Feedback Trading and Intermittent Market Turbulence

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Abstract

This paper studies the potential for complex asset return dynamics in a high-frequency, non-fundamental feedback trading model. Price adjustment is driven by the time-varying price impact of net order-flow. In tranquil times feedback trading has no impact on the price level. Given feedback trading intensities, as asset liquidity declines the market progressively becomes stressed and turbulent. Returns and absolute returns persistence are found to display power-law features, and episodes of turbulence are intermittent.

Keywords: Feedback trading, Price impact, Financial stability, Intermittence, Power law.

JEL classification: G12, G14

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1 Introduction

Market liquidity can evaporate in turbulent trading conditions, a phenomenon referred to as a "liquidity black hole" by Taleb (1997) and Persaud (2003). Sharp drops in liquidity associated with discontinuous rises in price impact create market stress and undermine investor confidence.[1] Negative sentiment can spread across financial markets, and what starts as an asset-specific crisis may end up having very costly systemic repercussions; see Borio (2000) and Cifuentes et al. (2004). In that regard, Morris and Shin (2004) show that net selling pressure can arise endogenously from the strategic interaction of long- and short-horizon traders. When the latter have privately-known liquidity constraints, these authors locate a unique trigger point at which a liquidity black hole emerges, and produce V-shaped price reversals consistent with Campbell, Grossman and Wang (1993) and Pastor and Stambaugh (2003). Understanding the onset of such catastrophic events is important for financial practitioners and policymakers alike.

In this paper I study the potential for high-frequency market stress and/or turbulence in the absence of information asymmetries or fundamental news. Specifically, market participants' asset demand is generated by two feedback rules at high frequency. First, according to *positive feedback*, investors buy (sell) the risky asset if its return over the last trading interval was positive (negative). Such trend-following amplifies price fluctuations in either direction and can become self-reinforcing in models with boundedly rational traders and short horizons; see Abreu and Brunnermeier (2003) and Brunnermeier and Pedersen (2005).

[1] Such headline episodes include the 1987 stock market crash, the collapse of the U.S. dollar against the yen on October 7, 1998, distressed trading in certain fixed income markets during the LTCM crisis in the summer of 1998, and the recent liquidity crisis driven by the U.S. sub-prime mortgage market in August 2007.

Second, following *risk feedback*, investors sell more (less) of the risky asset if its squared return in the last trading interval was high (low). Risk feedback trading always induces net selling pressure, and its absolute contribution to net orderflow rises (declines) during volatile (tranquil) periods. The information content of a larger absolute (square) return is that mean reversion to fundamentals-based price level—however defined—becomes more likely. In the presence of fundamental information, this may reflect Black's (1986) hypothesis that "the farther the price of a stock is from its [fundamental] value, the more aggressive the information traders become". In a continuous-time framework, Longstaff (2001) also shows that liquidity-constrained investors may optimally short-sell the asset if volatility is large enough.

In the absence of fundamental news, risk feedback is motivated by the observation that bad periods (negative returns) also tend to be more volatile. Risk feedback trading then amplifies net selling pressure and increases intraday returns persistence. Evidence to that effect has been found for the U.S. Treasury bond market by Cohen and Shin (2003). These authors report the presence of positive feedback in tick-by-tick trading on February 3, 2000, a particularly stressful day in the U.S. Treasury market: price declines (rises) elicit sales (purchases) at short horizons. Importantly, they also find positive feedback tends to be stronger when the market is under stress. To the extent that volatile market conditions tend to coincide with negative performance, the sales pressure on price will be growing in last period's volatility. The resulting conditional trading pattern can then be attributed to risk feedback.

Positive feedback trading is related to "momentum" strategies and herding behavior, while the sign asymmetry involved in risk feedback trading contributes to stabilize the market during an upswing—possibly extending the life of an asset price bubble—and to destabilize it during a downswing. Cohen and Shin (2003) argue that both short-term feedback types are more likely to arise in "pressurized" high-frequency environments where traders are subject to stop-loss constraints, such as those modeled by Morris and

Shin (2004). Both feedback rules are more relevant at high-frequency when there is no news about fundamentals. Indeed, Hwang and Salmon (2004) report that herding behavior is independent of fundamentals also at daily frequency.[2]

The paper's approach follows the microstructure literature investigating the presence of temporary but persistent imbalances between asset demand and supply at the transaction level. On the empirical side, Brandt and Kavajecz (2004) report that orderflow imbalances in the U.S. Treasury market can account for over 25 percent of the daily yield variation on days without major news, while Furfine and Remolona (2005) find that the market regains its composure quickly after experiencing discontinuous price changes.[3]

Feedback trading models with temporary orderflow disequilibrium start with Beja and Goldman (1980). These authors employ a linear pricing rule aggregating the demands of positive feedback traders (*chartists*, or directional traders) and information traders (*fundamentalists*, or value investors). The latter trade on subjective assessments of price in relation to value and their expectation formation matters for equilibrium behavior. A key finding then is that the equilibrium price can become unstable if the proportion of positive feedback traders is sufficiently large. More recently, Day and Huang (1990) introduce a market maker and find that price behavior can be chaotic if the price impact coefficient becomes too large. Using a nonlinear pricing rule, Chiarella (1992) shows that the price level reverts to a limit cycle following a shock away from equilibrium, and dynamic instability becomes more likely as feedback traders' horizons become shorter.[4]

[2]The asymmetry inherent in risk feedback is consistent with experimental evidence suggesting that economic agents are asymmetrically loss averse (Duxbury and Summers (2004)) and exhibit both herding and contrarian behavior in simulated trading conditions (Cipriani and Guarino (2005) and Drehmann et al. (2005)).

[3]On the price impact of order imbalances in other markets see Andersen et al. (2004), Chakravarty (2001) and Evans and Lyons (2002).

[4]Farmer and Joshi (2002) and Wieland and Westerhoff (2005) extend the Day and Huang framework with stochastic information and dynamic noise. Hommes (2006) and LeBaron (2006) survey agent-based models of feedback trading. For behavioral explana-

Liquidity is defined as *market depth*, which is inversely related to the price impact of trades (Kyle (1985)). High-frequency evidence suggests that market liquidity is time-varying, especially in stressful periods; see Dufour and Engle (2000), Farmer et al. (2005), and Goldreich et al. (2005). In view of this stylized fact, I assume that the price impact coefficient mapping net orderflow to prices follows a stationary AR(1) process characterized by a certain persistence and noise.

At the start of each trading period, a risk-neutral market maker receives the net orderflow and raises (lowers) the asset price if excess asset demand is positive (negative). The returns process generated by the combination of market participants' orderflow—assumed to be driven only by the most recent price fluctuation—and the market maker's linear price adjustment is shown to follow a logistic (quadratic) first-order difference equation. The logistic mapping has two fixed points, one of which is always zero while the other is generically nonzero.[5] Fixed point stability then becomes a function of price impact and feedback intensity. In this paper I consider the latter to be exogenously fixed and focus attention on time-varying liquidity.

The main results are as follows. First, as price impact grows and market liquidity declines, the dynamics progressively go through the following non-overlapping "states of the market": (i) stable, or *tranquil* markets, when the zero fixed point is stable and feedback trading does not affect asset returns; (ii) market *stress*, when the nonzero fixed point is unstable; and (iii) market *turbulence*, in which both fixed points are unstable. Beyond the turbulence range, the dynamics become chaotic and trading may be suspended. Hence, financial instability can be simply defined as the relative likelihood of market states (ii)-(iii), when feedback trading impacts upon asset returns, against the likelihood of tranquil state (i).

tions of herding behavior, including short-termism, see De Bondt and Thaler (1995).

[5]The logistic's potential for complex dynamics was realized by May (1973) and initially applied to evolutionary growth models, with positive feedback corresponding to the reproduction tendency and risk feedback to the inhibiting factor in a population.

Second, the magnitude of the price impact thresholds separating the three states is independent of risk feedback intensity and smoothly decreasing in positive feedback intensity. Therefore, the latter is potentially destabilizing. However, given any positive feedback, volatile market-making activity—in the sense of strong price impact persistence and noise—may always result in the return dynamics entering the stress state. A key implication, then, is that although low momentum is necessary for financial stability (state (i)), it is not sufficient because states (ii)-(iii) can always arise from extreme price impact realizations.

Third, the return dynamics are reversible: stochastic price impact may, in principle, cross a threshold from either direction and re-enter the relevant market state. Episodes of market stress and turbulence can thus be said to be *intermittent*, after Mandelbrot's (1974) original use of the term in fluid mechanics.[6] Intermittence amounts to "[...] the presence of irregular bursts in time series of a wide variety of volatility estimators" (Cont (2001)). I also find that as the market becomes stressed and turbulent, the distributions of returns and absolute return autocorrelations display power-law features ('Pareto tails'), and these become stronger as liquidity dries up.

The paper is related to the continuous-time framework of Bouchaud and Cont (1998). These authors posit behavioral-based trading rules—excess asset demand increases in positive feedback and decreases in returns volatility and loss aversion—and show that low and stable price impact can absorb big order imbalances with very small price changes. They also analyze the likelihood of a market crash as a function of the strength of feedback trading and liquidity conditions. I complement their approach by explicitly modeling time-varying price impact on a discrete time-scale, and by studying market dynamics in the absence of fundamental information other than that con-

[6] The potential unpredictability of a time series in the absence of shocks is central to the endogenous economic fluctuations literature; see Boldrin and Woodford (1990). Scheinkman and LeBaron (1989) presented evidence that U.S. stock returns may display chaotic behavior.

tained in prices themselves, i.e. assuming "pure feedback" akin to program trading at high-frequency.

In the remainder of the paper, Section 2 presents the feedback trading model, derives its steady states and classifies their stability; Section 3 obtains a correpondence between fixed point stability and the underlying state of the market; Section 4 illustrates the dynamic transition between states numerically, and estimates power-law exponents for returns and absolute return autocorrelations for different states; Section 6 discusses some implications of intermittence; and Section 7 concludes.

2 The model

2.1 Net order flow

Consider a single risky asset traded intraday, and denote its actual return at time t by $x_t = x(t, \Delta t) = \log P(t + \Delta t) - \log P(t)$, where $P(t)$ is the log price level at t. High-frequency applications often discretize the time between actual trades to $\Delta t = 5\,\text{min}$. For my purposes the frequency should be high enough so that fundamentals do not enter into consideration.

Asset demand is deterministically driven by the two short-term feedback trading rules discussed in Section 1. Under positive feedback, investors *buy* (sell) the asset at time t if they observe *positive* (negative) returns at $t-1$. Positive feedback generates orderflow $\omega_t^+ = \gamma^+ x_{t-1} \neq 0$, where $\gamma^+ > 0$ is a scalar mapping returns at $t-1$ to asset units demanded at t. The case $\gamma^+ < 0$, reflecting contrarian (buy low-sell high) strategies is excluded for analytical tractability.

Contrarian behavior is effectively captured by risk feedback trading: investors sell *more* (less) of the risky asset at t if they observe *higher* (lower) squared realized returns at $t-1$. Considering x_{t-1}^2 to be a volatility proxy, short-term contrarian traders employing some market equilibrium model will respond to higher x_{t-1}^2 and the associated positive expected returns for pe-

riod t by selling the asset, as they expect excess returns to mean-revert. This is related to Campbell and Hentschel's (1992) volatility-feedback hypothesis; after a bad and volatile period, such traders amplify the down movement while after a good (but less volatile) period they mitigate the up movement, so that "no news is good news".[7]

The orderflow component from risk feedback is $\omega_t^- = -\gamma^- x_{t-1}^2 < 0$, where scalar $\gamma^- > 0$ maps squared returns at $t-1$ to asset units sold at t. In principle, $\gamma^+ \neq \gamma^-$ and the intensity of each feedback type is expected to be time-varying. The *net orderflow* (signed trade volume) received by the market maker at time t then is

$$\begin{aligned} \omega_t &= \omega_t^+ + \omega_t^- \\ &= \gamma^+ x_{t-1} - \gamma^- x_{t-1}^2 \end{aligned} \tag{1}$$

Given γ^+ and γ^-, the sign of ω_t depends on the realized return and its square over the last trading interval. Negative return shocks at $t-1$ have a larger absolute impact on orderflow at t than symmetric positive shocks. As argued above, higher orderflow following a positive shock will be dampened as risk feedback cancels some of the associated momentum. This asymmetric contribution of risk feedback to asset demand is consistent with Longstaff's (2001) continuous-time asset pricing framework and the liquidity black hole model of Morris and Shin (2004), both assuming liquidity-constrained investors facing short-term trading limits.

2.2 Stochastic price impact

Assuming a risk-neutral market maker, so the price change depends only on the most recent orderflow and not on accumulated inventory, Farmer and Joshi (2002) derive a linear price adjustment mechanism as an approximation

[7] Campbell and Hentschel (1992) use a quadratic function of a GARCH(1,1) and find strong evidence of volatility feedback on stock market data; see also Campbell, Lo and MacKinlay (1997).

of a general price impact function. Upon receiving ω_t a single market maker adjusts prices from $t-1$ to t using the linear pricing rule

$$x_t = \lambda_t \, \omega_t = \lambda_t \gamma^+ x_{t-1} - \lambda_t \gamma^- x_{t-1}^2 \tag{2}$$

Equation (2) is consistent with order- rather than quote-driven markets. Coefficient $\lambda_t > 0$ measures the price impact of a unit change in net orderflow. Price impact is always positive and inversely related to market liquidity, and the sign of x_t changes with the aggregate volume imbalance. It is *positive* (negative) when there is net *buying* (selling) pressure. Further, following a negative return at $t-1$, net selling pressure has a greater absolute effect on returns at t than equal buying pressure because, given $\lambda_t > 0$, risk feedback reinforces the negative trend.

The price impact coefficient is assumed to follow an AR(1) process

$$\lambda_t = \overline{\lambda} + \theta \lambda_{t-1} + \eta_t \ , \quad \eta_t \sim iid(0, \sigma_\eta) \tag{3}$$

where the price impact innovations η_t are assumed to be iid. The constant term $\overline{\lambda} > 0$ is inversely related to average (intraday) market liquidity. Autoregressive representation (3) is similar to that used by Acharya and Pedersen (2005) for the time-varying cost of illiquidity.[8]

The dynamics of price impact are driven by $\overline{\lambda}$, θ, and the distribution of the innovation terms η. The properties of $\{\lambda_t\}$ are straightforward: it is covariance-stationary iff $\mid \theta \mid < 1$ and nonstationary otherwise; its unconditional mean and variance are $\overline{\lambda}/1-\theta$ and $\sigma_\eta^2/1-\theta^2$, respectively; its conditional mean and variance are $\overline{\lambda} + \theta\lambda_{t-1}$ and σ_η^2; and its persistence at lag k equals the relevant autocorrelation coefficient, $\rho(k) = \theta^k$.

[8] Esser and Mönch (2002) and Frey (2000) also model illiquidity as a mean-reverting state variable in continuous time. However, Bouchaud et al. (2004) and Upper and Werner (2002, 2007) find that the process of high-frequency price discovery displays breaks and is likely to be much more complex.

2.3 Fixed point existence and stability

Combining equations (1), (2) and (3) yields

$$x_t \;=\; h(x_{t-1}) \;=\; \lambda_t x_{t-1}\gamma^+\left[1-\frac{\gamma^-}{\gamma^+}x_{t-1}\right] \tag{4}$$

Equation (4) is a quadratic (*logistic*) first-order difference equation describing the nonlinear relationship between returns in adjacent time intervals. In principle, the resulting dynamics are independent of the time scale Δt used to discretize returns.

The implied covariance between successive return observations then is

$$E(x_t x_{t-1}) \;= \gamma^+ E(\lambda_t x_{t-1}^2) - \gamma^- E(\lambda_t x_{t-1}^3) \tag{5}$$

Equation (5) clearly suggests that skewness matters for returns persistence; ceteris paribus, short-term persistence grows in the asymmetry captured by risk feedback trading intensity γ^-. Section 4.3 discusses the implications of this property for high-frequency return autocorrelations.

The fixed points of logistic map h are defined as $x \equiv h(x_{t-1}) = x_t$. They are the equilibrium steady states of the underlying one-dimensional dynamical system; thus if asset returns equal x they will remain there unless perturbed. Specifically, the two fixed points are

$$x_t^{(1)} = 0 \quad , \; x_t^{(2)} = \frac{1}{\gamma^-}\left(\gamma^+ - \frac{1}{\lambda_t}\right) \neq 0 \tag{6}$$

Note that $x_t^{(1)}$ is always zero, hence I omit its time subscript, while $x_t^{(2)}$ is nonzero unless $\lambda_t = 1/\gamma^+$.[9] For given positive γ^+ and γ^-, fixed point $x_t^{(2)}$ is positive when $\lambda_t > 1/\gamma^+$ and negative if $\lambda_t < 1/\gamma^+$. In the first case, $x_t^{(2)} > 0$ increases in positive feedback intensity and price impact and decreases in risk feedback intensity. In the second case ($x_t^{(2)} < 0$), the nonzero fixed point is absolutely decreasing in γ^+, γ^- and in price impact.

[9] The two fixed points coincide when $\lambda_t = 1/\gamma^+$. If price impact is continuously distributed then $\lambda_t = 1/\gamma^+$ has zero measure, so $x_t^{(2)}$ is generically nonzero.

The evolution of asset returns turns on the two fixed points' relative stability. A fixed point is *stable* (unstable) if the absolute value of the slope of (4) at that fixed point is *smaller* (greater) than one. The slope of h with respect to x is $h'(x) = \lambda_t \gamma^+ (1 - \frac{2\gamma^-}{\gamma^+} x)$. The absolute value of $h'(\cdot)$ at $x^{(1)}$ and $x_t^{(2)}$ then is

$$\begin{aligned} &| \quad h'(x^{(1)}) \mid = \mid \lambda_t \gamma^+ \mid \\ &| \quad h'(x_t^{(2)}) \mid = \mid 2 - \lambda_t \gamma^+ \mid \end{aligned} \tag{7}$$

At most one fixed point can be stable at any time. Equations (7) then imply, first, that if $0 < \lambda_t < 1/\gamma^+$ then $x^{(1)} = 0$ is stable, while $x_t^{(2)} < 0$ is unstable. Second, if $1/\gamma^+ < \lambda_t < 3/\gamma^+$ then $x_t^{(2)} > 0$ is stable and $x^{(1)} = 0$ is unstable. In that case, the magnitude of the stable positive fixed point grows with price impact, as $\frac{\partial x_t^{(2)}}{\partial \lambda_t} = \frac{1}{\gamma^- \lambda_t^2} > 0$, and its sensitivity to λ_t declines with risk feedback intensity γ^-. Third, if $\lambda_t > 3/\gamma^+$ then $h'(\cdot)$ always exceeds one. Both fixed points are then unstable and the dynamics become turbulent. Finally, if $\lambda_t > 4/\gamma^+$, the logistic map can be shown to display sensitivity to initial conditions, there are periodic points of all orders and the dynamics become chaotic.

3 Intermittent financial (in)stability

3.1 Correspondence with market conditions

The above fixed point classification indicates that asset returs in each trading interval lie in one of three possible dynamic states, depending on the strength of positive feedback γ^+ and the determinants of price impact $\overline{\lambda}$, θ and σ_η. I introduce the following correspondence between fixed point stability and the "state of the market" at each point in time:

(i) *Tranquil* markets, corresponding to $0 < \lambda_t \leq 1/\gamma^+$. For such values, the nonzero fixed point is always negative and unstable. Consequently, the

presence of feedback trading does not displace asset returns from zero, their equilibrium level in the absence of fundamental news.

(ii) Market *stress*, corresponding to $1/\gamma^+ < \lambda_t \leq 3/\gamma^+$. The nonzero fixed point is now positive and stable, hence feedback trading impacts upon asset returns.

(iii) Market *turbulence*, where feedback trading generates unstable return dynamics ($3/\gamma^+ < \lambda_t < 4/\gamma^+$).

The price impact threshold separating tranquil markets from market stress is $1/\gamma^+$, while that between market stress and turbulence is $3/\gamma^+$. Note that each threshold is monotonically decreasing in the intensity of positive feedback but independent of risk feedback. Stronger momentum thus makes financial instability more likely, other things equal.

For price impact values beyond $\lambda_t = 4/\gamma^+$, the logistic map generates deterministic *chaos*; see Devaney (1989) and Holmgren (1997). Returns diverge and the corresponding state of the market has measure zero.

3.2 Simulation design

In this paper I restrict γ^+ and γ^- to 1 for illustration purposes; time-varying feedback intensities are discussed in Section 7. The subsequent simulations investigate the transition between states (i)-(iii) for Gaussian price impact innovations $\{\eta_t\}$.[10] The returns time series $\{x_t\}$ is initialized at $x_0 = \gamma^+/2\gamma^- = 0.5$, the logistic's unique critical point.[11] In turn, the price impact process is initialized at its unconditional mean: $\lambda_1 = \overline{\lambda}/1 - \theta$ where $\overline{\lambda} = 0.5$. Feeding the starting values x_0 and λ_0 to logistic map (4) yields x_1, the return in the next trading interval. The price impact coefficient for $t = 1$, λ_1 is then computed from equation (3), the values of x_1 and λ_1 are fed

[10] Using the uniform and other non-Gaussian shock pdf's did not affect the essence of the results; these are available upon request.

[11] The choice of x_0 is justified by Fatou's Theorem: if the trajectories of a quadratic polynomial have an attracting periodic point, then its critical point(s) are in the stable set of one of the points in each trajectory.

to equation (4) to determine x_2, etc. In this way, positive and risk feedback trading generate $t = 1, ..., T$ observations for $\{x_t\}$, each characterized by a different price impact realization $\{\lambda_t\}$.

I present $t = 1, ..., T = 500$ price impact and return realizations for states (i), (ii) and (iii). The choice of T reflects high-frequency trading in the U.S. Treasury market. Daily trading in U.S. Treasuries takes place between 7:00am and 5:00pm. Assuming, without loss of generality, that each minute in the 8-hour period contains a single trade yields 480 trading intervals.[12] The simulated asset returns can thus be viewed as snapshots of a trading day in a particular state of the market.

In order to assess the characteristics of the pdf's, the above 1-day process is run $n = 1, ..., N = 1,000$ times, yielding over 3 years of simulated high-frequency data. For each state, I report the simulated pdf's first four moments averaged over all N 1-day paths. Following standard practice in nonlinear dynamic simulations, the first 200 iterated returns of each path ($\{x_t\}_{t=1}^{200}$) are discarded to ensure convergence has occured. The simulated moments are then computed from the remaining ($\{x_{t=201}^{T=500}\}_{n=1}^{N}$) iterations.

4 Numerical simulations

4.1 Tranquil markets: $0 < \lambda_t \gamma^+ \leq 1$

When markets are tranquil ($0 < \lambda_t \leq 1$) feedback trading does not affect price dynamics. To illustrate, I set the persistence of price impact to $\theta = 0.2$, and assume that price impact innovations are iid distributed $\eta_t \sim N(0, 0.05)$. Price impact is initialized at its unconditional mean, $\lambda_1 = \overline{\lambda}/1 - \theta = 0.625$. Fig. 1 below shows the simulated $\{\lambda_t\}$ and $\{x_t\}$ series

FIGURE 1 HERE

[12] Fleming (2001) describes the microstructure of the U.S. Treasury market.

All price impact realizations are less than one, so the dynamics is in the stable range of $x^{(1)} = 0$.

The simulated returns' risk-adjusted mean (sample mean/standard deviation), skewness and kurtosis averaged over all $N = 1,000$ distributions, i.e. $\{x_{t=201}^{T=500}\}_{n=1}^{N}$, are reported in Table 1

Table 1: Tranquil markets

Simulated return pdf's

μ/s	0.000
Skewness	8.82
Kurtosis	87

Risk-adjusted expected returns are zero up to 8 decimal places. The simulated returns pdf is very skewed and fat-tailed. Excess kurtosis is broadly consistent with the high-frequency evidence: for example, Cont (2001) reports values of 15.95 for S&P 500, 74 for US$/DM, and 60 for returns on USD/SF exchange rate futures.

Next, I use bifurcation diagrams to assess the evolution of fixed point stability in the price impact domain. Initializing $\lambda_1 = 0.625$, $i = 1, ..., 400$ price impact realizations $\{\lambda_i\}$ are drawn from equation (3). For each λ_i, an asset return trajectory is initialized at $x_{i0} = 0.5$ as above. Each trajectory then consists of $t = 1, ..., 300$ iterations of logistic map (4): $(\lambda_i, h^i(0.5)_{t=1}^{300})$. The resulting behavior of return trajectories in tranquil market conditions is shown in the bifurcation diagram below

FIGURE 2 HERE

The key feature of Fig. 2 is that, at each choice of λ_i $(i = 1, ...100)$, the feedback return trajectories x_{it} $(t = 1, ..., 300)$ become dense towards zero. Price impact ranges from 0.45 to around 0.75, hence h is firmly in the stable range of $x^{(1)} = 0$ $(x_t^{(2)} \neq 0)$. Note that some simulated trajectories correspond

to positive returns. However, they are unstable because feedback trading—continued iteration of $\{x_{it}\}$ given i—attracts them back to zero. Also note that as price impact increases (liquidity declines), the unstable trajectories slope upwards and the dynamics edge towards the stress threshold.

4.2 Market stress: $1 < \lambda_t \gamma^+ \leq 3$

In the range $1 < \lambda_t \leq 3$, the zero fixed point $x^{(1)}$ becomes unstable (repelling) and $x_t^{(2)} > 0$ is stable (attracting). Asset returns generated by feedback trading then induce market stress: state (ii) in Section 3.1. Clearly, in order for some price impact realizations to break into the stress range, $\{\lambda_t\}$ has to become more persistent and/or noisy. To illustrate such dynamics, I increase the AR(1) coefficient to $\theta = 0.5$ and double the standard deviation of liquidity shocks, so that $\eta_t \sim N(0, 0.10)$. The price impact coefficient is now initialized at $\lambda_1 = \overline{\lambda}/1 - \theta = 1$. The simulated $\{\lambda_t\}$ and $\{x_t\}$ series are shown in Fig. 3

FIGURE 3 HERE

The market stress threshold is indicated with a horizontal line at $\lambda = 1$ in the bottom panel. Price impact is less than the threshold 45.2 percent of time (300 trading periods averaged over $1,000$ 1-day paths), and above it the remaining 54.8 percent. In the top panel, positive fixed point $x_t^{(2)}$ attracts neighboring points towards it and away from $x^{(1)} = 0$. Asset returns are still close to zero (note the 10^{-8} scale on the vertical axis) but much more volatile than in state (i).

The risk-adjusted mean, skewness and kurtosis averaged over $\{x_{t=201}^{T=500}\}_{n=1}^{N}$ simulated return distributions are reported in Table 2

Table 2: Market stress

Simulated return pdf's

μ/s	0.0011
Skewness	1.52
Kurtosis	5.21

Risk-adjusted expected returns are now positive, reflecting the fact that $x_t^{(2)} > 0$ when $\lambda_t \in (1, 3]$. There is much less right-skewness and leptokurtosis than in the tranquil state.

Beyond $\lambda = 1$, $x_t^{(2)}$ switches from being unstable to stable. This discontinuous dynamic transition is shown in the bifurcation diagram below

FIGURE 4 HERE

The price impact coefficient now ranges from 0.75 to 1.30. As in tranquil state (i), the return trajectories become more dense towards zero at each λ_i realization. Unlike state (i), however, as $\{\lambda_i\}$ exceeds 1 the return trajectories slope upwards because steady-state returns have turned positive.

4.3 Market turbulence: $3 < \lambda_t \gamma^+ < 4$

Asset returns enter state (iii), market turbulence, if both fixed points become unstable, which occurs when $\mid h'(\cdot) \mid > 1$. In turn, that requires price impact to be in the range $\lambda_t > 3$**.** To illustrate, I maintain $\overline{\lambda} = 0.5$ and $\eta_t \sim N(0, 0.10)$ but raise the AR(1) coefficient to $\theta = 0.85$, so $\lambda_1 = \overline{\lambda}/1-\theta = 3.33$. The simulated $\{\lambda_t\}$ and $\{x_t\}$ series are shown in Fig. 5

FIGURE 5 HERE

Only 0.4 percent of price impact realizations now falls in the tranquil state $0 < \lambda_t \leq 1$ (bottom panel). 6.8 percent is in the stress state $1 < \lambda_t \leq 3$, and the remaining 92.8 percent is in the turbulent range $3 < \lambda_t < 4$. There

are no outcomes in the chaotic range $\lambda_t \geq 4$, and there is very pronounced volatility clustering (top panel).

Table 3 reports the risk-adjusted mean, skewness and kurtosis averaged over all $\{x_{t=201}^{T=500}\}_{n=1}^{N}$ return distributions

Table 3: Market Turbulence

Simulated return pdf's

μ/s	0.0064
Skewness	-0.258
Kurtosis	1.837

Risk-adjusted returns are now very high; moreover, the simulated x_t distributions have become left-skewed and there is negative excess kurtosis. Left-skewness is particularly interesting because it relates to short-horizon returns persistence. From equation (5) recall that, given γ^-, the first-order (adjacent ticks) autocorrelation decreases in the skewness of returns. Therefore, the switch of skewness from positive to negative along the transition from tranquil markets to turbulence (via market stress) suggests that returns persistence progressively increases as market liquidity "dries up". This property appears consistent with Cohen and Shin's (2003) evidence for the U.S. Treasury market.

The bifurcation diagram for state (iii) is in Fig. 6

FIGURE 6 HERE

Note that the dynamics begin to change very rapidly beyond the turbulence threshold. In the range $3 < \lambda_t \leq 3.45$ there are period-doubling bifurcations. From $\lambda \simeq 3.33$, the return trajectories display a period-doubling bifurcation "cascade" familiar from chaotic dynamics. Specifically, every period-2 attracting orbit divides into an attracting period-4 trajectory and a repelling period-2 trajectory. Another period-doubling bifurcation occurs

when λ_i is near 3.5: period-4 trajectories then split into period-8 and period-16 trajectories, and so on. For $\lambda > 3.6$ it can be shown that there are periodic points of all orders if a period-3 periodic point exists. Finally, beyond $\lambda_i = 4$, the iterated return orbits can be shown to be period-k for all $k > 0$ and deterministic chaos emerges.

5 Power-law properties of returns

This Section investigates whether the asset returns generated by these dynamics are power-law distributed. I adopt the following nonparametric estimator of α, the power-law distribution's exponent for an empirical distribution consisting of $\{x_{t=1}^{T}\}_{n=1}^{N=1,000}$ return vectors, where each vector corresponds to a day consisting of $T = 500$ trading periods (ticks)[13]

$$\widehat{\alpha}_n = 1 + T \left[\sum_{t=1}^{T} \log \left(\frac{x_{tn}}{x_{\min}^{n}} \right) \right]^{-1} \tag{8}$$

where $x_{\min}^{n}$ is the minimum obervation in returns vector n that could be consistent with a power-law distribution. The power-law exponent estimator in (8) is always greater than 1, implying that all moments $m \geq \widehat{\alpha}_n + 1$ will diverge. Note that the summation terms are well-defined in the market stress and turbulence states, where the negative fixed point is unstable, but not in the tranquil state where returns can be negative. This α-estimator has the advantage of being less biased than regression methods, also in small samples, and is equivalent to the Hill estimator; see Hall (1982) and Newman (2005). However, it need not give the right estimate if the true dgp is not Pareto-type.

I use the same θ and σ values for each dynamic state as in Section 4, motivated by the U.S. Treasury market's microstructure. Averaging $\widehat{\alpha}_n$ over $n = 1, ..., N = 1,000$ 1-day paths for each parameter combination, the mean

[13] The power-law exponent is α, where the Pareto-tail probability of returns x is proportional to $P_\alpha(x) \sim \frac{A^\alpha}{|x|^{1+\alpha}}$ for $x \to \pm\infty$.

value of the power-law exponent is $\widehat{\alpha} = \left(\sum_{n=1}^{N} \widehat{\alpha}_n\right)/N$. The $\widehat{\alpha}$ values for each state are reported in Table 4

Table 4: Power-law exponents of returns[14]

Market state (θ, σ)	$\widehat{\alpha}$
Tranquil	1.01
Stress	1.43
Turbulence	2.54

The estimates suggest that returns are power-law distributed in all 3 states. Indeed, the exponent is closest to one in the tranquil and stress states, suggesting that all the corresponding moments diverge. However, this finding should be treated with caution because the minimum return realizations $x^n_{\min}$ are arbitrarily close to zero in each of the N paths, and particularly in state (i). This tends to lower all $\widehat{\alpha}_n$ estimates and their average.[15] By contrast, the 2.54 exponent estimate for state (iii) is close to the lower end of the 2–5 range reported by Bouchaud and Cont (2001), indicating that skewness, kurtosis and higher moments will likely diverge during turbulence.

I now turn to assess the persistence properties of returns. The autocorrelation coefficients of absolute return powers at lag τ are defined as $\rho(t,\tau) = corr[\mid x_t \mid^{\delta}, \mid x_{t-\tau} \mid^{\delta}]$. As there is sufficient consensus that autocorrelations decay slowest when $\delta = 1$, i.e. absolute returns, I report on that case only; see Bouchaud and Cont (2001). The absolute return autocorrelograms for $\tau = 1, ..., 100$ lags (ticks corresponding to 5 min) during market stress and turbulence are respectively shown in Figs. 7 and 8[16]

FIGURES 7 AND 8 HERE

[14] The (θ, σ) parameter values for each state are as before. The empirical histograms of $\widehat{\alpha}_n$ are available upon request.

[15] Another reason why α estimates in state (i) are unreliable is that excluding negative (tranquil) return observations reduces the effective sample size.

[16] For consistency with Fig. 3 above, Fig. 7 shows three autocorrelation paths. .

In both cases, the decay patterns of $\rho(t, \tau)$ clearly suggest the presence of long memory. During stress (Fig. 7) this appears to be declining smoothly and is more pronounced than the corresponding squared-returns autocorrelogram in Bouchaud and Cont (2001) for the same maximum lag number. During market turbulence (Fig. 8), persistence is even stronger but unexploitable because of the rapidly alternating signs.

The negative autocorrelations present a computational problem, in that the resulting power-law exponents estimated from equation (8) are complex-valued. I circumvent this by using the absolute value of the autocorrelation coefficient at all lags in the estimation; as persistence is symmetrical, this transformation does not affect its properties. Fig. 9 below reports the empirical histograms of α-exponents in states (ii) and (iii), respectively computed from $N = 1,000$ daily return paths

FIGURE 9 HERE

The wide range of α-exponents of absolute-return autocorrelation reflects the rapid oscillation of returns, particularly during turbulence. For example, the exponent for the particular autocorrelation pattern shown in Fig. 8 was 1.80. The first four moments of the power-law exponents included in the empirical histograms of Fig. 9 are reported below

Table 5: Absolute return autocorrelations

Power-law exponent descriptive statistics

Market state	Stress	Turbulence
mean	3.17	3.04
s.d.	2.39	2.88
skewness	2.26	3.44
kurtosis	10.85	18.14

Table 5 indicates that the mean α-exponent of absolute return autocorrelations is about 3 in both states, indicating that only kurtosis is undefined.

Both distributions are significantly right-skewed and fat-tailed, and more so for market turbulence. The tentative conclusion from the simulation evidence is that the persistence of absolute returns displays power-law features.

6 Discussion

The numerical simulations in Section 4 suggest a positive relationship between the values of θ and σ and the likelihood of a dynamic transition from tranquil to turbulent markets. Smaller price impact (greater liquidity) improves the zero fixed point's stability, while bigger price impact tends to destabilize it, all else equal. In other words, stable market-making is critical for maintaining financial stability. Given any intensity of feedback trading, liquidity should be sufficiently high and stable at each point in time—in the sense of weak persistence and small variance of price impact—to guarantee the tranquil market state.

However, the transition between two dynamic states can go in both directions. Turbulence is thus reversible, or *intermittent* in the sense of Mandelbrot (1974), related also to Giardina and Bouchaud (2003) and Heagy et al. (1994). To quote Shin (2004, p.150), "occasionally, financial markets experience episodes of turbulence of such an extreme kind that they appear to stop functioning". Accordingly, one interpretation of the high-frequency evidence offered by Morris and Shin (2004) is that active markets can undergo mini-liquidity gaps several times in a day. Intermittent turbulence is also consistent with Plantin et al. (2004), who find less liquid asset markets are more vulnerable to episodes of turbulence. In the same vein, Danielsson and Payne (2001) find intermittent "liquidity gaps" in electronic limit order book-based FX trading.

The framework also serves to illustrate the asymmetric effects of net orderflow on prices. On one hand, excess buying pressure through positive feedback is always mitigated by risk feedback. On the other hand, risk feed-

back acts to amplify excess selling pressure. This asymmetry is in line with evidence of relatively long periods of asset price rises—Alan Greenspan's *irrational exuberance*—followed by very steep price declines; Shiller (2005) considers such patterns to be asset price bubbles. The simplifying assumption that positive and risk feedback operate only on the last period's returns seems consistent with accounts of *short-termism* in financial markets. Trichet (2005) outlines alternative rationales for market participants' preoccupation with short-term results; these range from marking-to-market and benchmarking of performance against competitors, to the spread of index management and VaR considerations.

Finally, note that purely deterministic feedback trading cannot generate a collapsing asset price, because $x_t^{(2)} < 0$ is unstable for all $0 < \lambda_t \gamma^+ < 1$. Thus, the dynamics generated by logistic map (4) can only sustain zero or positive asset returns. Allowing for the possibility of falling price levels and negative returns requires introducing a noise term $\xi_t \sim iid(0, \sigma_\xi)$ affecting returns in period t. Such noise can be thought of as a stationary pricing error measuring the quality of the market microstructure; see Hasbrouck (1993). A noisy version of equation (4) would be

$$x_{t+1} = \lambda_t x_{t-1} \gamma^+ \left[1 - \frac{\gamma^-}{\gamma^+} x_{t-1} \right] + \xi_t \tag{9}$$

Simulating returns using stochastic first-order difference equation (9) is necessary for asset price bubbles to eventually burst. A straightforward numerical application of the noisy logistic map thus involves generating bubble paths for different parameterizations of λ_t (θ and σ_η), γ^+, γ^- and σ_ξ.

7 Concluding remarks

In this paper I investigated the potential for financial (in)stability and turbulence arising from high-frequency feedback trading in the absence of fundamental information. A logistic interaction of feedback portfolio strategies

generated stable (tranquil), unstable (stressful) or turbulent return dynamics depending on stochastic market liquidity and fixed feedback trading intensity. The dynamics were driven by price adjustment to short-term orderflow imbalances according to the combined strength of positive feedback trading—reflecting momentum strategies—and risk feedback trading, driven by liquidity-constrained investors' volatility-dependent strategies. I obtained a well-defined correspondence between the range of stochastic price impact and the likelihood of each dynamic state. Importantly, price impact can cross the thresholds separating the dynamic states in both directions, so the occurence of market turbulence is intermittent in Mandelbrot's sense.

The numerical calibration was motivated by high-frequency trading in the U.S. Treasury market and its microstructure. Empirically, the contrarian-based (volatility feedback) interpretation of risk feedback in Section 2 may offer a useful avenue for estimating feedback intensities by fitting difference equation (2) to lower-frequency data. However, for such applications feedback trading would have to be integrated within a market-clearing model. Analytically, the dynamics was kept tractable by fixing the intensities of both trading rules and limiting feedback to the last trading interval. In that regard, two important extensions involve allowing stochastic price impact to depend on the market's realized volatility, and introducing time-varying feedback intensities in response to the fluctuating heterogeneity of market opinion. Overall, the framework serves to underline the key role of stochastic price impact in driving high-frequency market dynamics, as well as influencing the long-memory properties of returns and their persistence.

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FIGURE 1
RETURN DYNAMICS: TRANQUIL MARKETS (theta=0.2, sigma=0.05)

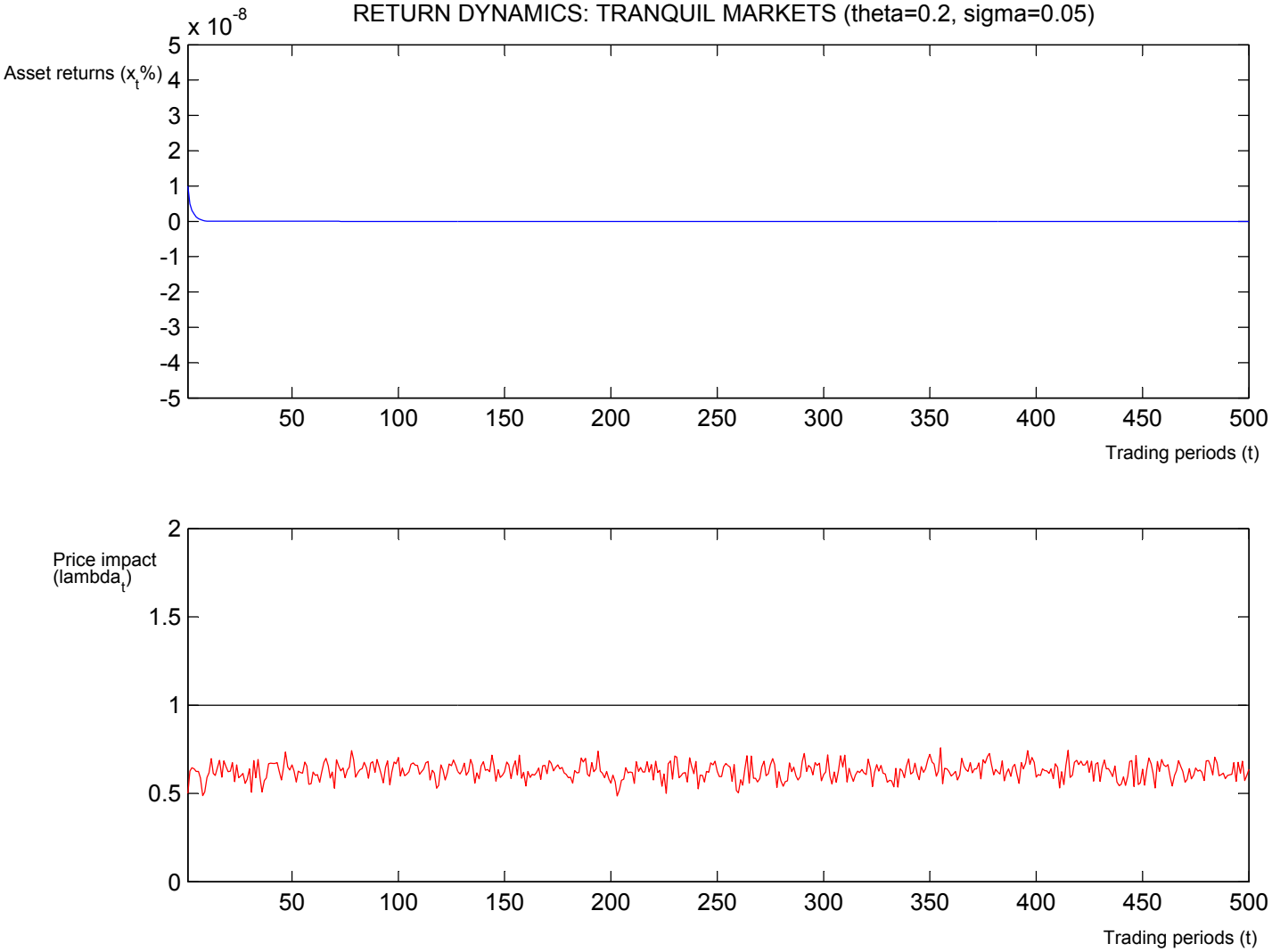
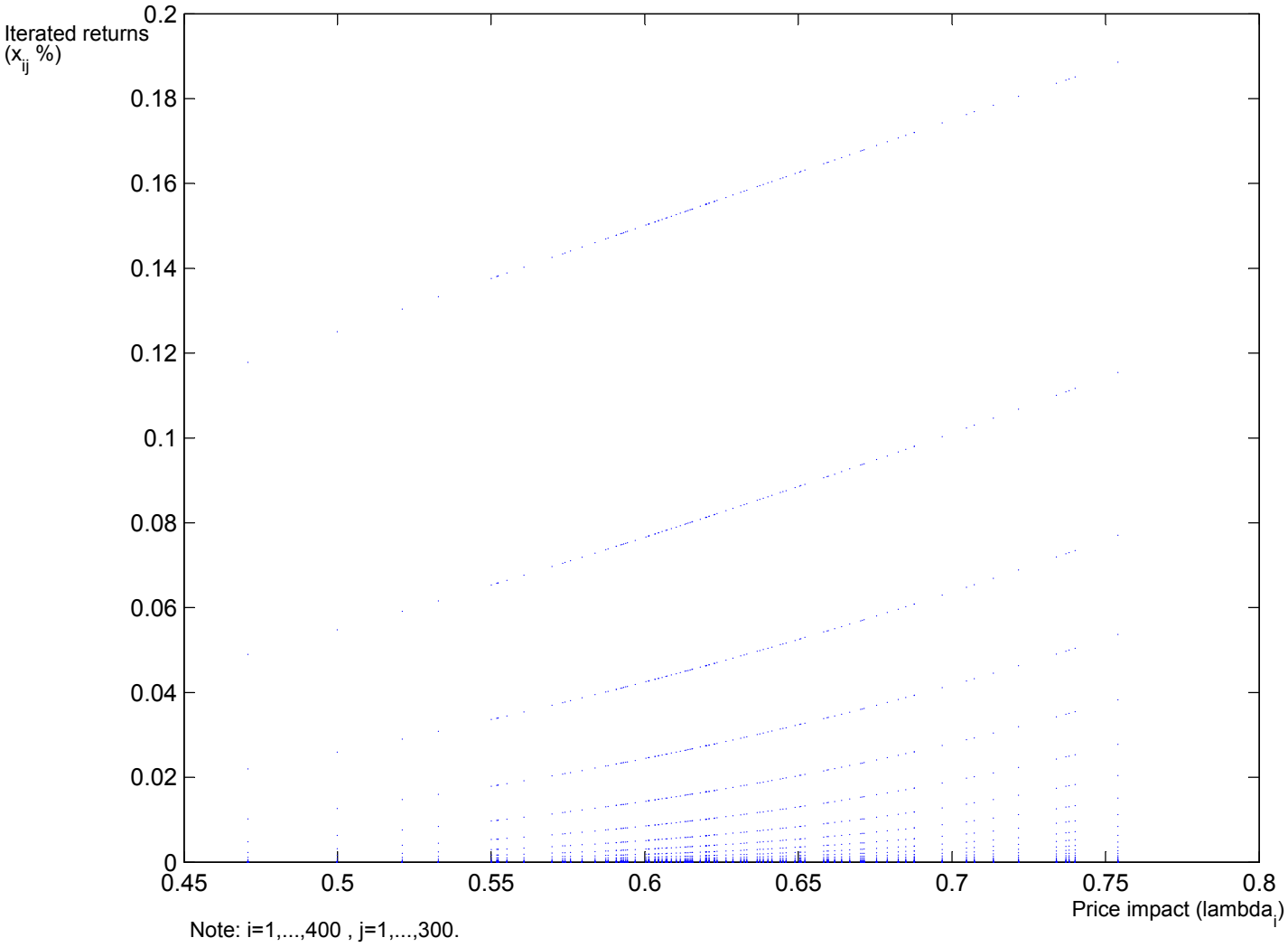


FIGURE 2

BIFURCATION DIAGRAM: TRANQUIL MARKETS (theta=0.2, sigma=0.05)



Note: i=1,...,400 , j=1,...,300.

FIGURE 3. RETURN DYNAMICS: MARKET STRESS (theta=0.5, sigma=0.10)

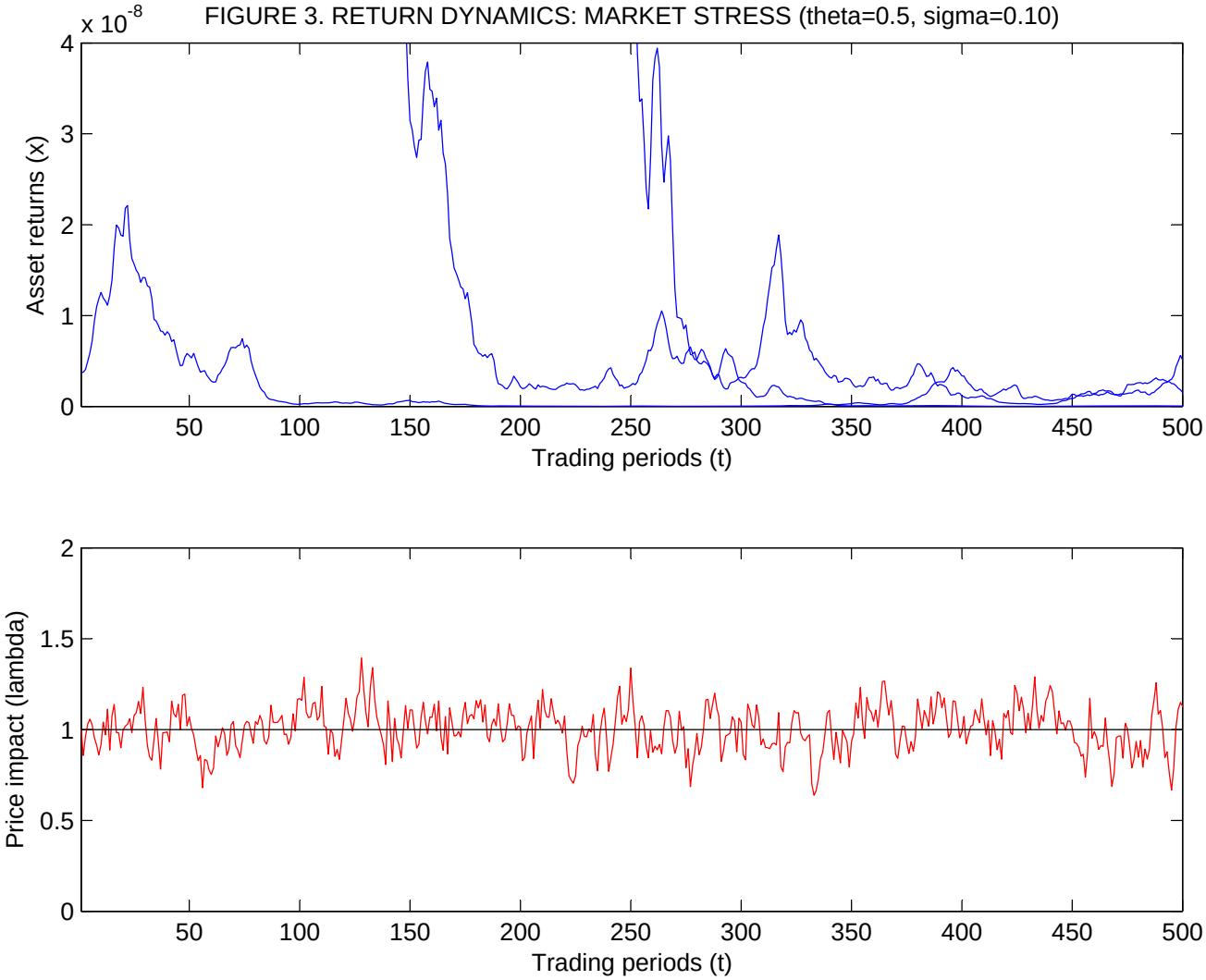
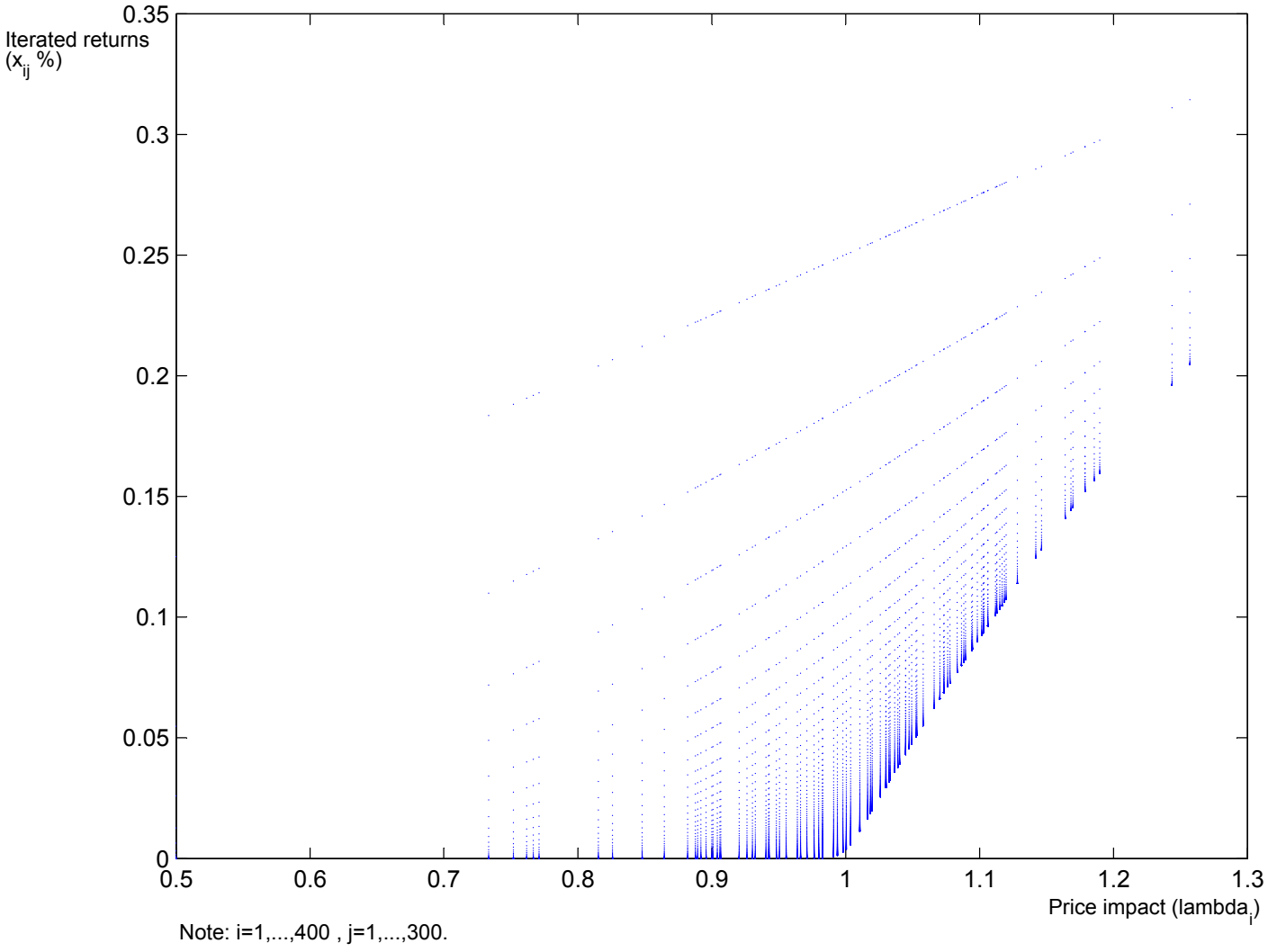


FIGURE 4

BIFURCATION DIAGRAM: MARKET STRESS (theta=0.5, sigma=0.10)



Note: i=1,...,400 , j=1,...,300.

FIGURE 5

RETURN DYNAMICS: MARKET TURBULENCE (theta=0.85, sigma=0.10)

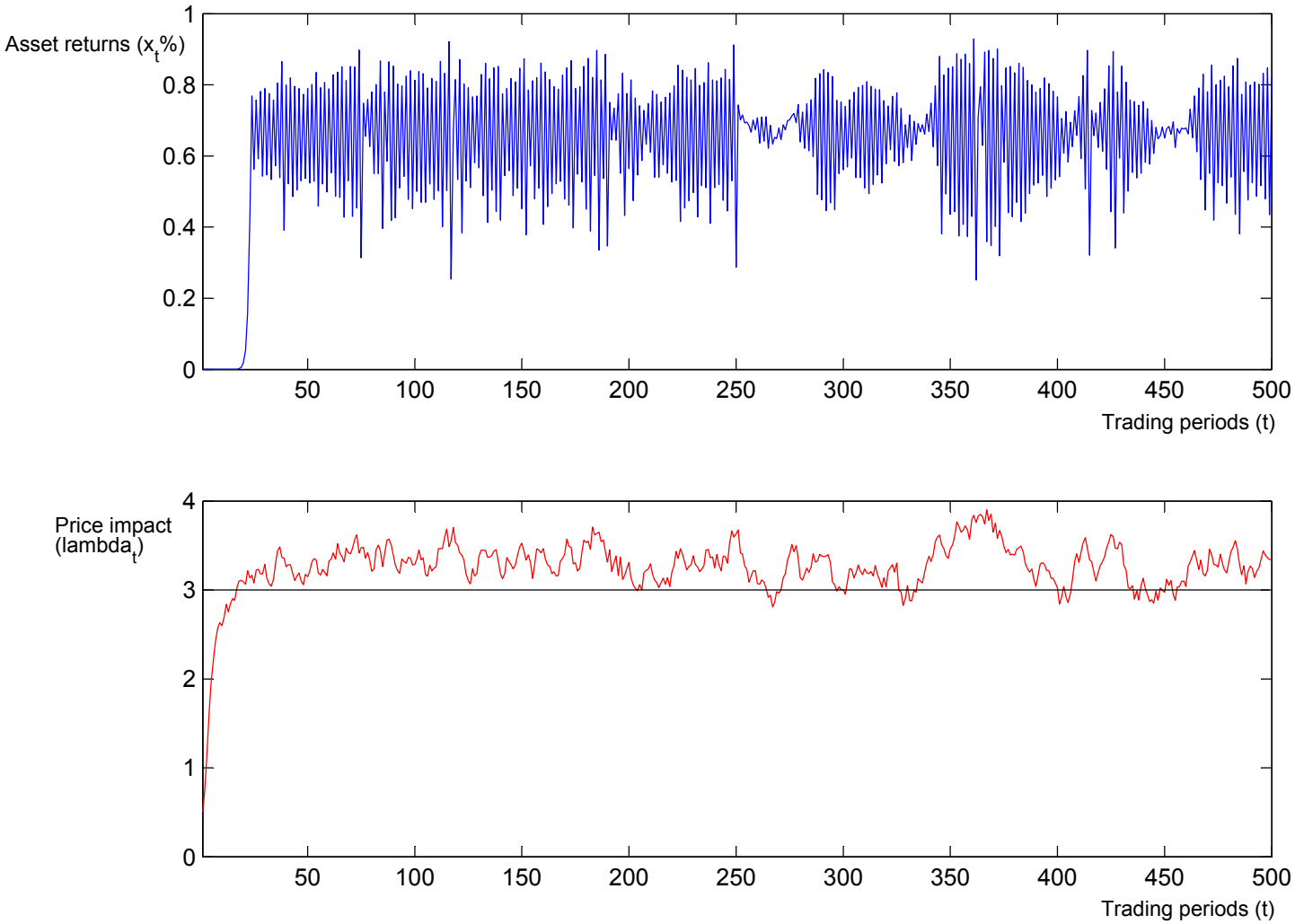
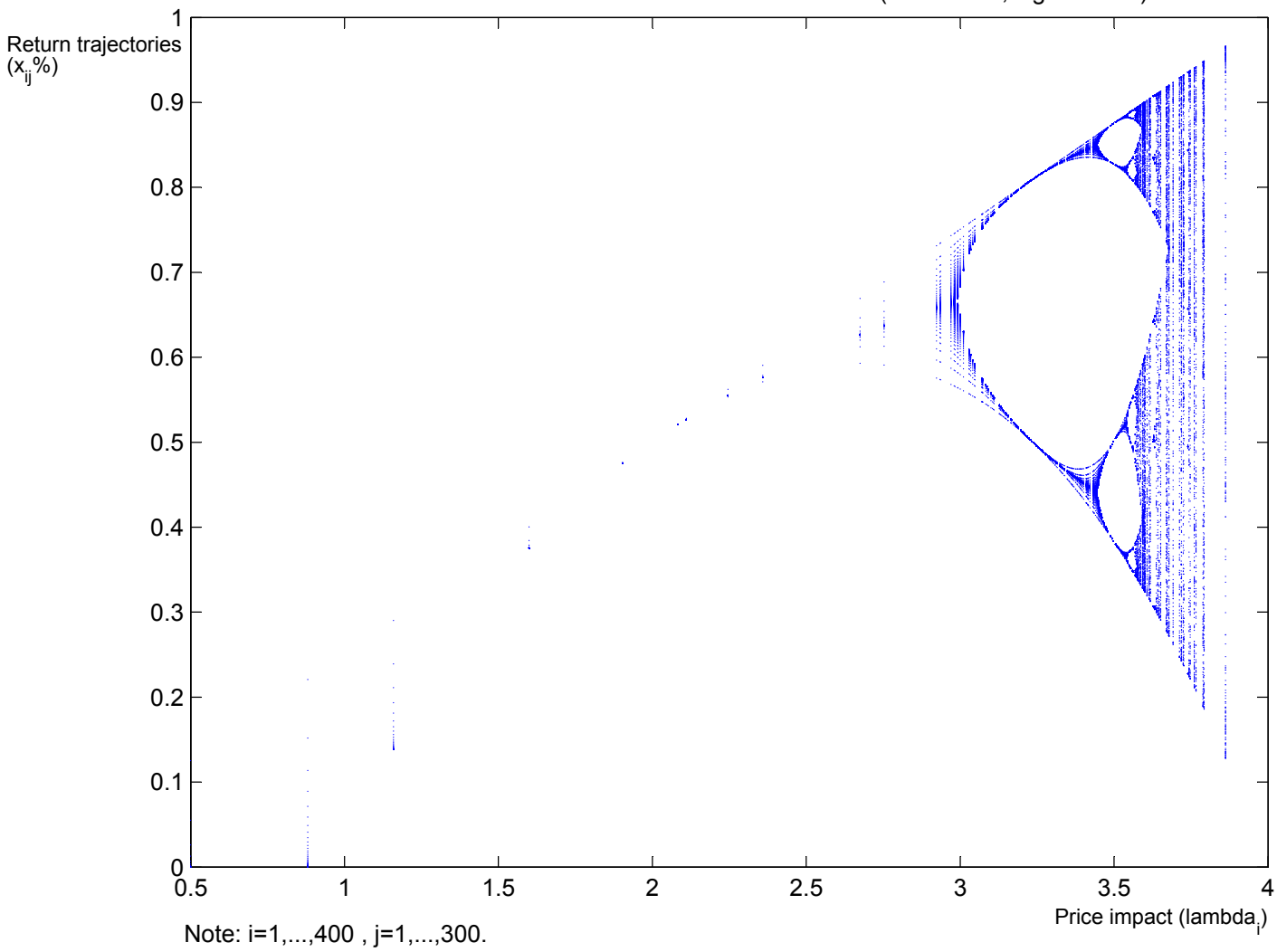


FIGURE 6

BIFURCATION DIAGRAM: MARKET TURBULENCE (theta=0.85, sigma=0.10)



Note: i=1,...,400 , j=1,...,300.

FIGURE 7. ABSOLUTE RETURN AUTOCORRELATIONS: MARKET STRESS (theta=0.5, sigma=0.10)

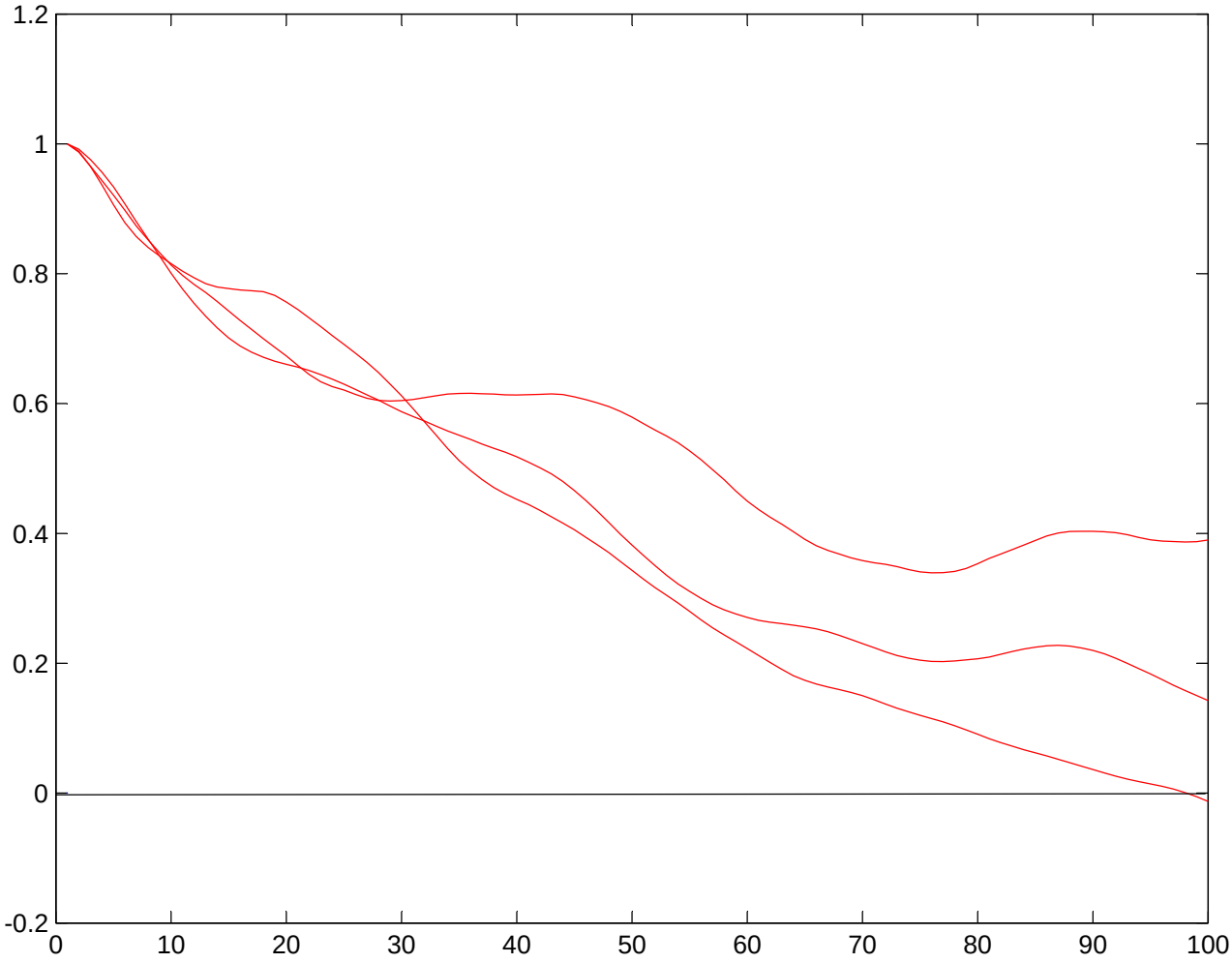


FIGURE 8. ABSOLUTE RETURN AUTOCORRELATIONS: MARKET TURBULENCE (theta=0.85, sigma=0.10)

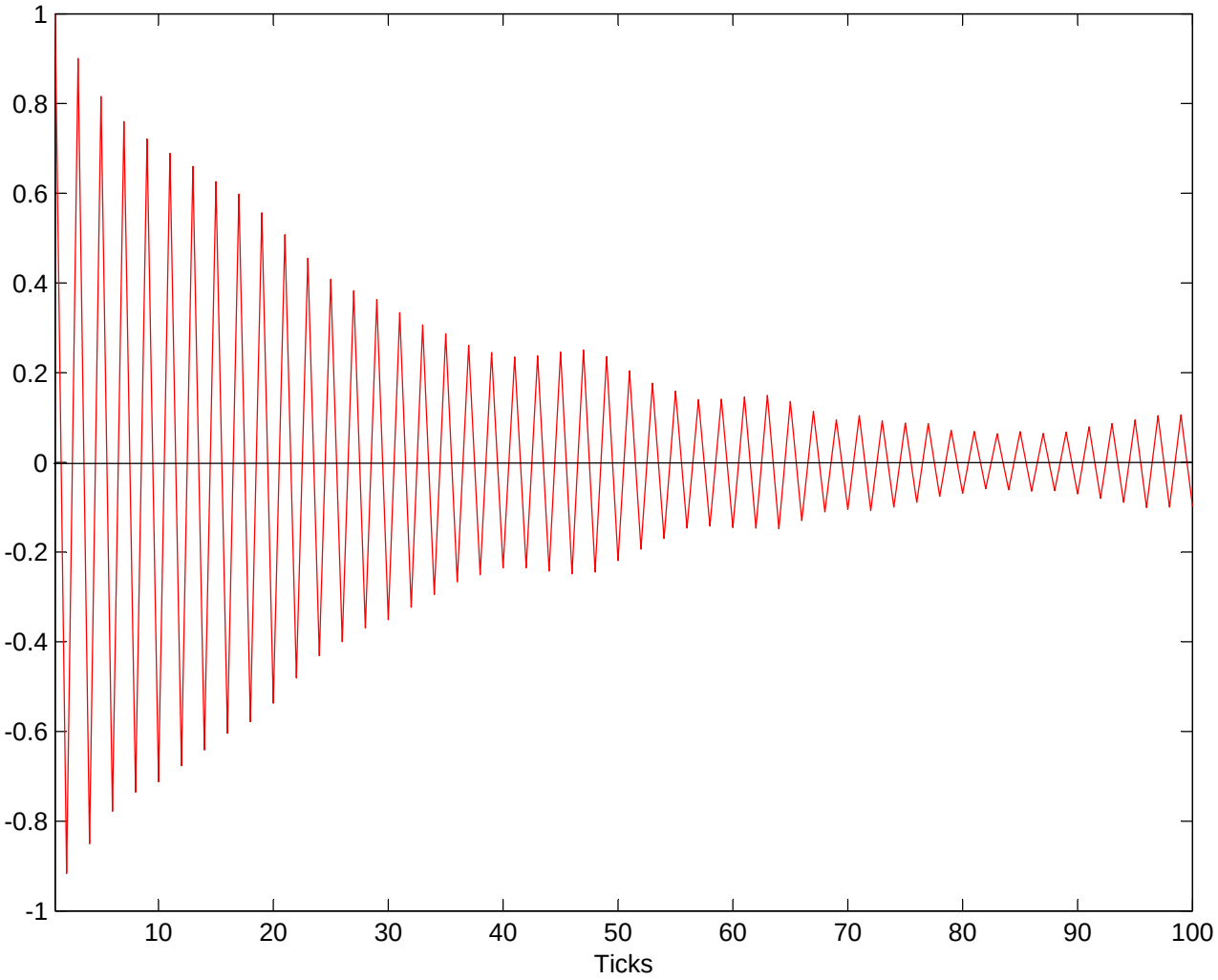
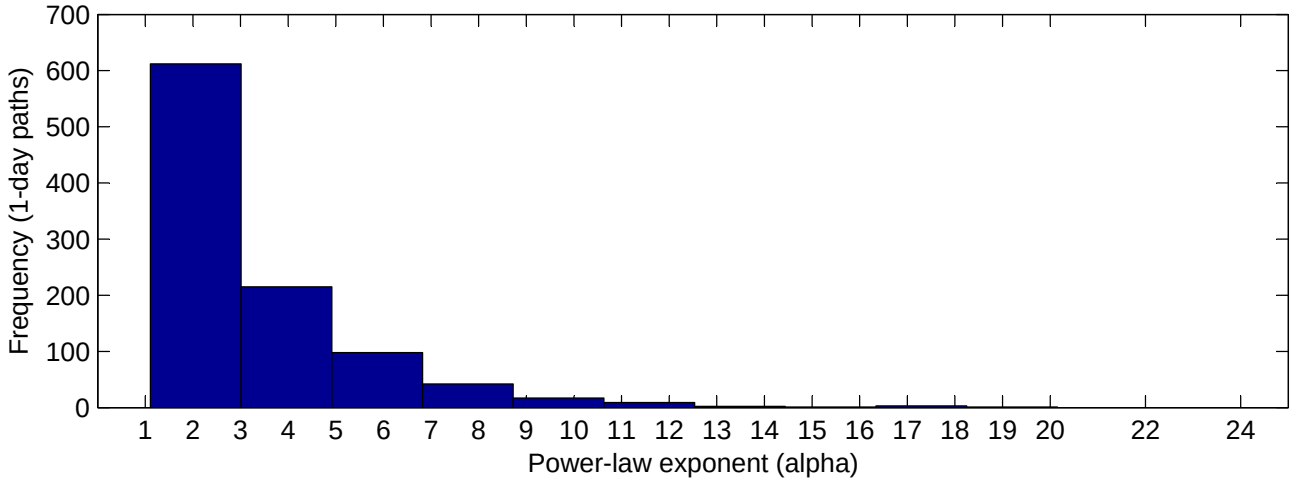


FIGURE 9. EMPIRICAL HISTOGRAMS OF ABSOLUTE RETURNS AUTOCORRELATION FUNCTION EXPONENT: STRESS



ABSOLUTE RETURNS AUTOCORRELATION FUNCTION EXPONENT: TURBULENCE