Groups and embeddings in $SL(2, \mathbb{C})$

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Abstract

We give results on when a finitely generated torsion free group does or does not embed in $SL(2, \mathbb{C})$. For instance if we glue two copies of the figure 8 knot along its torus boundary then the fundamental group of the resulting closed 3-manifold sometimes embeds in $SL(2, \mathbb{C})$ and sometimes does not, depending on the identification, whereas a graph of free groups with maximal cyclic edge groups need not embed, even if it is word hyperbolic.

Keywords: CSA group, graph of groups, limit group

1 Introduction

The group $SL(2, \mathbb{C})$ plays an important role in both algebra and geometry. In particular we obtain on quotienting out by $\{\pm I\}$ the group $PSL(2, \mathbb{C})$ of Möbius transformations which acts as the group of orientation preserving isometries on the hyperbolic space $\mathbb{H}^3$. A discrete subgroup Γ of $PSL(2, \mathbb{C})$ is a Kleinian group and these have been much studied because the quotient $\Gamma \backslash \mathbb{H}^3$ is a 3-orbifold, and even an orientable 3-manifold if Γ is torsion free, with a complete hyperbolic metric. Now if an abstract group G with no elements of order 2 is a discrete subgroup of $PSL(2,\mathbb{C})$ then [7] shows that G lifts to a discrete subgroup of $SL(2,\mathbb{C})$.

However we can approach this question from a group theoretic point of view and ask when a given finitely generated torsion free group does or does not embed in $SL(2,\mathbb{C})$, with no reference to discreteness. Of course there are many finitely generated, torsion free groups which do not embed in $SL(2,\mathbb{C})$. The first obstruction which comes to mind is that of being commutative transitive: a group G is commutative transitive (CT) if the relation of two elements commuting is an equivalence relation on $G - \{e\}$. It is straightforward to show and well known that a torsion free subgroup of $SL(2,\mathbb{C})$ (again this generalises to not containing an element of order 2) will be commutative transitive.

Meanwhile there are two well known sufficient conditions: being the fundamental group of an orientable hyperbolic 3-manifold as mentioned above, and being a limit group which here is defined to be a finitely generated group which is fully residually free. In this paper we show that this necessary condition is far from being sufficient and that these two sufficient conditions are far from being necessary, even within very restricted and well behaved classes of groups. One well behaved class that we consider consists of the fundamental groups of closed orientable irreducible 3-manifolds. Here we show that the fundamental group of any torus bundle embeds in $SL(2,\mathbb{C})$ provided it is CT, whereas the closed orientable 3-manifolds obtained by identifying two copies of the figure 8 knot complement along their boundary tori have fundamental groups that almost never embed in $SL(2,\mathbb{C})$. This is despite the latter groups having the property CSA which is strictly stronger than being CT, whereas the former groups do not.

We also look at the class of graphs of non abelian free groups with maximal cyclic edge groups. Here we have the Bestvina - Feighn combination theorems which tell us when such a group is word hyperbolic. Now all torsion free word hyperbolic groups are both CT and CSA, so we might expect that if the fundamental group of a graph of non abelian free groups with maximal cyclic edge groups happens to be word hyperbolic then it embeds in $SL(2,\mathbb{C})$. However we give an example to show that the answer to this is no, along with an example which does embed in $SL(2,\mathbb{C})$ but which is not a limit group, nor is the fundamental group of any orientable 3-manifold.

We introduce basic properties of CT and CSA groups in Section 2, including a characterisation of the CSA subgroups of $SL(2,\mathbb{C})$ in Proposition 2.1, then mention results in the literature that are known to preserve embeddings in $SL(2,\mathbb{C})$. It goes back to Nisnevič in 1940 that (on avoiding the element $-I$) a free product of subgroups of $SL(2,\mathbb{C})$ also embeds in $SL(2,\mathbb{C})$. This result of Nisnevič was rediscovered by Wehrfritz in [15] and by Shalen in [14], where each author also gives conditions for a free product with cyclic amalgamation of two subgroups of $SL(2,\mathbb{C})$ to embed in $SL(2,\mathbb{C})$.

In Section 3 we examine graphs of non abelian free groups with maximal cyclic edge groups. If this graph is a tree then it is known by results mentioned above that the resulting fundamental group is word hyperbolic and embeds in $SL(2,\mathbb{C})$. We first look at a very special case of this - the cyclically pinched groups, where we have a single amalgamation. We can therefore ask whether these groups are always limit groups. Although it is folklore that the answer is negative, no concrete example has been given. We prove in Theorem 3.2 that any cyclically pinched group formed by amalgamating a commutator on one side with a product of two proper powers of distinct commutators on the other side is not a limit group if the powers differ by at least 3. The proof is short and uses standard facts about stable commutator length, including the lower bound for stable commutator length in a free group. This means we can then give in Corollary 3.4 an explicit example of such a group which is also not the fundamental group of any orientable 3-manifold.

Graph of groups constructions also involve HNN extensions, but this case is less well behaved. For instance the Klein bottle group, which is an HNN extension of $\mathbb{Z}$ over maximal cyclic groups, is not even CT. We show in Proposition 3.5 that there is a construction using only HNN extensions of a graph of free groups with maximal cyclic edge groups where the fundamental group is word hyperbolic but which does not embed in $SL(2,\mathbb{C})$, as opposed to the case where only amalgamations are used.

In Section 4 we consider closed 3-manifolds which have a fundamental group that embeds in $SL(2,\mathbb{C})$. In particular, along with orientable hyperbolic 3-manifolds where the fundamental group will embed discretely, we give in Theorem 4.1 the complete list of torus bundles fibred over the circle with such fundamental groups: as well as the trivial bundle it is precisely those with Sol geometry. Therefore one might hope that the fundamental group of a closed orientable 3-manifold admitting a JSJ decomposition along tori with all pieces hyperbolic might embed, given that we would be amalgamating groups embedding in $SL(2,\mathbb{C})$ over abelian subgroups. However we show in Theorem 4.2 that if we glue two copies of the figure 8 knot along each boundary torus such that the meridians are identified then the fundamental group of the resulting 3-manifold does not embed in $SL(2,\mathbb{C})$ unless the longitudes are also glued to each other. In this latter case the group does embed, giving a genus 2 surface bundle fibred over the circle with fundamental group embedding in $SL(2,\mathbb{C})$ even though the homeomorphism is not pseudo-Anosov. This work utilises a result of Whittemore from 1973 giving all representations of the figure 8 knot group in $SL(2,\mathbb{C})$.

2 Embedding and non embedding results

We say that a group G is **commutative transitive** or **CT** for short if the relation of two elements commuting is transitive on $G - \{e\}$. The finite CT groups are known but there are many interesting examples of infinite CT groups. For instance Corollary 1 of [1] states that the free Burnside groups of sufficiently large odd period are also CT groups. However here our focus is on torsion free groups, in which case we have as examples free groups, limit groups and torsion free word hyperbolic groups. Moreover it is straightforward to see that any subgroup of $SL(2,\mathbb{C})$ not containing $-I$ (thus in particular a torsion free subgroup) is CT by looking at canonical forms of 2 by 2 matrices. Thus we can regard the absence of CT in a torsion free group as the first obstruction to having an embedding in $SL(2,\mathbb{C})$.

There is a condition that is stronger than being CT but which is often useful. We say that a subgroup H of a group G is **malnormal** (or conjugate separated) in G if $gHg^{-1} \cap H = \{e\}$ for all $g \in G - H$. A group G is then called **CSA** (standing for conjugate separated abelian) if every maximal abelian subgroup is malnormal, and this implies CT. Again free groups, limit groups and torsion free word hyperbolic groups are CSA, and both the CSA and the CT properties are preserved under taking subgroups, but now the situation for subgroups of $SL(2,\mathbb{C})$ not containing $-I$ is slightly different.

Proposition 2.1 *If G is a subgroup of $SL(2,\mathbb{C})$ such that the only element of G with trace in $\{-2, 2\}$ is I then G is a CSA group.*

More generally a non abelian subgroup G of $SL(2,\mathbb{C})$ is CSA if and only if it does not contain $-I$ and for any non identity element g with trace$(g) = \pm 2$ and $\gamma_g \in SL(2,\mathbb{C})$ such that $\gamma_g g \gamma_g^{-1} = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, the conjugate group $\gamma_g G \gamma_g^{-1}$ contains no elements with bottom left hand entry 0 other than those with trace ± 2.

Proof. On being given an element g of G with trace not equal to ± 2 we can assume by conjugation that g is a diagonal matrix not equal to $\pm I$. Clearly the centraliser $C_G(g)$ is equal to the abelian subgroup of diagonal elements in G and does not contain $-I$. Now it is easy to check that in a subgroup H of $SL(2,\mathbb{C})$ missing $-I$, the set of diagonal elements of H is a malnormal subgroup of H which completes the first case.

Now suppose (conjugating if necessary) that $g = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \neq 0$. Then $C_G(g)$ is also an abelian subgroup, consisting of all matrices in G with trace ± 2 and bottom left hand entry 0. However an element of $SL(2,\mathbb{C})$ can only conjugate one (non identity) element of this form into another if its bottom left hand entry is 0 as well. If this element is in G then it lies in $C_G(g)$ if and only if its trace is ± 2, by the given condition.

□

There is a useful situation which implies both the CSA and the CT conditions, which is when G is a group such that the centraliser of each non trivial element is infinite cyclic. Clearly G is CT and it is not hard to show that G is also CSA, for instance see [14] Lemma 3.3. In this case we say that an element g of G is **primitive** if it generates its own centraliser.

Two classes of (countable torsion free) groups which are known to embed in $SL(2,\mathbb{C})$ are the fundamental groups of orientable hyperbolic 3-manifolds and limit groups. However it has long been clear that there are various operations we can perform on subgroups of $SL(2,\mathbb{C})$ which will preserve the property of embedding in $SL(2,\mathbb{C})$. Using the direct product is doomed to failure because $G_1 \times G_2$ is not CT unless both G_1 and G_2 are abelian (or one is trivial). However there are known results concerning the free product.

Theorem 2.2 *(Nisnevič 1940 [12]) If k is a field of characteristic zero and $G_1, G_2 \leq GL(n,k)$ then $G_1 * G_2 \leq GL(n+1,k')$ for some other field k' of characteristic 0 and we can replace $n+1$ with n if there are no (non trivial) scalars in G_1 or in G_2.*

Similar versions were rediscovered in [15] Theorem 3 and in [14] Theorem 1, with any of these telling us

Corollary 2.3 *If we have two countable subgroups G_1 and G_2 of $SL(2,\mathbb{C})$, neither of which contain $-I$ then the free product $G_1 * G_2$ embeds in $SL(2,\mathbb{C})$ as well.*

Thus on taking closed orientable hyperbolic 3-manifolds M_1 and M_2 with $G_1 = \pi_1(M_1)$ and $G_2 = \pi_1(M_2)$, we have that $G_1 * G_2$ embeds in $SL(2,\mathbb{C})$ (and is word hyperbolic as the free product of two word hyperbolic factors) but is not the fundamental group of an orientable hyperbolic 3-manifold, as such a manifold is irreducible, but the Mayer-Vietoris sequence for the homology of a free product gives a contradiction for $H_3(M;\mathbb{Z})$. Moreover it is not a limit group (since this latter property is preserved on passing to finitely generated subgroups but G_1 and G_2 are not limit groups, as it was shown in [17] using work of Sela that the fundamental group of a closed orientable hyperbolic 3-manifold cannot be a limit group).

Given that linearity behaves well under free products, we now consider free products with amalgamation. However there would need to be restrictions on the factors and amalgamated subgroup because we might not even have residual finiteness in general. As we are concentrating here on torsion free groups, the first candidates for amalgamated subgroups ought to be those that are infinite cyclic. This is a fruitful pursuit for $SL(2,\mathbb{C})$, as we have:

Theorem 2.4 *(Wehrfritz 1973 [15] Theorem 5) If A and B are finite rank free groups with a a primitive element of A and b a primitive element of B then the amalgamated free product $A *_{a=b} B$ embeds in $SL(2,\mathbb{C})$.*

This was rediscovered by Shalen in [14] using a very similar proof, but resulting in a statement that can be applied recursively:

Theorem 2.5 *Let A and B be subgroups of $SL(2,\mathbb{C})$ with transcendental traces (meaning that every non identity element has a trace which is transcendental over $\mathbb{Q}$) and such that both groups satisfy the following property: the centraliser of every non identity element is infinite cyclic. Then the the amalgamated free product $A *_{a=b} B$ for any primitive $a \in A$ and $b \in B$ has an embedding into $SL(2,\mathbb{C})$ with transcendental traces and such that every non identity element has centraliser which is infinite cyclic.*

However we have no such result for HNN extensions with associated subgroup equal to a maximal infinite cyclic group, which can be seen just by looking at the Klein bottle group $\langle a, t | tat^{-1} = a^{-1} \rangle$ which is not CT, so certainly will not embed in $SL(2,\mathbb{C})$.

3 Graphs of non abelian free groups with maximal cyclic edge groups

By [3] due to Bestvina and Feighn we have that if A and B are torsion free word hyperbolic groups with $a \in A$ and $b \in B$ both non trivial elements then the amalgamated free product $G = A *_{a=b} B$ is word hyperbolic if and only if G contains no $\mathbb{Z} \times \mathbb{Z}$ subgroup, which is shown to occur if and only if either $\langle a \rangle$ is malnormal in A or $\langle b \rangle$ is malnormal in B. This is also the same as saying that either a or b is a primitive element in A or B respectively, because A and B are both word hyperbolic, hence CSA.

Thus the groups mentioned in Theorem 2.4 are all word hyperbolic. Moreover, as any finite rank free group embeds in $SL(2, \mathbb{C})$ with transcendental traces, we can take free groups $F_{r_1}, \ldots, F_{r_n}$ for any ranks r_i at least 2 and form repeated amalgamated free products over arbitrary cyclic subgroups generated by primitive elements to obtain word hyperbolic groups embedding in $SL(2, \mathbb{C})$ by Theorem 2.5.

A well studied similar construction is that of a graph of groups with non abelian free vertex groups and maximal cyclic edge groups. From this we obtain the fundamental group of a graph of groups by forming the amalgamated free product where an edge joins two distinct vertices, then contract this edge and continue until we are left with self loops which then give us HNN extensions. Consequently if the graph is a tree then only free product amalgamation occurs. The Bestvina - Feighn result mentioned above can be applied repeatedly to show word hyperbolicity of the resulting graph of groups (we also require that primitive elements remain primitive after each amalgamation, which can be shown easily using normal forms). Therefore by Theorem 2.5 and this we have:

Corollary 3.1 *If Γ is a graph of non abelian finite rank free groups with infinite cyclic edge groups which are maximal in each vertex group (namely generated by a primitive element on each side) and Γ is a tree then the fundamental group of Γ is word hyperbolic and embeds in $SL(2, \mathbb{C})$.*

We mentioned in the introduction that limit groups embed in $SL(2, \mathbb{C})$, for instance see Window 8 of [10] for a proof. Indeed [2] by B. Baumslag shows that a finitely generated residually free group is either a limit group or contains $F_2 \times \mathbb{Z}$. This last group is clearly not CT so limit groups are exactly the finitely generated residually free groups which embed in $SL(2, \mathbb{C})$.

Moreover limit groups can contain $\mathbb{Z}^k$ for $k \geq 2$ and so need not be word hyperbolic but a limit group which does not contain $\mathbb{Z}\times\mathbb{Z}$ is word hyperbolic. Therefore we should consider whether the groups in Corollary 3.1 are limit groups, in which case we already have the existence of an embedding into $SL(2,\mathbb{C})$.

A special case of the graph of free groups construction with $\mathbb{Z}$ edge groups is the amalgamated free product $G = F_{r_1} *_{w_1=w_2} F_{r_2}$ for F_{r_1}, F_{r_2} non abelian free groups and w_1, w_2 any two non trivial elements. This is called a **cyclically pinched** group if neither w_i is part of a free basis for F_{r_i} (otherwise we obtain the free group $F_{r_1+r_2-1}$). If neither element is primitive, say $w_i = u_i^{n_i}$ for $n_i \geq 2$, then it is well known that G cannot be a limit group. A quick way to see this is to note that G is not CT because $u_1u_2u_1^{-1}u_2^{-1}$ is non trivial but w_i commutes with u_1 and u_2. However if we let one or both of the w_i be primitive elements, it is still not known exactly when G is a limit group. Some cases are known, for instance doubles (where $r_1 = r_2$ and $w_1 = w_2$) have been shown to be limit groups but if only one element is primitive then examples of non limit groups go back to Lyndon in 1959. He showed that any solution to the equation $x^2 = y^2z^2$ in a free group has the property that x, y and z all commute. This means that if $G = F_{r_1} *_{u^2=v^2w^2} F_{r_2}$ for u, v, w any non trivial words then any homomorphism from G onto a free group F sends u, v, w to commuting elements in F. Thus if v and w do not commute in F_{r_2} then all homomorphisms from G to a free group send the commutator $[v, w]$ to the identity. Moreover the same property was shown in [11] to hold for the equation $x^l = y^mz^n$ for $l, m, n \geq 2$ so that an element of the form y^mz^n is not a proper power in a free group if y and z do not commute, and thus $G = F_{r_1} *_{u^l=v^mw^n} F_{r_2}$ is a word hyperbolic group which is not residually free.

Let us now assume that both w_1 and w_2 are primitive elements, so that we are in the case of Corollary 3.1 with a single edge. It is not true that we always obtain a limit group, as can be shown by the use of $\mathbb{R}$-trees. However we are unaware of any method in the literature that gives concrete examples, so will give the first ones here. Our reference on this question is [13] where it is mentioned in Part I (3) that "if the element w_1 is a commutator in the first free group and w_2 is a product of two "high" powers in the other free group then G is not a limit group" but no further details are given. However this is not true in full generality, even for any definition of "high": if G_1 is free on x, y; G_2 free on a, b and $G = G_1 *_{w_1=w_2} G_2$ is formed by setting $[x, y] = a^mb^n$ for $m, n \neq 0$ with highest common factor d then consider the homomorphism from G onto the free group $F(t,u)$ on t,u given by $x \mapsto t, y \mapsto u^{mn/d}$ and $a \mapsto tu^{n/d}t^{-1}, b \mapsto u^{-m/d}$. This sends both rank 2 free groups G_1 and G_2 to rank 2 free subgroups of $F(t,u)$, thus the restriction to each G_i is injective. Consequently G is an example of a generalised double over $F(t,u)$ as in Definition 4.4 of [6], with Proposition 4.7 of [6] showing that a generalised double over a limit group is also a limit group.

In fact the quote above becomes true if we change "powers" to "powers of commutators" with a suitably weak definition of "high". To prove this we will use stable commutator length, as described in [5]. This can be explained briefly as follows: a **length** on a group G is a function $l : G \to \mathbb{R}$ such that for $g, \gamma \in G$ we have

$$l(g\gamma) \leq l(g) + l(\gamma) \text{ and } l(g) = l(g^{-1}).$$

(Sometimes $l(e) = 0$ is required but this will not affect any of our results here.) From any length function l we obtain **stable length** $\sigma : G \to \mathbb{R}$ defined by $\sigma(g) = \lim_{n\to\infty} l(g^n)/n$. As for any $g \in G$ the sequence $a_n = l(g^n)$ is subadditive (meaning that $a_{n+m} \leq a_m + a_m$ for $n, m \in \mathbb{N}$), this limit exists provided only that there is $K \leq 0$ with $a_n \geq Kn$ for all n. It is straightforward to show using only the above properties that σ is constant on conjugacy classes, that $\sigma(g^k) = |k|\sigma(g)$ for all $k \in \mathbb{Z}$ and $g \in G$, and that $\sigma(gh) \leq \sigma(g) + \sigma(h)$ if g and h commute (although not in general). Given a group G, **commutator length** cl is a length on the commutator subgroup G' of G with $\text{cl}(g)$ defined to be the minimum number of commutators needed to form a product equal to g and **stable commutator length** is defined to be the stable length that results, denoted by $\text{scl}(g)$. A non trivial fact about scl in free groups F which we will need here is that every non identity element $w \in F'$ has $\text{scl}(w) \geq 1/2$ by [5] Theorem 4.111. Combining this with the point that in general a commutator $[w_1, w_2]$ has $\text{scl}([w_1, w_2]) \leq 1/2$, we see that non trivial commutators in free groups have stable commutator length exactly $1/2$.

Theorem 3.2 *If F_1 and F_2 are free non abelian groups then the cyclically pinched group $G = F_1 *_{\gamma=\delta^m\eta^n} F_2$, where γ is a non trivial commutator in F_1 and δ, η are both non trivial commutators in F_2 with $\delta \neq \eta^{\pm 1}$, is not residually free whenever $|m| - |n| \geq 3$ and $|m|, |n| \neq 0$ or 1, even though G embeds in $SL(2, \mathbb{C})$.*

Proof. We can assume by changing elements to inverses that $m, n > 0$. We know that $\gamma \neq e$ in G so suppose we have a homomorphism θ from G to a free group F where $\theta(\gamma) = a \neq e$. This gives us a non trivial commutator in F which is equal to $b^m c^n$ for two (possibly trivial) commutators $b = \theta(\delta)$ and $c = \theta(\eta)$ in F. First suppose that b is trivial in F then $a = c^n$ but in a free group a non trivial commutator cannot be a proper power. (This is due to Schützenberger but can also be seen here because in a free group $\mathrm{scl}(a) = 1/2 = |n|\mathrm{scl}(c) \geq |n|/2 \geq 1$, using the fact that c must be in the commutator subgroup F' too.) The same holds if c is trivial. As γ is primitive and we mentioned that $\delta^m \eta^n$ is also primitive, Theorem 2.4 tells us that G embeds in $SL(2, \mathbb{C})$. Therefore we are done by the next Proposition.

□

Proposition 3.3 *If the equation $a = b^m c^n$ holds in a free group F where a, b, c are all non trivial commutators in F then we cannot have $|m| - |n| \geq 3$.*

Proof. By Theorem 2.70 in [5] using Barvard duality, we have that for any g in G',

$$\mathrm{scl}(g) = \frac{1}{2}\mathrm{sup}_{\phi \in Q(G)} \frac{|\phi(g)|}{D(\phi)}$$

where $Q(G)$ is the space of homogeneous quasimorphisms on G and $D(\phi)$ is the defect of ϕ. This means that for all $a, b \in G$ and $k \in \mathbb{Z}$ we have $|\phi(ab) - \phi(a) - \phi(b)| \leq D(\phi)$ (the definition of a quasimorphism ϕ with defect D) and (homogeneity) $\phi(a^k) = k\phi(a)$.

Thus we can take a homogeneous quasimorphism $\phi_b \in Q$ with $|\phi_b(b)|/D(\phi_b)$ arbitrarily close to 1 as $\mathrm{scl}(b) = 1/2$. By rescaling, let us say that $D(\phi_b) = 1$ and we can assume $m\phi_b(b) > n + 2$ because $m - n \geq 3$. (In fact in a free group this supremum is obtained but we will not need to use that here.) Now for any homogeneous quasimorphism q on any group G we have that $|q(\gamma)| \leq D(q)$ if γ is a commutator in G. But as $a = b^m c^n$ holds in F we see that

$$m\phi_b(b) - n - 1 \leq m\phi_b(b) - n|\phi_b(c)| - |\phi_b(a)| \leq |m\phi_b(b) + n\phi_b(c) - \phi_b(a)| \leq 1$$

so $m\phi_b(b) \leq n + 2$, giving us a contradiction.

□

We note here that there exist explicit examples which are not the fundamental group of any orientable 3-manifold. For this we borrow an argument from the last section of [8]. Suppose we have an amalgamated free product $\Gamma = F_{r_1} *_{w_1=w_2} F_{r_2}$ as above, where w_1 and w_2 are primitive words. If $\Gamma = \pi_1(M)$ for M an orientable 3-manifold (assumed compact without loss of generality by the Scott compact core theorem) then we can take M to be prime as Γ is one ended. This means here that M is irreducible so the splitting over $\mathbb{Z}$ of $\pi_1(M)$ can be induced geometrically, giving orientable 3-manifolds M_1, M_2, which are irreducible because M is, with fundamental group free of rank r_1, respectively r_2. Now results in [9] imply that M_1 and M_2 must each be a cube with one handle, joined to form M_i by attaching along a neighbourhood of the curve w_i embedded in each boundary. But we can attach a thickened disc along an annulus neighbourhood of either of these curves as in [8]. Thus we obtain two orientable 3-manifolds with fundamental group $\langle F_{r_i}|w_i\rangle$. Now it is a pre-Geometrization fact that if we have an infinite order element x in the fundamental group of any 3-manifold M with x^i conjugate to x^j in $\pi_1(M)$ then we must have $|i| = |j|$. Thus on taking $|m| - 3 \geq |n| \geq 2$ in Theorem 3.2 with $F_2 = F(a, b)$ and $\delta = [a, b], \eta = [b^{-1}, a] = b^{-1}\delta b$, along with γ any non trivial commutator in F_1, we see that if the cyclically pinched group G were the fundamental group of an orientable 3-manifold then so is the group $\langle a, b|\delta^m b^{-1}\delta^n b\rangle$ which is a contradiction. We summarise these results as

Corollary 3.4 *There is a free product of two free groups of rank 2 amalgamated over a maximal cyclic subgroup on either side which is word hyperbolic (thus CT and CSA) and which embeds in $SL(2, \mathbb{C})$ but which is not a limit group, nor the fundamental group of an orientable 3-manifold.*

We now give a non embedding result for graphs of groups of non abelian free groups with maximal cyclic edge groups. Examples of such groups which are not subgroups of $SL(2, \mathbb{C})$ are easy to construct by making them contain the Klein bottle group. However these will fail the CT condition whereas we would like examples that are CT, CSA or even word hyperbolic.

We have a result in [4] giving conditions on when an HNN extension over a virtually cyclic group is word hyperbolic, alongside the theorem already mentioned in [3] for amalgamated free products. For a torsion free word hyperbolic group G, this states that if A, B are infinite cyclic subgroups of G then an HNN extension formed by identifying A and B is word hyperbolic if and only if for all $g \in G$ we have $gAg^{-1} \cap B = \{e\}$ and where at least one of A and B is generated by a primitive element. This is also equivalent to saying that the HNN extension does not contain a Baumslag Solitar subgroup. (This result is false if the primitive condition is removed, as was originally stated in [3], hence necessitating the appearance of [4].) We repeat this construction until too many elements are forced to be conjugate to each other.

Proposition 3.5 *Let F_2 be free on a, b and let Γ be the triple HNN extension formed using stable letters t, s, r so that $tat^{-1} = b, sas^{-1} = ab$ and $rar^{-1} = aba^{-1}b^{-1}$. Then Γ does not embed in $SL(2, \mathbb{C})$ but is a graph of non abelian free groups with maximal cyclic edge groups which is word hyperbolic.*

Proof. Suppose that G is a group where the centraliser of every non trivial element is infinite cyclic, and that a and b are primitive elements of G where no conjugate of $A = \langle a\rangle$ intersects $B = \langle b\rangle$ apart from in the identity. Then any primitive element of G is also primitive in the HNN extension $\Gamma = G*_{tat^{-1}=b}$ where t is the stable letter of the HNN extension. This can be seen directly using normal forms, whereupon we also have that if two primitive elements of G are conjugate in Γ then either they are conjugate in G or one is conjugate in G to $a^{\pm 1}$ and the other to $b^{\pm 1}$.

Now if Γ did embed in $SL(2, \mathbb{C})$ then a and b would generate a rank two free group where a, b, ab and $aba^{-1}b^{-1}$ all have the same trace, $z \in \mathbb{C}$ say. But using the well known trace identities in $SL(2, \mathbb{C})$ which go back to Fricke and Klein, we have that

$$\mathrm{tr}^2(a) + \mathrm{tr}^2(a) + \mathrm{tr}^2(ab) - 2 = \mathrm{tr}(a)\ \mathrm{tr}(b)\ \mathrm{tr}(ab)\ + \mathrm{tr}(aba^{-1}b^{-1})$$

so $(z-2)(z^2-z-1) = 0$, but $z = 2$ would imply that $\langle a, b\rangle$ is metabelian. If $z = (1 \pm \sqrt{5})/2$ then, as $\cos(\pi/5) = (1 + \sqrt{5})/4$ and $\cos(3\pi/5) = (1 - \sqrt{5})/4$ we have on diagonalising that any matrix in $SL(2, \mathbb{C})$ with trace either value for z has order 10, but Γ is torsion free.

Now Γ is formed using a graph of groups consisting of one vertex representing F_2 and three loops for the three pairs of edge groups, all of which are maximal cyclic in F_2. To see that Γ is word hyperbolic, we apply Bestvina and Feighn's result above each time, noting that $a, b, ab, aba^{-1}b^{-1}$ are non conjugate primitive elements of F_2 and using the comment at the start of this proof.

□

4 2 Dimensional linearity of 3-manifold groups

Following Perelman's positive solution to Thurston's Geometrization Conjecture, there has been very recent progress, using results of Wise and others on virtually special groups, which means that nearly all finitely generated 3-manifold groups are known to be linear: indeed the only compact orientable irreducible 3-manifolds for which this question is open are those closed graph 3-manifolds which do not admit a Riemannian metric of non positive curvature.

However we can ask which closed 3-manifolds have fundamental groups that embed in $SL(2,\mathbb{C})$. Clearly this holds for closed orientable hyperbolic 3-manifolds as their fundamental groups are discrete subgroups of $SL(2,\mathbb{C})$, but we can look at torus bundles, none of which have fundamental groups that embed discretely, or which are limit groups apart from $\mathbb{Z}^3$.

Theorem 4.1 *If M^3 is a closed 3-manifold which is a 2 dimensional torus bundle over the circle, so that $\pi_1(M^3)=\mathbb{Z}^2\rtimes_\alpha\mathbb{Z}$ where α is the automorphism of $\mathbb{Z}^2$ induced by the gluing map, then $\pi_1(M^3)$ is a subgroup of $SL(2,\mathbb{C})$ if and only if α is the identity or is a hyperbolic map (that is all powers of α fix only 0).*

Proof. Let $G=\pi_1(M^3)$ and $\mathbb{Z}^2=\langle a,b\rangle$, with conjugation by the stable letter t giving our automorphism α. If this is the identity then $G=\mathbb{Z}^3$ embeds in $SL(2,\mathbb{C})$. If some positive power α^n fixes $x\in\mathbb{Z}^2-\{0\}$ and G embeds in $SL(2,\mathbb{C})$ then t commutes with x because G is CT, but x commutes with all of the fibre subgroup $\mathbb{Z}^2$ so t does too and so we have the identity automorphism.

Otherwise the eigenvalues of α are not roots of unity and so not of modulus 1, because they satisfy a monic integer quadratic with constant term ±1. Say $\alpha(a)=tat^{-1}=a^ib^j$ and $\alpha(b)=tbt^{-1}=a^kb^l$. We set

$$a=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad b=\begin{pmatrix}1&x\\0&1\end{pmatrix}\qquad\text{and } t=\begin{pmatrix}\mu&0\\0&\mu^{-1}\end{pmatrix}$$

where λ and μ are non zero complex numbers to be determined. For the above two relations to be satisfied, we require $\mu^2=i+\lambda j$ and $\mu^2\lambda=k+\lambda l$. This just corresponds to the matrix $\begin{pmatrix}i&j\\k&l\end{pmatrix}\in GL(2,\mathbb{Z})$ having an eigenvalue μ^2 with the eigenvector $\begin{pmatrix}1\\\lambda\end{pmatrix}$ which does occur, and λ is not zero because

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ being an eigenvector implies that there was an eigenvalue of ± 1. Moreover $\lambda \in \mathbb{Q}$ implies that $\mu^2 \in \mathbb{Z}$ but the determinant being ± 1 would give $\mu^2 = \pm 1$ too, which has been eliminated. Therefore μ^2 not being a root of unity implies that the matrix t has infinite order, and the matrix $a^p b^q t^r = \begin{pmatrix} \mu^r & \mu^{-r}(p+\lambda q) \\ 0 & \mu^{-r} \end{pmatrix}$ can only be the identity if $p = q = r = 0$, so this representation of G in $SL(2,\mathbb{C})$ is faithful.

□

Note it does not matter here if α was orientation preserving or reversing and hence if M^3 is orientable or not. This result shows just how much variation there can be in the geometry of a 3-manifold and still the fundamental group embeds in $SL(2,\mathbb{C})$. For instance we can take the connected sum of a Lens space (with finite cyclic fundamental group of odd order), the 3-torus (with fundamental group $\mathbb{Z}^3$), a closed orientable hyperbolic 3-manifold and a torus bundle with a hyperbolic automorphism as in Theorem 4.1. The fundamental group of the resulting 3-manifold embeds in $SL(2,\mathbb{C})$ by Corollary 2.3 but there are pieces of this manifold which are positively curved, have zero curvature, have negative curvature and which do not admit a metric of nonpositive or of nonnegative curvature.

Based on this, one might hope that any finitely generated torsion free 3-manifold group $\pi_1(M^3)$ which is CT embeds in $SL(2,\mathbb{C})$. However this turns out not to be the case even if $\pi_1(M^3)$ is a CSA group, in spite of the fact that the groups in Theorem 4.1 are all (except when $\alpha = \mathrm{id}$) not CSA by Proposition 2.1.

Theorem 4.2 *Let M_1^3, M_2^3 be two copies of the figure eight knot complement with each boundary $\partial M_1^3, \partial M_2^3$ a torus, and let M^3 be the closed orientable irreducible 3-manifold formed by gluing the boundary tori together using any orientation reversing homeomorphism which identifies the two meridians, with the exception of the homeomorphism which also identifies the two longitudes. Then $\pi_1(M^3)$ does not embed in $SL(2,\mathbb{C})$ although it is CT and even a CSA group.*

Proof. Although M_1^3 has only one discrete faithful embedding in $SL(2,\mathbb{C})$ (because it has finite hyperbolic volume so Mostow rigidity applies), up to conjugation in $SL(2,\mathbb{C})$, replacing matrices with their negative and taking complex conjugates, there is a whole curve of representations. This curve was found in [16] where it was shown that if A and B are two non commuting elements of $SL(2,\mathbb{C})$ satisfying

$$r(A,B)=B^{-1}A^{-1}BAB^{-1}ABA^{-1}B^{-1}A=I$$

then we can conjugate A and B so that either

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix} \text{ and } B=\begin{pmatrix}1&0\\-\omega&1\end{pmatrix} \text{ where } \omega=e^{2\pi i/3} \text{ or } e^{-2\pi i/3} \tag{1}$$

(or A and B are both minus the above) or we can take

$$A=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix} \text{ and } B=\begin{pmatrix}\mu&1\\ \mu(x-\mu)-1&x-\mu\end{pmatrix} \tag{2}$$

(or both minus this) where $\lambda\in\mathbb{C}-\{0,\pm1\}$, $x=\lambda+\lambda^{-1}$,

$$z=\frac{1}{2}\left(1+x^2\pm\sqrt{(x^2-1)(x^2-5)}\right) \text{ and } \mu=\frac{\lambda z-x}{\lambda^2-1}.$$

The figure eight knot complement can also be thought of as the once punctured torus bundle given by taking the free group of rank 2 on a,b and forming the HNN extension

$$\langle t,a,b|tat^{-1}=aba,tbt^{-1}=ba\rangle$$

with stable letter t. Then $\langle t,aba^{-1}b^{-1}\rangle=\mathbb{Z}\times\mathbb{Z}$ which forms the boundary torus with t the meridian and $aba^{-1}b^{-1}$ the longitude. That this is isomorphic to the group $\langle A,B|r(A,B)\rangle$ given above can be seen by setting $a=BA^{-1}$, $b=B^{-1}ABA^{-1}$ and $t=A$, so that $A=t$ and $B=at$ which is also equal to $b^{-1}tb$. This means that the meridian m is equal to A and the longitude is $l=BA^{-1}B^{-1}A^2B^{-1}A^{-1}B$.

We can use the standard identities to find the trace τ of l in terms of the traces of A,B and AB, which here are equal to x,x and z respectively. The answer is $\tau(x,x,z)=2+x^2(z-2)(x^4+(z+2)(z+2-2x^2))$ but we also have the equation $z^2=(1+x^2)z-2x^2+1$ holding which is quadratic in z. If we now substitute this expression for z^2 into τ we find that z vanishes with the trace of l being a function just of x, namely x^4-5x^2+2.

We now suppose that we have an embedding of $G=\pi_1(M^3)$ in $SL(2,\mathbb{C})$ where $G=G_1*_{\mathbb{Z}\times\mathbb{Z}}G_2$ for $G_i=\pi_1(M_i^3)$ and $\mathbb{Z}\times\mathbb{Z}$ is the fundamental group of the boundary torus. We can conjugate G in $SL(2,\mathbb{C})$ so that without loss of generality we have $G_1 = \langle A_1, B_1\rangle$ with A_1, B_1 matrices in the form above, giving rise to the meridian $m_1 = A_1$ and the longitude $l_1(A_1, B_1)$.

First suppose that A_1 and B_1 are of the form in (2) for some parameter $\lambda_1 \neq 0, \pm 1$ (and other parameters x_1, z_1, μ_1 depending on λ_1). As the longitude $l_1(A_1, B_1)$ commutes with M_1, it will also be a diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_1^{-1} \end{pmatrix}$ say, with $\mathrm{tr}(l_1) = d_1 + d_1^{-1}$. We also have the meridian m_2 and longitude l_2 of G_2 and we are forcing m_2 to be equal to $m_1^{-1} = A_1^{-1}$, so that the two figure 8 knot complements are joined on either side of the boundary torus. Moreover the homeomorphism must identify the longitude l_2 of M_2^3 with the curve $m_1^n l_1$ in ∂M_1^3 to obtain an orientation reversing homeomorphism between the two boundary tori so as to match the orientations of the 3-manifolds and make M^3 orientable. In particular the longitude l_2 of G_2 must be a diagonal matrix $\begin{pmatrix} d_2 & 0 \\ 0 & d_2^{-1} \end{pmatrix}$ say, because it commutes with $m_2 = m_1^{-1}$, where $\mathrm{tr}(m_1) = \mathrm{tr}(m_2) = x_1$. Now although $\langle A_2, B_2\rangle$ is not now in the form (2), we have $\mathrm{tr}(l_1) = x_1^4 - 5x_1^2 + 2 = \mathrm{tr}(l_2)$, giving us that $d_2 = d_1^{\pm 1}$. But as $l_2 = m_1^n l_1$ is a product of diagonal matrices, we obtain $d_2 = \lambda_1^n d_1$. Thus we either have $d_1^2 \lambda_1^n = 1$, which is a contradiction because $\langle l_1, m_1\rangle = \mathbb{Z} \times \mathbb{Z}$, or $\lambda_1^n = 1$ which is also a contradiction unless $n = 0$.

The case in (1) is similar. We can assume by conjugating G in $SL(2, \mathbb{C})$, as well as taking minus signs and complex conjugation if necessary, that $A_1 = A$ and $B_1 = B$ in (1) for $\omega = e^{2\pi i/3}$ so a quick calculation tells us that $l_1 = \begin{pmatrix} -1 & -2\sqrt{3}i \\ 0 & -1 \end{pmatrix}$. Also we again have $X \in SL(2, \mathbb{C})$ so that $XG_2X^{-1} = \langle A_2, B_2\rangle$ for A_2, B_2 as in (1) but with $\omega = e^{\pm 2\pi i/3}$ (we can rule out having to put minus signs in front of A_2 and B_2 because $A_2 = XA_1X^{-1}$ so A_2 has trace 2). Thus Xl_2X^{-1} is equal to l_1 or its complex conjugate. Now $l_2 = m_1^n l_1 = \begin{pmatrix} -1 & -2\sqrt{3}i - n \\ 0 & -1 \end{pmatrix}$ but we know $A_2 = m_2 = m_1^{-1} = A_1^{-1}$ so X conjugates A_1 into its inverse and therefore can only be $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. But on comparing Xl_2X^{-1} and l_1, we again see that n can only be zero.

As for the last part, it is well known that discrete torsion free subgroups of $SL(2, \mathbb{C})$ are CSA groups. For instance this can be seen quickly by using Proposition 2.1 along with the straightforward fact that any group containing

$\pm\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \neq 0$ and any infinite order element of the form $\begin{pmatrix} \lambda & y \\ 0 & \lambda^{-1} \end{pmatrix}$ for $\lambda \neq 0, \pm 1$ and $y \in \mathbb{C}$ is non discrete. Thus G_1 and G_2 are CSA, so we are done by [6] Corollary A.8 which states as a special case that if $G = G_1 *_H G_2$ for G_1, G_2 CSA and H maximal abelian and malnormal in both G_1 and G_2 then G is CSA.

□

In the case where the longitudes are glued to each other we do in fact have an embedding of G in $SL(2,\mathbb{C})$, giving an example of a closed orientable 3-manifold M fibred over the circle with fibre a genus 2 surface and with a non pseudo-Anosov gluing homeomorphism but such that $\pi_1(M)$ still embeds in $SL(2,\mathbb{C})$. This follows from [14] Proposition 1.3, provided we can show that there are faithful embeddings of the fundamental group of the figure 8 knot complement of the form (2) in Theorem 4.2:

Proposition 4.3 *For any transcendental number $\lambda \in \mathbb{C}$, the matrices in (2) provide a faithful embedding in $SL(2,\mathbb{C})$ of the fundamental group of the figure 8 knot complement G.*

Proof. On taking λ to be any transcendental number, we have that x and z will be transcendental too (for either choice of z). Now it is well known that the trace of any element in $\langle A, B\rangle$ is given by a triple variable polynomial in the trace of A, of B, and of AB, having coefficients in $\mathbb{Z}$. Therefore if there is an element g in G which has trace ± 2 for this value of λ then we know that some polynomial f in $\mathbb{Z}[u,v]/\langle r(u,v)\rangle$ is zero at $(u,v) = (x(\lambda), z(\lambda))$ for one of the choices of z, where $r(u,v)$ is the irreducible polynomial $v^2-(1+u^2)v+2u^2-1$. Using this relation we can assume that $f(u,v)$ is of the form $vp(u)+q(u) = 0$ for $p, q \in \mathbb{Z}[u]$ which implies that $q^2-p^2+2u^2p^2+(1+u^2)pq$ is zero when evaluated at x. As x is transcendental, this polynomial must be identically zero and so $(pz+q)(pz-p(1+x^2)-q) = 0$ holds for all values of x and z satisfying $f(x,z) = 0$. Thus it must be a multiple of $r(x,z)$, which is an irreducible degree 2 polynomial in z, giving a contradiction unless $p = q = 0$. Hence the trace of g is constant in all representations. But on setting $x = 2$ so that $z = (5 \pm \sqrt{-3})/2 = 2 - \omega$, we have the faithful discrete representation in (1). In this case we know that the elements with trace ± 2 can all be conjugated to lie in the $\mathbb{Z} \times \mathbb{Z}$ subgroup $\langle m(A,B), l(A,B)\rangle$. Now $m(A,B) = A$ and $l(A,B)$ are both diagonal matrices in all other representations we are considering. Hence if $m^i l^j$ has (without loss of generality) trace equal to 2 when λ is transcendental, this element must be the identity and hence the identity for any λ. But on putting say $\lambda = 2$ we find that this cannot hold unless $i = j = 0$.

$\square$

In particular the faithful representations are dense in all representations of the fundamental group of the figure 8 knot complement and there exist faithful embeddings in $SL(2, \mathbb{R})$ by taking λ to be a real transcendental bigger than 2, say.

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