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A unified approach to NNLO soft and virtual corrections in electroweak, Higgs, QCD, and SUSY processes

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Abstract

I present a unified approach to calculating the next-to-next-to-leading order (NNLO) soft and virtual QCD corrections to cross sections for electroweak, Higgs, QCD, and SUSY processes. I derive master formulas that can be used for any of these processes in hadron-hadron and lepton-hadron collisions. The formulas are based on a unified threshold resummation formalism and can be applied to both total and differential cross sections for processes with either simple or complex color flows and for various factorization schemes and kinematics. As a test of the formalism, I rederive known NNLO results for Drell-Yan and Higgs production, deep inelastic scattering, and $W^+\gamma$ production, and I obtain expressions for several two-loop anomalous dimensions and other quantities needed in next-to-next-to-leading-logarithm (NNLL) resummations. I also present new results for the production of supersymmetric charged Higgs bosons; massive electroweak vector bosons; photons; heavy quarks in lepton-hadron and hadron-hadron collisions and in flavor-changing neutral current processes; jets; and squarks and gluinos. The NNLO soft and virtual corrections are often dominant, especially near threshold. Thus, a unified approach to these corrections is important in the search for new physics at present and future colliders.

1 Introduction

Progress in theoretical particle physics, from electroweak theory and Quantum Chromodynamics in the Standard Model to supersymmetry and beyond, often involves the comparison of predictions of theory with data from high-energy hadron-hadron and lepton-hadron collisions. Recent theoretical advances include an array of resummations and a few next-to-next-to-leading order (NNLO) calculations [1, 2]. These advances are necessary in many cases where next-to-leading order (NLO) calculations are not accurate enough, since higher-order corrections reduce the scale dependence and increase theoretical accuracy.

Next-to-next-to-leading order calculations are technically very challenging and have been completed only for a few processes, including Drell-Yan [3, 4] and Higgs production [5, 6, 7] and deep inelastic scattering [8, 9]. The corrections are usually split into hard, soft, and virtual parts, corresponding to contributions from energetic, soft, and virtual gluons, respectively. The soft and virtual corrections are an important component of the total result both theoretically and numerically. In fact, in some schemes and kinematical regions, e.g. threshold, they are the dominant part. Highlighting their importance is the fact that for Drell-Yan [10] and Higgs [11, 12] production the NNLO soft and virtual corrections were presented before the full NNLO result was calculated.

We will see that there is a universality in the form of these corrections, which becomes more evident from the techniques of threshold resummations, which arise from factorization properties of the cross sections. Threshold corrections can be resummed to all orders, and finite-order expansions of resummed cross sections have provided us with many cross sections at NNLO and next-to-next-to-leading logarithm (NNLL) accuracy [13]. Thus it is a worthwhile aim to see if a unified approach can be given for the calculation of NNLO total and differential cross sections for any process in the Standard Model and beyond in hadron-hadron and hadron-lepton colliders and in various factorization schemes and kinematics. This will not only increase and deepen theoretical understanding but will also help avoid effort duplication in calculations of NNLO corrections for new processes. It is important to note that new particles, such as in supersymmetry, will likely be discovered near threshold where the soft and virtual corrections are important. The new unified approach to NNLO soft and virtual corrections is the topic of this paper.

The calculation of cross sections in hadron-hadron or lepton-hadron collisions can be written schematically as

$$\sigma=\sum_f \int \left[\prod_i dx_i\, \phi_{f/h_i}(x_i,\mu_F^2)\right]\, \hat{\sigma}(s,t_i,\mu_F,\mu_R)\,, \tag{1.1}$$

where σ is the physical cross section, ϕ_{f/h_i} is the distribution function for parton f carrying momentum fraction x_i of hadron h_i, at a factorization scale μ_F, while μ_R is the renormalization scale. The parton-level hard scattering cross section is denoted by $\hat{\sigma}$, and s and t_i are standard kinematical invariants. In a lepton-hadron collision we obviously have one parton distribution ($i=1$) while in a hadron-hadron collision $i=1,2$. We note here that σ and $\hat{\sigma}$ are not restricted to be total cross sections; they can represent any relevant differential cross section of interest.

In general, $\hat{\sigma}$ includes plus distributions $\mathcal{D}_l(x_{th})$ and delta functions $\delta(x_{th})$ with respect to a kinematical variable x_{th} that measures distance from threshold, with $l\le 2n-1$ at nth order in α_s beyond the leading order. These are the soft and virtual corrections. In single-particle inclusive (1PI) kinematics, x_{th} is usually called s_4 (or s_2), $s_4 = s + t + u - \sum m^2$, and vanishes at threshold. Then

$$\mathcal{D}_l(s_4) \equiv \left[\frac{\ln^l(s_4/M^2)}{s_4}\right]_+ = \frac{\ln^l(s_4/M^2)}{s_4}\theta(s_4-\Delta) + \frac{1}{l+1}\ln^{l+1}\left(\frac{\Delta}{M^2}\right)\delta(s_4)\,, \tag{1.2}$$

where Δ is a small parameter introduced in order to separate the hard $x_{th} > \Delta$ and soft $x_{th} < \Delta$ gluon regions, and M^2 is a hard scale relevant to the process at hand, for example the mass m of a heavy quark, the transverse momentum p_T of a jet, etc. In pair-invariant-mass (PIM) kinematics, with Q^2 the invariant mass squared of the produced pair, x_{th} is usually called $1-x$ or $1-z$, with $z = Q^2/s \to 1$ at threshold. Then

$$\mathcal{D}_l(z) \equiv \left[\frac{\ln^l(1-z)}{1-z}\right]_+ = \frac{\ln^l(1-z)}{1-z}\theta(1-z-\Delta) + \frac{1}{l+1}\ln^{l+1}(\Delta)\,\delta(1-z)\,. \tag{1.3}$$

The highest powers of these distributions at each order in α_s are the leading logarithms (LL), the second highest are the next-to-leading logarithms (NLL), the third highest are the next-to-next-to-leading logarithms (NNLL), etc. These logarithms can be resummed to all orders in perturbation theory. By now there are several processes for which NLL resummations and NNLO-NNLL results (i.e. the NNLL terms at NNLO) have been presented [13].

In this paper I will present master formulas for the NLO and NNLO soft and virtual corrections for any process in hadron-hadron and lepton-hadron collisions. In the next section, I present a threshold resummation formula, that builds on and unifies previous work [13, 14, 15, 16, 17, 18]. I then present master formulas for the NLO and NNLO soft and virtual corrections in processes with simple color flow, that arise from the expansion of the resummation formula and matching to NLO. The formulas cover both the $\overline{\text{MS}}$ and DIS schemes, and 1PI and PIM kinematics. In Section 3, I present results for electroweak/Higgs processes as well as QCD and SUSY processes with simple color flows. I rederive known NNLO results for the Drell-Yan process and Higgs production, as well as for deep inelastic scattering and $W^+\gamma$ production, thus obtaing expressions for two-loop anomalous dimensions and other quantities that are universal in quark-antiquark and gluon-gluon scattering and are needed for NNLL resummations. I also present new results for the NNLO corrections for supersymmetric charged Higgs production, W, Z plus jet production, direct photon production, DIS heavy quark production, as well as single-top quark production mediated by flavor-changing neutral currents. In Section 4, I extend the master formulas to processes with complex color flows and I present results for heavy quark hadroproduction, jet production, and squark and gluino production. I close with a discussion of extensions to higher orders and conclusions in Section 5.

2 NNLO master formula for soft and virtual corrections

2.1 Soft corrections from threshold resummation

In hadron-hadron collisions we study processes where partons f_i collide and produce a specific final state. The partonic processes are of the form

$$f_1(p_1) + f_2(p_2) \to F + X\,, \tag{2.1}$$

where F represents a system in the final state, and X any additional allowed final-state particles. So F can represent a pair of heavy quarks, or a single heavy quark, or jet, or photon, a Higgs boson, squarks, etc.

In lepton-hadron collisions the processes are of the form

$$f_1(p_1) + l(p_2) \to F + X \,. \tag{2.2}$$

In either case $s = (p_1 + p_2)^2$ and t_i are the usual Mandelstam invariants formed by the four-momenta of the particles in the scattering.

Resummed cross sections have by now been studied for a variety of processes [13]. The resummation of threshold logarithms is carried out in moment space. We define moments of the partonic cross section by $\hat{\sigma}(N) = \int dz\, z^{N-1} \hat{\sigma}(z)$ or by $\hat{\sigma}(N) = \int (ds_4/s)\; e^{-Ns_4/s} \hat{\sigma}(s_4)$, with N the moment variable. The resummed partonic cross section in moment space is then given by

$$\begin{aligned}
\hat{\sigma}^{res}(N) \;=\; & \exp\left[\sum_i E^{(f_i)}(N_i)\right] \, \exp\left[\sum_j E'^{(f_j)}(N_j)\right] \\
& \times \exp\left[\sum_i 2\int_{\mu_F}^{\sqrt{s}} \frac{d\mu'}{\mu'} \left(\frac{\alpha_s(\mu'^2)}{\pi}\gamma_i^{(1)} + {\gamma'}_{i/i}\left(\alpha_s(\mu'^2)\right)\right)\right] \, \exp\left[2\, d_{\alpha_s} \int_{\mu_R}^{\sqrt{s}} \frac{d\mu'}{\mu'} \beta\left(\alpha_s(\mu'^2)\right)\right] \\
& \times \mathrm{Tr}\left\{ H\left(\alpha_s(\mu_R^2)\right) \, \bar{P} \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu'}{\mu'} \, {\Gamma'}_S^{\dagger}\left(\alpha_s(\mu'^2)\right)\right] \, \tilde{S}\left(\alpha_s(s/\tilde{N}_j^2)\right) \right. \\
& \qquad \left. \times P \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu'}{\mu'} \, {\Gamma'}_S\left(\alpha_s(\mu'^2)\right)\right] \right\} \,.
\end{aligned} \tag{2.3}$$

The sums over i run over incoming partons: in hadron-hadron colisions we have two partons in the initial state, so $i = 1, 2$; in lepton-hadron collisions we only have one parton (and one lepton). The sum over j is relevant only if we have massless partons in the final state at lowest order. Clearly the second exponent is then absent for processes such as Drell-Yan, Higgs, and top quark pair production.

Equation (2.3) is actually valid for both 1PI and PIM kinematics with appropriate definitions for N_i and N_j. In 1PI kinematics $N_i = N(-t_i/M^2)$ for incoming partons i, and $N_j = N(s/M^2)$ for outgoing massless partons j; here M^2 is any chosen hard scale relevant to the process at hand. The kinematical invariants t_i are assigned through $S_4/S = s_4/s - \sum_i (1 - x_i) t_i/s$ [17, 19] with S_4, S, the hadronic analogs of s_4, s; note that the t_i may include a $-m^2$ in case of massive particles. In PIM kinematics $N_i = N_j = N$. Often the resummed cross sections are given with the choice $M^2 = s$ in which case $N_j = N$ even in 1PI kinematics; here we keep our expressions more general. Also note that $\tilde{N} = N e^{\gamma_E}$, with γ_E the Euler constant.

The first exponent in Eq. (2.3) resums the N-dependence of incoming partons [20, 21] and is given in the $\overline{\mathrm{MS}}$ scheme by [18]

$$E^{(f_i)}(N_i) = -\int_0^1 dz \frac{z^{N_i-1} - 1}{1-z} \left\{ \int_{(1-z)^2 s}^{\mu_F^2} \frac{d\mu'^2}{\mu'^2} A^{(f_i)}\left(\alpha_s(\mu'^2)\right) + \nu_{f_i}\left[\alpha_s((1-z)^2 s)\right] \right\} \,, \tag{2.4}$$

with $A^{(f_i)}(\alpha_s) = C_{f_i}[\alpha_s/\pi + (\alpha_s/\pi)^2 K/2] + \cdots$. Here $C_{f_i} = C_F = (N_c^2-1)/(2N_c)$ for an incoming quark or antiquark, and $C_{f_i} = C_A = N_c$ for an incoming gluon, with N_c the number of colors, while $K = C_A\,(67/18 - \pi^2/6) - 5n_f/9$, where n_f is the number of quark flavors. Also $\nu_{f_i} = (\alpha_s/\pi)C_{f_i} + (\alpha_s/\pi)^2\nu_{f_i}^{(2)} + \cdots$.

The second exponent resums the N-dependence of any outgoing massless partons and is given by [17, 19]

$$E'^{(f_j)}(N_j) = \int_0^1 dz \frac{z^{N_j-1}-1}{1-z}\left\{\int_{(1-z)^2}^{1-z} \frac{d\lambda}{\lambda} A^{(f_j)}\left(\alpha_s\left(\lambda s\right)\right) - B'_j\left[\alpha_s((1-z)s)\right] - \nu_j\left[\alpha_s((1-z)^2 s)\right]\right\}, \tag{2.5}$$

where $B'_j = (\alpha_s/\pi)B'^{(1)}_j + (\alpha_s/\pi)^2 B'^{(2)}_j + \cdots$ with $B'^{(1)}_q = 3C_F/4$ and $B'^{(1)}_g = \beta_0/4$. We will determine $B'^{(2)}_q$ in Section 3.

The $\gamma_i^{(1)}$ in the third exponent are one-loop parton anomalous dimensions: $\gamma_q^{(1)} = 3C_F/4$ and $\gamma_g^{(1)} = \beta_0/4$ for quarks and gluons, respectively; it is important to note that in this specific form that the resummed cross section has been written, with μ_F^2 as the upper limit of the integral over $d\mu'^2$ in $E^{(f_i)}(N_i)$, only the one-loop γ_i appears in the third exponent. We also have defined ${\gamma'}_{i/i}$ as the moment-space anomalous dimension of the $\overline{\mathrm{MS}}$ density $\phi_{i/i}$, minus it's one-loop and its N-dependent two-loop components. This is again due to the specific form that we use for the resummed cross section. Thus, ${\gamma'}^{(2)}_{i/i}$ is the N-independent part of the two-loop anomalous dimension $\gamma_{i/i}$ [22, 23] and is given for quarks and gluons by

$${\gamma'}^{(2)}_{q/q} = C_F^2\left(\frac{3}{32} - \frac{3}{4}\zeta_2 + \frac{3}{2}\zeta_3\right) + C_F C_A\left(-\frac{3}{4}\zeta_3 + \frac{11}{12}\zeta_2 + \frac{17}{96}\right) + n_f C_F\left(-\frac{\zeta_2}{6} - \frac{1}{48}\right), \tag{2.6}$$

and

$${\gamma'}^{(2)}_{g/g} = C_A^2\left(\frac{2}{3} + \frac{3}{4}\zeta_3\right) - n_f\left(\frac{C_F}{8} + \frac{C_A}{6}\right), \tag{2.7}$$

respectively, with $\zeta_2 = \pi^2/6$ and $\zeta_3 = 1.2020569\cdots$.

The β function in the fourth exponent is given by $\beta(\alpha_s) \equiv \mu\, d\ln g/d\mu = -\beta_0\alpha_s/(4\pi) - \beta_1\alpha_s^2/(4\pi)^2 + \cdots$, with $\beta_0 = (11C_A - 2n_f)/3$ and $\beta_1 = 34C_A^2/3 - 2n_f(C_F + 5C_A/3)$. The constant $d_{\alpha_s} = 0, 1, 2$ if the Born cross section is of order α_s^0, α_s^1, α_s^2, respectively.

The trace appearing in the resummed expression is taken in the space of color exchanges. The symbols P and $\bar{P}$ denote path ordering in the same sense as the variable μ' and against it, respectively. The evolution of the soft function from scale $\sqrt{s}/\tilde{N}_j$ to $\sqrt{s}$ follows from its renormalization group properties and is given in terms of the soft anomalous dimension matrix Γ_S [15, 16, 13]. In Eq. (2.3) we actually use ${\Gamma'}_S$, which is given by Γ_S after dropping all gauge-dependent terms. We can do that because the gauge dependence has been shown to cancel out. At one loop, the gauge terms are of the form $C_{f_i}\ln(2\nu_i^n)$, where $\nu_i^n = (v_i\cdot n)^2/|n|^2$ with v_i a velocity vector and n the axial gauge vector [13, 15, 16]. In processes with simple color flow, Γ_S is a trivial 1×1 matrix. For the determination of Γ_S in processes with complex color flow, an appropriate choice of color basis has to be made. For gluon-gluon scattering, the most complex color flow encountered, Γ_S is an 8×8 matrix [16]. The process-dependent soft anomalous dimension matrices, evaluated through the calculation of eikonal vertex corrections [15], have by now been presented at one loop for practically all partonic processes [13]. They can be explicitly calculated for any process using the techniques and results in Refs. [15, 13]. Work is currently being done on two-loop calculations of these anomalous dimensions [24], but we note that we can extract the universal component of these anomalous dimensions for quark-antiquark and gluon-gluon initiated processes from the Drell-Yan and Higgs NNLO results as detailed in Section 3. In the color bases that we normally use, the soft matrices, S, are diagonal. At lowest order, the trace of the product of the hard and soft matrices reproduces the Born cross section for each partonic process. We also note that the Γ_S matrices are in general not diagonal in the color bases that we use for complex color flows. If we perform a diagonalization so that the Γ_S matrices do become diagonal, then the path-ordered exponentials of matrices in the resummed expression reduce to simple exponentials; however, this diagonalization procedure is complicated in practice [13]. A finite-order expansion bypasses the need for this diagonalization procedure. For the discussion below we expand Γ'_S as $\Gamma'_S = (\alpha_s/\pi)\Gamma'^{(1)}_S + (\alpha_s/\pi)^2\Gamma'^{(2)}_S + \cdots$. We will determine the universal component of $\mathrm{Re}\Gamma'^{(2)}_S - \nu^{(2)}$, where "Re" stands for "real part of," for quark-antiquark and gluon-gluon processes in Section 3.

Finally, we note that there are different ways of writing threshold resummation formulas that have been presented in the past for various processes, all consistent or equivalent at NLL accuracy; the expression presented here unifies those expressions for arbitrary processes and is superior in its simplicity and generality.

In the DIS scheme, the resummed cross section may be written as

$$\hat{\sigma}^{res}_{\mathrm{DIS}}(N) = \hat{\sigma}^{res}_{\overline{\mathrm{MS}}}(N) \exp\left\{-\sum_i \int_0^1 dz \frac{z^{N_i-1}-1}{1-z} \left[\int_1^{1-z} \frac{d\lambda}{\lambda} A^{(f_i)}\left(\alpha_s(\lambda\mu_F^2)\right) - B_{f_i}\left[\alpha_s((1-z)\mu_F^2)\right]\right]\right\} , \tag{2.8}$$

where $\hat{\sigma}^{res}_{\overline{\mathrm{MS}}}$ is the $\overline{\mathrm{MS}}$ cross section in Eq. (2.3), and $B_{f_i} = (\alpha_s/\pi)B^{(1)}_{f_i} + (\alpha_s/\pi)^2 B^{(2)}_{f_i} + \cdots$. For quarks, to which the scheme is usually applied, $B^{(1)}_q = 3C_F/4$. We will determine $B^{(2)}_q$ in Section 3.

We first expand the resummed formulas in Eqs. (2.3) and (2.8) for processes with simple color flow, i.e. when H, S and Γ_S are 1×1 matrices, to next-to-leading order and present NLO master formulas in the $\overline{\mathrm{MS}}$ and DIS schemes, which reproduce the NLO results for a variety of processes. We then present NNLO master formulas for soft and virtual corrections in processes with simple color flow in both the $\overline{\mathrm{MS}}$ and DIS schemes.

2.2 NLO master formula for soft and virtual corrections - simple color flow

2.2.1 $\overline{\mathrm{MS}}$ scheme

At next-to-leading order, the expansion of Eq. (2.3) gives the NLO soft and virtual corrections in the $\overline{\mathrm{MS}}$ scheme:

$$\hat{\sigma}^{(1)} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3\, \mathcal{D}_1(x_{th}) + c_2\, \mathcal{D}_0(x_{th}) + c_1\, \delta(x_{th})\right\} , \tag{2.9}$$

where σ^B is the Born term,

$$c_3 = \sum_i 2C_{f_i} - \sum_j C_{f_j} , \tag{2.10}$$

and

$$c_2 = 2\,\mathrm{Re}\Gamma'^{(1)}_S - \sum_i \left[C_{f_i} + 2C_{f_i}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right) + C_{f_i}\ln\left(\frac{\mu_F^2}{s}\right)\right] - \sum_j \left[B'^{(1)}_j + C_{f_j} + C_{f_j}\,\delta_K\,\ln\left(\frac{M^2}{s}\right)\right]\,, \tag{2.11}$$

where $\mathrm{Re}\Gamma'^{(1)}_S$ is the real part of the one-loop Γ_S after dropping all gauge-dependent terms, and δ_K is 0 for PIM kinematics and 1 otherwise. We remind the reader that the sums over i run over incoming partons and the sums over j run over any massless partons in the final state at lowest order. For future use we will write $c_2 = c_2^\mu + T_2$, where c_2^μ represents the scale term $-\sum_i C_{f_i}\ln(\mu_F^2/s)$ and T_2 is the remainder. Also, $c_1 = c_1^\mu + T_1$, with

$$c_1^\mu = \sum_i \left[C_{f_i}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right) - \gamma_i^{(1)}\right]\ln\left(\frac{\mu_F^2}{s}\right) + d_{\alpha_s}\frac{\beta_0}{4}\ln\left(\frac{\mu_R^2}{s}\right)\,. \tag{2.12}$$

T_1 denotes the terms in c_1 that do not involve the factorization and renormalization scales, and it can be read off from a full calculation of the NLO virtual corrections for any specified process; a formal expression for these terms is given in Section 4.1 in the more general case of complex color flow.

We note that our formula passes a number of tests. As we will see, its predictions agree with exact NLO soft plus virtual results for all processes where those results are already available. Also the renormalization and factorization scale dependence in the physical cross section (after convoluting the partonic cross section with the parton distributions) cancels explicitly, i.e. $d\sigma/d\mu_F = 0$ and $d\sigma/d\mu_R = 0$ at NLO.

2.2.2 DIS scheme

In the DIS scheme, the NLO soft and virtual corrections are

$$\hat{\sigma}^{(1)}_{\rm DIS} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi}\left\{c'_3\,\mathcal{D}_1(x_{th}) + c'_2\,\mathcal{D}_0(x_{th}) + c'_1\,\delta(x_{th})\right\}\,, \tag{2.13}$$

with $c'_3 = c_3 - \sum_i C_{f_i}$,

$$c'_2 = c_2 + \sum_i \left[C_{f_i}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right) + B^{(1)}_{f_i}\right]\,, \tag{2.14}$$

and

$$c'_1 = c_1 - \sum_i \left[C_{f_i}\,\delta_K\,\frac{1}{2}\ln^2\left(\frac{-t_i}{M^2}\right) + B^{(1)}_{f_i}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right) - D_{f_i}\right]\,, \tag{2.15}$$

with c_3, c_2, c_1 the $\overline{\rm MS}$ results given in the previous subsection. For future use we will write $c'_2 = c'^{\,\mu}_2 + T'_2$, and $c'_1 = c'^{\,\mu}_1 + T'_1$ as we did for the $\overline{\rm MS}$ corrections. Note that the changes in going from the $\overline{\rm MS}$ scheme to the DIS scheme are all in the scale-independent parts of the c_i's, i.e. c_2^μ and c_1^μ remain unchanged while T_2 and T_1 (and c_3) are modified when changing schemes. The DIS scheme is normally applied to quarks. For quarks the term D_{f_i} in c'_1 is $D_q = C_F\zeta_2 + (9/4)C_F$. We note that our formula passes the same tests as we outlined in the previous subsection for the $\overline{\rm MS}$ scheme.

2.3 NNLO master formula for soft and virtual corrections - simple color flow

2.3.1 $\overline{\mathrm{MS}}$ scheme

At next-to-next-to-leading order, the expansion of Eq. (2.3), with matching to the full NLO soft-plus-virtual result, gives the NNLO soft and virtual corrections in the $\overline{\mathrm{MS}}$ scheme

$$\hat{\sigma}^{(2)} = \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \hat{\sigma'}^{(2)} \tag{2.16}$$

with

$$\begin{aligned}
\hat{\sigma'}^{(2)} &= \frac{1}{2}c_3^2\,\mathcal{D}_3(x_{th}) + \left[\frac{3}{2}c_3\,c_2 - \frac{\beta_0}{4}c_3 + \sum_j C_{f_j}\frac{\beta_0}{8}\right]\mathcal{D}_2(x_{th}) \\
&+ \left\{c_3\,c_1 + c_2^2 - \zeta_2\,c_3^2 - \frac{\beta_0}{2}\,T_2 + \frac{\beta_0}{4}c_3\ln\left(\frac{\mu_R^2}{s}\right) + \sum_i C_{f_i}\,K \right. \\
&\quad \left. + \sum_j C_{f_j}\left[-\frac{K}{2} + \frac{\beta_0}{4}\,\delta_K\,\ln\left(\frac{M^2}{s}\right)\right] - \sum_j \frac{\beta_0}{4}{B'}_j^{(1)}\right\}\mathcal{D}_1(x_{th}) \\
&+ \left\{c_2\,c_1 - \zeta_2\,c_2\,c_3 + \zeta_3\,c_3^2 - \frac{\beta_0}{2}T_1 + \frac{\beta_0}{4}\,c_2\ln\left(\frac{\mu_R^2}{s}\right) + 2\,\mathrm{Re}{\Gamma'}_S^{(2)} - \sum_i \nu_{f_i}^{(2)} \right. \\
&\quad + \sum_i C_{f_i}\left[\frac{\beta_0}{8}\ln^2\left(\frac{\mu_F^2}{s}\right) - \frac{K}{2}\ln\left(\frac{\mu_F^2}{s}\right) - K\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right)\right] - \sum_j\left({B'}_j^{(2)} + \nu_j^{(2)}\right) \\
&\quad \left. + \sum_j C_{f_j}\,\delta_K\left[\frac{\beta_0}{8}\ln^2\left(\frac{M^2}{s}\right) - \frac{K}{2}\ln\left(\frac{M^2}{s}\right)\right] - \sum_j\frac{\beta_0}{4}{B'}_j^{(1)}\,\delta_K\,\ln\left(\frac{M^2}{s}\right)\right\}\mathcal{D}_0(x_{th}) \\
&+ \left\{\frac{1}{2}c_1^2 - \frac{\zeta_2}{2}\,c_2^2 + \frac{1}{4}\zeta_2^2\,c_3^2 + \zeta_3\,c_3\,c_2 - \frac{3}{4}\zeta_4\,c_3^2 + \frac{\beta_0}{4}c_1\ln\left(\frac{\mu_R^2}{s}\right) + 2\,\mathrm{Re}{\Gamma'}_S^{(2)}\,\delta_K\,\ln\left(\frac{M^2}{s}\right)\right. \\
&\quad - \frac{\beta_0}{2}\,\delta_K\,T_1\ln\left(\frac{M^2}{s}\right) + \frac{\beta_0}{4}\,\delta_K\,T_2\ln^2\left(\frac{M^2}{s}\right) + \frac{d_{\alpha_s}}{16}\left[-\frac{\beta_0^2}{2}\ln^2\left(\frac{\mu_R^2}{s}\right) + \beta_1\ln\left(\frac{\mu_R^2}{s}\right)\right] \\
&\quad + \sum_i\frac{\beta_0}{8}\left[\gamma_i^{(1)} - C_{f_i}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right)\right]\ln^2\left(\frac{\mu_F^2}{s}\right) + \sum_i C_{f_i}\frac{K}{2}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right)\ln\left(\frac{\mu_F^2}{s}\right) \\
&\quad - \sum_i {\gamma'}_{i/i}^{(2)}\ln\left(\frac{\mu_F^2}{s}\right) + \sum_i C_{f_i}\,\delta_K\left[\frac{\beta_0}{6}\ln^3\left(\frac{-t_i}{M^2}\right) + \left(\frac{\beta_0}{4} + \frac{K}{2}\right)\ln^2\left(\frac{-t_i}{M^2}\right)\right] \\
&\quad + \sum_i C_{f_i}\frac{\beta_0}{2}\,\delta_K\,\ln\left(\frac{M^2}{s}\right)\left[\ln^2\left(\frac{-t_i}{M^2}\right) - \ln\left(\frac{-t_i}{M^2}\right)\ln\left(\frac{M^2}{s}\right) - \frac{1}{2}\ln\left(\frac{M^2}{s}\right) + \ln\left(\frac{-t_i}{M^2}\right)\right] \\
&\quad + \sum_i \nu_{f_i}^{(2)}\,\delta_K\,\ln\left(\frac{-t_i}{M^2}\right) - \sum_j\left({B'}_j^{(2)} + \nu_j^{(2)}\right)\,\delta_K\,\ln\left(\frac{M^2}{s}\right) \\
&\quad \left. + \sum_j\left[\frac{\beta_0}{8}C_{f_j}\ln\left(\frac{M^2}{s}\right) - \frac{K}{4}C_{f_j} - \frac{\beta_0}{8}{B'}_j^{(1)}\right]\,\delta_K\,\ln^2\left(\frac{M^2}{s}\right) + R\right\}\delta(x_{th})\,,
\end{aligned} \tag{2.17}$$

where the last term R, for which a formal expression is given in Section 4.1 in the more general case of complex color flow, can only be known from a full two-loop calculation. We will derive in Section 3 the universal components of R for processes with quark-antiquark and gluon-gluon collisions by comparing our predictions to the full NNLO corrections for Drell-Yan and Higgs production.

The quantities d_{α_s}, β_0, β_1, K, ζ_2, and ζ_3 have all been defined in Section 2.1, and $\zeta_4 = \pi^4/90$. We note that ${\rm Re}{\Gamma'}_S^{(2)}$ is the real part of the two-loop Γ_S after dropping all gauge-dependent terms. The universal part of $2{\rm Re}{\Gamma'}_S^{(2)} - \sum_i \nu_{f_i}^{(2)}$ in quark-antiquark and gluon-gluon collisions will be derived in Section 3 from Drell-Yan and Higgs production. ${B'}_q^{(2)}$ will also be derived in Section 3. There is also current work on full two-loop evaluations for general processes [24]. Also note, again, that the sum over i involves summing over the two incoming partons in hadron-hadron collisions, or one parton in lepton-hadron collisions, and that there is no sum over j if there are no massless partons in the final state at lowest order. Also note that δ_K in the formula again indicates which terms vanish in PIM kinematics. Finally, we note that the choice $M^2 = s$ further reduces the number of terms in the master formula in 1PI kinematics; however, we keep our expression as general as possible.

As we will see below, the NNLO master formula passes many rigorous tests. It reproduces the exact NNLO soft plus virtual results for Drell-Yan and Higgs production, and deep inelastic scattering, as well as the NNLO-NNLL results that have been derived already for many different processes from threshold resummation studies. Also I have checked explicitly that at NNLO the renormalization and factorization scale dependence in the physical cross section (after convoluting the partonic cross section with the parton distributions) cancels out for both hadron-hadron and lepton-hadron collisions.

2.3.2 DIS scheme

In the DIS scheme the NNLO soft and virtual corrections are

$$\hat{\sigma}^{(2)}_{\rm DIS} = \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \, \hat{\sigma'}^{(2)}_{\rm DIS} \tag{2.18}$$

with

$$\begin{aligned}
\hat{\sigma'}^{(2)}_{\rm DIS} &= \hat{\sigma'}^{(2)}|_{c'_i} - \sum_i \frac{\beta_0}{8} C_{f_i} \, \mathcal{D}_2(x_{th}) \\
&+ \left[\frac{\beta_0}{4}(T'_2 - T_2) - \sum_i C_{f_i} \frac{K}{2} + \sum_i C_{f_i} \frac{\beta_0}{4} \ln\left(\frac{\mu_F^2}{s}\right)\right] \mathcal{D}_1(x_{th}) \\
&+ \left\{\frac{\beta_0}{4}(T'_1 - T_1) + \sum_i C_{f_i}\, \delta_K \, \frac{K}{2} \ln\left(\frac{-t_i}{M^2}\right) - \frac{\beta_0}{4} \ln\left(\frac{\mu_F^2}{s}\right)(T'_2 - T_2) + \sum_i B_{f_i}^{(2)}\right\} \mathcal{D}_0(x_{th}) \\
&+ \left\{-\sum_i \frac{\beta_0}{24} C_{f_i}\, \delta_K \, \ln^3\left(\frac{-t_i}{M^2}\right) - \sum_i C_{f_i}\, \delta_K \, \frac{K}{4} \ln^2\left(\frac{-t_i}{M^2}\right) - \sum_i B_{f_i}^{(1)}\, \delta_K \, \frac{\beta_0}{8} \ln^2\left(\frac{-t_i}{M^2}\right)\right. \\
&\quad - \frac{\beta_0}{4} \ln\left(\frac{\mu_F^2}{s}\right)(T'_1 - T_1) + \frac{\beta_0}{2}\, \delta_K \, (T'_1 - T_1) \ln\left(\frac{M^2}{s}\right) + \sum_i \frac{\beta_0}{12} C_{f_i}\, \delta_K \, \ln^3\left(\frac{M^2}{s}\right) \\
&\quad \left. - \frac{\beta_0}{4}\, \delta_K \, (T'_2 - T_2) \ln^2\left(\frac{M^2}{s}\right) - \sum_i B_{f_i}^{(2)}\, \delta_K \, \ln\left(\frac{-t_i}{M^2}\right)\right\} \delta(x_{th}) \,.
\end{aligned} \tag{2.19}$$

We note that the T_i''s are the DIS quantities while the unprimed T_i's are the $\overline{\rm MS}$ counterparts. Also $\hat{\sigma}'^{(2)}|_{c_i'}$ denotes the cross section in Eq. (2.17) after replacing all the c_i, T_i, and R, by their DIS counterparts c_i', T_i', R'. Our formula reproduces the exact known NNLO corrections for Drell-Yan and $W^+\gamma$ production and deep inelastic scattering in the DIS scheme, as we will see in the next section. We will derive the expression for $B_q^{(2)}$ in the next section by matching to the known NNLO corrections for the Drell-Yan process. We also note that again we have checked the scale independence of the physical cross section at NNLO.

3 Electroweak/Higgs, QCD, and SUSY processes with simple color flows

We now apply our NLO and NNLO master formulas to a variety of processes with simple color flow which are of electroweak, QCD, or SUSY origin at lowest order.

3.1 The Drell-Yan process

Our first application is the Drell-Yan process, i.e. lepton pair production in hadron-hadron collisions, for which the NNLO corrections have been calculated in Refs. [3, 4, 10]. Also a comparison of the expansion of the resummed cross section in [20] with the NNLO corrections in [10] was presented in Ref. [25]. The partonic process we discuss is $q\bar{q} \to V + X$, where V is a vector boson (γ, Z, W) which later decays to a lepton pair $V \to l_1 l_2$ with invariant mass $\sqrt{Q^2}$, and X denotes any additional partons in the final state. At threshold $s = Q^2$, where s is the center-of-mass energy squared of the incoming quark-antiquark pair. The NNLO corrections for the cross section $d\sigma/dQ^2$ in the $\overline{\rm MS}$ scheme are given in Ref. [3]. The soft and virtual corrections are given the label $\Delta_{q\bar{q}}^{(n),S+V}$ and are given explicitly at NLO by Eq. (B.3) in Ref. [3], and at NNLO by Eq. (B.8) plus the renormalization scale term in Eq. (B.7) of Ref. [3]. Here the plus distributions are $D_l(x)$ with $x = Q^2/s$.

We are able to reproduce these results using our master NLO and NNLO formulas. Evidently at lowest order there are no final-state massless partons and the cross section is given in PIM kinematics. We note that for this process ${\rm Re}\Gamma_S'^{(1)} = C_F$ and we choose the hard scale $M^2 = Q^2$.

At NLO our master formula, Eq. (2.9), gives

$$\hat{\sigma}^{(1)}_{q\bar{q}\to V} = \sigma^B_{q\bar{q}\to V} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(x) + c_2 \mathcal{D}_0(x) + c_1 \delta(1-x)\right\} \tag{3.1}$$

with $c_3 = 4C_F$,

$$c_2 = -2C_F \ln\left(\frac{\mu_F^2}{Q^2}\right), \qquad c_1^\mu = -\frac{3}{2} C_F \ln\left(\frac{\mu_F^2}{Q^2}\right), \tag{3.2}$$

which reproduces Eq. (B.3) in Ref. [3], and we identify the non-scale $\delta(1-x)$ terms as $T_1 = 2C_F\zeta_2 - 4C_F$.

At NNLO our master formula, Eq. (2.17), reproduces the $D_3(x), D_2(x), D_1(x), D_0(x)$, and $\delta(1-x)$ terms. Note that our results use explicitly the beta function β_0, thus simplifying the expression in Eq. (B.8) of [3]. We also note that we use the full NNLO soft-plus-virtual result in [3] to derive the two-loop soft anomalous dimension, that appears in the $D_0(x)$ term in our master formula, for the Drell-Yan process :

$$\mathrm{Re}\Gamma'^{(2)}_{S,q\bar{q}} - \nu_q^{(2)} = C_F C_A \left(\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{299}{54}\right) + n_f C_F \left(-\frac{2}{3}\zeta_2 + \frac{25}{27}\right) . \tag{3.3}$$

Note that this term is universal for all processes with quark-antiquark annihilation. We also determine for the Drell-Yan process in the $\overline{\mathrm{MS}}$ scheme the term R that appears in the $\delta(1-x)$ terms in our master formula and which is also universal in all $q\bar{q}$ processes:

$$\begin{aligned} R_{q\bar{q}} &= C_F^2 \left(-\frac{59}{10}\zeta_2^2 + \frac{29}{8}\zeta_2 - \frac{15}{4}\zeta_3 + 12\zeta_4 - \frac{1}{64}\right) + C_F C_A \left(-\frac{3}{20}\zeta_2^2 + \frac{37}{9}\zeta_2 + \frac{7}{4}\zeta_3 - \frac{1535}{192}\right) \\ &\quad + n_f C_F \left(\frac{\zeta_3}{2} - \frac{7}{9}\zeta_2 + \frac{127}{96}\right) . \end{aligned} \tag{3.4}$$

The NNLO soft and virtual corrections for Drell-Yan production have also been calculated in the DIS scheme in Ref. [4]. Using our NLO formula in the DIS scheme, Eq. (2.13), we find $c_3' = 2C_F$, $c_2' = 3C_F/2 - 2C_F \ln(\mu_F^2/Q^2)$, $c'^{\mu}_1 = -(3/2)C_F \ln(\mu_F^2/Q^2)$, and $T_1' = 4C_F\zeta_2 + C_F/2$, which agress with Eq. (A.3) in [4].

Using our NNLO master formula in the DIS scheme, Eq. (2.18), we are also able to rederive the NNLO result in Eq. (A.8) (plus the renormalization scale terms in Eq. (A.7)) of Ref. [4], and thus we identify the $B_q^{(2)}$ term in the DIS scheme master formula:

$$B_q^{(2)} = C_F^2 \left(\frac{3}{32} - \frac{3}{4}\zeta_2 + \frac{3}{2}\zeta_3\right) + C_F C_A \left(-\frac{5}{2}\zeta_3 + \frac{4937}{864}\right) - \frac{409}{432} n_f C_F . \tag{3.5}$$

Finally, we also determine for the Drell-Yan process in the DIS scheme the term R' that appears in the $\delta(1-x)$ term in our master formula:

$$\begin{aligned} R'_{q\bar{q}} &= C_F^2 \left(-\frac{43}{20}\zeta_2^2 - \frac{17}{16}\zeta_2 + \frac{9}{2}\zeta_3 + 3\zeta_4 - \frac{1}{8}\right) + C_F C_A \left(-\frac{77}{40}\zeta_2^2 + \frac{1049}{72}\zeta_2 - \frac{49}{12}\zeta_3 + \frac{215}{144}\right) \\ &\quad + n_f C_F \left(\frac{\zeta_3}{3} - \frac{85}{36}\zeta_2 - \frac{19}{72}\right) . \end{aligned} \tag{3.6}$$

This is also universal in all $q\bar{q}$ processes in the DIS scheme, as we will verify below for $W^+\gamma$ production.

3.2 Standard Model Higgs production

Our next application is Higgs production in the Standard Model in hadron-hadron collisions, for which the full NNLO corrections, using an effective Lagrangian for the Higgs-gluon interaction, have been calculated in Refs. [5, 6, 7]. The partonic process we discuss is $gg \to H + X$, where H is the Higgs boson. At threshold $s = M_H^2$, where s is the center-of-mass energy squared of the incoming gluon pair and M_H is the Higgs mass. Evidently at lowest order there are no final-state massless partons and the cross section is given in PIM kinematics. Here the plus distributions are $D_l(x)$ with $x = M_H^2/s$, $\mathrm{Re}\Gamma'^{(1)}_S = C_A$, and we choose the hard scale $M^2 = M_H^2$. The soft and virtual corrections for the total cross section $\sigma_{gg\to H}$ are given explicitly at NNLO in Refs. [5, 6, 7, 11, 12]. We reproduce and generalize those results by keeping the factorization and renormalization scales distinct, using the beta function β_0 explicitly, and keeping the color factors C_A explicit in our results.

At NLO, our $\overline{\rm MS}$ scheme master formula gives

$$\hat{\sigma}^{(1)}_{gg\to H} = \sigma^B_{gg\to H} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{ c_3 \mathcal{D}_1(x) + c_2 \mathcal{D}_0(x) + c_1 \delta(1-x) \right\} \tag{3.7}$$

with $c_3 = 4C_A$,

$$c_2 = -2C_A \ln\left(\frac{\mu_F^2}{M_H^2}\right), \qquad c_1^\mu = \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{\mu_F^2}\right), \tag{3.8}$$

which reproduces the results in [5, 6, 7, 11, 12]. We also identify $T_1 = 11/2 + 2C_A\zeta_2$. At NNLO, our master formula reproduces the $D_3(x), D_2(x), D_1(x), D_0(x)$, and $\delta(1-x)$ terms. We note that we use the full NNLO soft-plus-virtual result in these references to derive the two-loop soft anomalous dimension, that appears in the $D_0(x)$ term in our master formula, for Higgs production:

$${\rm Re}\Gamma'^{(2)}_{S,gg} - \nu_g^{(2)} = C_A^2 \left(\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{41}{216}\right) + n_f C_A \left(-\frac{2}{3}\zeta_2 - \frac{5}{108}\right). \tag{3.9}$$

This anomalous dimension is universal for all processes with gluon-gluon fusion.

We also determine for Higgs production in the $\overline{\rm MS}$ scheme the term R that appears in the $\delta(1-x)$ terms in our master formula and which is also universal in all gg processes:

$$R_{gg} = \frac{9221}{144} + \frac{67}{2}\zeta_2 - \frac{1089}{20}\zeta_2^2 - \frac{165}{4}\zeta_3 + 108\,\zeta_4 + n_f\left(-\frac{1189}{144} - \frac{5}{3}\zeta_2 + \frac{5}{6}\zeta_3\right). \tag{3.10}$$

3.3 Deep inelastic scattering

Our methods can also be applied to the coefficient functions in deep inelastic scattering, $\gamma^* q \to q$, where the distributions are $D_l(z)$. We note that here we have a massless parton (quark) in the final state. In the $\overline{\rm MS}$ scheme, our NLO formula gives: $c_3 = C_F$, $c_2 = -3C_F/4 - C_F \ln(\mu_F^2/Q^2)$, and $c_1 = -(3C_F/4)\ln(\mu_F^2/Q^2) - C_F\zeta_2 - 9C_F/4$. The NNLO corrections are then given by our NNLO master formula. Our NLO and NNLO corrections agree with the results for the coefficient functions in Refs. [8, 9] (see Appendix B in [8] and Appendix A in [9]) and we identify the two-loop B'_q that appears in the $D_0(x)$ terms in our master formula:

$$B'^{(2)}_q = C_F^2\left(\frac{3}{32} - \frac{3}{4}\zeta_2 + \frac{3}{2}\zeta_3\right) + C_F C_A\left(\zeta_3 + \frac{55}{12}\zeta_2 + \frac{319}{864}\right) + n_f C_F\left(-\frac{5}{6}\zeta_2 - \frac{59}{432}\right). \tag{3.11}$$

I have checked that in the DIS scheme the NLO and NNLO corrections that do not involve the scale vanish, as expected (after all this is the definition of the DIS scheme, that the corrections to deep inelastic scattering in that scheme vanish). This involves checking that $2\Gamma'^{(2)}_{S,qq} - 2\nu_q^{(2)} - B'^{(2)}_q + (\beta_0/4)D_q + B_q^{(2)} = 0$ which is a further test of the correctness of the expressions for the various two-loop quantities that we have derived.

3.4 $W^+\gamma$ production

We now discuss the cross section $s^2 d^2\hat{\sigma}/(dt\,du)$ for $W^+\gamma$ production in $p\bar{p}$ collisions for which the NNLO soft-plus-virtual corrections in the DIS scheme have been presented in Ref. [26]. The lowest-order partonic process is $q\bar{q} \to W^+\gamma$ and ${\rm Re}\Gamma'^{(1)}_S = C_F$. We define $s_4 = s + t + u - m_W^2$, with $s = (p_q + p_{\bar{q}})^2$, $t = (p_q - p_\gamma)^2$, $u = (p_{\bar{q}} - p_\gamma)^2$, and $t_1 = t - m_W^2$, $u_1 = u - m_W^2$. The plus distributions are $D_l(s_4)$, and we choose $M^2 = s$.

In the DIS scheme at NLO, our master formula gives

$$\hat{\sigma}^{(1)}_{q\bar{q}\to W^+\gamma} = \sigma^B_{q\bar{q}\to W^+\gamma} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3' \mathcal{D}_1(s_4) + c_2' \mathcal{D}_0(s_4) + c_1' \delta(s_4)\right\} \tag{3.12}$$

with $c_3' = 2C_F$,

$$c_2' = \frac{3}{2}C_F - C_F \ln\left(\frac{t_1 u_1}{s^2}\right) - 2C_F \ln\left(\frac{\mu_F^2}{s}\right), \tag{3.13}$$

and

$$c'^{\mu}_1 = C_F\left[-\frac{3}{2} + \ln\left(\frac{t_1 u_1}{s^2}\right)\right] \ln\left(\frac{\mu_F^2}{s}\right). \tag{3.14}$$

Our NLO expansion agrees with Eq. (3.2) in [26] and we identify

$$T_1' = \frac{1}{2}C_F \ln^2\left(\frac{-t_1}{s}\right) + \frac{1}{2}C_F \ln^2\left(\frac{-u_1}{s}\right) - \frac{3}{4}C_F \ln\left(\frac{t_1 u_1}{s^2}\right) + 4C_F\zeta_2 + \frac{C_F}{2}. \tag{3.15}$$

Using our NNLO master formula in the DIS scheme, we are also able to rederive the NNLO soft and virtual corrections for that process in Eqs. (B1) and (B2) of Ref. [26]. Since this is the only process previously calculated at NNLO not involving PIM kinematics, and hence the $\ln(-t_i/M^2)$ terms are explicit, this re-derivation provides an additional highly non-trivial test of our NNLO master formula.

Finally, we note that we can easily derive the NNLO corrections for this process in the $\overline{\rm MS}$ scheme for which no results have been given in the literature. Now, $c_3 = 4C_F$, $c_2 = -2C_F \ln(t_1u_1/s^2) - 2C_F \ln(\mu_F^2/s)$, $c_1^\mu = c'^\mu_1$, and $T_1 = C_F \ln^2(-t_1/s) + C_F \ln^2(-u_1/s) + 2C_F\zeta_2 - 4C_F$. The NNLO corrections in the $\overline{\rm MS}$ scheme are then given by our master formula.

3.5 Charged Higgs production

We now present new results for processes for which no full NNLO calculations have ever been done.

We first consider the production of a charged Higgs boson in the Minimal Supersymmetric Standard Model, for which there have been recent NLO calculations [27, 28]. The lowest-order partonic process is $\bar{b}(p_{\bar{b}}) + g(p_g) \to H^+(p_{H^+}) + \bar{t}(p_{\bar{t}})$. Again there are no final-state massless partons and in this case we work in 1PI kinematics. Here $\hat{\sigma}$ can stand, for example, for $E_{H^+}\, d\sigma/d^3p_{H^+}$. The relevant hard scale we choose is $M = m_{H^+}$ and we define the Mandelstam invariants $s = (p_{\bar{b}} + p_g)^2$, $t = (p_{H^+} - p_{\bar{b}})^2$, and $u = (p_{H^+} - p_g)^2$. The threshold variable is $s_2 = s + t + u - m_{H^+}^2 - m_{\bar{t}}^2 - m_{\bar{b}}^2$ and the plus distributions are $D_l(s_2)$. Also, $t_g = t_1 = t - m_t^2$ and $t_{\bar{b}} = u_1 = u - m_t^2$. The one-loop soft anomalous dimension is ${\rm Re}\Gamma'^{(1)}_S = C_F[\ln(-u_1/s) + (1/2)\ln(s/m_t^2)] + (C_A/2)[\ln(t_1/u_1) + 1]$.

The NLO soft-plus-virtual corrections in the $\overline{\rm MS}$ scheme are

$$\hat{\sigma}^{(1)}_{\bar{b}g\rightarrow H^+\bar{t}}=\sigma^{B}_{\bar{b}g\rightarrow H^+\bar{t}}\frac{\alpha_s(\mu_R^2)}{\pi}\left\{c_3\mathcal{D}_1(s_2)+c_2\mathcal{D}_0(s_2)+c_1\delta(s_2)\right\} \tag{3.16}$$

with $c_3=2(C_F+C_A)$,

$$c_2=C_F\left[\ln\left(\frac{m_H^4}{s m_t^2}\right)-1-\ln\left(\frac{\mu_F^2}{s}\right)\right]+C_A\left[\ln\left(\frac{m_H^4}{t_1 u_1}\right)-\ln\left(\frac{\mu_F^2}{s}\right)\right], \tag{3.17}$$

and

$$c_1^{\mu}=\left[C_F\ln\left(\frac{-u_1}{m_H^2}\right)+C_A\ln\left(\frac{-t_1}{m_H^2}\right)-\frac{3C_F}{4}-\frac{\beta_0}{4}\right]\ln\left(\frac{\mu_F^2}{s}\right)+\frac{\beta_0}{4}\ln\left(\frac{\mu_R^2}{s}\right). \tag{3.18}$$

The NNLO soft and virtual corrections are then explicitly given by our master formula.

3.6 W, Z plus jet production

W, Z plus jet production in hadron colliders has been studied at NLO in Refs. [29, 30], and a resummed cross section and NNLO-NNLL corrections have been presented in Ref. [19]. Here we follow the notation of Ref. [19] and discuss the 1PI cross section $E_Q\, d\sigma/d^3Q$ with Q the momentum of the electroweak boson. At lowest order there are two partonic processes, $q(p_a)+g(p_b)\rightarrow q(p_c)+V(Q)$ and $q(p_a)+\bar{q}(p_b)\rightarrow g(p_c)+V(Q)$, where V stands for W or Z. We note that we have final-state massless partons in both processes. We define the kinematic invariants $s=(p_a+p_b)^2$, $t=(p_a-Q)^2$, and $u=(p_b-Q)^2$. We choose $M^2=Q^2$, and the plus distributions are $D_l(s_2)$ with $s_2=s+t+u-Q^2$. We discuss the $\overline{\rm MS}$ corrections for the two partonic processes in turn.

3.6.1 $q\bar{q}\rightarrow gV$

Here $t_q=u$, $t_{\bar{q}}=t$, and ${\rm Re}\Gamma'^{(1)}_S=C_F+(C_A/2)\ln(tu/s^2)+C_A/2$ [19]. The NLO soft and virtual corrections are

$$\hat{\sigma}^{(1)}_{q\bar{q}\rightarrow gV}=\sigma^{B}_{q\bar{q}\rightarrow gV}\frac{\alpha_s(\mu_R^2)}{\pi}\left\{c_3\mathcal{D}_1(s_2)+c_2\mathcal{D}_0(s_2)+c_1\delta(s_2)\right\} \tag{3.19}$$

with $c_3=4C_F-C_A$,

$$c_2=-\frac{\beta_0}{4}-2C_F\ln\left(\frac{\mu_F^2}{Q^2}\right)-(2C_F-C_A)\ln\left(\frac{tu}{sQ^2}\right), \tag{3.20}$$

and

$$c_1^{\mu}=C_F\left[-\frac{3}{2}+\ln\left(\frac{tu}{Q^4}\right)\right]\ln\left(\frac{\mu_F^2}{s}\right)+\frac{\beta_0}{4}\ln\left(\frac{\mu_R^2}{s}\right). \tag{3.21}$$

These are in agreement with the NLO result in [30].

3.6.2 $qg \to qV$

Here $t_q = u$, $t_g = t$, and $\mathrm{Re}\Gamma'^{(1)}_S = C_F \ln(-u/s) + C_F + (C_A/2)\ln(t/u) + C_A/2$ [19]. The NLO soft and virtual corrections are

$$\hat{\sigma}^{(1)}_{qg\to qV} = \sigma^B_{qg\to qV} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(s_2) + c_2 \mathcal{D}_0(s_2) + c_1 \delta(s_2)\right\} \tag{3.22}$$

with $c_3 = C_F + 2C_A$,

$$c_2 = -\frac{3}{4}C_F - (C_F + C_A)\ln\left(\frac{\mu_F^2}{Q^2}\right) - C_A \ln\left(\frac{tu}{sQ^2}\right), \tag{3.23}$$

and

$$c_1^\mu = \left[-\frac{\beta_0}{4} - \frac{3}{4}C_F + C_F \ln\left(\frac{-u}{Q^2}\right) + C_A \ln\left(\frac{-t}{Q^2}\right)\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{4}\ln\left(\frac{\mu_R^2}{s}\right). \tag{3.24}$$

They are in agreement with the NLO result in [30].

For both partonic subprocesses the NNLO corrections are derived from our master formula and are in agreement with the NNLO-NNLL results in Ref. [19] (apart from a term $\sigma^B[\alpha_s^2(\mu_R^2)/\pi^2](\beta_0/4)c_3\ln(Q^2/s)\mathcal{D}_1$ that was missing in that reference).

3.7 Direct photon production

Direct photon production is often recognised as a process that can aid determinations of the gluon distribution. The NLO cross section for direct photon production has been given in Refs. [31, 32]. At lowest order, the parton-parton scattering subprocesses are $q(p_a) + g(p_b) \to \gamma(p_\gamma) + q(p_J)$ and $q(p_a) + \bar{q}(p_b) \to \gamma(p_\gamma) + g(p_J)$, so there are final-state massless partons in both subprocesses. We define the Mandelstam invariants $s = (p_a + p_b)^2$, $t = (p_a - p_\gamma)^2$, and $u = (p_b - p_\gamma)^2$, which satisfy $s_4 \equiv s+t+u = 0$ at threshold. Here we choose $M^2 = p_T^2 = tu/s$, and we work in 1PI kinematics in the $\overline{\mathrm{MS}}$ scheme with the cross section $E_\gamma\, d^3\sigma/d^3p_\gamma$. The threshold logarithms $D_l(s_4)$ have been resummed and NNLO-NNLL corrections have been presented in Ref. [33].

3.7.1 $q\bar{q} \to \gamma g$

We start with the process $q\bar{q} \to \gamma g$ for which $t_q = u$, $t_{\bar{q}} = t$, and $\mathrm{Re}\Gamma'^{(1)}_S = C_F + (C_A/2)\ln(tu/s^2) + C_A/2$. The NLO soft plus virtual corrections are

$$\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma g} = \sigma^B_{q\bar{q}\to\gamma g} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(s_4) + c_2 \mathcal{D}_0(s_4) + c_1 \delta(s_4)\right\} \tag{3.25}$$

with $c_3 = 4C_F - C_A$,

$$c_2 = -\frac{\beta_0}{4} - 2C_F \ln\left(\frac{\mu_F^2}{p_T^2}\right), \tag{3.26}$$

and $c_1 = c_1^\mu + T_1$ with

$$c_1^\mu = C_F \left[-\frac{3}{2} - \ln\left(\frac{p_T^2}{s}\right)\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right) . \tag{3.27}$$

The term T_1 is given in Eq. (3.11) of Ref. [33] (where it is called $c_1'^{q\bar{q}}$). The NNLO soft plus virtual corrections are then explicitly given by our master formula. We note that the terms $D_3(s_4)$, $D_2(s_4)$, and $D_1(s_4)$ were already derived from a resummation study in Section III of Ref. [33] and are in agreement with our formula.

3.7.2 $qg \to \gamma q$

We continue with the process $qg \to \gamma q$ for which $t_q = u$, $t_g = t$, and $\mathrm{Re}\Gamma'^{(1)}_S = C_F \ln(-u/s) + C_F + (C_A/2)\ln(t/u) + C_A/2$. The NLO soft plus virtual corrections are

$$\hat{\sigma}^{(1)}_{qg\to\gamma q} = \sigma^B_{qg\to\gamma q} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(s_4) + c_2 \mathcal{D}_0(s_4) + c_1 \delta(s_4)\right\} \tag{3.28}$$

with $c_3 = C_F + 2C_A$,

$$c_2 = -\frac{3}{4} C_F - (C_F + C_A) \ln\left(\frac{\mu_F^2}{p_T^2}\right) , \tag{3.29}$$

and $c_1 = c_1^\mu + T_1$ with

$$c_1^\mu = \left[-\frac{\beta_0}{4} - \frac{3}{4} C_F - C_F \ln\left(\frac{-t}{s}\right) - C_A \ln\left(\frac{-u}{s}\right)\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right) . \tag{3.30}$$

The term T_1 is given in Eq. (3.8) of Ref. [33] (where it is called $c_1'^{qg}$). The NNLO soft plus virtual corrections are then explicitly given by our master formula. Again, we note that the terms $D_3(s_4)$, $D_2(s_4)$, and $D_1(s_4)$ were already derived from a resummation study in Section III of Ref. [33] and are in agreement with our formula.

3.8 DIS heavy quark production

The NNLO corrections to heavy quark production in deep inelastic scattering may be needed, together with various resummations [34, 35], in explaining the discrepancy between NLO theory [36] and experiment for bottom quark production. Here the lowest-order partonic process is $\gamma^* g \to Q\bar{Q}$, so there are no final-state massless partons, and we are working in 1PI kinematics in the $\overline{\mathrm{MS}}$ scheme with the cross section $d^2\sigma/(dt_1 du_1)$. The soft anomalous dimension is $\mathrm{Re}\Gamma'^{(1)}_S = (C_A/2 - C_F)(\mathrm{Re}L_\beta + 1) + (C_A/2)\ln(t_1 u_1/(m^2 s))$, where $s = (p_{\gamma^*} + p_g)^2$, $t_1 = (p_g - p_Q)^2 - m^2$, and $u_1 = (p_{\gamma^*} - p_Q)^2 - m^2$, with m the heavy quark mass. The singular distributions are $D_l(s_2)$ with $s_2 = s + t_1 + u_1$, and we use $M = m$.

The NLO corrections are

$$\hat{\sigma}^{(1)}_{\gamma^* g\to Q\bar{Q}} = \sigma^B_{\gamma^* g\to Q\bar{Q}} \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(s_2) + c_2 \mathcal{D}_0(s_2) + c_1 \delta(s_2)\right\} \tag{3.31}$$

with $c_3 = 2C_A$,

$$c_2 = -2C_F(\mathrm{Re}L_\beta + 1) + C_A\left[\mathrm{Re}L_\beta + \ln\left(\frac{t_1}{u_1}\right) - \ln\left(\frac{\mu_F^2}{m^2}\right)\right], \tag{3.32}$$

and

$$c_1^\mu = \left[-\frac{\beta_0}{4} + C_A \ln\left(\frac{-u_1}{m^2}\right)\right]\ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{4}\ln\left(\frac{\mu_R^2}{s}\right). \tag{3.33}$$

The NNLO soft and virtual corrections are then given by our master formula. We note that the $D_3(s_2)$ and $D_2(s_2)$ terms were derived in Ref. [34] (with $\mu_F = \mu_R$) and are in agreement with our results.

3.9 FCNC single-top production

The last process with simple color flow that we consider is single-top production mediated by flavor-changing neutral currents (FCNC). The QCD corrections to the 1PI cross section $d\sigma/(dt\,du)$ in the $\overline{\mathrm{MS}}$ scheme for FCNC single-top-quark production in ep collisions at the HERA collider were studied at NLO in the eikonal approximation in Ref. [37]. Here the partonic process is $eu \to et$, so there are no final-state massless partons, and $\mathrm{Re}\Gamma'^{(1)}_S = C_F \ln[(m_t^2 - t)/(\sqrt{s}m_t)]$, with $s = (p_e + p_u)^2$, $t = (p_t - p_u)^2$, and $u = (p_t - p_e)^2$. The singular distributions are $D_l(s_2)$ with $s_2 = s + t + u - m_t^2 - 2m_e^2$, and we choose $M = m_t$.

The NLO corrections are

$$\hat{\sigma}^{(1)}_{eu\to et} = \sigma^B_{eu\to et}\frac{\alpha_s(\mu_R^2)}{\pi}\left\{c_3\mathcal{D}_1(s_2) + c_2\mathcal{D}_0(s_2) + c_1\delta(s_2)\right\} \tag{3.34}$$

with $c_3 = 2C_F$,

$$c_2 = C_F\left[-1 - 2\ln\left(\frac{-u + m_e^2}{m_t^2}\right) + 2\ln\left(\frac{m_t^2 - t}{m_t^2}\right) - \ln\left(\frac{\mu_F^2}{m_t^2}\right)\right], \tag{3.35}$$

and

$$c_1^\mu = \left[-\frac{3}{4} + \ln\left(\frac{-u + m_e^2}{m_t^2}\right)\right] C_F \ln\left(\frac{\mu_F^2}{s}\right). \tag{3.36}$$

These agree with the results in [37]. The NNLO corrections are then given by our master formula.

4 NNLO master formula and applications - complex color flow

4.1 NLO and NNLO master formulas for complex color flow

When the lowest-order cross section involves already a complex color flow that is expressed in terms of non-trivial color matrices, then the master formulas given in Section 2 have to be extended. Now not all the NLO corrections are proportional to the Born term; only the leading logarithms and terms involving the scale are. Therefore at NLO our master formula for soft and virtual corrections in the $\overline{\rm MS}$ scheme is extended for the case of complex color flow as

$$\hat{\sigma}^{(1)} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \{c_3\, \mathcal{D}_1(x_{th}) + c_2\, \mathcal{D}_0(x_{th}) + c_1\, \delta(x_{th})\} + \frac{\alpha_s^{d_{\alpha_s}+1}(\mu_R^2)}{\pi} \left[A^c\, \mathcal{D}_0(x_{th}) + T_1^c\, \delta(x_{th})\right] , \tag{4.1}$$

where c_3 is defined as before in Eq. (2.10), and c_2 is defined by

$$c_2 = -\sum_i \left[C_{f_i} + 2C_{f_i}\, \delta_K\, \ln\left(\frac{-t_i}{M^2}\right) + C_{f_i} \ln\left(\frac{\mu_F^2}{s}\right)\right] - \sum_j \left[B'^{(1)}_j + C_{f_j} + C_{f_j}\, \delta_K\, \ln\left(\frac{M^2}{s}\right)\right] , \tag{4.2}$$

i.e. is the same as in the simple color flow case except without the term $2{\rm Re}\Gamma'^{(1)}_S$. The function A^c is process-dependent and depends on the color structure of the hard-scattering, for which specific examples are given in the next subsections. It is defined by

$$A^c = {\rm Tr}\left(H^{(0)} \Gamma'^{(1)\,\dagger}_S S^{(0)} + H^{(0)} S^{(0)} \Gamma'^{(1)}_S\right) . \tag{4.3}$$

Note that we use the expansions for the hard and soft matrices: $H = \alpha_s^{d_{\alpha_s}} H^{(0)} + (\alpha_s^{d_{\alpha_s}+1}/\pi) H^{(1)} + (\alpha_s^{d_{\alpha_s}+2}/\pi^2) H^{(2)} + \cdots$ and $S = S^{(0)} + (\alpha_s/\pi) S^{(1)} + (\alpha_s/\pi)^2 S^{(2)} + \cdots$. The Born term is then given by $\sigma^B = {\rm Tr}[H^{(0)} S^{(0)}]$.

With respect to the $\delta(x_{th})$ terms, we split them into a term $c_1 = c_1^\mu + T_1$, with c_1^μ defined in Eq. (2.12) as before, that is proportional to the Born cross section, and a term T_1^c that is not. T_1^c is also process-dependent, is formally defined by

$$\begin{aligned} T_1^c \;\; = \;\; & {\rm Tr}\left(H^{(1)} S^{(0)} + H^{(0)} S^{(1)}\right) + A^c\, \delta_K\, \ln\left(\frac{M^2}{s}\right) + \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \left[-T_1 + \frac{c_3}{2} \delta_K \ln^2\left(\frac{M^2}{s}\right) \right. \\ & \left. + T_2\, \delta_K \ln\left(\frac{M^2}{s}\right)\right] , \end{aligned} \tag{4.4}$$

and can also be derived by matching to a full NLO calculation.

In the DIS scheme the corresponding terms c_3', c_2', and c_1' are given as in Section 2.2.2.

At NNLO the master formula for soft and virtual corrections in the $\overline{\rm MS}$ scheme is extended for the case of complex color flow as

$$\begin{aligned} \hat{\sigma}^{(2)} \;\; = \;\; & \hat{\sigma}^{(2)}_{\rm simple} + \frac{\alpha_s^{d_{\alpha_s}+2}(\mu_R^2)}{\pi^2} \left\{ \frac{3}{2} c_3\, A^c\, \mathcal{D}_2(x_{th}) + \left[\left(2c_2 - \frac{\beta_0}{2}\right) A^c + c_3 T_1^c + F\right] \mathcal{D}_1(x_{th}) \right. \\ & + \left[\left(c_1 - \zeta_2 c_3 + \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right)\right) A^c + \left(c_2 - \frac{\beta_0}{2}\right) T_1^c + F\, \delta_K\, \ln\left(\frac{M^2}{s}\right) + G\right] \mathcal{D}_0(x_{th}) \\ & + \left[(\zeta_3 c_3 - \zeta_2 c_2) A^c + \left(c_1 + \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right)\right) T_1^c + \frac{1}{2}\left(\delta_K\, \ln^2\left(\frac{M^2}{s}\right) - \zeta_2\right) F \right. \\ & \left.\left. - \frac{\beta_0}{2} \delta_K\, T_1^c \ln\left(\frac{M^2}{s}\right) + \frac{\beta_0}{4} \delta_K\, A^c \ln^2\left(\frac{M^2}{s}\right) + G\, \delta_K\, \ln\left(\frac{M^2}{s}\right) + R^c\right] \delta(x_{th}) \right\} . \end{aligned} \tag{4.5}$$

Here $\hat{\sigma}^{(2)}_{\rm simple}$ denotes the expression in Eq. (2.16) after using the new c_2 of Eq. (4.2) everywhere in that expression and deleting all Γ'_S, and R, from that expression. Also, we have used

$$F = {\rm Tr}\left[H^{(0)}\left({\Gamma'}_S^{(1)\,\dagger}\right)^2 S^{(0)} + H^{(0)}S^{(0)}\left({\Gamma'}_S^{(1)}\right)^2 + 2H^{(0)}{\Gamma'}_S^{(1)\,\dagger}S^{(0)}{\Gamma'}_S^{(1)}\right] , \tag{4.6}$$

$$\begin{aligned} G \;=\; & {\rm Tr}\left[H^{(1)}{\Gamma'}_S^{(1)\,\dagger}S^{(0)} + H^{(1)}S^{(0)}{\Gamma'}_S^{(1)} + H^{(0)}{\Gamma'}_S^{(1)\,\dagger}S^{(1)} + H^{(0)}S^{(1)}{\Gamma'}_S^{(1)}\right. \\ & \left. + H^{(0)}{\Gamma'}_S^{(2)\,\dagger}S^{(0)} + H^{(0)}S^{(0)}{\Gamma'}_S^{(2)}\right] , \end{aligned} \tag{4.7}$$

and

$$R^c = {\rm Tr}\left(H^{(2)}S^{(0)} + H^{(0)}S^{(2)} + H^{(1)}S^{(1)}\right) - \frac{1}{2}T_1^2 - T_1 T_1^c \, . \tag{4.8}$$

I have checked explicitly that the renormalization and factorization scale dependence cancels in the physical cross section at NNLO.

In the DIS scheme the NNLO corrections are given by Eq. (2.18), after replacing the term $\sigma^B(\alpha_s^2(\mu_R^2)/\pi^2){\hat{\sigma}'}^{(2)}|_{c'_i}$ by the cross section in Eq. (4.5), having also replaced in Eq. (4.5) all the c_i, T_i, G, and R^c, by their DIS counterparts c'_i, T'_i, G', and $R^{c\prime}$.

We now apply the NLO and NNLO master formulas to a variety of processes with complex color flow.

4.2 Heavy quark hadroproduction

The production cross sections of top, bottom, and charm quarks in hadron colliders can be considerably enhanced near threshold. The NLL resummation of threshold corrections was derived in Ref. [15] and the NNLO-NNLL corrections for heavy quark total cross sections and differential distributions were calculated in Refs. [18, 38] in both 1PI and PIM kinematics. There are two partonic channels involved at lowest order, $q\bar{q} \to Q\bar{Q}$ and $gg \to Q\bar{Q}$. We choose the hard scale $M = m$, with m the heavy quark mass. The 1PI cross section is $s^2\, d^2\hat{\sigma}/(dt_1\, du_1)$, with $t_1 = t - m^2$, $u_1 = u - m^2$, and the singular distributions are ${\cal D}_l(s_4)$ with $s_4 = s + t_1 + u_1$. The PIM cross section is $s\, d^2\hat{\sigma}/(dM^2_{Q\bar{Q}} d\cos\theta)$, with $M_{Q\bar{Q}}$ the heavy quark pair mass and θ the scattering angle in the partonic center-of-mass frame, and the singular distributions are ${\cal D}_l(z)$ with $z = M^2_{Q\bar{Q}}/s$. The one-loop soft anomalous dimension matrices $\Gamma_S^{(1)}$ were calculated for both partonic processes in Ref. [15]. Explicit results for the matrices $H^{(0)}$ and $S^{(0)}$ can be found in Refs. [18, 38] for both partonic channels.

4.2.1 $q\bar{q} \to Q\bar{Q}$ channel

We begin with the quark-antiquark annihilation channel. We note that Γ'_S, H, and S are 2×2 matrices. However, the triviality of the hard matrix $H^{(0)}$ (only one non-zero element) leads to almost simple-color-flow-like expressions.

The NLO soft plus virtual corrections in 1PI kinematics in the $\overline{\rm MS}$ scheme are given by Eq. (4.1) with $c_3 = 4C_F$,

$$c_2 = -2C_F - 2C_F \ln\left(\frac{t_1 u_1}{m^4}\right) - 2C_F \ln\left(\frac{\mu_F^2}{s}\right) , \tag{4.9}$$

$$c_1^\mu = C_F \left[\ln\left(\frac{t_1 u_1}{m^4}\right) - \frac{3}{2}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{s}\right) , \tag{4.10}$$

and $A^c = (\sigma^B/\alpha_s^2)\, 2\, \mathrm{Re}\Gamma'^{(1)}_{S,22}$, and are in agreement with the NLO results in [39]. Here the real part of the one-loop 22 element of the soft anomalous dimension matrix is $\mathrm{Re}\Gamma'^{(1)}_{S,22} = C_F[4\ln(u_1/t_1) - \mathrm{Re}L_\beta] + (C_A/2)[-3\ln(u_1/t_1) - \ln(m^2 s/(t_1 u_1)) + \mathrm{Re}L_\beta]$ with $L_\beta = (1-2m^2/s)/\beta \cdot [\ln((1-\beta)/(1+\beta)) + \pi i]$ and $\beta = \sqrt{1-4m^2/s}$. Explicit expressions for T_1, T_1^c can be extracted from Ref. [39].

The corresponding NLO $\overline{\mathrm{MS}}$ corrections in PIM kinematics are given by similar expressions after striking out the $\ln(t_1 u_1/m^4)$ terms from the 1PI c_2 and c_1^μ, as per our NLO master formula, and using the relevant PIM Born term and T_1, T_1^c [38].

The NNLO soft and virtual corrections are then given by our master formula for complex color flows for either kinematics and are in agreement with the NNLO-NNLL results in [18, 38]. We note that the term F in Eq. (4.6) has a relatively simple form, $F = (\sigma^B/\alpha_s^2)[4(\mathrm{Re}\Gamma'^{(1)}_{S,22})^2 + 4\Gamma'^{(1)}_{S,12}\Gamma'^{(1)}_{S,21}]$ with $\Gamma'^{(1)}_{S,12} = (C_F/C_A)\ln(u_1/t_1)$ and $\Gamma'^{(1)}_{S,21} = 2\ln(u_1/t_1)$.

In the DIS scheme, we have in 1PI kinematics $c_3' = 2C_F$,

$$c_2' = -\frac{C_F}{2} - C_F \ln\left(\frac{t_1 u_1}{m^4}\right) - 2C_F \ln\left(\frac{\mu_F^2}{s}\right) , \tag{4.11}$$

$$c_1' = c_1 - \frac{1}{2}C_F \left[\ln^2\left(\frac{-t_1}{m^2}\right) + \ln^2\left(\frac{-u_1}{m^2}\right)\right] - \frac{3}{4}C_F \ln\left(\frac{t_1 u_1}{m^4}\right) + 2C_F\zeta_2 + \frac{9}{2}C_F. \tag{4.12}$$

The corresponding corrections in PIM kinematics are again given by similar expressions after striking out the $\ln^2(-t_1/m^2)$, $\ln^2(-u_1/m^2)$, and $\ln(t_1 u_1/m^4)$ terms from the 1PI c_2' and c_1', and using the relevant PIM Born term and $T_1^{c\prime}$ [38]. The NNLO soft and virtual corrections in the DIS scheme are then given by our master formula for either kinematics.

4.2.2 $gg \to Q\bar{Q}$ channel

We continue with the $gg \to Q\bar{Q}$ channel whose color structure is considerably more complex. Here Γ'_S, H, and S are 3×3 matrices. The NLO soft plus virtual corrections are given in 1PI kinematics in the $\overline{\mathrm{MS}}$ scheme by Eq. (4.1) with $c_3 = 4C_A$,

$$c_2 = -2C_A - 2C_A \ln\left(\frac{t_1 u_1}{m^4}\right) - 2C_A \ln\left(\frac{\mu_F^2}{s}\right) , \tag{4.13}$$

$$c_1^\mu = \left[C_A \ln\left(\frac{t_1 u_1}{m^4}\right) - \frac{\beta_0}{2}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{s}\right) , \tag{4.14}$$

and

$$\begin{aligned} A^c \;=\;& \pi K_{gg} B_{QED} (N_c^2 - 1) \left\{ N_c \frac{t_1^2 + u_1^2}{s^2} \left[\left(-C_F + \frac{C_A}{2}\right)(\mathrm{Re}L_\beta + 1) + \frac{N_c}{2} + \frac{N_c}{2} \ln\left(\frac{t_1 u_1}{m^2 s}\right) \right] \right. \\ & \left. + \frac{1}{N_c}(C_F - C_A)(\mathrm{Re}L_\beta + 1) - \ln\left(\frac{t_1 u_1}{m^2 s}\right) + \frac{N_c^2}{2} \frac{t_1^2 - u_1^2}{s^2} \ln\left(\frac{u_1}{t_1}\right) \right\} , \end{aligned} \tag{4.15}$$

and are in agreement with the NLO results in [40]. Explicit expressions for T_1, T_1^c can be extracted from Ref. [40]. As for the $q\bar{q}$ channel, the corresponding NLO corrections in PIM kinematics are obtained after striking out the $\ln(t_1 u_1/m^4)$ terms from the 1PI c_2 and c_1^μ (note that A^c is not affected by the kinematics choice) and using the relevant PIM Born term and T_1, T_1^c [38].

The term F can be explicitly calculated using Eq. (4.6). The NNLO soft and virtual corrections for either kinematics are then given by our master formula for complex color flow and are in agreement with the NNLO-NNLL results in [18, 38].

4.3 Jet production

Threshold resummation for jet production has been studied in Refs. [16, 41]. There are many partonic subprocesses for which explicit results for the NNLO corrections at NLL accuracy were given in [41]. Here we are able to extend those results by employing our master formula. We note that the color structure for these processes is quite complex, and the one-loop soft anomalous dimension matrices [16, 13] along with the lowest-order hard and soft matrices can be found in Ref. [41]. Recently, the complete two-loop virtual corrections have been calculated [1, 2, 42, 43].

Here we discuss the single-jet inclusive cross section $E_J\, d^3\hat{\sigma}/d^3p_J$ in the $\overline{\mathrm{MS}}$ scheme. The NLO soft and virtual corrections with $M^2 = p_T^2 = tu/s$ and $s_4 = s + t + u$ can be written for each subprocess as

$$\hat{\sigma}^{(1)} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \left\{c_3 \mathcal{D}_1(s_4) + c_2 \mathcal{D}_0(s_4) + c_1 \delta(s_4)\right\} + \frac{\alpha_s^{d_{\alpha_s}+1}(\mu_R^2)}{\pi} \left[A^c\, \mathcal{D}_0(s_4) + T_1^c\, \delta(s_4)\right] . \quad (4.16)$$

The expressions are symmetric in t and u except for the $qg \to qg$ channel. Full NLO expressions have been given in [44]. We discuss next the individual partonic processes in jet production.

4.3.1 $q\bar{q} \to q\bar{q}$ and $qq \to qq$

There are several processes involving distinct or identical quarks and antiquarks: $q_j\bar{q}_j \to q_j\bar{q}_j$, $q_j\bar{q}_j \to q_k\bar{q}_k$, $q_j\bar{q}_k \to q_j\bar{q}_k$, $q_jq_j \to q_jq_j$, $q_jq_k \to q_jq_k$, and the corresponding ones with antiquarks.

The NLO soft and virtual corrections are given by Eq. (4.16) with $c_3 = 2C_F$,

$$c_2 = -2C_F \ln\left(\frac{\mu_F^2}{s}\right) - \frac{11}{2} C_F \, , \quad (4.17)$$

$$c_1^\mu = -C_F \left[\ln\left(\frac{p_T^2}{s}\right) + \frac{3}{2}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{s}\right) . \quad (4.18)$$

The expressions for σ^B and A^c depend on the specific process. The expressions for σ^B can be found in Appendix A (for $q\bar{q} \to q\bar{q}$ processes) and Appendix B (for $qq \to qq$ processes) of Ref. [41]. The expressions for A^c can be easily derived from Eq. (4.3) or by comparing Eq. (4.16) with the expressions in Appendix A or Appendix B of [41] for the NLO corrections for the various subprocesses.

4.3.2 $q\bar{q} \rightarrow gg$ and $gg \rightarrow q\bar{q}$

The NLO soft and virtual corrections are given by Eq. (4.16) with $c_3 = 4C_F - 2C_A$,

$$c_2 = -2C_F \ln\left(\frac{\mu_F^2}{p_T^2}\right) - \frac{\beta_0}{2} - 2C_F - 2C_A - 2C_A \ln\left(\frac{p_T^2}{s}\right) , \tag{4.19}$$

$$c_1^\mu = -C_F \left[\ln\left(\frac{p_T^2}{s}\right) + \frac{3}{2}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{s}\right) , \tag{4.20}$$

for the process $q\bar{q} \rightarrow gg$, and $c_3 = 4C_A - 2C_F$,

$$c_2 = -2C_A \ln\left(\frac{\mu_F^2}{p_T^2}\right) - \frac{7}{2}C_F - 2C_A - 2C_F \ln\left(\frac{p_T^2}{s}\right) , \tag{4.21}$$

$$c_1^\mu = -C_A \ln\left(\frac{p_T^2}{s}\right) \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{\mu_F^2}\right) , \tag{4.22}$$

for the process $gg \rightarrow q\bar{q}$. The expressions for σ^B depend on the specific process and can be found in Appendix C of Ref. [41]. The expressions for A^c can be easily derived from Eq. (4.3) or by comparing Eq. (4.16) with the expressions in Appendix C of [41] for the NLO corrections for these subprocesses.

4.3.3 $qg \rightarrow qg$

The NLO soft and virtual corrections, using $t_q = u$ and $t_g = t$, are given by Eq. (4.16) with $c_3 = C_F + C_A$,

$$c_2 = -(C_F + C_A) \ln\left(\frac{\mu_F^2}{p_T^2}\right) - \frac{11}{4}C_F - 2C_A - \frac{\beta_0}{4} - 2C_F \ln\left(\frac{-u}{s}\right) - 2C_A \ln\left(\frac{-t}{s}\right) , \tag{4.23}$$

and

$$c_1^\mu = -\left[C_F \ln\left(\frac{-t}{s}\right) + C_A \ln\left(\frac{-u}{s}\right) + \frac{3}{4}C_F + \frac{\beta_0}{4}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{s}\right) . \tag{4.24}$$

The expression for σ^B can be found in Appendix D of Ref. [41]. The expression for A^c can be easily derived from Eq. (4.3) or by comparing Eq. (4.16) with the expression in Appendix D of [41] for the NLO corrections for this subprocess.

4.3.4 $gg \rightarrow gg$

The NLO soft and virtual corrections are given by Eq. (4.16) with $c_3 = 2C_A$,

$$c_2 = -2C_A \ln\left(\frac{\mu_F^2}{s}\right) - \frac{\beta_0}{2} - 4C_A , \tag{4.25}$$

and

$$c_1^\mu = -C_A \ln\left(\frac{p_T^2}{s}\right) \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2} \ln\left(\frac{\mu_R^2}{\mu_F^2}\right) . \tag{4.26}$$

The expression for σ^B can be found in Appendix E of Ref. [41]. The expression for A^c can be easily derived from Eq. (4.3) or by comparing Eq. (4.16) with the expression in Appendix E of [41] for the NLO corrections for this subprocess.

The term F in Eq. (4.6) can be easily derived for each subprocess using the matrices $H^{(0)}$, $S^{(0)}$, and $\Gamma_S^{(1)}$ as given in Ref. [41]. For all subprocesses the NNLO soft and virtual corrections are given by our master formula for complex color flows and are in agreement with the NNLO-NLL expressions in [41].

4.4 Squark and gluino production

Our last application is the production of squarks and gluinos in hadron colliders. The complete NLO corrections for squark and gluino production in the Minimal Supersymmetric Standard Model have been given in [45]. There are several partonic subprocesses involved.

We begin with squark production. For the channel $q\bar{q} \to \tilde{q}\bar{\tilde{q}}$ the NLO terms c_3, c_2, and c_1^μ in the $\overline{\rm MS}$ scheme are the same as for the $q\bar{q} \to Q\bar{Q}$ channel in heavy quark production in Section 4.2.1 in either 1PI or PIM kinematics, with $M = m$ the mass of the squark. The same holds for the channel $qq \to \tilde{q}\tilde{q}$, and analogous results hold in the DIS scheme. For the channel $gg \to \tilde{q}\bar{\tilde{q}}$ in the $\overline{\rm MS}$ scheme, c_3, c_2, and c_1^μ are the same as for the $gg \to Q\bar{Q}$ channel in heavy quark production in Section 4.2.2 in either kinematics.

We continue with gluino production. Here we choose $M = m$, with m the mass of the gluino. For the process $q\bar{q} \to \tilde{g}\tilde{g}$, c_3, c_2, and c_1^μ are the same as for the process $q\bar{q} \to \tilde{q}\bar{\tilde{q}}$ in either 1PI or PIM kinematics and either factorization scheme; for the process $gg \to \tilde{g}\tilde{g}$ in the $\overline{\rm MS}$ scheme they are the same as for $gg \to \tilde{q}\bar{\tilde{q}}$ in either kinematics.

We finally have the process of squark-gluino production, $qg \to \tilde{q}\tilde{g}$. We choose $M = m$, with m the mass of the squark or the mass of the gluino. In 1PI kinematics in the $\overline{\rm MS}$ scheme we have $c_3 = 2(C_F + C_A)$,

$$c_2 = -C_F - C_A - 2C_F \ln\left(\frac{-u_1}{m^2}\right) - 2C_A \ln\left(\frac{-t_1}{m^2}\right) - (C_F + C_A)\ln\left(\frac{\mu_F^2}{s}\right) , \tag{4.27}$$

and

$$c_1^\mu = \left[C_F \ln\left(\frac{-u_1}{m^2}\right) + C_A \ln\left(\frac{-t_1}{m^2}\right) - \frac{3}{4}C_F - \frac{\beta_0}{4}\right] \ln\left(\frac{\mu_F^2}{s}\right) + \frac{\beta_0}{2}\ln\left(\frac{\mu_R^2}{s}\right) . \tag{4.28}$$

In PIM kinematics the expressions for c_2 and c_1^μ are obtained after striking out the $\ln(-t_1/m^2)$ and $\ln(-u_1/m^2)$ terms from the 1PI c_2 and c_1^μ.

For all these processes, A^c, T_1, T_1^c can be read off the full NLO calculation. Also A^c and F can be calculated explicitly once the lowest-order H, S, and Γ_S matrices have been constructed. The NNLO soft and virtual corrections are then given by our master formula.

5 Conclusions and outlook to higher orders

In this paper, I presented a unified approach to calculating the NNLO soft and virtual QCD corrections for any process in hadron-hadron and lepton-hadron collisions in either the $\overline{\rm MS}$ or

DIS schemes and in either 1PI or PIM kinematics. The master formulas given in the paper are based on a unified threshold resummation formalism and they allow explicit calculations for any process, with either simple or complex color flows, keeping in general the factorization and renormalization scales separate and the beta function and color factors explicit. I verified that the scale-dependence of the physical cross section cancels out at NNLO for any process. Detailed results, illustrating the use of the master formulas, were given for various electroweak, Higgs, QCD, and SUSY processes in various factorization schemes, kinematics, and colliders. As tests of the master formulas, I reproduced the previously known NNLO corrections for Drell-Yan and Higgs production, deep inelastic scattering, and $W^+\gamma$ production, thus also determining a number of two-loop anomalous dimensions and other quantities which are needed in NNLL resummations. Furthermore, I presented new results for several other processes in the Standard Model and beyond.

The NNLO corrections increase theoretical accuracy and diminish the dependence on the factorization and renormalization scales, and thus are essential in further testing QCD and particularly in searching for the Higgs boson and supersymmetric particles as well as other processes, such as flavor-changing neutral currents, that signal new physics beyond the Standard Model.

The unified approach employed in this paper can be extended to higher orders. For a sketch of how this extension may be carried out through next-to-next-to-next-to-next-to-leading order in the specific context of heavy quark hadroproduction see Ref. [18]. A complete calculation of next-to-next-to-next-to-leading and even higher-order soft and virtual corrections is a formidable task and unlikely to be necessary at least in the forseeable future. The accuracy attainable at NNLO should be sufficient for the high-energy colliders of our era. The unified master formulas for the NNLO soft and virtual corrections for any process, which are an important component of the full NNLO calculation, should serve as a milestone in the push for ever-increasing theoretical accuracy and understanding of high-energy processes.

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