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# On the motive of some hyperKähler varieties

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**Abstract.** We show that the motive of the Hilbert scheme of length-$n$ subschemes on a K3 surface or on an abelian surface admits a decomposition similar to the decomposition of the motive of an abelian variety obtained by Shermenev, Beauville, and Deninger and Murre.

## Introduction

In this work, we fix a field $k$ and all varieties are defined over this field $k$. Chow groups are always meant with rational coefficients and $\mathrm{H}^*(-,\mathbb{Q})$ is Betti cohomology with rational coefficients. Up to replacing Betti cohomology with a suitable Weil cohomology theory (for example $\ell$-adic cohomology), we may and we will assume that $k$ is a subfield of the complex numbers $\mathbb{C}$. We use freely the language of (Chow) motives as is described in [11].

Work of Shermenev [18], Beauville [2], and Deninger and Murre [7] unravelled the structure of the motives of abelian varieties:

**Theorem** (Beauville, Deninger–Murre, Shermenev). *Let $A$ be an abelian variety of dimension $g$. Then the Chow motive $\mathfrak{h}(A)$ of $A$ splits as*

$$\mathfrak{h}(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A) \tag{1}$$

*with the following properties:*

(i) $\mathrm{H}^*(\mathfrak{h}^i(A),\mathbb{Q}) = \mathrm{H}^i(A,\mathbb{Q})$,

(ii) *the multiplication morphism (as defined in* (5)*)*

$$\mathfrak{h}(A) \otimes \mathfrak{h}(A) \to \mathfrak{h}(A)$$

*factors through the direct summand $\mathfrak{h}^{i+j}(A)$ when restricted to $\mathfrak{h}^i(A) \otimes \mathfrak{h}^j(A)$,*

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(iii) *for any integer $n$, the morphism*

$$[n]^* : \mathfrak{h}^i(A) \to \mathfrak{h}^i(A)$$

*induced by the multiplication by $n$ morphism $[n] : A \to A$ is multiplication by $n^i$. In particular, $\mathfrak{h}^i(A)$ is an eigen-submotive for the action of $[n]$.*

For an arbitrary smooth projective variety $X$, it is expected that a decomposition of the motive $\mathfrak{h}(X)$ as in (1) satisfying (i) should exist; see [13]. Such a decomposition is called a *Chow–Künneth decomposition*. However, in general, there is no analogue of the multiplication by $n$ morphisms, and the existence of a Chow–Künneth decomposition of the motive of $X$ satisfying (ii) (in that case, the Chow–Künneth decomposition is said to be *multiplicative*) is very restrictive. We refer to [16, Section 8] for some discussion on the existence of such a multiplicative decomposition.

Nonetheless, inspired by the seminal work of Beauville and Voisin [3,4,19], we were led to ask in [16] whether the motives of hyperKähler varieties admit a multiplicative decomposition similar to that of the motive of abelian varieties as in the theorem of Beauville, Deninger and Murre, and Shermenev. Here, by hyperKähler variety we mean a simply connected smooth projective variety $X$ whose space of global 2-forms $\mathrm{H}^0(X, \Omega^2_X)$ is spanned by a nowhere degenerate 2-form. When $k = \mathbb{C}$, a hyperKähler variety is nothing but a projective irreducible holomorphic symplectic manifold [1].

**Conjecture 1.** Let $X$ be a hyperKähler variety of dimension $2n$. Then the Chow motive $\mathfrak{h}(X)$ of $X$ splits as

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(X)$$

with the property that

(i) $\mathrm{H}^*(\mathfrak{h}^i(X), \mathbb{Q}) = \mathrm{H}^i(X, \mathbb{Q})$,

(ii) the multiplication morphism $\mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$ factors through the direct summand $\mathfrak{h}^{i+j}(X)$ when restricted to $\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X)$.

An important class of hyperKähler varieties is given by the Hilbert schemes $S^{[n]}$ of length-$n$ subschemes on a K3 surface $S$; see [1]. The following theorem shows in particular that the motive of $S^{[n]}$ for $S$ a K3 surface admits a decomposition with properties (i) and (ii) and thus answers affirmatively the question raised in Conjecture 1 in that case.

**Theorem 1.** *Let $S$ be either a K3 surface or an abelian surface, and let $n$ be a positive integer. Then the Chow motive $\mathfrak{h}(S^{[n]})$ of $S^{[n]}$ splits as*

$$\mathfrak{h}(S^{[n]}) = \bigoplus_{i=0}^{4n} \mathfrak{h}^i(S^{[n]})$$

*with the property that*

(i) $\mathrm{H}^*(\mathfrak{h}^i(S^{[n]}), \mathbb{Q}) = \mathrm{H}^i(S^{[n]}, \mathbb{Q})$,

(ii) *the multiplication $\mathfrak{h}(S^{[n]}) \otimes \mathfrak{h}(S^{[n]}) \to \mathfrak{h}(S^{[n]})$ factors through the direct summand $\mathfrak{h}^{i+j}(S^{[n]})$ when restricted to $\mathfrak{h}^i(S^{[n]}) \otimes \mathfrak{h}^j(S^{[n]})$.*

Theorem 6 of [16] can then be improved by including the Hilbert schemes of length-$n$ subschemes on K3 surfaces. Theorem 1 is due for $S$ a K3 surface and $n = 1$ to Beauville and Voisin [4] (see [16, Proposition 8.14] for the link between the original statement of [4] (recalled in Theorem 3.4) and the statement given here), and was established in [16] for $n = 2$. Its proof in full generality is given in Section 3. Note that, as explained in Section 1, the existence of a Chow–Künneth decomposition for the Hilbert scheme $S^{[n]}$ of any smooth projective surface $S$ goes back to de Cataldo and Migliorini [5] (the existence of such a decomposition for $S$ is due to Murre [12]). Our main contribution is the claim that by choosing the Beauville–Voisin decomposition of K3 surfaces [4], the induced Chow–Künneth decomposition of Hilbert schemes of K3 surfaces established by de Cataldo and Migliorini [5] is multiplicative, i.e., it satisfies (ii).

Let us then define for all $i \geq 0$ and all $s \in \mathbb{Z}$

$$\mathrm{CH}^i(S^{[n]})_s := \mathrm{CH}^i(\mathfrak{h}^{2i-s}(S^{[n]})).$$

We have the following corollary to Theorem 1:

**Theorem 2.** *The Chow ring* $\mathrm{CH}^*(S^{[n]})$ *admits a multiplicative bigrading*

$$\mathrm{CH}^*(S^{[n]}) = \bigoplus_{i,s} \mathrm{CH}^i(S^{[n]})_s$$

*that is induced by a Chow–Künneth decomposition of the diagonal (as defined in Section* 1*). Moreover, the Chern classes* $c_i(S^{[n]})$ *belong to the graded-zero part* $\mathrm{CH}^i(S^{[n]})_0$ *of* $\mathrm{CH}^i(S^{[n]})$.

Theorem 2 answers partially a question raised by Beauville in [3]: the filtration $F^\bullet$ defined by $\mathrm{F}^l\mathrm{CH}^i(X) := \bigoplus_{s \geq l} \mathrm{CH}^i(X)_s$ is a filtration on the Chow ring $\mathrm{CH}^*(S^{[n]})$ that is split. Moreover, this filtration is expected to be the one predicted by Bloch and Beilinson (because it is induced by a Chow–Künneth decomposition – conjecturally all such filtrations coincide). For this filtration to be of Bloch–Beilinson type, one would need to establish Murre's conjectures, namely that $\mathrm{CH}^i(S^{[n]})_s = 0$ for $s < 0$ and that $\bigoplus_{s>0} \mathrm{CH}^i(S^{[n]})_s$ is exactly the kernel of the cycle class map $\mathrm{CH}^i(S^{[n]}) \to \mathrm{H}^{2i}(S^{[n]}, \mathbb{Q})$. Note that for $i = 0, 1, 2n-1$ or $2n$, it is indeed the case that $\mathrm{CH}^i(S^{[n]})_s = 0$ for $s < 0$ and that

$$\bigoplus_{s>0} \mathrm{CH}^i(S^{[n]})_s = \mathrm{Ker}\{\mathrm{CH}^i(S^{[n]}) \to \mathrm{H}^{2i}(S^{[n]}, \mathbb{Q})\}.$$

Therefore, we have

**Corollary 1.** *Let* $i_1, \dots, i_m$ *be positive integers such that* $i_1 + \dots + i_m = 2n-1$ *or* $2n$, *and let* $\gamma_l$ *be cycles in* $\mathrm{CH}^{i_l}(S^{[n]})$ *for* $l = 1, \dots, m$ *that sit in* $\mathrm{CH}^{i_l}(S^{[n]})_0$ *for the grading induced by the decomposition of Theorem* 1*. Then,* $[\gamma_1] \cdot [\gamma_2] \cdots [\gamma_m] = 0$ *in* $\mathrm{H}^*(S^{[n]}, \mathbb{Q})$ *if and only if* $\gamma_1 \cdot \gamma_2 \cdots \gamma_m = 0$ *in* $\mathrm{CH}^*(S^{[n]})$.

Let us mention that Theorem 1 and Theorem 2 (and a fortiori Corollary 1) are also valid for hyperKähler varieties that are birational to $S^{[n]}$, for some K3 surface $S$. Indeed, Riess [15] showed that birational hyperKähler varieties have isomorphic Chow rings and isomorphic Chow motives (as algebras in the category of Chow motives); see also [16, Section 6]. As for more evidence as why Conjecture 1 should be true, Mingmin Shen and I showed [16] that the variety of lines on a very general cubic fourfold satisfies the conclusions of Theorem 2.

Finally, we use the notion of multiplicative Chow–Künneth decomposition to obtain new decomposition results in the spirit of [20]; see Theorem 4.3.

**Notations.** A morphism denoted $\mathrm{pr}_r$ will always denote the projection on the $r$-th factor and a morphism denoted $\mathrm{pr}_{s,t}$ will always denote the projection on the product of the $s$-th and $t$-th factors. The context will usually make it clear which varieties are involved. Chow groups $\mathrm{CH}^i$ are with rational coefficients. If $X$ is a variety, the cycle class map sends a cycle $\sigma \in \mathrm{CH}^i(X)$ to its cohomology class $[\sigma] \in \mathrm{H}^{2i}(X, \mathbb{Q})$. If $Y$ is another variety and if $\gamma$ is a correspondence in $\mathrm{CH}^i(X \times Y)$, its transpose ${}^t\gamma \in \mathrm{CH}^i(Y \times X)$ is the image of $\gamma$ under the action of the permutation map $X \times Y \to Y \times X$. If $\gamma_1, \dots, \gamma_n$ are correspondences in $\mathrm{CH}^*(X \times Y)$, then the correspondence $\gamma_1 \otimes \cdots \otimes \gamma_n \in \mathrm{CH}^*(X^n \times Y^n)$ is defined as

$$\gamma_1 \otimes \cdots \otimes \gamma_n := \prod_{i=1}^{n} (\mathrm{pr}_{i,n+i})^* \gamma_i .$$

## 1. Chow–Künneth decompositions

A Chow motive $M$ is said to have a *Chow–Künneth decomposition* if it splits as

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

with $\mathrm{H}^*(M^i, \mathbb{Q}) = \mathrm{H}^i(M, \mathbb{Q})$. In other words, $M$ admits a Künneth decomposition that lifts to rational equivalence. Concretely, if $M = (X, p, n)$ with $X$ a smooth projective variety of pure dimension $d$ and $p \in \mathrm{CH}^d(X \times X)$ an idempotent and $n$ an integer, then $M$ has a Chow–Künneth decomposition if there exist finitely many correspondences $p^i \in \mathrm{CH}^d(X \times X)$, $i \in \mathbb{Z}$, such that $p = \sum_i p^i$, $p^i \circ p^i = p^i$, $p \circ p^i = p^i \circ p = p^i$, $p^i \circ p^j = 0$ for all $i \in \mathbb{Z}$ and all $j \neq i$ and such that $p^i_* \mathrm{H}^*(X, \mathbb{Q}) = p_* \mathrm{H}^{i+2n}(X, \mathbb{Q})$.

A smooth projective variety $X$ of dimension $d$ has a Chow–Künneth decomposition if its Chow motive $\mathfrak{h}(X)$ has a Chow–Künneth decomposition, that is, there exist correspondences $\pi^i \in \mathrm{CH}^d(X \times X)$ such that $\Delta_X = \sum_{i=0}^{2d} \pi^i$, with $\pi^i \circ \pi^i = \pi^i$, $\pi^i \circ \pi^j = 0$ for $i \neq j$ and $\pi^i_* \mathrm{H}^*(X, \mathbb{Q}) = \mathrm{H}^i(X, \mathbb{Q})$. A Chow–Künneth decomposition $\{\pi^i : 0 \leq i \leq 2d\}$ of $X$ is said to be *self-dual* if $\pi^{2d-i} = {}^t\pi^i$ for all $i$.

If $\mathfrak{o}$ is the class of a rational point on $X$ (or more generally a zero-cycle of degree 1 on $X$), then $\pi^0 := \mathrm{pr}_1^* \mathfrak{o} = \mathfrak{o} \times X$ and $\pi^{2d} = \mathrm{pr}_2^* \mathfrak{o} = X \times \mathfrak{o}$ define mutually orthogonal idempotents such that $\pi^0_* \mathrm{H}^*(X, \mathbb{Q}) = \mathrm{H}^0(X, \mathbb{Q})$ and $\pi^{2d}_* \mathrm{H}^*(X, \mathbb{Q}) = \mathrm{H}^{2d}(X, \mathbb{Q})$. Note that pairs of idempotents with the property above are certainly not unique: a different choice (modulo rational equivalence) of zero-cycle of degree 1 gives different idempotents in the ring of correspondences $\mathrm{CH}^d(X \times X)$. From the above, one sees that every curve $C$ admits a Chow–Künneth decomposition: one defines $\pi^0$ and $\pi^2$ as above and then $\pi^1$ is simply given by $\Delta_C - \pi^0 - \pi^2$. It is a theorem of Murre [12] that every smooth projective surface $S$ admits a Chow–Künneth decomposition $\Delta_S = \pi_S^0 + \pi_S^1 + \pi_S^2 + \pi_S^3 + \pi_S^4$.

The notion of Chow–Künneth decomposition is significant because when it exists it induces a filtration

$$\mathrm{F}^l \,\mathrm{CH}^i(X) := \bigoplus_{s \geq l} (\pi^{2i-s})_* \,\mathrm{CH}^i(X)$$

on the Chow group $\mathrm{CH}^*(X)$ which should not depend on the choice of the Chow–Künneth decomposition $\Delta_X = \sum_{i=0}^{2d} \pi^i$ and which should be of Bloch–Beilinson type; cf. [10, 13].

Let now $S^{[n]}$ denote the Hilbert scheme of length-$n$ subschemes on a smooth projective surface $S$. By Fogarty [9], the scheme $S^{[n]}$ is in fact a smooth projective variety, and it comes equipped with a morphism $S^{[n]} \to S^{(n)}$ to the $n$-th symmetric product of $S$, called the *Hilbert–Chow morphism*. De Cataldo and Migliorini [5] have given an explicit description of the motive of $S^{[n]}$. Let us introduce some notations related to this description. Let $\mu = \{A_1, \dots, A_l\}$ be a partition of the set $\{1, \dots, n\}$, where all the $A_i$ are nonempty. The integer $l$, also denoted $l(\mu)$, is the *length* of the partition $\mu$. Let $S^\mu \simeq S^l \subseteq S^n$ be the set

$$\{(s_1, \dots, s_n) : s_i = s_j \text{ if } i, j \in A_k \text{ for some } k\}$$

and let

$$\Gamma_\mu := (S^\mu \times_{S^{(n)}} S^{[n]})_{\mathrm{red}} \subset S^\mu \times S^{[n]},$$

where the subscript "red" means the underlying reduced scheme. It is known that $\Gamma_\mu$ is irreducible of dimension $n + l(\mu)$. The subgroup $\mathfrak{S}_\mu$ of $\mathfrak{S}_n$ that acts on $\{1, \dots, n\}$ by permuting the $A_i$ with same cardinality acts on the first factor of the product $S^\mu \times S^{[n]}$, and the correspondence $\Gamma_\mu$ is invariant under this action. We can therefore define

$$\hat{\Gamma}_\mu := \Gamma_\mu / \mathfrak{S}_\mu \in \mathrm{CH}^*(S^{(\mu)} \times S^{[n]}) = \mathrm{CH}^*(S^\mu \times S^{[n]})^{\mathfrak{S}_\mu},$$

where $S^{(\mu)} := S^\mu / \mathfrak{S}_\mu$. Since for a variety $X$ endowed with the action of a finite group $G$ we have $\mathrm{CH}^*(X/G) = \mathrm{CH}^*(X)^G$ (with rational coefficients), the calculus of correspondences and the theory of motives in the setting of smooth projective varieties endowed with the action of a finite group is similar in every way to the usual case of smooth projective varieties. We will therefore freely consider actions of correspondences and motives of quotient varieties by the action of a finite group.

The symmetric groups $\mathfrak{S}_n$ acts naturally on the set of partitions of $\{1, \dots, n\}$. By choosing one element in each orbit for the above action, we may define a subset $\mathfrak{B}(n)$ of the set of partitions of $\{1, \dots, n\}$. This set is isomorphic to the set of partitions of the integer $n$.

**Theorem 1.1** (de Cataldo and Migliorini [5]). *Let $S$ be a smooth projective surface defined over an arbitrary field. The morphism*

$$\bigoplus_{\mu \in \mathfrak{B}(n)} {}^t\hat{\Gamma}_\mu : \mathfrak{h}(S^{[n]}) \xrightarrow{\simeq} \bigoplus_{\mu \in \mathfrak{B}(n)} \mathfrak{h}(S^{(\mu)})(l(\mu) - n) \tag{2}$$

*is an isomorphism of Chow motives. Moreover, its inverse is given by the correspondence $\sum_{\mu \in \mathfrak{B}(n)} \frac{1}{m_\mu} \hat{\Gamma}_\mu$ for some nonzero rational numbers $m_\mu$ that are independent of $S$.*

Let now $\Delta_S = \pi_S^0 + \pi_S^1 + \pi_S^2 + \pi_S^3 + \pi_S^4$ be a Chow–Künneth decomposition of $S$. For all nonnegative integers $m$, the correspondences

$$\pi_{S^m}^i := \sum_{i_1 + \dots + i_m = i} \pi_S^{i_1} \otimes \dots \otimes \pi_S^{i_m} \quad \text{in } \mathrm{CH}^{2m}(S^m \times S^m) \tag{3}$$

define a Chow–Künneth decomposition of $S^m$ that is clearly $\mathfrak{S}_m$-equivariant. Therefore, these correspondences can be seen as correspondences of $\mathrm{CH}^{2m}(S^{(m)} \times S^{(m)})$ and they do define a Chow–Künneth decomposition of the $m$-th symmetric product $S^{(m)}$. Let us denote this decomposition

$$\Delta_{S^{(m)}} = \pi^0_{S^{(m)}} + \cdots + \pi^{4m}_{S^{(m)}} \quad \text{in } \mathrm{CH}^{2m}(S^{(m)} \times S^{(m)}).$$

Since $S^l$ is endowed with a $\mathfrak{S}_l$-equivariant Chow–Künneth decomposition as above and since $\mathfrak{S}_\mu$ is a subgroup of $\mathfrak{S}_l$, $S^\mu \simeq S^l$ is endowed with a $\mathfrak{S}_\mu$-equivariant Chow–Künneth decomposition. Therefore $S^{(\mu)}$ is endowed with a natural Chow–Künneth decomposition

$$\Delta_{S^{(\mu)}} = \pi^0_{S^{(\mu)}} + \cdots + \pi^{4l}_{S^{(\mu)}} \quad \text{in } \mathrm{CH}^{2l}(S^{(\mu)} \times S^{(\mu)})$$

coming from that of $S$. In particular, the isomorphism of de Cataldo and Migliorini gives a natural Chow–Künneth decomposition for the Hilbert scheme $S^{[n]}$ coming from that of $S$. Precisely, this Chow–Künneth decomposition is given by

$$\pi^i_{S^{[n]}} = \sum_{\mu \in \mathfrak{B}(n)} \frac{1}{m_\mu} \hat{\Gamma}_\mu \circ \pi^{i-2n+2l(\mu)}_{S^{(\mu)}} \circ {}^t\hat{\Gamma}_\mu. \tag{4}$$

Note that if the Chow–Künneth decomposition $\{\pi^i_S\}$ of $S$ is self-dual, then the Chow–Künneth decomposition $\{\pi^i_{S^{[n]}}\}$ of $S^{[n]}$ is also self-dual.

We will show that when $S$ is either a K3 surface or an abelian surface the Chow–Künneth decomposition above induces a decomposition of the motive $\mathfrak{h}(S^{[n]})$ that satisfies the conclusions of Theorem 1 for an appropriate choice of Chow–Künneth decomposition for $S$.

## 2. Multiplicative Chow–Künneth decompositions

Let $X$ be a smooth projective variety of dimension $d$ and let $\Delta_3 \in \mathrm{CH}_d(X \times X \times X)$ be the small diagonal, that is, the class of the subvariety

$$\{(x,x,x) : x \in X\} \subset X \times X \times X.$$

If we view $\Delta_3$ as a correspondence from $X \times X$ to $X$, then $\Delta_3$ induces the *multiplication morphism*

$$\mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X). \tag{5}$$

Note that if $\alpha$ and $\beta$ are cycles in $\mathrm{CH}^*(X)$, then $(\Delta_3)_*(\alpha \times \beta) = \alpha \cdot \beta$ in $\mathrm{CH}^*(X)$.

If $X$ admits a Chow–Künneth decomposition

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2d} \mathfrak{h}^i(X), \tag{6}$$

then this decomposition is said to be *multiplicative* if the multiplication morphism

$$\mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X) \to \mathfrak{h}(X)$$

factors through the direct summand $\mathfrak{h}^{i+j}(X)$ for all $i$ and $j$. For a variety to be endowed with a multiplicative Chow–Künneth decomposition is very restrictive; we refer to [16], where this notion was introduced, for some discussions. For instance, a very general curve of genus $\geq 3$ does not admit such a decomposition. Examples of varieties admitting a multiplicative

Chow–Künneth decomposition are provided by [16, Theorem 6] and include hyperelliptic curves, K3 surfaces, abelian varieties, and their Hilbert squares.

If one writes $\Delta_X = \pi_X^0 + \cdots + \pi_X^{2d}$ for the Chow–Künneth decomposition (6) of $X$, then by definition this decomposition is multiplicative if

$$\pi_X^k \circ \Delta_3 \circ (\pi_X^i \otimes \pi_X^j) = 0 \quad \text{in } \mathrm{CH}_d(X^3) \text{ for all } k \neq i + j,$$

or equivalently if

$$({}^t\pi_X^i \otimes {}^t\pi_X^j \otimes \pi_X^k)_* \Delta_3 = 0 \quad \text{in } \mathrm{CH}_d(X^3) \text{ for all } k \neq i + j.$$

If the Chow–Künneth decomposition $\{\pi_X^i\}$ is self-dual, then it is multiplicative if

$$(\pi_X^i \otimes \pi_X^j \otimes \pi_X^k)_* \Delta_3 = 0 \quad \text{in } \mathrm{CH}_d(X^3) \text{ for all } i + j + k \neq 4d.$$

Note that the above three relations always hold modulo homological equivalence.

Given a multiplicative Chow–Künneth decomposition $\pi_S^i$ for a surface $S$, one could expect the Chow–Künneth decomposition (4) of $S^{[n]}$ to be multiplicative. This was answered affirmatively when $n = 2$ for any smooth projective variety $X$ with a self-dual Chow–Künneth decomposition (under some additional assumptions on the Chern classes of $X$) in [16], and a similar result when $n = 3$ can be found in [17]. (For $n > 3$, $X^{[n]}$ is no longer smooth if $X$ is smooth of dimension $> 2$.) Here we deal with the case when $S$ is a K3 surface or an abelian surface and will prove Theorem 1. By the isomorphism (2) of de Cataldo and Migliorini, it is enough to check that

$$(\hat{\Gamma}_{\mu_1} \otimes \hat{\Gamma}_{\mu_2} \otimes \hat{\Gamma}_{\mu_3})^* (\pi_{S^{[n]}}^i \otimes \pi_{S^{[n]}}^j \otimes \pi_{S^{[n]}}^k)_* \Delta_3 = 0$$

for all $i + j + k \neq 8n$ and for all partitions $\mu_1, \mu_2$ and $\mu_3$ of $\{1, \ldots, n\}$, or equivalently for all $i, j, k$ such that $(\pi_{S^{[n]}}^i \otimes \pi_{S^{[n]}}^j \otimes \pi_{S^{[n]}}^k)_* [\Delta_3] = 0$ in $\mathrm{H}^*(S^{[n]} \times S^{[n]} \times S^{[n]}, \mathbb{Q})$ and all partitions $\mu_1, \mu_2$ and $\mu_3$. By (4), it is even enough to show that

$$\begin{aligned} (7) \qquad &\big(\hat{\Gamma}_{\mu_1} \otimes \hat{\Gamma}_{\mu_2} \otimes \hat{\Gamma}_{\mu_3}\big)^* \big( (\hat{\Gamma}_{\nu_1} \circ \pi_{S^{(\nu_1)}}^i \circ {}^t\hat{\Gamma}_{\nu_1}) \\ &\qquad \otimes \big(\hat{\Gamma}_{\nu_2} \circ \pi_{S^{(\nu_2)}}^j \circ {}^t\hat{\Gamma}_{\nu_2}\big) \otimes \big(\hat{\Gamma}_{\nu_3} \circ \pi_{S^{(\nu_3)}}^k \circ {}^t\hat{\Gamma}_{\nu_3}\big)\big)_* \Delta_3 \end{aligned}$$

is zero in $\mathrm{CH}^*(S^{(\mu_1)} \times S^{(\mu_2)} \times S^{(\mu_3)})$ for all partitions $\mu_1, \mu_2, \mu_3$, and all partitions $\nu_1, \nu_2, \nu_3$ and all $i, j, k$ such that

$$\big(\big(\hat{\Gamma}_{\nu_1} \circ \pi_{S^{(\nu_1)}}^i \circ {}^t\hat{\Gamma}_{\nu_1}\big) \otimes \big(\hat{\Gamma}_{\nu_2} \circ \pi_{S^{(\nu_2)}}^j \circ {}^t\hat{\Gamma}_{\nu_2}\big) \otimes \big(\hat{\Gamma}_{\nu_3} \circ \pi_{S^{(\nu_3)}}^k \circ {}^t\hat{\Gamma}_{\nu_3}\big)\big)_* [\Delta_3] = 0$$

in $\mathrm{H}^*((S^{[n]})^3, \mathbb{Q})$. Note that the expression (7) is equal to

$$\begin{aligned} &\big[\big({}^t\hat{\Gamma}_{\mu_1} \otimes {}^t\hat{\Gamma}_{\mu_2} \otimes {}^t\hat{\Gamma}_{\mu_3}\big) \circ \big(\hat{\Gamma}_{\nu_1} \otimes \hat{\Gamma}_{\nu_2} \otimes \hat{\Gamma}_{\nu_3}\big) \\ &\qquad \circ \big(\pi_{S^{(\nu_1)}}^i \otimes \pi_{S^{(\nu_2)}}^j \otimes \pi_{S^{(\nu_3)}}^k\big) \circ \big({}^t\hat{\Gamma}_{\nu_1} \otimes {}^t\hat{\Gamma}_{\nu_2} \otimes {}^t\hat{\Gamma}_{\nu_3}\big)\big]_* \Delta_3. \end{aligned}$$

But it is clear from Theorem 1.1 that

$$\begin{aligned} &\big({}^t\hat{\Gamma}_{\mu_1} \otimes {}^t\hat{\Gamma}_{\mu_2} \otimes {}^t\hat{\Gamma}_{\mu_3}\big) \circ \big(\hat{\Gamma}_{\nu_1} \otimes \hat{\Gamma}_{\nu_2} \otimes \hat{\Gamma}_{\nu_3}\big) \\ &\qquad = \begin{cases} 0 & \text{if } (\nu_1, \nu_2, \nu_3) \neq (\mu_1, \mu_2, \mu_3), \\ m_{\mu_1} m_{\mu_2} m_{\mu_3} \, \Delta_{S^{(\mu_1)} \times S^{(\mu_2)} \times S^{(\mu_3)}} & \text{if } (\nu_1, \nu_2, \nu_3) = (\mu_1, \mu_2, \mu_3). \end{cases} \end{aligned}$$

Thus we have proved the following criterion for the Chow–Künneth decomposition (4) to be multiplicative.

**Proposition 2.1.** *The Chow–Künneth decomposition* (4) *is multiplicative (equivalently, the motive of $S^{[n]}$ splits as in Theorem* 1*) if for all partitions $\mu_1$, $\mu_2$ and $\mu_3$ of the set $\{1,\dots,n\}$*

$$\big(\pi^i_{S^{\mu_1}} \otimes \pi^j_{S^{\mu_2}} \otimes \pi^k_{S^{\mu_3}}\big)_*\big(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3}\big)^* \Delta_3 = 0 \quad \text{in } \mathrm{CH}^*(S^{\mu_1} \times S^{\mu_2} \times S^{\mu_3})$$

*as soon as*

$$\big(\pi^i_{S^{\mu_1}} \otimes \pi^j_{S^{\mu_2}} \otimes \pi^k_{S^{\mu_3}}\big)_*\big(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3}\big)^* [\Delta_3] = 0 \quad \text{in } \mathrm{H}^*(S^{\mu_1} \times S^{\mu_2} \times S^{\mu_3}, \mathbb{Q}).$$

## 3. Proof of Theorem 1 and Theorem 2

The proof is inspired by the proof of Claire Voisin's [21, Theorem 5.1]. In fact, because of [16, Proposition 8.12], Theorem 1 for K3 surfaces implies [21, Theorem 5.1]. The first step towards the proof of Theorem 1 is to understand the cycle $(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3})^*\Delta_3$. The following proposition, due to Voisin [21] (see also [19]), builds on the work of Ellingsrud, Göttsche and Lehn [8]. Here, $S$ is a smooth projective surface and $\Delta_k$ is the class of the small diagonal inside $S^k$ in $\mathrm{CH}_2(S^k)$.

**Proposition 3.1** (Voisin [21, Proposition 5.6]). *For any set of partitions*

$$\boldsymbol{\mu} := \{\mu_1, \dots, \mu_k\}$$

*of $\{1,\dots,n\}$, there exists a universal (i.e., independent of $S$) polynomial $P_{\boldsymbol{\mu}}$ with the following property:*

$$\big(\Gamma_{\mu_1} \otimes \cdots \otimes \Gamma_{\mu_k}\big)^* \Delta_k = P_{\boldsymbol{\mu}}\big(\mathrm{pr}_r^* c_2(S), \mathrm{pr}_{r'}^* K_S, \mathrm{pr}_{s,t}^* \Delta_S\big) \quad \text{in } \mathrm{CH}^*(S^{\boldsymbol{\mu}}),$$

*where the $\mathrm{pr}_r$ are the projections from $S^{\boldsymbol{\mu}} := \prod_i S^{\mu_i} \simeq S^N$ to its factors, and the $\mathrm{pr}_{s,t}$ are the projections from $S^{\boldsymbol{\mu}}$ to the products of two of its factors.*

In fact, Proposition 3.1 is a particular instance of [21, Theorem 5.12]. Another consequence of [21, Theorem 5.12], which will be used to prove Theorem 2, is

**Proposition 3.2** (Voisin). *For any partition $\mu$ of $\{1,\dots,n\}$ and any polynomial $P$ in the Chern classes of $S^{[n]}$, the cycle $(\Gamma_\mu)^* P$ of $S^\mu$ is a universal (i.e., independent of $S$) polynomial in the variables $\mathrm{pr}_r^* c_2(S), \mathrm{pr}_{r'}^* K_S, \mathrm{pr}_{s,t}^* \Delta_S$, where the $\mathrm{pr}_r$ are the projections from $S^\mu \simeq S^N$ to its factors, and the $\mathrm{pr}_{s,t}$ are the projections from $S^\mu$ to the products of two of its factors.*

We first prove Theorems 1–2 for $S$ a K3 surface and then for $S$ an abelian surface. Note that clearly a multiplicative Chow–Künneth decomposition $\{\pi^i_{S^{[n]}} : 0 \leq i \leq 4n\}$ induces a multiplicative bigrading on the Chow ring $\mathrm{CH}^*(S^{[n]})$ :

$$\mathrm{CH}^*(S^{[n]}) = \bigoplus_{i,s} \mathrm{CH}^i(S^{[n]})_s, \quad \text{where } \mathrm{CH}^i(S^{[n]})_s = (\pi^{2i-s}_{S^{[n]}})_* \mathrm{CH}^i(S^{[n]}).$$

Thus once Theorem 1 is established it only remains to show that the Chern classes of $S^{[n]}$ sit in $\mathrm{CH}^*(S^{[n]})_0$ in order to conclude.

**3.1. The Hilbert scheme of points on a K3 surface.** Let $S$ be a smooth projective surface and let $\mathfrak{o}$ be a zero-cycle of degree 1 on $S$. Let $m$ be a positive integer and consider the $m$-fold product $S^m$ of $S$. Let us define the idempotent correspondences

$$\pi_S^0 := \mathrm{pr}_1^*\mathfrak{o} = \mathfrak{o} \times S, \quad \pi_S^4 = \mathrm{pr}_2^*\mathfrak{o} = S \times \mathfrak{o}, \quad \pi_S^2 := \Delta_S - \pi_S^0 - \pi_S^4. \tag{8}$$

(Note that the idempotent correspondence $\pi_S^2$ is not quite a Chow–Künneth projector, it projects onto $\mathrm{H}^1(S, \mathbb{Q}) \oplus \mathrm{H}^2(S, \mathbb{Q}) \oplus \mathrm{H}^3(S, \mathbb{Q})$.) In this case, the idempotents

$$\pi_{S^m}^i := \sum_{i_1+\cdots+i_n=i} \pi_S^{i_1} \otimes \cdots \otimes \pi_S^{i_n}$$

given in (3) are clearly sums of monomials of degree $2m$ in $\mathrm{pr}_r^*\mathfrak{o}$ and $\mathrm{pr}_{s,t}^*\Delta_S$. By Proposition 3.1, it follows that for any smooth projective surface $S$ and any zero-cycle $\mathfrak{o}$ of degree 1 on $S$

$$\big(\pi_{S^{\mu_1}}^i \otimes \pi_{S^{\mu_2}}^j \otimes \pi_{S^{\mu_3}}^k\big)_* \big(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3}\big)^* \Delta_3$$

is a polynomial $Q_{\boldsymbol{\mu},i,j,k}$ in the variables $\mathrm{pr}_r^*c_2(S), \mathrm{pr}_{r'}^*K_S, \mathrm{pr}_{r''}^*\mathfrak{o}$ and $\mathrm{pr}_{s,t}^*\Delta_S$.

We now have the following key result which is due to Claire Voisin [21, Corollary 5.9] and which relies in an essential way on a theorem due to Qizheng Yin [22] that describes the cohomological relations among the cycles $\mathrm{pr}_{r,s}^*\pi_S^2$.

**Proposition 3.3** (Voisin [21]). *For all smooth projective surfaces $S$ and any degree-1 zero-cycle $\mathfrak{o}$ on $S$, let $P$ be a polynomial (independent of $S$) in the variables $\mathrm{pr}_r^*[c_2(S)]$, $\mathrm{pr}_{r'}^*[K_S]$, $\mathrm{pr}_{r''}^*[\mathfrak{o}]$ and $\mathrm{pr}_{s,t}^*[\Delta_S]$ with value an algebraic cycle of $S^n$. If $P$ vanishes for all smooth projective surfaces with $b_1(S) = 0$, then the polynomial $P$ belongs to the ideal generated by the relations:*

(a) $[c_2(S)] = \chi_{\mathrm{top}}(S)[\mathfrak{o}]$,

(b) $[K_S]^2 = \deg(K_S^2)[\mathfrak{o}]$,

(c) $[\Delta_S] \cdot \mathrm{pr}_1^*[K_S] = \mathrm{pr}_1^*[K_S] \cdot \mathrm{pr}_2^*[\mathfrak{o}] + \mathrm{pr}_1^*[\mathfrak{o}] \cdot \mathrm{pr}_2^*[K_S]$,

(d) $$[\Delta_3] = \mathrm{pr}_{1,2}^*[\Delta_S] \cdot \mathrm{pr}_3^*[\mathfrak{o}] + \mathrm{pr}_{1,3}^*[\Delta_S] \cdot \mathrm{pr}_2^*[\mathfrak{o}] + \mathrm{pr}_{2,3}^*[\Delta_S] \cdot \mathrm{pr}_1^*[\mathfrak{o}] - \mathrm{pr}_1^*[\mathfrak{o}] \cdot \mathrm{pr}_2^*[\mathfrak{o}] - \mathrm{pr}_1^*[\mathfrak{o}] \cdot \mathrm{pr}_3^*[\mathfrak{o}] - \mathrm{pr}_2^*[\mathfrak{o}] \cdot \mathrm{pr}_3^*[\mathfrak{o}],$$

(e) $[\Delta_S]^2 = \chi_{\mathrm{top}}(S)\,\mathrm{pr}_1^*[\mathfrak{o}] \cdot \mathrm{pr}_2^*[\mathfrak{o}]$,

(f) $[\Delta_S] \cdot \mathrm{pr}_1^*[\mathfrak{o}] = \mathrm{pr}_1^*[\mathfrak{o}] \cdot \mathrm{pr}_2^*[\mathfrak{o}]$.

We may then specialize to the case where $S$ is a K3 surface. Consider then a K3 surface $S$ and let $\mathfrak{o}$ be the class of a point lying on a rational curve of $S$. Note that by definition of a K3 surface $K_S = 0$. The following theorem of Beauville and Voisin shows that the relations (a)–(f) listed above actually hold modulo rational equivalence.

**Theorem 3.4** (Beauville–Voisin [4]). *Let $S$ be a K3 surface and let $\mathfrak{o}$ be a rational point lying on a rational curve on $S$. The following relations hold:*

(i) *in* $\mathrm{CH}^2(S)$,

$$c_2(S) = \chi_{\mathrm{top}}(S)\mathfrak{o}\ (= 24\mathfrak{o}),$$

(ii) *in* $\mathrm{CH}_2(S \times S \times S)$,

$$\Delta_3 = \mathrm{pr}_{1,2}^*\Delta_S \cdot \mathrm{pr}_3^*\mathfrak{o} + \mathrm{pr}_{1,3}^*\Delta_S \cdot \mathrm{pr}_2^*\mathfrak{o} + \mathrm{pr}_{2,3}^*\Delta_S \cdot \mathrm{pr}_1^*\mathfrak{o} - \mathrm{pr}_1^*\mathfrak{o} \cdot \mathrm{pr}_2^*\mathfrak{o} - \mathrm{pr}_1^*\mathfrak{o} \cdot \mathrm{pr}_3^*\mathfrak{o} - \mathrm{pr}_2^*\mathfrak{o} \cdot \mathrm{pr}_3^*\mathfrak{o}.$$

The proof of Theorem 1 in the case when $S$ is a K3 surface is then immediate: by the discussion above if the cycle

$$\delta := \left(\pi^i_{S^{\mu_1}} \otimes \pi^j_{S^{\mu_2}} \otimes \pi^k_{S^{\mu_3}}\right)_* \left(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3}\right)^* \Delta_3$$

is zero in $\mathrm{H}^*(S^{\mu_1} \times S^{\mu_2} \times S^{\mu_3}, \mathbb{Q})$, then by Proposition 3.3 it belongs to the ideal generated by the relations (a)–(f). By Theorem 3.4, the relations (a)–(f) actually hold modulo rational equivalence. Therefore, the cycle $\delta$ is zero in $\mathrm{CH}^*(S^{\mu_1} \times S^{\mu_2} \times S^{\mu_3})$. We may then conclude by invoking Proposition 2.1.

It remains to prove that the Chern classes $c_i(S^{[n]})$ sit in $\mathrm{CH}^i(S^{[n]})_0$. It suffices to show that

$$(\pi^j_{S^{[n]}})_* c_i(S^{[n]}) = 0 \quad \text{in } \mathrm{CH}^i(S^{[n]})$$

as soon as

$$(\pi^j_{S^{[n]}})_*[c_i(S^{[n]})] = 0 \quad \text{in } \mathrm{H}^{2i}(S^{[n]}, \mathbb{Q})$$

(equivalently as soon as $j \neq 2i$). By de Cataldo and Migliorini's theorem, it is enough to show for all partitions $\mu$ of $\{1, \dots, n\}$ that

$$(\Gamma_\mu)^*(\pi^j_{S^{[n]}})_* c_i(S^{[n]}) = 0 \quad \text{in } \mathrm{CH}^*(S^\mu)$$

as soon as

$$(\Gamma_\mu)^*(\pi^j_{S^{[n]}})_*[c_i(S^{[n]})] = 0 \quad \text{in } \mathrm{H}^*(S^\mu, \mathbb{Q}).$$

Proceeding as in Section 2, it is even enough to show that, for all partitions $\mu$ of $\{1, \dots, n\}$, $(\pi^j_{S^\mu})_*(\Gamma_\mu)^* c_i(S^{[n]}) = 0$ in $\mathrm{CH}^*(S^\mu)$ as soon as $(\pi^j_{S^\mu})_*(\Gamma_\mu)^*[c_i(S^{[n]})] = 0$ in $\mathrm{H}^*(S^\mu, \mathbb{Q})$. By Proposition 3.2, $(\Gamma_\mu)^* c_i(S^{[n]})$ is a universal polynomial in the variables $\mathrm{pr}_r^* c_2(S)$, $\mathrm{pr}_{r'}^* K_S$, $\mathrm{pr}_{s,t}^* \Delta_S$. It follows that $(\pi^j_{S^\mu})_*(\Gamma_\mu)^* c_i(S^{[n]})$ is also a universal polynomial in the variables $\mathrm{pr}_r^* c_2(S)$, $\mathrm{pr}_{r'}^* K_S$, $\mathrm{pr}_{s,t}^* \Delta_S$. We can then conclude thanks to Proposition 3.3 and Theorem 3.4.

**3.2. The Hilbert scheme of points on an abelian surface.** Let $A$ be an abelian surface. In that case, the Chow–Künneth projectors $\{\pi^i_A\}$ given by the theorem of Deninger and Murre are *symmetrically distinguished* in the Chow ring $\mathrm{CH}^*(A \times A)$ in the sense of O'Sullivan [14]. (We refer to [16, Section 7] for a summary of O'Sullivan's theory of symmetrically distinguished cycles on abelian varieties.) Let us mention that the identity element $O_A$ of $A$ plays the role of the Beauville–Voisin cycle $\mathfrak{o}$ in the case of K3 surfaces, e.g. $\pi^0_A = O_A \times A$. By O'Sullivan's theorem, the Chow–Künneth projectors $\pi^i_{A^m}$ given in (3) are symmetrically distinguished for all positive integers $m$. By Proposition 3.1, the cycle $(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3})^* \Delta_3$ is a polynomial in the variables $\mathrm{pr}_r^* c_2(A)$, $\mathrm{pr}_{r'}^* K_A$ and $\mathrm{pr}_{s,t}^* \Delta_A$. Since $c_2(A) = 0$ and $K_A = 0$, this cycle is in fact symmetrically distinguished. It immediately follows that

$$\left(\pi^i_{A^{\mu_1}} \otimes \pi^j_{A^{\mu_2}} \otimes \pi^k_{A^{\mu_3}}\right)_* \left(\Gamma_{\mu_1} \otimes \Gamma_{\mu_2} \otimes \Gamma_{\mu_3}\right)^* \Delta_3$$

is symmetrically distinguished. Thus by O'Sullivan's theorem [14], this cycle is rationally trivial if and only if it is numerically trivial. By Proposition 2.1, we conclude that $A^{[n]}$ has a multiplicative Chow–Künneth decomposition. The proof of Theorem 1 is now complete.

It remains to prove that the Chern classes $c_i(A^{[n]})$ sit in $\mathrm{CH}^i(A^{[n]})_0$. As in the case of K3 surfaces, it suffices to show that, for all partitions $\mu$ of $\{1, \dots, n\}$,

$$(\pi^j_{A^\mu})_*(\Gamma_\mu)^* c_i(A^{[n]}) = 0 \quad \text{in } \mathrm{CH}^*(A^\mu)$$

as soon as

$$(\pi^j_{A^\mu})_*(\Gamma_\mu)^*[c_i(A^{[n]})] = 0 \quad \text{in } \mathrm{H}^*(A^\mu, \mathbb{Q}).$$

By Proposition 3.2, $(\Gamma_\mu)^* c_i(A^{[n]})$ is a polynomial in the variables $\mathrm{pr}_r^* c_2(A) = 0$, $\mathrm{pr}_{r'}^* K_A = 0$, $\mathrm{pr}_{s,t}^* \Delta_A$. It follows that the cycle $(\pi^j_{A^\mu})_*(\Gamma_\mu)^* c_i(A^{[n]})$ is symmetrically distinguished. We can then conclude thanks to O'Sullivan's theorem.

## 4. Decomposition theorems for the relative Hilbert scheme of abelian surface schemes and of families of K3 surfaces

In this section, we generalize Voisin's decomposition theorem [20, Theorem 0.7] for families of K3 surfaces to families of Hilbert schemes of points on K3 surfaces or abelian surfaces.

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. Deligne's decomposition theorem states the following:

**Theorem 4.1** (Deligne [6]). *In the derived category of sheaves of $\mathbb{Q}$-vector spaces on $B$, there is a decomposition (which is noncanonical in general)*

$$R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i]. \tag{9}$$

Both sides of (9) carry a cup-product: on the right-hand side the cup-product is the direct sum of the usual cup-products $R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q} \to R^{i+j}\pi_*\mathbb{Q}$ defined on local systems, while on the left-hand side the derived cup-product $R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ is such that it induces the usual cup-product in cohomology. As explained in [20], the isomorphism (9) does not respect the cup-product in general. Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, Voisin [20, Question 0.2] asked if there exists a decomposition as in (9) which is multiplicative, i.e., which is compatible with cup-product. By Deninger–Murre [7], there does exist such a decomposition for an abelian scheme $\pi : \mathcal{A} \to B$. The main result of [20] is:

**Theorem 4.2** (Voisin [20]). *For any smooth projective family $\pi : \mathcal{X} \to B$ of K3 surfaces, there exist a decomposition isomorphism as in equation* (9) *and a nonempty Zariski open subset $U$ of $B$ such that this decomposition becomes multiplicative for the restricted family $\pi|_U : \mathcal{X}|_U \to U$.*

Our main result in this section is the following extension of Theorem 4.2:

**Theorem 4.3.** *Let $\pi : \mathcal{X} \to B$ be either an abelian surface over $B$ or a smooth projective family of K3 surfaces. Consider $\pi^{[n]} : \mathcal{X}^{[n]} \to B$ the relative Hilbert scheme of length-$n$ subschemes on $\mathcal{X} \to B$. Then there exist a decomposition isomorphism for $\pi^{[n]} : \mathcal{X}^{[n]} \to B$ as in* (9) *and a nonempty Zariski open subset $U$ of $B$ such that this decomposition becomes multiplicative for the restricted family $\pi^{[n]}|_U : \mathcal{X}^{[n]}|_U \to U$.*

*Proof.* The proof follows the original approach of Voisin [20] (after reinterpreting, as in [16, Proposition 8.14], the vanishing of the modified diagonal cycle of Beauville–Voisin [4] as the multiplicativity of the Beauville–Voisin Chow–Künneth decomposition).

First, we note that there exist a nonempty Zariski open subset $U$ of $B$ and relative Chow–Künneth projectors

$$\Pi^i := \Pi^i_{\mathcal{X}^{[n]}|_U/U} \in \mathrm{CH}^{2n}(\mathcal{X}^{[n]}|_U \times_U \mathcal{X}^{[n]}|_U),$$

which means that $\Delta_{\mathcal{X}|_U/U} = \sum_i \Pi^i$, $\Pi^i \circ \Pi^i = \Pi^i$, $\Pi^i \circ \Pi^j = 0$ for $i \neq j$, and $\Pi^i$ acts as the identity on $R^i(\pi^{[n]}|_U)_*\mathbb{Q}$ and as zero on $R^j(\pi^{[n]}|_U)_*\mathbb{Q}$ for $j \neq i$. Indeed, let $X$ be the generic fiber of $\pi : \mathcal{X} \to B$. If $X$ is a K3 surface, then we consider the degree 1 zero-cycle

$$\mathfrak{o} := \frac{1}{24} c_2(X) \in \mathrm{CH}_0(X).$$

We then have a Chow–Künneth decomposition for $X$ given by $\pi^0_X := \mathrm{pr}_1^* \mathfrak{o}$, $\pi^4_X := \mathrm{pr}_2^* \mathfrak{o}$ and $\pi^2_X := \Delta_X - \pi^0_X - \pi^4_X$. If $X$ is an abelian surface, we may consider the Chow–Künneth decomposition of Deninger–Murre [7]. In both cases, these Chow–Künneth decompositions induce as in (4) a Chow–Künneth decomposition $\Delta_{X^{[n]}} = \sum_i \pi^i_{X^{[n]}}$ of the Hilbert scheme of points $X^{[n]}$. By spreading out, we obtain the existence of a sufficiently small but nonempty open subset $U$ of $B$ such that this Chow–Künneth decomposition spreads out to a relative Chow–Künneth decomposition $\Delta_{\mathcal{X}|_U/U} = \sum_i \Pi^i$.

By [20, Lemma 2.1], the relative idempotents $\Pi^i$ induce a decomposition in the derived category

$$R\pi_*\mathbb{Q} \cong \bigoplus_{i=0}^{4n} \mathrm{H}^i(R\pi_*\mathbb{Q})[-i] = \bigoplus_{i=0}^{4n} R^i\pi_*\mathbb{Q}[-i]$$

with the property that $\Pi^i$ acts as the identity on the summand $\mathrm{H}^i(R\pi_*\mathbb{Q})[-i]$ and acts as zero on the summands $\mathrm{H}^j(R\pi_*\mathbb{Q})[-j]$ for $j \neq i$. Thus, in order to show the existence of a decomposition as in (9) that is multiplicative, it is enough to show, up to further shrinking $U$ if necessary, that the relative Chow–Künneth decomposition $\{\Pi^i\}$ above satisfies

$$(10)\quad \Pi^k \circ \Delta_3 \circ (\Pi^i \otimes \Pi^j) = 0 \quad \text{in } \mathrm{CH}^{4n}((\mathcal{X}^{[n]} \times_B \mathcal{X}^{[n]} \times_B \mathcal{X}^{[n]})|_U), \quad k \neq i + j.$$

Here, $\Delta_3$ is the class of the relative small diagonal inside $\mathrm{CH}^{4n}(\mathcal{X}^{[n]} \times_B \mathcal{X}^{[n]} \times_B \mathcal{X}^{[n]})$. But then, by Theorem 1, the relation (10) holds generically. Therefore, by spreading out, (10) holds over a nonempty open subset of $B$. This concludes the proof of the theorem. □

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