# Abstract

Bayesian statistical methods are naturally oriented towards pooling in a rigorous way information coming from separate sources. It has been suggested that both historical and implied volatilities convey information about future volatility. However, typically in the literature implied and return volatility series are fed separately into models to provide rival forecasts of volatility or options prices. We develop a formal Bayesian framework where we can merge the backward looking information as represented in historical daily return data with the forward looking information as represented in implied volatilities of reported options prices. We apply our theory in forecasting (in- and out- of sample) the prices of FTSE 100 European Index options. We find that for forecasting option prices out of sample (i.e. one-day ahead) our Bayesian estimators outperform standard forecasts that use implied or historical volatilities. For explaining the observed market prices of options we find, as expected, no evidence to suggest that standard procedures that use implied volatility estimates are redundant.

# BAYESIAN FORECASTING OF OPTIONS PRICES: A NATURAL FRAMEWORK FOR POOLING HISTORICAL AND IMPLIED VOLATILITY INFORMATION

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## 1. INTRODUCTION

The purpose of this paper is to propose a Bayesian approach for forecasting (in- and out-of sample) the prices of European options. In the classical Black-Scholes (1973) framework and its subsequent extensions, this basically requires estimating ex-ante the volatility of financial asset returns. As Engle and Mustafa (1992) suggest there are two approaches available to the analyst undertaking this task:

i) The direct one, although backward looking in nature, is to use high frequency historical data on the behaviour of asset prices to either calculate some statistic, such as the standard deviation of returns, or explicitly estimate the stochastic process of volatility via maximum likelihood or other methods. The former usually applies when a constant volatility option pricing model is used, while the latter is more relevant for a (discrete-time) stochastic volatility option pricing model.

ii) The indirect one, first introduced by Latane and Rendleman (1976), is forward looking in nature, and uses the market prices of traded options together with an option pricing model (e.g. the Black-Scholes) to infer expectations about future volatility. This is done by exploiting the monotonicity of the option price with respect to volatility to invert the option pricing formula in terms of the volatility parameter. It is the so-called implied volatility approach, which has proved very popular amongst market practitioners but has also helped uncover the limitations of the Black-Scholes model.

Indeed it is now widely recognised that the Black-Scholes constant volatility assumption is no longer sufficient to capture modern market (i.e. post 1987 crash) phenomena (see for example Rubinstein (1994) for a discussion of the observed pattern of implied volatilities known as the "smile" effect). Although there has been a lot of work done in modifying the specification of volatility to make it a stochastic process there has not yet been a model of stochastic volatility that enjoys the popularity of Black-Scholes. This is partly due to the many challenges that arise under stochastic volatility. First of all, volatility is a "hidden" process and therefore the process parameters are hard to estimate in a continuous framework with only discrete observations. Moreover stochastic volatility introduces a new source of

randomness that cannot be hedged and typically induces market incompleteness.[1] This in turn implies that there is not a unique, arbitrage free, price for a contingent claim and further assumptions about investors preferences need to be made to restore market completeness.

There are many widely cited papers on stochastic volatility option pricing. Just to name a few: Scott (1987), Wiggins (1987), Hull and White (1987), Stein and Stein (1991), Heston (1993) Hobson and Rogers (2000) address the issue in a continuous time framework, while Satchell and Timmermann (1993), Duan (1995)[2], Heston and Nandi (2000) attack the problem in discrete time. Very briefly, the former class of papers model the volatility process as a diffusion while the latter model volatility as a GARCH process. Interestingly since the work of Nelson (e.g. Nelson (1990)) there has been a lot of theoretical interest in the convergence of discrete-time heteroskedastic volatility models to continuous time stochastic volatility models particularly since the former present the comparative advantage of ease of estimation of the process parameters.

Due to the difficulties associated with the empirical application of stochastic volatility models (on top of the aforementioned they often lead to intractable results or require cumbersome numerical analysis and lengthy simulations), financial practitioners, in as much as they announce what they do, seem to use GARCH to predict volatility but use the traditional Black-Scholes coupled with GARCH to price the option. This hybrid procedure, whilst lacking theoretical rigor, can be partially justified by the arguments of Amin and Jarrow (1991), and by the empirical results of Baillie and Bollerslev (1992), Duan (1995), Engle and Mustafa (1992), and Satchell and Timmermann (1993). In this sort of framework, Noh, Engle, and Kane (1994) assess the effectiveness of ARCH models for pricing options. Their study compares predictions of S&P 500 index options prices from GARCH with predictions of the same options prices from forecasting implied volatility. They suggest that both methods can effectively forecast prices well enough to profit by trading if transaction costs are not too high. However they also claim that volatilities incorporated in option prices do not fully utilise historical information and that GARCH volatility forecasts could add value.

Another ad hoc procedure often used by practitioners to deal with the (implied) volatility smile and term structure is to regress the past Black-Scholes implied volatility of an option to its strike prices and maturities. This estimated relationship then serves as the basis for calibrating the future implied volatilities for options with different strikes and maturities. Harvey and Whaley (1992) use time-series regressions of option implied volatilities to forecast the one-day-ahead volatility of S&P 100 index options. They reject the null hypothesis that volatility changes are unpredictable on a daily basis. However, after accounting for transaction costs, a trading strategy based upon out-of-sample volatility forecasts does not generate abnormal returns. Day and Lewis (1992) introduce implied

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[1] An exception occurs if one (unrealistically) assumes that the volatility process is either uncorrelated with the underlying (this in effect implies a market price of volatility risk equal to zero) or consider a model where the volatility and the underlying processes are perfectly correlated. (see for example Hull and White (1987) for a discussion).
[2] Recently Duan and Zhang (2001) used the GARCH option pricing model of Duan (1995) to forecast in- and out-of-sample the prices of Hang Seng Index options and concluded that it outperforms the Black-Scholes model even after allowing for a smile/smirk adjustment.

volatilities into a GARCH and EGARCH model and find that they have some explanatory power for predicting variance in most models, but that in no case are they adequate for predicting implied volatilities.

One sub-class of stochastic volatility models that enjoys popularity is the class of what Rebonato (1999) calls "restricted stochastic volatility models" or otherwise commonly known as "deterministic (level-dependent) volatility models". These models describe the stochastic evolution of the state variable by means of a volatility term that is a deterministic function of the stochastic underlying stock price.[3] (see for example Cox and Ross (1976)). The advantage of these models is that they preserve market completeness since the (stochastic) volatility functionally depends on the underlying. Assuming a "restricted stochastic volatility model", Dupire (1994), Rubinstein (1994) and Derman and Kani (1998) provide tree based algorithms to extract from observed option prices of different strikes and maturities a volatility function that is capable of fitting the cross section of option prices (i.e. reproducing the smile). However, Dumas, Fleming and Whaley (1998) test the predictive and hedging performance of these models and find that it is no better than an ad hoc procedure that merely smoothes Black-Scholes implied volatilities across exercise prices and times to expiration. This is interpreted as evidence that more complex (than the constant) volatility specifications overfit the observed structure of option prices.

Turning now to the application of Bayesian methods in the valuation of options, Karolyi (1993) utilises prior information extracted from the cross sectional patterns in the return volatilities for groups of stocks sorted either by size or financial leverage or trading volume, together with the sample information, to derive the posterior density of the variance. He reports improved prediction accuracy for estimates of option prices calculated using the Bayesian volatility estimates relative to those computed using implied volatility, standard historical volatility, or even the actual ex-post volatility that occurred during each option's life. More recently Bauwens and Lubrano (2000) show how option prices can be evaluated from a Bayesian viewpoint using a GARCH model for the dynamics of the volatility of the underlying asset. Their methodology delivers (via a numerical algorithm) the predictive distribution of the payoff function of the underlying. The authors suggest that this predictive distribution can be utilised by market participants to compare the Bayesian predictions to realised market prices or to other predictions. Our paper differs from Bauwens and Lubrano's in that we follow the log-normal Black-Scholes structure (which allows us to derive the posterior distribution of the option price in analytic form) whilst they follow a GARCH discrete-time structure similar to the one suggested in Duan (1995) or more recently in Hafner and Herwartz (1999).

In particular, the contribution of this paper is twofold:

i) Bayesian statistical methods are naturally oriented towards pooling in a rigorous way information coming from separate sources. It has been suggested that both historical and

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[3] Or in other words, the volatility of the underlying $\sigma$ should only exhibit the stochastic behaviour allowed by the functional dependence on the stock price $S$. Therefore under a restricted volatility model the underlying process is given by: $dS(t) = \mu(S,t)dt + \sigma(S,t)dW_t$. This equation describes the most general set-up that goes beyond the case of a purely deterministic (time-dependent) volatility, and still allows risk-neutral valuation without introducing other hedging instruments apart from the underlying itself.

implied volatilities convey information about future volatility. However, typically in the literature implied and return volatilities series are fed separately into models to provide rival forecasts of volatility or options prices.[4] We develop a formal Bayesian framework where we can merge the backward looking information as represented in historical daily return data with the forward looking information as represented in implied volatilities of reported options prices. In a recent paper (Darsinos and Satchell (2001)) we have derived the prior and posterior densities of the Black-Scholes option price. In this paper we extend our previous analysis from a modelling context to a forecasting context by deriving the predictive density of the Black-Scholes option price. We also apply our theory in forecasting the prices of FTSE 100 European Index options.

ii) Option prices are substantially influenced by the volatility of underlying asset prices as well as the price itself. The majority of existing theories of (out-of-sample) option price forecasting (e.g. Noh, Engle, and Kane (1994), Harvey and Whaley (1992)) use only the implied volatility or historical volatility (e.g. GARCH) to forecast option prices while keeping the price of the underlying fixed. That is, the closing price of the underlying today is used as a forecast of tomorrow's value. In our Bayesian framework we treat both the underlying and its volatility as random variables and the predictive density introduced in this paper implicitly incorporates randomness in both price and volatility. Regarding the stochastic or non-stationary character of volatility, we are able by the very nature of our approach to introduce this parameter in a probabilistic rather than deterministic way. Bayesian statistics treat the parameters of distributions of random variables as random variables themselves and assign to them probability distributions. This adds an important element of flexibility to our method.[5]

The organisation of the paper is as follows: In section 2 we outline the classical distributional assumptions behind the Black-Scholes model and its estimation. In section 3 we work towards establishing a Bayesian option pricing framework and extend our previous work (Darsinos and Satchell (2001)) from a modelling context to a forecasting context. The posterior and predictive densities of the Black-Scholes option price are derived. Section 4 deals with the empirical implementation of our model. We test the predictive performance of our Bayesian distributions when applied to the market of FTSE 100 European index options. Concluding remarks follow in section 5.

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[4] For an exception see however Day and Lewis (1992). There the authors add the implied volatility as an exogenous variable to GARCH-type models to examine the incremental information content of implied volatilities.

[5] Indeed as noted by Bauwens and Lubrano (2000) the predictive method, which incorporates an additional source of uncertainty, is a better alternative to using a marginal measure to be plugged in the Black-Scholes formula which can be very dangerous, particularly at times of near nonstationarity.

## 2. A CLASSICAL FRAMEWORK FOR OPTION PRICING

We start with the classical Black-Scholes assumption that the stock price $P_t$ follows a Geometric Brownian Motion. This then implies the following formula for the stock price:

$$(2.1) \qquad P_t = P_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

where $W_t$ is a standard Brownian motion ($W_0 = 0$), $P_0$ is the initial price at time $0$, $\mu$ is the instantaneous mean and $\sigma^2$ the instantaneous variance. The Black-Scholes option price for a European call option is then given by

$$(2.2) \qquad C_t = C_{BS}(P_t, \sigma) = P_t \Phi(d_1) - K \exp(-r\tau)\Phi(d_2),$$

where 
$$d_1 = \frac{\log(P_t / K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

and 
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-\frac{y^2}{2})dy$$

$K$ is the exercise price at the expiry date $T$, $r$ is the risk-free rate of interest, and $\tau = T - t$ the time to maturity. Note that the only unobservable parameter entering the valuation formula (2.2) is the variance parameter $\sigma^2$. The next step in the valuation problem is therefore to estimate $\sigma^2$.

### 2.1. Historical Information (Sample Information)

The classical minimum-variance unbiased estimator of $\sigma^2$ for $t$ observations of (daily) continuously compounded returns is then given by the sample variance $s^2 = \sum_{i=1}^{t}(x_i - \overline{x})^2 /(t-1)$ where $x$ is the log-return between two consecutive time intervals (i.e. $x_j = \log(P_j / P_{j-1})$) and $\overline{x}$ is the sample mean return (i.e. $\overline{x} = (1/t)\sum_{j=1}^{t} x_j$). It is well known that the statistic $(t-1)s^2 / \sigma^2$ has a $\chi^2$ (chi-square) distribution with $t$-1 degrees of freedom[6]. From this we can obtain the probability density function of the sample variance (or the likelihood function of the true variance). It is given by:

$$(2.3) \qquad f(s^2 \backslash \sigma^2, t) \equiv L(\sigma^2 \backslash s^2, t) \equiv (\frac{t-1}{2})^{\frac{t-1}{2}} \frac{(s^2)^{\frac{(t-1)}{2}-1}}{\Gamma(\frac{t-1}{2})(\sigma^2)^{\frac{t-1}{2}}} \exp(-\frac{(t-1)s^2}{2\sigma^2}).$$

$L(\ )$ denotes a likelihood function.

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[6] The probability density function of a variable $z$ that is distributed chi-squared with $t$-1 degrees of freedom is given by $f(z) = \frac{1}{\Gamma((t-1)/2)2^{(t-1)/2}} z^{((t-1)/2)-1} \exp(-z/2)$.

Since we want to work with standard deviations rather than variances we transform equation (2.3) to represent the distribution of the sample standard deviation:

$$(2.4) \qquad f(s \setminus \sigma, t) = 2(\frac{t-1}{2})^{\frac{t-1}{2}} \frac{s^{t-2}}{\Gamma(\frac{t-1}{2})\sigma^{t-1}} \exp(-\frac{(t-1)s^2}{2\sigma^2}).$$

Note that here and throughout we will use $f(\ )$ to denote probability density functions generally and not one specific probability density. The argument of $f(\ )$ as well as the context in which it is used will identify the particular probability density being considered.

Also from equation (2.1) we have that the stock price $P_t$ is lognormally distributed. Its probability density function is given by:

$$(2.5) \qquad f(P_t \setminus \mu, \sigma, t) = \frac{1}{P_t \sqrt{2\pi t}\sigma} \exp\left\{ \frac{-[\ln(\frac{P_t}{P_0}) - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t} \right\}$$

Note that for notational simplicity we will ignore dependence on $P_0$.


# 3. A BAYESIAN FRAMEWORK FOR OPTION PRICING

In the classical Black-Scholes framework the drift and diffusion parameters are regarded as constants. In a Bayesian framework these parameters are introduced in a probabilistic rather than deterministic way and are treated as random variables. We should therefore identify probability distributions for the drift and diffusion parameters. In identifying these distributions we follow standard Bayesian methodology as presented in Raiffa and Schlaifer (1961), Zellner (1971) and more recently in Hamilton (1994), and Bauwens, Lubrano and Richard (1999).

## 3.1 Drift Information (Non-Informative Prior)

Following Darsinos and Satchell (2001) we have that the conditional probability density function of the expected rate of return $\mu$ is given by:

$$(3.1) \qquad f(\mu \setminus \sigma, t, m) = \frac{\sqrt{t}}{\sqrt{2\pi}\sigma} \exp\left( -\frac{t(\mu - m)^2}{2\sigma^2} \right)$$

where $m$ is a hyperparameter.

In calculating $m$ we intend to use the "empirical" Bayes approach. That is we estimate the hyperparameter from the sample information $(1,2,\ldots,t)$. This can be viewed as incorporating a non-informative prior for the expected rate of return of the underlying. Alternatively one could use prior sample information or information in the form of analysts forecasts.

## 3.2 Implied Volatility Information (Informative Prior)

As a source of prior information we use the implied volatilities or the at-the-money implied volatilities of reported option prices. We assume that the variance has an Inverted-Gamma-1 distribution with prior parameters $(t's'^2/2, t'/2)$.

$$(3.2) \qquad f(\sigma^2 \backslash s'^2, t') = \left(\frac{t's'^2}{2}\right)^{\frac{t'}{2}} \frac{\exp(-\frac{t's'^2}{2\sigma^2})}{\Gamma(\frac{t'}{2})(\sigma^2)^{\frac{t'}{2}+1}} \qquad 7$$

Transforming the above equation to represent the probability density of volatility we have:

$$(3.3) \qquad f(\sigma \backslash s', t') = 2\left(\frac{t's'^2}{2}\right)^{\frac{t'}{2}} \frac{\exp(-\frac{t's'^2}{2\sigma^2})}{\Gamma(\frac{t'}{2})(\sigma)^{t'+1}}$$

Note that $s'$ represents the implied volatility estimate, and $t'$ the weight attached to it. For example the analyst might use a month of implied volatility data and calculate $s'$ as the sample mean of this data (alternatively $s'$ can represent a composite measure of implied volatilities). The weight she attaches to $s'$ can then be for example a month ($t'=30$). However this may not necessarily be the case and the analyst can use any weight she feels appropriate. To illustrate consider the example where she uses the previous day's implied volatility estimate as prior information. Here if she chooses $t'=1$ (i.e. 1 day) the prior (implied volatility) information will be absorbed by the sample information since if we assume that she used one month (say $t = 30$ days) of daily historical returns to estimate the sample standard deviation $s$ the weight attached to the sample information will be $t$-1 =29 (1 degree of freedom is lost in estimating $s$) while the weight attached to the implied volatility information will be just 1. To sum up, if one believes strongly about the prior information she can use as much weight as she feels it merits. This can be a powerful tool in the hand of the analyst since different point (or interval) estimates can be obtained using different weights. Bayesian methods are in a way both a science and an art!

## 3.3 The Posterior Density of the Black-Scholes Option Price

In contrast to classical analysis where the main piece of output is a point estimate, Bayesian analysis produces as its main piece of output the so-called posterior density. This posterior density can then be combined with a loss or utility function to allow a decision to be made on the basis of minimising expected loss or maximising expected utility. For example, for positive definite quadratic loss functions the mean of the posterior distribution is an

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[7] *Remark*: when the distributions are conditional on any prior parameters (i.e. $s', t'$ and $m$) and on $t$ (it is not unreasonable to assume that the sample size is known before the sample is drawn) we will refer to these distributions as prior or unconditional.

optimal point estimate. If the loss is proportional to the absolute value of the difference between the true and the estimated values, the median is chosen, while a zero loss for a correct estimate and a constant loss for an incorrect estimate leads to the choice of the mode.

We now illustrate how we can derive the posterior density of the Black-Scholes option price by using equations (2.2 - 2.5) and (3.1 - 3.3) above. Note that we will use only symbolic notation. The analytical formulae for all the densities involved in the calculations are exhibited for reference in the Appendix. For their derivations however, see Darsinos and Satchell (2001). i) Since the option price as an unconditional random variable depends both on the underlying and its volatility, let us first obtain the posterior density of price and volatility. Then the posterior density of the option price follows after (ii) applying a non-linear transformation and (iii) dividing by the conditional (on the sample and prior information) density of the asset price.

i) We start from the densities of the drift (i.e. equation (3.1)) and of volatility (i.e equation (3.3)). Then the joint density of drift and volatility is given by:

(3.4) $\qquad f(\mu, \sigma \setminus t, s', t', m) = f(\mu \setminus \sigma, s', t', m) f(\sigma \setminus s', t', (m))$ [8]

Now using equation (3.4) and the distribution of the underlying (i.e. equation (2.5)) we get:

(3.5) $\qquad f(P_t, \mu, \sigma \setminus t, s', t', m) = f(\mu, \sigma \setminus t, s', t', m) f(P_t \setminus \mu, \sigma, t, (s'), (t'), (m))$

Integrating out the drift rate we get the "prior" density of price and volatility (see also Footnote 6):

(3.6) $\qquad f(P_t, \sigma \setminus t, s', t', m) = \int_{-\infty}^{\infty} f(P_t, \mu, \sigma \setminus t, s', t', m) \partial \mu$

Then applying Bayes rule the posterior density of price and volatility is given by:

(3.7) $\qquad f(P_t, \sigma \setminus s, t, s', t', m) = \dfrac{f(P_t, \sigma \setminus t, s', t', m) f(s \setminus (P_t), \sigma, t, (s'), (t'), (m))}{f(s \setminus t, s', t', (m))}$

$$= \dfrac{f(P_t, \sigma, s \setminus t, s', t', m)}{f(s \setminus t, s', t', (m))}$$

Observe that the numerator of the above equation (i.e. $f(P_t, \sigma, s \setminus t, s', t', m)$) is readily obtained by multiplying equations (3.6) and (2.4), while the denominator (i.e. $f(s \setminus t, s', t', (m))$) is derived from the following calculations:

Multiplying equation (3.2) with equation (2.3) we get:

$$f(s^2, \sigma^2 \setminus t, s'^2, t') = f(\sigma^2 \setminus (t), s'^2, t') f(s^2 \setminus \sigma^2, t, (s'^2), (t'))$$

Then

$$f(s^2 \setminus t, s', t') = \int_0^{\infty} f(s^2, \sigma^2 \setminus t, s'^2, t') \partial \sigma^2 .$$

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[8] Observe that in the density of volatility $f(\sigma \setminus s', t', (m))$, $(m)$ appears in parenthesis. Here and below when a parameter is exhibited in parenthesis we take this to mean that it does not actually appear in the analytic formula for that specific density but for coherence of the argument we include it in the symbolic notation.

Finally

(3.8)     $f(s \setminus t, s', t', (m)) = 2sf(s^2 \setminus t, s'^2, t')$ .

ii) Having obtained the posterior density $f(P_t, \sigma \setminus s, t, s', t', m)$ and remembering that $C_t = C_{BS}(P_t, \sigma)$ (defined in (2.2)) we now apply the non-linear transformation:

$$P_t = P_t$$
$$\sigma = C_{BS}^{-1}(C_t) \equiv C_{BS}^{-1}(P_t, C_t) \Leftrightarrow C_t = C_{BS}(P_t, \sigma)$$

We invert the Black-Scholes option pricing formula in terms of $\sigma$ for fixed $P_t$, thus obtaining $\sigma$ as a function of $C_t$ and $P_t$. This is the so-called implied volatility of the option price and there is known to be a unique (one-to-one) inverse function from the monotonicity of the option price as a function of volatility. Note however that there is no analytic expression (with the exception of an at-the-money option) for $\sigma = C_{BS}^{-1}(P_t, C_t)$ and it will have to be evaluated numerically using a Newton-Raphson iterative procedure.

Applying the transformation we get

(3.9)     $f(P_t, C_t \setminus s, t, s', t', m) = f(P_t, \sigma = C_{BS}^{-1}(P_t, C_t) \setminus s, t, s', t', m)|J|$

where $J$ is the Jacobian of the non-linear transformation and is given by:

$$\frac{1}{J} = \begin{vmatrix} \partial P_t / \partial P_t & \partial P_t / d\sigma \\ \partial C_t / \partial P_t & \partial C_t / \partial \sigma \end{vmatrix} = \frac{\partial C_t}{\partial \sigma} = Vega$$

where $Vega = \phi\left(\dfrac{\ln(\dfrac{P_t}{Ke^{-r\tau}}) + \dfrac{C_{BS}^{-1}(P_t, C_t)^2 \tau}{2}}{C_{BS}^{-1}(P_t, C_t)\sqrt{\tau}}\right) P_t \sqrt{\tau}$ .

$\phi(...) = \Phi'(...)$ denotes the standard normal probability density function.

iii) Finally to obtain the posterior density of the Black-Scholes option price we divide the joint density of price and option price (i.e. equation (3.9)) with the conditional (on the sample information) density of the underlying price:

(3.10)     $f(C_t \setminus P_t, s, t, s', t', m) = \dfrac{f(P_t, C_t \setminus s, t, s', t', m)}{f(P_t \setminus s, t, s', t', m)}$

From Darsinos and Satchell (2001) the analytic expression for the posterior density of the Black-Scholes option price is given by:

$$f(C_t \setminus P_t, s, t, s', t', m) = \frac{|J|\left(\dfrac{2(t's'^2 + (t-1)s^2)t + (\ln(P_t / P_0) - mt)^2}{64}\right)^{\frac{t'+t}{4}}}{K_{\frac{t'+t}{2}}\left(\dfrac{1}{4}\sqrt{2(t's'^2 + (t-1)s^2)t + (\ln(\dfrac{P_t}{P_0}) - mt)^2}\right) C_{BS}^{-1}(P_t, C_t)^{t'+t+1} (\dfrac{t}{16})^{\frac{t'+t}{2}}}$$

$$\times \exp(-\frac{t's'^2 + (t-1)s^2}{2C_{BS}^{-1}(P_t,C_t)^2} + \frac{\ln(P_t/P_0) - mt}{4})$$

$$\times \exp\left(-\frac{1}{4C_{BS}^{-1}(P_t,C_t)^2 t}[\ln(\frac{P_t}{P_0}) - (m - \frac{1}{2}C_{BS}^{-1}(P_t,C_t)^2)t]^2\right)$$

where $K_{\frac{t'+t}{2}}(...)$ represents the modified Bessel function of the second kind of order $(t'+t)/2$.

Although seemingly complicated the above density is in fact very simple and fully operational. We subject it to an immediate numerical test to verify that it is a proper density. We are able to confirm that it does integrate to one.[9]

A number of different point or interval estimators can be obtained from the above density. As a point estimator the most popular is probably the mean of the posterior density which is optimal under square error loss. However as already mentioned above the median and the mode can also prove useful point estimators. The derived density is also ideal for quantile estimation and Value-at-Risk (VaR) calculations. Darsinos and Satchell (2001) show that the posterior probability distribution for a long call option generally exhibits excess kurtosis and is positively skewed. In particular for an at-the money option the distribution is close to normal, however as we move progressively out- or in- the money the distribution of the option price exhibits an increasingly thinner left tail than the normal distribution. The VaR for an asset or portfolio of assets is critically dependent on the left tail of their distributions. If for example one assumes that the distribution of a long call option is normal (s)he will tend to calculate a VaR that is higher than the true VaR. Similarly for a short call the calculated VaR will be too low.

### 3.4 The Predictive Density of the Black-Scholes Option Price

We now extend the work of Darsinos and Satchell (2001) from a modelling context to a forecasting context. On many occasions given our sample information, we are interested in making inferences about other observations that are still unobserved, one part of the problem of prediction. In the Bayesian approach the probability density function for the as yet unobserved observations given our sample information can be obtained and is known in the Bayesian literature as the predictive density.

In our case, given our sample information $(P_t, s)$, (note that $s$ is computed from $P_0...P_t$), we are interested in making inferences about future option values $C_T = C(P_T, \sigma)$ for some $T > t$. To obtain the predictive density of the option price we proceed as follows. In equation (3.7)

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[9] Just to report a set of trial values that we used (in daily format):

$t' = 30$, $s' = 0.010174$, $t = 30$, $s = 0.0076$, $m = 0.0005$, $P_t = 3157$, $P_0 = 3148$, $K = 3025$, $r = 0.00022$, $\tau = 18$.

For a detailed discussion on the numerical evaluation of the posterior density see Darsinos and Satchell (2001), p. 16-18. Here it suffices to report that $C_{BS}^{-1}(P_t,C_t)$ is an $n$-dimensional vector of implied volatilities: we evaluate $C_{BS}^{-1}(P_t,C_i)$ for $i = 1,....,n$ spanning (with the desired degree of accuracy) the whole range of values of $C_t$, thus generating an $n$-dimensional vector of implied volatilities.

above we have shown how to derive the posterior density of price and volatility given the sample and prior information. In analytic form it is given by:

$$f(P_t, \sigma \setminus P_0, s, t, s', t', m) = \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{\frac{t'+t-1}{2}}}{\sqrt{\pi t}\, P_t \Gamma(\frac{t'+t-1}{2})} \frac{1}{\sigma^{t'+t+1}} \exp\left(-\frac{t's'^2 + (t-1)s^2}{2\sigma^2}\right)$$

$$\times \exp\left(-\frac{1}{4\sigma^2 t}[\ln(\frac{P_t}{P_0}) - (m - \frac{1}{2}\sigma^2)t]^2\right)$$

Without loss of generality we can rewrite this as the joint distribution of the yet unobserved underlying price $P_T$ and of volatility, given the sample (i.e. $P_t, s$) and prior information:

(3.11)

$$f(P_T, \sigma \setminus P_t, s, T, t, s', t', m) = \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{\frac{t'+t-1}{2}}}{\sqrt{\pi(T-t)}\, P_T \Gamma(\frac{t'+t-1}{2})} \frac{1}{\sigma^{t'+t+1}} \exp\left(-\frac{t's'^2 + (t-1)s^2}{2\sigma^2}\right)$$

$$\times \exp\left(-\frac{1}{4\sigma^2(T-t)}[\ln(\frac{P_T}{P_t}) - (m - \frac{1}{2}\sigma^2)(T-t)]^2\right)$$

Our next step is to derive the joint distribution of the unobserved option price at time $T$ and of the future stock price. Again remembering that $C_T = C_{BS}(P_T, \sigma)$, take $f(P_T, \sigma \setminus P_t, s, T, t, s', t', m)$ and consider the transformation:

$$P_T = P_T$$
$$\sigma = C_{BS}^{-1}(C_T) \equiv C_{BS}^{-1}(P_T, C_T)$$

with Jacobian:

$$1/J^* = \begin{vmatrix} \partial P_T / \partial P_T & \partial P_T / d\sigma \\ \partial C_T / \partial P_T & \partial C_T / \partial \sigma \end{vmatrix} = \phi\left(\frac{\ln(\frac{P_T}{Ke^{-r(\tau-(T-t))}}) + \frac{[C_{BS}^{-1}(P_T,C_T)]^2(\tau-(T-t))}{2}}{C_{BS}^{-1}(P_T,C_T)\sqrt{\tau-(T-t)}}\right) P_T \sqrt{\tau-(T-t)}$$

Then

(3.12)

$$f(P_T, C_T \setminus P_t, s, T, t, s', t', m) = f\left(P_T, \sigma = C_{BS}^{-1}(P_T, C_T) \setminus P_t, s, T, t, s', t', m\right) |J^*|$$

Finally, to obtain the predictive density of the option price we require a single integration:

(3.13)

$$f(C_T \setminus P_t, s, T, t, s', t', m) = \int f(P_T, C_T \setminus P_t, s, T, t, s', t', m) \partial P_T$$

$$= \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{\frac{t'+t-1}{2}}}{\sqrt{\pi(T-t)}\Gamma(\frac{t'+t-1}{2})} \times$$

$$\int_0^\infty \frac{|J^*|\exp\left(-\frac{t's'^2+(t-1)s^2}{2[C_{BS}^{-1}(P_T,C_T)]^2} - \frac{1}{4[C_{BS}^{-1}(P_T,C_T)]^2(T-t)}[\ln(\frac{P_T}{P_t}) - (m-\frac{1}{2}[C_{BS}^{-1}(P_T,C_T)]^2)(T-t)]^2\right)}{[C_{BS}^{-1}(P_T,C_T)]^{t'+t+1}P_T} \partial P_T$$

In this case as well we are able to confirm that this is a proper density. Regarding the numerical evaluation procedure that could be applied for the calculation of the predictive density see again Darsinos and Satchell (2001). Here it suffices to note that $C_{BS}^{-1}(P_T,C_T)$ is a $n \times m$ matrix of implied volatilities. (We evaluate $C_{BS}^{-1}(P_i,C_j)$ for $i=1,...,n$ and $j=1,...,m$ spanning the whole range of attainable values of $P_T$ and $C_T$, thus generating a $n \times m$ matrix of implied volatilities).

The estimation of both the posterior (i.e. equation (3.10)) and predictive (i.e. equation (3.13) distributions is relatively simple. Both densities can be evaluated solely on the basis of observables. Estimation of the parameters of the distributions requires only historical data on the underlying and reported option price data. In the forthcoming section we illustrate how the derived densities can be used for explaining and forecasting the market prices of FTSE 100 Index European call options.

# 4. EMPIRICAL IMPLEMENTATION

We aim to forecast call option prices both in- and out- of sample. This effectively means that in the former case we will use the posterior density of section (3.3) to *explain* the observed market prices of call options and compare the precision of the Bayesian estimates with some benchmark forecasts that use historical or implied volatility. In the latter case we aim to *forecast* the prices of call option prices one day in the future using the predictive density of section (3.4) above.

We use daily data from the London International Financial Futures and Options Exchange (LIFFE) for the period from September 1992 to December 2000. Our data concern FTSE 100 Index European call option contracts. The data record for each contract contains the price of the underlying, the exercise price, the expiration date, the settlement price, the trading volume, the corresponding implied volatility and the at-the-money implied volatility. To proxy for the risk-free rate, the rate on a T-bill of comparable maturity is used. Note that the underlying on the European contract is the price of the corresponding index future. One slight modification that therefore has to be made is that instead of the Black-Scholes model of equation (2.2) we will use Black's (1976) model for options on futures:

(4.1) $$C_t = e^{-r\tau}[F_t\Phi(d_1) - K\Phi(d_2)],$$

where $d_1 = \dfrac{\log(F_t/K) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

$F_t$ represents the corresponding future's price.

We are aware that on data where there is little or no volume, the exchange uses artificially generated prices based on a system called Autoquote, which effectively uses Black's formula. To minimise such an effect we limit our range of investigation to option maturities ranging from one week to seven weeks. For maturities within this range there is always a reasonable volume of trading. Similarly since for each contract we have a variety of exercise prices we keep the contracts where the option was at some stage during the period of investigation close to the money. Our remaining data represent a variety of options out-of-, at- and in-the-money with maturities varying from one week to seven weeks.

## 4.1 In-Sample Forecasting

This section assesses the extent to which the Bayesian call option value estimators improve upon standard classical procedures in describing actual market prices of call options. By standard procedures we mean forecasts based on Black's model for options on futures (i.e. equation (4.1)). We use five different estimates of volatility to be plugged into Black's model and thus provide five benchmark estimates of the price of the option today. In particular as an estimate of volatility we use (i) the sample standard deviation from daily returns over the 15 days preceding the date of each option price reported (ii) the sample standard deviation from daily returns over the 30 days preceding the date of each option price reported, (iii) the mean from implied volatilities over the 15 days preceding the date of each option price reported,

(iv) the mean from implied volatilities over the 30 days preceding the date of each option price reported, and (v) the previous days' implied volatility value.

For the Bayesian estimators we take the mean of the posterior distribution as a point estimate. This is optimal under quadratic loss. We use four different Bayesian estimates. These are distinct in the sense that we use two different estimation horizons when estimating the parameters of the distribution (i.e. $t, s, t', s', m$). In particular two estimates are derived using a 15-day estimation horizon and the other two using a 30-day estimation horizon. This effectively means that in the former case $t$ and $t'$ are both equal to 15 (i.e. we use 15 days of prior and sample information) while in the latter case they are equal to 30 (i.e. we use 30 days of prior and sample information). The parameter $s$ is estimated as the sample standard deviation from daily returns over the 15 and 30 days respectively preceding the date of each option price reported. The parameter $m$ is estimated as the mean of daily returns over the last 15 and 30 days respectively preceding each reported option price.

The two estimates that belong to the same estimation horizon are in turn distinct since we either use the implied volatility or the at-the-money implied volatility to estimate $s'$. Hence $s'$ is estimated as the mean of implied volatilities or the mean of at-the-money implied volatilities over the specified time horizon preceding the reported option price.

We measure overall fit to market data in terms of Mean Mispricing Error (MME). The Mispricing Error (ME) of each estimate is computed by $(\hat{C}_{it} - C_{it}) / C_{it}$ where $C_{it}$ is the market call option price and $\hat{C}_{it}$ is the estimated Bayesian or other (benchmark) option value. Then the Mean Mispricing Error (MME) is given by:

(4.2)
$$MME = (1/N) \sum_{i=1}^{N} (\hat{C}_{it} - C_{it}) / C_{it} .$$

We also assess the Relative Mispricing Error (RME) of the option price estimates with respect to the time to maturity of the option and the degree to which the option is in- or out- of the money (moneyness). This is done by averaging the Mispricing Error (ME) within the different subgroups. Note that the "moneyness" of an option is measured as $(F_t / K) - 1$ where $F_t$ is the underlying FTSE 100 index future price and $K$ the exercise price of the option. Hence a negative value indicates an out-of-the money option and a positive value an in-the-money option.

We use four randomly/arbitrarily selected European FTSE 100 index call option contracts. Namely these are the June 1998, March 1999, September 2000 and December 2000 contracts. The moneyness of options in these contracts ranged from 5% out of the money to 8% in the money. The maturities considered were 1 - 7 weeks. Table 1 summarises our results.

TABLE 1

I) Mean Mispricing Error (MME) of Bayesian and Other Call Option Price Estimates

Mean Mispricing Error (MME)

*Benchmark Models:*

| | MME |
|---|---|
| Historical Volatility (15-Days) | 0.114 |
| Historical Volatility (30-Days) | 0.108 |
| Implied Vol. (15-Day moving average) | 0.060 |
| Implied Vol. (30-Day moving average) | 0.066 |
| Implied Vol. (Previous Day) | 0.042 |

*15-Day Estimation Horizon*

| | |
|---|---|
| Bayesian (Implied V. - Hist. V.) | 0.050 |
| Bayesian (A-T-M Implied V. - Hist. V.) | 0.060 |

*30-Day Estimation Horizon*

| | |
|---|---|
| Bayesian (Implied V. - Hist. V.) | 0.053 |
| Bayesian (A-T-M Implied V. - Hist. V.) | 0.072 |

II) Relative Mispricing Errors of Bayesian and Other Call Option Price Estimates with Different Times to Maturity

| Time to maturity: | 1-week | 2-weeks | 3-weeks | 4-weeks | 7-weeks |
|---|---|---|---|---|---|
| *Benchmark Models:* | | | | | |
| Historical Volatility (15-Days) | 0.088 | 0.028 | 0.122 | 0.218 | 0.112 |
| Historical Volatility (30-Days) | 0.135 | 0.093 | 0.097 | 0.073 | 0.140 |
| Implied Vol. (15-Day moving average) | 0.108 | 0.057 | 0.026 | 0.081 | 0.030 |
| Implied Vol. (30-Day moving average) | 0.063 | 0.076 | 0.031 | 0.115 | 0.043 |
| Implied Vol. (Previous Day) | 0.098 | 0.028 | 0.026 | 0.027 | 0.032 |
| *15-Day Estimation Horizon* | | | | | |
| Bayesian (Implied V. - Hist. V.) | 0.092 | 0.008 | 0.054 | 0.040 | 0.055 |
| Bayesian (A-T-M Implied V. - Hist. V.) | 0.111 | 0.023 | 0.061 | 0.040 | 0.065 |
| *30-Day Estimation Horizon* | | | | | |
| Bayesian (Implied V. - Hist. V.) | 0.074 | 0.049 | 0.026 | 0.051 | 0.067 |
| Bayesian (A-T-M Implied V. - Hist. V.) | 0.089 | 0.046 | 0.025 | 0.054 | 0.072 |

III) Relative Mispricing Errors of Bayesian and Other Call Option Price Estimates with Different Degrees of Moneyness

| Moneyness ($M$) | $-5\%<M<-2\%$ | $-2\%<M<0\%$ | $0\%<M<3\%$ | $3\%<M<8\%$ |
|---|---|---|---|---|
| *Benchmark Models:* | | | | |
| Historical Volatility (15-Days) | 0.116 | 0.124 | 0.169 | 0.048 |
| Historical Volatility (30-Days) | 0.253 | 0.086 | 0.112 | 0.049 |
| Implied Vol. (15-Day moving average) | 0.188 | 0.063 | 0.041 | 0.016 |
| Implied Vol. (30-Day moving average) | 0.116 | 0.068 | 0.081 | 0.023 |
| Implied Vol. (Previous Day) | 0.139 | 0.050 | 0.021 | 0.008 |
| *15-Day Estimation Horizon* | | | | |
| Bayesian (Implied V. - Hist. V.) | 0.121 | 0.050 | 0.049 | 0.015 |
| Bayesian (A-T-M Implied V - Hist. V) | 0.161 | 0.051 | 0.054 | 0.019 |
| *30-Day Estimation Horizon* | | | | |
| Bayesian (Implied V. - Hist. V.) | 0.145 | 0.052 | 0.048 | 0.013 |
| Bayesian (A-T-M Implied V - Hist. V) | 0.156 | 0.053 | 0.050 | 0.018 |

Even before this exercise was undertaken the evidence was overwhelming that implied volatilities fit option prices much better than historical volatilities. Our results from Table 1 confirm the aforementioned statement. The Mean Mispricing Error (MME) for the historical volatility estimates ranged between 10.8% and 11.4% against a range of 4.2% and 6.6% for the implied volatility estimates. The estimate that used the reported implied volatility value of the previous day to be plugged into Black's model outperformed all other estimates with a MME of 4.2%. The performance of our Bayesian estimates paralleled that of the implied volatility estimates with MME ranging between 5.0% -7.2%. However we cannot claim that the Bayesian method is superior in explaining the observed market prices of FTSE 100 European call options. We might have achieved better results had we used the previous day's reported implied volatility as prior information rather than the sample mean over the 15 or 30 days preceding the date of each reported option price. After all
it turned out that the previous day's implied volatility conveyed enough information to outperform the other estimates.[10] Other potential approaches that might have yielded better results could include using solely implied volatilities as a source of prior and sample information.

## 4.2 Out-of-Sample Forecasting

Implied volatilities by definition perform very well in explaining the observed market prices of options. For example analysts or even exchanges often calculate implied volatilities from actively traded options on a certain stock and use them to calculate the price of a less actively traded option on the same stock. The evidence however is not clear whether implied volatilities can on their own provide adequate forecasts of future volatility or indeed option prices.

Hence we now turn to the more interesting exercise of forecasting the prices of options one-day ahead. For this exercise we assume that there are five agents, each following a particular forecasting method to predict the price of the FTSE 100 Index European call of tomorrow. In particular *Agents* 1, 2, and 3 use Black's (1976) model for options on futures. As an estimate of volatility to be plugged into the model: *Agent* 1 uses the sample standard deviation from daily returns over the 30 days preceding the date of each option price being reported. *Agent* 2 uses the mean of the implied volatilities over the 30 days preceding the date of each option price reported. *Agent* 3 uses today's reported implied volatility value. As a forecast for tomorrow's value for the underlying all three agents use today's price. The approach of *Agent* 3 is quite popular amongst market practitioners. Gemmill and Saflekos (2000) estimate the implied distribution for stock index options in London as a mixture of two lognormals and find that this method is somewhat better that the Black-Scholes (one-lognormal) approach at predicting out-of-sample option prices. However according to the authors an ad-hoc model in which today's implied volatilities are applied to tomorrow's options does even better.

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[10] Using the previous day's implied volatility as prior information requires taking into account our discussion in the last paragraph of section (3.2).

*Agents* 4 and 5 use the Bayesian predictive density of section (3.4). They use the sample standard deviation from daily returns over the 30 days preceding the date of each option price reported to estimate the sample parameter *s* of the distribution. However, to estimate the prior parameter *s'* of the distribution *Agent* 4 uses the mean of the implied volatilities over the 30 days preceding the date of each option price reported while *Agent* 5 uses the mean of the *at-the-money* implied volatilities over the 30 days preceding the date of each option price reported. Finally the parameter *m* is estimated by both agents as the mean of daily returns over the last 30 days preceding each reported option price. As a point estimate of the option price the agents take the mean of the predictive distribution.

We use 12 randomly selected European FTSE 100 Index call option contracts. Namely these are the September 1992, June 1993, December 1994, December 1995, December 1996, June 1997, June 1998, December 1998, March 1999, June 1999, September 2000 and December 2000 contracts. Then for *each* contract we fix eight dates for which we want to forecast the price of the option. Specifically we obtain forecasts for the value of the option 1 week, 2 weeks, 18 days, 3 weeks, 25 days, 4 weeks, 30 days, and 7 weeks before maturity. This effectively means that (for each contract) each agent applies his forecasting rule 8 days, 15 days, 19 days, 22 days, 26 days, 29 days, 31 days, and 50 days respectively before maturity. Note here that the above dates were pre-specified arbitrarily/randomly.

In this exercise we measure overall forecasting performance in terms of the Mean Forecasting Error (MFE):

(4.3) $$MFE = (1/N)\sum_{i=1}^{N}(\hat{C}_{i(t+1)} - C_{i(t+1)})/C_{i(t+1)}.$$

We also report the Relative Forecasting Error (RFE) of the option price forecasts with respect to the time to maturity of the options. Our results are exhibited in Table 2.

TABLE 2

| I) 1- Day Ahead Mean Forecast Error of Bayesian and Other Call Option Price Estimates |
|---|

Mean Forecast Error (MFE)

*Benchmark Models:*
*Agent* 1: (Historical Volatility - 30 days)          0.231
*Agent* 2: (Implied Volatility - 30 days)            0.225
*Agent* 3: (Today's Implied Volatility)              0.220

*30-Day Estimation Horizon*
*Agent* 4: (Bayesian - Implied V - Hist V)           0.196
*Agent* 5: (Bayesian - A-T-M IV - Hist V)            0.198

II) 1- Day Ahead Relative Forecast Error (RFE) of Bayesian and Other Call Option Price Estimates with Different Times to Maturity

| Time to maturity: | 1-week | 2-weeks | 18-days | 3-weeks |
|---|---|---|---|---|
| *Benchmark Models:* | | | | |
| *Agent* 1: (Historical Volatility - 30 days) | 0.311 | 0.370 | 0.228 | 0.209 |
| *Agent* 2: (Implied Volatility - 30 days) | 0.406 | 0.351 | 0.257 | 0.167 |
| *Agent* 3: (Today's Implied Volatility) | 0.443 | 0.340 | 0.195 | 0.186 |
| *30-Day Estimation Horizon* | | | | |
| *Agent* 4: (Bayesian - Implied V - Hist V) | 0.197 | 0.323 | 0.256 | 0.173 |
| *Agent* 5: (Bayesian - A-T-M IV - Hist V) | 0.194 | 0.321 | 0.258 | 0.174 |

| Time to maturity: | 25-days | 4-weeks | 30-days | 7-weeks |
|---|---|---|---|---|
| *Benchmark Models:* | | | | |
| *Agent* 1: (Historical Volatility - 30 days) | 0.197 | 0.094 | 0.208 | 0.234 |
| *Agent* 2: (Implied Volatility - 30 days) | 0.140 | 0.133 | 0.193 | 0.155 |
| *Agent* 3: (Today's Implied Volatility) | 0.161 | 0.095 | 0.169 | 0.168 |
| *30-Day Estimation Horizon* | | | | |
| *Agent* 4: (Bayesian - Implied V - Hist V) | 0.154 | 0.124 | 0.167 | 0.173 |
| *Agent* 5: (Bayesian - A-T-M IV - Hist V) | 0.156 | 0.138 | 0.162 | 0.180 |

This time the performance gap between the estimates that use either implied or historical volatilities is significantly reduced. *Agent* 1 who uses the historical volatility estimate still produces the poorest forecasts (with an MFE of 23.1%) closely followed by *Agents* 2 and 3 who use the two implied volatility forecasts (with an MFE of 22.5% for *Agent* 2 and 22.0% for *Agent* 3). *Agents* 4 and 5 who use the Bayesian predictive density forecasts outperform all others with the lowest MFE, at 19.6% and 19.8% respectively. In terms of the Relative Forecast Error (RFE) we observe that the Bayesian forecasts dramatically outperform all other forecasts for close-to- maturity options (i.e. 1 week) and generally are as good as, or better, than the benchmark models.

To assess further the performance of the five agents we now devise the following simple trading rule.[11] As mentioned above, during the sample period (September 1992 to December 2000), at the pre-specified dates, each agent applies his forecasting rule to get a forecast of the

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[11] Our trading rule is similar in spirit to Noh, Engle, and Kane (1994).

FTSE 100 Index call option price of tomorrow. If the option price forecast is greater than the market price of the option today, the call option is bought. If the option price forecast is less than the market option price today, the call option is sold. Note however that we apply a filtering strategy where each agent trades only when the price change is expected to exceed 2% of today's price: i.e.

(4.4)
$$(\hat{C}_{i(t+1)} - C_{it})/C_{it} > 2\%$$

The positions of the traders last only a day. Thus every trader is forced to close his position tomorrow. Hence for a long position the trader sells the option at tomorrow's settlement price. For a short position the trader buys the option at tomorrow's settlement price.

Each agent is given £100 to invest each time. When a call option is sold we allow the agent to invest the proceeds plus £100 in a risk free asset. Also if the forecasting rule indicates that no trade should take place, the sum of £100 is invested in a risk free asset. For simplicity we assume that the rate on the risk free asset is zero.

Thus, the rate of return (RT) (per trade) on buying call options is computed as:

(4.5)
$$RT = \frac{100}{C_t}(C_{t+1} - C_t)$$

The rate of return (RT) on selling call options is computed as:

(4.6)
$$RT = \frac{100}{C_t}(-(C_{t+1} - C_t))$$

The above rates however are without taking into account transaction costs. We need to incorporate this. Hence we assume that the transaction costs (per trade) incurred by the agent amount to 3% of the amount invested. Hence the net rate of return (NRT) from the trading of options is computed by:

(4.7)
$$NRT = RT - 100 * 3\% = RT - 3$$

Noh, Engle and Kane (1994) assume that the transaction cost for trading a straddle (i.e. a call and a put option) is $0.25 per straddle. Inspired from that we also calculate an alternative net rate of return were we assume that the cost of trading a call option is £0.50.

(4.8)
$$NRT = RT - \frac{100}{C_t} * 0.50$$

Discussions with option traders suggest that our transaction costs are not accurate, as they do not capture the huge spreads and extreme illiquidity that can occur in these markets. Thus our trading profits should be seen as a measure of economic worth and not necessarily as an attainable amount of money.

We can now compare the performance of the agents with different forecasting algorithms. In Table 3 we report the mean return of each agent per trade (or day).

TABLE 3

Mean Daily Rate of Return (or Mean Per Trade Rate of Return) from Trading FTSE 100 Index European Call Options.

|  | Mean | SD | t-Ratio |
|---|---|---|---|
| *Before Transaction Costs* | | | |
| *Agent 1* (Historical Volatility) | 1.4% | 29.0 | 0.46 |
| *Agent 2* (Implied Volatility) | 2.3% | 19.8 | 1.11 |
| Agent 3 (Today's Implied Volatility) | 0.4% | 25.5 | 0.15 |
| *Agent 4* (Bayesian) | 6.6% | 25.0 | 2.52 |
| *Agent 5* (Bayesian A-T-M) | 6.2% | 26.7 | 2.22 |
| | | | |
| *3% Transaction Costs* | | | |
| *Agent 1* (Historical Volatility) | -1.5% | 29.0 | 0.49 |
| *Agent 2* (Implied Volatility) | 0.4% | 19.8 | 0.19 |
| Agent 3 (Today's Implied Volatility) | -1.4% | 25.5 | 0.52 |
| *Agent 4* (Bayesian) | 4.6% | 24.7 | 1.78 |
| *Agent 5* (Bayesian A-T-M) | 3.9% | 26.5 | 1.40 |
| | | | |
| *£0.50 per call option Transaction Costs* | | | |
| *Agent 1* (Historical Volatility) | 0.4% | 28.2 | 0.14 |
| *Agent 2* (Implied Volatility) | 1.7% | 19.6 | 0.83 |
| Agent 3 (Today's Implied Volatility) | -0.4% | 26.2 | 0.15 |
| *Agent 4* (Bayesian) | 5.8% | 24.2 | 2.29 |
| *Agent 5* (Bayesian A-T-M) | 5.4% | 26.0 | 1.98 |

Table 3 shows the daily rate of return from trading call options before and after transaction costs. It is clear that *Agents* 4 and 5 outperform the others with average daily rates of return between 4.6% and 6.2% and 3.9% and 6.2% respectively depending on the transaction costs incurred. It should be noted though that the profits of all agents are far from certain since the corresponding standard deviations range from 19.6 to 29. However t-ratios higher than 2 indicate that profits from the Bayesian forecasting method are significantly greater than zero.[12] This argument is supported by Table 4, which shows the cumulative rate of return of the 5 agents.

TABLE 4

Cumulative Returns from Call option Trading

| | Before Transaction Costs | 3% Transaction Costs | £0.50 Per Call Option Transaction Costs |
|---|---|---|---|
| *Agent 1* | 128.4% | -138.6% | 39.9% |
| *Agent 2* | 212.2% | 38.2% | 156.2% |
| *Agent 3* | 38.8% | -123.2% | -33.7% |
| *Agent 4* | 603.1% | 417.1% | 531.7% |
| *Agent 5* | 567.5% | 354.5% | 490.7% |

Note that Table 4 is for comparative purposes among the different performances of the agents and is not representative for assessing the return of each method over the 8-year period.

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[12] Since rates of return from call trading are assumed to be independent, the t-ratio is computed as a ratio of mean to standard deviation divided by the square root of the number of observations.

What is striking from Tables 3 & 4 is not the absolute performance of the agents, which as already mentioned may not necessarily represent real returns, but their comparative performance. Indeed *Agents* 4 & 5 outperform comfortably the agents that use the standard routines. Noh, Engle and Kane (1994) perform a similar study. They feed an asset's return series into a GARCH model to obtain a forecast of volatility to be plugged into the Black–Scholes model. They compare this method against forecasts obtained from implied volatility regressions that are also to be plugged into the Black-Scholes model. They assess the performance of these two volatility prediction models for S&P 500 index options over the April 1986 to December 1991 period. They find that the average daily rate of return from trading near-the-money straddles (before transaction costs) is 1.36% for the GARCH forecasting method and 0.44% for the Implied Volatility forecasting method. The standard deviations that they obtain are also quite large (in the range of 10 – 12). Although because of the large standard deviations we cannot really be sure, observe that what we find in Table 3 is not that dissimilar. *Agent* 1 who uses historical returns delivers (before transaction costs) a mean return of 1.4% and *Agents* 2 and 3 who use Implied Volatilities deliver 2.3% and 0.4% respectively.

Finally in Table 5 we report the percentage of times where the forecast of each agent resulted in a profit being made, a no-trade situation, or a loss being made.

TABLE 5

Percentage of times where the forecast of each agent resulted in a profit being made, a no-trade situation, or a loss being made.

|  | Profit | No Trade | Loss |
|---|---|---|---|
| *Agent* 1: (Historical Volatility - 30 days) | 52.7% | 4.4% | 42.9% |
| *Agent* 2: (Implied Volatility - 30 days) | 34.0% | 37.4% | 28.6% |
| *Agent* 3: (Today's Implied Volatility) | 30.8% | 41.7% | 27.5% |
| *Agent* 4: (Bayesian - Implied Vol.) | 46.2% | 34.1% | 19.7% |
| *Agent* 5: (Bayesian - A-T-M Implied V.) | 49.5% | 24.2% | 26.3% |

In this case as well *Agents* 4 and 5 outperform the others. They make a profit 70% and 65% respectively of the times they *trade*. (The loss-making times being of course the remaining percentage). Although *Agent* 1 in absolute terms makes a profit more times than every other agent (i.e. 52.7%), in real terms her profit rises to only 55% of the times she trades. Finally *Agents* 2 and 3 make a profit 54% and 53% of the times they trade.

## 5. CONCLUSION

It has been suggested that both historical and implied volatilities convey information about future volatility. In this paper we have developed a formal Bayesian framework to simultaneously exploit the information content of historical data as represented in moving averages of daily squared returns with the information content of options prices as represented in moving averages of reported implied volatilities. To this end, we have derived the posterior

and predictive distributions of the Black Scholes option price. We have used the FTSE 100 Index European options market to compare our model's forecasting performance with standard models that use historical or implied volatility forecasts. All such benchmark forecasts are plugged into Black's (1976) model.

Our approach gives a modest outperformance relative to the usual volatility schemes when measured in terms of MFE (Mean Forecasting Error), and RFE (Relative Forecasting Error). However, when we assess our model in economic terms, i.e. in terms of the profit to a trading strategy based on our forecasts versus the other benchmark forecasts, we find quite substantial outperformance. We do not claim guaranteed excess risk adjusted returns for practitioners who might wish to follow our strategy. We recognise that option markets tend to have high and varying transaction costs and high illiquidity at unpredictable times which makes real-time back-testing extremely difficult. Our results should be interpreted as an alternative measure of forecasting performance. With such an interpretation our results indicate a clear superiority over the other methods.

We have not experimented a great deal with different weighting schemes for the prior and sample information, which given our results might deserve more attention. For example a useful extension might be to attach weights according to the forecasting performance that each source of information has. Likewise we have only used the simplest of time series models (i.e. moving average) to capture time-variation in historical and implied volatilities. The sole reason for doing so was for higher theoretical consistency with our benchmark option-pricing model (i.e. the Black-Scholes). Although we did not pursue this point, we could have just as easily merged EWMA (exponentially weighted moving average) or GARCH volatility estimates with the implied volatility information. Given the success of GARCH when compared with other time-series models in capturing time-varying risk and the wealth of information about price risk contained in options the combination of the two might have produced even better results. This would however lead us to the Duan (1995) GARCH option-pricing framework and to the Bauwens and Lubrano (2000) Bayesian GARCH approach. We leave such an analysis for future research.

# 6. APPENDIX

## Analytic Formulae for the Densities Required in Deriving the Posterior Density of the Black-Scholes Option Price:

Equation (3.4):

$$f(\mu,\sigma \setminus t,s',t',m) = \sqrt{\frac{2t}{\pi}} \frac{(t's'^2/2)^{t'/2}}{\Gamma(t'/2)} \frac{1}{\sigma^{t'+2}} \exp\left(-\frac{t's'^2 + t(\mu-m)^2}{2\sigma^2}\right)$$

Equation (3.5):

$$f(P_t,\mu,\sigma \setminus t,s',t',m) = \frac{1}{\pi P_t} \frac{(t's'^2/2)^{t'/2}}{\Gamma(t'/2)} \frac{1}{\sigma^{t'+3}}$$

$$\times \exp\left\{-\frac{t's'^2}{2\sigma^2} - \frac{1}{2\sigma^2 t}\left([\ln(\frac{P_t}{P_0}) + \frac{\sigma^2 t}{2} - \mu t]^2 + t^2(\mu-m)^2\right)\right\}.$$

Equation (3.6):

$$f(P_t,\sigma \setminus t,s',t',m) = \frac{1}{\sqrt{\pi}P_t} \frac{(t's'^2/2)^{t'/2}}{\Gamma(t'/2)} \frac{1}{\sigma^{t'+2}} \exp\left(-\frac{t's'^2}{2\sigma^2} - \frac{1}{4\sigma^2 t}[\ln(\frac{P_t}{P_0}) - (m-\frac{1}{2}\sigma^2)t]^2\right)$$

Equation (3.7):

$$f(P_t,\sigma \setminus s,t,s',t',m) = \frac{1}{\sqrt{\pi}P_t} \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{(t'+t-1)/2}}{\Gamma(\frac{t'+t-1}{2})} \frac{1}{\sigma^{t'+t+1}}$$

$$\times \exp\left(-\frac{t's'^2 + (t-1)s^2}{2\sigma^2} - \frac{1}{4\sigma^2 t}[\ln(\frac{P_t}{P_0}) - (m-\frac{1}{2}\sigma^2)t]^2\right)$$

Equation (3.8):

$$f(s \setminus t,s',t',(m)) = \frac{2}{B(\frac{t-1}{2},\frac{t'}{2})}\left(\frac{t's'^2}{2}\right)^{t'/2}(\frac{t-1}{2})^{\frac{t-1}{2}} \frac{s^{t-2}}{(\frac{t's'^2 + (t-1)s^2}{2})^{\frac{t+t'-1}{2}}}$$

Equation (3.9):

$$pdf(P_t,C_t \setminus s,t,s',t',m) = \frac{|J|}{\sqrt{\pi}P_t\Gamma(\frac{t'+t-1}{2})} \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{\frac{t'+t-1}{2}}}{C_{BS}^{-1}(P_t,C_t)^{2\theta+v+2}} \exp(-\frac{t's'^2 + (t-1)s^2}{2C_{BS}^{-1}(P_t,C_t)^2})$$

$$\times \exp\left(-\frac{1}{4C_{BS}^{-1}(P_t,C_t)^2 t}[\ln(\frac{P_t}{P_0}) - (m-\frac{1}{2}C_{BS}^{-1}(P_t,C_t)^2)t]^2\right)$$

<u>Denominator of Equation (3.10):</u>

$$pdf(P_t \setminus s,t,s',t',m) = K_{\frac{t'+t}{2}}\left(\frac{1}{4}\sqrt{2(t's'^2 + (t-1)s^2)t + (\ln(\frac{P_t}{P_0}) - mt)^2}\right)$$

$$\times \frac{(\frac{t's'^2 + (t-1)s^2}{2})^{(t'+t-1)/2}}{\sqrt{\pi t}P_t\Gamma(\frac{t'+t-1}{2})}(\frac{t}{16})^{\frac{t'+t}{2}}$$

$$\times [\frac{2(t's'^2 + (t-1)s^2)t + (\ln(\frac{P_t}{P_0}) - mt)^2}{64}]^{-\frac{t'+t}{4}} \exp(-\frac{\ln(P_t/P_0) - mt}{4})$$

where $K_{\frac{t'+t}{2}}\left(\frac{1}{4}\sqrt{2(t's'^2 + (t-1)s^2)t + (\ln(\frac{P_t}{P_0}) - mt)^2}\right)$ is the modified Bessel function of the

second kind of order $\frac{t'+t}{2}$ .

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