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A CONTINUOUS/DISCONTINUOUS GALERKIN FORMULATION FOR A STRAIN GRADIENT-DEPENDENT DAMAGE MODEL

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Abstract. *The numerical solution of strain gradient-dependent continuum problems has been hindered by continuity demands on the basis functions. The presence of terms in constitutive models which involve gradients of the strain field means that the C^0 continuity of standard finite element shape functions is insufficient. In this work, a continuous/discontinuous Galerkin formulation is developed to solve a strain gradient-dependent damage problem in a rigorous manner. Potential discontinuities in the strain field across element boundaries are incorporated in the weak form of the governing equations. The performance of the formulation is tested in one dimension for various interpolations, which provides guidance for two-dimensional simulations.*

1 INTRODUCTION

An increasing range of important mechanical phenomena cannot be described by classical continuum theories, such as conventional elasticity and plasticity. Two important such phenomena are size effects and strain localisation. Size effect is the phenomenon by which 'smaller is stronger'. Specimens appear to 'gain' mechanical strength as they become smaller. Strain localisation involves the concentration of deformations in narrow bands. When classical continuum models are applied for this problem by incorporating strain softening, the governing equations become ill-posed. An approach to model both size effects and strain localisation is the addition of strain gradient terms to the continuum constitutive model. A broad range of models have been proposed, a small selection of which can be found in References [1–6].

The inclusion of strain gradient terms in a constitutive model leads immediately to the difficulty that the underlying governing equation is of a higher order than for classical theories (typically fourth-order, instead of second-order). It is no longer sufficient to adopt a C^0 interpolation of the primal field (the displacement) to solve the boundary value problem of interest using the finite element method. The complex nature of governing equations also prevents the use of a mixed formulation in the traditional sense. Typically, adoption of a mixed formulation cannot avoid the need for a C^1 interpolation [2, 7].

A method is developed here for the solution of a strain gradient dependent damage model by drawing on developments in discontinuous Galerkin methods for elliptic problems (see Arnold et al. [8] for an overview) and continuous/discontinuous Galerkin methods [9]. Unlike the linear problems addressed in Engel et al. [9], the nonlinear nature and the complex structure of the governing equations for the damage model prevents the straightforward application of integration by parts to derive a 'conventional' weak form. A mixed-type formulation is adopted in which discontinuities across element boundaries are accounted, allowing C^0 interpolations to be used. The formulation is elucidated, and then several different elements are tested numerically. Particular attention is paid to the imposition of non-standard boundary conditions.

2 VARIATIONAL FORMULATION

2.1 Preliminaries

Consider a body Ω in $\mathbb{R}^n$, with boundary $\Gamma = \partial\Omega$. The outward normal vector to the body is denoted $\boldsymbol{n}$. The strong form of the equilibrium equation for the body Ω, in the absence of body forces, and associated standard boundary conditions, are given by:

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \qquad \text{in } \Omega \tag{1}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \qquad \text{on } \Gamma_h \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_g \tag{3}$$

where ∇ is the gradient operator, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{h}$ is the prescribed traction on Γ_h and $\boldsymbol{g}$ is the prescribed displacement on the boundary Γ_g ($\Gamma_g \cup \Gamma_h = \Gamma$, $\Gamma_g \cap \Gamma_h = \emptyset$).

The stress $\boldsymbol{\sigma}$ at a material point $\boldsymbol{x} \in \Omega$ is given by:

$$\boldsymbol{\sigma} = (1 - \omega)\, \mathcal{C} : \nabla^{\mathrm{s}} \boldsymbol{u} \tag{4}$$

where the scalar $\omega \in [0, 1]$ is the 'damage', $\mathcal{C}$ is the usual linear, isotropic elasticity tensor, and $\nabla^{\mathrm{s}}(\cdot)$ represents the symmetric gradient of $(\cdot)$. The damage ω is a function of a scalar history parameter κ, which in turn is related to a scalar 'equivalent strain' measure, $\bar{\epsilon}$. In a classical formulation, $\bar{\epsilon}$ is simply an invariant of the strain tensor. For a strain gradient dependent damage model, a gradient dependency (of the form proposed by Aifantis [1]) is introduced,

$$\bar{\epsilon} = \epsilon_{\mathrm{eq}} + c^2 \Delta \epsilon_{\mathrm{eq}} \tag{5}$$

where ϵ_{eq} is an invariant of the local strain tensor, and Δ is the Laplacian operator. For dimensional consistency, a length scale c is included. The chosen invariant ϵ_{eq} reflects the mechanical processes which drive damage in a particular material. Importantly, the form of equation (5) is common to a wide range of strain gradient-dependent continuum models. In this sense, the examined model can be considered a prototype for a range of different models.

The history parameter κ is equal to the largest (positive) value of $\bar{\epsilon}$ reached at a material point during loading. It is akin to the equivalent plastic strain, and its evolution obeys the well-known Kuhn-Tucker conditions,

$$\dot{\kappa} \geq 0, \qquad f \leq 0, \qquad \dot{\kappa} f = 0. \tag{6}$$

where f is a loading function, $f = \bar{\epsilon} - \kappa$. Both the invariant of the strain tensor ϵ_{eq}, and the dependency of the damage ω on κ reflect properties of the material being modelled. The dependency of ω on κ is typically complex. Upon insertion of the constitutive model, the fundamental problem is locally fourth-order (in regions where damage is developing). This requires higher-order boundary conditions. The most commonly accepted boundary condition is

$$\nabla \epsilon_{\mathrm{eq}} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma. \tag{7}$$

2.2 Galerkin formulation

For a linear fourth-order equation, the standard procedure is to integrate the governing equation by parts twice, leading to weak form which involves second-oder derivatives. The difficulty with a strain gradient-dependent continuum model is that the nonlinear nature of the governing equation and the complex dependency on the higher-order derivatives prevents the straightforward application of integration by parts to derive a weak from which involves second-order derivatives only. Hence, here both equations (1) and (5) are addressed, leading to a coupled set of equations.

In a conventional setting, a mixed approach for this type of problem still demands that one of the two interpolated fields be C^0 continuous, and the other C^1 (see, for example, de Borst and Mühlhaus [2]). A detailed discussion of this issue can be found in Wells et al. [10]. To avoid the need for C^1 continuity, which has led to serious numerical difficulties, a formulation is presented here which requires only C^0 continuity for the displacement field, and C^0 or C^{-1} for $\bar{\epsilon}$.

Before proceeding, it is necessary to consider a partition of the domain Ω into finite elements Ω_e such that

$$\bar{\Omega} = \bigcup_{e=1}^{n_{el}} \bar{\Omega}_e. \tag{8}$$

where $\bar{\Omega}_e$ is a closed set (i.e., it includes the boundary of the element). A domain $\tilde{\Omega}$ is also defined

$$\tilde{\Omega} = \bigcup_{e=1}^{n_{el}} \Omega_e \tag{9}$$

where $\tilde{\Omega}$ does not include element boundaries. It is also useful to define the 'interior' boundary $\tilde{\Gamma}$,

$$\tilde{\Gamma} = \bigcup_{i=1}^{n_b} \Gamma_i \tag{10}$$

where Γ_i is the ith interior element boundary and n_b is the number of internal inter-element boundaries. The function spaces $\mathcal{S}^h$, $\mathcal{V}^h$ and $\mathcal{W}^h$ are introduced:

$$\mathcal{S}^h = \left\{ u_i^h \in H_0^1(\Omega) \;\middle|\; u_i^h|_{\Omega_e} \in P_{k_1}(\Omega_e)\, \forall e,\ u_i = g_i \text{ on } \Gamma_g \right\} \tag{11}$$

$$\mathcal{V}^h = \left\{ w_i^h \in H_0^1(\Omega) \;\middle|\; w_i^h|_{\Omega_e} \in P_{k_1}(\Omega_e)\, \forall e,\ w_i = 0 \text{ on } \Gamma_g \right\} \tag{12}$$

$$\mathcal{W}^h = \left\{ q^h \in L_2(\Omega) \;\middle|\; q^h|_{\Omega_e} \in P_{k_2}(\Omega_e)\, \forall e \right\} \tag{13}$$

where P_k represents the space of polynomial finite element shape functions (of polynomial order k). The spaces $\mathcal{S}^h$ and $\mathcal{V}^h$ represent usual C^0 continuous finite element shape functions. The space $\mathcal{W}^h$ can contain discontinuous functions.

Inspired by work on discontinuous Galerkin formulations for second-order elliptic problems (a comprehensive review of which can be found in Arnold et al. [8]), the following Galerkin problem is considered: find $\boldsymbol{u}^h \in \mathcal{S}^h$ and $\bar{\epsilon}^h \in \mathcal{W}^h$ such that

$$\int_{\Omega} \nabla \boldsymbol{w}^h : \left(1 - \omega\left(\bar{\epsilon}^h\right)\right) \mathcal{C} : \nabla^{\mathrm{s}} \boldsymbol{u}^h \, d\Omega - \int_{\Gamma_h} \boldsymbol{w}^h \cdot \boldsymbol{h} \, d\Gamma = 0 \qquad \forall \boldsymbol{w}^h \in \mathcal{V}^h \tag{14}$$

$$\begin{aligned} \int_{\Omega} q^h \bar{\epsilon}^h \, d\Omega - \int_{\Omega} q^h \epsilon_{\mathrm{eq}}^h \, d\Omega + \int_{\tilde{\Omega}} \nabla q^h \cdot c^2 \nabla \epsilon_{\mathrm{eq}}^h \, d\Omega - \int_{\Gamma} q^h \nabla \epsilon_{\mathrm{eq}}^h \cdot \boldsymbol{n} \, d\Gamma \\ - \int_{\tilde{\Gamma}} [\![q^h]\!] \cdot c^2 \left\langle \nabla \epsilon_{\mathrm{eq}}^h \right\rangle d\Gamma - \int_{\tilde{\Gamma}} \left\langle \nabla q^h \right\rangle \cdot c^2 [\![\epsilon_{\mathrm{eq}}^h]\!] \, d\Gamma \\ + \int_{\tilde{\Gamma}} \frac{\alpha_1 c^2}{h_e} [\![q^h]\!] \cdot [\![\epsilon_{\mathrm{eq}}^h]\!] \, d\Gamma = 0 \quad \forall q^h \in \mathcal{W}^h \end{aligned} \tag{15}$$

where α_1 is a numerical parameter, h_e is a measure of element length, and, adopting the notation from Arnold et al. [8], the jump operator is defined as:

$$[\![\boldsymbol{a}]\!] = \boldsymbol{a}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{a}_2 \cdot \boldsymbol{n}_2 \tag{16}$$

and the average operator as:

$$\langle \boldsymbol{a} \rangle = \frac{\boldsymbol{a}_1 + \boldsymbol{a}_2}{2} \tag{17}$$

where the subscripts '1' and '2' denote sides of the surface across which the relevant quantities are being computed. This formulation is valid for the case in which damage does not reach Γ (as will be discussed in Section 2.3).

The above Galerkin formulation requires a C^0 interpolation of the displacement $\boldsymbol{u}^h$, and allows a discontinuous interpolation of $\bar{\epsilon}^h$. However, by requiring that $\bar{\epsilon}^h$ be C^0 continuous, a yet simpler formulation is possible, as the terms involving the jump in q^h vanish. The problem is then of the form: find $\boldsymbol{u}^h \in \mathcal{S}^h$ and $\bar{\epsilon}^h \in \mathcal{W}^h$ such that

$$\int_\Omega \nabla \boldsymbol{w}^h : \left(1 - \omega\left(\bar{\epsilon}^h\right)\right) \mathcal{C} : \nabla^{\mathrm{s}} \boldsymbol{u}^h \, d\Omega - \int_{\Gamma_h} \boldsymbol{w}^h \cdot \boldsymbol{h} \, d\Gamma = 0 \qquad \forall \boldsymbol{w}^h \in \mathcal{V}^h \tag{18}$$

$$\int_\Omega q^h \bar{\epsilon}^h \, d\Omega - \int_\Omega q^h \epsilon^h_{\mathrm{eq}} \, d\Omega + \int_{\tilde{\Omega}} \nabla q^h \cdot c^2 \nabla \epsilon^h_{\mathrm{eq}} \, d\Omega - \int_\Gamma q^h \nabla \epsilon^h_{\mathrm{eq}} \cdot \boldsymbol{n} \, d\Gamma \\ - \int_{\tilde{\Gamma}} \left\langle \nabla q^h \right\rangle \cdot c^2 \left[\!\left[\epsilon^h_{\mathrm{eq}}\right]\!\right] \, d\Gamma = 0 \qquad \forall q^h \in \mathcal{W}^h \tag{19}$$

This formulation will be termed the 'continuous formulation'. It has several advantages over the more general formulation. Firstly, fewer degrees of freedom are necessary at element interfaces (this also reduces the practical complexity of the implementation). Secondly, the penalty-like stabilisation term vanishes. The choice of the stabilisation term is one of the issues which leads to ambiguity in the development of discontinuous Galerkin methods for elliptic problems.

By applying integration by parts to the proposed weak forms, and after some manipulations, consistency with the strong governing equations can be proven [10].

2.3 Application of non-standard boundary conditions

An issue which requires special attention is the prescription of higher-order boundary conditions. The physical significance of these boundary conditions is an issue of vigorous debate, but is not of key importance here. Rather, of crucial importance is how a given boundary condition can be enforced. The application of higher-order boundary conditions in the previous section was not addressed, and is now considered.

The governing equation is fourth-order only in regions where damage is developing – elsewhere the problem is governed by a second-order PDE. Therefore, extra boundary conditions are required on the boundary of the damaging (fourth-order) regions. If these boundaries are internal, to Ω, boundary conditions do not need to be explicitly supplied as they are implied by continuity (or at least weak continuity) between the damaging and undamaged (or unloading) regions. Special attention is however required when the boundary of a damaging region coincides with the boundary of Ω. At this point, extra boundary conditions must be supplied on the boundary Γ_κ, which is defined as:

$$\Gamma_\kappa = \{\boldsymbol{x} \in \Gamma \mid \dot{\kappa} > 0\} \tag{20}$$

From the form of the specific problem considered, boundary conditions for $\nabla\epsilon_{\mathrm{eq}}$ or $\Delta\epsilon_{\mathrm{eq}}$ are required on Γ_κ. Choosing to apply a boundary condition for $\nabla\epsilon_{\mathrm{eq}}$ ($h_{\nabla\epsilon}$) on Γ_κ, the Galerkin problem in equation (14) is modified to: find $\boldsymbol{u}^h \in \mathcal{S}^h$ such that

$$\begin{aligned}\int_\Omega \nabla\boldsymbol{w}^h : \left(1-\omega\left(\bar{\epsilon}^h\right)\right)\mathcal{C} : \nabla^{\mathrm{s}}\boldsymbol{u}^h \, d\Omega + \int_{\Gamma_\kappa} \alpha_2 \nabla w^h_{\mathrm{eq}} \cdot \boldsymbol{n}\, Ec^2 \nabla\epsilon^h_{\mathrm{eq}} \cdot \boldsymbol{n} \, d\Gamma \\ = \int_{\Gamma_h} \boldsymbol{w}^h \cdot \boldsymbol{h} \, d\Gamma + \int_{\Gamma_\kappa} \alpha_2 \nabla w^h_{\mathrm{eq}} \cdot \boldsymbol{n}\, Ec^2 h_{\nabla\epsilon} \, d\Gamma \qquad \forall \boldsymbol{w}^h \in \mathcal{V}^h\end{aligned} \tag{21}$$

where α_2 is a penalty parameter. Equation (15) remains unaltered. The incorporation of boundary conditions on Γ_κ is a delicate issue and requires further careful study.

3 NUMERICAL PERFORMANCE

The proposed formulation is tested here for a series of simple tests. Nonlinear problems are linearised exactly and solved using a full Newton-Raphson procedure. The logical choice for ϵ_{eq} in one dimension is $\epsilon_{\mathrm{eq}} = \epsilon$. Details of the finite element formulation and linearisation can be found in Wells et al. [10]. The motivation behind the gradient-dependent damage model studied herein is the provision regularisation to avoid the well-known mesh dependency due to the classical (second-order) formulation being ill-posed upon the introduction of strain softening. Therefore, examination of the damaging bar focuses upon objectivity with respect to mesh refinement.

3.1 Elastic bar

The response of the proposed formulation is first examined for a quadratically tapering elastic bar in one dimension (Figure 1). The term $\bar{\epsilon}$ is composed of two components: one due to ϵ_{eq} and one due to $\epsilon_{\mathrm{eq},xx}$. The convergence of the proposed formulation in terms of $\epsilon_{,xx}$ is studied in this section. Effectively, the numerical model is reconstructing $\epsilon_{,xx}$ using the solution from the elastic equilibrium equation. For Young's modulus $E = 1$, the exact solution of $\varepsilon_{,xx}$ for the bar is:

$$\varepsilon_{,xx} = \frac{450000\left(27x^2 - 2700x + 65000\right)}{\left(9x^2 - 900x + 25000\right)^3} \tag{22}$$

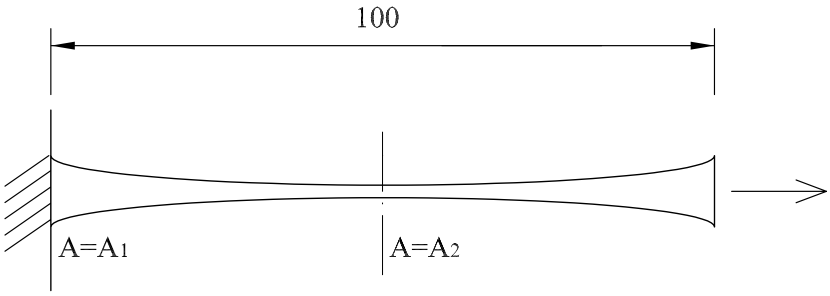


Figure 1: Quadratically tapering bar. $A_1 = 1$ and $A_2 = 0.1$.

The convergence rate for different elements is examined now by defining the error as:

$$e = \|\bar{\epsilon} - \varepsilon^{\text{exact}}_{,xx}\|_{\Omega} \tag{23}$$

The convergence results are shown in Figure 2. For elements which utilise a discontinuous interpolation for $\bar{\epsilon}$, $\alpha_1 = 1$. Numerically, all of the tested elements converge, although the discontinuous $\bar{\epsilon}$ elements perform less predictably. This is partly related to a less than optimal choice of α_1. The absence of the parameter α_1 is a distinct advantage of the continuous formulation. A number of issues in terms of the convergence rate are surprising and require further study.

3.2 Damaging bar

The formulation is now tested for a damaging bar. Damage does not reach the boundaries of the bar, hence no non-standard boundary conditions are supplied. The performance of an element based on a continuous, piecewise linear interpolation of u^h and discontinuous, piecewise constant $\bar{\epsilon}^h$ was studied in Wells et al. [10], where the formulation was shown to perform well against a benchmark solution. Here, the performance of some higher-order interpolations for the continuous formulation are examined.

A linearly tapering bar, similar to that in Figure 1, with unit area at the ends and an area of 0.8mm^2 at the centre, is now examined for three different elements with continuous u^h and continuous $\bar{\epsilon}^h$. The commonly adopted dependency:

$$\omega = \begin{cases} 0 & \text{if} \quad \kappa \leq \kappa_0 \\ 1 - \dfrac{\kappa_0 \left(\kappa_c - \kappa\right)}{\kappa \left(\kappa_c - \kappa_0\right)} & \text{if} \quad \kappa_0 < \kappa < \kappa_c \\ 1 & \text{if} \quad \kappa \geq \kappa_c \end{cases} \tag{24}$$

is applied, where κ_0 is the value of the history parameter at which damage begins to develop and κ_c is the value at which $\omega = 1$. The evolution of ω in equation (24) yields a

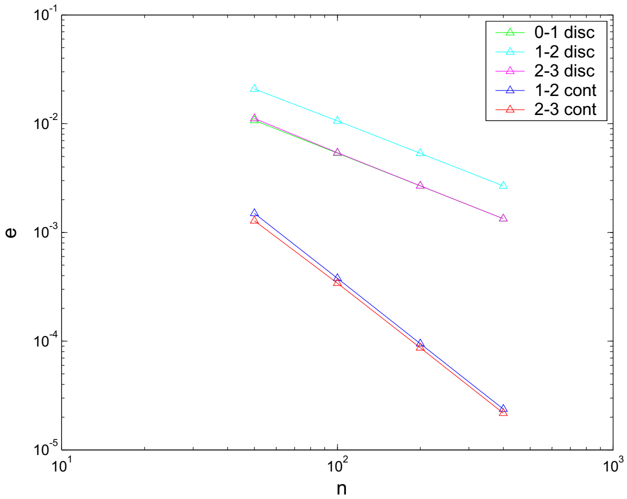


Figure 2: Convergence of different elements on the domain $10 < x < 90$ for the problem $\bar{\varepsilon} = \varepsilon_{,xx}$. In the legend, the first number represents the polynomial order for $\bar{\epsilon}^h$, and the second the polynomial order for u^h.

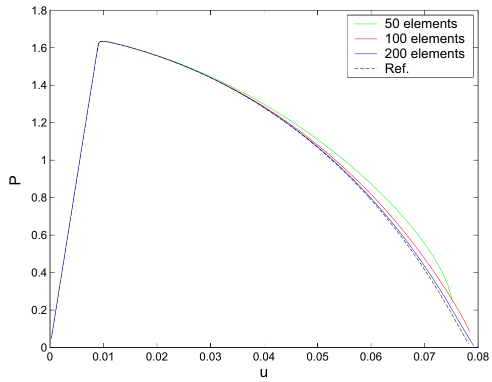
linear softening response for a uniaxial test in the absence of strain gradient effects. The adopted material parameters are: Young's modulus $E = 20 \times 10^3$ MPa, $\kappa_0 = 0.0001$,$\kappa_C = 0.0125$, and $c = 1$ mm.

The load–displacement responses for three different elements are shown in Figure 3. Elements are differentiated from each other on the basis of the interpolation order for u^h, the interpolation order for $\bar{\varepsilon}^h$. For example, an element with C^0 cubic shape functions for u^h and C^0 quadratic shape functions for $\bar{\varepsilon}^h$ is denoted $P^3/P^2(C^0)$. From the computed load–displacement responses for various meshes, it is clear that the formulation is objective with respect to spatial discretisation.

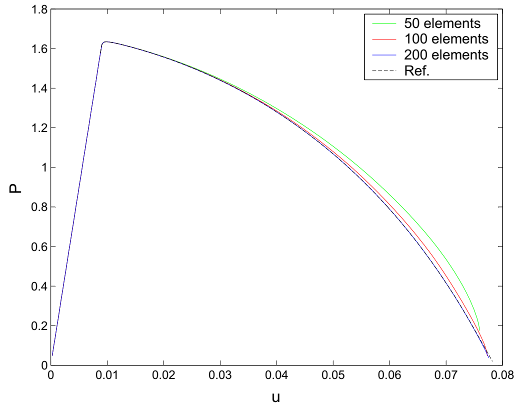
A study of discontinuous varieties of elements for this problem can be found in Molari et al. [11]. The discontinuous case is complicated by the presence of the numerical parameter α_1.

4 CONCLUSIONS

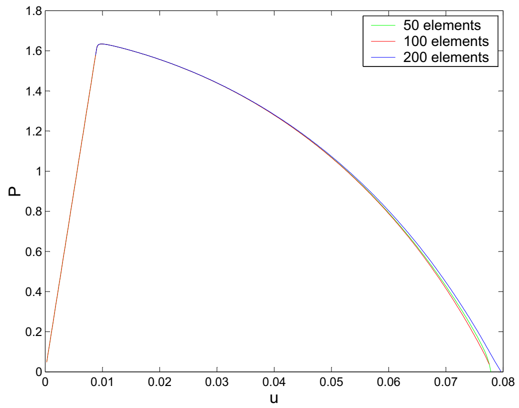
A Galerkin formulation has been presented for a strain gradient-dependent damage model. The formulation allows for discontinuities in derivatives of the displacement field, hence allowing the application of conventional C^0 finite element shape functions. Unlike previous formulations, it allows the rigorous use of C^0 finite element shape functions for all unknown fields. The effective performance of the formulation for continuous u^h and $\bar{\epsilon}^h$ in one dimension provides guidance for the extension to two dimensions which is currently underway. The relative simplicity of the 'continuous' approach makes it attractive for application in higher spatial dimensions.



(a)



(b)



(c)

Figure 3: Load–displacement response for (a) $P^2/P^1\ (C^0)$, (b) $P^3/P^1\ (C^0)$ and (c) $P^3/P^2\ (C^0)$ elements. The dashed line represents the reference $P^3/P^2\ (C^0)$ response with 200 elements.

While the results are promising for the prototype model examined, a number of important issues remain for the proposed formulation. While convergence has been shown numerically, a more detailed theoretical analysis would enhance confidence and possibly explain some unexpected results. A particularly sensitive issue remains the application of non-standard boundary conditions, which still requires further close examination and numerical testing.

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