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Abstract

We consider Sharpe's one factor model of asset returns and its extension to K factors in order to explain theoretically why diversification can fail. This model can be used to explain nonlinear dependence amongst the assets in a portfolio. The result is intimately related to the tail distribution of the driving factor, the market. We study these properties for general classes of distribution functions. We find asymptotic conditions on the tails of the distribution which determine whether diversification will succeed or fail in the presence of a market fall. Turning to exact analysis, we characterize the only distribution having constant correlation when the market falls, namely the exponential distribution.

Keywords: Distribution Function; Factor Model; Portfolio Diversification; Truncated Variance.

JEL Classification: C16, G11.

1 Introduction

It is a well known fact that the correlation of assets changes during periods of high market volatility. There is a considerable literature on the changing nature of correlation under different market conditions. Either for local (e.g. US) or global markets, volatility is found to be asymmetric and the correlation is high when the aggregate market is down, but when it is up the correlation is little different from the normal period (e.g. Longin and Solnik, 1995, Ramchmand and Susmel, 1998). Taken together, these results suggest that high positive correlation reduces the benefit of market diversification and the effects are worse when the market is down (see Silvapulle and Granger, 2001, for further references on the subject).

These phenomena have been seen as result of financial complexity. None of the above authors work with asset pricing models. They prefer sophisticated econometric models such as the regime switching ARCH model of Ramchmand and Susmel (1998), kernel methods of Silvapulle and Granger (2001), or the use of copulae to capture nonlinear dependence (e.g. Sancetta and Satchell, 2001).

The purpose of this paper is to present a simple asset pricing model which can capture all these phenomena: Sharpe's (1964) market model which is compatible with the capital asset pricing model (CAPM).

The use of a factor model allows us to explain changing correlation amongst the assets which are driven by the same factor. We show that the nature of the change in the correlation is intimately related to the tail distribution of the factor. As a by-product, we see that the asymmetry in the correlation is a consequence of asymmetric tail behaviour of the factor. Therefore, non-linearity in correlation is not a consequence of a non-linear relation between the factor and the assets, but a consequence of asymmetric behaviour of the tail distribution of the factor itself. For this reason, we consider the variance of the factor conditional on being either above or below some threshold in order to verify under what distributional assumptions diversification looses importance. This is the key concept for understanding nonlinear dependence in the context of Sharpe's market model.

The plan for the paper is as follows. The theoretical results are presented in Section 2. A detailed study of the conditional variance for the factor being below (or above) some threshold is provided in Section 3. Further remarks are contained in Section 4. Proofs of results can be found in the appendix.

2 Factor Models and Correlation and Diversification

We consider a one factor model first. Then we extend the result to K factor models. We introduce some notation. $(Z_t)_{t\in\mathbb{Z}_+}$ is used to describe a factor process. If $K>1$ factors are considered, we may write $(\mathbf{Z}_t)_{t\in\mathbb{Z}_+}\equiv(Z_{1,t},...,Z_{K,t})'_{t\in\mathbb{Z}_+}$, where $(Z_{j,t})_{t\in\mathbb{Z}_+}$ is the j^{th} factor process. The factors are restricted to be orthogonal to each other, although this is not essential. This is no real restriction as we can always orthogonalize if this were not true. Further, $(\mathbf{X}_t)_{t\in\mathbb{Z}_+}\equiv(X_{1,t},...,X_{N,t})'_{t\in\mathbb{Z}_+}$ is a vector valued process of assets' arithmetic returns, e.g. $(X_{i,t})_{t\in\mathbb{Z}_+}$ is the i^{th} asset's returns process. Finally, $(\boldsymbol{\varepsilon}_t)_{t\in\mathbb{Z}_+}\equiv(\varepsilon_{1,t},...,\varepsilon_{N,t})'_{t\in\mathbb{Z}_+}$ is a martingale difference vector valued process which satisfies the following orthogonal condition $E(Z_{i,t}\varepsilon_{j,t})$, $\forall i,j,t$, where expectation is taken with respect to their joint distribution. We shall also use the following notation: $\mathbf{a}=(\alpha_1,...,\alpha_N)'$, $\mathbf{A}=(\mathbf{a}_1,...,\mathbf{a}_K)$, for arbitrary Roman and Greek letters.

2.1 One Factor Model

Recall that Sharpe's one factor model is given by

$$\mathbf{X}_t=\mathbf{a}+Z_t\mathbf{b}+\boldsymbol{\varepsilon}_t, \tag{1}$$

where

$$E(\mathbf{X}_t)=\mathbf{a}+E(Z_t)\,\mathbf{b}.$$

In the context of the CAPM, $\mathbf{X}_t$ is the excess returns on the stocks, Z_t is the excess returns on the market. The CAPM asserts that $\mathbf{a}=\mathbf{0}$ (e.g. Ingersoll, 1987, ch. 4, for derivations). Sharpe's model, generalizable to many factors, allows for closed form expressions for the inverse of the unconditional covariance matrix. It follows that

$$cov\left(\mathbf{X}_t|Z_t\right) = diag\left\{\sigma^2_{\varepsilon,1},...,\sigma^2_{\varepsilon,N}\right\},$$

where $diag\{...\}$ stands for the diagonal matrix. However, our concern is the unconditional covariance matrix when Z_t is restricted in various ways:

$$cov\left(\mathbf{X}_t\right) = var\left(Z_t\right)\mathbf{bb}' + diag\left\{\sigma^2_{\varepsilon,1},...,\sigma^2_{\varepsilon,N}\right\},$$

and conditioning on $Z_t \in S$ (S a Borel set) we have

$$cov\left(\mathbf{X}_t|Z_t \in S\right) = var\left(Z_t|Z_t \in S\right)\mathbf{bb}' + diag\left\{\sigma^2_{\varepsilon,1},...,\sigma^2_{\varepsilon,N}\right\},$$

e.g.

$$var\left(X_{i,t}|Z_t \in S\right) = \beta_i^2 var\left(Z_t|Z_t \in S\right) + \sigma^2_{\varepsilon,i}.$$

Consider $S \equiv \{Z_t : Z_t \le c\}$, and write $\sigma^2_{z(t)}(c) \equiv var\left(Z_t|Z_t \le c\right)$. Then, the conditional correlation matrix has upper and lower triangular entries, say (i,j), $i \ne j$, given by

$$\rho_{ij}(c) = \frac{\beta_i\beta_j}{\sqrt{\beta_i^2 + \sigma^2_{\varepsilon,i}/\sigma^2_{z(t)}(c)}\sqrt{\beta_j^2 + \sigma^2_{\varepsilon,j}/\sigma^2_{z(t)}(c)}}. \quad (2)$$

The last two displays reveal an interesting common feature. Both the variance and the correlation are functions of the conditional variance of Z_t. If we take limits in the left tail, i.e. $c \to -\infty$, and $\sigma^2_{z(t)}(c) \to \infty$, where, usually, $\sigma^2_{z(t)}(c) \to \infty$ is a non-linear function of c, then

$$\lim_{c\to-\infty}\rho_{ij}(c) = \frac{\beta_i\beta_j}{\sqrt{\beta_i}\sqrt{\beta_j}} = 1.$$

This explains the fact that asset diversification can become useless for assets driven by the same factor, under specific conditions on the truncated variance of the factor. Therefore, we can formally state the following.

Theorem 1. *Let asset returns follow (1) and* $\beta_i, \beta_j \neq 0$. *Then,*

$$\lim_{c \to -\infty} \rho_{ij}(c) = 1$$

where $\rho_{ij}(c)$ *is as in (2), if*

$$\lim_{c \to -\infty} \sigma^2_{z(t)}(c) = \infty.$$

The reason can also be grasped intuitively: as the variance of the factor increases, the variance of the assets are completely dominated by the factor if the residuals are independent of it.

2.2 Multifactor Models

We can extend the one factor model to the K factor case as follows

$$\mathbf{X}_t = \mathbf{a} + \mathbf{B}\mathbf{Z}_t + \boldsymbol{\varepsilon}_t,$$

where

$$E(\mathbf{X}_t) = \mathbf{a} + \mathbf{B}E(\mathbf{Z}_t),$$

and

$$cov(\mathbf{X}_t) = \mathbf{B} diag\left\{\sigma^2_{z(t),1}, ..., \sigma^2_{z(t),K}\right\}\mathbf{B}' + diag\left\{\sigma^2_{\varepsilon,1}, ..., \sigma^2_{\varepsilon,N}\right\},$$

assuming orthogonality of the factors. The $(i,j)^{th}$ entry is given by

$$cov(X_{it}, X_{jt}) = \sum_{k=1}^{K} \beta_{ik}\beta_{jk}\sigma^2_{z(t),k} + \delta_{ij}\sigma^2_{\varepsilon,i},$$

$\delta_{ij} = 1$ if $i = j$, 0 otherwise. Then the $(i,j)^{th}$, entry, $i \neq j$, in the correlation matrix is given by

$$\rho_{i,j} = \left(\sum_{k=1}^{K} \beta_{ik}\beta_{jk}\sigma^2_{z(t),k}\right) \Bigg/ \sqrt{\left(\sum_{k=1}^{K} \beta^2_{ik}\sigma^2_{z(t),k} + \sigma^2_{\varepsilon,j}\right)\left(\sum_{k=1}^{K} \beta^2_{jk}\sigma^2_{z(t),k} + \sigma^2_{\varepsilon,j}\right)}.$$

Conditioning on $Z_l < c$, for the l^{th} factor, we have

$$\begin{aligned}\rho_{i,j}(c) &= \left(\beta_{il}\beta_{jl}\sigma^2_{z(t),l}(c) + \sum_{\substack{k=1\\k\neq l}}^{K}\beta_{ik}\beta_{jk}\sigma^2_{z(t),k}\right)\left(\beta^2_{il}\sigma^2_{z(t),l}(c) + \sum_{\substack{k=1\\k\neq l}}^{K}\beta^2_{ik}\sigma^2_{z(t),k} + \sigma^2_{\varepsilon,i}\right)^{-\frac{1}{2}} \\ &\quad \times \left(\beta^2_{jl}\sigma^2_{z(t),l}(c) + \sum_{\substack{k=1\\k\neq l}}^{K}\beta^2_{jk}\sigma^2_{z(t),k} + \sigma^2_{\varepsilon,j}\right)^{-\frac{1}{2}}.\end{aligned}$$

As in the previous subsection, if $\sigma^2_{z(t),l}(c) \to \infty$ as $c \to -\infty$, then

$$\begin{aligned}\lim_{c\to-\infty}\rho_{i,j}(c) &\to \left(\beta_{il}\beta_{jl}\sigma^2_{z(t),l}(c)\right)/\sqrt{\left(\beta^2_{il}\sigma^2_{z(t),l}(c)\right)\left(\beta^2_{jl}\sigma^2_{z(t),l}(c)\right)} \\ &= 1.\end{aligned}$$

The key issue is that one factor goes to minus infinity while the others are kept fixed. For this to happen, orthogonality of factors is helpful. The more general case of correlated factors can also be dealt with by singular value decomposition. Also, the case of omitted factors, i.e. when the conditional correlation matrix is non-diagonal, can be dealt with in a similar manner. Since notation would become more untidy, we do not consider these cases.

2.3 Portfolio Diversification

One of the implications of the literature is that diversification could fail in period of extreme market crashes. This needs clarification, and so we present a brief discussion of diversification in a factor model context as in (1). Let $\mathbf{w} \in \mathbb{R}^N$, such that $\mathbf{w}'\mathbf{1}_N = 1$, where prime stands for transposition and $\mathbf{1}_N$ is a $N \times 1$ vector of ones. Then,

$$\mathbf{w}'\mathbf{X}_t = \mathbf{w}'\mathbf{a} + Z_t\mathbf{w}'\mathbf{b} + \mathbf{w}'\boldsymbol{\varepsilon}_t,$$

which can be rewritten as

$$Y_{pt} = a_p + b_p Z_t + \nu_t,$$

so that the residual risk (non-systematic risk) ν_t is decreased, while the market risk (systematic risk) is unaffected. Thus on this market model, at least, we would not expect there to be any impact on unconditional correlation via diversification, but only on conditional correlation; this is exactly what Silvapulle and Granger (2001) find. It is worth noticing that to eliminate market exposure, and hence unconditional correlation requires selling the market or selling a completion fund, i.e. a fund designed to hedge factor exposure.

3 The Truncated Variance

From Theorem 1, we know that the behaviour of the truncated variance is of fundamental importance for its implications on risk. For convenience, we consider the right tail of the distribution, i.e. $var\left(Z_t|Z_t > c\right)$. Let $F_{z_t}\left(z\right)$ and $f_{z_t}\left(z\right)$ be, respectively, the distribution and the density of Z_t. Then

$$\begin{aligned} var\left(Z_t|Z_t > c\right) &= E\left(Z_t^2|Z_t > c\right) - \left[E\left(Z_t|Z_t > c\right)\right]^2 \\ &= \frac{E\left(Z_t^2 I\left\{Z_t > c\right\}\right)}{1 - F_{z_t}\left(c\right)} - \left[\frac{E\left(Z_t I\left\{Z_t > c\right\}\right)}{1 - F_{z_t}\left(c\right)}\right]^2. \end{aligned}$$

In order to derive more general results, we consider two classes of random variables: random variables with nonpower tails and random variables with power tails.

3.1 Non-Power Tailed Distributions

Here, we consider random variables having the following arbitrary parametric specification

$$\omega z^{\alpha} \exp\left\{-\lambda z^{\beta}\right\} \quad \left(z \geq 0\right), \tag{3}$$

where ω is a constant of integration. In this case, z would be a loss and treated as a positive random variable. For $\alpha = 0$, $\beta = 2$ we have the (half) normal distribution, for $\alpha > -1$, $\beta = 1$, the gamma with the exponential being a special case, for

$\alpha = \beta - 1$, the Weibull. Power laws can be recovered for $\alpha < 0$, $\beta = 0$. However, the following result is valid for $\beta > 0$. Therefore, we will treat this case separately in the next subsection. For the next result we need to recall some familiar notation.

Notation.

$$\Gamma(a, x) \equiv \int_x^\infty z^a \exp\{-z\} \, dz,$$

which is the complementary incomplete gamma function.

Theorem 2. *Let Z have density with respect to the Lebesgue measure given by (3) with $\beta > 0$. Then, for $c > 0$*

$$var(Z|Z > c) = \lambda^{-\frac{2}{\beta}} \left(\frac{\Gamma\left((\alpha+3)/\beta, (\lambda c)^\beta\right)}{\Gamma\left((\alpha+1)/\beta, (\lambda c)^\beta\right)} - \left[\frac{\Gamma\left((\alpha+2)/\beta, (\lambda c)^\beta\right)}{\Gamma\left((\alpha+1)/\beta, (\lambda c)^\beta\right)} \right]^2 \right),$$

and for $c \to \infty$

$$var(Z|Z > c) \sim \beta^{-2} \lambda^{-\frac{2\beta^2 - 2\beta + 2}{\beta}} c^{2(1-\beta)}.$$

Corollary 1. *Let Z be as in Theorem 1. For $c \to \infty$ and*

i. for $\beta < 1$,

$$var(Z|Z > c) \to \infty;$$

ii. for $\beta = 1$,

$$var(Z|Z > c) \sim \frac{1}{\lambda^2};$$

iii. for $\beta > 1$,

$$var(Z|Z > c) \to 0.$$

Theorem 2 allows us to characterize the limiting behaviour of the conditional variance for a wide class of loss distributions.

3.2 The Exponential Distribution

The exponential distribution represent an important limiting case for the behaviour of the truncated variance. In particular, this distribution is the only one to have truncated variance independent of the truncation level, i.e. constant. While this can be expected because of the memoryless properties of the exponential distribution, the fact that this is the only distribution satisfying this property is not necessarily so obvious. For this reason, we deal with it separately in this subsection. This property is summarized in the following result.

Theorem 3. *$var\left(Z|Z>c\right)$ is independent of c if and only if Z is an exponential random variable.*

Using Theorem 2 and 3, the following is obvious.

Corollary 2. *Let $c\in\mathbb{R}_+$. Then,*

$$var\left(Z|Z>c\right)=\frac{1}{\lambda^2}$$

if and only if Z is an exponential random variable.

As opposed to Corollary 1, this result is exact and not an asymptotic one.

3.3 Power Laws

Results for power laws are expected to follow the same result of Corollary 1 for $\beta<1$. To gain some more generality and cover more cases of interest, we impose the following semiparametric specification

$$f_z\left(z\right)\sim L\left(z\right)z^{-\theta-1},\text{ as }z\to\infty, \tag{4}$$

where $L\left(x\right)$ is slowly varying at infinity, i.e. $\lim_{x\to\infty}L\left(\lambda x\right)/L\left(x\right)\to 1$, $\forall\lambda>0$ (e.g. Feller, 1971, ch. VIII.8 and XIII.5). As obvious integrability condition we have $\theta>0$. It is worth noticing that (4) includes the t-distribution as a special case, and in general all distributions with leading power term at infinity.

Theorem 4. *Let Z have density with respect to the Lebesgue measure given by*

(4). Then, for $\theta > 2$,

$$var\left(Z_t|Z_t > c\right) \sim \frac{\theta c^2}{\left(\theta-2\right)\left(\theta-1\right)^2}, \text{ as } c \to \infty.$$

3.4 Examples

We provide a few illustrative examples on the behaviour of the truncated variance. The first two examples do not satisfy the condition of Theorem 1, while the last two do. Further, examples 2 and 4 contain some exact results, not in terms of special functions.

Example 1 *For simplicity, let us assume that* Z *is standard normal with density* $\phi\left(z\right)$ *and distribution* $\Phi\left(z\right)$. *Then, we have the following known result (e.g. Greene, 2000, p.899),*

$$var\left(Z_t|Z_t > c\right) = \left[1 - \frac{\phi\left(c\right)}{1-\Phi\left(c\right)}\left(\frac{\phi\left(c\right)}{1-\Phi\left(c\right)} - c\right)\right],$$

and by Corollary 1, we have, as $c \to \infty$,

$$var\left(Z_t|Z_t > c\right) \to 0.$$

Example 2 *Let* Z *have a gamma distribution with shape parameter* $\alpha > -1$ *and scaling parameter* $\lambda = 1$. *Then, by Theorem 2*

$$var\left(Z_t|Z_t > c\right) = \frac{\Gamma\left(\alpha+2,c\right)}{\Gamma\left(\alpha,c\right)} - \frac{\Gamma\left(\alpha+1,c\right)^2}{\Gamma\left(\alpha,c\right)^2},$$

and, by Corollary 2, as $c \to \infty$,

$$var\left(Z_t|Z_t > c\right) \sim 1.$$

An exact closed form solution is obtained when $\alpha \in \mathbb{Z}_+$. *Then, integrating by parts the complementary incomplete gamma function (e.g. Olver, 1974, p. 110)*

$$var\left(Z_t|Z_t > c\right) = c^2\left(\frac{1+\sum_{s=1}^{\alpha+1}\frac{(\alpha+1)!}{(\alpha+1-s)!c^s}}{1+\sum_{s=1}^{\alpha-1}\frac{(\alpha-1)!}{(\alpha-1-s)!c^s}}\right) - c^2\left(\frac{1+\sum_{s=1}^{\alpha}\frac{\alpha!}{(\alpha-s)!c^s}}{1+\sum_{s=1}^{\alpha-1}\frac{(\alpha-1)!}{(\alpha-1-s)!c^s}}\right)^2.$$

Example 3 *Let Z have a Weibull distribution with parameters β and λ. Then, by Theorem 2, as $c \to \infty$,*

$$var\left(Z_t | Z_t > c\right) \sim \eta c^{2(1-\beta)},$$

where $\eta \equiv \beta^{-2} \lambda^{-\frac{2\beta^2 - 2\beta + 2}{\beta}}$.

Example 4 *Let Z have a Pareto distribution $z^{-\theta}$, $\theta > 2$. Then, by Theorem 4,*

$$var\left(Z_t | Z_t > c\right) = \frac{\theta c^2}{\left(\theta - 2\right)\left(\theta - 1\right)^2}.$$

Clearly, this is not an asymptotic result, as (4) holds for $\forall z \in \mathbb{R}_+$.

4 Further Remarks

We have shown that simple factor models can generate a kind of nonlinear dependence under certain conditions which depend on the tail distribution of the factors. Further, we have shown that the well known asymmetric behaviour of correlation is a consequence of the tail of the driving factor being asymmetric.

One important conclusion about our study of the conditional variance of the factor is that the factor should not have super exponential tails if Sharpe's market model holds. In this case the model cannot explain the many stylized facts discussed in the introduction. Further, by our Theorem 3, the exponential distribution is the least suited candidate for the study of changing correlation in financial portfolios. This statement should be compared with the remark in Silvapulle and Granger (2001, section 2.3) and references therein. On the other hand, common parametric distributions like the t-distribution and the Weibull with $\beta < 1$ appear to be adequate choices.

Further, the asymmetric behaviour of the correlation is essentially due to tail asymmetry of the factor. This asymmetry can be modelled by the family of distributions proposed by Knight et al. (1995), i.e.

$$\begin{aligned} pdf\left(z\right) &= \left(1-p\right) f^{-}\left(-z\right), \text{ if } z < 0 \\ &= p f^{+}\left(z\right), \text{ if } z \geq 0, \end{aligned}$$

where for our purposes, $f^{-}(z)$ is a density such that $\sigma_z^2(c) \to \infty$, for $c \to -\infty$ (e.g. a t-distribution), while $f^{+}(z)$ should be chosen such that $\sigma_z^2(c) = O(1)$ for $c \to \infty$, and $p = \Pr(Z \geq 0)$. This would lead to assets which become highly correlated as the factor falls (e.g. the market), while being unaffected when the market becomes bullish.

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A Proofs

Proof of Theorem 2. We solve the following integral,

$$\begin{aligned}\int_c^\infty \omega z^{\alpha+\tau}\exp\left\{-\lambda z^\beta\right\}dz &= \rho\int_{(\lambda c)^\beta}^\infty y^{(\alpha+\tau+1-\beta)/\beta}\exp\{-y\}\,dy\\ &= \rho\Gamma\left(\frac{\alpha+\tau+1}{\beta},(\lambda c)^\beta\right),\end{aligned}\tag{5}$$

for $\alpha+\tau>-1$, where we made the following change of variables, $\lambda z^\beta = y \Longrightarrow |dy/dz| = \lambda\beta z^{\beta-1}$, defining $\rho\equiv\omega\lambda^{-\frac{\alpha+\tau+1}{\beta}}\beta^{-1}$. Therefore,

$$\begin{aligned}var\left(Z_t|Z_t>c\right) &= \frac{\int_c^\infty \omega z^{\alpha+2}\exp\left\{-\lambda z^\beta\right\}dz}{\int_c^\infty \omega z^{\alpha}\exp\left\{-\lambda z^\beta\right\}dz} - \frac{\left[\int_c^\infty \omega z^{\alpha+1}\exp\left\{-\lambda z^\beta\right\}dz\right]^2}{\left[\int_c^\infty \omega z^{\alpha}\exp\left\{-\lambda z^\beta\right\}dz\right]^2}\\ &= \frac{\Gamma\left((\alpha+3)/\beta,(\lambda c)^\beta\right)}{\Gamma\left((\alpha+1)/\beta,(\lambda c)^\beta\right)} - \left[\frac{\Gamma\left((\alpha+2)/\beta,(\lambda c)^\beta\right)}{\Gamma\left((\alpha+1)/\beta,(\lambda c)^\beta\right)}\right]^2,\end{aligned}$$

using (5). Notice that, as $x\to\infty$,

$$\Gamma(a,x) = \exp\{-x\}x^{a-1}\left\{1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^2}+O\left(x^{-3}\right)\right\},$$

which also implies

$$1/\Gamma(a,x) = \exp\{x\}x^{1-a}\left\{1-\frac{a-1}{x}+\frac{a-1}{x^2}+O\left(x^{-3}\right)\right\},$$

(e.g. Olver, 1974, p. 20). Now,

$$\begin{aligned}\frac{\Gamma\left(a+\delta,x\right)}{\Gamma\left(a,x\right)} &\sim x^{\delta}\left\{1+\frac{a+\delta-1}{x}+\frac{\left(a+\delta-1\right)\left(a+\delta-2\right)}{x^{2}}\right\} \\ &\quad\times\left\{1-\frac{a-1}{x}+\frac{a-1}{x^{2}}+O\left(x^{-3}\right)\right\} \\ &= x^{\delta}\Bigg\{1-\frac{a-1}{x}+\frac{a-1}{x^{2}}+\frac{a+\delta-1}{x}-\frac{\left(a+\delta-1\right)\left(a-1\right)}{x} \\ &\quad+\frac{\left(a+\delta-1\right)\left(a+\delta-2\right)}{x^{2}}+O\left(x^{-3}\right)\Bigg\} \\ &= x^{\delta}\left\{1+\frac{\delta}{x}+\frac{\delta^{2}+\left(a-2\right)\delta}{x^{2}}+O\left(x^{-3}\right)\right\},\end{aligned}$$

and

$$\left(\frac{\Gamma\left(a+\delta,x\right)}{\Gamma\left(a,x\right)}\right)^{2}\sim x^{2\delta}\left\{1+\frac{2\delta}{x}+\frac{3\delta^{2}+\left(a-2\right)\delta}{x^{2}}+O\left(x^{-3}\right)\right\}.$$

Therefore,

$$\frac{\Gamma\left(\left(\alpha+3\right)/\beta,\left(\lambda c\right)^{\beta}\right)}{\Gamma\left(\left(\alpha+1\right)/\beta,\left(\lambda c\right)^{\beta}\right)}\sim\left(\lambda c\right)^{2}\left\{1+\frac{2/\beta}{\left(\lambda c\right)^{\beta}}+\frac{4/\beta^{2}+2\left(a-2\right)/\beta}{\left(\lambda c\right)^{2\beta}}+O\left(c^{-3\beta}\right)\right\},$$

and

$$\begin{aligned}\frac{\Gamma\left(\left(\alpha+2\right)/\beta,\left(\lambda c\right)^{\beta}\right)}{\Gamma\left(\left(\alpha+1\right)/\beta,\left(\lambda c\right)^{\beta}\right)} &\sim \left[\lambda c\left\{1+\frac{1/\beta}{\left(\lambda c\right)^{\beta}}+\frac{1/\beta^{2}+\left(\alpha-1\right)/\beta}{\left(\lambda c\right)^{2\beta}}+O\left(c^{-3\beta}\right)\right\}\right]^{2} \\ &= \left(\lambda c\right)^{2}\left\{1+\frac{2/\beta}{\left(\lambda c\right)^{\beta}}+\frac{3/\beta^{2}+2\left(\alpha-1\right)/\beta}{\left(\lambda c\right)^{2\beta}}+O\left(c^{-3\beta}\right)\right\}.\end{aligned}$$

The two last displays together with (5) imply, as $c\to\infty$,

$$\begin{aligned}var\left(Z_{t}|Z_{t}>c\right) &\sim \lambda^{-\frac{2}{\beta}}\frac{\left(\lambda c\right)^{2}}{\beta^{2}\left(\lambda c\right)^{2\beta}} \\ &= \beta^{-2}\lambda^{-\frac{2\beta^{2}-2\beta+2}{\beta}}c^{2(1-\beta)}.\end{aligned}$$

■

In order to prove Theorem 3, we need some notation and a lemma.

Notation.

$$\mu_s(c) \equiv \frac{\int_c^\infty z^s f_{z_t}(z)\,dz}{[1-F(c)]},$$

which is the truncated s^{th} *moment, i.e.* f *and* F *are, respectively, the density and distribution function of* Z*, and*

$$h(c) \equiv \frac{f(c)}{[1-F(c)]} = -\frac{d\ln S(c)}{dc},$$

is the hazard rate, where $S(c) = [1-F(c)]$ *is the survival function.*

We have the following.

Lemma 1.

$$\frac{d\mu_s(c)}{dc} = h(c)\left[\mu_s(c) - c^s\right].$$

Proof of Lemma 1. Using the Leibnitz integral rule,

$$\begin{aligned}\frac{d\mu_s(c)}{dc} &= -c^s\frac{f(c)}{[1-F(c)]} + f(c)\frac{\int_c^\infty z^s f_{z_t}(z)\,dz}{[1-F(c)]^2} \\ &= -c^s\frac{f(c)}{[1-F(c)]} + \mu_s(c)\frac{f(c)}{[1-F(c)]}.\end{aligned}$$

■

Proof of Theorem 3. Write

$$var(Z_t|Z_t > c) \equiv \sigma^2_{z(t)}(c) = \mu_2(c) - \mu_1(c)^2 = \theta, \tag{6}$$

where we used the notation above. Differentiating (6) we derive the following equations

$$\begin{aligned}\frac{d\sigma^2_{z(t)}(c)}{dc} &= \mu_2'(c) - 2\mu_1(c)\mu_1'(c) \\ &= h(c)\left[\mu_2(c) - c^2\right] - 2\mu_1(c)h(c)\left[\mu_1(c) - c\right],\end{aligned}$$

$$\begin{aligned}\frac{d^2\sigma^2_{z(t)}(c)}{dc^2} &= \mu_2''(c) - 2\left[\mu_1(c)\mu_1''(c) + \mu_1'(c)^2\right] \\ &= h'(c)\left[\mu_2(c) - c^2\right] + h(c)\left[\left(\mu_2(c) - c^2\right) - 2c\right] \\ &\quad -2\mu_1(c)h'(c)\left[\mu_1(c) - c\right] - 2\mu_1(c)h(c)\left[\left(\mu_1(c) - c\right) - 1\right] \\ &\quad -2\left(h(c)\left[\mu_1(c) - c\right]\right)^2.\end{aligned}$$

where $\mu_s'(c) = d\mu_s(c)/dc$ and similarly for the hazard rate and higher derivatives. Since the hazard rate cannot be zero, from the first display we derive the following condition

$$\left[\mu_2(c) - c^2\right] - 2\mu_1(c)\left[\mu_1(c) - c\right] = 0, \tag{7}$$

and plugging in this condition in the second display we get the additional condition

$$c + \mu_1(c) + h(c)\left[\mu_1(c) - c\right]^2 = 0. \tag{8}$$

Using (7) and (6), we find $\mu_1(c) = c \pm \sqrt{\theta}$, and plugging in (8), we get

$$h(c)\,\theta = \pm\sqrt{\theta},$$

which, from the definition of $h(c)$, implies the following differential equation

$$-\frac{d\ln S(c)}{dc} = \theta^{-\frac{1}{2}},$$

as the hazard rate must be positive. Therefore, the only possible solution is

$$\ln S(c) = -\theta^{-\frac{1}{2}}c + K,$$

or precisely

$$S(c) = \exp\left\{-\theta^{-\frac{1}{2}}c\right\},$$

as K must be equal zero by the condition $S(0) = 1$. ■

Proof of Theorem 4. By a slight modification of Lemma 3(ii) in Robinson (1994), we infer that

$$\begin{aligned} \int_c^\infty f_{z_t}(z)\,dx &= 1 - F_{z_t}(c) \\ &\sim \frac{1}{\theta}L(c)\,c^{-\theta}, \end{aligned} \tag{9}$$

for z very large, where we require $\theta > 0$, an obvious integrability condition. Assuming the variance existence condition $\theta > 2$, this implies that

$$\begin{aligned}
var\left(Z_t|Z_t > c\right) &= \frac{\int_c^\infty z^2 f_{z_t}\left(z\right)dz}{\left[1 - F_{z_t}\left(c\right)\right]} - \frac{\left[\int_c^\infty z f_{z_t}\left(z\right)dz\right]^2}{\left[1 - F_{z_t}\left(c\right)\right]^2} \\
&\sim \frac{L\left(c\right)c^{-\theta+2}/\left(\theta - 2\right)}{\left[L\left(c\right)c^{-\theta}/\theta\right]} - \frac{\left[L\left(c\right)c^{-\theta+1}\backslash\left(\theta - 1\right)\right]^2}{\left[L\left(c\right)c^{-\theta}/\theta\right]^2} \\
&= c^2\left[\frac{\theta}{\theta - 2} - \frac{\theta^2}{\left(\theta - 1\right)^2}\right] \\
&= \frac{\theta c^2}{\left(\theta - 2\right)\left(\theta - 1\right)^2},
\end{aligned}$$

where in the second step we made use of (9).