth schemes which is a powerful generalization of the relative Gromov-Witten theory because there are a lot more log smooth schemes out there that are not smooth. For example, the classical relative invariants fail to be defined when the divisor D in a pair (X, D) is normal crossings but log invariants are very well defined. One complication in the classical theory is the change of the target for compactifying the naive space, which makes the deformation-obstruction theory complicated. In the log Gromov-Witten theory, we can proceed to construct the moduli space by imitating what we have done for the absolute Gromov-Witten theory without doing any surgery on the target space with contact orders well defined even if come components of the source curve get fully mapped into the divisor. Besides, the corresponding logarithmic cotangent complex developed by Martin Olsson in [Ol3] can be applied in a more straightforward way.

In Section 2.1, we are going to recap a basic theory of logarithmic algebraic geometry in order for us to dive into the logarithmic Gromov-Witten theory.

1.3 Gross-Siebert Mirror Rings

In [GS2], Gross and Siebert associated to a simple normal crossings log Calabi-Yau pair (W, D) a ring, $R(W, D)$. In the case that D is maximally degenerate, i.e., has a zero-dimensional stratum, the expectation is that $R(W, D)$ is the coordinate ring of the mirror to the pair (W, D). Here we consider a very different case, namely, the situation where D is a smooth anti-canonical divisor on W. The first interesting case is when $W = \mathbb{P}^2$ and D is a smooth cubic curve.

The product rule on the ring $R(W, D)$ is defined using *punctured invariants*, introduced in [ACGS2]. There are a generalization of the logarithmic Gromov-Witten invariants introduced in [GS1], [Ch], [AC]. Punctured invariants allow maps with negative contact order with the divisor D. In general, they may be very difficult to calculate and as yet not many techniques for their calculation have been developed.

One of the main results of this paper gives a relationship between the punctured invariants necessary to define the ring $R(W, D)$ and logarithmic Gromov-Witten invariants. These logarithmic invariants in turn have been proven by Abramovich, Marcus and Wise in [AMW] to coincide with Jun Li's relative invariants. Thus, once this relationship is established, we obtain a description of $R(W, D)$ in terms of relative Gromov-Witten invariants.

Punctured Gromov-Witten invariants appear in the Gross-Siebert mirror construction for a general log Calabi-Yau pair as the structure coefficients of the coordinate ring of the mirror degeneration, with the space of non-negative contact orders representing generators. The whole construction has been studied in [GS2]. Roughly speaking, the relevant punctured Gromov-Witten invariants are defined using the moduli space of stable punctured logarithmic maps with 2 marked points, and exactly one non-trivial punctured point with a point constraint at this latter point.

In order to obtain the initial results, we will need to studying the splitting and gluing behavior of those moduli spaces relevant to the construction of $R(W, D)$. In order to speak of splitting and gluing phenomena properly, [ACGS2] stratifies moduli spaces of punctured maps by tropical types. Morally speaking, for any punctured logarithmic map $f : C/T \to W$ over a log scheme T, there exists a functorial way associating to it a family of maps in the category of generalized cone complexes. This process is called *tropicalization*, and we write $\Sigma(f) : \Sigma(C)/\Sigma(T) \to \Sigma(W)$ for the corresponding tropical map. Associated to this tropical map is certain combinatorial data, the *tropical type* of the map, which we denote by $\boldsymbol{\tau}$. There exists moduli spaces $\mathscr{M}(W, \boldsymbol{\tau})$ of punctured maps marked by the tropical type $\boldsymbol{\tau}$, and these give a stratification of the corresponding moduli space $\mathscr{M}(W, \boldsymbol{\beta})$ of punctured log maps of type $\boldsymbol{\beta}$.

In the context of the construction of $R(W, D)$, the type $\boldsymbol{\beta}$ indicates a curve class of the punctured map and contact orders p, q and $-r$ for the three marked or punctured points on the domain curve. We then have variant moduli spaces $\mathscr{M}(W, \boldsymbol{\beta}, z)$ and $\mathscr{M}(W, \boldsymbol{\tau}, z)$ after imposing requiring that the punctured point with contact order $-r$ maps to a suitably chosen point $z \in W$. See §2.7 for more details.

In the case that D is smooth, the ring $R(W, D)$ can now be defined as follows. First, any non-negative contact order is an integral point of $\Sigma(W) = \mathbb{R}_{\geq 0}$, i.e., all contact orders lie in $\mathbb{N}$. Choose a finitely generated, saturated submonoid $P \subseteq H_2(W, \mathbb{Z})$ containing all effective curve classes on W. Then we have (in the case that D is ample)

$$R(W, D) := \bigoplus_{p \in \mathbb{N}} k[P]\vartheta_p$$

where the ϑ_p are generators. We then define the product structure via the formula

$$\vartheta_p \cdot \vartheta_q = \sum_{\beta \in P} \sum_{r \in \mathbb{N}} N^{\boldsymbol{\beta}}_{pqr} t^{\beta} \vartheta_r$$

where

$$N^{\boldsymbol{\beta}}_{pqr} = \deg[\mathscr{M}(W, \boldsymbol{\beta}, z)]^{\mathrm{virt}}.$$

Here $\boldsymbol{\beta}$ is the punctured type with curve class β and contact orders $p, q, -r$ at the three marked or punctured points. A special case is the case that $r = 0$ in which we will get back to the ordinary 2-pointed relative Gromov-Witten invariants with contact orders p and q having an interior point-constraint imposed.

If D is not ample, then the above product rule may not be a finite sum, and in this case one can avoid this problem by choosing an appropriate completion. See [GS2] for details on this point.

1.4 The Main Results

Recall that for a log Calabi-Yau pair (W, D) with D a smooth divisor, there is an associated so-called Gross-Siebert mirror ring

$$R(W, D) := \bigoplus_{p\in\mathbb{N}} k[P]\vartheta_p$$

where

$$N^{\boldsymbol{\beta}}_{pqr} = \deg[\mathscr{M}(W, \boldsymbol{\beta}, z)]^{\mathrm{virt}}.$$

Here $\boldsymbol{\beta}$ is the punctured type with curve class β and contact orders $p, q, -r$ at the three marked or punctured points. $\mathscr{M}(W, \boldsymbol{\beta}, z)$ denotes the moduli space of genus 0 stable punctured log maps to W of type $\boldsymbol{\beta}$ where $\boldsymbol{\beta}$ indicates a curve class β of punctured log map and contact orders p, q and $-r$ for the three punctured points on the domain curve.

Then, by exploring gluing and splitting, we obtain (see Corollary 3.3.13):

Theorem 1.4.1. *Let (W, D) be as above, $p, q, r \in \mathbb{N}$, $r > 0$ and $\beta \neq 0$. Then we have*

$$N^{\boldsymbol{\beta}}_{pqr} = (q - r)N_{p,q-r} + (p - r)N_{q,p-r},$$

where $N_{a,b}$ is the logarithmic or relative Gromov-Witten invariant counting two-pointed rational curves with contact orders a, b with D, and satisfying a point constraint at the second point.

Remark 1.4.2. Neither $\beta \neq 0$ nor $r > 0$ can be dropped since there will not be gluing happening if either of these two conditions fails. Moreover, These punctured invariants will become ordinary 2-pointed relative Gromov-Witten invariants with an interior point-constraint.

Next, set $e = \beta \cdot D$, then we realize that the invariants $N_{e-1,1}$ for any smooth log Calabi-Yau pair (W, D) with an extra hypothesis for D being nef have a close relation with closed Gromov-Witten invariants $n_{\beta+fh}$ defined and studied in [Cha] where h is the fiber class of the anti-canonical bundle of W over W and f is a natural number, and further investigated in [LLW] and [Lau]. Their closed GW invariants involve a moduli space $\mathscr{M}_{0,1}(X, \beta + h, s)$, that is, the moduli space of genus 0, 1-marked relative stable maps to $X = \mathbb{P}(\omega_W \oplus \mathcal{O})$ with the curve class $\beta + h$ passing through a fixed point s in X. To compare Chan's invariants with the numbers we require, we use a slightly modified version of deformation to the normal cone and degeneration formula, following a similar strategy as in [vGGR]. We will prove the following theorem in §4.3 showing an equational relation between the closed invariants $n_{\beta+h}$ and $N_{e-1,1}$:

Theorem 1.4.3. $(-1)^{e-1} \cdot (e-1) \cdot p_*[\mathscr{M}(X, \beta+h, s)]^{vir} = [\mathscr{M}(W(logD), \beta, s)]^{vir}$ *where* β *is an effective curve class in* W *and* h *is the fiber class of* $p : X \to W$*.*

Then, as a direct consequence of theorem 1.4.3, we have:

Corollary 1.4.4. $(-1)^e \cdot (e-1) \cdot n_{\beta+h} = N_{e-1,1}$ *where* β *is an effective curve class in* W *and* h *is the fiber class.*

Finally, we can apply Theorem 1.4.1 and Corollary 1.4.4 to the case where $W = \mathbb{P}^2$ and D is an elliptic curve to give a full understanding of the mirror ring $R(\mathbb{P}^2, D)$. Note that in this case, $e = 3d$ when $\beta = dH$ where H is a hyperplane class in $\mathbb{P}^2$.

First of all, as an application of the recursion formula, we can abstractly describe an enumerative behavior of 2-pointed relative Gromov-Witten invariants of $(\mathbb{P}^2, D)$ with a point condition where D is a smooth cubic curve. Let us briefly preview the results deduced for $(\mathbb{P}^2, D)$.

Proposition 1.4.5. *Given any positive integer* d*, for* $a + b = 3d$*, the invariants* N^d_{ab0} *and* $N_{a,b}$ *are completely determined by the number* $N_{3d-1,1}$ *plus those lower degree invariants.*

Then to complete an understanding of $R(\mathbb{P}^2, D)$ in this case, we can directly apply Theorem 1.4.3 to have the following corollary

Corollary 1.4.6. $(-1)^{3d} \cdot (3d-1) \cdot n_{\beta+h} = N_{3d-1,1}$ *where* $\beta = dH$ *as a curve class in* $\mathbb{P}^2$ *and* h *is the fiber class.*

Moreover, using these tools, we computed out all the degree 2 punctured Gromov-Witten invariants and all degree 2, 2-pointed relative Gromov-Witten invariants with a point condition for $(\mathbb{P}^2, D)$ as follows

Corollary 1.4.7. *We have* $N_{1,5} = 1, N_{5,1} = 25, N_{2,4} = 7/2, N_{4,2} = 14,$ $N_{3,3} = 9, N^2_{240} = N^2_{420} = 42, N^2_{150} = N^2_{510} = 30$ *and* $N^2_{330} = 54$*.*

Remark 1.4.8. The numbers N^d_{ab0} are just the number of degree d rational curves tangent to order a, b at two unspecified points of D respectively, passing through a specified point away from D, see Remark 3.1.3.

Remark 1.4.9. In [GRZ], T. Grafnitz, H. Ruddat and E. Zaslow also computed various 2-point Gromov-Witten invariants for $(\mathbb{P}^2, D)$, even in higher genus by the tropical correspondence results for smooth del Pezzo Calabi-Yau pairs proven in [Gra] and [Gra1] and our results here agree with their calculations. Roughly speaking, their result is the computation of the broken line expansion of theta functions ϑ_1 for a toric del Pezzo surface with smooth D in terms of tropical invariants, log invariants, and hence an explicit computation of the Landau-Ginzburg potential ϑ_1. The method shown in this thesis might also be applied to calculations of some higher genus Gromov-Witten invariants but it is going to involve in some complicated analysis of geometry of moduli spaces when the genus is larger than 0.

1.5 Outline of the thesis

The thesis is organized as follows. Chapter 2 contains a detailed preliminaries of logarithmic algebraic geometry, the theory of moduli space of stable punctured maps and its tropical interpretation. Section 2.1 reviews a basic theory of log geometry including often used properties and operations in the field. Section 2.2 gives an overview of tropicalizations of log schemes. In Section 2.3, we will quickly recap the theory of stable log maps and their basicness condition. Section 2.4 contains a brief introduction about the theory of punctured logarithmic maps and the stability condition. Then section 2.5 gives an tropical interpretations of punctured log maps and basicness condition and section 2.6 will introduce a notion called *Artin fan* which is a finer stack parametrizing log structures and describing many properties than the stack we introduced in section 2.1. In section 2.7, we briefly talk about the relevant moduli spaces that we are going to use later.

Chapter 3 contains a detailed description of the process of splitting-gluing punctured log maps and give a proof to one of our main gluing theorem. Section 3.1 reviews the process of splitting punctured log maps and how this splitting operation affects the behaviour of related moduli spaces. Section 3.2 basically describes a reversing process of the splitting process that is described in section 3.1. Then section 3.3 states our main gluing theorem in details and gives a complete proof to the theorem.

Chapter 4 begins a kind of new story in which we give a full comparison between the 2-pointed Gromov-Witten invariants $N_{e-1,1}$ and the closed Gromov-Witten invariants defined in [LLW] and [Cha], which implicitly yields a computation of these 2-pointed invariants. In section 4.1, we clarify the basic setup and recap the definition of closed Gromov-Witten invariants. Section 4.2 will be dedicated to review the degeneration formulae for relative Gromov-Witten invariants originally given by J. Li in [Li2] and for stable log maps developed by B. Kim, H. Lho and H. Ruddat in [KLR]. Then section 4.3 contains a full description of the comparison theorem and a detailed proof to it.

Chapter 5 is basically an application of a combination of the gluing theorem in chapter 3 and the comparison theorem in chapter 4. Section 5.1 is dedicated to give an abstract description of calculations of 2-pointed invariants on smooth log Calabi-Yau pairs using the associativity of θ functions and the gluing theorem. Section 5.2 contains a full calculations of 2-pointed invariants of degree 2 of $\mathbb{P}^2$ relative to an elliptic curve as a concrete example showing how powerful the gluing formula together with the associativity is.

1.6 Notations

Throughout the whole paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0, and we assume that all relevant logarithmic structures are Zariski. For a logarithmic scheme X and a map $f : X \longrightarrow Y$ between logarithmic schemes X and Y, $\underline{X}$ and $\underline{f}$ represent the underlying scheme of X and the underlying map of f respectively. For a point $\mathrm{Spec}(\Bbbk)$, $(\mathrm{Spec}(\Bbbk), Q)$ means the logarithmic point with the logarithmic structure whose ghost sheaf is Q.

For any monoid Q, set $Q^* := \mathrm{Hom}_{\mathrm{Mon}}(Q, \mathbb{Z}), Q^\vee := \mathrm{Hom}_{\mathrm{Mon}}(Q, \mathbb{N})$ and $Q^\vee_{\mathbb{R}} := \mathrm{Hom}_{\mathrm{Mon}}(Q, \mathbb{R}_{\geq 0})$, here **Mon** represents the category of monoids.

Chapter 2

Preliminaries and review

2.1 Log Geometry

A key technical tool in the Gross-Siebert program is log geometry, which formalizes the open sector of the Gromov-Witten theory and systematizes the study of toric degenerations. Log structures were discovered by J.-M. Fontaine and L. Illusie on Sunday, July 17, 1988 during a discussion in a train on their travel to Oberwolfach workshop "Aritmetische Algebraische Geometrie". Later, a basic theory of log geometry was first studied and written down by K. Kato in the paper [Kk]. The foundational properties were explored in [Og] which form a basis for the entire theory. A modern guiding philosophy that has emerged is that log geometry provides a connecting link between algebraic and tropical geometry. We shall start off by introducing these log schemes.

Definition 2.1.1. Let X be a scheme. A *pre-logarithmic structure on* X is a sheaf of monoids $\mathcal{M}_X$ together with a morphism of sheaves of monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ with the monoid structure on $\mathcal{O}_X$ given by the multiplication. A *logarithmic structure* on X is a pre-logarithmic structure such that the induced morphism $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.

A scheme together with a (pre-)log structure is called a (pre-)log scheme.

Remark 2.1.2. To any pre-log scheme $(X, \mathcal{M}_X)$, there is an associated log structure defined by taking the amalgamated sum

$$\mathcal{M}_X \oplus_{\alpha^{-1}\mathcal{O}_X} \mathcal{O}_X^*$$

Definition 2.1.3. A morphism of log schemes $\phi : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a pair $(\underline{\phi}, \phi^\#)$ with $\underline{\phi} : X \to Y$ a morphism of schemes and $\phi^\# : \underline{\phi}^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ a morphism of the sheaves of monoids fitting into a commutative diagram

$$\begin{tikzcd}
\underline{\phi}^{-1}\mathcal{M}_Y \arrow[r, "\phi^{\#}"] \arrow[d, "\phi^{-1}\alpha_Y"] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\
\underline{\phi}^{-1}\mathcal{O}_Y \arrow[r, "\phi^*"] & \mathcal{O}_X
\end{tikzcd}$$

Remark 2.1.4. Given a morphism of schemes $\phi : X \to Y$ and a log structure $\mathcal{M}_Y$ on Y, the inverse image sheaf $\phi^{-1}\mathcal{M}_Y$ on X is naturally a pre-log structure. Then by the remark 2.1.2, it yields a log structure on X called the *pull-back* log structure, denoted by $\phi^*\mathcal{M}_Y$.

Definition 2.1.5. A morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is called *strict* if the induced morphism $f^{\#} : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

There is a closely related object called *ghost sheaf* or *characteristic sheaf* bridging algebraic geometry and combinatorial geometry not seen by classical algebraic geometry. It is defined to be the quotient sheaf $\mathcal{M}_X/\alpha^{-1}\mathcal{O}_X^*$.

In the thesis, we will mainly focus on the log structure given by the divisor for log Calabi-Yau pairs. In general, if we have a pair (X, D) where D is a divisor (not necessarily smooth), there is an associated log structure called the *divisorial log structure.* Let us explore several examples to have a taste of log schemes.

Example 2.1.6 (*divisorial log structure*)**.** Let X be a scheme and D be a divisor. Write U for the complement of D in X and let j be the inclusion of U in X. Then, define $\mathcal{M}_X := j_*\mathcal{O}_U^* \cap \mathcal{O}_X$ and α_X to be the inclusion. It turns out that this defines a log structure on X called the divisorial log structure.

Example 2.1.7 (*standard log point*)**.** The *standard log point* over a field $\Bbbk$ is defined to be the pair $(\operatorname{Spec}\Bbbk, \Bbbk^* \oplus \mathbb{N})$ with the morphism $\alpha : \Bbbk^* \oplus \mathbb{N} \to \Bbbk$ defined to be

$$(a, n) \mapsto \begin{cases} 0 & n \neq 0 \\ a & n = 0 \end{cases}$$

or simply we can write it as $(a, n) \mapsto a \cdot 0^n$ with the convention that $0^n = 0$ if $n \neq 0$ and $0^0 = 1$.

One can easily check that this indeed defines a log structure on the point $\operatorname{Spec}\Bbbk$. Usually, the standard log point is denoted by $\operatorname{Spec}\Bbbk^\dagger$ or $(\operatorname{Spec}\Bbbk, \mathbb{N})$. More generally, we can substitute any monoid Q for $\mathbb{N}$ to get a general *log point*, usually denoted by $(\operatorname{Spec}\Bbbk, Q)$ where Q indicates the ghost sheaf.

Example 2.1.8. Let Q be a monoid. Then the scheme $\operatorname{Spec}(k[Q])$ has a natural log structure associated to the morphism $Q \to k[Q]$. More specifically, if Q is a toric monoid meaning the monoid appears as a dual cone of a strongly convex rational polyhedral cone, this log structure on the toric variety $\operatorname{Spec}(k[Q])$ agrees with the divisorial log structure defined by the toric boundary. The log schemes of this form play a crucial role in the category of log schemes as the role of affine schemes in the category of schemes, i.e. the log structure on an arbitrary log scheme can be étale locally modeled on such a log scheme. As a matter of fact, they form *charts* in the category of log schemes.

Definition 2.1.9. Let $(X, \mathcal{M}_X)$ be a log scheme and Q be a monoid. A *chart* for $\mathcal{M}_{\mathcal{X}}$ is a morphism $f : X \to \mathrm{Spec}(\mathbb{Z}[Q])$ such that $f^{\#}$ is an isomorphism.

Lemma 2.1.10. *The morphism*

$$\mathrm{Hom}_{\mathrm{LSch}}(X, \mathrm{Spec}(\mathbb{Z}[Q])) \to \mathrm{Hom}_{\mathrm{Mon}}(Q, \Gamma(X, \mathcal{M}_X))$$

associating to f the composition

$$Q \to \Gamma(X, Q_X) \xrightarrow{\Gamma(f^{\#})} \Gamma(X, \mathcal{M}_X)$$

is an isomorphism where Q_X is the constant sheaf on X associated to Q.

Remark 2.1.11. Based on the lemma above, a chart for $\mathcal{M}_X$ is equivalent to a map $Q \to \Gamma(X, \mathcal{M}_X)$ such that the induced morphism of sheaves $Q^a \to \mathcal{M}_X$ is an isomorphism where Q^a is the log structure associated to the pre-log structure given by $Q \to \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)$.

We can also consider charts for log morphisms.

Definition 2.1.12. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of log schemes. A *chart* for f is triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ where P_X, Q_Y are the constant sheaves of monoids associated to the monoids P, Q respectively, which satisfy the following conditions

- $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are charts.
- the morphism of monoids $Q \to P$ makes the following diagram commute

$$\begin{array}{ccc} f^*Q_Y & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^*\mathcal{M}_Y & \longrightarrow & \mathcal{M}_X \end{array}$$

Arbitrary log schemes can be very wild to manipulate. They are roughly analogous to general ringed spaces in classical algebraic geometry. We need to narrow down to a smaller category in which objects are more geometric. Fine log schemes, fine and saturated log schemes are important classes of log schemes that we need to take into account. They are analogous to integral schemes and normal schemes respectively in the classical theory.

Given a monoid P, we can associate a group

$$P^{\mathrm{gp}} = \{(a,b) \in P \times P | (a,b) \sim (c,d) \,\text{if}\, \exists s \in P \,\text{such that}\, s+a+d = s+b+c\}.$$

There is a universal morphism $P \to P^{\mathrm{gp}}$ such that any morphism from P to a group uniquely factors through this morphism.

Definition 2.1.13. A monoid P is said to be *integral* if the universal morphism is injective. It is called *saturated* if it is integral and for all $p \in P^{\mathrm{gp}}$ such that $n \cdot p \in P$ for some positive integer n, then $p \in P$.

Definition 2.1.14. A log scheme $(X, \mathcal{M}_X)$ is *fine* if étale locally there is a chart $P \to \Gamma(X, \mathcal{M}_X)$ where P is a finitely generated integral monoid. If moreover P can be choosen to be saturated, then $(X, \mathcal{M}_X)$ is called *fine* and *saturated* (abbreviated as fs). Finally, if $P \cong \mathbb{N}^r$ for some non-negative integer r, we call the log structure *locally free.*

From now on, without being explicitly stated, we will always assume our log schemes are fine throughout the entire rest of the thesis.

Log smoothness, étaleness and flatness. One powerful application of log geometry is to study log smooth schemes. Many nice properties fulfilled by smooth schemes in classical algebraic geometry theory admit a log version of these properties which are fulfilled by log smooth schemes. Furthermore, K. Kato gives a combinatorial criterion to determine if a log morphism is log smooth, which makes everything easier for the sake of computations.

Definition 2.1.15. Let $i : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes. i is said to be a *closed immersion* if $\underline{i} : X \to Y$ is a closed immersion as the usual sense and $i^{\#} : f^*\mathcal{N} \to \mathcal{M}$ is surjective. If, in addition, $i^{\#}$ is an isomorphism, then i is called an *strict* closed immersion.

Definition 2.1.16. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes. f is said to be *logarithmically smooth* (resp. *logarithmically étale*) if $\underline{f}$ is of finite presentation and for any commutative diagram of fine log schemes

$$\begin{array}{ccc} (T', \mathcal{L}') & \longrightarrow & (X, \mathcal{M}) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle f} \\ (T, \mathcal{L}) & \longrightarrow & (Y, \mathcal{N}) \end{array}$$

with i a strict closed immersion and $T' \subset T$ closed subscheme defined by a square-zero ideal I, there exists étale locally on T a (resp. unique) lifting $g : (T, \mathcal{L}) \to (X, \mathcal{M})$ such that everything commutes.

Proposition 2.1.17. *A strict morphism of fine log schemes* $(X, \mathcal{M}) \to (Y, \mathcal{N})$ *is log smooth (resp. log étale) if and only if the underlying morphism* $\underline{f}$ *is smooth (resp. étale).*

K. Kato gave a combinatorial way to determine if a log morphism is a log smooth (resp. log étale).

Theorem 2.1.18. *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of fine log schemes. Then the following are equivalent:*

- *f is log étale (log smooth)*
- *étale locally on X and Y, there exist charts $(P_X \to \mathcal{M}, Q_Y \to \mathcal{N}, Q \to P)$ of f satisfying*
  - *the kernel and (resp. the torsion part of) the cokernel of $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ are finite groups of order invertible on X.*
  - *the induced morphism of schemes*

$$X \xrightarrow{\tilde{f}} Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$$

is étale. (For the smooth part of the theorem, it is sufficient to require the map to be smooth.)

As a matter of fact, the induced map $\tilde{f}$ can be used to describe log flatness as well.

Definition 2.1.19. A log morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is said to be *log flat* if étale locally on X and Y, there exist charts $(P_X \to \mathcal{M}, Q_Y \to \mathcal{N}, Q \to P)$ of f such that the induced map $\tilde{f}$ shown in the above theorem is flat.

A flat morphism has to make all fibers equal-dimensional. The analogue for a log flat morphism is that their *log fiber dimensions* are the same where the notion of log fiber dimension is defined as follows. Let us fix a ground field $\Bbbk$ of characteristic 0.

Definition 2.1.20. Let $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of log schemes with the underlying morphism $\underline{f}$ of schemes of finite presentation. Let $x \in X$ and $y = f(x)$, and X_y be the scheme-theoretic fiber over y with the pull-back log structure from the ambient space X. Let $\kappa(x)$ and $\kappa(y)$ be the residue fields respectively at x and y. Choose geometric points $\overline{x} \to x, \overline{y} \to y$. Then we define the *log fiber dimension* by

$$\begin{aligned}\dim^{\log} f^{-1}(y) = &\dim \mathcal{O}_{X_y,\overline{x}}/\langle \alpha_X(\mathcal{M}_{X,\overline{x}} \backslash \mathcal{O}^*_{X,\overline{x}}\rangle + \mathrm{tr.deg.}\,\kappa(x)/\kappa(y) + \mathrm{rank}\,\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\overline{x}} \\ &- \mathrm{rank}\,\overline{\mathcal{M}}^{\mathrm{gp}}_{Y,\overline{y}}\end{aligned}$$

The above definition was first due to A. Abbes and T. Saito in [AS] with a slight difference. Later, M. Gross and B. Siebert spotted that Lemma 3.10, 2 is not true with the original definition and modified the definition to the one shown above to make the lemma fully true. In [AS], they proved the invariance of the log fiber dimensions of a log flat morphism for fs log schemes:

Proposition 2.1.21. *Let $f : X \to Y$ be a log flat morphism of fs log schemes, with the underlying morphism $\underline{f}$ finite of presentation. Then* $\dim^{\log} f^{-1}f(x)$ *is a locally constant function in x.*

There are other good properties that a flat morphism satisfies and for more discussions, reader can refer to [AS] or the appendix A2 of [GS2].

Let us get our hands dirty on calculating examples using this theorem in order to have a taste of what's going on here.

Example 2.1.22 (log smooth curve)**.** Let $X = \operatorname{Spec} \Bbbk[x, y]/(xy)$ with the log structure $\mathcal{M}$ induced by the chart

$$P = \mathbb{N}^2 \to \Bbbk[x, y]/(xy) : (a, b) \mapsto x^a y^b$$

and Y be the standard log point $\operatorname{Spec} \Bbbk^\dagger$ shown in the example 2.1.7 with chart

$$Q = \mathbb{N} \to \Bbbk : a \mapsto 0^a$$

Let f be the morphism $(X, \mathcal{M}) \to \operatorname{Spec} \Bbbk^\dagger$ induced by the diagonal $\Delta : \mathbb{N} \to \mathbb{N}^2$. Then, $\ker(\Delta^{\mathrm{gp}}) = (0)$ and the torsion of $\operatorname{coker}(\Delta^{\mathrm{gp}}) = (0)$ are finite and their orders are 1 which is invertible. Moreover, the induced map

$$X \to \operatorname{Spec} \Bbbk \times_{\operatorname{Spec} \Bbbk[t]} \operatorname{Spec} \Bbbk[x, y] = \operatorname{Spec} \Bbbk[x, y]/(xy)$$

is the identity map. By the theorem, it is log étale, also log smooth. In the next chapter, we will see a general structure theorem of log smooth curves.

Example 2.1.23 (toroidal embedding)**.** Let $\Bbbk$ be a field, and X a scheme locally of finite type over $\Bbbk$, with fine log structure $\mathcal{M}$. Then the theorem says $(X, \mathcal{M})$ is log smooth over Spec $\Bbbk$ (with trivial log structure, i.e. take chart with $Q = \{1\}$) if and only if étale locally on X, there exists a f.g. integral monoid P and étale morphism $X \to \operatorname{Spec} \Bbbk[P]$ such that $(P_X)^a \cong \mathcal{M}$ and the torsion part of P^{gp} is finite of order invertible in $\Bbbk$.

Hence, such $(X, \mathcal{M})$ corresponds to a toroidal embedding, which is étale locally given by the open immersion

$$X \times_{\operatorname{Spec} \Bbbk[P]} \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}] \hookrightarrow X.$$

Fiber products. Next, we are going to talk about fiber products in the categories of log schemes, fine log schemes and fs log schemes respectively because these are the core operations in the theory of splitting and gluing logarithmic maps. First of all, the fiber product of the diagram

$$\begin{tikzcd} & (X_1,\mathcal{M}_1) \arrow[d] \\ (X_2,\mathcal{M}_2) \arrow[r] & (X_0,\mathcal{M}_0) \end{tikzcd}$$

in the category of log schemes can be constructed as follows.

Étale locally, choose charts for these log structures $P_i \to \mathcal{M}_i (i = 0, 1, 2)$, then we have a diagram

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d] & P_1 \\ P_2 & \end{tikzcd}$$

Let P be the pushout in the category of monoids and X be the fiber product of underlying schemes of X_1 and X_2 over X_0 in the category of schemes. Then, étale locally, we have a canonical map $P \to \mathcal{O}_X$. Hence, it creates a log structure $\mathcal{M}$ on X, étale locally on X given by the morphism $X \to \operatorname{Spec}\mathbb{Z}[P]$. $(X, \mathcal{M})$ is defined to be the fiber product in the category of log schemes. Furthermore, if $(X_i, \mathcal{M}_i)(i = 0, 1, 2)$ are fine log schemes, i.e. P_i can be chosen to be integral monoids, let P^{int} be the image of $P \to P^{\text{gp}}$, then étale locally, the fiber product of $(X_1, \mathcal{M}_1)$ and $(X_2, \mathcal{M}_2)$ over $(X_0, \mathcal{M}_0)$ in the category of fine log schemes is defined to be $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{\text{int}}]$ with the log structure pulled back from $\operatorname{Spec}\mathbb{Z}[P^{\text{int}}]$. Analogously, if $(X_i, \mathcal{M}_i)(i = 0, 1, 2)$ are fine and saturated log schemes, let P^{sat} be the saturation of the image of $P \to P^{\text{gp}}$. Then, the fiber product of $(X_i, \mathcal{M}_i)(i = 1, 2)$ in the category of fs log schemes is defined to be $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{\text{sat}}]$ with the log structure pulled back from $\operatorname{Spec}\mathbb{Z}[P^{\text{sat}}]$. We denote these three fiber prodcuts by $X_1 \times_{X_0} X_2, X_1 \times^{\text{f}}_{X_0} X_2$ and $X_1 \times^{\text{fs}}_{X_0} X_2$.

Note that in general, if $(X_i, \mathcal{M}_i)(i = 0, 1, 2)$ are fine and saturated schemes, these three fiber products do not coincide. Geometrically, the fine fiber product is going to pick a main irreducible component of the fiber product and the fs fiber product is to normalize the main component. Thus, a general principle is

$$X_1 \times^{\text{fs}}_{X_0} X_2 \xrightarrow{\text{normalization}} X_1 \times^{\text{f}}_{X_0} X_2 \subset X_1 \times_{X_0} X_2.$$

As we mentioned earlier, log geometry is a bridge connecting algebraic geometry and combinatorial geometry known as *tropical geometry* nowadays. Recall that in toric geometry, a toric blow up of a toric variety corresponds to a subdivision of the fan which is the tropicalization of the toric variety regarded as a log scheme with the induced log structure by the toric bundary. We will define the *tropicalization* map from the category of log schemes to the category of rational polyhedral cone complexes. In this sense, log geometry can be treated as a vast generalization of toric geometry with no map being defined backwards meaning that there is not an inverse map from tropical geometry back to log geometry in general. Many notions in log geometry such as log étaleness, flatness have nice combinatorial descriptions by using tropical geometry. As a very crucial example, *log blow up* is one operation often used in the study of moduli space of stable log maps.

Definition 2.1.24. Let $(X, \mathcal{M})$ be a log scheme. By a *log ideal* we mean a coherent sheaf of ideals $J \subset \mathcal{M}$ where coherence means that locally around any geometric point $\overline{x}$, the ideal is generated by $J_{\overline{x}}$. We call a log ideal *invertible* if it is locally generated by a single element.

In the following, we are going to define the *log blow up* Y of $(X, \mathcal{M})$ against J by showing an explicit construction, denoted by $\operatorname{LogBl}_J X$.

- First of all, if X is of the form $\operatorname{Spec} \mathbb{Z}[P]$ and let I be $\mathbb{Z}[J]$. Define the underlying scheme of $\operatorname{LogBl}_J X$ to be $\operatorname{Bl}_I X$ the ordinary blow up of X at the ideal I and the log structure on the affine chart $Y_s = \operatorname{Spec}(\mathbb{Z}[P[J - s]])$ is induced by $P[J - s]$ the submonoid of P^{gp} generated by P and the elements of $J - s$. Therefore, it is fairly obvious that Y_s is the universal log scheme over X such that the pull back of J to Y_s is generated by s.

- Étale locally, when X admits a chart $X \to \operatorname{Spec} \mathbb{Z}[P]$ and J is generated by $J_0 \subset P$, then we define $\operatorname{LogBl}_J X = \operatorname{LogBl}_{J_0} \operatorname{Spec} \mathbb{Z}[P] \times_{\operatorname{Spec} \mathbb{Z}[P]} X$. It is not hard to show that such a local construction gives rise to a global scheme which is defined to be the log blow up.

From the construction, we implicitly proved that the log blow up satisfies the following universal property

Proposition 2.1.25. *$f : \operatorname{LogBl}_J X \to X$ is a universal log morphism such that $f^{-1}J$ is invertible.*

Remark 2.1.26. Note that any log blowup is a log étale morphisms. By base change and étale descent, we can reduce to the case where everything is affine. Then it is not so hard to verify the log étaleness using Kato's combinatorial criterion for log étale morphisms.

Obvious examples of log blowups are toric blowups.

The stack of log structures. The next part is devoted to a modern and important perspective of looking at log schemes, which was introduced by M. Olsson in [Ol1] with a strong influence of ideas of Luc Illusie. It turns out fine log schemes T over a base fine log scheme S are classified by an Artin stack $\mathbf{Log}_S$. Working with such a stack allows us to re-interpret various notions in log geometry in terms of classical algebraic geometry language.

To any fine logarithmic scheme $(S, \mathcal{M}_S)$, Olsson assigns the category $\mathbf{Log}_S$ fibered in groupoids over the category of S-schemes as follows. The objects of $\mathbf{Log}_S$ are fine logarithmic S-schemes and morphisms are strict log morphisms over S. The fiber functor is just the forgetful functor forgetting log structures.

Theorem 2.1.27. ***Log***$_S$ *is an Artin stack of locally finite type over S.*

Remark 2.1.28. (1) Let T be a S-scheme, then $\mathbf{Log}_S(T)$ is actually the groupoid in which objects are log schemes with the underlying scheme T, i.e. it parametrizes the ways in which one can enhance S-schemes into log schemes.

(2) Note that the stack $\mathbf{Log}_S$ possesses a natural log structure. Indeed, for any S-scheme T with the structure map $f : T \to S$, a morphism $T \to \mathbf{Log}_S$ endows T with a log structure $\mathcal{M}_T$ and a morphism $f^*\mathcal{M}_S \to \mathcal{M}_T$ of log structures.

Recall that a log structure on a stack $\mathscr{F}$ is defined in the way that to every morphism $\underline{T} \to \underline{\mathscr{F}}$ from a scheme $\underline{T}$ to the underlying stack $\underline{\mathscr{F}}$, there is a log structure $\mathcal{M}_T$ and morphisms are compatible with log structures.

Therefore, it defines a log structure on the stack $\mathbf{Log}_S$. Hence, giving a log structure on a scheme T is equivalent to giving a morphism from T to the stack $\mathbf{Log}_{\Bbbk}$. In particular, there is a tautological strict morphism $(S, \mathcal{M}_S) \to \mathbf{Log}_S$.

By Remark 2.1.28 (2), one can easily see that any morphism of log schemes $\phi : (T, \mathcal{M}_T) \to (S, \mathcal{M}_S)$ will induce a morphims of stacks $\mathbf{Log}(\phi) : \mathbf{Log}_T \to \mathbf{Log}_S$. Martin Olsson has shown that the properties of ϕ being log smooth, log étale, log flat originally defined by K. Kato is equivalent to the morphism $\mathbf{Log}(\phi)$ being smooth, étale and flat.

2.2 Tropicalization

Tropical geometry associates to an algebraic variety X a "polyhedral shadow" known as its *tropicalization* whose polyhedral geometry surprisingly reflect a lot of enumerative properties of the variety. Many astonishing correspondences between algebraic geometry and tropical geometry have been found such as the count of rational curves of degree d in the projective plane can also be calculated by the corresponding count of tropical curves of degree d in the corresponding tropical surface. Another application in Gromov-Witten theory is shown in the paper [GPS] in which they have shown that the count of certain tropical curves in a cocharacter lattice corresponds to some Gromov-Witten invariants of the induced toric variety.

Classically, one has to choose an embedding of X into a toric variety in order to define a tropicalization map, and in general this map does depend on the choice of the embedding. Standard references for the definition of a variety admitting an embedding to a toric variety are [MS], [Ka] and [Pay] etc.. For log varieties, a construction of the tropicalization map was first given by M. Gross and B. Siebert using the characteristic sheaf in the appendix B of [GS1] that is analogous to the notion of the tropical part of an exploded manifold of

Parker's work [Par]. A functoriality can be shown immediately by the definition of log morphisms meaning that if we have a log morphism $X \to Y$, we can then have a morphism of generalized polyhedral cone complexes. Besides, an alternative construction was given functorially by Martin Ulirsch using Berkovich analytic spaces in the paper [Ul] in which M. Ulirsch proved that his tropicalization as a generalized polyhedral cone complex is homeomorphic to the one given by Gross and Siebert. We will exhibit the construction by Gross-Siebert and not discuss the Ulirsch's construction of tropicalization map because it requires a quite bit of knowledge of non-Archimedean geometry and Berkovich analytic spaces that are completely off-topic for this thesis.

- *Generalized cone complexes.* We consider the category of rational polyhedral cones, denoted by **Cone**. The objects of **Cone** are pairs (σ, N) where N is a lattice and σ is a top-dimensional strictly convex rational polyhedral cone in $N_{\mathbb{R}} = N \otimes \mathbb{R}$. A morphism in **Cone** is a lattice map $\varphi : N_1 \to N_2$ such that $\varphi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$ where $\varphi_{\mathbb{R}}$ is the $\mathbb{R}$-extension of φ. Such a morphism φ is called a *face morphism* if $\varphi_{\mathbb{R}}$ identifies σ_1 as a face of σ_2. Then recall from [KKMS] and [ACP] that a *generalized polyhedral cone complex* is a topological space with a presentation as the colimit of a finite diagram in **Cone** with all morphisms being face morphisms.

Let Σ be such a generalized polyhedral cone complex. We write $\sigma \in \Sigma$ if σ appears as a cone in the diagram defining Σ. Write $|\Sigma|$ for its underlying topological space. A morphism $\varphi : \Sigma_1 \to \Sigma_2$ of generalized polyhedral cone complexes is a continuous map $\varphi : |\Sigma_1| \to |\Sigma_2|$ such that for each cone $\sigma_1 \subset \Sigma_1$, the induced map $\sigma_1 \to |\Sigma_2|$ factors through a cone σ_2 as $\sigma_1 \to \sigma_2 \in \Sigma_2$. Then it yields the category of generalized polyhedral cone complexes.

- *Tropicalization.* Given a log scheme $(X, \mathcal{M})$ with the log structure in the Zariski topology, we set

$$\Sigma(X) := \left(\coprod_{x \in X} \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \right) / \sim$$

where the disjoint union is taken over all scheme-theoretic points of X and the equivalence relation is generated by the identifications of faces given by dualizing generization maps $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,x'}$ where x is the specilization of x'. One then obtains for each x a map

$$i_x : \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \Sigma(X)$$

In general, the maps i_x may not necessarily be injective because it could cause some wired self-identified faces.

The above definition actually gives a covariant functor from the category of log schemes equipped with Zariski topology to the category of generalized polyhedral cone complexes.

Definition 2.2.1. Let $(X, \mathcal{M}_X)$ be an fs log scheme. The tropicalization $\Sigma(X)$ is said to be *monodromy-free* if for each geometric point x, the map i_x is injective on the interior of any face of the cone $\mathrm{Hom}(\mathcal{M}_{X,x}, \mathbb{R}_{\geq 0})$.

Remark 2.2.2. (1) In this thesis, we will only deal with a target having monodromy-free tropicalization particularly when we try to split and glue punctured log maps later.

(2) In the next chapter, we can going to see more about how tropical geometry can be applied in studying moduli spaces such as stratifying the moduli space of punctured log maps.

Example 2.2.3. In the example 2.1.22, we can easily see that $\Sigma(X) \cong \mathbb{R}^2_{\geq 0}$ where the two components correspond to the rays $x \geq 0$ and $y \geq 0$ respectively. Moreover, the tropicalization of the standard log point is the positive ray $\mathbb{R}_{\geq 0} \cong (t \geq 0)$ and by the functoriality, there is supposed to be a map $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$. Indeed, readers can calculate this map directly by means of the definition, which is the map $(a, b) \mapsto a + b$.

Example 2.2.4. Now, we have a better explanation for log blowups of log smooth log schemes. It turns out that a log blowup $\mathrm{LogBl}_J X \to X$ for X log smooth will give rise to a map between the corresponding generalized polyhedral cone complexes which is a subdivision of the tropicalization of the target and the way to see it is basically from toric geometry because locally the log structure is pulled back from a toric variety using a chart. In this sense, it is a natural generalization of toric blowups.

Remark 2.2.5. The feature that a log blowup of a log smooth log scheme induces a subdivision of a generalized polyhedral cone complexes is in fact true for proper, birational, surjective log étale morphisms in general. It is even good enough for us to take this combinatorial property as our definition for *logarithmic modification* whenever we deal with proper surjective log étale morphims.

Definition 2.2.6. A proper, birational log étale morphism is called a *logarithmic modification.*

2.3 Basics on log/Relative GW invariants

In this subsection, we will quickly review the basics about logarithmic Gromov-Witten theory introduced by Abramovich-Chen and Gross-Siebert in [GS1], [Ch] and [AC]. The results from the paper [AMW] show that relative Gromov-Witten invariants constructed by using the method of so called expanded pair due to Jun Li are equivalent to logarithmic Gromov-Witten invariants in terms of curve counting. Therefore, we will use the terminologies logarithmic and relative interchangeably but we are not going to talk about relative Gromov-Witten theory in details, and readers can refer to the paper [Li1] for detailed theory about relative invariants. For foundations of logarithmic geometry, reader can go back to section 2.1 or refer to K.Kato's paper [Kk] and [Og].

A logarithmic version of the theory of stable curves has been studied by F. Kato in [Kf], which turns out to be very powerful to investigate the smoothing property of a nodal curve around its nodes by putting an appropriate logarithmic structure.

Remark 2.3.1. Recall that the *ghost sheaf* or *characteristic sheaf* of a logarithmic structure $\mathcal{M}_X$ on a scheme X is the quotient sheaf $\overline{\mathcal{M}}_X := \mathcal{M}_X/\alpha^{-1}(\mathcal{O}_X^*)$.

In our paper, all logarithmic structures are assumed to be fine unless otherwise stated. For the precise definition of fine or saturated logarithmic structure, reader can refer to [Kk]. A fine and saturated log scheme is also simply called a fs log scheme.

Definition 2.3.2. A *logarithmic curve* is a flat and logarithmically smooth morphism of fs log schemes $\pi : X \to S$ such that all geometric fibers are reduced and connected schemes of pure dimension 1 satisfying the following. If $\underline{U} \subset \underline{C}$ is the non-singular locus of $\underline{\pi}$, then there exist sections $x_1, \ldots, x_n$ of π such that

$$\overline{\mathcal{M}}_C|_{\underline{U}} \cong \pi^*\overline{\mathcal{M}}_S \oplus \bigoplus_{i=1}^{n} (x_i)_*\mathbb{N}.$$

Hence, the ghost sheaf $\overline{\mathcal{M}}_C$ has three different possibilities shown in the following theorem:

Theorem 2.3.3. *Assume that $\pi : C \to S$ is a log curve. Then*

1. *fibers have at worst nodes as singularities.*

2. *étale locally on S, we can choose disjoint sections $x_i : S \to C$ in the smooth locus of π whose images are called marked points such that:*

   (a) If η is a general point in C away from the marked points and nodes, then

   $$\overline{\mathcal{M}}_{C,\eta} \cong \overline{\mathcal{M}}_{S,\pi(\eta)}.$$

   (b) If p is a marked point in C, then

   $$\overline{\mathcal{M}}_{C,p} \cong \overline{\mathcal{M}}_{S,\pi(p)} \oplus \mathbb{N}.$$

   (c) If q is a node of $\pi^{-1}\pi(q)$ and $Q := \overline{\mathcal{M}}_{S,\pi(q)}$, then

   $$\overline{\mathcal{M}}_{C,q} \cong Q \oplus_{\mathbb{N}} \mathbb{N}^2.$$

where the map $\mathbb{N} \to \mathbb{N}^2$ is the diagonal map and the map $\mathbb{N} \to Q$ given by $1 \mapsto \rho$ is some homomorphism of monoids uniquely determined by the map π with $\rho \neq 0$

Roughly speaking, based on the theorem above, the benefit of applying log geometry to the theory of moduli space of stable curves is that log smoothness often allows mild singularities (e.g. nodes) to occur. In a nutshell, log geometry techniques sometimes magically put us back in category of smooth spaces when we deal with something with mild singularities.

With the preparation above, we are finally ready for the definition of stable log maps.

Definition 2.3.4 ([GS1])**.** Let $g : X \to W$ be a morphism of log schemes. A *pre-stable log map with n markings* to X is a commutative diagram of morphisms of log schemes

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle g} \\ S & \xrightarrow{f_S} & W \end{array}$$

where $\pi : C \to S$ is a log curve with n mutually disjoint sections $x_1, \ldots, x_n$ such that the image of each x_i lies in the smooth locus of π.
Furthermore, a pre-stable log map is called *stable* if the underlying pre-stable map of schemes forgetting the log structures is stable in the ordinary sense.

In general, the moduli space of stable log maps to a log scheme X will not be of finite type. Hence we have to get rid of some less important log maps to make moduli space of finite type. The notion of *basic* stable log maps due to Abramovich-Chen and Gross-Siebert in [Ch],[AC] and [GS1] now appear, and the idea is to only keep those kind of stable log maps which become "universal" tropical map after tropicalization. We will quickly recall this and for more details, the reader can refer to section 1 of [GS1].

Suppose we are given a stable log map $(C/S, \mathbf{p}, f)$ where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a collection of markings with $S = (\operatorname{Spec} \Bbbk, Q)$ where Q is an arbitrary sharp, fs monoid. We will use the convention that p_i will always represent some marked point and a node will be denoted by q. Then the log morphisms π and f will induce morphisms of the sheaves of monoids $\psi = \pi^\# : Q \to \overline{\mathcal{M}}_C$ and $\varphi = f^\# : f^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ respectively. Let us analyze these two maps in a little more details.

• *Structure of* ψ: At first, the homomorphism ψ will be an isomorphism when restricted to a general point complementary to the special points. By the theorem 2.3.3, we know that the sheaf $\overline{\mathcal{M}}_C$ has stalks $Q \oplus \mathbb{N}$ and $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ at marked points and nodes respectively. The latter fiber sum is determined by the map $\mathbb{N} \to Q : 1 \mapsto \rho_q$, and the diagonal map $\mathbb{N} \to \mathbb{N}^2$. Thus, the map ψ is given by the inclusion $Q \to Q \oplus \mathbb{N}$ and $Q \to Q \oplus_{\mathbb{N}} \mathbb{N}^2$ into the first components respectively.

• *Structure of* φ: For a point $x \in C$, the map φ induces a map of monoids $\varphi_x : P_x \to \overline{\mathcal{M}}_{C,x}$ for $P_x := \overline{\mathcal{M}}_{X,f(x)}$. By Theorem 2.3.3, we have the following behaviour at three types of points on C:

- if $x = \eta$ is a general point away from the special points on C, then we have a local homomorphism of monoids:
$$P_\eta \to Q.$$
By a local homomorphism $f : M \to N$ of monoids, we mean that it is a homomorphism of monoids such that $f^{-1}(N^*) = M^*$.

- if $x = p$ is a smooth marked point, then it gives the composition:
$$u_p : P_p \xrightarrow{\varphi_p} Q \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}$$
where u_p is an element of $\mathrm{Hom}(\overline{\mathcal{M}}_{X,f(p)}, \mathbb{N})$, called the *contact order* at p.

- when $x = q$ a node of two irreducible components with two generic points η_1 and η_2 respectively, let $\chi_i : P_q \to P_{\eta_i}$ $(i = 1, 2)$ are the generization maps, then there exists a map
$$u_q : P_q \to \mathbb{Z}$$
called the *contact order* at q such that
$$\varphi_{\eta_2}(\chi_2(m)) - \varphi_{\eta_1}(\chi_1(m)) = u_q(m)\rho_q$$
with $\rho_q \neq 0$ given by the structure of ψ.

Based on the analysis above, we can define a *basic* monoid Q by first defining its dual:
$$Q^\vee := \{((V_\eta)_\eta, (e_q)_q) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \mid \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q\}$$
Here, the sum is taken over all generic points and nodes. We then set
$$Q := \mathrm{Hom}(Q^\vee, \mathbb{N}).$$

Then, the basic monoid Q associated to the the contact orders $\{u_p, u_q\}$ is "universal" in the following sense:

Given a stable log map $(C'/S', \mathbf{p}, f')$ over a log point $S = (\mathrm{Spec}\, \Bbbk, Q')$ with the same contact orders as above, one then obtains a map
$$Q \to Q'$$

which is defined as the transpose of the map:

$$(Q')^\vee \to Q^\vee \subset \bigoplus_\eta P_\eta \oplus \bigoplus \mathbb{N} : m \mapsto \big((\varphi_\eta^t(m))_\eta, m(\rho_q)_q\big)$$

Definition 2.3.5 (basicness)**.** Let $(C/S, \mathbf{p}, f)$ be a stable log map. We call f *basic* if at every geometric point $s \in S$, the map $Q \to Q' = \overline{\mathcal{M}}_{S,s}$ defined by the restriction $(C_s/s, \mathbf{p}_s, f_s)$ is an isomorphism.

Remark 2.3.6. We will see the tropical interpretation of the basic monoid of a stable log map later. Roughly speaking, the basic monoid of a *combinatorial type* is the monoid over which a family of tropical maps of the same combinatorial type is universal.

Once we impose the basicness condition on stable log maps, we obtain a moduli space. Then we have the following theorem due to Gross and Siebert.

Theorem 2.3.7 (Proposition 5.1, [GS1])**.** *If $g : X \to W$ is log smooth, then the moduli space of basic stable log maps to X with fixed contact orders and curve class is a Deligne-Mumford stack and carries a relative perfect obstruction theory relative to the moduli stack of pre-stable logarithmic curves, and therefore possesses a virtual fundamental class. Moreover, if the map g is proper, then the moduli stack is proper.*

2.4 Stable punctured log maps

Definition 2.4.1 ([ACGS2])**.** Let $Y = (\underline{Y}, \mathcal{M}_Y)$ be a fine and saturated logarithmic scheme with a decomposition $\mathcal{M}_Y = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. A *puncturing* of Y along $\mathcal{P} \subset \mathcal{M}_Y$ is a fine sub-sheaf of monoids

$$\mathcal{M}_{Y^\circ} \subset \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$$

containing $\mathcal{M}_Y$ with a structure map $\alpha_{Y^\circ} : \mathcal{M}_{Y^\circ} \to \mathcal{O}_Y$ such that

1. The inclusion $p^\flat : \mathcal{M}_Y \to \mathcal{M}_{Y^\circ}$ is a morphism of logarithmic structures on $\underline{Y}$.

2. For any geometric point $\overline{x}$ of $\underline{Y}$, let $s_{\overline{x}} \in \mathcal{M}_{Y^\circ,\overline{x}}$ be such that $s_{\overline{x}} \notin \mathcal{M}_{\overline{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}_{\overline{x}}$. Representing $s_{\overline{x}} = (m_{\overline{x}}, p_{\overline{x}}) \in \mathcal{M}_{\overline{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}_{\overline{x}}^{\mathrm{gp}}$, we have $\alpha_{Y^\circ}(s_{\overline{x}}) = \alpha_{\mathcal{M}_Y}(m_{\overline{x}}) = 0$ in $\mathcal{O}_{Y,\overline{x}}$.

We call a puncturing $\mathcal{M}_{Y^\circ}$ *trivial* if the induced map $p^\flat$ is an isomorphism. Write $Y^\circ = (\underline{Y}, \mathcal{M}_{Y^\circ})$.

Remark 2.4.2. Note that unlike stable logarithmic curves/maps, the logarithmic structure put on a punctured log curve is **not** necessarily saturated, in other words, C° is in general only a fine logarithmic scheme.

Remark 2.4.3. Readers can easily see that a puncturing of a log structure is not unique. Nonetheless, once a log scheme with a choice of puncturing is equipped with a log morphism to another log scheme, there is in fact a smallest choice for puncturing. More precisely, we have the following definiton.

Proposition 2.4.4. *Let X be a fine log scheme and Y be as in Definition 2.4.1, with a choice of puncturing Y° and a morphism $f : Y^\circ \to X$. In addition, let $\tilde{Y}^\circ$ be the puncturing of Y given by the sub-sheaf of $\mathcal{M}_{Y^\circ}$ generated by $\mathcal{M}_Y$ and $f^\flat(f^*\mathcal{M}_X)$. Then*

- *We have $\mathcal{M}_{\tilde{Y}^\circ}$ is a sub-logarithmic structure of $\mathcal{M}_{Y^\circ}$.*
- *There is a factorization*

$$\begin{tikzcd} Y^\circ \arrow[r] \arrow[d] & X \\ \tilde{Y}^\circ \arrow[ur] & \end{tikzcd}$$

- *Given $Y_1^\circ \to Y_2^\circ \to Y$ with both Y_1°, Y_2° puncturings of Y, and compatible morphisms $f_i : Y_i^\circ \to X$, $\tilde{Y}_1^\circ = \tilde{Y}_2^\circ$.*

Proof. The proof is very straightforward and follows immediately from the definitions. $\square$

This proposition motivates the following definition:

Definition 2.4.5. Let X be a log scheme. A morphism $f : Y^\circ \to X$ from a punctured log scheme is said to be *pre-stable* if the puncturing $\mathcal{M}_{Y^\circ}$ is generated as a sheaf of fine monoids by $\mathcal{M}_Y$ and $f^\flat(f^*\mathcal{M}_X)$.

Furthermore, a puncturing can be pulled back in the following sense:

Proposition 2.4.6. *Let X and Y be fine and saturated log schemes with log structures $\mathcal{M}_X$ and $\mathcal{M}_Y$ respectively and suppose given a morphism $g : X \to Y$. Suppose also given a fine and saturated log structure $\mathcal{P}_Y$ on Y and an induced log structure $\mathcal{P}_X := g^*\mathcal{P}_Y$ on X. Set*

$$X' = (X, \mathcal{M}_X \oplus_{\mathcal{O}_X^\times} \mathcal{P}_X), Y' = (Y, \mathcal{M}_Y \oplus_{\mathcal{O}_Y^\times} \mathcal{P}_Y)$$

Further, let Y° be a puncturing of Y' along $\mathcal{P}_Y$. Then, there is a diagram

$$\begin{tikzcd} X^\circ \arrow[r, "g^\circ"] \arrow[d] & Y^\circ \arrow[d] \\ X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

with all squares Cartesian in the category of underlying schemes, the lower square Cartesian in the category of fs log schemes, and the top square Cartesian in the category of fine log schemes. Furthermore, X° is a puncturing of X' along $\mathcal{P}_X$.

Remark 2.4.7. For the proof of the above proposition, readers can refer to [Proposition 2.7, [ACGS2]]

Throughout the paper, we will essentially only be interested in the case that Y° is a punctured log scheme with the underlying log scheme a logarithmic curve Y over a fine and saturated log scheme S.

Note that when we are given a log curve $\pi : C \to S$, then by Theorem 2.3.3, we have $\mathcal{M}_C = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$ where $\mathcal{M}$ is the the log structure on C with no marked points and $\mathcal{P}$ is the logarithmic structure associated to the marked points. Therefore, it yields the definition of *punctured logarithmic curve.*

Definition 2.4.8. A *punctured logarithmic curve* parametrized by a fine and saturated log scheme S is the following data:

$$(C^\circ \xrightarrow{p} C \xrightarrow{\pi} S, \mathbf{x} = (x_1, x_2, \dots, x_n))$$

where

1. $(C \xrightarrow{\pi} S, \mathbf{x} = (x_1, x_2, \dots, x_n))$ is a logarithmic curve with n disjoint marked points $x_1, \dots, x_n$.
2. $\mathcal{M}_{C^\circ}$ is a choice of a puncturing of $\mathcal{M}$ along the log structure $\mathcal{P}$ associated to the marked points.

Furthermore, given a punctured logarithmic curve as defined above, a pre-stable *punctured logarithmic map* is a diagram

$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d, "p"] & X \\ C \arrow[d, "\pi"] & \\ S & \end{tikzcd}$$

where f is pre-stable in the sense of Definition 2.4.5, and the pre-stable punctured logarithmic map is called *stable* if forgetting about all logarithmic structures, the diagram above is a stable map in the ordinary sense.

Since we can pull back puncturing along log morphisms in general, it indicates that punctured log curves can be pulled back along log morphisms as well. Consider a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism of fine and saturated log schemes $T \to W$. Denote by $(C_T \to T, \mathbf{p}_T)$ the pullback of the log curve $C \to W$ via $T \to W$. Proposition 2.4.6 gives rise to a diagram

$$\begin{tikzcd} C_T^\circ \arrow[r] \arrow[d] & C^\circ \arrow[d] \\ C_T \arrow[r] \arrow[d] & C \arrow[d] \\ T \arrow[r] & W \end{tikzcd}$$

Then, $C_T^\circ \to C_T \to T$ is the pullback of the punctured log curve $C^\circ \to C \to W$.

Example 2.4.9. Let $W = \operatorname{Spec} \Bbbk$ with the trivial log structure. Let C be a smooth curve over W. Choose a point $x \in C$ and a puncturing $\mathcal{M}_{C^\circ}$ at x. In this case, $\overline{\mathcal{M}}_{C^\circ} = \mathcal{P}$, which means all puncturings must be trivial. Indeed, if $s_x = (m_x, p_x) \in \mathcal{M}_{C^\circ,x}$, then $\alpha_{C^\circ}(s_x) = 0$ in $\mathcal{O}_C$ but this is not allowed to occur since m_x must be a unit in $\mathcal{O}_C$.

Example 2.4.10. Let $W = \operatorname{Spec} \Bbbk^\dagger$ the standard log point, and C be a non-singular curve over W with the structure morphism π such that $\mathcal{M}_C = \mathcal{O}_C^\times \oplus \underline{\mathbb{N}}$ where $\underline{\mathbb{N}}$ denotes the constant sheaf on C with the stalk $\mathbb{N}$. Again, choose a punctured point $p \in C$. Let $\mathcal{M}_{C^\circ} \subset \pi^*\mathcal{M}_W \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$ be a puncturing. Let s be a local section of $\mathcal{M}_{C^\circ}$ near p, then s is of the form $((\varphi, n), m)$ with $\varphi \in \mathcal{O}_C^\times, n \in \mathbb{N}$ and $m \in \mathbb{Z}$. If $m < 0$, the condition 2 of Definition 2.4.1 implies that $\alpha_{\pi^*\mathcal{M}_W}(\varphi, n) = 0$, which implies that $n > 0$. Therefore, one has an inclusion that

$$\overline{\mathcal{M}}_{C^\circ} \subset \{(n,m) \in \mathbb{N} \oplus \mathbb{Z} | m \geq 0 \,\mathrm{if}\, n = 0\}.$$

Conversely, any fine submonoid of the right hand side of the above inclusion which contains $\mathbb{N}^2$ can be realized as the stalk of the ghost sheaf at p for a puncturing.

Remark 2.4.11. The notion of puncturing of a stable log map along the marked points allows us to talk about negative contact orders. More concretely, suppose given a stable punctured log map $f : C^\circ \to X$ and $x \in \underline{C}^\circ$ a marked point. Then we have a chain of maps of monoids, denoting the composition by u_x:

$$P_x := \mathcal{M}_{X,f(x)} \to \mathcal{M}_{C^\circ,x} \hookrightarrow \mathcal{M}_{S,\pi(x)} \oplus_{\mathcal{O}^\times} \mathbb{N}^{\mathrm{gp}} \xrightarrow{pr_2} \mathbb{Z}$$

where pr_2 is the second projection map and $u_x \in P_x^*$ is the *contact order* of f at the point x.

Notice that x is sometimes called a *marked point* if $u_x \in P_x^\vee$, otherwise, it is called a *punctured point.*

Remark 2.4.12. In general, imposing well-defined contact orders at punctured points is a quite subtle thing. For a full discussion, we refer readers to [ACGS2]. Roughly speaking, given a family of punctured logarithmic maps $f : C^\circ/W \to X$, at each geometric point $w \in W$ and for each punctured point $x \in C^\circ_w$, we have the contact order $u_x : P_x \to \mathbb{Z}$ defined as above, i.e. we specify an integral tangent vector u_x to $\sigma_{f(x)} \in \Sigma(X)$ (see section 2.5). Then as w varies on W, the cones $\sigma_{f(x)}$ might vary, hence we have to consider the notion so-called *family of contact orders* and its *connected components*.

However, in this paper, especially for the main gluing theorem, we will stick to the case where X is a smooth projective variety with the divisorial logarithmic structure given by a smooth divisor. So, the tropicalization $\Sigma(X)$ is just $\mathbb{R}_{\geq 0}$, and there is not any issue to impose contact orders at punctures.

Remark 2.4.13 (Geometric interpretation of negative contact orders). Let $f : C^\circ/W \to X$ be a punctured log map with $W = (\operatorname{Spec} \Bbbk, Q)$. Suppose $p \in C$ is a non-trivial punctured point and let C' be the irreducible component containing p with the generic point η. Then if C' has negative tangency order with some strata in X, it forces the image of C' under f to be fully contained in those strata.

Indeed, let $P_p := \overline{\mathcal{M}}_{X,f(p)}$ and u_p be the contact order $P_p \to \mathbb{Z}$ described above, which is the composition $\overline{f}^\#_p : P_p \to \overline{\mathcal{M}}_{C^\circ,p} \subset Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$. For any element $t \in P_p$ with $u_p(t) < 0$, we must have $\mathrm{pr}_1 \circ \overline{f}^\#_p(t) \neq 0$, otherwise we have $\alpha_{C^\circ}(0, u_p(t)) \in \mathcal{O}_C^\times$ (one can see this by pulling back everything onto W), which contradicts to the condition 2 of the definition 2.4.1. Thus, if $\chi : P_p \to \overline{\mathcal{M}}_{X,f(\eta)}$ is the generization map, we must have $u_p^{-1}(\mathbb{Z}_{<0}) \cap \chi^{-1}(0) = \emptyset$. This restricts the strata in which C' can lie.

For example, the target is a pair (X, D) with X a scheme and D a smooth divisor and we endow X with the divisorial log structure given by the divisor D. Now, if $f(C') \nsubseteq D$, i.e. $f(\eta) \notin D$, then $\overline{\mathcal{M}}_{X,f(\eta)} = 0$ which implies that $u_p^{-1}(\mathbb{Z}_{<0}) \cap \chi^{-1}(0) = u_p^{-1}(\mathbb{Z}_{<0}) \neq \emptyset$, contradiction! This forces the image of C' under f to be contained in the divisor D.

The punctured points which are not usual marked points impose extra constraints on the possible deformations of a punctured curve, hence of a punctured log map, captured by the *puncturing log ideal* in the base monoid.

Recall from [Og] that a *log ideal* $\mathcal{K}$ on a fine log scheme W is a sheaf of ideals of $\mathcal{M}_W$. It is invariant under the multiplication action $\mathcal{O}^\times$. Hence, every log ideal $\mathcal{K}$ is the pullback of $\overline{\mathcal{K}} := \mathcal{K}/\mathcal{O}^\times$ under the map $\mathcal{M}_W \to \overline{\mathcal{M}}_W$. Moreover, a log ideal $\mathcal{K}$ is called *coherent* if for any chart $Q \to \Gamma(U, \mathcal{M}_W)$, there exists a finite set $S \subset Q$ generating $\mathcal{K}|_U$.

Let $(C^\circ \to W, \mathbf{p})$ be a family of punctured log curves. For each punctured point $p : W \to C^\circ$, consider the composition of maps

$$u_p^\circ : p^*\mathcal{M}_{C^\circ} \to \mathcal{M}_W \oplus \mathbb{Z} \to \mathbb{Z}$$

of sheaves of fine monoids with the first map inclusion and the second map the second projection. Denote by $\mathcal{I}_p \subset p^*\mathcal{M}_{C^\circ}$ the sheaf of ideals generated by $(u_p^\circ)^{-1}(\mathbb{Z}_{\leq 0})$. In [ACGS2], they define the notion of *puncturing log ideal* and explore some properties of it in Section 2.5. Let us review some core properties in the following.

Definition 2.4.14. The *puncturing log ideal* $\mathcal{K} \subset \mathcal{M}_W$ of a family of punctured log curves $(\pi : C^\circ \to W, \mathbf{p})$ is the sheaf of ideals generated by

$$\bigcup_p (\pi^\#)^{-1}(\mathcal{I}_p) \subset \mathcal{M}_W,$$

where p runs over all punctured points.

Remark 2.4.15. In the definition, we abuse the notation when writing $\pi^\#$ for the composition

$$\mathcal{M}_W \to \pi_*\mathcal{M}_{C^\circ} \to \pi_*p_*p^*\mathcal{M}_{C^\circ} = p^*\mathcal{M}_{C^\circ}.$$

Lemma 2.4.16. *The puncturing log ideal of a family of punctured log curves is coherent.*

Here also comes with the vanishing property putting restrictions on deformations of punctured curves, which also makes a punctured log curve an idealized log scheme.

Proposition 2.4.17. *Let $(C^\circ \to W, \boldsymbol{p})$ be a family of punctured log curves and $\mathcal{K}$ be its puncturing log ideal. Then,*

$$\alpha_W(\mathcal{K}) = 0.$$

Proof. For each punctured point $p \in C^\circ$, by the definition of puncturing, one has $p^*\alpha_{C^\circ}(\mathcal{I}_p) = 0$. Pulling back via $\pi^\# : \mathcal{M}_W \to p^*\mathcal{M}_{C^\circ}$ then yields

$$\alpha_W\left((\pi^\#)^{-1}(\mathcal{I}_p)\right) = (p^*\alpha_{C^\circ})(\mathcal{I}_p) = 0.$$

Then, running over all punctured points will give us the vanishing property. □

Definition 2.4.18. An *idealized log scheme* is a triple $(X, \mathcal{M}, \mathcal{K})$ such that $(X, \mathcal{M})$ is a log scheme and $\mathcal{K} \subset \mathcal{M}$ is a sheaf of ideals satisfying $\mathcal{K} \subset \alpha_X^{-1}(0)$. It is called *coherent* if $\mathcal{M}$ and $\mathcal{K}$ are both coherent.

Remark 2.4.19. Proposition 2.4.17 says that a punctured log curve with its puncturing log ideal forms an idealized log scheme.

Example 2.4.20. Let $\operatorname{Spec} \Bbbk^\dagger = (\operatorname{Spec} \Bbbk, \mathbb{N})$ be the standard log point and $(C^\circ/W, \mathbf{p})$ be a punctured log curve over $\operatorname{Spec} \Bbbk^\dagger$ with the underlying curve C smooth and connected, and with only one punctured point p such that

$$\overline{\mathcal{M}}_{C^\circ,p} = \mathbb{N}^2 + \mathbb{N} \cdot (a, -1) \subset \mathbb{N} \oplus \mathbb{Z}.$$

for $0 \neq a \in \mathbb{N}$. Then the punctured log ideal $\overline{\mathcal{K}}$ is $\mathbb{N} \cdot a$. This actually implies that if we want to fit this curve into an one-parameter family with W as a subscheme of the affine line $\operatorname{Spec} \Bbbk[t]$, unlike a log smooth curve over $\operatorname{Spec} \Bbbk^\dagger$, the maximal closed subscheme of $\mathbb{A}^1$ to which this punctured curve can extend is given by the ideal $(t^a) \subset \Bbbk[t]$. This can be checked using the commutative diagram in the definition of log morphisms, and one can realize that a subscheme of $\mathbb{A}^1$ to which the punctured curve can be extended must be on the zero loci of the puncturing log ideal.

Definition 2.4.21. The *puncturing log ideal $\mathcal{K}_W$* of a pre-stable punctured log map $(C^\circ/W, \mathbf{p}, f)$ is the puncturing log ideal of the punctured domain curve.

We will end this section with an example reflecting a new subtlety that the natural bases in punctured Gromov-Witten theory could possibly be reducible and even not pure dimensional.

Example 2.4.22. Let $W = \operatorname{Spec} \Bbbk$ and consider a target X a smooth surface equipped with the divisorial log structure given by a smooth rational curve D such that $D^2 = 2$. Consider a punctured log map $(C^\circ/W, \mathbf{p} = (p_1, p_2, p_3, p_3), f : C^\circ \to X)$ with the contact orders $-1, -1, 2, 2$ respectively and the domain curve having three irreducible components $C_1 \cup C_2 \cup C_3$ in which $q_1 = C_1 \cap C_2$ and $q_2 = C_1 \cap C_3$ are two nodes. In addition, assume that $p_1, p_3 \in C_2$ and $p_2, p_4 \in C_3$, and f contracts the components C_2 and C_3 and identifies C_1 with D. Orient the node q_i from C_i to $C_{i+1} (i = 1, 2)$. We also require that the contact orders at the nodes $q_i (i = 1, 2)$ are $u_{q_1} = u_{q_2} = 1$. It is not so hard to check such a map does exist.

The tropical map. The corresponding tropical curve Γ has three vertices v_1, v_2, v_3 with v_i corresponding to the component C_i $(i = 1, 2, 3)$, edges E_{q_1}, E_{q_2} and legs $E_{p_1}, E_{p_2}, E_{p_3}, E_{p_4}$. Then the moduli space of tropical maps of this type is $\mathbb{R}^3_{\geq 0}$ with coordinates ρ, l_1, l_2 where ρ represents the distance of the image of v_1 to the origin of the tropicalization $\Sigma(X) = \mathbb{R}_{\geq 0}$, and l_1, l_2 represent the lengths of E_{q_1}, E_{q_2} respectively. In particular, the basic monoid Q of this type is $\mathbb{N}^3$ generated by ρ, l_1, l_2.

The punctured log ideal. Let us calculate the punctured log ideal $\overline{\mathcal{K}}$ in this case. For each punctured point p_i $(i = 1, 2)$, we have chains of maps

$$u_{p_i} : P_i := \overline{\mathcal{M}}_{X, f(p_i)} \to \overline{\mathcal{M}}_{C^\circ, p_i} \subset Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$$

and

$$\varphi_{p_i} : P_i := \overline{\mathcal{M}}_{X, f(p_i)} \to \overline{\mathcal{M}}_{C^\circ, p_i} \subset Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_1} Q.$$

Notice that φ_i are dual to the evaluation map $\mathrm{ev}_i : Q^\vee_{\mathbb{R}} \to P_i^\vee = \mathbb{R}_{\geq 0}$ evaluating the tropical curve parameterized by $Q^\vee_{\mathbb{R}}$ at the vertices v_2, v_3. For each $m \in Q^\vee_{\mathbb{R}}$, $\mathrm{ev}_i(m) = \rho(m) + l_i(m)$. Therefore, φ_i is the map given by $1 \mapsto \rho + l_i$. Since $u_{p_i}(1) = -1 < 0$, by the definition of puncturing log ideal, one has that $\overline{\mathcal{K}}_W$ is generated by $\rho + l_1, \rho + l_2$. If we write $\Bbbk[Q] = \Bbbk[x, y, z]$ with the three variables corresponding to ρ, l_1, l_2 respectively, then we can see that $\operatorname{Spec} \Bbbk[Q]/\mathcal{K} \cong \operatorname{Spec} \Bbbk[x, y, z]/(xy, xz)$ which has two components of different dimensions.

We can be instructed from the example that this reflects a smooth local structure of the moduli space of punctured log maps with certain tropical type, so we can see that they behave very badly in general because of the existence of non-trivial puncturing.

2.5 Tropical interpretations

Tropical geometry supplies an efficient way for us to capture the combinatorial data of stable log maps or stable punctured log maps. In most cases, grouping stable (punctured) log maps by means of their tropical data inside the corresponding moduli space will result in a good stratification of the moduli space. Moreover, the notion *basicness* for stable (punctured) log maps can be extracted naturally by looking at the tropical picture.

Recall that for any logarithmic scheme X, we can associate to it its tropicalization $\Sigma(X)$ functorially which is a generalized cone complex in general, see the section 1.4 or [ACGS2]. Thus, when we have a stable log curve

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ S & \end{tikzcd}$$

we have a corresponding tropical picture

$$\begin{tikzcd} \Sigma(C) \arrow[r, "\Sigma(f)"] \arrow[d, "\Sigma(\pi)"] & \Sigma(X) \\ \Sigma(S) & \end{tikzcd}$$

Throughout the entire paper, we are mainly interested in the case that either S is a log point or it is a stacky point with a log structure (e.g. $B\mathbb{G}_m^{\dagger}$).

Here we will focus on the first case, that is, if $S = (\operatorname{Spec}(\Bbbk), Q)$, then $\Sigma(S) = Q_{\mathbb{R}}^{\vee}$. Let us recap how we get the cone complex $\Sigma(C)$.

First of all, at the generic point η of each irreducible component of C, by Theorem 2.3.3, we have $\overline{\mathcal{M}}_{C,\eta} \cong Q$, and thus it produces a cone $Q_{\mathbb{R}}^{\vee}$.

Secondly, at each marked point p, again by Theorem 2.3.3, $\overline{\mathcal{M}}_{C,p} \cong Q \oplus \mathbb{N}$. Then $\operatorname{Hom}(\overline{\mathcal{M}}_{C,p}, \mathbb{R}_{\geq 0}) \cong Q_{\mathbb{R}}^{\vee} \times \mathbb{R}_{\geq 0}$. So, each marked point offers a cone $Q_{\mathbb{R}}^{\vee} \times \mathbb{R}_{\geq 0}$.

Thirdly, at each node q, Theorem 2.3.3 tells us that $\overline{\mathcal{M}}_{C,q} \cong Q \oplus_{\mathbb{N}} \mathbb{N}^2$ where the map $\mathbb{N} \to Q$ maps 1 to some non-zero element ρ and the map $\mathbb{N} \to \mathbb{N}^2$ maps 1 to $(1, 1)$. Then we have $\mathrm{Hom}(\overline{\mathcal{M}}_{C,q}, \mathbb{R}_{\geq 0}) \cong Q^\vee_{\mathbb{R}} \times_{\mathbb{R}_{\geq 0}} \mathbb{R}^2_{\geq 0}$ where the map $Q^\vee_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ is given by evaluation at ρ and the map $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ maps (a, b) to $a + b$. Thus, we can easily notice that we have an isomorphism $Q^\vee_{\mathbb{R}} \times_{\mathbb{R}_{\geq 0}} \mathbb{R}^2_{\geq 0} \cong \{(m, \lambda) \in Q^\vee_{\mathbb{R}} \times \mathbb{R}_{\geq 0} | \lambda \leq m(\rho)\}$.

Finally, $\Sigma(C)$ is obtained by gluing all possible cones described above using natural corresponding generization maps. More details are shown in [ACGS1].

In the meanwhile, it is also shown in [ACGS1], Prop. 2.25, that the map $\Sigma(\pi)$ together with data described above actually gives rise to a family of abstract tropical curves over $Q^\vee_{\mathbb{R}}$ written as a triple $\Gamma = (G, \mathbf{g}, l)$ where G is the dual intersection graph of C with sets $V(G), E(G), L(G)$ of vertices, edges and legs, and the maps

$$\mathbf{g} : V(G) \to \mathbb{N}, \text{and } l : E(G) \to \mathrm{Hom}(Q^*, \mathbb{N}) \setminus 0.$$

where $\mathbf{g}$ is the genus function and l is the "length" function such that for any edge E and $m \in Q^*$, $l(E)(m) = m(\rho)$ in which ρ is the non-zero element explained above for the node to which the edge E corresponds.

Conversely, given such a triple $\Gamma = (G, \mathbf{g}, l)$ of a family of tropical curves, we are able to construct a generalized cone complex, denoted by $\Gamma(G, l)$. This has one cone ω_x for each $x \in V(G) \cup E(G) \cup L(G)$, with $\omega_v = Q^\vee_{\mathbb{R}}$ for $v \in V(G)$.

There is a more or less parallel tropical interpretation for punctured logarithmic maps in which the only change is that a leg may have finite length, in other words, we only need to do a slight modification for punctured points. Indeed, suppose $L \in L(G)$ corresponds to a punctured point (not marked point) with a puncturing $Q^\circ \subset Q \oplus \mathbb{Z}$ that contains $Q \oplus \mathbb{N}$ as its proper submonoid. Set $\omega_L = \mathrm{Hom}(Q^\circ, \mathbb{R}_{\geq 0})$. By the finite generation property of Q, one can quickly show that there is a piecewise linear function $l(L) : Q^*_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ such that

$$\omega_L = \{(s, \lambda) \in Q^*_{\mathbb{R}} \times \mathbb{R}_{\geq 0} | \lambda \leq l(L)(s)\}.$$

Therefore, this tiny difference motivates the following definition of *punctured tropical curves* over a monoid Q.

Definition 2.5.1. A family of *punctured tropical curves* over a monoid T is a graph G together with two maps

$$\mathbf{g} : V(G) \to \mathbb{N}, \quad \text{and } l : E(G) \to \mathrm{Hom}(T, \mathbb{R}_{\geq 0}).$$

Remark 2.5.2. Therefore, according to this definition, the tropicalization of a family of punctured log curves over a base $S = (\mathrm{Spec}(\Bbbk), Q)$ is a family of tropical curves over the monoid Q^*.

Furthermore, a family of *punctured tropical maps* over a monoid T is a map of cone complexes $h : \Gamma \to \Sigma(X)$ for X a logarithmic scheme and Γ associated to the family of tropical curves $(G, \mathbf{g}, l)$.

From this definition, we can directly see that tropicalization of punctured log maps yields a family of punctured tropical maps to $\Sigma(X)$. More importantly, the corresponding punctured tropical maps of a punctured log map contains a collection of combinatorial information encoded by log structures.

In summary, we can extract the following combinatorial data from a punctured log maps over $S = (\mathrm{Spec}(\Bbbk), Q)$ out of its tropicalization:

1. A family of punctured tropical curves $\Gamma = (G, \mathbf{g}, l)$.

2. A map $\sigma : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$ which associates to each object of G the minimal cone of $\Sigma(X)$ that the object gets mapped to.

3. For each edge E_q corresponding to a node q, we have a *weight vector* $u_q \in N_{\sigma(E_q)}$(the lattice of integral vectors to the cone $\sigma(E_q)$).

4. For each E_p corresponding to a marked point or punctured point, we have a contact order $u_L \in N_{\sigma(L)}$.

5. A map of cone complexes $h : \Gamma(G, l) \to \Sigma(X)$ such that if $L \in L(G)$ is a leg with a vertex v, then $h(\mathrm{Int}(\omega_L)) \subset \mathrm{Int}(\sigma(L))$ and from Proposition 2.5.3 below or more specifically, according to the pre-stability condition, one has

$$h(\omega_L) = (h(\omega_v) + \mathbb{R}_{\geq 0} u_L) \cap \sigma(L) \subset N_{\sigma(L)} \otimes_{\mathbb{Z}} \mathbb{R}.$$

   If $h(\omega_v) + \mathbb{R}_{\geq 0} u_L \subseteq \sigma(L)$, then we call the leg L a marked leg; otherwise it is a punctured leg.

6. For each fiber $\Sigma(\pi)^{-1}(t)$ for all $t \in Q^*$, the restriction of h onto the fiber satisfies that if v_1, v_2 are vertices of an edge E_q from v_1 to v_2, then $h(\mathrm{Int}(E_q)) \subset \mathrm{Int}(\sigma(E_q))$, and satisfy the equation

$$h(v_2) - h(v_1) = l(E_q)(t) u_q.$$

   Note that we abuse the notation h for the restriction map.

Sometimes, a collection of data described as above for a family of (punctured) logarithmic maps is called *combinatorial type* of the (punctured) logarithmic maps, denoted by

$$\tau = (G, \boldsymbol{g}, \sigma, \boldsymbol{u})$$

in which $\boldsymbol{u}$ is the package $\{u_p, u_q\}$ collecting all the information of contact orders at punctures or nodes, and define $\boldsymbol{\tau} := (\tau, \beta)$ to be the combinatorial type τ with curve class β.

There is a simple tropical interpretation of pre-stability [ACGS2] saying that the images of punctured legs can extend exactly as far as the image cone allows:

Proposition 2.5.3. *If $(C^\circ/W, \boldsymbol{p}, f)$ is a pre-stable punctured log map over a log point $(\operatorname{Spec} \Bbbk, Q)$ and $h = \Sigma(f) : \Gamma(G, l) \to \Sigma(X)$ is its tropicalization, then for each punctured leg L and $s \in Q_{\mathbb{R}}^\vee$, it holds that*

$$h(s, l(L)(s)) \in \partial\sigma(L),$$

while $h(s, l(L)(s)) + \epsilon u_L \notin \sigma(L)$ for all $\epsilon > 0$.

Proof. Let $p \in C^\circ$ be the punctured point defined by L, and write $\omega = Q_{\mathbb{R}}^\vee, \boldsymbol{\sigma} = P_{\mathbb{R}}^\vee$ for $P = \overline{\mathcal{M}}_{X,f(p)}$. Then the map $h_L : \omega_L \to \boldsymbol{\sigma}$ induced by h is dual to

$$\overline{f}^\#_{f(p)} : P \to \overline{\mathcal{M}}_{C^\circ,p} = Q^\circ \subset Q \oplus \mathbb{Z}.$$

Note that $\omega_L \subset \omega \times \mathbb{R}_{\geq 0}$. Hence, by pre-stability, Q° is generated by $Q \oplus \mathbb{N}$ and by the image $\overline{f}^\#_{f(p)}$. Dually, one has

$$\omega_L = (Q_{\mathbb{R}}^\circ)^\vee = (\omega \times \mathbb{R}_{\geq 0}) \cap h^{-1}(\boldsymbol{\sigma}),$$

which indicates that ω_L is the convex hull of $\omega \times \{0\}$ and $\{(s, l(L)(s)) \in \omega \times \mathbb{R}_{\geq 0}\}$. Thus, this shows that points $(s, l(L)(s))$ get only mapped to the boundary points of $\boldsymbol{\sigma}$. □

Now we are in position to introduce a very crucial notion in punctured Gromov-Witten theory, which is the notion of *basicness* of a punctured log map.

Definition 2.5.4. A pre-stable punctured logarithmic map $(C/S, \mathbf{p}, f : C \to X)$ over a log point $S = (\operatorname{Spec}(\Bbbk), Q)$ is said to be *basic* if the associated family of tropical maps

$$h : \Gamma(G, l) \to \Sigma(X)$$

over $Q_{\mathbb{R}}^\vee$ is universal among all tropical maps of the same combinatorial type, meaning that the associated family of tropical maps of any other pre-stable punctured logarithmic map of the same tropical type is the pullback of $(C/S, \mathbf{p}, f : C \to X)$.

Remark 2.5.5. We have defined basicness using the notion of basic monoid in the previous section and the definition we give here is merely a re-phrase in the language of tropical geometry.

Remark 2.5.6. We can also restrict $\Sigma(f)$ to fibers of $\Sigma(\pi)$ to get maps between cone complexes

$$\Sigma_m := \Sigma(f)|_{\Sigma(\pi)^{-1}(m)} : \Sigma(\pi)^{-1}(m) \to \Sigma(X),$$

for each $m \in \Sigma(S)$.

2.6 Artin fans

In general, an *Artin fan* is a logarithmic Artin stack that is logarithmically étale over a point Spec($\Bbbk$) with the trivial log structure. In logarithmic algebraic geometry, there is a way of associating to any logarithmic scheme X a canonical *Artin fan* $\mathcal{X}$ such that there is an initial factorization $X \to \mathcal{X} \to \mathbf{Log}_{\Bbbk}$ for the canonical map $X \to \mathbf{Log}_{\Bbbk}$ introduced by M. Olsson in [Ol1]. Here $\mathbf{Log}_{\Bbbk}$ is Olsson's stack parametrizing all fine logarithmic structures defined in Section 2.2. The reason that the Artin fan $\mathcal{X}$ is better used than $\mathbf{Log}_{\Bbbk}$ is that the latter is too big and contains a lot of redundant information, whilst the Artin fan just keeps all tropical information of X which is very practical for the sake of studying moduli problems. The detailed construction of $\mathcal{X}$ can be found in [ACMW] and we will briefly outline how to construct the Artin fan for a log scheme in the following.

Theorem 2.6.1. *Let X be a logarithmic algebraic stack over* Spec($\Bbbk$) *which is locally connected in the smooth topology. Then there is an initial strict étale morphism $\mathcal{X} \to \boldsymbol{Log}_{\Bbbk}$ over which $X \to \boldsymbol{Log}_{\Bbbk}$ factors. Moreover, the morphism $\mathcal{X} \to \boldsymbol{Log}_{\Bbbk}$ is representable by an algebraic space.*

Proof. The proof can be found in [[ACMW], Prop.3.1.1] □

Let us quickly sketch the construction of the Artin fan $\mathcal{X}$ of a log scheme/Deligne-Mumford stack X in the following

Given a log scheme/Deligne-Mumford stack X, tropicalization operation yields a generalized polyhedral cone complexes $\Sigma(X)$ shown in section 2.2. Then, for any cone $(\sigma, N) \in \Sigma(X)$, let $P = \sigma^\vee \cap M$ where $M = N^*$. We write $\mathcal{A}_\sigma$ for the following stack quotient

$$\mathcal{A}_\sigma := [\operatorname{Spec}\Bbbk[P]/\operatorname{Spec}\Bbbk[P^{\mathrm{gp}}]]$$

This stack carries the standard toric log structure induced by the descent from the global chart $P \to \Bbbk[P]$. Then the canonical Artin fan of X is defined as follows:

$$\mathcal{X} := \varinjlim \mathcal{A}_\sigma,$$

where the colimit is taken over all cones $\sigma \in \Sigma(X)$ in the category of sheaves over $\mathbf{Log}_{\Bbbk}$.

The appearance of $\mathcal{X}$ can rephrase many properties about X in logarithmic geometry. For instance, X is logarithmically smooth if and only if the associated map $X \to \mathcal{X}$

is smooth. Furthermore, the following proposition in [ACGS1] reflects an importance of Artin fans in the sense of encoding combinatorial data of log schemes given by their tropicalizations.

Proposition 2.6.2 ([ACGS1],Prop. 2.10)**.** *Let X be a Zariski fs log scheme log smooth over* $\mathrm{Spec}(\Bbbk)$*. Then for any fs log scheme T, there is a canonical bijection*

$$\mathrm{Hom}_{\mathrm{fs}}(T, \mathcal{X}) \to \mathrm{Hom}_{\mathrm{Cones}}(\Sigma(T), \Sigma(X)),$$

which is functorial in T.

Remark 2.6.3. There is another definition for Artin fan introduced by D. Abramovich and J. Wise [AW] for log smooth log schemes. When X is log smooth, then the tautological map $X \to \mathbf{Log}_{\Bbbk}$ is smooth, and we define the Artin fan to be $\pi_0(X/\mathbf{Log}_{\Bbbk})$ the connected components of the fibers.

Remark 2.6.4. Unlike tropicalization, the construction of Artin fan of log schemes is **not** functorial in general and you should not expect it to be true for all morphisms whatsoever. This is because for any log morphism $f: X \to Y$, the map $X \to \mathbf{Log}_{\Bbbk}$ does not factor through Y unless f is strict. However, if the domain is Zariski log smooth, then the functoriality holds.

Example 2.6.5. Let $X = \mathbb{A}^1$ the affine line with the toric log structure. Then an easy computation shows that $\mathcal{X} = \mathcal{A}_{\mathbb{N}} = [\mathbb{A}^1/\mathbb{G}_m]$. Given an ordinary scheme T, a map $f: T \to \mathcal{A}_{\mathbb{N}}$ endows T with the pullback log structure $f^*\mathcal{M}_{\mathcal{A}_{\mathbb{N}}}$. Note that in this case, the universal $\mathbb{G}_m$-torsor corresponds to the $\mathbb{G}_m$-torsor subsheaf of $\mathcal{M}_{\mathcal{A}_{\mathbb{N}}}$ defined by the generating section of $\overline{\mathcal{M}}_{\mathcal{A}_{\mathbb{N}}}$. Thus, the pullback log structure on T corresponds to a line bundle with a section.

Therefore, for any arbitrary log scheme $(T, \mathcal{M}_T)$, a morphism $(T, \mathcal{M}_T) \to \mathcal{A}_{\mathbb{N}}$ is the same data as the restriction of $\mathcal{M}_T \to \mathcal{O}_T$ to a $\mathbb{G}_m$-torsor subsheaf of $\mathcal{M}_T$. The pair $\mathcal{A}_{\mathbb{N}}$ with its universal line bundle has a crucial application in Li's relative Gromov-Witten theory since it simplifies relative obstruction theory a lot, and thus yields a much easier way to define a virtual fundamental class on the corresponding moduli space.

Proposition 2.6.6. *Let $X \to Y$ be a morphism of log schemes and suppose X is log smooth with Zariski log structure. Then it induces a morphism of Artin fans $\mathcal{X} \to \mathcal{Y}$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Proof. We will just sketch a proof here and detailed discussion can be found in [ACGS1]. The proof roughly splits into two steps:

- First of all, we need to show that the Artin fan $\mathcal{X}$ admits a Zariski open cover $\{\mathcal{A}_\sigma \mid \sigma \in \Sigma(X)\}$. Since the log structure is Zariski, we may choose a Zariski open cover $\{U \to X\}$ such that $U \to \mathcal{A}_U$ is the Artin fan of U. Note that X is log smooth, then $X \to \mathcal{X}$ is smooth, hence the image $\tilde{U} \subset \mathcal{X}$ of U is an open substack of $\mathcal{X}$. Then it amounts to show that $\tilde{U}$ is the Artin fan of U. Now, we need to use the other definition of Artin fan, see 2.6.3. So it remains to show that $\tilde{U}$ parameterizes the connected components of the map $U \to \mathbf{Log}_{\Bbbk}$. Since every map encountered here is smooth, it is sufficient to check at a geometric point, which is not hard to verify.

- Next, we will show that any morphism $X \to \mathcal{A}_\tau$ factors through a map $\mathcal{X} \to \mathcal{A}_\tau$ for any cone τ. By the above discussion, we have a Zariski open cover $\{\mathcal{A}_\sigma \subset \mathcal{X}\}$ of $\mathcal{X}$ such that $\{U_\sigma := \mathcal{A}_\sigma \times_{\mathcal{X}} X \subset X\}$ forms a Zariski open cover of X.

  Locally, a morphism $U_\sigma \to \mathcal{A}_\tau$ induces a homomorphism $\tau^\vee \to \sigma^\vee$, hence a canonical morphism $\mathcal{A}_\sigma \to \mathcal{A}_\tau$ through which $U_\sigma \to \mathcal{A}_\tau$ factors. Then, one can verify that these local factorizations glue to a global one.

- Note that the statement can be checked étale locally on $\mathcal{Y}$, hence we may assume that $\mathcal{Y} = \mathcal{A}_\tau$ for some cone τ. Then the statement reduces to exactly the discussion above.

□

One important application of Artin fans is in the study of relative Gromov-Witten invariants. Recall from Section 1.2 that a relative stable map might have an expanded target rather than the original one, where an expanded target for a pair (X, D) developed by J. Li in [Li1] is of the following form

$$X[n] := X \sqcup_D P \sqcup_D P \sqcup_D \ldots \sqcup_D P$$

here $P = \mathbb{P}(\mathcal{O} \oplus N_{D/X})$ and the gluing over D attaches 0-sections to ∞-sections. Notice that not every deformation of an expanded target is an expanded target, which makes deformation-obstruction theory much more complicated than the absolute case.

In [ACFM], following the idea in [Ca], the pair (X, D) induces a canonical map $X \to \mathcal{X}$ where X is endowed with the divisorial log structure given by the divisor D and $\mathcal{X}$ is the associated Artin fan. In the case, $\mathcal{X} = [\mathbb{A}^1/\mathbb{G}_m]$ which is also the classifying space of line bundles with a section and there is a divisor $\mathcal{D} := [*/\mathbb{G}_m]$. Note that any deformation of an expanded target of the pair $(\mathcal{X}, \mathcal{D})$ is indeed again an expanded target. As a matter of fact, any expanded target $X[n] \to X$ is the pullback $X[n] = \mathcal{X}' \times_{\mathcal{X}} X$ where $\mathcal{X}' \to \mathcal{X}$ is an expanded target of the pair $(\mathcal{X}, \mathcal{D})$.

Using expanded targets, J. Li defined the moduli space $\mathscr{M}_\Gamma(X, D)$ of relative stable maps to (X, D) with a fixed enumerative data containing the genus of source curve, the number of marked points, a curve class of X and a set of contact orders to D and a virtual fundamental class on this moduli space is constructed by bare hands. Nowadays, we know that there is just a very natural relative obstruction theory for the map $\mathscr{M}_\Gamma(X, D) \to \mathscr{M}_\Gamma(\mathcal{X}, \mathcal{D})$ where the latter is the moduli space of pre-stable maps which is pure-dimensional. Therefore, by [Ma], the virtual fundamental class of $\mathscr{M}_\Gamma(X, D)$ is just the virtual pullback of the fundamental class of $\mathscr{M}_\Gamma(\mathcal{X}, \mathcal{D})$.

2.7 Relevant moduli spaces

Throughout this sub-section, we fix a log smooth morphism $X \to W$ where X carries a Zariski logarithmic structure and $W = (\mathrm{Spec}(\Bbbk), Q)$ is a log point. We further assume that the log structure $\overline{\mathcal{M}}_X$ is globally generated, that is, the natural map $\Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,p}$ is surjective for every $p \in X$. In this section, we essentially follow the notions and terminologies used in [GS2], section 3.

Remark 2.7.1. Let r be an integral point in a cone of $\Sigma(X)$; we write $r \in \Sigma(X)(\mathbb{Z})$. We may view $-r$ as a contact order of a punctured log map to X at a punctured point, and let σ be the minimal cone in $\Sigma(X)$ containing r. Then σ corresponds to a locally closed stratum Z° in X. Let Z be the closure of Z° in X. Thus, r can be viewed as an element of $\mathrm{Hom}(\overline{\mathcal{M}}_Z, \mathbb{N})$ given as follows. Let η be the generic point of Z, so we have $\sigma = \sigma_\eta = \mathrm{Hom}(\overline{\mathcal{M}}_{Z,\eta}, \mathbb{R}_{\geq 0})$. Hence, for any section $s \in \overline{\mathcal{M}}_Z$, we obtain its germ $s_\eta \in \overline{\mathcal{M}}_{Z,\eta}$. So we just define $r(s) := r(s_\eta)$.

In this subsection, we are going to introduce several related moduli spaces, and then recall the procedure of imposing a point-constraint at a punctured point on moduli spaces. Here, we give the basic setup for our target space X.

Moduli of punctured maps. First of all, we fix a type $\boldsymbol{\beta}$ of punctured map, that is, a curve class $\beta \in \mathrm{H}_2(X, \mathbb{Z})$, a genus assigned to each irreducible component, a number of punctured points with a choice of contact order for each of punctured points. Since we are going to only consider genus 0 case, we fix genus to be 0 for source curves once and for all.

Therefore, we have a moduli space of stable basic punctured log maps to X denoted by $\mathscr{M}(X, \boldsymbol{\beta})$, and then if we forget the curve class β from the type $\boldsymbol{\beta}$, then we get a moduli space of pre-stable punctured log maps to $\mathcal{X}$ denoted by $\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta})$. Moreover, we have a natural map

$$\mathscr{M}(X, \boldsymbol{\beta}) \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta})$$

via the composition of stable punctured log maps with the canonical map $X \to \mathcal{X}$. Based on the theory of punctured log maps developed in [ACGS2], the map above possesses a perfect relative obstruction theory and hence a virtual pullback of cycles via [Ma]. However, in general, $\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta})$ might behave very badly, e.g. it is not purely dimensional, see Example 2.4.22, so $\mathscr{M}(X, \boldsymbol{\beta})$ might not possess a virtual fundamental class.

More specifically, we can talk about punctured logarithmic maps marked by a combinatorial type τ or $\boldsymbol{\tau}$, and consider the corresponding moduli stacks $\mathscr{M}(X, \tau)$ or $\mathscr{M}(X, \boldsymbol{\tau})$. However, defining $\mathscr{M}(X, \tau)$ or $\mathscr{M}(X, \boldsymbol{\tau})$ is a bit subtler than that of $\mathscr{M}(X, \boldsymbol{\beta})$, and we refer the reader to §3 of [ACGS2] for the precise definitions of $\mathscr{M}(X, \tau)$ and $\mathscr{M}(X, \boldsymbol{\tau})$.

Evaluation space. Next, we are going to construct a moduli space parametrizing punctures in X with a given contact order $-r$ where $r \in \Sigma(X)(\mathbb{Z})$ and its universal family of such punctures. We will follow [GS2] for the notations of the space parametrizing punctures with contact order $-r$ and its universal family, which are denoted by $\mathscr{P}(X, r)$ and $\widetilde{\mathscr{P}}(X, r)$ respectively.

Let us recall the constructions of $\mathscr{P}(X, r)$ and $\widetilde{\mathscr{P}}(X, r)$ from [GS2]. First, let $Z := Z_r$ be the closed stratum indexed by $r \in \Sigma(X)(\mathbb{Z})$ as in Remark 2.7.1. Then we set

$$\widetilde{\mathscr{P}}(X, r) := Z \times B\mathbb{G}_m^\dagger$$

where Z inherits an induced log structure from X and $B\mathbb{G}_m^\dagger$ is the classifying stack $B\mathbb{G}_m$ equipped with the log structure pulled back from the divisorial log structure on $[\mathbb{A}^1/\mathbb{G}_m]$ with respect to $B\mathbb{G}_m$.

Next, we define the moduli space $\mathscr{P}(X, r)$ parametrizing punctures with contact order $-r$. We define $\mathscr{P}(X, r)$ to have the same underlying stack as $\widetilde{\mathscr{P}}(X, r)$ with the log structure defined as follows. Firstly, we define $\overline{\mathcal{M}}_{\mathscr{P}(X,r)}$ as the sub-sheaf of $\overline{\mathcal{M}}_{\widetilde{\mathscr{P}}(X,r)} = \overline{\mathcal{M}}_Z \oplus \mathbb{N}$ given by

$$\overline{\mathcal{M}}_{\mathscr{P}(X,r)}(U) := \{(m, r(m)) \mid m \in \overline{\mathcal{M}}_Z(U)\}$$

where r can be viewed as an element of $\mathrm{Hom}(\overline{\mathcal{M}}_Z, \mathbb{N})$ according to Remark 2.7.1.
Now we may define a logarithmic structure on $\mathscr{P}(X, r)$ as

$$\mathcal{M}_{\mathscr{P}(X,r)} := \overline{\mathcal{M}}_{\mathscr{P}(X,r)} \times_{\overline{\mathcal{M}}_{\widetilde{\mathscr{P}}(X,r)}} \mathcal{M}_{\widetilde{\mathscr{P}}(X,r)}.$$

Analogously, we can define $\mathscr{P}(\mathcal{X}, r)$. The whole story about moduli space of punctures can also be developed in parallel for the associated Artin stack $\mathcal{X}$ of X. We just substitute $\mathcal{X}$ for X everywhere and keep everything else unchanged.
The next proposition reflects the universal property of $\mathscr{P}(X, r)$.

Proposition 2.7.2 ([GS2], Proposition 3.3)**.** *Let $f : C^\circ/W \to X$ be a pre-stable punctured* *log map with a punctured point $x : \underline{W} \to \underline{C}$ with contact order $-r$. Then there exists a canonical morphism $ev : W \to \mathscr{P}(X,r)$ with the property that*

$$W^\circ := W \times^{fine}_{\mathscr{P}(X,r)} \widetilde{\mathscr{P}}(X,r)$$

agrees with $(\underline{W}, x^*\mathcal{M}_{C^\circ})$.
The analogous statements for punctured log maps $f : C^\circ/W \to \mathcal{X}$ also hold.

Remark 2.7.3. In fact, in §3 of [GS2], they give $\mathscr{P}(\mathcal{X},r)$ the structure of an idealized logarithmic stack as follows. We can view r as an element of $\mathrm{Hom}(\overline{\mathcal{M}}_{\mathscr{P}(\mathcal{X},r)}, \mathbb{N})$. Then let $\mathcal{I} \subset \overline{\mathcal{M}}_{\mathscr{P}(\mathcal{X},r)}$ be the ideal sheaf given by $r^{-1}(\mathbb{Z}_{>0})$. The we can lift it to an ideal sheaf $\mathcal{I} \subset \mathcal{M}_{\mathscr{P}(\mathcal{X},r)}$. This turns out to be a coherent idealized log structure on $\mathscr{P}(\mathcal{X},r)$. For details about idealized log structure, see §III.1.3 of [Og].

We fix a type $\boldsymbol{\beta}$ of punctured log maps, and further assume that there is one punctured point x_{out} with the contact order $-r$ where $r \in \Sigma(X)(\mathbb{Z})$. According to the proposition above, we have two maps of stacks

$$\mathrm{ev}_X : \mathscr{M}(X,\boldsymbol{\beta}) \to \mathscr{P}(X,r), \text{and } \mathrm{ev}_{\mathcal{X}} : \mathfrak{M}(\mathcal{X},\boldsymbol{\beta}) \to \mathscr{P}(\mathcal{X},r).$$

Then we define a moduli space as follows.

Definition 2.7.4. We define

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\beta}) := \mathfrak{M}(\mathcal{X},\boldsymbol{\beta}) \times_{\underline{\mathcal{X}}} \underline{X}$$

where the map $\mathfrak{M}(\mathcal{X},\boldsymbol{\beta}) \to \underline{\mathcal{X}}$ is the evaluation map at x_{out}.

Remark 2.7.5. Note that there is a factorization

$$\mathscr{M}(X,\boldsymbol{\beta}) \xrightarrow{\epsilon} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\beta}) \to \mathfrak{M}(\mathcal{X},\boldsymbol{\beta})$$

and we have a relative perfect obstruction theory for ϵ which is described in detail in [ACGS2],§4.

Remark 2.7.6. In general, notice that, for any subset $\boldsymbol{S}$ of {nodal sections, marked sections, punctured sections}, we have a evaluation map $\mathfrak{M}(\mathcal{X},\boldsymbol{\beta}) \to \prod_{S\in\boldsymbol{S}} \underline{\mathcal{X}}$ given by

$$(C^\circ/W, \boldsymbol{p}, f) \mapsto (\underline{f} \circ p_S)_{S\in\boldsymbol{S}},$$

Therefore, we can define the corresponding moduli space $\mathfrak{M}^{\mathrm{ev},\boldsymbol{S}}(\mathcal{X},\boldsymbol{\beta})$ associated to $\boldsymbol{S}$ as the following:

$$\mathfrak{M}^{\mathrm{ev},\boldsymbol{S}}(\mathcal{X},\boldsymbol{\beta}) := \mathfrak{M}(\mathcal{X},\boldsymbol{\beta}) \times_{\prod_{S\in\boldsymbol{S}} \underline{\mathcal{X}}} \prod_{S\in\boldsymbol{S}} \underline{X}.$$

If there is no danger of confusion, we just leave out the superscript $\boldsymbol{S}$ and write it simply as $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta})$.

Point-constrained moduli spaces. We start off by stating a proposition.

Proposition 2.7.7 ([GS2], Proposition 3.8)**.** *Fix $r \in \Sigma(X)(\mathbb{Z})$ and a closed point $z \in Z^\circ$ where Z° is the corresponding locally closed stratum determined by r. Then there is a morphism*

$$B\mathbb{G}_m^\dagger \to \mathscr{P}(X, r)$$

with image $z \times B\mathbb{G}_m^\dagger$ which, on the level of the ghost sheaves, is given by

$$r : \overline{\mathcal{M}}_{Z,z} \cong \overline{\mathcal{M}}_{\mathscr{P}(X,r),z} \to \mathbb{N}.$$

Now we are in position to define the so-called point-constrained moduli space.

Definition 2.7.8 ([GS2], Definition 3.10)**.** We define

$$\begin{aligned} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z) :&= \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}) \times_{\mathscr{P}(X,r)} B\mathbb{G}_m^\dagger \\ &= \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}) \times_{\mathscr{P}(\mathcal{X},r)} B\mathbb{G}_m^\dagger \end{aligned}$$

where $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}) \to \mathscr{P}(X, r)$ is the evaluation map at x_{out} and the map $B\mathbb{G}_m^\dagger \to \mathscr{P}(X, r)$ is given by the proposition above.

Similarly, we can define

$$\mathscr{M}(X, \boldsymbol{\beta}, z) := \mathscr{M}(X, \boldsymbol{\beta}) \times_{\mathscr{P}(X,r)} B\mathbb{G}_m^\dagger.$$

Then, we can easily see that there is a cartesian diagram (in all categories)

$$\begin{tikzcd} \mathscr{M}(X, \boldsymbol{\beta}, z) \arrow[r] \arrow[d, "\epsilon'"] & \mathscr{M}(X, \boldsymbol{\beta}) \arrow[d, "\epsilon"] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z) \arrow[r] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}) \end{tikzcd}$$

Therefore, the relative perfect obstruction theory for ϵ pulls back to a relative perfect obstruction theory for ϵ', so the virtual pullback is defined via [Ma]. However, as we already mentioned before, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z)$ may be very bad and not equi-dimensional in general, thus $\mathscr{M}(X, \boldsymbol{\beta}, z)$ might not possess a virtual fundamental class. However, since our goal is to illustrate some enumerative properties of punctured invariants, the moduli spaces that we will consider will be all nice and equipped with virtual cycles.

Chapter 3

The main gluing theorem

3.1 Splitting punctured logarithmic maps

Let us explain the setup in which we will be for the rest of the thesis. We will fix an arbitrary smooth log Calabi-Yau pair (X, D) where X is a smooth projective variety with a smooth divisor D satisfying $K_X + D = 0$. Choose a general point $z \in D$ such that there is not any rational curve going through z inside D. We can achieve this since D, as a smooth variety, is a (weak) Calabi-Yau variety. However, if for any point on D, there existed a rational curve passing through that point, then it would imply that D is uniruled by 1.3 Proposition of §4 in [Ko]. So, by 1.11 Corollary of §4 in [Ko], we could conclude that D has Kodaira dimension $-\infty$, which would tell us that D is by no chance a Calabi-Yau variety.

Fix a type of curve $\boldsymbol{\beta}$ for X where $\boldsymbol{\beta}$ consists of non-zero curve class $\beta \in \mathrm{H}_2(X, \mathbb{Z})$, 2 distinct ordinary marked points x_1, x_2 with prescribed contact orders $p, q \in \Sigma(X)(\mathbb{Z})$ respectively, and exactly one punctured point x_{out} with contact order $-r \neq 0 \in \Sigma(X)(\mathbb{Z})$. By the point-constraint imposing process described above, we can form the point-constrained moduli spaces $\mathscr{M}(X, \boldsymbol{\beta}, z)$ and $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z)$. The result [Proposition 3.19, [GS2]] shows that the space $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z)$ is actually pure-dimensional of dimension 0. Further, $\mathscr{M}(X, \boldsymbol{\beta}, z)$ possesses a virtual fundamental class defined by the virtual pullback of the fundamental class of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\beta}, z)$. The relative virtual dimension is $\beta \cdot c_1(\Theta_{X/\Bbbk}) = \beta \cdot (K_X + D) = 0$, and so the virtual dimension of $\mathscr{M}(X, \boldsymbol{\beta}, z)$ is 0. See Proposition 3.12 in [GS2] for more details. Then these facts naturally yield the following definition of punctured Gromov-Witten invariants relevant for the cosntruction.

Definition 3.1.1. Let $p_1, p_2, r \in \Sigma(X)(\mathbb{Z})$, and let $\boldsymbol{\beta}$ be a type of punctured curve with underlying curve class β and three punctured points $x_1, x_2, x_{\mathrm{out}}$ with contact order p, q

and $-r$ respectively. We define

$$N_{pqr}^{\boldsymbol{\beta}} := \int_{[\mathscr{M}(X,\boldsymbol{\beta},z)]^{\mathrm{vir}}} 1.$$

Remark 3.1.2. Notice that, by [Corollary 1.14, [GS2]], one can see that $p_1 + p_2 - r = \beta \cdot D$.

Remark 3.1.3. In fact, in the definition of punctured Gromov-Witten invariants, we could allow r to be 0. Then the punctured invariants $N_{pq0}^{\boldsymbol{\beta}}$ are just standard 3-point relative Gromov-Witten invariants with one point-constraint away from D, having tangency order p, q respectively with D.

Next, we are going to briefly recap splitting and gluing operations in the theory of punctured logarithmic maps. In §5 of [ACGS2], they defined the notion *splitting type* for nodal sections of punctured logarithmic maps.

Definition 3.1.4 (Definition 5.1, [GS2])**.** A *nodal section* of a family of nodal curves $\underline{\pi} : \underline{C} \to \underline{W}$ is a section $s : \underline{W} \to \underline{C}$ of $\underline{\pi}$ that étale locally in $\underline{W}$ factors over the closed embedding defined by the ideal (x, y) in the domain of an étale map

$$\operatorname{Spec} \mathcal{O}_W[x, y]/(xy) \to \underline{C}.$$

The *partial normalization* of $\underline{C}/\underline{W}$ along s is the map

$$\underline{\kappa} : \underline{\tilde{C}} \to \underline{C}$$

that étale locally is given by base change from the normalization of the plane nodal curve $\operatorname{Spec} \Bbbk[x, y]/(xy)$. We say s is of *splitting type* if the two-fold unbranched cover $\underline{\kappa}^{-1}(\operatorname{im}(s)) \to \operatorname{im}(s)$ is trivial.

Roughly speaking, a nodal section of a family of punctured logarithmic maps is of splitting type if étale locally, after partially normalizing this nodal section, we will end up with a trivial two-fold unbranched cover of the original nodal section. Then any nodal section will allow us to split the type of punctured logarithmic map, and then result in a splitting map which breaks our moduli space into many pieces. More precisely, by Proposition 5.15 of [ACGS2], for a nodal section of splitting type, let τ be the original tropical type, and τ_1 and τ_2 be the tropical types after splitting along the nodal section respectively, we have a Cartesian diagram

$$\begin{tikzcd} \mathscr{M}(X,\tau) \arrow[r, "\delta"] \arrow[d] & \mathscr{M}(X,\tau_1) \times \mathscr{M}(X,\tau_2) \arrow[d] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau) \arrow[r, "\delta^{\mathrm{ev}}"] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau_1) \times \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau_2) \end{tikzcd} \tag{3.1.1}$$

with compatible obstruction theories and both maps being finite and representable.

Remark 3.1.5. In the situation we are interested in, all nodal sections are automatically of splitting type since our source curve is always rational. Hence, both δ^{ev} and δ always exist in our setup.

Remark 3.1.6. Notice that after splitting a punctured logarithmic map of combinatorial type τ at a nodal section into punctured logarithmic maps of combinatorial type τ_1, τ_2 respectively, we actually get two extra punctured points $w \in \tau_1, w' \in \tau_2$ such that the contact orders at these two points are negative to each other.

Since we are interested in some enumerative properties of punctured invariants, we will always take the point-constraint into account. Thus, we need a version of splitting maps with point-constraint. The following lemma deals with such a slight modification.

Lemma 3.1.7. *Let τ be a combinatorial type with two input legs representing ordinary marked points and one output leg representing a punctured point such that one of the chosen input legs is adjacent to the same vertex v_{out} as the output leg. Assume given an edge E also adjacent to v_{out}. Given a punctured map of type τ, let $\boldsymbol{S}$ be the 2-point set consisting of the nodal section corresponding to E and the punctured point x_{out}. Let τ_1 and τ_2 be the resulting types of punctured curve after splitting $\boldsymbol{\tau}$ at E, with x_{out} in the component corresponding to $\boldsymbol{\tau}_2$. Then we have the following cartesian diagram*

$$\begin{array}{ccc}
\mathfrak{M}^{ev}(\mathcal{X}, \tau, z) & \xrightarrow{\widetilde{\delta}^{ev}} & \mathfrak{M}^{ev}(\mathcal{X}, \tau_1) \times \mathfrak{M}^{ev}(\mathcal{X}, \tau_2, z) \\
\downarrow & & \downarrow \\
\mathfrak{M}^{ev}(\mathcal{X}, \tau) & \xrightarrow{\delta^{ev}} & \mathfrak{M}^{ev}(\mathcal{X}, \tau_1) \times \mathfrak{M}^{ev}(\mathcal{X}, \tau_2)
\end{array}$$

and an analogous statement holds for $\mathscr{M}(X, \tau, z)$. Note that these evaluation spaces are with respect to $\boldsymbol{S}$.
Hence, finiteness and representability of δ^{ev} imply that the point-constraint splitting map $\widetilde{\delta}^{ev}$ is finite and representable. More importantly, they have compatible obstruction theories.

Proof. The lemma follows directly from the definition of point-constrained moduli space and (3.1.1). Indeed, note that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_2, z) = \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_2) \times_{\mathscr{P}(\mathcal{X}, r)} B\mathbb{G}_m^{\dagger}$. Then the conclusion follows directly by taking the fiber product. $\square$

3.2 Gluing punctured logarithmic maps

A general framework of gluing arbitrary punctured logarithmic maps has been developed by Abramovich, Chen, Gross and Siebert in [ACGS2], and in this subsection, we are going to just apply it to our setup, in other words, we want to reverse the splitting process which has been explained in the diagram (3.1.1).

Consider base schemes W_1, W_2 with maps $W_1 \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_1)$ and $W_2 \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_2)$, i.e., consider families of punctured logarithmic maps $(C_1^\circ/W_1, \tau_1, f_1 : C_1^\circ \to X), (C_2^\circ/W_2, \tau_2, f_2 : C_2^\circ \to X)$ parametrized by W_1 and W_2 respectively, and two sections $\underline{w} : \underline{W}_1 \to \underline{C}_1^\circ, \underline{w}' : \underline{W}_2 \to \underline{C}_2^\circ$.

Obviously, if we want to glue these two families, we need to be able to glue them schematically. Therefore, it is reasonable to assume that $\underline{f}_1 \circ \underline{w} = \underline{f}_2 \circ \underline{w}'$. The chief difficulty that differs from the ordinary gluing situation of ordinary stable maps is that we do not have evaluation maps in logarithmic category. So the method used in [ACGS2] is to enlarge the logarithmic structures of W_1 and W_2 as follows:

Let $W_1^E = (\underline{W_1}, w^*\mathcal{M}_{C_1^\circ})$ and $\widetilde{W}_1$ be the saturation of W_1^E (see [Og], III Prop. 2.1.5), and similarly we can define $\widetilde{W}_2$ as well. Then, it is not so hard to see that we have the following evaluation map

$$\widetilde{W}_i \longrightarrow W_i^E \longrightarrow X$$

for $i = 1, 2$. Then we have the following gluing proposition

Proposition 3.2.1. *There exists a Cartesian diagram in the category of fs log stacks*

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \widetilde{W}_1 \times \widetilde{W}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

such that there is a logarithmic scheme $W = (\underline{\widetilde{W}}, \mathcal{M}_W)$ with $\mathcal{M}_W \subset \mathcal{M}_{\widetilde{W}}$, equipped with morphisms $\psi_i : W \to W_i$, $(i = 1, 2)$ and a universal glued family $(\pi : C^\circ \to W, \boldsymbol{\beta}, f : C^\circ \to X)$.

Proof. This is just a special case of Theorem 2.5 in [Gro]. □

Remark 3.2.2. In fact, there is no barrier at all to extend such a gluing method to the situation with point-constraint at x_{out}.

Recall that we have a splitting map $\tilde{\delta}^{\mathrm{ev}}$ in Lemma 3.1.7. The next proposition will give us a factorization of $\tilde{\delta}^{\mathrm{ev}}$ which can be used to relate punctured invariants to 2-pointed relative/logarithmic Gromov-Witten invariants with one-point constraint.

Proposition 3.2.3. *In the situation of Lemma 3.1.7 above, further assume that $r \neq 0$. We then have a diagram*

$$
\begin{tikzcd}
\mathscr{M}(X,\tau,z) \arrow[r, "\tilde{\phi}"] \arrow[d] & (\mathscr{M}(X,\tau_1)\times_{\underline{D}} z)\times \mathscr{M}(X,\tau_2,z) \arrow[r] \arrow[d, "\epsilon"] & \mathscr{M}(X,\tau_1)\times \mathscr{M}(X,\tau_2,z) \arrow[d] \\
\mathfrak{M}^{ev}(\mathcal{X},\tau,z) \arrow[r, "\phi"] & (\mathfrak{M}^{ev}(\mathcal{X},\tau_1)\times_{\underline{D}} z)\times \mathfrak{M}^{ev}(\mathcal{X},\tau_2,z) \arrow[r, "\tilde{\Delta}"] \arrow[d] & \mathfrak{M}^{ev}(\mathcal{X},\tau_1)\times \mathfrak{M}^{ev}(\mathcal{X},\tau_2,z) \arrow[d] \\
 & \underline{D} \arrow[r, "\Delta"] & \underline{D}\times\underline{D}
\end{tikzcd}
$$

with all squares Cartesian in the category of fine log schemes. Furthermore, ϕ is a finite surjective morphism.

Proof. Firstly, note that the types of curve class τ_1 and τ_2 incorporate an ordinary marked point p and a punctured point q respectively coming from splitting the chosen node. Note the component which contains x_{out} gets mapped into the divisor D because the contact order at x_{out} namely $-r$, is negative. But for a generic $z \in D$, there is no rational curve passing through z in D because D, by the definition, is a Calabi-Yau variety. Hence, we can conclude that the map restricting to the component will be just the constant map with the value z. Therefore, the map ϕ is just the splitting map from any punctured logarithmic map of type τ into two punctured logarithmic maps of type τ_1 and τ_2 respectively, each of which has been equipped with a point-constraint about z. Further, $\tilde{\Delta}$ is just the map forgetting the point-constraint given on $\mathfrak{M}^{\text{ev}}(\mathcal{X},\tau_1)$. Then checking Cartesianess of the square involved in Δ and $\tilde{\Delta}$ becomes not math. Secondly, the fact that the top two diagrams are Cartesian follows directly from Lemma 3.1.7 and the diagram (3.1), and the finiteness of ϕ follows from the finiteness of $\delta^{\text{ev}} = \tilde{\Delta}\circ\phi$.

Finally, we need to show that ϕ is surjective. In other words, we need to show that given $W_1 \to \mathfrak{M}^{\text{ev}}(\mathcal{X},\tau_1)$ and $W_2 \to \mathfrak{M}^{\text{ev}}(\mathcal{X},\tau_2,z)$, based on Proposition 3.2.1, the fine and saturated fiber product $\widetilde{W}_1 \times^{fs}_X \widetilde{W}_2$ is not empty.

Let us observe that the only possibility to make the fs fiber product empty is that the fine fiber product $\widetilde{W}_1 \times^{f}_X \widetilde{W}_2$ is empty since going from the fine to the fs fibre product is saturation, which is always surjective, see [Og], III,Prop. 2.1.5. However, in this case, $\widetilde{W}_1 \times^{f}_X \widetilde{W}_2$ is not empty since the morphisms $\widetilde{W}_i \to X$ are integral because any non-zero morphism from the free rank 1 monoid (that comes from the log structure of X) is integral. □

Remark 3.2.4. Roughly speaking, the factorization diagram in the proposition above indicates that gluing punctured logarithmic maps splits into 2 steps, that is, gluing them schematically at first and then gluing the logarithmic structures. We remark that the map ϕ showing up in the diagram above needs not be surjective in general and the case in which the map ϕ is surjective is called *tropical transverse*, see [Gro] for a detailed discussion about tropical transversality, and calculating the degree of ϕ is literally the key point of relating punctured invariants to usual logarithmic invariants.

3.3 The main gluing formula

Prior to a technical calculation of the degree of ϕ that appears in Proposition 3.2.3, we need to do a bit of analysis about virtual fundamental classes to figure out what kinds of tropical types $\boldsymbol{\tau}$ will contribute to punctured invariants.

Note that there is an extreme situation in which the graph of $\boldsymbol{\tau}$ has only one vertex carrying three legs $p, q, -r$ with $r \neq 0$. In this case, there is nothing we can glue and there is only one punctured log map to W realizing it, which is the constant map to the point at which we impose point-constraint, in other words, the curve class β of $\boldsymbol{\tau}$ in this case is just 0. So, this situation will not yield anything interesting by [GS2], Lemma 1.15. Henceforth, throughout this entire subsection, we are going to assume that the curve class β is not 0, which means we will no longer take this extreme situation into account.

The following lemma is one of the key observations to simplify the gluing problem.

Lemma 3.3.1. *In the situation of Lemma 3.1.7, we further assume that both x_1 and x_2 lie in $\boldsymbol{\tau}_2$, and let $\mathscr{M}_z(X, \boldsymbol{\tau}_1) := \mathscr{M}(X, \boldsymbol{\tau}_1) \times_{\underline{D}} z$ and $\mathfrak{M}^{ev}_z(\mathcal{X}, \boldsymbol{\tau}_1) := \mathfrak{M}^{ev}(\mathcal{X}, \boldsymbol{\tau}_1) \times_{\underline{D}} z$ be the moduli spaces appearing in the diagram of Proposition 3.2.3. Then*

$$dim(\mathfrak{M}^{ev}_z(\mathcal{X}, \boldsymbol{\tau}_1)) = -1.$$

Hence, $[\mathscr{M}_z(X, \boldsymbol{\tau}_1)]^{\mathrm{vir}} = 0$ and the moduli space $\mathscr{M}_z(X, \boldsymbol{\tau_1})$ has no contribution at all to virtual class and curve counting, in other words,

$$[\mathscr{M}_z(X, \boldsymbol{\tau}_1) \times \mathscr{M}(X, \boldsymbol{\tau}_2, z)]^{\mathrm{vir}} = 0.$$

Remark 3.3.2. Remember that after splitting $\boldsymbol{\tau}$, $\boldsymbol{\tau}_1$ acquires an extra punctured point w and $\boldsymbol{\tau}_2$ gets an extra punctured point w' such that the contact orders at w, w' are opposite to each other.

Proof. The key point is to compute the dimension of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$. Recall that, by the definition of evaluation space, we have the following Cartesian diagram

$$\begin{tikzcd}
\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_1) \arrow[r] & \mathcal{X}
\end{tikzcd}$$

By [ACGS2],Prop. 3.28, we have $\dim(\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_1)) \leq -2$. So, by the fiber diagram above, we have $\dim(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)) \leq \dim(X) - 2$. Then by the definition of $\mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1)$, we then get that $\dim(\mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1)) = \dim(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)) - \dim(D) \leq \dim(X) - 2 - (\dim(X) - 1) = -1$.

On the other hand, obviously the map $\mathscr{M}_z(X, \boldsymbol{\tau}_1) \to \mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1)$ admits a perfect relative obstruction theory and $[\mathscr{M}_z(X, \boldsymbol{\tau}_1)]^{\mathrm{vir}} = \epsilon^![\mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1)]$ (ϵ is the map which appears in the proposition 3.2.3), then by Riemann-Roch, we know the relative virtual dimension is $\beta_1 \cdot c_1(\Theta_{X/\Bbbk}) = \beta_1 \cdot (K_X + D) = \beta_1 \cdot 0 = 0$, thus the virtual dimension of $\mathscr{M}_z(X, \boldsymbol{\tau}_1) = -1$, so $[\mathscr{M}_z(X, \boldsymbol{\tau}_1)]^{\mathrm{vir}} = 0$ since $\mathscr{M}_z(X, \boldsymbol{\tau}_1)$ is a Deligne-Mumford stack. $\square$

Corollary 3.3.3. *In the situation of Lemma 3.3.1, we have*

$$[\mathscr{M}(X, \boldsymbol{\tau}, z)]^{\mathrm{vir}} = 0.$$

Proof. According to [[ACGS2], Theorem 3.10], we know $\mathscr{M}(X, \boldsymbol{\tau}, z)$, $\mathscr{M}_z(X, \boldsymbol{\tau}_1)$ and $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ are all Deligne-Mumford stacks because they are obtained by base change via the representable morphism $B\mathbb{G}_m^\dagger \to \mathscr{P}(X, r)$. Also, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z)$ is pure dimensional by [[GS2], Proposition 3.19]. All these stacks are stratified by quotient stacks. Further, the lemma above tells us that $\dim(\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)) \leq -1$ and $\mathscr{M}_z(X, \boldsymbol{\tau}_1) \to \dim(\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1))$ has relative virtual dimension $\beta_1 \cdot c_1(\Theta_{X/\Bbbk}) = \beta_1 \cdot (K_X + D) = 0$. Therefore, the vanishing of the virtual fundamental class in the corollary follows directly from Theorem A.13 of [GS2]. $\square$

Remark 3.3.4. By the lemma and the corollary we just saw, the virtual fundamental class of $\mathscr{M}(X, \boldsymbol{\tau}, z)$ can be non-zero only if the $\boldsymbol{\tau}_1$ contains at least one of x_1 and x_2. However, if $\boldsymbol{\tau}_1$ contains both x_1, x_2 at the same time, in order to maintain the stability of our punctured logarithmic maps, the component that contains x_{out} must contain an additional node, then we can further split $\boldsymbol{\tau}_2$ at this additional node to get two new types of curves $\boldsymbol{\tau}_2'$ and $\boldsymbol{\tau}_3$ with $x_{\mathrm{out}} \in \boldsymbol{\tau}_2'$ and $\boldsymbol{\tau}_3$ not having any punctured points, which means we can split $\boldsymbol{\tau}$ from the beginning so that one of the resulting class contains no punctured points, and then the virtual fundamental class corresponding to this type $\boldsymbol{\tau}$ is actually 0 by applying Lemma 3.3.1 and Corollary 3.3.3 again.

Corollary 3.3.5. *The virtual fundamental class* $[\mathscr{M}(X, \boldsymbol{\tau}, z)]^{\mathrm{vir}}$ *non-zero implies that* $\boldsymbol{\tau}$*, as a tropical type of punctured log maps, is such that the vertex* V *that contains the leg* x_{out} *contains exactly one of legs* x_1 *and* x_2*, and any vertices other than* V *get mapped to the origin by the corresponding tropical map.*

Proof. First, if the graph G of $\boldsymbol{\tau}$ has at least two vertices mapping into the interior of the tropicalization $\Sigma(X)(= \mathbb{R}_{\geq 0})$ of X, then the family of corresponding tropical maps is at least two-dimensional as these vertices are free to move. This implies that the virtual dimension of the moduli space $\mathscr{M}(X, \boldsymbol{\tau}, z)$ will be negative for such a tropical type $\boldsymbol{\tau}$. Indeed, by [[GS2], Proposition 3.19], we know that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \beta, z)$ is pure dimensional of dimension 0. However, an open stratum of this moduli space corresponds to a type with a one-dimensional tropical moduli space. Thus, $\mathscr{M}(X, \boldsymbol{\tau}, z)$ is of virtual dimension -1.

Since any irreducible component corresponding to a vertex mapping into the interior of the tropicalization corresponds to a contracted component, balancing holds at these vertices, see e.g., [ACGS2], Proposition 2.25. From this, it follows that the component which contains the leg x_{out} must also contain at least one of the legs x_1, x_2. Then the conclusion follows from Corollary 3.3.3 and Remark 3.3.4. $\square$

We now note a virtual decomposition for the moduli spaces $\mathscr{M}(X, \boldsymbol{\beta}, z)$ giving the following formula, which is proved in the same way as [GS3], §6.

Lemma 3.3.6. *There is a decomposition*

$$[\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}, z)] = \sum_{\boldsymbol{\tau}} \frac{m_{\boldsymbol{\tau}}}{\mathrm{Aut}(\boldsymbol{\tau})} [\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}, z)],$$

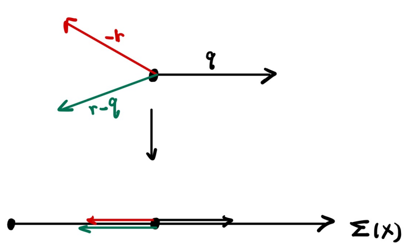
where the sum is over all decorated types $\boldsymbol{\tau}$ of punctured tropical maps which are degenerations of $\boldsymbol{\beta}$ and with one-dimensional moduli space of tropical maps. Further, $m_{\boldsymbol{\tau}}$ is the multiplicity of the (union of) irreducible component of $\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}, z)$ which is the image of $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}, z)$ in $\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}, z)$. This yields an equality on virtual fundamental classes via pull-back:

$$[\mathscr{M}(X, \boldsymbol{\beta}, z)]^{\mathrm{vir}} = \sum_{\boldsymbol{\tau}} \frac{m_{\boldsymbol{\tau}}}{\mathrm{Aut}(\boldsymbol{\tau})} [\mathscr{M}(X, \boldsymbol{\tau}, z)]^{\mathrm{vir}}.$$

Next, we are going to deduce our main gluing theorem about 2-point Gromov-Witten invariants and punctured Gromov-Witten invariants. By Corollary 3.3.5, the only non-zero virtual cycle in the summation of Lemma 3.3.6 is given by the type satisfying the following assumption.

Assumption 3.3.7. The type of curve class $\boldsymbol{\tau}$ that we consider satisfies the basic setup described in the beginning of this section such that the vertex V containing the puncturing leg x_{out} contains exactly one of legs x_1, x_2. Moreover, we assume that any other vertices other than V get mapped to the origin by the corresponding tropical map.

Remark 3.3.8. Once $\boldsymbol{\tau}$ fulfills the assumption above, by the proof of Lemma 3.3.1, we can deduce that the moduli space $\mathscr{M}_z(X, \boldsymbol{\tau}_1)$ has virtual dimension 0, and we assume further that the rational tail contains x_2 in which by a rational tail, we mean a smooth component of genus 0 having only one intersection point with the closure of its complement. Recall from Remark 3.1.6 that $\boldsymbol{\tau}_1$ acquires an extra punctured point w with positive contact order $q - r$ after the splitting by the balancing condition of tropical curves. The picture of the tropical type τ_2 after splitting τ is shown in the following



Readers can refer to [ACGS2] for more details about the balancing condition for punctured log maps. Then we can define 2-point invariants with a point-constraint as follows

$$N_{p,q-r} := \int_{[\mathscr{M}_z(X,\boldsymbol{\tau}_1)]^{\mathrm{vir}}} 1.$$

When the rational tail contains x_1 instead of x_2, we can analogously define $N_{q,p-r}$. Roughly speaking, $N_{p,q-r}$ virtually counts rational curves having curve class $\beta_1 \in \mathrm{H}_2(X,\mathbb{Z})$ and intersecting the divisor D at two points x, y with tangency order $p, q-r$ respectively in which we put a point constraint at y. More generally, we can define $N_{a,b}$ for any a, b such that $a + b = \beta \cdot D$.

Proposition 3.3.9. *Let $\boldsymbol{\tau}_2$ be the type after splitting $\boldsymbol{\tau}$ which satisfies Assumption 3.3.7 such that $x_2, x_{out} \in \boldsymbol{\tau}_2$. Then the canonical projection map $\mathfrak{M}^{ev}(\mathcal{X}, \boldsymbol{\tau}_2, z) \to B\mathbb{G}_m^{\dagger}$ is logarithmically smooth.*

Proof. Recall from Remark 2.7.3 that $\mathscr{P}(\mathcal{X}, r)$ is an idealized log stack with the ideal sheaf $\overline{\mathcal{I}} \subset \overline{\mathcal{M}}_{\mathscr{P}(\mathcal{X},r)}$ given by $r^{-1}(\mathbb{Z}_{>0})$. In our case, since $\overline{\mathcal{M}}_Z$ is generated by $\mathbb{N}$, the ideal is generated by $\mathbb{Z}_{>0}$. For any parametrizing space $W \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_2)$, note that the basic monoid on $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_2)$ is $\mathbb{N}$, so the pullback of the ideal to W under the map $W \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_2) \to \mathscr{P}(\mathcal{X}, r)$ is also generated by $\mathbb{Z}_{>0}$. Let $C^\circ \to \mathcal{X}$ the the map parameterized by W.

By the pre-stability condition, $\overline{\mathcal{M}}_{C^\circ} \subseteq \mathbb{N} \oplus \mathbb{Z}_{x_1} \oplus \mathbb{Z}_{x_2} \oplus \mathbb{Z}_{x_{\mathrm{out}}}$ is generated by $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,0,r-q)$ and $(1,0,-r,0)$. Note that the quantity $r - q$ is always negative by balancing condition of tropical maps (keep in mind that every vertex other than V is mapped to the origin by the corresponding tropical map). Then, by the definition of puncturing log-ideal given in Definition 2.30 of [ACGS2], once we project it into $\overline{\mathcal{M}}_W$, the puncturing log-ideal is generated by $\mathbb{Z}_{>0}$. Thus, the map $\mathrm{ev}_{\mathcal{X}} : \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_2) \to \mathscr{P}(\mathcal{X}, r)$ is actually ideally strict (see Section 1.3, Chapter 3 of [Og]).

By Theorem 3.15 of [GS2], we know that the map $\mathrm{ev}_{\mathcal{X}}$ is idealized log smooth. Thus, it is in fact log smooth by [Og], IV Variant 3.1.22. Note that the projection $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z) \to B\mathbb{G}_m^{\dagger}$ is just a base change of $\mathrm{ev}_{\mathcal{X}}$, hence it is log smooth. □

Corollary 3.3.10. *In the situation of Proposition 3.3.9, the moduli space $\mathfrak{M}^{ev}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ is a reduced algebraic stack.*

Proof. According to the tropical interpretation of the map $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z) \to B\mathbb{G}_m^{\dagger}$ (see Lemma 3.22 of [GS2]), smooth locally at a generic point w, we have a chart for this map

$$\begin{tikzcd} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z) \arrow[r] \arrow[d] & {[\mathrm{Spec}(\Bbbk[\mathbb{N}])/\mathbb{G}_m]} \arrow[d, "\eta"] \\ B\mathbb{G}_m^{\dagger} \arrow[r] & {[\mathrm{Spec}(\Bbbk[\mathbb{N}])/\mathbb{G}_m]} \end{tikzcd} \tag{3.3.1}$$

where the map between monoids corresponding to η is sending 1 to δ such that $\Sigma_1(v_{\text{out}}) = \delta \cdot r$ (see Remark 2.5.6 for the definition of Σ_m). Here v_{out} is the vertex corresponding to the component containing x_{out} (don't forget that we only have one component in this case) and the tropical map

$$\begin{tikzcd} \omega_{v_{\text{out}}} \arrow[r, "\Sigma"] \arrow[d] & \mathbb{R}_{\geq 0} \\ \sigma_y & \end{tikzcd}$$

corresponds to the image y of w in $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}_2)$. In fact, $\delta = 1$, as we can find a tropical map Σ' of type $\boldsymbol{\tau}_2$ with $\Sigma'(v_{\text{out}}) = r$. Indeed, the underlying graph of $\boldsymbol{\tau}_2$ has only one vertex, and we are free to send that vertex where we would like.

Note that by Proposition 3.3.9, the map $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z) \to B\mathbb{G}_m^{\dagger}$ is log smooth, then in the diagram (3.3.1) above, let $T = B\mathbb{G}_m^{\dagger} \times_{[\mathbb{A}^1/\mathbb{G}_m]} [\mathbb{A}^1/\mathbb{G}_m]$, the induced map $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z) \to T$ is smooth in the usual sense and δ is the multiplicity of T. Hence δ is literally just the multiplicity of the moduli space. So, $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ is reduced and even smooth over $B\mathbb{G}_m^{\dagger}$. $\square$

Lemma 3.3.11. *Let $\boldsymbol{\tau}_2$ be the type after splitting $\boldsymbol{\tau}$ which satisfies the assumption 3.3.7 such that $x_{out} \in \boldsymbol{\tau}_2$. Then the corresponding moduli space $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is a reduced point, i.e. $\mathscr{M}(X, \boldsymbol{\tau}_2, z) = \operatorname{Spec}(\Bbbk)$.*

Proof. At first, note that $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is a Deligne-Mumford stack according to [GS2]. Any point $\operatorname{Spec}(\Bbbk) \to \mathscr{M}(X, \boldsymbol{\tau}_2, z)$ of $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ corresponds to a (family of) punctured logarithmic map

$$\begin{tikzcd} C^{\circ} \arrow[r, "f"] \arrow[d] & D \subset X \\ \operatorname{Spec}(\Bbbk) & \end{tikzcd}$$

Note that f is a constant map to $z \in D$ by the genericness of z, and $\underline{C}^{\circ} = \mathbb{P}^1$ has three punctured points. According to [[GS2], Claim 3.22], such a basic punctured log map is unique. Thus the $\Bbbk$-points of $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is just the single point shown as above with trivial stabilizer, so $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is an algebraic space.

Furthermore, obviously there is a map $\mathscr{M}(X, \boldsymbol{\tau}_2, z) \to \operatorname{Spec}(\Bbbk)$ which is of finite type, quasi-finite, and separated. So, by Theorem 7.2.10 of Olsson in [Ol2], $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is a scheme. So, it is just a (possibly non-reduced) point.

However, the map $\mathscr{M}(X, \boldsymbol{\tau}_2, z) \to \mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ is smooth because, as f is constant, $f^*\Theta_X$ (where Θ_X is the logarithmic tangent bundle) is trivial, so $H^1(C, f^*\Theta_X(-x_{\text{out}})) = 0$, i.e. the obstruction space is 0. We know that $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ is reduced by the corollary above. Hence, $\mathscr{M}(X, \boldsymbol{\tau}_2, z)$ is just a reduced point. $\square$

Theorem 3.3.12. *Let (X, D) be a smooth log Calabi-Yau pair, i.e. X is a smooth projective variety with a smooth divisor $D \subset X$. Let $\boldsymbol{\tau}$ be the type of curve class fulfilling Assumption 3.3.7, and suppose that the vertex V containing the leg x_{out} contains x_2 (resp. x_1). Then the degree of the map ϕ which appears in the diagram of Proposition 3.2.3 has degree $q - r$ (resp. $p - r$), in other words, the multiplicity $m_{\boldsymbol{\tau}} = q - r$ (resp. $p - r$).*

Proof. In order to calculate the degree of ϕ, we need to choose general points in $\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ respectively and then compute the glued family of punctured logarithmic maps to see how many connected components will be produced after gluing.

Step 1: At first, let us choose a general point $\underline{W_1} = \mathrm{Spec}(\Bbbk) \to \mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$ of $\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$ with the pullback logarithmic structure on $\underline{W_1}$ from the basic logarithmic structure on $\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$. Without loss of generality, we can assume that the choice of point corresponds to a punctured log map $(C_1^\circ/W_1, \boldsymbol{\tau}_1, f_1 : C_1^\circ \to \mathcal{X})$ such that the source curve C_1° is smooth and isn't mapped into $\mathcal{D}$ by f_1. In this case, the basic log structure at this point is just trivial, i.e., $W_1 = (\mathrm{Spec}(\Bbbk), 0)$. (We can make such an assumption because such a kind of punctured log maps form a dense open subset of $\mathfrak{M}_z^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_1)$ equipped with the basic log structure).

Step 2: Choose a generic point $\underline{W_2'} = \mathrm{Spec}(\Bbbk) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2)$, so $\mathrm{Spec}(\Bbbk)$ is equipped with the logarithmic structure pulled back from the basic logarithmic structure on $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2)$. Since $\boldsymbol{\tau}_2$ contains x_{out} having negative contact order, the whole component will be completely mapped into D, so the corresponding tropical map is parametrized by $\mathbb{R}_{\geq 0}$. Hence, $W_2' = (\mathrm{Spec}(\Bbbk), \mathbb{N})$.

Next, we need to compute $W_2 := W_2' \times_{\mathscr{P}(X,r)} B\mathbb{G}_m^\dagger$. Note that W_2' parametrizes the punctured logarithmic map $(C^\circ/W_2', \boldsymbol{\tau}_2, f' : C^\circ \to \mathcal{X})$ where C° is just $\mathbb{P}^1$ and we chose $z \in D$ generically enough such that no rational curves pass through z inside D, thus the f' is the constant map to z. Thus, without loss of generality, we can assume that

$$\widetilde{\mathscr{P}}(X, r) = (\mathrm{Spec}(\Bbbk), \mathbb{N}) \times B\mathbb{G}_m^\dagger = (B\mathbb{G}_m^\dagger, \mathbb{N} \oplus \mathbb{N}).$$

Thus, by the definition of $\mathscr{P}(X, r)$, we get $\underline{\mathscr{P}}(X, r) = B\mathbb{G}_m$, and the ghost sheaf of the logarithmic structure is the sub-monoid $\{(m, rm) | m \in \mathbb{N}\} \subset \mathbb{N} \oplus \mathbb{N}$ which is isomorphic to $\mathbb{N}$. So, we have

$$\mathscr{P}(X, r) = (B\mathbb{G}_m, \mathbb{N})$$

where the torsor associated to $1 \in \mathbb{N}$ is the r^{th} tensor power of the universal torsor. Then, if we impose the point-constraint, $W_2 = W_2' \times_{B\mathbb{G}_m^\dagger} (B\mathbb{G}_m, \mathbb{N})$ with the corresponding maps between ghost sheaves

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\mathrm{id}"] \arrow[d, "\cdot r"] & \mathbb{N} \\ \mathbb{N} & \end{tikzcd}$$

where the vertical multiplication-by-r map is the map from the ghost sheaf of $\mathscr{P}(\mathcal{X}, r) = (B\mathbb{G}_m, \mathbb{N})$ to the ghost sheaf of $B\mathbb{G}_m^{\dagger}$ and the horizontal map is given by the map from W_2' to $B\mathbb{G}_m^{\dagger}$. Thus then push-out of the diagram above is $\mathbb{N}$. Thus, we have $W_2 = (\mathrm{Spec}(\Bbbk), \mathbb{N})$ with the corresponding map from the ghost sheaf of $\mathscr{P}(\mathcal{X}, r)$ to the ghost sheaf of W_2 being the multiplication-by-r map. Denote by $(C_2^{\circ}/W_2, \boldsymbol{\tau}_2, f_2 : C_2^{\circ} \to X)$ the family of punctured logarithmic maps parametrized by W_2.

Step 3: Let $w_1 \in \boldsymbol{\tau}_1$ and $w_2 \in \boldsymbol{\tau}_2$ be the two punctured points that we glue together to form the glued family of punctured logarithmic maps. Thus, $W_1^E = (\underline{W_1}, w_1^*\overline{\mathcal{M}}_{C_1^{\circ}, w_1}) = (\mathrm{Spec}(\Bbbk), \mathbb{N})$, and $W_2^E = (\underline{W_2}, w_1^*\overline{\mathcal{M}}_{C_2^{\circ}, w_2})$. So we need to figure out $\overline{\mathcal{M}}_{C_2^{\circ}, w_2}$ first. Note that we have a map $\overline{\mathcal{M}}_{X,z} = \mathbb{N} \to \overline{\mathcal{M}}_{C_2^{\circ}, w_2} \subset \mathbb{N} \oplus \mathbb{Z}$, and by the calculations we did in the step 2, we know this map is defined by sending the generator 1 to $(r, r-q)$. By the pre-stability condition, $\overline{\mathcal{M}}_{C_2^{\circ}, w_2}$ is supposed to be generated by $\overline{\mathcal{M}}_{C_2}$ and the image of $\overline{\mathcal{M}}_{X,z}$. Hence, $\overline{\mathcal{M}}_{C_2^{\circ}, w_2} = \langle (1,0), (0,1), (r, r-q) \rangle =: R \subset \mathbb{N} \oplus \mathbb{Z}$. Therefore, $W_2^E = (\underline{W_2}, R) = (\mathrm{Spec}(\Bbbk), R)$.

Step 4: This step is to saturate both W_1^E and W_2^E. For W_1^E, there is nothing to saturate. So, $\widetilde{W}_1 = W_1^E = (\mathrm{Spec}(\Bbbk), \mathbb{N})$. For W_2^E, the saturation $\widetilde{W}_2 = \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\Bbbk[R])} \mathrm{Spec}(\Bbbk[R^{\mathrm{sat.}}])$ where $R^{\mathrm{sat.}}$ is the saturation monoid of R inside $\mathbb{N} \oplus \mathbb{Z}$. Note that $\Bbbk \otimes_{\Bbbk[R]} \Bbbk[R^{\mathrm{sat.}}] \cong \Bbbk[R^{\mathrm{sat.}}]/I$ where is the ideal generated by $(1,0), (0,1), (r, r-q)$ in $\Bbbk[R^{\mathrm{sat.}}]$. Then, $\widetilde{W}_2 = (\mathrm{Spec}(\Bbbk[R^{\mathrm{sat.}}]/I), R^{\mathrm{sat.}})$.

Step 5: In this step, we compute the number of connected components of the glued family $\widetilde{W}_1 \times_X^{\mathrm{fs}} \widetilde{W}_2$ where the fiber product of $\widetilde{W}_1$ and $\widetilde{W}_2$ is taken in the category of fine and saturated logarithmic schemes. We know the corresponding maps between logarithmic structures:

$$\begin{tikzcd} \mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \\ R^{\mathrm{sat.}} & \end{tikzcd}$$

where the vertical map sends 1 to $(r, r-q)$ and the horizontal map sends 1 to $q-r$.

We are hoping to apply Theorem 4.4 in [Gro], so we define a map as follows:

$$\begin{aligned} \theta := (\theta_1, -\theta_2) : \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus (R^{\mathrm{sat.}})^{\mathrm{gp}} \\ 1 &\longmapsto (q-r, -r, q-r) \end{aligned}$$

where the superscript gp. means groupification. Then, by a direct computation, we assert that $\#(\mathrm{coker}(\theta)_{\mathrm{tor.}}) = \mathrm{g.c.d}(r, q-r)$ where $\#(\mathrm{coker}(\theta)_{\mathrm{tor.}})$ means the cardinality of the torsion elements of the cokernel of θ and $\mathrm{g.c.d}(r, q-r)$ represents the greatest common divisor of r and $q-r$.

So, by Theorem 4.4 in [Gro], since the fiber product is not empty, the number of connected component of $\widetilde{W}_1 \times^{\mathrm{fs}}_X \widetilde{W}_2$ is $\mathrm{g.c.d}(r, q-r)$. This calculation is in fact giving us the degree of the map $\phi_{\mathrm{red}} : \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z)_{\mathrm{red}} \longrightarrow \mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1) \times \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ where the subscript red means the reduction of the original space.

Step 6: The last step is to figure out the non-reduced structure on $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z)$. By Proposition B.2 in [ACGS2], the map $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z) \to B\mathbb{G}_m^{\dagger}$ is logarithmically smooth. So, at a generic point, we can choose a chart for this map around the point shown as follows

$$\begin{tikzcd} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z) \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\eta"] \\ B\mathbb{G}_m^{\dagger} \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

and by Lemma 3.22 in [GS2], the corresponding map between monoids of η is

$$\begin{aligned} \eta^* : \mathbb{N} &\longrightarrow \mathbb{N} \\ 1 &\longmapsto \delta \end{aligned}$$

Here, δ is the smallest natural number such that $\Sigma_1(v_{\mathrm{out}}) = \delta \cdot r = l \cdot (q-r)$ for some integer l in which Σ_1 is the corresponding tropical map and v_{out} is the component containing x_{out}. Then, it is easy to see that $\delta = (q-r)/\mathrm{g.c.d}(r, q-r)$. By the definition of logarithmic smoothness, δ is the multiplicity of $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}, z)$.
Moreover, by Corollary 3.3.10, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_2, z)$ is reduced, and $\mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X}, \boldsymbol{\tau}_1)$ is obviously reduced since it has no any non-trivial punctured points.

Hence, the degree of ϕ is $\mathrm{g.c.d}(r, q-r) \cdot \delta = q - r$ as desired. □

Corollary 3.3.13. *Let X be a smooth log Calabi-Yau pair. Then we have the following formula relating punctured Gromov-Witten invariants to 2-point Gromov-Witten invariants with point-constraint*

$$N^{\boldsymbol{\beta}}_{pqr} = (q-r)N_{p,q-r} + (p-r)N_{q,p-r}$$

where $r > 0$ and $\beta \neq 0$.

Remark 3.3.14. In fact, one can conjecture that $N_{w,1} = w^2 N_{1,w}$ where $w + 1 = \beta \cdot D$ and this can be proven via the formula in the corollary.

Proof. Recall the part of the Cartesion diagram involved in ϕ and $\tilde{\phi}$ from Proposition 3.2.3

$$
\begin{tikzcd}
\mathscr{M}(X,\boldsymbol{\tau},z) \arrow[r, "\tilde{\phi}"] \arrow[d] & \mathscr{M}_z(X,\boldsymbol{\tau}_1)\times\mathscr{M}(X,\boldsymbol{\tau}_2,z) \arrow[d, "\epsilon"] \\
\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau},z) \arrow[r, "\phi"] & \mathfrak{M}^{\mathrm{ev}}_z(\mathcal{X},\boldsymbol{\tau}_1)\times\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau}_2,z)
\end{tikzcd}
$$

where ϕ is a finite surjective map. By Lemma 3.3.11, $\mathscr{M}(X,\boldsymbol{\tau}_2,z)=\mathrm{Spec}(\Bbbk)$. There are two situations that could happen.

- If $\boldsymbol{\tau}_2$ contains x_1, then by Theorem 3.3.12, we know that the degree of $\tilde{\phi}$ is $p-r$. Hence, we have
$$\tilde{\phi}_*[\mathscr{M}(X,\boldsymbol{\tau},z)]^{\mathrm{vir}}=(p-r)\cdot[\mathscr{M}_z(X,\boldsymbol{\tau}_1)]^{\mathrm{vir}}$$
- For the same reason, if $\boldsymbol{\tau}_2$ contains x_2, we have
$$\tilde{\phi}_*[\mathscr{M}(X,\boldsymbol{\tau},z)]^{\mathrm{vir}}=(q-r)\cdot[\mathscr{M}_z(X,\boldsymbol{\tau}_1)]^{\mathrm{vir}}$$

Then Lemma 3.3.6 gives rise to the formula we want to prove. □

Remark 3.3.15. In [FWY], the authors define a new kind of relative Gromov-Witten invariants $\tilde{N}^{\beta}_{pqr}$ using orbifold Gromov-Witten theory, in which the new theory also allows negative tangency order to occur. In [You], F. You found the same formula for their new relative invariants $\tilde{N}^{\beta}_{pqr}$ and it is easy to argue that our invariants $N^{\boldsymbol{\beta}}_{pqr}$ agree with their $\tilde{N}^{\beta}_{pqr}$. Moreover, he can proceed to calculate 2-pointed invariants using their invariants and I-function and J-function technique.

Chapter 4

A general calculation for the invariants $N_{e-1,1}$

Given a smooth log Calabi-Yau pair (W, D) with D nef, as the title of this section indicates, we are going to calculate the invariants $N_{e-1,1}$ by comparing it with closed Gromov-Witten invariants defined in [LLW] and [Cha]. The methods we will apply here are the standard deformation to normal cone and degeneration formula. Therefore, let us quickly review those closed invariants and clarify the setup at first, and then get our hands on comparison of these invariants.

As a matter of fact, for a given curve class $\beta \in H_2(W, \mathbb{Z})$, it is sufficient to just require that $\beta_i \cdot D \geq 0$ for any decomposition $\beta = \Sigma_i \beta_i$ of β into effective curve classes for Theorem 1.4.3 to be true. The nefness of D is clearly a uniform condition ensuring this property for all β.

4.1 Basic setup

Denoting by ω_W the canonical bundle of W, we set $X = \mathbb{P}(\omega_W \oplus \mathcal{O})$, the projective completion of the canonical bundle with the structure map $p : X \to W$. Next, we are going to construct a degeneration of X. We let $\mathcal{X} = \mathrm{Bl}_{D\times 0}(W \times \mathbb{A}^1)$ and let $\mathcal{D}$ be the strict transform of $D \times \mathbb{A}^1$ in $\mathcal{X}$. So, a general fiber of $\mathcal{X} \to \mathbb{A}^1$ is W and the special fiber is $W \sqcup_D Y$, the union of W and Y. Here, Y is the projective completion of the normal bundle to D in W over D, i.e., $Y = \mathbb{P}(N_{D/W} \oplus \mathcal{O})$. We view $D \subset Y$ as the 0-section D_0 of the normal bundle of D, and in particular, the normal bundle of D in Y is the dual of $N_{D/W}$.

Set $\mathcal{L} = \mathbb{P}(\mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \oplus \mathcal{O})$. It is easy to check for the degeneration $\pi : \mathcal{L} \to \mathbb{A}^1$ that a general fiber $\pi^{-1}(\{t\})$ is X for all $t \neq 0$ and the special fiber $\mathcal{L}_0$ is $W \times \mathbb{P}^1 \sqcup_{D\times\mathbb{P}^1} \mathbb{P}(\mathcal{O}_Y(-D_\infty) \oplus \mathcal{O})$. Moreover, there exists a natural projection map $q : \mathcal{L}/\mathbb{A}^1 \to W \times \mathbb{A}^1$.

For simplicity, we let $\mathcal{L}_X = W \times \mathbb{P}^1, \mathcal{L}_D = D \times \mathbb{P}^1$ and $\mathcal{L}_Y = \mathbb{P}(\mathcal{O}_Y(-D_\infty) \oplus \mathcal{O})$. We endow $\mathcal{X}$ with the divisorial log structure given by the central fiber $\mathcal{X}_0$.

Remark 4.1.1. We abuse the notation $\mathcal{O}$ to mean structure sheaf or trivial line bundle on whatever space we have. So, the symbol $\mathcal{O}$ could mean different structure sheaves depending on context.

Remark 4.1.2. It is worthwhile mentioning that our idea in this subsection is analogous to that presented in [vGGR]. So, we will collect some results proven in that paper later on.

We consider the moduli space $\mathscr{M}_{0,1}(X, \beta + h)$ parametrizing genus 0 stable maps $f : C \to X$ with one marked point such that $f_*[C] = \beta + h$ where β is a chosen effective curve class in W (viewed as the 0-section sitting inside X) satisfying $c_1(X) \cdot \beta = 0$ and h is the fiber class $[\mathbb{P}^1]$. The invariants defined in [Cha] are just

$$\int_{[\mathscr{M}_{0,1}(X,\beta+h)]^{\mathrm{vir}}} \mathrm{ev}^*[\mathrm{pt}]$$

where [pt] is a point class in X of which the effect is to impose a point constraint on stable maps and ev is the evaluation map at the marked point.

Remark 4.1.3. In fact, if we consider the corresponding moduli space with a point constraint $\mathscr{M}_{0,1}(X, \beta + h, \sigma) := \mathscr{M}_{0,1}(X, \beta + h) \times_X \{\mathrm{pt}\}$, here σ can be understood as a geometric point $\sigma : \{\mathrm{pt}\} \to X$, then $\mathscr{M}_{0,1}(X, \beta + h, \sigma)$ possesses a virtual fundamental class and degree of the virtual class is precisely the same as the integration shown above.

Lemma 4.1.4. *Let $Q : \mathscr{M}(\mathcal{L}/\mathbb{A}^1, \beta + h) \to \mathscr{M}(W \times \mathbb{A}^1/\mathbb{A}^1, \beta)$ be the natural map from moduli space of stable log maps to the families $\mathcal{L}/\mathbb{A}^1$ to moduli space of log stable maps to $W \times \mathbb{A}^1/\mathbb{A}^1$, which is induced by $q : \mathcal{L} \to W \times \mathbb{A}^1$. Then, we have, for any $t \neq 0 \in \mathbb{A}^1$, the following equality*

$$(Q_t)_*[\mathscr{M}(X, \beta + h)]^{vir} = (Q_0)_*[\mathscr{M}(\mathcal{L}_0, \beta + h)]^{vir}$$

where Q_0 and Q_t are the restrictions of the map Q to the special fiber and any generic fiber respectively, and either side of the equality is a cycle class in $A_*(\mathscr{M}(W, \beta); \mathbb{Q})$.

Proof. Notice that we have, for any $t \neq 0$, the following four Cartesian diagrams

$$\begin{tikzcd}
\mathscr{M}(\mathcal{L}_0, \beta + h) \arrow[r] \arrow[d, "Q_0"] & \mathscr{M}(\mathcal{L}/\mathbb{A}^1, \beta + h) \arrow[d, "Q"] & \mathscr{M}(X, \beta + h) \arrow[l] \arrow[d, "Q_t"] \\
\mathscr{M}(W, \beta) \arrow[r] \arrow[d] & \mathscr{M}(W \times \mathbb{A}^1, \beta + h) \arrow[d, "p_2"] & \mathscr{M}(W, \beta) \arrow[l] \arrow[d] \\
\{0\} \arrow[r, hook, "i_0"] & \mathbb{A}^1 & \{t\} \arrow[l, hook', "i_t"']
\end{tikzcd}$$

So, by commutativity of Gysin pullback with proper pushforward, we have

$$\begin{aligned}(Q_0)_*[\mathscr{M}(\mathcal{L}_0,\beta+h)]^{\mathrm{vir}} &= i_0^! Q_*[\mathscr{M}(\mathcal{L}/\mathbb{A}^1,\beta+h)]^{\mathrm{vir}}\\ &= i_t^! Q_*[\mathscr{M}(\mathcal{L}/\mathbb{A}^1,\beta+h)]^{\mathrm{vir}}\\ &= (Q_t)_*[\mathscr{M}(X,\beta+h)]^{\mathrm{vir}}\end{aligned}$$

in which the middle equality holds because the family p_2 is a trivial family. □

4.2 Degeneration formula

Since our method involves calculating invariants of the special fiber, which is a normal crossing union of two spaces, the appearance of the degeneration formula becomes inevitable. In this subsection, we forget our setup described above for the time being and give a more general description of the formula dealing with a normal crossing union of two spaces. Let X be a logarithmically smooth variety over a standard log point $b^\dagger$ with irreducible components X_0 and Y_0. The central fiber of the degeneration described in the previous subsection is an example. A degeneration formula for stable log maps was first given by B. Kim, H. Lho and H. Ruddat in [KLR]. In principle, their degeneration formula is enough for the sake of our calculations. However, as we are going to spend more time later dealing with moduli spaces with a point constraint, a decomposition formula for moduli spaces with point constraints given in [Theorem 5.4, [ACGS1]] can be applied first in order to simplify the terms in Kim et al's degeneration formula. We refer readers to [KLR] and [ACGS1] for further details about the degeneration formula and decomposition formula respectively.

We recall that given a curve class β in X, we can consider moduli spaces $\mathscr{M}(X,\boldsymbol{\tau})$ of stable log maps marked by $\boldsymbol{\tau}=(\tau,\beta)$ where $\tau=(G,\boldsymbol{g},\sigma,\boldsymbol{u})$, a decorated type of a rigid tropical map. See [ACGS1] for the definition of rigid tropical map. However, rigid tropical maps in our setting becomes much easier to describe. Proposition 5.1 in [ACGS1] actually tells us that a decorated type $\boldsymbol{\tau}$ of tropical maps is rigid in our setting if and only if every vertex will represent an irreducible component which gets mapped to either X_0 or Y_0, and we call a vertex V an X_0-vertex if the corresponding component gets mapped into X_0 and likewise for Y_0-vertex. Furthermore, no edge connects two vertices which are both X_0-vertices or Y_0 vertices, i.e., the endpoints of any edge will be of different type. On the other hand, a choice of decorated rigid tropical map in our situation is exactly what Jun Li terms an *admissible triple* in [Li1] and [Li2]. More precisely, giving a rigid tropical map is equivalent to giving a *bipartite* graph Γ in which the edges are enumerated $e_1,e_2.\ldots,e_r$ with each edge decorated with a positive integer w_e and each vertex V is decorated with a set n_V thought of as the set of markings and a class β_V that is an effective curve class in either X_0 or Y_0. For more details, readers can refer to [Li1], [Li2] and [vGGR].

So, we are actually free to go back and forth between our tropical language and Jun Li's degeneration formula.

We incorporate a point constraint by defining

$$\mathscr{M}(X,\boldsymbol{\tau},s) := \mathscr{M}(X,\boldsymbol{\tau}) \times_X \{\mathrm{pt}\}, \mathscr{M}(X,\beta,s) := \mathscr{M}(X,\beta) \times_X \{\mathrm{pt}\}$$

where $s : \{\mathrm{pt}\} \to X$ is a geometric point. Note that there is a finite map $j_{\boldsymbol{\tau}*} : \mathscr{M}(X,\boldsymbol{\tau},s) \to \mathscr{M}(X,\beta,s)$. Now we have the following decomposition for a point condition, refer to [[ACGS1], Theorem 3.11] for the proof.

Theorem 4.2.1. *Suppose X is a logarithmically smooth variety. Then*

$$[\mathscr{M}(X,\beta,s)]^{\mathrm{vir}} = \sum_{\boldsymbol{\tau}=(\tau,\beta)} \frac{m_\tau}{|Aut(\tau)|} j_{\boldsymbol{\tau}*}[\mathscr{M}(X,\boldsymbol{\tau},s)]^{\mathrm{vir}}$$

Proof. Notice that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)$ is logarithmically smooth over the standard log point $b^\dagger$ according to [Proposition 3.3 (2), [ACGS1]], then by writing down a chart for the point-constrained moduli space $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta,s)$, we can easily check that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta,s)$ is also logarithmically smooth over $b^\dagger$. Therefore, by [Corollary 3.8, [ACGS1]], we have the following equality of top-dimensional algebraic cycles in the pure dimensional algebraic stack $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta,s)$:

$$[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta,s)] = \sum_{\boldsymbol{\tau}=(\tau,\beta)} m_\tau j_{\tau*}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau},s)]$$

Then, emulating the proof of [Theorem 3.11, [ACGS1]], we will get the decomposition of our theorem. $\square$

Furthermore, we will apply Kim, Lho and Ruddat's degeneration formula for the spaces $\mathscr{M}(X,\boldsymbol{\tau},s)$. Note that there is a diagram

$$\begin{tikzcd}
\mathscr{M}(X,\boldsymbol{\tau}) \arrow[dr, "\phi"] & & \\
& \bigodot_V \mathscr{M}_V \arrow[r] \arrow[d] & \prod_V \mathscr{M}_V \arrow[d] \\
& \prod_e D \arrow[r, "\Delta"] & \prod_V \prod_{V\in e} D \times D
\end{tikzcd}$$

where V's and e's represent the vertices and the edges in $\boldsymbol{\tau}$ respectively and the space $\bigodot_V \mathscr{M}_V$ is defined by the Cartesian square diagram. By [KLR], Equation (1.4), we know that ϕ is a finite étale map and the degree of ϕ is $\dfrac{\prod_e w_e}{\mathrm{l.\,c.\,m}(w_e)}$. So we have the following equality

$$[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{vir}} = \frac{\prod_e w_e}{\mathrm{l.\,c.\,m}(w_e)} \phi^* \Delta^! \prod_V [\mathscr{M}_V]^{\mathrm{vir}},$$

where w_e is the contact order to the divisor $\underline{D}$ at the relative marking corresponding to e.

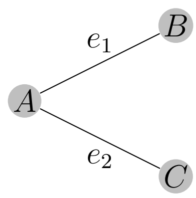
Moreover, as analogous to the proof of [Theorem 3.11, [ACGS1]] and the proof of Theorem 4.2.1, we can impose point constraints at vertices of $\boldsymbol{\tau}$ and an analogous decomposition will hold as well.

4.3 The main comparison

Now, we are in position to deduce our comparison result by applying the degeneration formula to $\mathcal{L}_0$ which is the union of $\mathcal{L}_X = W \times \mathbb{P}^1$ and $\mathcal{L}_Y = \mathbb{P}(\mathcal{O}_Y(-D_\infty) \oplus \mathcal{O})$ along the shared divisor $\mathcal{L}_D = D \times \mathbb{P}^1$. Then we need to impose a point condition for the family. It is easy to choose a section $s : \mathbb{A}^1 \to \mathcal{L}$ of π such that $s(t)$ lies on the fiber of X for $t \neq 0$ and $s(0) \in \mathcal{L}_Y$ passing through a $\mathbb{P}^1$-fiber over Y. For instance, let us select any point $z \in D$, according to the construction of the space $\mathcal{L}$, we can choose the section $s' : \mathbb{A}^1 \to W \times \mathbb{A}^1$ given by $t \mapsto (z,t)$. We then take the strict transform of this section to the blow-up $\mathcal{X}$, and then choose a general section of the $\mathbb{P}^1$-bundle $\mathcal{L} \to \mathcal{X}$ over this strict transform.

From now on, we can always assume that $s(0) \in \mathcal{L}_Y$, and then there exists a fiber of $\mathcal{L}_Y \to Y$ passing through $s(0)$. The class of the fiber is denoted by h. Put in another way, when we have a decorated tropical type of rigid tropical maps $\boldsymbol{\tau}$ with the graph G, we can always assume that there is only some $\mathcal{L}_Y$-vertex V such that it carries the point constraint and the associated curve class β_V contains h. Then we can consider the related moduli spaces with this point condition. The key step is to work out what kind of graphs will non-trivially contribute to the virtual fundamental class in the degeneration formula. The following theorem basically deals with this problem.

Theorem 4.3.1. *For a decorated type of rigid tropical maps $\boldsymbol{\tau}$, let $P := Q_0$ from Lemma 4.1.4. Then we have $P_* j_{\boldsymbol{\tau}*}[\mathscr{M}(\mathcal{L}_0,\boldsymbol{\tau},s)]^{\mathrm{vir}} = 0$ unless the graph G of $\boldsymbol{\tau}$ is the following*



with A an $\mathcal{L}_X$-vertex, B and C both $\mathcal{L}_Y$-vertices and furthermore, $w_{e_1} + w_{e_2} = \beta \cdot \mathcal{L}_D$,

$$\begin{aligned}\beta_A =& \beta,\\ \beta_B =& w_{e_1} F + h,\\ \beta_C =& w_{e_2} F.\end{aligned}$$

Here β is the curve class β on W viewed as a curve class on $\mathcal{L}_X = W \times \mathbb{P}^1$ and F is a fibre of $p : Y \to D$ viewed as a curve class on $\mathcal{L}_Y$ via the inclusion of Y in $\mathcal{L}_Y$ as the 0*-section. Note B carries the point constraint.*

Before jumping to the proof directly, we need a lemma as preparation.

Lemma 4.3.2. *For a decorated type of rigid tropical maps $\boldsymbol{\tau}$ with the graph G of $\boldsymbol{\tau}$ having an $\mathcal{L}_X$-vertex V, let r be the number of edges adjacent to V connecting to $\mathcal{L}_Y$-vertices. Then $P_* j_{\boldsymbol{\tau}*}[\mathscr{M}(\mathcal{L}_0, \boldsymbol{\tau}, s)]^{\text{vir}} = 0$ if $r > 2$.*

Proof. Let us fix this $\mathcal{L}_X$-vertex V and $r + u$ be the number of the edges of $\boldsymbol{\tau}$. By the choice of the section s, we know that β_V contains no fiber class of $\mathcal{L}_X \to X$ attached to V. Since a map from a proper curve to $\mathbb{P}^1$ which is not surjective is just a constant map, by separating out the factors for V, the gluing diagram factors as follows

$$\begin{tikzcd}
\mathscr{M}_V \times_{\mathcal{L}_D} \bigodot_{V' \neq V} \mathscr{M}_{V'} \arrow[r] \arrow[d, "ev"] & \mathscr{M}_V \times_{\mathcal{L}_D} \prod_{V' \neq V} \mathscr{M}_{V'} \arrow[d, "ev"] \\
(D^r \times \mathbb{P}^1) \times (D \times \mathbb{P}^1)^u \arrow[r, "\Delta'"] \arrow[d, "(id \times diag) \times id =: \delta"] & (D^r \times \mathbb{P}^1)^2 \times (D \times \mathbb{P}^1)^{2u} \arrow[d, "(id \times diag) \times id"] \\
(D \times \mathbb{P}^1)^r \times (D \times \mathbb{P}^1)^u \arrow[r, "\Delta"] & (D \times \mathbb{P}^1)^{2r} \times (D \times \mathbb{P}^1)^{2u}
\end{tikzcd}$$

where the two squares are both Cartesian.

Let N denote the normal bundle of the embedding Δ which has rank $(r+u)(\dim D+1)$ and N' denote that of Δ' which has the rank $r\dim D + 1 + u(\dim D + 1)$. Set $E := (\delta^* N)/N'$ which is of rank $r - 1$. Let $c_{r-1}(E)$ be its top Chern class. For any k and $\alpha \in A_k(\mathscr{M}_V \times_{\mathcal{L}_D} \prod_{V' \neq V} \mathscr{M}_{V'})$, the excess intersection formula says that

$$\Delta^! \alpha = c_{r-1}(E) \cap (\Delta')^! \alpha.$$

Note that $c_{r-1}(E) = 0$ when $r \geq 3$ since the bundle E is in fact the pullback of the corresponding bundle from the diagram

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "\Delta'"] \arrow[d, "diag"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "diag"] \\
(\mathbb{P}^1)^r \arrow[r, "\Delta"] & (\mathbb{P}^1 \times \mathbb{P}^1)^r
\end{tikzcd}$$

and taking Chern classes commutes with the pullback operation. Then applying this to the virtual cycle $\alpha = [\mathscr{M}_V]^{\mathrm{vir}} \times_{D_0} \prod_{V' \neq V} [\mathscr{M}_{V'}]^{\mathrm{vir}}$ gives the conclusion of this lemma. $\square$

Proof of the theorem. For a decorated type of rigid tropical maps $\boldsymbol{\tau}$ with the graph G, let us collect what is implied for G if the pushforward to $\mathscr{M}(W, \beta)$ of the corresponding virtual cycles is non-trivial. Firstly, by Lemma 4.3.2, each of the $\mathcal{L}_X$-vertices of the graph G has no more than 2 adjacent edges.

Let V be any $\mathcal{L}_Y$-vertex in G. Firstly, by [vGGR, Proposition 5.3], we know that $\pi_{Y*}\beta_V$ must be a multiple of the fiber class of the projective bundle $Y \to D$ where π_Y is just the natural projection $\mathcal{L}_Y \to Y$, otherwise $P_* j_{\boldsymbol{\tau}*}[\mathscr{M}(\mathcal{L}_0, \boldsymbol{\tau}, s)]^{\mathrm{vir}} = 0$. Hence, the curve class β_V associated to the vertex V is either $\beta' + h$ or β' in which β' is a multiple of the fiber class F of the projective bundle $Y \to E$ and h is the fiber class of the projective bundle $\mathcal{L}_Y \to Y$.

Furthermore, we need to argue that V has only a single adjacent edge to it. This is easily verified using [vGGR, Lemma 5.4]. Indeed, let $\mathscr{M}^\circ := \prod_{V' \neq V} \mathscr{M}_{V'}$ and G_V be the subgraph of G at the vertex X, then the evaluation map from $\mathscr{M}_{G_V}(\mathcal{L}_Y(\log \mathcal{L}_D), \beta_V) \times_{(D \times \mathbb{P}^1)^{r_V}} \mathscr{M}^\circ$ to $\mathscr{M}(W, \beta)$ factors through $D \times_{(D \times \mathbb{P}^1)^{r_V}} \mathscr{M}^\circ$ in either case of $\beta_V = \beta'$ or $\beta_V = \beta' + h$. Therefore, by [vGGR, Lemma 5.4], we can immediately conclude that there are at most 2 Y_0-vertices existing in $\boldsymbol{\tau}$ or otherwise $P_* j_{\boldsymbol{\tau}*}[\mathscr{M}(X, \boldsymbol{\tau}, s)]^{\mathrm{vir}} = 0$. $\square$

Proposition 4.3.3. *For a decorated type of rigid tropical maps $\boldsymbol{\tau}$ with the graph G as in the theorem, we have $P_* j_{\boldsymbol{\tau}*}[\mathscr{M}(\mathcal{L}_0), \boldsymbol{\tau}, s)]^{vir} = 0$ unless $w_{e_1} = 1$ and $w_{e_2} = e - 1$.*

Proof. Let V denote the vertex B which carries the point constraint, and the curve class $\beta_V = \beta' + h$ where β' is the fiber class of $Y \to D$ and h is the fiber class of $\mathcal{L}_Y \to Y$. Consider the surface S which is obtained by restricting $\mathcal{L}_Y$ onto a fiber of $Y \to D$ such that it contains the point constraint $s(0)$. In fact, it is not so hard to see that S is just the Hirzebruch surface F_1. Indeed, recall that $\mathcal{L}_Y = \mathbb{P}(\mathcal{O}_Y(-D_\infty) \oplus \mathcal{O})$, and note that the restriction of $\mathcal{O}_Y(-D_\infty) \oplus \mathcal{O}$ to any fiber of $Y \to D$ is actually $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ since the intersection of $-D_\infty$ and the fiber is -1.

Note that any curve realizing the curve class $\beta' + h$ and containing the point $s(0)$ will lie in this surface S.

Assume that $w_{e_1} \geq 2$. Let C be any curve which lies in S and realizes the class β', that is, C is a fiber of $Y \to D$. Then for any curve in $C' \subset S$ such that $[C'] = w_{e_1}\beta' + h$, we find that $C \cdot C' = \beta' \cdot (w_{e_1}\beta' + h) = -w_{e_1} + 1 < 0$. Therefore, C' has to contain a fiber of $Y \to D$ which is contained in S as its one irreducible component. The evaluation map from $\mathscr{M}_V$ to $\mathcal{L}_D$ is a constant map. Hence, the vanishing result follows immediately by applying the degeneration formula. $\square$

Theorem 4.3.4. $(-1)^{e-1} \cdot (e-1) \cdot p_*[\mathscr{M}(X, \beta + h, s)]^{vir} = [\mathscr{M}(W(\log D), \beta, s)]^{vir}$ *where β is an effective curve class in W and h is the fiber class of $p : X \to W$.*

Proof. The proof is now just an application of degeneration formula. By Theorem 4.3.1 and the degeneration formula, we can conclude that

$$[\mathscr{M}(\mathcal{L}_0, \beta + h, s)]^{\mathrm{vir}} = \frac{m_{\boldsymbol{\tau}}}{|\mathrm{Aut}(\boldsymbol{\tau})|} j_{\boldsymbol{\tau}*}\phi^*\Delta^! \prod_{V\in\boldsymbol{\tau}} [\mathscr{M}_V]^{\mathrm{vir}}$$

where the decorated tropical type of rigid tropical maps $\boldsymbol{\tau}$ is exactly the one depicted in Theorem 4.3.1. So, in our situation, $|\mathrm{Aut}(\boldsymbol{\tau})| = 1$ and $m_{\boldsymbol{\tau}} = e - 1$.

Let $P : \mathcal{L}_0 \to W$ be the natural projection. Then, by Lemma 4.1.4, we have

$$p_*[\mathscr{M}(X, \beta + h, s)]^{\mathrm{vir}} = (e-1) \cdot P_* j_{\boldsymbol{\tau}*}\phi^*\Delta^! \prod_{V\in\boldsymbol{\tau}} [\mathscr{M}_V]^{\mathrm{vir}}$$

By [vGGR, Proposition 2.4], we know that the right hand side becomes

$$(e-1) \cdot \frac{(-1)^e}{(e-1)^2} [\mathscr{M}(W(\log D), \beta, s)]^{\mathrm{vir}}$$

where the space $\mathscr{M}(W(\log D), \beta, s)$ is exactly the moduli space of genus 0 stable logarithmic maps to W with the curve class β such that image of map intersects D at one specified point to tangent order 1 and at one unspecified point to tangent order $e-1$, that is, we have an equality of cycles

$$(-1)^{e-1} \cdot (e-1) \cdot p_*[\mathscr{M}(X, \beta + h, s)]^{\mathrm{vir}} = [\mathscr{M}(W(\log D), \beta, s)]^{\mathrm{vir}}.$$

□

As a direct consequence of Theorem 4.3.4, we find:

Corollary 4.3.5. $(-1)^e \cdot (e-1) \cdot n_{\beta+h} = N_{e-1,1}$ *where* β *is an effective curve class in* W *and* h *is the fiber class of* $X \to W$*.*

Remark 4.3.6. In the case that $W = \mathbb{P}^2$, all those invariants $n_{\beta+h}$ have been calculated by Siu-Cheong Lau in [Lau]. Therefore, after identifying $N_{3d-1,1}$ with $n_{\beta+h}$ using the equality in Corollary 4.3.5, we can calculate all 2-pointed relative GW invariants for any degree d.

Chapter 5

Calculations for $(\mathbb{P}^2, D)$

Throughout this section, our smooth log Calabi-Yau pair is always $(\mathbb{P}^2, D)$ where D is a smooth cubic curve.

5.1 A relation between invariants $N_{a,b}$

In this subsection, we will investigate how those punctured invariants and 2-point relative invariants with a point-constraint abstractly determine each other. Besides the formula we proved in Corollary 3.3.13, the other key intermediate tool we are going to use is the "degree 0 part of relative quantum cohomology ring" defined by Gross and Siebert in [GS2, Construction 1.8] where the terminology "degree 0" means that we only take genus 0 curves into account.

As we mentioned in Remark 1.4.9 that T. Grafnitz, H. Ruddat and E. Zaslow also computed various 2-pointed invariants for $(\mathbb{P}^2, D)$ in [GRZ] using broken line counting technique. However, their convention about $N_{a,b}$ is opposite to mine in this thesis, i.e. $N_{a,b}$ in this thesis are supposed to be $N_{b,a}$ in [GRZ].

Roughly speaking, our degree 0 part of relative quantum cohomology is a $\mathbb{Q}[t]-$algebra structure on

$$R := \bigoplus_{p=0}^{\infty} \theta_p \mathbb{Q}[t],$$

where θ_p can be viewed as a symbol, and the multiplication law is

$$\theta_p \cdot \theta_q := \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} N^d_{pqr} t^d \theta_r.$$

Here, for the precise definition of N^d_{pqr}, see Definition 3.1.1. However, roughly speaking, it is the virtual count of degree d genus zero stable maps with three punctured points x_1, x_2, x_{out}, with x_1 having contact order p with D, x_2 having contact order q, and x_{out}

having contact order $-r$ with D such that x_{out} gets mapped to a generically fixed point $z \in D$.

Then, we can replace N^d_{pqr} with those 2-point invariants N^d_{pqr} by means of the formula in Corollary 3.3.13, and we have

$$\theta_p \cdot \theta_q = \sum_{d=0}^{\infty} N^d_{pq0} t^d \theta_0 + \sum_{r \geq 1} N^0_{pqr} t^0 \theta_r + \sum_{r=1}^{\infty} \sum_{d=1}^{\infty} ((q-r)N_{p,q-r} + (p-r)N_{q,p-r}) t^d \theta_r.$$

Notice that firstly, $N_{p,i}$ are the number of rational curves with two marked points, one tangent to order p at an unspecified point of D and one tangent to order i at a specified point of D. So, if $p \leq 0$ or $i \leq 0$, we have $N_{p,i} = 0$. Also, note that these rational curves must be of degree $(p+i)/3$. Therefore, we have the convention that $N^{(p+q)/3}_{pq0}$ and $N_{p,q}$ vanish if three does not divide $p+q$ since there will not be such a curve of degree $p+q$ with $p+q$ is not divided by three.

Moreover, by Lemma 1.15 in [GS2], we have

$$N^0_{pqr} = \begin{cases} 1 & r = p+q \\ 0 & \text{otherwise.} \end{cases}$$

So, combining all aforementioned results and separating the degree 0 term, we can write the multiplication law as follows

$$\theta_p \cdot \theta_q = \theta_{p+q} t^0 + N^{(p+q)/3}_{pq0} t^{(p+q)/3} \theta_0 + \sum_{r=1}^{\min(p,q)} [(q-r)N_{p,q-r} + (p-r)N_{q,p-r}] t^{(p+q-r)/3} \theta_r.$$

Note that the right-hand side is actually a finite sum.

Let's see what relations we can get from associativity. We can calculate the product $\theta_p \cdot \theta_q \cdot \theta_r$ in two ways. Recall that θ_0 is the identity in the ring, so we will not gain anything if one of p, q, r are 0. Thus we assume $p, q, r > 0$.

We have

$$(\theta_p \cdot \theta_q)\cdot\theta_r = \left(\theta_{p+q} + N_{pq0}^{(p+q)/3} t^{(p+q)/3}\theta_0 + \sum_{s=1}^{\min(p,q)} [(q-s)N_{p,q-s} + (p-s)N_{q,p-s}]t^{(p+q-s)/3}\theta_s\right)\cdot\theta_r$$

$$\begin{aligned}
=&\theta_{p+q+r} + N_{p+q,r,0}^{(p+q+r)/3} t^{(p+q+r)/3}\theta_0 + \\
&+ \sum_{s'=1}^{\min(p+q,r)} [(r-s')N_{p+q,r-s'} + (p+q-s')N_{r,p+q-s'}]t^{(p+q+r-s')/3}\theta_{s'} \\
&+ N_{pq0}^{(p+q)/3} t^{(p+q)/3}\theta_r + \sum_{s=1}^{\min(p,q)} [(q-s)N_{p,q-s} + (p-s)N_{q,p-s}]t^{(p+q-s)/3}\cdot \\
&\cdot\left(\theta_{s+r} + N_{sr0}^{(s+r)/3} t^{(s+r)/3}\theta_0 + \sum_{w=1}^{\min(s,r)} [(r-w)N_{s,r-w} + (s-w)N_{r,s-w}]t^{(s+r-w)/3}\theta_w\right)
\end{aligned}$$

and associating the other way gives:

$$\theta_p\cdot(\theta_q \cdot \theta_r) = \theta_p \cdot \left(\theta_{q+r} + N_{qr0}^{(q+r)/3} t^{(q+r)/3}\theta_0 + \sum_{s=1}^{\min(q,r)} [(r-s)N_{q,r-s} + (q-s)N_{r,q-s}]t^{(q+r-s)/3}\theta_s\right)$$

$$\begin{aligned}
=&\theta_{p+q+r} + N_{p,q+r,0}^{(p+q+r)/3} t^{(p+q+r)/3}\theta_0 + \\
&\sum_{s'=1}^{\min(p,q+r)} [(q+r-s')N_{p,q+r-s'} + (p-s')N_{q+r,p-s'}]t^{(p+q+r-s')/3}\theta_{s'} \\
&+ N_{qr0}^{(q+r)/3} t^{(q+r)/3}\theta_p + \sum_{s=1}^{\min(q,r)} [(r-s)N_{q,r-s} + (q-s)N_{r,q-s}]t^{(q+r-s)/3}\cdot \\
&\cdot\left(\theta_{p+s} + N_{ps0}^{(p+s)/3} t^{(p+s)/3}\theta_0 + \sum_{w=1}^{\min(p,s)} [(s-w)N_{p,s-w} + (p-w)N_{s,p-w}]t^{(p+s-w)/3}\theta_w\right)
\end{aligned}$$

Let us extract the highest degree contribution to the products above. Suppose that $p+q+r \equiv i \bmod 3$, with $i \in \{0,1,2\}$. We have the following possibilities:

- If $i=0$, then we get the coefficient of $t^{(p+q+r)/3}$ of the two products being

$$\begin{aligned}
&N_{p+q,r,0}^{(p+q+r)/3} + \sum_{s=1}^{\min(p,q)} [(q-s)N_{p,q-s} + (p-s)N_{q,p-s}]N_{sr0}^{(s+r)/3} \\
&= N_{p,q+r,0}^{(p+q+r)/3} + \sum_{s=1}^{\min(q,r)} [(r-s)N_{q,r-s} + (q-s)N_{r,q-s}]N_{ps0}^{(s+p)/3}
\end{aligned} \tag{5.1.1}$$

- If $i=1$ or 2, then the highest degree power of t is $t^{(p+q+r-i)/3}$, and the coefficient of this is θ_i times the following:

$$\begin{aligned}
&[(r-i)N_{p+q,r-i}+(p+q-i)N_{r,p+q-i}]+N_{pq0}^{(p+q)/3}\delta_{ri}\\
&+\sum_{s=1}^{\min(p,q)}[(q-s)N_{p,q-s}+(p-s)N_{q,p-s}][\delta_{s,i-r}+(r-i)N_{s,r-i}+(s-i)N_{r,s-i}]\\
&=[(q+r-i)N_{p,q+r-i}+(p-i)N_{q+r,p-i}]+N_{qr0}^{(q+r)/3}\delta_{pi}\\
&+\sum_{s=1}^{\min(q,r)}[(r-s)N_{q,r-s}+(q-s)N_{r,q-s}][\delta_{s,i-p}+(s-i)N_{p,s-i}+(p-i)N_{s,p-i}].
\end{aligned} \tag{5.1.2}$$

Note that $N_{pq0}^{(p+q)/3}$ and $N_{p,q}$ are both zero if 3 doesn't divide $p+q$.

Before showing any relations between these invariants, there is an easy but interesting observation shown in the following lemma.

Lemma 5.1.1. *For any positive integer d, we have $N_{3d-1,1}=(3d-1)^2N_{1,3d-1}$.*

Proof. The equality holds for the corresponding orbifold Gromov-Witten invariants as a direct corollary of the formula given by Cadman and Chen in [CC], and according to the comparison result proven by Abramovich-Cadman-Wise in [ACW], the equality holds in our setting. $\square$

By the equations (5.1.1) and (5.1.2) we derived above, we can prove the following proposition.

Proposition 5.1.2. *For $a+b=3d$, the invariants N_{ab0}^d and $N_{a,b}$ are completely determined by the number $N_{1,3d-1}$ plus those lower degree invariants using associativity.*

Remark 5.1.3. There will be many complex terms popping up while playing the whole algebraic game with the equations (4.1) and (4.2), and the indices will be a mess. So, in the following proof, we decide to denote any lower degree terms or the combination of lower degree terms just by a single symbol S meaning "some lower degree terms". Thus, **we cannot generally cancel S out even if we see the symbol S appears at the same time on the both sides of an equation**.

Proof. Note that the conclusion is true when $d=1$, see §5.2. Therefore, we can assume $d\geq 2$.

First of all, notice that, by the equation (4.1), all the invariants N_{ab0}^d are actually determined by $N_{3d-1,1,0}^d$ and lower degree invariants. Indeed, set $r=1$, then $p+q=3d-1$, so by (4.1), we have equations

$$N_{3d-1,1,0}^d+S=N_{p,q+1,0}^d+S.$$

Then for any a, b such that $a + b = 3d$, set $p = a$ and $q = b - 1$, we can solve for N^d_{ab0} by using $N^d_{3d-1,1,0}$ and those lower degree terms.

Secondly, in the equation (4.2), we set $r = i = 1$, so $p + q = 3d$, then we get

$$(3d-1)N_{1,3d-1} + N^d_{pq0} + S = qN_{p,q} + (p-1)N_{q+1,p-1} + N^{(q+1)/3}_{q10}\delta_{p,1} + S. \tag{5.1.3}$$

If we set $r = i = 2$, we have

$$(3d-2)N_{2,3d-2} + N^d_{pq0} + S = qN_{p,q} + (p-2)N_{q+2,p-2} + N^{(q+2)/3}_{q20}\delta_{p,2} + S. \tag{5.1.4}$$

Then in (5.1.3), set $p = 3d - 1, q = 1$, and by Lemma 5.1.1, we get

$$(3d-1)N_{1,3d-1} + N^d_{3d-1,1,0} + S = (3d-1)^2 N_{1,3d-1} + (3d-2)N_{2,3d-2} + S, \tag{5.1.5}$$

and in (5.1.4), set $p = 1, q = 3d - 1$, then we get

$$(3d-2)N_{2,3d-2} + N^d_{1,3d-1,0} + S = (3d-1)N_{1,3d-1} + S. \tag{5.1.6}$$

Obviously, the equations (4.5) and (4.6) are independent and we can solve for both $N^d_{3d-1,1,0}$ and $N_{2,3d-2}$ by using $N_{1,3d-1}$ (bearing in mind that $N^d_{ab0} = N^d_{ba0}$ for every a, b). Henceforth, all numbers N^d_{ab0} can be solved by $N_{1,3d-1}$ and lower degree invariants, and $N_{2,3d-2}$ can be solved by $N_{1,3d-1}$ plus lower degree invariants.

Furthermore, (4.3) subtract (4.4) tells us that $(p-1)N_{q+1,p-1} - (p-2)N_{q+2,p-2}$ can be solved by using $N_{1,3d-1}$ and those lower degree invariants for any p, q such that $p + q = 3d$. Then, just by an easy algebra, we can conclude that all $N_{a,b}$ can be expressed by $N_{1,3d-1}$ and lower degree invariants. $\square$

Remark 5.1.4. The same conclusion should also hold for general $(\mathbb{P}^n, D)$ where D is a smooth anti-canonical divisor, that is, all degree d, 2-pointed relative Gromov-Witten invariants with a point condition should be determined by $N_{(n+1)d-1,1}$ plus the lower degree invariants, and we can proceed the proof in an analogous fashion as we did in the proof above with a much more complex algebraic deduction.

Remark 5.1.5. In joint work [WY] with F. You, we are able to generalize my result 4.3.4 to relative Gromov-Witten invariants with any number of relative markings and then we are able to do a lot more calculations.

5.2 Calculations of degree 2 relative invariants

In this subsection, we are going to compute all the relevant degree 2 invariants. Before doing that, let us see a warm-up computation for degree 1 invariants.

The first non-trivial case to consider is $(p, q, r) = (2, 1, 1)$, so $i = 1$, and we get

$$2N_{1,2} + N^1_{120} = 2N_{2,1}.$$

If we take $(p, q, r) = (3, 1, 1)$, then we get

$$4N_{1,2} = N_{2,1}.$$

It is well-known that the number of lines tangent to order 2 at a specified point of E is 1, in other words, $N_{1,2} = 1$. So, $N_{2,1} = 4$ and $N^1_{120} = 6$.

Proposition 5.2.1. *We have relations* $25N_{1,5} = N_{5,1}, 2N_{2,4} = 5N_{1,5} + 2, N_{3,3} = 5N_{1,5} + 4$ *and* $N_{4,2} = 10N_{1,5} + 4$.

Proof. The way to get these relations is very straightforward. We just try to pick some special values for (p, q, r) and plug them into equations (4.1) and (4.2), and then get a series of equations about these invariants. First of all, let $(p, q, r) = (1, 2, 3)$, by (4.1), we have $N^2_{330} = N^2_{150} + 24$; Let $(p, q, r) = (3, 1, 2)$, again by (4.1), we have $N^2_{420} + 12 = N^2_{330}$. So we have an equation $N^2_{420} = N^2_{150} + 12$.

- Let $(p, q, r) = (4, 2, 1)$, we have

$$5N_{1,5} + N^2_{420} + 8 = 2N_{4,2} + 3N_{3,3} \tag{5.2.1}$$

- Let $(p, q, r) = (2, 1, 4)$, we have

$$3N_{3,3} + 2N_{4,2} = 4N_{2,4} + N_{5,1} + 16 \tag{5.2.2}$$

- $(p, q, r) = (5, 2, 1)$:

$$5N_{1,5} + 4N_{2,4} + 8 = 3N_{3,3} \tag{5.2.3}$$

- let $(p, q, r) = (1, 5, 2)$ and $N^2_{420} = N^2_{150} + 12$, we have

$$N^2_{420} = 5N_{1,5} + N_{5,1} + 12 \tag{5.2.4}$$

- $(p, q, r) = (4, 1, 3)$:

$$N_{5,1} + 3N_{3,3} + 4 = 4N_{4,2} \tag{5.2.5}$$

It is fairly easy to check that these 5 equations are linearly independent and any other equation in above invariants given by choosing (p, q, r) which are different from above will be recovered by the 5 equations above. So, by an easy algebra, we can get the relations that we want to derive. $\square$

Corollary 5.2.2. *We have* $N_{1,5} = 1, N_{5,1} = 25, N_{2,4} = 7/2, N_{4,2} = 14,$ $N_{3,3} = 9, N^2_{240} = N^2_{420} = 42, N^2_{150} = N^2_{510} = 30$ *and* $N^2_{330} = 54$.

Proof. By Corollary 4.3.5 and computations of $n_{\beta+h}$ in [Lau], [LLW] and [GS], we know that $N_{5,1} = 25$. Then, just by some easy algebra. So, other numbers can be computed out by the equations shown in the proof of Proposition 5.2.1. $\square$

Remark 5.2.3. As Theorem 1.1 in [Lau] indicates, open Gromov-Witten invariants of a toric Calabi-Yau manifold are determined by Gross-Siebert slab functions. For the anti-canonical bundle of $\mathbb{P}^2$, the slab function is

$$1 + z^{(1,0,0,0)} + z^{(0,1,0,0)} + z^{(-1,-1,0,0)} - 2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 + \dots$$

as shown in [GS], §5. Hence, in theory, we have succeeded in calculating degree d, 2-pointed, relative Gromov-Witten invariants with a point condition for $\mathbb{P}^2$ with an elliptic curve for any degree d.

Bibliography

[AC0] D. Abramovich, Q. Chen: *Stable logarithmic maps to Deligne-Faltings pairs I*, Annals of Mathematics 180 (2014), 455–521.

[AC] D. Abramovich, Q. Chen: *Stable logarithmic maps to Deligne-Faltings pairs II*, Asian J. Math. 18(2014), 465-488.

[ACFM] D. Abramovich, C. Cadman, B. Fantechi, J. Wise: *Expanded degenerations and pairs*, Comm. Algebra 41 (2013), no. 6, 2346–2386. MR 3225278.

[ACGS1] D. Abramovich, Q. Chen, M. Gross, B. Siebert: *Decomposition of degenerate Gromov-Witten invariants*, Compos. Math.156 (2020), no.10,2020-2075.

[ACGS2] D. Abramovich, Q. Chen, M. Gross, B. Siebert: Punctured logarithmic maps.

[ACMW] D. Abramovich, Q. Chen, S. Marcus, J. Wise: *Boundedness of the space of stable logarithmic maps*, J. Eur. Math. Soc.(JEMS), 2017

[ACP] D. Abramovich, L. Caporaso, and S. Payne: The tropicalization of the moduli space of curves, Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), 765–809.

[ACW] D. Abramovich, C. Cadman, J. Wise: *Relative and orbifold Gromov-Witten invariants*, Algebr. Geom. 4 (2017), no.4, 472-500

[AMW] D. Abramovich, S. Marcus, J. Wise: *Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenrations*,Annales de l'Institut Fourier, 2004.

[AS] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields II*, Doc. Math. 2003, Extra Vol., 5–72.

[AW] D. Abramovich and J. Wise: *Birational invariance in logarithmic Gromov-Witten theory*, Compos. Math. 154 (2018), 595–620.

[BF] K. Behrend, B. Fantechi: The *intrinsic normal cone*, Invent. Math, 1997.

[Ca] C. Cadman: *Using stacks to impose tangency conditions on curves*, Amer. J. Math. 129 (2007), no. 2, 405–427. MR MR2306040 (2008g:14016)

[Co] K.Costello: *Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products*, Ann. of Math.(2) **164**(2006) no.2, 561-601.

[CC] C.Cadman, L. Chen: *Enumeration of rational plane curves tangent to a smooth cubic*, Adv. Math. 219 (2008), 316-343.

[Cha] K. Chan: *A formula equating open and closed Gromov-Witten invariants and its application to mirror symmetry*, Pacific. J. Math. **254**(2011), no.2, 275-293. MR2900016

[Ch] Q. Chen: *Stable logarithmic maps to Deligne-Faltings pairs I*, Ann. of Math. (2), 2014.

[CLR] P. Candelas, M. Lynker, R. Schimmirgk: *Calabi-Yau manifolds in weighted* $\mathbb{P}_4$, Nuclear Phys. B 341 (1990), 383–402.

[Fu] W.Fulton: *Intersection theory.* Second edition. Ergebinsse der Mathematik und ihrer Grenzgebiete. 3 Folge. Springer-Verlag, Berlin, 1998.

[FWY] H. Fan, L. Wu, F. You: *Structures in genus-zero relative Gromov-Witten theory*, J. Topol. 13 (2020), no. 1, 269-307.

[Ga] A. Gathmann: *Absolute and relative Gromov-Witten invariants of very ample hypersurfaces*, Duke Math. J., 115 (2002), pp. 171–203. 2

[GP] B. Greene, M. Plesser: *Duality in Calabi-Yau moduli space*, Nuclear Phys. B 338 (1990), 15–37.

[GPS] M. Gross, R. Pandharipande, and B. Siebert: *The tropical vertex*, Duke Math. J. 153 (2010), no. 2, 297–362.

[GS0] M. Gross, B. Siebert: *Mirror symmetry via logarithmic degeneration data I*, J. Differential Geom., 72, (2006), 169-338.

[GS] M. Gross, B. Siebert: *From real affine geometry to complex geometry*, Ann. of Math. (2) **174** (2011), no. 3, 1301-1428.

[GS1] M. Gross, B. Siebert: *Logarithmic Gromov-Witten invariants.* J. Amer. Math. Soc. 2013

[GS2] M. Gross, B. Siebert: Intrinsic mirror symmetry, 2019.

[GS3] M. Gross, B. Siebert: *The canonical wall structure and intrinsic mirror symmetry*, Inventiones Mathematicae **229**, 1101-1202 (2022)

[Gro] M. Gross: Remarks on gluing punctured logarithmic maps.

[Gra] T. Grafnitz: *Tropical correspondence for smooth del Pezzo Calabi-Yau pairs*, J. Algebraic Geom. 31 (2022), 687-749.

[Gra1] T. Grafnitz: Theta functions, broken lines and 2-marked log Gromov-Witten invariants, 2022.

[GRZ] T. Grafnitz, H. Ruddat, E. Zaslow: The proper Landau-Ginzburg potential is the open mirror map, 2022.

[IP] E-N. Ionel, T. Parker: *Relative Gromov-Witten invariants*, Annals of Mathematics, 157 (2003), 45–96.

[Ka] T. Kajiwara: *Tropical toric geometry*, Toric topology, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 197–207. MR 2428356 (2010c:14078)

[Kk] K. Kato: *Logarithmic structures of Fontaine-Illusie*. Algebraic analysis, geometry and number theory(Baltimore, MD, 1988). Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Kf] F. Kato: *Log smooth deformation and moduli of log smooth curves*. International J. Math, 2000

[Ko] János Kollár: *Rtional Curves on Algebraic Varieties*. Volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1996.

[Kon] M. Kontsevich: *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), 1–23.

[Kon1] M. Kontsevich: *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhuser, Basel, 1995.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat: *Toroidal Embeddings I*, Springer, LNM 339, 1973.

[KLR] B. Kim, H. Lho, H. Ruddat: *The degeneration formula for stable log maps*, Manuscripta Mathematica, 2021

[Lau] S. Lau: *Gross-Siebert's slab functions and open GW invariants for toric Calabi-Yau manifolds*, Mathematical Research Letters 22 (2015), no.3, 881-898.

[Li1] J. Li: *stable morphisms to singular schemes and relative stable morphisms.* J. Differential Geometry, 2001

[Li2] J. Li: *A degeneration formula of GW-invariants.* J. Differential Geom, 2002

[LLW] S.Lau, N. Leung, B.Wu: *A relation for Gromov-Witten invariants of local Calabi-Yau threefolds*, Mathematical Research Letters 18 (2011), pp.943-956.

[LMB] G.Laumon, L.Moret-Bailly: Champs algébriques, Springer 2000.

[LR] A-M. Li, Y. Ruan: *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math., 145 (2001), pp. 151–218.

[LT] J. Li, G. Tian: *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. Volume 11, Number 1, January 1998, Pages 119–174 S 0894-0347(98)00250-1

[Ma] C. Manolache: *Virtual pull-backs.* J. Algebraic Geometry, 21, (2012) 201–245.

[MS] D. Maclagan, B. Sturmfels: *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015. MR 3287221

[Ol1] M. Olsson: *Logarithmic geometry and algebraic stacks*, Ann. Sci. Ecole Norm. Sup 36 (2003), 747-791.

[Ol2] M. Olsson: *Algebraic spaces and Stacks*, American Mathematical Society, 2016.

[Ol3] M. Olsson: *The logarithmic cotangent complex*, Math. Ann. 333 (2005), no. 4 859-931.

[Og] A. Ogus: *Lectures on logarithmic Algebraic Geometry*, Cambridge University Press, 2018.

[Pay] S. Payne: *Analytification is the limit of all tropicalizations*, Math. Res. Lett. 16 (2009), no. 3, 543–556. MR 2511632 (2010j:14104)

[Par] B. Parker: *Exploded manifolds*, Adv. Math., 229:3256–3319, 2012.

[SYZ] A. Strominger, S.-T. Yau, and E. Zaslow: *Mirror Symmetry is T -duality*, Nucl. Phys. B479, (1996) 243–259.

[Ul] M. Ulirsch: Functorial tropicalization of logarithmic schemes: The case of constant co
Proc. Lond. Math. Soc. (3) 114 (2017), no. 6, 1081–1113.

[vGGR] M. van Garrel, T. Graber, H. Ruddat: *Local Gromov-Witten invariants are log invariants*, Advances in Mathematics 350 (2019), 860-876.

[WY] Y. Wang, F. You: *Generalized local-global relative Gromov-Witten invariants for smooth log Calabi-Yau pairs*, in preparation.

[You] F. You: *The proper Landau-Ginzburg potential, intrinsic mirror symmetry and the relative mirror map*, in preparation.