Random conformally covariant metrics in the plane

Liam Hughes

Department of Pure Mathematics and Mathematical Statistics

University of Cambridge

St Edmund's College

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Declaration

This thesis is based on research conducted while a graduate student in the Department of Pure Mathematics and Mathematical Statistics at the University of Cambridge between October 2019 and September 2023, under the supervision of Jason Miller and supported by the University of Cambridge Harding Distinguished Postgraduate Scholars Programme. This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted before, or is concurrently being submitted, for any degree or other qualification.

Summary

This thesis is in the broad area of random conformal geometry, combining tools from probability and complex analysis.

We mainly consider *Liouville quantum gravity* (LQG), a model introduced in the physics literature in the 1980s by Polyakov in order to provide a canonical example of a random surface with conformal symmetries and formally given by the Riemannian metric tensor "$e^{\gamma h}(dx^2 + dy^2)$" where h is a Gaussian free field (GFF) on a planar domain and $\gamma \in (0, 2)$. Duplantier and Sheffield constructed the γ-LQG area and boundary length measures, which fall under the framework of Kahane's Gaussian multiplicative chaos. Later, a conformally covariant distance metric associated to γ-LQG was constructed for whole-plane and zero-boundary GFFs.

In this thesis we describe the γ-LQG metric corresponding to a free-boundary GFF and derive basic properties and estimates for the boundary behaviour of the metric using GFF techniques. We use these to show that when one uses a conformal welding to glue together boundary segments of two appropriate independent LQG surfaces to get another LQG surface decorated by a *Schramm–Loewner evolution* (SLE) curve, the LQG metric on the resulting surface can be obtained as a natural metric space quotient of those on the two original surfaces. This generalizes results of Gwynne and Miller in the special case $\gamma = \sqrt{8/3}$ (for which the LQG metric can be explicitly described in terms of Brownian motion) to the entire subcritical range $\gamma \in (0, 2)$. Moreover, we show that LQG metrics are infinite-dimensional (in the sense of Assouad) and thus that their embeddings into the plane cannot be quasisymmetric.

We also consider chemical distance metrics associated to *conformal loop ensembles*, the loop version of SLE, using the imaginary geometry coupling to the GFF to bound the exponent governing the conformal symmetries of such a metric.

Preface

Chapter 1 introduces the topics covered in this thesis and describes the main results of Chapters 3–5.

Chapter 2 gives preliminary results required for the proofs in Chapters 3–5.

Chapter 3 is based on [HM22], a work in collaboration with Jason Miller (University of Cambridge) that has been submitted for publication.

Chapter 4 is my own work.

Chapter 5 is based on joint work in progress with Valeria Ambrosio and Jason Miller (both University of Cambridge).

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Contents

Chapter 1

Introduction

1.1 Liouville quantum gravity

1.1.1 Quantum surfaces

A γ-*Liouville quantum gravity* (LQG) surface is a random surface parametrized by a domain $D \subseteq \mathbb{C}$ given, in a formal sense, by the random metric tensor

$$e^{\gamma h(z)} (dx^2 + dy^2) \tag{1.1.1}$$

where γ is a parameter in $(0, 2)$, h is some form of the Gaussian free field (GFF) on the domain D and $dx^2 + dy^2$ is the Euclidean metric. These surfaces have been shown to arise as scaling limits of several random planar map models ([She16b, KMSW19, LSW17, GM21b, GM21d, GKMW18]; see also [GHS23] and the references therein).

Since h is not sufficiently regular to be a random function on D (it is only a distribution on D, in the sense of Schwartz), the expression (1.1.1) for the LQG metric tensor does not make literal sense; in order to rigorously define an LQG surface one must take a limit of regularized versions of $e^{\gamma h(z)} dz$. This was done for the volume form in [DS11], resulting in the γ-*LQG measure* μ_h, a random measure on D, and the γ-*LQG boundary length* ν_h, a random measure on ∂D, each of which fall under the general framework of Kahane's *Gaussian multiplicative chaos*, as introduced in [Kah85]. These measures are conformally covariant in the following sense: given a conformal map $\psi : \widetilde{D} \to D$, if we set

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}, \quad \widetilde{h} = h \circ \psi + Q \log |\psi'|, \tag{1.1.2}$$

then by [DS11, Prop. 2.1], almost surely we have $\mu_{\widetilde{h}} = \mu_h \circ \psi$ and (provided ψ extends to a homeomorphism between the closures of $\widetilde{D}$ and D in the Riemann sphere) $\nu_{\widetilde{h}} = \nu_h \circ \psi$.

We can then consider various types of *quantum surfaces*, random surfaces which can be parametrized by (D, h) with D a domain in $\mathbb{C}$ and h some form of the GFF on D, with (random) quantum area and boundary length measures given by μ_h and ν_h, and which are defined as equivalence classes of pairs (D, h) that are related by conformal reparametrizations as described by (1.1.2). A particular one-parameter family of such surfaces are the *α-quantum wedges* for $\alpha \leq Q$. An α-quantum wedge is parametrized by $\mathbb{H}$ with marked points at 0 and ∞, and is given by $h - \alpha \log|\cdot|$ where h is a variant of the free-boundary GFF on $\mathbb{H}$ chosen so that the law of the resulting surface is invariant under the operation of replacing h with $h+c$ for $c \in \mathbb{R}$. For any $\alpha \in (-\infty, Q)$, this surface is homeomorphic to $\mathbb{H}$, and is referred to as a *thick* quantum wedge, as in [DMS21, §4.2]. The starting point for an alternative but equivalent definition [DMS21, Def. 4.15] is a Bessel process of dimension

$$\delta := 2 + \frac{2(Q-\alpha)}{\gamma};$$

this can be used to extend the definition to include $\alpha \in (Q, Q+\gamma/2)$. For such α, the Bessel process has dimension in $(1,2)$ and thus hits zero, and one no longer obtains a single surface homeomorphic to $\mathbb{H}$; for each excursion of the Bessel process away from 0 one obtains a surface with the topology of the disc, and concatenating all these surfaces (the *beads* of the wedge) gives a *thin* quantum wedge, as seen in [DMS21, §4.4]. Instead of using the parameters α or δ, it is often more convenient to consider the value

$$\mathfrak{w} = \gamma\left(\frac{\gamma}{2} + Q - \alpha\right),$$

called the *weight* of the wedge.

Different kinds of quantum surfaces include *quantum cones*, which are homeomorphic to $\mathbb{C}$, and *quantum spheres*, which are homeomorphic to the Riemann sphere (and can thus be parametrized by the bi-infinite cylinder $\mathscr{C}$ given by $\mathbb{R} \times [0, 2\pi]$ with $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\pi\}$ identified and the points $-\infty$ and $+\infty$ added). By [DMS21, Thm 1.5], a quantum cone of weight $\mathfrak{w}$ is the surface that results when the two sides $(-\infty, 0)$ and $(0, \infty)$ of a quantum wedge of weight $\mathfrak{w}$ are conformally welded together. As with wedges, there are choices of parameter other than the weight parameter $\mathfrak{w}$. A quantum cone of weight $\mathfrak{w}$ can be referred to as an α-quantum cone, where the parameter α corresponding to the log singularity of the field and the weight $\mathfrak{w}$ are related by $\alpha = Q - \mathfrak{w}/(2\gamma)$. A quantum sphere of weight $\mathfrak{w}$ is a compact finite-volume surface constructed so as to look like a quantum cone of weight $\mathfrak{w}$ near each of its endpoints $-\infty$ and $+\infty$.

It was proven in [MS20, MS21a, MS21b] for $\gamma = \sqrt{8/3}$, and later in [GM21c] for $\gamma \in (0, 2)$, there is a unique random metric $\mathfrak{d}_h$, measurable w.r.t. the GFF h, that satisfies a certain list of axioms associated with LQG ($\mathfrak{d}_h$ is required to induce the Euclidean topology and to transform appropriately under affine coordinate changes and adding a continuous function to h, and must also be a length metric locally determined by h). This metric arises as a subsequential limit of *Liouville first passage percolation* (LFPP), a family of random metrics obtained from a regularized version of the GFF; existence of such subsequential limits was established in [DDDF20], and building on [DFG⁺20], the article [GM21c] then showed that the limit is unique and satisfies the requisite axioms. (More recently in [DG23] the *critical* LQG metric corresponding to $\gamma = 2$ was constructed and proven to be unique, as were *supercritical* LQG metrics corresponding to complex values of γ with $|\gamma| = 2$.)

The result [She16a, Thm 1.8] (later generalized by [DMS21, Thm 1.2]) says that when a certain quantum wedge $\mathcal{W}$ is cut by an appropriate independent random curve η, the regions to the left and right of η (call them $\mathcal{W}^-$, $\mathcal{W}^+$ respectively) are independent quantum wedges; moreover, the original wedge $\mathcal{W}$ and curve η may be reconstructed by *conformally welding* the right side of $\mathcal{W}^-$ to the left side of $\mathcal{W}^+$ according to γ-LQG boundary length. The curve η is a variant of Schramm's [Sch00] SLE – more specifically it is an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$, as first defined in [LSW03, §8.3].

Though we will not need it in this work, we briefly discuss what is meant here by "conformal welding". Given a homeomorphism between boundary arcs of two topological surfaces, one can obtain a new surface by gluing along the boundary arcs; if the two original surfaces are each endowed with a conformal structure, the problem of *conformally welding* them is that of obtaining a conformal structure on the glued surface compatible with those on the original surfaces. In the setting of the previous paragraph, it turns out [She16a, Thm 1.3] that the LQG boundary length measures on the boundaries of $\mathcal{W}^-$ and $\mathcal{W}^+$ agree for segments of η. This allows us to recover the original surface $\mathcal{W}$ from the surfaces $\mathcal{W}^-$ and $\mathcal{W}^+$. Indeed, if $\mathcal{W}^-$ and $\mathcal{W}^+$ are reparametrized by $\mathbb{H}$ with corresponding fields h^-

and h^+, then we can define a homeomorphism

$$\psi : [0,\infty) \to (-\infty, 0]$$

from the right-hand boundary arc of $\mathcal{W}^-$ to the left-hand boundary arc of $\mathcal{W}^-$ via the equation

$$\nu_{h^-}([0,x]) = \nu_{h^+}([\psi(x), 0]), \quad x \in (0,\infty).$$

Crucially, ψ is uniquely determined by $\mathcal{W}^-$ and $\mathcal{W}^+$ as surfaces (i.e., modulo reparametrization as in (1.1.2)). We can glue the surfaces together by identifying each point $x \in [0,\infty) \subset \partial\mathcal{W}^-$ with its corresponding point $\psi(x) \in (-\infty, 0] \subset \partial\mathcal{W}^+$; then by a **conformal welding** of $\mathcal{W}^-$ and $\mathcal{W}^+$ along ψ we mean a map from the resulting space into $\mathbb{H}$ that is conformal on the interiors of $\mathcal{W}^-$ and $\mathcal{W}^+$. In this case the glued space is the original surface $\mathcal{W}$, so such a map is given by a parametrization of $\mathcal{W}$ by $\mathbb{H}$. In this case, this map is in fact (up to conformal automorphisms of $\mathbb{H}$) the *unique* conformal welding of $\mathcal{W}^-$ and $\mathcal{W}^+$ along ψ, so that both the original surface $\mathcal{W}$ and the SLE-type interface η can be recovered from $\mathcal{W}^-$ and $\mathcal{W}^+$ (see [She16a, Thm 1.4]).

1.1.2 Metric gluing

Since these conformal welding uniqueness results do not give an explicit way to reconstruct the original surface, for applications a more explicit way to glue surfaces together may be required. In the case $\gamma = \sqrt{8/3}$, the theorem [GM19, Thm 1.5] states that the γ-LQG metric on $\mathcal{W}$ can be obtained by *metrically gluing* those on $\mathcal{W}^-$ and $\mathcal{W}^+$ along the conformal welding interface η according to γ-LQG boundary length, i.e. as a quotient of the two metric spaces $\mathcal{W}^-$ and $\mathcal{W}^+$ under the identification of points given by the welding homeomorphism. This theorem – stating that conformal welding and metric gluing give the same result – was an essential input into the proof in [GM21b] that the self-avoiding walk (SAW) on random quadrangulations converges to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG. Indeed, one can construct a SAW-decorated random quadrangulation by performing a discrete graph gluing of two quadrangulations with boundary, and [GM21b] shows that this construction converges to an analogous one in the continuum using quantum wedges; the result of [GM19] then applies to show that we get the same surface by *first* passing to the scaling limit of each of the two original quadrangulations and *then* performing the metric gluing in the continuum. The importance of [GM19, Thm 1.5] here is that, whilst metric gluing and conformal welding both provide ways to show that an LQG surface is determined by the two surfaces formed by cutting along an independent SLE, metric gluing recovers the original surface via a construction that has a direct discrete analogue, namely the graph gluing.

The notion of "metric gluing" here is the natural way to define the quotient space obtained from identifying two metric spaces along a common subset; we define it below.

Definition 1.1.1 (metric gluing)**.** Let (X, d_X) and (Y, d_Y) be *pseudometric spaces* (that is, d_X satisfies all the conditions to be a metric on X except that it need not be positive definite, and likewise for d_Y on Y). Let f be a function from a subset of X to a subset of Y. Let $\sim$ be the finest equivalence relation on $X \sqcup Y$ such that $x \sim f(x)$ for each x in the domain of f, and for each $x \in X \sqcup Y$ let $[x]$ be the equivalence class of x under $\sim$. Define d' on $(X \times X) \sqcup (Y \times Y)$ to equal d_X on $X \times X$ and d_Y on $Y \times Y$. Then the **metric gluing** of X and Y along f is the quotient space $(X \sqcup Y)/\sim$ equipped with the *gluing pseudometric* d defined by

$$d([x],[y]) = \inf \sum_{i=1}^{n} d'(x_i, y_i)$$

where the infimum is over all $n \in \mathbb{N}$ and all sequences $x_1, y_1, x_2, y_2 \ldots, x_n, y_n$ in $X \sqcup Y$ such that $x_1 \in [x]$, $y_n \in [y]$, and $x_{i+1} \sim y_i$ for each $i \in \{1, \ldots, n-1\}$, such that the sum is defined (so for each i we must have x_i and y_i either both in X or both in Y). If (X_i, d_i) are pseudometric spaces for $i \in I$, we can define the **metric quotient** of the X_i by an equivalence relation $\sim$ on $X := \bigsqcup_{i \in I} X_i$ by defining the partial function d' on $\bigsqcup_{i \in I}(X_i \times X_i)$ and the gluing pseudometric d on $X/\sim$ in the same way as above.

Note that this d is easily verified to be a pseudometric; in fact, it is the largest pseudometric on the quotient space which is bounded above by d'. In the case of [GM19, Thm 1.5], the gluing function f sends a point z on the right-hand part of $\partial\mathcal{W}^-$ to the point w on the left-hand part of $\partial\mathcal{W}^+$ such that the boundary segments from 0 to z and from 0 to w have equal γ-LQG boundary length.

1.2 Conformal loop ensembles

We will also consider random metrics associated to *conformal loop ensembles* (CLE). The conformal loop ensemble CLE_κ is a conformally invariant probability measure on countable families of non-crossing loops in a simply connected planar domain, defined for each choice of the parameter $\kappa \in (8/3, 8)$ by Sheffield in [She09]. CLE$_\kappa$ is the loop version of SLE$_\kappa$ – one can use SLE$_\kappa$ to construct CLE$_\kappa$, and CLE has the same phases as SLE [RS05], with the loops of a CLE$_\kappa$ locally looking like SLE$_\kappa$ curves. In particular, for $\kappa \in (8/3, 4]$ the loops of a CLE$_\kappa$ are simple and disjoint, and do not hit the boundary of the domain, whereas for $\kappa \in (4, 8)$, the loops can intersect (though not cross) themselves, each other, and the domain boundary. In either regime, the set of points not surrounded by any CLE loop is a closed connected set with zero Lebesgue measure, and is called the CLE$_\kappa$ *carpet* (for $\kappa \in (8/3, 4]$) or the CLE$_\kappa$ *gasket* (for $\kappa \in (4, 8)$), by analogy with the Sierpiński carpet and gasket, respectively. The set is fractal, and its dimension has been shown to equal $2-(8-\kappa)(3\kappa-8)/(32\kappa)$ [SSW09, NW11, MSW14].

Just as SLE$_\kappa$ is the scaling limit of a single interface in many two-dimensional discrete lattice models at criticality, CLE$_\kappa$ is either proven or conjectured to arise as the scaling limit of the collection of *all* of the interfaces in many such models. Of particular interest are the cases $\kappa = 3, 16/3, 6, 8$, which have respectively been shown to describe the scaling limit (at criticality) of the Ising model, FK Ising model, percolation, and the uniform spanning tree [BH19, CDCH$^+$14, KS19, Smi10, Smi01, CN08, LSW04].

One can ask whether, for a discrete model that converges to CLE in the scaling limit, the chemical distance metric also has a scaling limit, which should be the "natural" conformally covariant random metric associated to the CLE carpet or gasket. In Chapter 5 we will study the properties such a "CLE chemical distance metric" would have to have. We emphasize that such a metric has not yet been constructed, although in the case $\kappa \in (8/3, 4)$ a sequence of random metrics that should approximate this chemical distance metric were proven to be tight in [Mil21]. In Chapter 5 we will consider a metric which is defined on the CLE$_\kappa$ carpet (for $\kappa \in (8/3, 4]$) or gasket (for $\kappa \in (4, 8)$) satisfying a list of natural assumptions (see Assumption 1.3.10) which should be satisfied by the CLE chemical distance metric. Though we conjecture that there exists a unique metric satisfying the assumptions (which we will call the CLE$_\kappa$ *metric*), we will not address the problems of existence and uniqueness here; instead we will assume the metric exists and derive some of its properties.

1.3 Main results

1.3.1 Equivalence of metric gluing and conformal welding

In the light of the construction of the γ-LQG metric for all $\gamma \in (0, 2)$, the main result of Chapter 3 extends [GM19, Thm 1.5], giving the analogous statement for the γ-LQG metric for all values of $\gamma \in (0,2)$. In order to state the result, we need to define what it means for a metric defined on a subspace to extend by continuity to a larger set:

Definition 1.3.1. Let (X,τ) be a topological space and Y a subset of X. If d is a metric on Y that is continuous w.r.t. the subspace topology induced by τ on Y and $Z \subseteq X \setminus Y$, then we say d **extends by continuity (w.r.t. τ) to** Z if there exists a metric d' on $Y \cup Z$ which agrees with d on Y and is continuous w.r.t. the subspace topology induced by τ on $Y \cup Z$.

Note that if Y is dense in $Y \cup Z$ then there can be at most one metric d' extending d by continuity to Z.

Theorem 1.3.2. *Let $\gamma \in (0,2)$, $\mathfrak{w}^-, \mathfrak{w}^+ > 0$ and $\mathfrak{w} = \mathfrak{w}^- + \mathfrak{w}^+$. Let $(\mathbb{H}, h, 0, \infty)$ be a quantum wedge of weight $\mathfrak{w}$ if $\mathfrak{w} \geq \gamma^2/2$, or a single bead of a quantum wedge of weight $\mathfrak{w}$ with area $\mathfrak{a} > 0$ and left and right boundary lengths $\mathfrak{l}^-, \mathfrak{l}^+ > 0$ otherwise. Let η be an independently sampled* $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}^- - 2; \mathfrak{w}^+ - 2)$ *process from 0 to ∞ in $\mathbb{H}$ with force points at 0^- and 0^+. Denote the regions to the left and right of η by W^- and W^+ respectively, and let $\mathcal{W}^\pm$ be the quantum surface obtained by restricting h to $W^\pm$. Let $\mathcal{U}^\pm$ be the ordered sequence of connected components of the interior of $W^\pm$, and let $\mathfrak{d}_h$, $\mathfrak{d}_{h|_{\mathcal{U}^-}}$ and $\mathfrak{d}_{h|_{\mathcal{U}^+}}$ respectively be the γ-LQG metrics induced by h, $h|_{\mathcal{U}^-}$ and $h|_{\mathcal{U}^+}$. Then $\mathfrak{d}_{h|_{\mathcal{U}^-}}$ and $\mathfrak{d}_{h|_{\mathcal{U}^+}}$ respectively extend by continuity (w.r.t. the Euclidean topology) to $\partial\mathcal{U}^-$ and $\partial\mathcal{U}^+$ and $(\mathbb{H}, \mathfrak{d}_h)$ is obtained by metrically gluing $(\overline{\mathcal{U}^-}, \mathfrak{d}_{h|_{\mathcal{U}^-}})$, and $(\overline{\mathcal{U}^+}, \mathfrak{d}_{h|_{\mathcal{U}^+}})$ along η according to γ-LQG boundary length.*

Although we have no specific application in mind, this result is potentially useful in proving convergence of a path-decorated lattice model in the scaling limit to γ-LQG decorated by an SLE_{γ^2}-type curve, as it would play the role of [GM19, Thm 1.5] in an argument along the lines of [GM21b].

A statement weaker than Theorem 1.3.2 follows straightforwardly from a *locality* property in the definition of the LQG metric, which gives that the $\mathfrak{d}_{h|_{\mathcal{U}^-}}$-distance between points in $\mathcal{U}^-$ coincides with the infimum of the $\mathfrak{d}_h$-lengths of paths between the points that stay in $\mathcal{U}^-$, and likewise for $\mathcal{U}^+$. It is important to note that this property does *not* imply that the metric gluing recovers $(\mathbb{H}, \mathfrak{d}_h)$; Thm 1.3.2 is stronger because it rules out certain pathologies which can arise from metric gluings along badly behaved interfaces (note that the interfaces along which we are gluing are SLE-type curves and thus fractal).

One such pathology can occur when the function used to identify boundary segments is insufficiently well behaved: for instance, using a Cantor-type function can collapse the gluing interface to a point (see [GM19, Lemma 2.2]). This kind of behaviour does not occur in our setting, since we know that $\mathfrak{d}_h$ is a pseudometric on the glued space that is bounded above by the partial function d' constructed from $\mathfrak{d}_{h|_{\mathcal{U}^-}}$ and $\mathfrak{d}_{h|_{\mathcal{U}^+}}$, whereas the gluing pseudometric is always the *largest* such pseudometric (and thus in this case is a bona fide metric). The main issue for us is that the definition of the gluing metric only considers paths which cross the gluing interface η finitely many times, whereas paths in $(\mathbb{H}, \mathfrak{d}_h)$ which cross η infinitely many times might a priori be significantly shorter.

Though both this metric gluing result and the result [DMS21, Thm 1.2] on conformal welding for the same surfaces can be thought of as saying that we can recover the original surface from the two pieces it is cut into by the SLE-type curve η, the conformal welding result is more of an abstract measurability statement, whereas the metric gluing result shows concretely how one can reconstruct the metric on the original surface. Nevertheless, we might intuitively expect that, since one result is true, so should the other be – so that we avoid the pathologies that are generally liable to arise from metric gluing along fractal curves. In our setting, we aim to rule out pathological behaviour of $\mathfrak{d}_h$-geodesics hitting η. An analogous problem in the conformal setting is to show that a curve is *conformally removable*, i.e. that any homeomorphism of $\mathbb{C}$ that is conformal off the image of the curve must in fact be conformal everywhere. Indeed, the reason that the conformal welding is unique – that η can be recovered from the two surfaces on either side of it – is that η is conformally removable. Thus, if there were another welding along the same boundary arc homeomorphism which produced a different interface, the two weldings would differ by a homeomorphism of $\mathbb{H}$ that was conformal off the image of η, which by removability would have to be a conformal automorphism of $\mathbb{H}$. The fact [RS05, Thm 5.2] [JS00, Cor. 2] that an SLE$_\kappa$ curve with $\kappa \in (0, 4)$ is conformally removable follows from the fact that it is the boundary of a *Hölder domain*, i.e. a domain which can be uniformized by a Hölder-continuous map from the unit disc. Proving conformal removability of a curve involves controlling how much a straight line segment near the curve is distorted by such a homeomorphism, whereas as mentioned above our task is to establish control on the extent to which an LQG geodesic is affected by its crossings of η. Though the two problems are similar in flavour, in the metric gluing setting we do not have a simple sufficient criterion analogous to the Hölder domain condition for conformal removability.

We also obtain the appropriate generalizations for the other main theorems in [GM19].

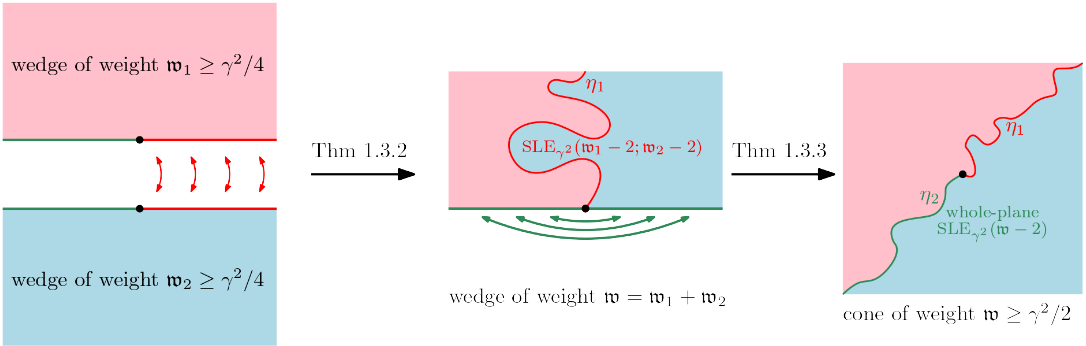

Figure 1.1: An illustration of Theorems 1.3.2 and 1.3.3 in the case of two thick wedges ($\mathfrak{w}_1$, $\mathfrak{w}_2 \geq \gamma^2/4$) which are glued along half their boundaries to yield a wedge of weight $\mathfrak{w} = \mathfrak{w}_1 + \mathfrak{w}_2$, then along the other half to yield a cone of weight $\mathfrak{w}$.

Our version of [GM19, Thm 1.6], concerning gluing the two boundary arcs of a quantum wedge together to create a quantum cone, is as follows:

Theorem 1.3.3. *Fix $\gamma \in (0,2)$ and $\mathfrak{w} \geq 0$. Let $(\mathbb{C}, h, 0, \infty)$ be a quantum cone of weight $\mathfrak{w}$ and let $\mathfrak{d}_h$ be the γ-LQG metric induced by h. Let η be an independent whole-plane* $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ *from 0 to ∞ and let $U = \mathbb{C} \setminus \eta$. Then $\mathfrak{d}_{h|_U}$ almost surely extends by continuity to ∂U (seen as a set of prime ends), and $(\mathbb{C}, \mathfrak{d}_h)$ almost surely agrees with the metric quotient of $(U, \mathfrak{d}_{h|_U})$ under identifying the two sides of η in the obvious way (i.e., two prime ends corresponding to the same point in $\mathbb{C}$ are identified).*

Here the surface $(U, h|_U)$ is a quantum wedge of weight $\mathfrak{w}$ by [DMS21, Thm 1.5], and this result tells us that we can recover the original cone from this wedge via metric gluing. We also generalize [GM19, Thm 1.7], which says that a quantum cone cut by a space-filling variant of SLE into countable collection of beads of thin wedges can be recovered by metrically gluing the beads along their boundaries:

Theorem 1.3.4. *Fix $\gamma \in (0,2)$ and let $(\mathbb{C}, h, 0, \infty)$ be a γ-quantum cone with associated γ-LQG metric $\mathfrak{d}_h$. Let η' be an independent whole-plane space-filling* $\mathrm{SLE}_{16/\gamma^2}$ *from ∞ to ∞ through 0, as defined in [DMS21, Footnote 4], and reparametrize η' by* quantum time *(so that $\mu_h(\eta([a,b])) = b - a$), with $\eta'(0) = 0$. Then let $\mathcal{U}^-$ (resp. $\mathcal{U}^+$) be the set of connected components of the interior of $\eta'((-\infty, 0])$ (resp. $\eta'([0,\infty))$) and for each $U \in \mathcal{U}^- \cup \mathcal{U}^+$ let $\mathfrak{d}_{h|_U}$ be the γ-LQG metric induced by $h|_U$. Then almost surely, each $\mathfrak{d}_{h|_U}$ extends continuously (w.r.t. the Euclidean metric) to ∂U, and $(\mathbb{C}, \mathfrak{d}_h)$ is the metric quotient (under the obvious identification) of*

$$\bigsqcup_{U \in \mathcal{U}^- \cup \mathcal{U}^+} (\overline{U}, \mathfrak{d}_{h|_U}).$$

By [DMS21, Thm 1.2, Thm 1.5], when $\gamma > \sqrt{2}$ the surfaces $(U, h|_U)$ here are single beads of thin wedges of weight $2 - \gamma^2/2$, whereas for $\gamma \leq \sqrt{2}$ they are thick wedges. Finally, we generalize [GM19, Thm 1.8], in which we recover a quantum sphere as a quotient of a set of surfaces into which it is cut by a space-filling $\mathrm{SLE}_{16/\gamma^2}$.

Theorem 1.3.5. *Fix* $\gamma \in (0,2)$ *and let* $(\mathscr{C}, h, -\infty, \infty)$ *be a unit area quantum sphere with associated* γ*-LQG metric* $\mathfrak{d}_h$*. Let* η' *be an independent whole-plane space-filling* $\mathrm{SLE}_{16/\gamma^2}$ *from* $+\infty$ *to* $+\infty$ *and reparametrize* η' *by quantum time. Let* T *be a* $U[0,1]$ *variable independent of everything else, and let* $\mathcal{U}^-$ *(resp.* $\mathcal{U}^+$*) be the set of connected components of the interior of* $\eta'([0,T])$ *(resp.* $\eta'([T,1])$*). For each* $U \in \mathcal{U}^- \cup \mathcal{U}^+$ *let* $\mathfrak{d}_{h|_U}$ *be the* γ*-LQG metric induced by* $h|_U$*. Then almost surely, each* $\mathfrak{d}_{h|_U}$ *extends continuously (w.r.t. the Euclidean metric) to* ∂U*, and* $(\mathbb{C}, \mathfrak{d}_h)$ *is the metric quotient (under the obvious identification) of*

$$\bigsqcup_{U \in \mathcal{U}^- \cup \mathcal{U}^+} (\overline{U}, \mathfrak{d}_{h|_U}).$$

In [GM19] many of the preliminary results are proved using the results in [MS21a]. In that paper, the metric $\mathfrak{d}_h$ is constructed in the case $\gamma = \sqrt{8/3}$ (the more general $\gamma \in (0,2)$ result was not established until later). It is then shown that for $\gamma = \sqrt{8/3}$, there almost surely exists an isometry from the quantum sphere to another object, the *Brownian map* introduced by Le Gall [LG13] (whose law intuitively describes that of a metric space chosen "uniformly at random" from those spaces with the topology of a sphere), and further that this isometry almost surely pushes forward the LQG measure μ_h to the natural measure on the Brownian map. Similar isomorphisms of metric measure spaces are established between other quantum and Brownian surfaces. Distances in these surfaces have explicit formulae in terms of Brownian motion-type processes.

Since the equivalence between quantum and Brownian surfaces only holds for $\gamma = \sqrt{8/3}$, the techniques used in [GM19, §3.2] to establish estimates on areas, distances and boundary lengths are not available in this more general setting. We instead obtain analogues of these estimates largely via GFF methods, as well as the conformal welding properties of quantum wedges, which let us transfer our understanding of the interior behaviour of our surfaces to their boundaries (sometimes using existing results about the SLE curves that form the welding interfaces). In fact, in the case $\gamma \neq \sqrt{8/3}$ the existing literature only addresses LQG metrics associated to whole-plane or *zero-boundary* GFFs; our work provides the first treatment of the metric on surfaces with free boundary conditions for the complete subcritical case $\gamma \in (0,2)$. In particular we establish that the LQG metric given by a free-boundary GFF actually does extend continuously to the boundary:

Proposition 1.3.6. *Fix $\gamma \in (0,2)$. Let h be a free-boundary GFF on $\mathbb{H}$ with the additive constant fixed so that the semicircle average $h_1(0)$ equals zero, and let $\mathfrak{d}_h$ be the associated γ-LQG metric on $\mathbb{H}$. Then $\mathfrak{d}_h$ almost surely extends by continuity to a metric on $\overline{\mathbb{H}}$ that induces the Euclidean topology on $\overline{\mathbb{H}}$.*

Some of our other results about the γ-LQG metric on the boundary may be of independent interest. For one, we establish local bi-Hölder continuity w.r.t. the Euclidean metric:

Proposition 1.3.7. *In the setting of Prop. 1.3.6, there are exponents $\alpha_1, \alpha_2 > 0$ such that, almost surely, for each compact $K \subset \overline{\mathbb{H}}$, there exists $C > 0$ finite such that*

$$C^{-1}|z-w|^{\alpha_1} \leq \mathfrak{d}_h(z,w) \leq C|z-w|^{\alpha_2}$$

for each $z, w \in K$.

It should be noted that, although we obtain the right-hand inequality for arbitrary $\alpha_2 < \xi(Q-2)$ which is the optimal exponent even away from the boundary [DFG+20, Thm 1.7], we make no attempt to obtain the optimal exponent for the left-hand inequality, and we do not expect that the value for α_1 resulting from our proof is optimal. During the proof we establish a new regularity estimate for SLE$_\kappa$ curves with $\kappa \in (0,4)$. Namely, we combine the “non-self-tracing” result in [MMQ21] for SLE$_\kappa$ curves with $\kappa \in (0,8)$ with an argument based on conformal covariance of the LQG measure that rules out large bottlenecks to establish that, in the case that $\kappa \in (0,4)$, the (Euclidean) diameter of an SLE$_\kappa$ segment is at most polynomial in the distance between its endpoints. (Recall that $\kappa \leq 4$ is the range for which SLE$_\kappa$ is simple, though we do not investigate the critical value $\kappa = 4$ here.)

Proposition 1.3.8. *For each $\kappa \in (0,4)$ there is an exponent $\zeta > 0$ such that the following holds. Let η be an SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ (with any parametrization). For each compact $K \subset \mathbb{H}$, there almost surely exists $C \in (0,\infty)$ such that*

$$\operatorname{diam} \eta([s,t]) \leq C|\eta(s)-\eta(t)|^{\zeta}$$

whenever $\eta(s), \eta(t) \in K$.

1.3.2 LQG metrics are not doubling

In Chapter 4 we will study the embeddability of the LQG metric into Euclidean space. LQG is known to describe the scaling limit of certain discrete conformal embeddings of certain kinds of random planar map (e.g., the Tutte embedding of the mated-CRT map [GMS21]). Given a quantum surface $\mathcal{S}$ and an embedding $\mathcal{S} \to D$ into the plane (i.e., a particular choice of parametrization (D, h)) obtained via such a scaling limit, one might therefore expect the embedding $\mathcal{S} \to D$ to somehow retain the conformality of the discrete embeddings. It is meaningless to ask directly whether the embedding $\mathcal{S} \to D$ is conformal, as the complex structure on $\mathcal{S}$ comes from the embedding in the first place. However, since $\mathcal{S}$ is a metric space, one could ask whether the embedding is *quasisymmetric*. Quasisymmetric mappings are embeddings of metric spaces in which the distortion of the metric is uniformly controlled; in the case where both the domain and the target space are open subsets of $\mathbb{R}^n$, locally quasisymmetric mappings are equivalent to locally quasiconformal mappings.

As mentioned, in [MS21a] it is shown that, in the particular case $\gamma = \sqrt{8/3}$, the quantum sphere is almost surely isomorphic as a metric measure space to Le Gall's [LG13] Brownian map. The law of the Brownian map is, intuitively, that of a "uniform random element" from the set of metric spaces that are homeomorphic to the sphere S^2, and it was proven independently by Le Gall [LG13] and Miermont [Mie13] that the Brownian map is the scaling limit of uniform random planar quadrangulations.

The Brownian map can be constructed using a continuous process (the *Brownian snake*) parametrized by the *continuum random tree* (CRT), a random metric space introduced by Aldous [Ald91a, Ald91b, Ald93] that arises as the scaling limit of uniform discrete plane trees. The CRT is constructed from the graph of a Brownian excursion by identifying points connected by horizontal line segments that stay underneath the graph.

In [Tro21], Troscheit proved that the continuum random tree and the Brownian map almost surely cannot be embedded quasisymmetrically into $\mathbb{R}^n$ for any n. The method was to show that those spaces have the property that for every N one can find sets of N points all roughly equidistant from each other. Any quasisymmetric image of such a space has infinite *Assouad dimension*. The Assouad dimension of a metric space is defined somewhat similarly to the upper box-counting dimension, but can be strictly greater – intuitively, this happens when, in each covering by boxes of a given scale, disproportionately many boxes are required to cover certain particularly thick parts of the space. Spaces of infinite Assouad dimension can be equivalently characterized as those that are not *doubling*, i.e. those in which, for every N, there exists a metric ball that cannot be covered by N balls of half its radius; this property is preserved by quasisymmetric mappings.

Although the equivalence with Brownian surfaces only holds for $\gamma = \sqrt{8/3}$, we will instead use GFF techniques to find approximately equidistant sets of points, proving that, for all $\gamma \in (0,2]$, no γ-Liouville quantum gravity metric space $(D, \mathfrak{d}_h)$ can be embedded quasisymmetrically into $\mathbb{R}^n$, or indeed into any complete Riemannian manifold with non-negative Ricci curvature.

Theorem 1.3.9. *Let $D \subseteq \mathbb{C}$ be a domain and h some variant of the GFF on D. Let $\gamma \in (0,2]$ and let $\mathfrak{d}_h$ be the γ-LQG metric on D associated to h. Then the metric space $(D, \mathfrak{d}_h)$ almost surely cannot be embedded quasisymmetrically into any doubling metric space (in particular, into any complete n-dimensional Riemannian manifold with non-negative Ricci curvature for any $n \in \mathbb{N}$).*

1.3.3 CLE metrics

In Chapter 5 we study properties of metrics in the carpet (for $\kappa \in (8/3,4]$) or gasket (for $\kappa \in (4,8)$) of a CLE_κ which satisfy the following natural hypotheses (such a metric will be called a CLE_κ **metric**).

Assumption 1.3.10. *Suppose that $\kappa \in (8/3,8)$. We assume that there exists a collection (μ_D) of probability measures indexed by the set of simply connected proper domains $D \subsetneq \mathbb{C}$ such that, for each D, μ_D is a measure on pairs $(\Gamma, \mathfrak{d}(\cdot,\cdot;\Gamma))$ where the marginal law of Γ is that of a CLE_κ on D and $\mathfrak{d}(\cdot,\cdot;\Gamma)$ is a metric on the carpet (resp. gasket) Υ of Γ when $\kappa \in (8/3,4]$ (resp. $\kappa \in (4,8)$) which satisfies the following additional properties.*

(i) (Geodesic.) For every $x, y \in \Upsilon \setminus \partial D$ there exists a $\mathfrak{d}(\cdot,\cdot;\Gamma)$-geodesic γ from x to y.

(ii) (Locality.) Suppose that $U \subseteq D$ is a domain. Given $U \cap \Upsilon$, the internal metric induced by $\mathfrak{d}(\cdot,\cdot;\Gamma)$ on $U \cap \Upsilon$ is conditionally independent of the internal metric induced by $\mathfrak{d}(\cdot,\cdot;\Gamma)$ on $\Upsilon \cap (D \setminus U)$.

(iii) (Conformal covariance.) There exists a constant $\alpha > 0$ so that the following is true. Suppose that $\widetilde{D}$ is a simply connected domain and $\varphi \colon D \to \widetilde{D}$ is a conformal transformation. Then the joint law of $\widetilde{\Gamma} = \varphi(\Gamma)$ and the metric on $\widetilde{\Upsilon} = \varphi(\Upsilon)$ defined by

$$\inf_{\gamma\colon \varphi^{-1}(x) \to \varphi^{-1}(y)} \int |\varphi'(\gamma(t))|^\alpha dt,$$

where the infimum is over all $\mathfrak{d}(\cdot,\cdot;\Gamma)$-finite length paths parameterized at unit speed in Υ connecting $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$, is $\mu_{\widetilde{D}}$.

In order to simplify our proofs, we will consider the following additional assumption on the regularity of a CLE metric. We expect that in fact this additional assumption can be deduced from Assumption 1.3.10 but we will not address this in the present work.

Assumption 1.3.11. *If $\kappa \in (8/3,4)$, let D be the (random) simply connected domain whose boundary is the right side of a whole-plane two-sided SLE_κ process η from ∞ to ∞ through 0. Then, when η is given the natural parametrization, $\sup_{t\in(0,1)}\mathfrak{d}(\eta(0),\eta(t);\Gamma)$ is almost surely positive and has finite expectation. If instead $\kappa \in (4,8)$, let D be the (random) simply connected domain whose boundary is the right side of a two-sided whole-plane $\mathrm{SLE}_{16/\kappa}$ process η from ∞ to ∞ through 0. Then, when η is given the natural parametrization, $\sup_{t\in(0,1)}\mathfrak{d}(\eta(0),\eta(t);\Gamma)$ is almost surely positive and has finite expectation. Moreover, if $\rho > -2$ and $\tau_1,\tau_2 \le 0$ are respectively stopping times for the processes $\{\eta|_{(-\infty,t]} : t \in \mathbb{R}\}$ and $\{\eta|_{[-t,\infty)} : t \in \mathbb{R}\}$, then the same is true with η replaced by the curve $\widetilde{\eta}$ obtained by first sampling $\eta|_{(-\infty,\tau_1]}$ and $\eta|_{[-\tau_2,\infty)}$, then sampling η', an $\mathrm{SLE}_\kappa(\rho)$ (if $\kappa \in (8/3,4)$) or $\mathrm{SLE}_{16/\kappa}(\rho)$ (if $\kappa \in (4,8)$) from η_{τ_1} to $\eta_{-\tau_2}$ in the domain $\mathbb{C}\setminus\eta|_{(-\infty,\tau_1]\cup[-\tau_2,\infty)}$, then concatenating $\eta|_{(-\infty,\tau_1]}$, η' and $\eta|_{[-\tau_2,\infty)}$.*

We conjecture that there exists a unique metric which satisfies Assumption 1.3.10 and that, when $\kappa \neq 4$, this metric satisfies Assumption 1.3.11. We will not consider the critical case $\kappa = 4$ in this work.

In Assumption 1.3.10 we only assume the existence of a conformal covariance exponent $\alpha > 0$ – in particular we do not assume uniqueness (i.e., we do not assume that two different metrics d, d' associated to the same CLE will necessarily have the same conformal covariance exponent), nor does the assumption describe how α might depend on κ. The main purpose of Chapter 5 is to obtain an upper bound for any such α associated to a CLE_κ metric:

Theorem 1.3.12. *Fix $\kappa \in (8/3,8)\setminus\{4\}$ and suppose that $\mathfrak{d}(\cdot,\cdot;\cdot)$ satisfies Assumptions 1.3.10 and 1.3.11. Let $\alpha > 0$ be the conformal covariance exponent from Assumption 1.3.10. If we set*

$$d = \begin{cases} 1+\kappa/8 & \kappa \in (8/3,4) \\ 1+2/\kappa & \kappa \in (4,8) \end{cases}$$

then $\alpha < d$.

For the upper bound in Theorem 1.3.12 when $\kappa \in (8/3, 4)$, we consider a domain whose boundary is a two-sided whole-plane SLE_κ and use that the law of this curve (call it η) is preserved [Zha21] by the scaling map

$$S_r : \eta(\cdot) \mapsto r^{-1/(1+\kappa/8)}\eta(r\cdot) \tag{1.3.1}$$

when η is given the *natural parametrization* (equivalently, when η is parametrized by $(1+\kappa/8)$-dimensional Minkowski content). The fact that $1+\kappa/8$ is the exponent for this scale invariance property readily implies that we must have $\alpha \leq 1+\kappa/8$. To rule out $\alpha = 1+\kappa/8$ we establish ergodicity for this scaling map:

Proposition 1.3.13. *Let $\kappa \in (0,4)$ and let η be a two-sided whole-plane SLE_κ with the natural parametrization. Then for $r > 0$, the map S_r defined in (1.3.1) is ergodic w.r.t. the law of η.*

We then argue that if $\alpha = 1 + \kappa/8$, this ergodicity would cause CLE metric distances between points on η to be determined by η alone, from which we derive a contradiction. (The same argument works for $\kappa \in (4,8)$ except that we work with a domain whose boundary is an $\mathrm{SLE}_{16/\kappa}$.)

1.4 Notation

If $(E(C))_{C\in S}$ is a family of events indexed by a set $S \subseteq \mathbb{R}$ which is unbounded above, we say that $E(C)$ happens with **superpolynomially high probability** as $C \to \infty$ if for any $N \in \mathbb{N}$ we have $\mathbb{P}[E(C)^c] = O(C^{-N})$ as $C \to \infty$. If $E(C)$ depends on other parameters, we say $E(C)$ happens with superpolynomially high probability *at a rate which is uniform in* some subset of those parameters if the bounds on $\mathbb{P}[E(C)^c]C^N$ can be chosen not to depend on that subset of those parameters. Similarly, we say that a function f **decays superpolynomially** if for all N we have $f(x) = O(x^N)$ as $x \to 0$.

For $z \in \mathbb{C}$ and $r > 0$, $B(z,r)$ and $\overline{B}(z,r)$ will always mean, respectively, the *Euclidean* open and closed balls of radius r centred at z; we will define notation ad hoc for balls of other metrics. For $z \in \mathbb{C}$ and $R_2 > R_1 > 0$, we write $\mathbb{A}_{R_1,R_2}(z)$ for the open annulus $B(z,R_2) \setminus \overline{B}(z,R_1)$.

Chapter 2

Preliminaries

2.1 The Gaussian free field

The **Gaussian free field** (GFF) is a random process analogous to Brownian motion, where the analogue of the time parameter ranges over a domain in the complex plane. We recall the definition of the *zero-boundary* GFF from [She07, Def. 2.10], which begins with an open set $D \subset \mathbb{C}$ with *harmonically non-trivial* boundary (meaning that a Brownian motion started from $z \in D$ will almost surely hit ∂D). We let $H_s(D)$ be the set of smooth functions with compact support contained in D, equipped with the *Dirichlet inner product*

$$(f,g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(x) \cdot \nabla g(x)\, dx,$$

and complete this inner product space to a Hilbert space $H(D)$. Taking an orthonormal basis (φ_n) of $H(D)$ and letting (α_n) be i.i.d. $N(0,1)$ variables, the zero-boundary GFF in D is then defined as a random linear combination of elements of $H(D)$ given by

$$h = \sum_n \alpha_n \varphi_n. \tag{2.1.1}$$

It can be shown (see [She07, Prop. 2.7]) that this sum converges almost surely in the space of distributions and in the fractional Sobolev space $H^{-\varepsilon}(D)$ for each $\varepsilon > 0$ (even though it does *not* converge pointwise or in $H(D)$ itself) and that the law of the limit h does not depend on the choice of basis (φ_n). This limiting distribution h is the **zero-boundary Gaussian free**

field. Writing $(\cdot,\cdot)$ for the usual L^2 inner product, we can define for each $f \in H_s(D)$

$$(h, f) := \lim_{n\to\infty} \left(\sum_n \alpha_n \varphi_n, f \right).$$

Note that, for each $f \in H_s(D)$, this sum converges almost surely as an L^2-bounded martingale. Indeed, the limit almost surely exists for all $f \in H_s(D)$ simultaneously, and is such that $f \mapsto (h, f)$ is a continuous functional on $H_s(D)$.

Moreover, one can define the L^2 pairing of h with certain other measures. Most importantly for us, if h is a zero-boundary GFF, $\varepsilon > 0$ and $B(z,\varepsilon) \subset D$, we denote by $h_\varepsilon(z)$ the **circle average** of h on the circle $\partial B(z,\varepsilon)$, defined as

$$(h, \rho_{z,\varepsilon}) = -2\pi(h, \Delta^{-1}\rho_{z,\varepsilon})_\nabla$$

where $\rho_{z,\varepsilon}$ is the uniform probability measure on $\partial B(z,\varepsilon)$. In [HMP10, Prop. 2.1], it is shown that for each fixed $z \in D$, the process $\{h_{e^{-t}}(z) \colon B(z,e^{-t}) \subset D\}$ has the covariance structure of a standard Brownian motion on the interval $\{t \colon B(z,e^{-t}) \subset D\}$, and that the circle average process $\{h_{e^{-t}}(z) \colon B(z,e^{-t}) \subset D\}$ has a version that is continuous in both t and z.

Given a function g on ∂D such that there exists a unique function $\mathfrak{h}$ on $\overline{D}$ which is continuous at all but finitely many points of $\overline{D}$, equals g on ∂D and is harmonic in D, we define the law of a GFF in D with (Dirichlet) boundary data g to be the law of $h + \mathfrak{h}$ where h is a zero-boundary GFF in D.

We can instead set D to be all of $\mathbb{C}$. In this case, as in [MS17, §2.2.1], we define the **whole-plane Gaussian free field** h in the same way, except that we consider h *modulo additive constant*. This means that we consider the equivalence relation $\sim$ on the space of distributions defined by the condition that $h_1 \sim h_2$ if and only if $h_1 - h_2$ is a constant distribution, i.e. if and only if there exists $a \in \mathbb{R}$ such that $(h_1, f) - (h_2, f) = a \int_{\mathbb{C}} f(z)\, dz$ for all $f \in H_s(\mathbb{C})$. We take (φ_n) to be a fixed orthonormal basis for $H(\mathbb{C})$ and sample i.i.d. $N(0,1)$ variables α_n, and define h as the equivalence class of $\sim$ containing $\sum_n \alpha_n \varphi_n$. Equivalently, for $f \in H_s(\mathbb{C})$, we only consider (h, f) to be defined if $f \in H_{s,0}$, the subspace of those functions in $H_s(\mathbb{C})$ whose integral over $\mathbb{C}$ is zero. Observe that the circle average process $(h_{e^{-t}}(z) - h_1(z))_{t\in\mathbb{R}}$ is well-defined, since $h_{e^{-t}}(z) - h_1(z) = (h, \rho_{z,e^{-t}} - \rho_{z,1})$ and $\int_{\mathbb{C}} d(\rho_{z,e^{-t}} - \rho_{z,1}) = 0$. It turns out that the process $(h_{e^{-t}}(z) - h_1(z))_{t\in\mathbb{R}}$ has a version which is a standard two-sided Brownian motion starting from 0.

We can also *fix* the additive constant, i.e. choose a representative of the equivalence class under $\sim$. For example, we can stipulate that $h_1(0) = 0$, obtaining a random distribution *not* modulo additive constant. Note that we consider two *random* distributions (on the same probability space) to be the same modulo additive constant if their difference is almost surely a constant distribution, i.e. constant in the spatial variable z; this constant need not be deterministic. Thus, if we have two ways of fixing the additive constant of a whole-plane GFF – say, the normalizations $h_1(0) = 0$ and $h_e(0) = 0$ – their difference need not be a deterministic constant (indeed, in this case it is a standard Gaussian). We say that a random distribution $\widehat{h}$ on $\mathbb{C}$ (*not* modulo additive constant) is a **whole-plane Gaussian free field plus a continuous function** if there exists a coupling of $\widehat{h}$ with a whole-plane GFF h (with the additive constant fixed in some way) such that $\widehat{h} - h$ is almost surely a continuous function; note that this definition does not depend on how the additive constant for h is fixed.

Instead of fixing the additive constant, we can consider the **whole-plane GFF modulo** $r > 0$ by changing the equivalence relation to only identify distributions that differ by a constant $a \in r\mathbb{Z}$. Fixing $\varphi_0 \in H_s(\mathbb{C})$ with $\int_{\mathbb{C}} \varphi_0 = 1$, an instance h of the whole-plane GFF modulo r is constructed by first sampling h a whole-plane GFF modulo additive constant, and independently choosing $U \in [0, r)$ uniformly at random and requiring that $(h, \varphi_0) \in U + r\mathbb{Z}$.

We will also need the notion of the **free-boundary Gaussian free field** on a domain D with harmonically non-trivial boundary, as defined in [She16a, §3.2]. The free-boundary GFF is defined in the same way as the zero-boundary GFF but with $H(D)$ replaced by the Hilbert space closure $H^F(D)$ of the space of smooth functions whose gradients are in $L^2(D)$, considered modulo additive constant (these functions need not be compactly supported). Note that we have to consider functions only modulo additive constant in order for the Dirichlet inner product to be positive definite on this space. Note also that, since it is constructed as a limit (in a Sobolev space or space of distributions) of functions modulo additive constant, the free-boundary GFF is a distribution modulo additive constant.

We note for later reference some key properties of the GFF. Firstly, it is straightforward to check that the Dirichlet inner product is conformally invariant in two dimensions, from which it follows that the GFF is also conformally invariant. In particular the whole-plane

GFF, and the free-boundary GFF on $\mathbb{H}$, are invariant under scalings and translations (when considered modulo additive constant). Secondly, one has the *domain Markov property* [She07, §2.6]; for a zero-boundary GFF in D, this states that if $U \subseteq D$ is open, then we can write $h = h_1 + h_2$ where h_1 is a zero-boundary GFF on U and h_2 is a random harmonic function independent of h_1. This holds because $H(D)$ is the orthogonal direct sum of the space $H(U)$ and the subspace $H_{\mathrm{harm}}(U)$ of $H(D)$ given by functions that are harmonic in U, so that one can define h_1 and h_2 as the orthogonal projections of h onto, respectively, $H(U)$ and $H_{\mathrm{harm}}(U)$. Independence of h_1 and h_2 follows by taking the basis for $H(D)$ in (2.1.1) to be a union of bases for $H(U)$ and $H_{\mathrm{harm}}(U)$. Note that the domain Markov property also holds if h is instead a whole-plane GFF, or if h is a free-boundary GFF on $\mathbb{H}$ and $U = \mathbb{H}$. In these cases h_2 will only be defined modulo additive constant – this will be discussed further at the beginning of §3.1.

If h is a free-boundary GFF on $\mathbb{H}$, $x \in \partial\mathbb{H}$ and $\varepsilon > 0$, we denote by $h_\varepsilon(x)$ the **semicircle average** of h on the semicircular arc $\partial B(x, \varepsilon) \cap \mathbb{H}$, defined as $(h, \rho^+_{x,\varepsilon}) = -2\pi(h, \Delta^{-1}\rho^+_{x,\varepsilon})_\nabla$ where $\rho^+_{x,\varepsilon}$ is the uniform probability measure on $\partial B(x, \varepsilon) \cap \mathbb{H}$. Note that we are using the same notation for semicircle and circle averages, since for fields defined on $\mathbb{H}$ we will usually consider semicircle averages and for fields defined on $\mathbb{C}$ we will usually consider circle averages. If h is defined on D, we can simply define $h_\varepsilon(z)$ to mean the average of h on $\partial B(z, \varepsilon) \cap D$, which covers both these cases. However, for a field h defined on $\mathbb{H}$, we will make it clear when we are considering circle averages as opposed to semicircle averages by writing $h^{\mathrm{circ}}_\varepsilon(z)$ for the average of h over the circle $\partial B(z, \varepsilon)$ (when $B(z, \varepsilon) \subset \overline{\mathbb{H}}$).

By [DMS21, Lemma 4.9], another orthogonal decomposition of $H(\mathbb{C})$ is given by the **radial–lateral decomposition** into the space $H_{\mathrm{rad}}(\mathbb{C})$ of radially symmetric functions and the space $H_{\mathrm{lat}}(\mathbb{C})$ of functions with mean zero on all circles centred at 0. We define the *radial part* $h^{\mathrm{wp}}_{\mathrm{rad}}$ of a whole-plane GFF h^{wp}, given by the projection of h^{wp} onto $H_{\mathrm{rad}}(\mathbb{C})$, as the function $h^{\mathrm{wp}}_{|\cdot|}(0)$ whose value on each circle centred at 0 is simply given the average of h on that circle (and is only defined modulo additive constant). We also define the *lateral part* $h^{\mathrm{wp}}_{\mathrm{lat}}$ of h^{wp} as the projection of h^{wp} onto $H_{\mathrm{lat}}(\mathbb{C})$, which is given by $h^{\mathrm{wp}} - h^{\mathrm{wp}}_{|\cdot|}(0)$ and is well-defined not just modulo additive constant. Then the radial–lateral decomposition implies that $h^{\mathrm{wp}}_{\mathrm{rad}}$ and $h^{\mathrm{wp}}_{\mathrm{lat}}$ are independent.

One also has a radial–lateral decomposition for the free-boundary GFF on $\mathbb{H}$. Indeed, $H^F(\mathbb{H})$ is the orthogonal sum of the space $H^F_{\mathrm{rad}}(\mathbb{H})$ of functions that are radially symmetric about 0 and $H^F_{\mathrm{lat}}(\mathbb{H})$ of functions that have the same average on all semicircles centred at 0 (recall that elements of $H^F(\mathbb{H})$ are only defined modulo additive constant). Note that the radial part, i.e. the projection h^{rad} of h onto $H^F_{\mathrm{rad}}(\mathbb{H})$, whose values are given by the semicircle average process centred at 0, is only defined modulo additive constant, but we can consider the lateral part $h^{\mathrm{lat}} = h - h^{\mathrm{rad}}$ as a function not just modulo additive constant, whose average is zero on every semicircle centred at 0. Again h^{rad} and h^{lat} are independent.

Finally one can consider the radial–lateral decomposition for the free-boundary GFF $\widetilde{h}$ on the bi-infinite strip $\mathscr{S} = \mathbb{R} \times [0, \pi]$, which by conformal invariance can be obtained as $\widetilde{h}(\cdot) = h(\exp(\cdot))$ for h a free-boundary GFF on $\mathbb{H}$. In this case the orthogonal decomposition of $H^F(\mathscr{S})$ is given [DMS21, Lemma 4.3] by the space $H^F_{\mathrm{rad}}(\mathscr{S})$ of functions that are constant on the vertical line $u + [0, i\pi]$ for each $u \in \mathbb{R}$ and the space $H^F_{\mathrm{lat}}(\mathscr{S})$ of functions that have the same average on all such vertical lines. A similar decomposition holds for the bi-infinite cylinder $\mathscr{C}$ given by $\mathbb{R} \times [0, 2\pi]$ with $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\pi\}$ identified.

We will show in Lemma 3.1.1 that, for x fixed, the process $(h_{e^{-t}}(x) - h_1(x))_{t\in\mathbb{R}}$ has the covariance structure of $\sqrt{2}$ times a standard two-sided Brownian motion, and that the semicircle average process $(h_{e^{-t}}(x) - h_1(x))_{t,x\in\mathbb{R}}$ has a version that is continuous (in both t and x). This is a straightforward adaptation of [HMP10, Prop. 2.1], the analogous result for *circle* averages of a zero-boundary GFF.

2.2 Liouville quantum gravity

Given a domain $D \subseteq \mathbb{C}$ and h some form of the GFF on D (with the additive constant fixed in some way if necessary), and $\gamma \in (0, 2)$, we define the random area measure μ_h on D as a Gaussian multiplicative chaos measure in the sense of [Kah85], given by the weak limit of the regularized measures

$$\mu_h^\varepsilon := \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz$$

as $\varepsilon \to 0$ along powers of two, where dz is Lebesgue measure on D and $h_\varepsilon(z)$ is the average of h on the circle of radius ε centred at z (or on the intersection of this circle with D when $z \in \partial D$). This limit was shown to exist almost surely in [DS11]. Likewise, [DS11] shows the almost sure existence of the corresponding weak limit ν_h of the measures

$$\nu_h^\varepsilon := \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(x)/2}\, dx$$

where dx is Lebesgue measure on a linear segment of ∂D.. The regularization procedure implies the conformal coordinate change rule (1.1.2) given in the introduction, under which, by [DS11, Prop. 2.1], almost surely we have $\mu_h = \mu_{\widetilde{h}} \circ \psi$ and $\nu_h = \nu_{\widetilde{h}} \circ \psi$. In particular, we can use this to define ν_h when ∂D is not piecewise linear by conformally mapping to, for example, the upper half-plane (provided the conformal map extends to a homeomorphism $\partial D \to \mathbb{R} \cup \{\infty\}$).

2.2.1 Quantum wedges and cones

We define a **quantum surface** as an equivalence class of objects of the form (D, h) where D is a planar domain and h is a random distribution on D, where $(\widetilde{D}, \widetilde{h})$ and (D, h) are considered equivalent if and only if there exists a conformal map $\psi : \widetilde{D} \to D$ such that $\widetilde{h}$ and h satisfy the rule (1.1.2). Often one also wants to keep track of certain *marked points*; to this end we define a **quantum surface with k marked points** as an equivalence class of objects of the form $(D, h, z_1, \ldots, z_k)$ where $z_i \in \overline{D}$, so that two quantum surfaces $(D, h, z_1, \ldots, z_k)$ and $(\widetilde{D}, \widetilde{h}, \widetilde{z}_1, \ldots, \widetilde{z}_k)$ such that the conformal map $\psi : \widetilde{D} \to D$ satisfies the rule (1.1.2) are only considered equivalent as surfaces with k marked points if in addition we have $\psi(\widetilde{z}_i) = z_i$ for $i = 1, \ldots, k$.

We will now define the notion of "quantum wedge"; the idea is that we would like to define a quantum surface homeomorphic to $\mathbb{H}$, whose law is invariant under scaling and under the operation of adding a constant to the field, and thus a good candidate for infinite-volume scaling limits. As a warm-up we will define an "unscaled quantum wedge", for which the field is only defined modulo additive constant, but keep in mind that the ordinary quantum wedge does not arise by fixing this constant, since such a surface would not have the desired invariance properties.

An **unscaled α-quantum wedge** is given by $(\mathbb{H}, h^F - \alpha \log|\cdot|, 0, \infty)$ where h^F is an instance of the free-boundary GFF on $\mathbb{H}$. (Note that this h^F is only defined modulo additive constant, meaning that μ_h and ν_h are only defined modulo multiplicative constant and thus the unscaled wedge is *not* a quantum surface by our definition above.) The definition arises (as does the nomenclature) by considering a free-boundary GFF on an infinite wedge $W_\vartheta = \{z \in \mathbb{C} : \arg z \in [0, \vartheta]\}$ (viewed as a Riemann surface, so that the parametrization is not single-valued if $\vartheta \geq 2\pi$), and then using (1.1.2) to reparametrize by $\mathbb{H}$ via the conformal map $z \mapsto z^{\pi/\vartheta}$, where $\vartheta = \pi(1 - \alpha/Q)$.

We can reparametrize by the infinite strip $\mathscr{S} = \mathbb{R} \times [0, \pi]$ instead of by $\mathbb{H}$. If we use an appropriate branch of log to map $\mathbb{H}$ to $\mathscr{S}$, so that 0 maps to $-\infty$ whilst ∞ maps to $+\infty$, then the conformal coordinate change formula (1.1.2) gives that the mean of the resulting field $\widetilde{h}$ on the vertical segment $\{t\} \times [0, \pi]$ is given by $B_{2t} + (Q - \alpha)t$, where B is a standard two-sided Brownian motion, defined *modulo additive constant*. We next define an ordinary **quantum wedge** [DMS21, Def. 4.5] by replacing the process $B_{2t} + (Q - \alpha)t$ by a related but different process, in such a way that we fix the additive constant and thus obtain a genuine quantum surface, whose *law* will nonetheless be invariant under the operation of adding a constant to the field. Namely, define an α-quantum wedge by $(\mathscr{S}, \widehat{h}, -\infty, +\infty)$ where $\widehat{h}$ is obtained from $\widetilde{h}$ by replacing the process $B_{2t} + (Q - \alpha)t$ by $(A_t)_{t \in \mathbb{R}}$, where for $t \leq 0$ we define $A_t = B_{-2t} + (Q - \alpha)t$ for B a standard Brownian motion started from 0, and for $t > 0$ we define $A_t = \widehat{B}_{2t} + (Q - \alpha)t$ where $\widehat{B}$ is a standard Brownian motion started from 0 independent of B and conditioned on the event that $\widehat{B}_{2t} + (Q - \alpha)t > 0$ for all $t > 0$.

This is called the **circle average embedding** since it has the property that, when we use $z \mapsto \exp(z)$ to map from $\mathscr{S}$ back to $\mathbb{H}$ to produce a different parametrization of the surface, namely $(\mathbb{H}, h, 0, \infty)$ where $h = \widehat{h} \circ \log - Q \log |\cdot|$, we have

$$0 = \sup\{t \in \mathbb{R} : h_{e^t}(0) + Qt = 0\},$$

where $h_r(z)$ is the semicircle average on $\partial B(z, r)$. One can next construct the circle average embedding of $h + C$ where C is a constant by spatially rescaling by e^{t^C}, where we define $t^C = \sup\{t \in \mathbb{R} : h_{e^t}(0) + Qt + C = 0\}$ – note that by (1.1.2) this corresponds to replacing the field $h + C$ by $h(e^{t^C} \cdot) + Qt^C + C$. From the properties of Brownian motion with drift, one can then check [DMS21, Prop. 4.7(i)] that a quantum wedge has the key property that its law as a quantum surface is invariant under the operation of adding a constant to the field, i.e. the circle average embeddings of h and $h + C$ have the same law for a constant $C > 0$. One can also observe the convenient property that if $(\mathbb{H}, h, 0, \infty)$ is the circle average embedding of an α-quantum wedge, then the restriction of h to $\mathbb{H} \cap \mathbb{D}$ (where $\mathbb{D}$ is the unit disc) has the same law as the restriction of $h^{F,0} - \alpha \log |\cdot|$ to $\mathbb{H} \cap \mathbb{D}$, where $h^{F,0}$ is a free-boundary GFF on $\mathbb{H}$ with the additive constant fixed so that the semicircle average $h_1^{F,0}(0)$ is 0.

Since the conditioning event has probability zero, some care is needed to define the process $\widehat{B}$; the details, given in [DMS21, Remark 4.4], are as follows. The process can be constructed by setting $\widehat{B}_{2t} + (Q - \alpha)t = \widetilde{B}_{2(t+\tau)} + (Q - \alpha)(t + \tau)$ for all $t \geq 0$, where $\widetilde{B}$ is a standard Brownian motion started from 0 and τ is the last time that $\widetilde{B}_{2t} + (Q-\alpha)t$ hits 0. Note that $\tau < \infty$ almost surely since $Q > \alpha$. Then $\widehat{B}$ is characterized by the property that, for each $\varepsilon > 0$, if τ_ε is the hitting time of ε by $\widehat{B}$ then $(\widehat{B}_{2(t+\tau_\varepsilon)} + (Q-\alpha)(t+\tau_\varepsilon))_t$ has the law of a Brownian motion with drift $Q-\alpha$ started from ε and conditioned not to hit 0, which makes the law of $\widehat{B}$ the only sensible choice for the required conditional law. The reason this property characterizes the law of $\widehat{B}$ is that if X is another process with the same property and, for each $\varepsilon > 0$, $\widetilde{\tau}_\varepsilon$ is the hitting time of ε by X, then for each $\varepsilon > 0$ there is a coupling of $\widehat{B}$ and X so that $(\widehat{B}_{2(t+\tau_\varepsilon)})_t = (X_{2(t+\widetilde{\tau}_\varepsilon)})_t$, whereas $\tau_\varepsilon, \widetilde{\tau}_\varepsilon \to 0$ as $\varepsilon \to 0$ almost surely, so that in any subsequential limit of such couplings as $\varepsilon \to 0$ we have $\widehat{B} = X$ almost surely. In fact, one can also define A_t (see [DMS21, §1.1.2]) as the log of a Bessel process of dimension $2 + 2(Q-\alpha)/\gamma$, parametrized by quadratic variation; this definition also makes sense for $\alpha = Q$. A surface constructed as above (with $\alpha \leq Q$) is an α-**quantum wedge**; we refer to such wedges, which are homeomorphic to $\mathbb{H}$, as being **thick**.

The Bessel process construction generalizes further, to the case $\alpha \in (Q, Q+\gamma/2)$. In this case the Bessel process has dimension between 1 and 2 so will hit 0; we obtain one surface for each excursion of the Bessel process away from 0, and thus by concatenating all these surfaces (see [DMS21, §1.1.2]) we get an infinite chain called a **thin quantum wedge** (in this case, the horizontal translation is fixed by requiring the process to attain a maximum at $t = 0$). More formally, we use the fact [RY99, Ch. XI, XII] that the excursions of a Bessel process X form a Poisson point process when indexed by local time at 0. Specifically, for each excursion $\mathbf{e}$ of X (say, over the time interval $(a_\mathbf{e}, b_\mathbf{e})$), if $s_\mathbf{e}$ is the local time at 0 accumulated by $X|_{[0,a]}$, then the $(s_\mathbf{e}, \mathbf{e})$ form a Poisson point process with mean measure $ds \otimes N$ where ds is Lebesgue measure on $[0,\infty)$ and N is an infinite measure, the so-called **Itô excursion measure** corresponding to X (on the space of excursions translated back in time so as to start at time 0). It is known that this process determines X. We thus define an α-quantum wedge for $\alpha \in (Q, Q+\gamma/2)$ as a point process where the points are of the form $(s_\mathbf{e}, \mathbf{e}, h_\mathbf{e})$ where each $h_\mathbf{e}$ is a quantum surface defined on the strip $\mathscr{S}$ as for a thick quantum wedge but using $\mathbf{e}$ parametrized by quadratic variation (and, for concreteness, with the parametrization chosen so that the maximum is attained at time 0) in place of A_t, and where the lateral parts of the $h_\mathbf{e}$ for different excursions $\mathbf{e}$ are independent. Each doubly marked surface $(\mathscr{S}, h_\mathbf{e}, -\infty, +\infty)$ is a **bead** of the wedge, with the two marked points referred to as the *opening point* $(-\infty)$ and the *closing point* $(+\infty)$.

Since such *beaded quantum surfaces* are no longer parametrized by domains in $\mathbb{C}$, we need to slightly amend the notion of equivalence for such surfaces: a beaded quantum surface is parametrized by a closed set D such that each component of the interior of D together with its prime-end boundary is homeomorphic to a closed disc, and we regard two surfaces parametrized by such sets $\widetilde{D}$, D as equivalent if they are related by the formula (1.1.2) for $\psi : \widetilde{D} \to D$ a homeomorphism that is conformal on each component of the interior of $\widetilde{D}$.

As mentioned in the introduction, we will often refer to an α-quantum wedge, in either the thick or thin regimes, as a **quantum wedge of weight $\mathfrak{w}$** where the weight parameter $\mathfrak{w} > 0$ is defined as

$$\mathfrak{w} = \gamma\left(\frac{\gamma}{2} + Q - \alpha\right).$$

Note that the wedge is thick when $\mathfrak{w} \geq \gamma^2/2$ and thin otherwise. We use the weight parameter because it is additive under the operation of conformally welding two independent wedges according to LQG boundary length to obtain another wedge. Specifically, [DMS21, Thm 1.2] states that if $\mathfrak{w}_1, \mathfrak{w}_2 > 0$ and $\mathfrak{w} = \mathfrak{w}_1 + \mathfrak{w}_2$, when a wedge $\mathcal{W}$ of weight $\mathfrak{w}$ is decorated by η, an independent $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}_1 - 2; \mathfrak{w}_2 - 2)$ from 0 to ∞ (or if $\mathfrak{w} < \gamma^2/2$, a concatenation of independent $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}_1 - 2; \mathfrak{w}_2 - 2)$ curves from the opening point to the closing point of each bead), then the region $\mathcal{W}_1$ (resp. $\mathcal{W}_2$) to the left (resp. right) of η is a wedge of weight $\mathfrak{w}_1$ (resp. $\mathfrak{w}_2$) and $\mathcal{W}_1$ and $\mathcal{W}_2$ are independent as quantum surfaces. Moreover, by [DMS21, Thm 1.4] there is a unique conformal welding of the right-hand side of $\mathcal{W}_1$ to the left-hand side of $\mathcal{W}_2$ according to γ-LQG boundary length, which recovers $\mathcal{W}$ and η.

These results from [DMS21] build on the earlier result [She16a, Thm 1.8] that in the case of a wedge $(\mathbb{H}, h, 0, \infty)$ of weight 4 cut by an SLE_κ η into two wedges of weight 2, for each $t > 0$ the law of the pair of surfaces to the left and right of η is invariant under both the operation $\mathcal{Z}_{-t}$ of cutting only along $\eta([0,t])$ (where η is parametrized by LQG boundary length) and the operation $\mathcal{Z}_t$ of conformally welding the boundary segments $(x_t^-, 0]$ and $[0, x_t^+)$ according to LQG boundary length, where $x_t^\pm$ are defined so that $\nu_h((x_t^-, 0]) = \nu_h([0, x_t^+)) = t$ (in particular, [She16a, Thm 1.8] states that this welding is almost surely unique). The group of transformations $\{\mathcal{Z}_t : t \in \mathbb{R}\}$ is called the **(length) quantum zipper**: for $t > 0$, $\mathcal{Z}_t$ "zips up" the pair of surfaces by t units of LQG boundary length whilst $\mathcal{Z}_{-t}$ "unzips" by t units of LQG boundary length.

We next define the whole-plane analogue of the quantum wedge. An **α-quantum cone** is intuitively the doubly marked quantum surface corresponding to a GFF on the surface homeomorphic to $\mathbb{C}$ obtained by gluing together the sides of W_θ, where $\theta = 2\pi(1-\alpha/Q)$, according to Lebesgue measure. It is given by $(\mathbb{C}, h, 0, \infty)$, where the field h is defined in [DMS21, Definition 4.10] for $\alpha < Q$ by taking the process A_t as for an α-quantum wedge, except with B_{2t} and $\widehat{B}_{2t}$ replaced by B_t and $\widehat{B}_t$ respectively, and then setting h to be the field on $\mathbb{C}$ whose radial part is given by A_t on the circle of radius e^{-t} around 0, and whose lateral part is that of an independent whole-plane GFF. Note that the radial part is only defined modulo additive constant, but we generally fix the constant as we do for a wedge, i.e. by requiring $A_0 = 0$. As before, the law of a quantum cone is invariant under the operation of adding a constant to the field (i.e., the circle average of the resulting cone will have the same law as that of the original one); analogously to the case with wedges, the restriction of the circle average embedding of an α-quantum cone h to the unit disc $\mathbb{D}$ is equal in law to the restriction of $h^{\mathrm{wp}} - \alpha \log|\cdot|$ to $\mathbb{D}$ where h^{wp} is a whole-plane GFF with the additive constant chosen so that the circle average $h_1^{\mathrm{wp}}(0)$ is 0.

Again, instead of using the parameter $\alpha < Q$, we will often refer to an α-quantum cone as a **quantum cone of weight $\mathfrak{w}$** where this time the weight parameter $\mathfrak{w} > 0$ is given by

$$\mathfrak{w} = 2\gamma(Q - \alpha).$$

This choice is convenient because cones of weight $\mathfrak{w}$ are the whole-plane analogues of wedges of weight $\mathfrak{w}$. Specifically, [DMS21, Thm 1.5] states that if a cone $C = (\mathbb{C}, h, 0, \infty)$ of weight $\mathfrak{w}$ is decorated with η, an independent whole-plane $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ from 0, then the surface $\mathcal{W}$ described by $(\mathbb{C} \setminus \eta, h, 0, \infty)$ is a wedge of weight $\mathfrak{w}$, and there is a unique conformal welding of left-hand and right-hand boundary segments of $\mathcal{W}$ according to γ-LQG boundary length, which recovers C and η.

In order to construct a probability measure on finite-volume surfaces, we can first consider the "law" on finite-volume surfaces corresponding in the above constructions to that of a single Bessel excursion. (Note that the "law" of a Bessel excursion is an infinite measure, so the words "law" and "sample" do not have their literal meanings in this setting.) To define a **quantum sphere of weight $\mathfrak{w}$** we first define an infinite measure $\mathcal{N}_{\mathfrak{w}}$ on fields parametrized by the bi-infinite cylinder $\mathscr{C}$ with marked points at $-\infty$ and $+\infty$ as follows. (Recall that $\mathscr{C}$ is given by $\mathbb{R} \times [0, 2\pi]$ with $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{2\pi\}$ identified.) A "sample" h from $\mathcal{N}_{\mathfrak{w}}$ can be obtained by "sampling" a Bessel excursion Z of dimension $2 - 2\mathfrak{w}/\gamma^2$, setting the radial part of h (i.e., the projection of h onto the space of functions that are constant on vertical lines $\{r\} \times [0, 2\pi]$) to be given by $\frac{2}{\gamma} \log Z$ parametrized by quadratic variation, and setting the lateral part of h (i.e., the projection of h onto the space of functions that have the same mean on all vertical lines $\{r\} \times [0, 2\pi]$) to be given by the corresponding projection of a GFF on $\mathscr{C}$ (so that the lateral part has mean zero on all vertical lines). One can show that, for $0 < a < b < \infty$, the measure $\mathcal{N}_{\mathfrak{w}}$ assigns finite mass to the event $\mu_h(\mathscr{C}) \in [a, b]$, allowing us to construct the probability measure $\mathcal{N}_{\mathfrak{w}}(\cdot | \mu_h(\mathscr{C}) = r)$ as a regular conditional probability for almost every $r > 0$. The scaling properties of $\mathcal{N}_{\mathfrak{w}}$ mean that in fact this measure must exist for every $r > 0$, and we can thus define the law of a **unit area quantum sphere** as the law of a quantum sphere of weight $4 - \gamma^2$ conditioned to have unit area, i.e. as the probability measure $\mathcal{N}_{4-\gamma^2}(\cdot | \mu_h(\mathscr{C}) = 1)$. (The weight $4 - \gamma^2$, corresponding to a γ-quantum cone, is special because in this case the marked points at 0 and ∞ "look like" *quantum typical* points, i.e. ones sampled according to the measure μ_h – see [DMS21, Lemma A.10].) This argument for the existence of the conditional law appears in the discussion after [DMS21, Definition 4.21].

2.2.2 The subcritical Liouville quantum gravity metric

In [GM21c, Thm 1.2] it is proven that for $\gamma \in (0, 2)$ there exists a measurable map $h \mapsto \mathfrak{d}_h$, from the space of distributions on $\mathbb{C}$ with its usual topology to the space of metrics on $\mathbb{C}$ that induce the Euclidean topology, that is characterized by satisfying the following axioms whenever h is a whole-plane GFF plus a continuous function:

Length space Almost surely, the $\mathfrak{d}_h$-distance between any two points of $\mathbb{C}$ is the infimum of the $\mathfrak{d}_h$-lengths of continuous paths between the two points.

Locality If $U \subseteq \mathbb{C}$ is deterministic and open, then the **internal metric $\mathfrak{d}_h(\cdot, \cdot; U)$ of $\mathfrak{d}_h$ on U**, defined between two points of U by taking the infimum of the $\mathfrak{d}_h$-lengths of continuous paths between the two points that lie entirely in U, is almost surely determined by $h|_U$.

Weyl scaling Let $\xi = \gamma / d_\gamma$ where d_γ is the fractal dimension defined in [DG20]. Then for $f : \mathbb{C} \to \mathbb{R}$ continuous and $z, w \in \mathbb{C}$, define

$$(e^{\xi f} \cdot \mathfrak{d}_h)(z, w) = \inf_P \int_0^{\text{length}(P; \mathfrak{d}_h)} e^{\xi f(P(t))} \, dt$$

where P ranges over all continuous paths from z to w parametrized at unit $\mathfrak{d}_h$-speed. Then, almost surely, $e^{\xi f} \cdot \mathfrak{d}_h = \mathfrak{d}_{h+f}$ for all continuous f.

Affine coordinate change For each fixed deterministic $r > 0$ and $z \in \mathbb{C}$ we almost surely have, for all $u, v \in \mathbb{C}$,

$$\mathfrak{d}_h(ru + z, rv + z) = \mathfrak{d}_{h(r\cdot+z)+Q\log r}(u, v).$$

This map is unique in the sense that for any two such objects $\mathfrak{d}$, $\widetilde{\mathfrak{d}}$, there is a deterministic constant C such that whenever h is a whole-plane GFF plus a continuous function, almost surely we have $\mathfrak{d}_h = C\widetilde{\mathfrak{d}}_h$. We refer to this unique (modulo multiplicative constant) object as the **(whole-plane) γ-LQG metric**. Following [GM21c] we fix the constant so that the median distance between the left and right boundaries of $[0, 1]^2$ is 1 when h is a whole-plane GFF normalized so that $h_1(0) = 0$. Existence is proven by constructing the metric as a subsequential limit of the **ε-Liouville first passage percolation metric** defined by

$$\mathfrak{d}_h^\varepsilon(z, w) = \inf_P \int_0^1 e^{\xi(h*p_{\varepsilon^2/2})(P(t))}|P'(t)|\,dt$$

where the infimum is over all piecewise C^1 paths from z to w, and $p_{\varepsilon^2/2}$ is the heat kernel with variance $\varepsilon^2/2$ (so we are using a mollified version of h). Existence of such subsequential limits was shown in [DDDF20]; subsequently the paper [GM21c] proved that such subsequential limits are unique and characterized by the above axioms, and in [GM21a] it was established that the resulting metric $\mathfrak{d}$ has a *conformal covariance* property. Noting that we can, for instance, use the domain Markov property to write a zero-boundary GFF $\mathring{h}$ on a proper domain $U \subset \mathbb{C}$ as the restriction of a whole-plane GFF h to U plus a continuous function f, we can define the γ-LQG metric $\mathfrak{d}_{\mathring{h}}$ on U corresponding to $\mathring{h}$ as the internal metric $\mathfrak{d}_{h+f}(\cdot, \cdot; U)$, and thus also define $\mathfrak{d}_{\mathring{h}+g}$ for g continuous on U via Weyl scaling. We will review this construction in more detail at the beginning of §3.1. Then, if U, V are domains and $\phi : U \to V$ is conformal, and h is a GFF on U plus a continuous function, the conformal covariance property states that almost surely

$$\mathfrak{d}_{h^U}(z, w) = \mathfrak{d}_{h^U\circ\phi^{-1}+Q\log|(\phi^{-1})'|}(\phi(z), \phi(w))$$

for all $z, w \in U$.

The reason the scaling in the axiomatic definition of $\mathfrak{d}_h$ is controlled by ξ, rather than γ, is that, since adding a constant C to h scales μ_h by $e^{\gamma C}$, it should be true that $\mathfrak{d}_h$ is scaled by $e^{\xi C}$, where $\xi := \gamma/d_\gamma$ and d_γ is the Hausdorff dimension of the γ-LQG metric. In order to define the metric $\mathfrak{d}_h$, a candidate d_γ was needed to state the scaling axiom. For each $\gamma \in (0, 2)$ there is such a value, defined in [DG20], which describes distances in certain discrete approximations of γ-Liouville quantum gravity. A posteriori, it was shown in [GP22] that d_γ is indeed the Hausdorff dimension of the γ-LQG metric.

2.2.3 The critical and supercritical cases

As mentioned, existence and uniqueness of the γ-LQG metric was extended to the critical case $\gamma = 2$ in [DG23], as well as the supercritical case corresponding to $\gamma \in \mathbb{C}$ with $|\gamma| = 2$. We do not treat the supercritical case here, since it has *singular points* that are at distance ∞ from all other points and thus, while lower semicontinuous, fails to induce the Euclidean topology; however, the critical 2-LQG metric was shown to induce the Euclidean topology in [DG21] and satisfies the same axiomatic characterization as in the subcritical case, except with ξ replaced by $\xi_c := \lim_{\gamma \uparrow 2} \gamma/d_\gamma$.

2.3 Schramm–Loewner evolutions

2.3.1 Chordal SLE

Firstly we recall (e.g., from [Law05, Def. 6.1]) the construction of chordal SLE from 0 to ∞ in $\mathbb{H}$ using the *chordal Loewner equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z \tag{2.3.1}$$

where $U\colon [0, \infty) \to \mathbb{R}$ is a continuous function. Here U is the so-called *(Loewner) driving function*. For each fixed $z \in \mathbb{H}$ the *Loewner flow*, i.e. the solution to (2.3.1), is defined up to $\tau(z) = \inf\{t \geq 0 : \operatorname{Im}(g_t(z)) = 0\}$. If we define the compact hull $K_t = \overline{\{z \in \mathbb{H} : \tau(z) \leq t\}}$, then g_t is the unique conformal map from $\mathbb{H} \setminus K_t$ to $\mathbb{H}$ that satisfies the *hydrodynamic normalization* $g_t(z) - z \to 0$ as $z \to \infty$. (We also say that a conformal map $f : D \to \widehat{D}$ between unbounded domains "looks like the identity at ∞" if it satisfies $f(z) - z \to 0$ as $z \to \infty$.)

When $U_t = \sqrt{\kappa} B_t$ for some multiple $\kappa > 0$ of a standard Brownian motion (B_t), there almost surely exists a curve η parametrized by $t \in [0, \infty)$ such that for each t, $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \eta([0, t])$ and $g_t(\eta(t)) = U_t$; we say that η *generates* the family of hulls $(K_t)_{t\geq 0}$. Moreover, the curve η is determined by U. This was proven for $\kappa \neq 8$ in [RS05]; the case $\kappa = 8$ was proven in [LSW03] as a consequence of the convergence of the uniform spanning tree Peano curve, but a proof has since been given in [AM22] for the $\kappa = 8$ case which does not rely on discrete models. The law of η is, by definition, that of a **chordal** SLE_κ **in** $\mathbb{H}$ **from** 0 **to** ∞. The one-parameter family of SLE_κ laws for $\kappa > 0$ has three distinct phases. When $\kappa \in (0,4)$, the curve η is almost surely simple and does not hit $\partial\mathbb{H}$ other than at its endpoints. When $4 < \kappa < 8$, η almost surely does hit $\partial\mathbb{H}$ infinitely often, and has a dense set of double points, but does not cross itself [RS05]; in this phase η *swallows* points, i.e. disconnects them from ∞ without hitting them. When $\kappa \geq 8$, η is almost surely space-filling.

The Markov property of Brownian motion implies that SLE_κ has a *conformal Markov property* [RS05, Thm 2.1(ii)]: given $\eta|_{[0,t]}$, the conditional law of the image of $\eta|_{[t,\infty)}$ under the map $g_t - U_t$ is the same as the law of the whole curve η. The scale invariance of Brownian motion, and the fact that the only conformal automorphisms of $\mathbb{H}$ that fix 0 and ∞ are scalings, imply that SLE is conformally invariant up to time reparametrization, so that by applying a conformal map chordal SLE can be defined (up to time reparametrization) between any two distinct boundary points in any simply connected proper domain.

This definition can be generalized [LSW03, §8.3] to the $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ processes where $\kappa > 0$ and $\rho_1,\rho_2 > -2$, a variant where one additionally keeps track of marked points known as *force points*. The $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ process (with force points at 0^- and 0^+) is defined from 0 to ∞ in $\mathbb{H}$ using (2.3.1), where this time U_t satisfies the SDE

$$dU_t = \sqrt{\kappa}dB_t + \left(\frac{\rho_1}{U_t - V_t^1} + \frac{\rho_2}{U_t - V_t^2}\right)dt, \quad dV_t^1 = \frac{2}{V_t^1 - U_t}dt, \quad dV_t^2 = \frac{2}{V_t^2 - U_t}dt$$

with initial conditions $V_0^1 = U_0 = V_0^2 = 0$ and the further condition that $V_t^1 \leq U_t \leq V_t^2$ for all $t \geq 0$. To motivate the equations for V_t^1 and V_t^2, observe that for any Loewner flow (g_t) from 0 in $\mathbb{H}$ driven by a continuous function U_t, if we define

$$x_t = \sup\{g_t(x) : x < 0, x \notin K_t\}, \quad y_t = \inf\{g_t(x) : x > 0, x \notin K_t\}$$

(noting that the x values quantified over are simple boundary points of $\mathbb{H} \setminus K_t$ and thus g_t

extends continuously to them), then (2.3.1) gives

$$\partial_t x_t = \frac{2}{x_t - U_t}, \quad \partial_t y_t = \frac{2}{y_t - U_t},$$

which means in this case that V_t^1 and V_t^2 can be seen as the images of 0^- and 0^+ (i.e. the left-hand and right-hand prime ends of $\mathbb{H} \setminus K_t$ corresponding to 0) under g_t. We think of the two extra terms in the SDE for U_t as providing "forces" causing the force points to either repel (for positive ρ values) or attract (for negative ρ values) the driving function U_t.

As before, the resulting family of hulls turns out to be generated by a continuous curve η, with $g_t(\eta(t)) = U_t$ [MS16a, Thm 1.3]. This defines the law of an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ curve. If $x_L \leq 0 \leq x_R$, one obtains the same result for the $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ *process with force points at* x_L *and* x_R given by replacing the initial conditions with

$$x_L = V_0^1 \leq U_0 = 0 \leq V_0^2 = x_R.$$

If $\rho_1 = 0$ (resp. $\rho_2 = 0$), the process is known as $\mathrm{SLE}_\kappa(\rho)$ where $\rho = \rho_2$ (resp. $\rho = \rho_1$). (Note that if $\rho_1 = \rho_2 = 0$ we have an ordinary SLE_κ.)

The driving function U_t of an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ process still satisfies Brownian scaling, and thus we have conformal invariance and can define $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ between any two distinct boundary points of any simply connected proper domain. The conformal Markov property changes slightly: given $\eta|_{[0,t]}$, the conditional law of the image of $\eta|_{[t,\infty)}$ under the map $g_t - U_t$ is that of an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ process with the force points at $V_t^1 - U_t$ and $V_t^2 - U_t$.

Although in the case $\kappa \leq 4$ ordinary SLE_κ cannot intersect the boundary except at its endpoints, force points with sufficiently negative weights can make $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ processes hit the boundary (although they still do not self-intersect). In particular, an $\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ process from 0 to ∞ in $\mathbb{H}$ almost surely hits $(0, \infty)$ if $\rho_2 < \kappa/2 - 2$, but almost surely does not hit $(0, \infty)$ if $\rho_2 \geq \kappa/2 - 2$ (see [MW17, Lemma 2.1]). The analogous result holds with ρ_2 replaced by ρ_1 and $(0, \infty)$ replaced by $(-\infty, 0)$.

2.3.2 Radial, whole-plane and space-filling SLE

As well as chordal SLE, which goes from one boundary point to another, one can consider *radial* SLE, which grows from a boundary point towards an interior point. First we define radial SLE_κ in the unit disc $\mathbb{D}$ targeted at 0 [Law05, Def. 6.20] to be the set of hulls K_t

associated to the family of conformal maps $(g_t)_{t\geq 0}$ solving the **radial Loewner equation**

$$\partial_t g_t(z) = g_t(z)\frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z \tag{2.3.2}$$

driven by $W_t = e^{i\sqrt{\kappa}B_t}$ where B is a standard Brownian motion. This time the maps $g_t : \mathbb{D} \setminus K_t \to \mathbb{D}$ are normalized by requiring $g_t(0) = 0$ and $g_t'(0) > 0$, which as in the chordal case defines a unique choice for each g_t. As with chordal SLE, radial SLE has a generalization with a force point; we define

$$\Psi(z, w) = -z\frac{z + w}{z - w}, \quad \widetilde{\Psi}(z, w) = \frac{\Psi(z, w) + \Psi(1/\overline{z}, w)}{2},$$

and let (W, O) be the solution to the equations

$$\begin{aligned} dW_t &= \left[-\frac{\kappa}{2}W_t + \frac{\rho}{2}\widetilde{\Psi}(O_t, W_t)\right] dt + i\sqrt{\kappa}W_t\, dB_t, \\ dO_t &= \Psi(W_t, O_t)\, dt \end{aligned} \tag{2.3.3}$$

(this solution exists and is unique – see [MS17, §2.1.2]). We can then define a **radial** SLE$_\kappa(\rho)$ as the process associated to the solution (g_t) of (2.3.2) with this driving function W. As before, the family of hulls (K_t) is generated by a continuous curve.

Moreover, we can define a version of radial SLE$_\kappa(\rho)$ in bi-infinite time: the radial SLE$_\kappa(\rho)$ equations (2.3.3) with B a *two-sided* Brownian motion still have a unique solution. If we take W to be the resulting driving function, then there is a family $(\widetilde{g}_t)_{t\in\mathbb{R}}$ of conformal maps onto $\mathbb{C} \setminus \overline{\mathbb{D}}$ that each fix ∞, have positive spatial derivative at ∞, and satisfy (2.3.2) (without the initial condition) [MS17, §2.3]. If we define the hull K_t as the complement of the domain of g_t, then the family $(K_t)_{t\in\mathbb{R}}$ is generated by a **whole-plane** SLE$_\kappa(\rho)$ **from** 0 **to** ∞. Moreover, for $\kappa \in (0, 8)$, one can define a **two-sided whole-plane** SLE$_\kappa$ **from** ∞ **to** ∞ **through** 0 by first sampling a whole-plane SLE$_\kappa(2)$ from 0 to ∞ (call this $\eta|_{[0,\infty)}$) then sampling a chordal SLE$_\kappa$ from 0 to ∞ in an unbounded component of $\mathbb{C} \setminus \eta|_{[0,\infty)}$ (call the time-reversal of this curve $\eta|_{(-\infty,0]}$).

For $\kappa > 4$ and $\rho_1, \rho_2 \in (-2, \kappa/2-2)$, the **space-filling** SLE$_\kappa(\rho_1; \rho_2)$ process was defined in [MS17]. When $\kappa \geq 8$, this coincides with ordinary SLE$_\kappa(\rho_1; \rho_2)$, which as mentioned above is almost surely space-filling. When $\kappa \in (4, 8)$ one starts with an ordinary SLE$_\kappa(\rho_1; \rho_2)$ η' and extends it by sequentially "filling in" the regions η' disconnects from ∞. Indeed, for each component C_i of the complement of η', there is a first time t_i such that $\eta'|_{[0,t_i]}$ disconnects C_i from ∞. We then define the space-filling SLE$_\kappa(\rho_1; \rho_2)$ to hit the points in the range of η' in the same order that η' does, but so that immediately after hitting $\eta'(t_i)$ it traces a C_i-filling SLE$_\kappa$-type loop beginning and ending at $\eta'(t_i)$, constructed using a coupling with the Gaussian free field. This construction is described in [MS17, §1.2.3]. Finally, one can define a **whole-plane space-filling** SLE$_\kappa$ from ∞ to ∞ via the chordal version as explained in [DMS21, Footnote 4]. For $\kappa \in (4, 8)$, one first uses the SLE/GFF coupling to draw SLE-type curves partitioning the plane into a countable collection of pockets, and then concatenates chordal space-filling SLE$_\kappa$ curves in each pocket.

2.3.3 Branchable SLE

If $\kappa \in (2, 8)$ and $\rho \in (-2, \kappa-4)$, SLE$_\kappa(\rho; \kappa-6-\rho)$ has an additional special property, namely that of *target invariance* [SW05]. This means that, given a domain D and a point $z \in \partial D$, one can construct a family of curves $\{\gamma_y : y \in \partial D \setminus \{z\}\}$ such that, for every $y \in \partial D \setminus \{z\}$, γ_y is an SLE$_\kappa(\rho; \kappa - 6 - \rho)$ processes from z to y, with the γ_y coupled in such a way that for distinct points y and y' in $\partial D \setminus \{z\}$, the curve γ_y targeted at y run until the first time γ_y disconnects y from y' almost surely coincides (up to time reparametrization) with the curve $\gamma_{y'}$ targeted at y' run until the first time $\gamma_{y'}$ disconnects y' from y. The family $\{\gamma_y : y \in \partial D \setminus \{z\}\}$ is known as the SLE$_\kappa(\rho; \kappa-6-\rho)$ **branching tree** in D rooted at z and targeted at all boundary points, or as a **branchable** SLE$_\kappa(\rho)$ process or bSLE$_\kappa(\rho)$ process.

This definition can be generalized to define the process SLE$_\kappa(\kappa - 6)$ from 0 to ∞ in $\mathbb{H}$ when $\kappa \in (8/3, 4)$, for which we have $\kappa - 6 < -2$ so that the previous definition does not apply. Here the SDE becomes

$$dU_t = \sqrt{\kappa} dB_t + \frac{\kappa - 6}{U_t - V_t} dt, \quad dV_t = \frac{2}{V_t - U_t} dt$$

with $U_0 = V_0 = 0$. This suggests that $Z_t := (U_t - V_t)/\sqrt{\kappa}$ satisfies the SDE

$$dZ_t = dB_t + \frac{\kappa - 4}{\kappa Z_t} dt. \tag{2.3.4}$$

Since $-B$ is also a Brownian motion, (2.3.4) should also describe the evolution of $|Z_t|$ *when it is away from the origin*, but $|Z_t|$ will not actually solve (2.3.4) since it has to stay non-negative – one can think of this condition as imposing an infinitesimal upward push whenever Z hits the origin (compare the case of a reflected Brownian motion, whose evolution away from the origin is described by the equation $dY_t = dB_t$ but which does not solve this equation).

A **Bessel process of dimension** δ is a process X_t whose law is characterized [WW13, §2.1] by the conditions that X is almost surely non-negative and continuous, that the Lebesgue measure of the zero set of X is zero almost surely, and that while X is away from 0 its evolution is described by the *δ-dimensional Bessel SDE*

$$dX_t = dB_t + \frac{\delta - 1}{2X_t}\, dt. \tag{2.3.5}$$

The term "dimension" is used because if $\delta \in \mathbb{N}$ then the L^2 norm of a standard δ-dimensional Brownian motion is a Bessel process of dimension δ – one can check it satisfies (2.3.5) by applying Itô's formula and using Lévy's characterization of Brownian motion to identify the martingale term.

In our case, $|Z|$ must be a Bessel process of dimension $1 + 2(\kappa - 4)/\kappa = 3 - 8/\kappa$. More generally, a force point of weight ρ corresponds to a Bessel process of dimension $1+2(\rho+2)/\kappa$. This is why the range of ρ for which $\mathrm{SLE}_\kappa(\rho)$ intersects the boundary is given by $\rho < \kappa/2-2$; this range corresponds to the regime $\delta < 2$ for which a Bessel process almost surely hits 0, and the SLE hits the boundary when U_t ad V_t collide.

The reason $\kappa - 6 < -2$ causes an issue in defining the SLE process is that it corresponds to $\delta < 1$. In this case the integral of $1/|Z_t|$ blows up, which is a problem since one wants to define

$$U_t = \sqrt{\kappa} Z_t - 2\int_0^t \frac{ds}{\sqrt{\kappa} Z_s}. \tag{2.3.6}$$

We therefore need to make sense of this integral. It is possible [MSW17, §3.3.1–3.3.2] to make $1/(U_t - V_t)$ well-behaved by introducing *side-swapping*. The **side-swapping** $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ **process**, where $\beta \in [-1, 1]$, is defined by making each excursion of $U_t - V_t$ away from 0 positive with probability $(1 + \beta)/2$ and negative with probability $(1 - \beta)/2$, with the sign of each excursion chosen independently. Positive (resp. negative) excursions correspond to the process trying to grow to the right (resp. left) of the marked point. One then uses a compensation described in [WW13, §2.2] to make sense of the integral of $1/Z_t$ and thus define U_t and V_t via (2.3.6).

Again, this side-swapping $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ has a target-invariance property which allows us to define the $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ **branching tree** (also known as **branchable** $\mathrm{SLE}_\kappa^\beta$, or $\mathrm{bSLE}_\kappa^\beta$).

2.3.4 Natural parametrization

In the above constructions of chordal and radial SLE the curve is said to be *parametrized by capacity*, which is a certain complex-analytic notion of size for the hull K_t. For instance, for a chordal SLE from 0 to ∞ in $\mathbb{H}$, we have $\operatorname{hcap}(K_t) := \lim_{z\to\infty} z(g_t(z) - z) = 2t$, where hcap is the so-called **half-plane capacity**, which has a representation in terms of Brownian motion given by

$$\operatorname{hcap}(K_t) = \lim_{y\to\infty} \mathbb{E}_{iy}[\operatorname{Im} B_\tau],$$

where the expectation $\mathbb{E}_{iy}$ is w.r.t. the law of a complex Brownian motion B started from iy and τ is the first exit time of B from $\mathbb{H} \setminus K_t$. The capacity time parametrization is natural given the construction via the Loewner equation (if the g_t were parametrized differently then, for instance, the 2 in the numerator of the half-plane chordal Loewner equation would be replaced by $\partial_t \operatorname{hcap}(K_t)$). However, other time parametrizations are possible; for instance, when an LQG surface with field h is decorated with an independent space-filling SLE η, one can parametrize by **quantum time**, i.e. so that $\mu_h(\eta([a,b])) = b - a$.

Another possibility is the **natural parametrization**, which is conjecturally the one that arises in the scaling limit of a discrete model converging to SLE in which one parametrizes the discrete interface by the number of edges it traverses. In the space-filling case $\kappa \geq 8$, the natural parametrization is simply given by $m(\eta([a,b])) = b - a$ where m is two-dimensional Lebesgue measure. When $\kappa < 8$, and SLE_κ has Hausdorff dimension $1 + \kappa/8$ [Bef08], the natural parametrization is (a constant multiple of) the $1 + \kappa/8$-dimensional Minkowski content of the curve, as proven in [LR15] – which in particular proves the non-trivial fact that this Minkowski content exists. The natural parametrization had earlier been constructed indirectly, first for $\kappa < 4(7 - \sqrt{33})$ in [LS11] and later for all $\kappa < 8$ in [LZ13], whereas it was later shown to arise as the expectation of LQG boundary measure along the curve w.r.t. an independent GFF in the quantum zipper construction (first for $\kappa < 4$ in [Ben18], then for $\kappa \in (4,8)$ in [MS23b] and finally for $\kappa = 4$ in [MS23a]). Note that when $\kappa \geq 8$ the SLE_κ trace has dimension 2 so the appropriate Minkowski content is just Lebesgue measure.

The natural parametrization has the appropriate conformal covariance property to be a $(1 + \kappa/8)$-dimensional Minkowski content: if η is an SLE_κ in D with the natural parametrization and $\psi\colon D \to \widetilde{D}$ is conformal, then the amount of time the natural parametrization assigns to $\psi(\eta([a,b]))$ is equal to

$$\int_a^b |\psi'(\eta(t))|^{1+\kappa/8}\,dt.$$

Another important property is that the natural parametrization makes the law of two-sided whole-plane SLE shift-invariant. To be precise, let $\kappa \in (0,8)$ and let η be a two-sided whole-plane SLE_κ from ∞ to ∞ through 0 with the natural parametrization normalized such that $\eta(0) = 0$. Then by [Zha21, Cor. 4.7], for each $r > 0$ the scaling map

$$S_r : \eta(\cdot) \mapsto r^{-1/(1+\kappa/8)}\eta(r\cdot)$$

and the translation map

$$T_r : \eta(\cdot) \mapsto \eta(\cdot + r) - \eta(r)$$

are measure-preserving w.r.t. the law of η. In fact, we will show (Prop. 1.3.13) that S_r is ergodic for every $r > 0$.

2.4 Conformal loop ensembles

2.4.1 The branching tree construction

For $\kappa \in (4,8)$, one definition of CLE_κ in a domain D (see [MSW20, §2.3]) begins with the $\mathrm{SLE}_\kappa(\kappa-6)$ branching tree in D rooted at some $x \in \partial D$. Given $z \in D$, let η_z be the branch of this tree targeted at z. Let τ_z be the first time at which η_z surrounds z clockwise (i.e., the first time t at which the harmonic measure of the right-hand side of $\eta_z([0,t])$ seen from z is 1). Let σ_z be the largest time t before τ_z at which the harmonic measure seen from z of the right-hand side of $\eta_z([0,t])$ is 0. Then we can form a loop from the exploration tree by concatenating $\eta_z|_{[\sigma_z,\tau_z]}$ with the branch of the tree from $\eta_z(\tau_z)$ to $\eta_z(\sigma_z)$. These loops, for a countable dense collection of points $z \in D$, form a CLE_κ in D. It was proved in [She09] that the loops of a CLE_κ correspond to continuous curves conditionally on the continuity of the $\mathrm{SLE}_\kappa(\kappa-6)$ processes which was subsequently proved in [MS16a]. It was also proved in [She09] that the law of the CLE_κ does not depend on the choice of the root x for the exploration tree conditionally on the reversibility of SLE_κ for $\kappa \in (4,8)$ which was subsequently proved in [MS16c].

When $\kappa \in (8/3,4)$, we can consider the $\mathrm{SLE}_\kappa^\beta(\kappa-6)$ branching tree rooted at some $x \in D$, for some $\beta \in [-1, 1]$, and then define a set of continuous loops as above (again, see [MSW20, §2.3]). Again, the law of this set of loops is independent of the choice of root x; it also turns out to be independent of the choice of the side-swapping parameter β [WW13, Prop. 3]. This is the definition of CLE_κ in D.

Intuitively, the loops correspond to excursions of the Bessel process used in the definition of the $\mathrm{SLE}_\kappa^\beta(\kappa - 6)$ away from 0; the times at which the Bessel process is at 0 correspond to a "trunk" in the SLE branching tree which the loops hang off. (See [MSW17, §4].)

2.4.2 The loop-soup construction

For $\kappa \in (8/3, 4)$ there is an alternate construction of CLE_κ. Given a simply connected planar domain D, one can consider the *Brownian loop-soup* as defined in [LW04]. This is obtained by sampling a Poisson point process of loops in $\mathbb{C}$ with intensity $c\mu$, where $c > 0$ is a constant and μ is the *Brownian loop measure*, a conformally invariant infinite measure on the set of continuous loops in the plane, and then taking those loops which lie entirely in D. These loops can intersect, forming clusters, some of which can be contained inside (i.e., disconnected from ∞ by) other clusters. Sheffield and Werner [SW12] showed that, when $\kappa \in (8/3, 4)$ and $c = (3\kappa - 8)(6 - \kappa)/2\kappa$, the collection of outer boundaries of outermost clusters forms a CLE_κ.

Note that the loop-soup construction satisfies a *restriction* property: if $U \subseteq D$ is a simply connected domain, then the restriction of a Brownian loop-soup in D to U (i.e., the collection of loops of the soup that are entirely contained in U) is a Brownian loop-soup in U. Thus, by using the same point process, we can couple CLE_κ processes on all simply connected planar domains.

2.4.3 Boundary conformal loop ensembles

We will sometimes use an iterative construction of CLE_κ based on the so-called **boundary conformal loop ensemble** processes $\mathrm{BCLE}_\kappa(\rho)$, which are constructed using $\mathrm{bSLE}_\kappa(\rho)$ processes, and which are defined for $\kappa \in (2, 8)$ (even though CLE_κ is not defined everywhere in this range) and ρ in a certain range depending on κ. These $\mathrm{BCLE}_\kappa(\rho)$ processes are supposed to describe the scaling limits of collections of boundary-intersecting loops in certain discrete models, where the parameter ρ corresponds to a kind of boundary condition.

For $\kappa \in (2, 4]$, $\mathrm{BCLE}_\kappa(\rho)$ is defined when $-2 < \rho < \kappa - 4$ as follows [MSW17, §7.1.2]. Take the domain D to be the unit disc (we can define the process for other domains by conformal invariance). The $\mathrm{BCLE}_\kappa(\rho)$ process will be the set of boundary-touching loops traced by a $\mathrm{bSLE}_\kappa(\rho)$ branching tree started from a boundary point x and targeted at all other boundary points. This tree is the union of countably many disjoint arcs from the boundary to itself, each of which has a natural orientation (going from x towards the other boundary points). A fixed point $z \in D$ will then almost surely be surrounded by (i.e., the boundary of its connected component in the complement of the tree will be) a concatenation of such arcs forming either a clockwise or a counterclockwise loop. We will refer to the collection of all such clockwise loops as (the *true loops* of) a $\mathrm{BCLE}^{\circlearrowright}_\kappa(\rho)$ process; the collection of all the counterclockwise loops is the set of *false loops* of this $\mathrm{BCLE}^{\circlearrowright}_\kappa(\rho)$ process.

The result of reversing the orientation of all the (true) loops of a $\mathrm{BCLE}^{\circlearrowright}_\kappa(\rho)$ process will be called (the set of true loops of) a $\mathrm{BCLE}^{\circlearrowleft}_\kappa(\rho)$ process. It follows from the definition of $\mathrm{bSLE}_\kappa(\rho)$ that the collection of false loops of a $\mathrm{BCLE}^{\circlearrowright}_\kappa(\kappa-6-\rho)$ also has the law of the set of true loops of a $\mathrm{BCLE}^{\circlearrowleft}_\kappa(\rho)$ process as just defined (note that $\mathrm{BCLE}_\kappa(\rho)$ is defined if and only if $\mathrm{BCLE}_\kappa(\kappa-6-\rho)$ is). In the light of this, reversibility for $\mathrm{SLE}_\kappa(\rho)$ implies that the law of $\mathrm{BCLE}_\kappa(\rho)$ does not depend on the choice of the root x of the branching tree, making it invariant under conformal automorphisms of the disc; thus the definition extends to other simply connected domains to define a conformally invariant process.

For $\kappa \in (4, 8)$ the definition [MSW17, §7.1.3] is similar, except that we now strengthen the condition on ρ to $\kappa/2-4 < \rho < \kappa/2-2$, which ensures that the $\mathrm{bSLE}_\kappa(\rho)$ does not trace the domain boundary. Note that although, as before, the true and false loops of the $\mathrm{BCLE}^{\circlearrowright}_\kappa(\rho)$ process are obtained by concatenating the boundary-to-boundary arcs of the $\mathrm{bSLE}_\kappa(\rho)$, these loops are no longer simple (since the arcs locally look like SLE_κ in the interior), so that the loops can no longer be viewed as the boundaries of components of the complement of the tree. As in the $\kappa \in (2, 4]$ case, though, it remains true that $\mathrm{BCLE}_\kappa(\rho)$ is defined if and only if $\mathrm{BCLE}_\kappa(\kappa-6-\rho)$ is, and that the set of true loops of a $\mathrm{BCLE}^{\circlearrowleft}_\kappa(\rho)$ process has the same law as the set of false loops of a $\mathrm{BCLE}^{\circlearrowright}_\kappa(\kappa-6-\rho)$ process.

For $\kappa \in (8/3, 4)$, a coupling with the GFF can be used to prove [MSW17, Thm 7.8] that the following procedure constructs a CLE_κ in a simply connected domain D: set $\kappa' = 16/\kappa$ and sample a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$ in D. Then sample independent $\mathrm{BCLE}^{\circlearrowleft}_\kappa(-\kappa/2)$ processes in each of the true (clockwise) loops of the $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$. The true loops of these $\mathrm{BCLE}^{\circlearrowleft}_\kappa(-\kappa/2)$ processes will be part of the CLE_κ, but there are more CLE loops to discover in the hitherto unexplored regions: those bounded by the false loops of the $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$ and $\mathrm{BCLE}^{\circlearrowleft}_\kappa(-\kappa/2)$

processes. Inside each of these unexplored regions we iterate the construction, independently sampling a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$ in each region and then sampling independent $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(-\kappa/2)$ processes in each of its true loops, and so on. (For a given point $z \in D$, the number of such iterations required to find the CLE loop surrounding z is almost surely finite – in fact it is a geometric random variable, since by conformal invariance the probability of success at each iteration step is independent of z.) The entire CLE_κ is given by all the true loops of all the $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(-\kappa/2)$ processes.

For $\kappa \in (4,8)$ there is a slightly simpler construction. The set of true loops of a $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(0)$ process in D has the same law as the set of loops of a CLE_κ in D that intersect the boundary, so to generate the full CLE_κ we can just iteratively sample independent $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(0)$ processes in each of the false loops (see [MSW20, §2.3]).

2.4.4 Continuous percolation interfaces

For $\kappa \in (8/3, 4)$ the trunk in the branching tree construction corresponds to a continuous analogue of a "critical percolation interface", as shown in [MSW17, Prop. 4.1]. Given a CLE_κ Γ in D produced by the $\mathrm{SLE}^{\beta}_{\kappa}(\kappa-6)$ construction, we label a loop *open* (resp. *closed*) if it is traced counterclockwise (resp. clockwise) by the SLE process. (For $\beta = 1$ all loops are traced counterclockwise and for $\beta = -1$ all loops are traced clockwise.) The law of the labelled set of loops (called a $\mathrm{CLE}^{\beta}_{\kappa}$) is that of a CLE_κ in which each loop has been labelled as open (resp. closed) independently with probability $(1+\beta)/2$ (resp. $(1-\beta)/2$).

Then the trunk γ of the $\mathrm{SLE}^{\beta}_{\kappa}(\kappa-6)$ process is a *continuous percolation interface* (CPI), meaning that γ does not cross itself or intersect the interior of any loop, but keeps the open loops on one side and the closed loops on the other, and that a certain *conformal Markov property* is satisfied, which is as follows. Suppose γ goes from x to y. Say we explore $\gamma([0,t])$ and all the loops of Γ it intersects, then obtain a new domain D^0_t from D by removing $\gamma([0,t])$ and all these loops and their interiors. Then if we conformally map the connected component D_t of D^0_t whose boundary contains y back to D with a map that looks like the identity at y, the law of the image of the restriction of (Γ, γ) to D_t given the exploration so far is simply the original law of (Γ, γ). Moreover, given the exploration, the conditional law of Γ (with labels) in each other component D'_t of D^0_t is that of a $\mathrm{CLE}^{\beta}_{\kappa}$ in D'_t.

2.5 Imaginary geometry

In what follows we will often use the fact that SLE and CLE processes in a domain D can be coupled with instances of the Gaussian free field in D with suitable boundary data. The simplest of these constructions is the coupling of SLE_4 as a level line of the GFF [SS13], but we will not need that coupling here and therefore will not discuss it. What we will need is the *imaginary geometry* introduced in [MS16a, MS16b, MS16c, MS17], so called because the objects of study are (formally) the flow lines of the vector field $e^{ih/\chi}$ where h is a GFF and $\chi > 0$, i.e. solutions of the ODE

$$\partial_t \eta(t) = e^{ih\eta(t)/\chi}. \tag{2.5.1}$$

Note that if h were a smooth function in a domain D and η were a flow line of h in the sense of satisfying the above ODE, and we had a conformal map $\psi : D \to \widetilde{D}$, then $\psi \circ \eta$ would be a flow line of the smooth function on $\widetilde{D}$ defined by (the complex exponential of)

$$h \circ \psi^{-1} - \chi \arg (\psi^{-1})'. \tag{2.5.2}$$

Since h is not defined pointwise, the ODE (2.5.1) does not make literal sense, but note that if h were a smooth function and η a solution of (2.5.1), then η would determine the values of h on the trace of η, and changing the values of h off η would not change the fact that η solves (2.5.1). This is the sense in which we can couple a GFF h and a random curve η in a domain D: we first sample η, then sample a GFF h in the complement of the trace of η in D with certain boundary conditions. For appropriate η and suitable boundary conditions, when this h is viewed as a field on all of D, it is in fact a GFF in D with certain boundary data, and η is almost surely determined by h.

For now briefly explain the SLE/GFF couplings in the half-plane case; we will explain the corresponding CLE/GFF couplings in Chapter 5. We will first need the notion of a *local set* for the GFF as introduced in [SS13]. If (A, h) is a coupling of a GFF h on a domain $D \subseteq \mathbb{C}$ and a random non-empty closed set A such that $\partial D \subseteq A \subseteq \overline{D}$, we say that A is a **local set** of h if there exists a law μ, supported on pairs (A, h_1) of a subset of $\overline{D}$ and a distribution on D that is harmonic in $D \setminus A$, such that one can produce a sample from the law of (A, h) by first sampling (A, h_1) from μ, then sampling a zero-boundary GFF h_2 on $D \setminus A$ and setting

$h = h_1 + h_2$. In particular, the domain Markov property of the GFF implies that *deterministic* closed sets are local; indeed local sets are those random sets for which an analogous Markov property holds.

For $\kappa \in (0,4)$, set $\lambda = \pi/\sqrt{\kappa}$ and $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$. Fix $\rho_1, \rho_2 > -2$. Then an $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ process η from 0 to ∞ in $\mathbb{H}$, with associated conformal maps (g_t) and hulls (K_t), and driving function U, can be coupled [MS16a, Thm 1.1] with a GFF h in $\mathbb{H}$ with boundary data

$$\mathfrak{h}_0(x) = \begin{cases} -\lambda(1+\rho_1) & x \in (-\infty, 0), \\ \lambda(1+\rho_2) & x \in (0,\infty), \end{cases}$$

so that K_t is local for h, and if $f_t = g_t - U_t$ (so that $f_t(\eta(t)) = 0$), then the conditional law of h on $\mathbb{H} \setminus \eta([0,t])$ given $\eta|_{[0,t]}$ is the same as the law of

$$h \circ f_t - \chi \arg(f_t'). \tag{2.5.3}$$

Although it is not obvious from the definition, in this coupling, η is almost surely determined by h [MS16a, Thm 1.2]. We say that η is a **flow line of h starting from 0 and targeted at ∞**. Note that (2.5.3) is the coordinate change formula (2.5.2) with $\psi = f_t^{-1}$, so this definition captures the intuition that if η is a flow line of h in $\mathbb{H}$, then the flow line of h in the domain $\mathbb{H} \setminus \eta([0,t])$ started from $\eta(t)$ with the same angle as η should just be the continuation of η, and thus the flow line in $\mathbb{H}$ of the field (2.5.3) started from 0 should be a flow line of this field in the same sense that η is a flow line of h.

This definition extends to domains other than $\mathbb{H}$ by using the conformal coordinate change formula (2.5.2); this can also be used to define flow lines targeting other boundary points. We can also consider *flow lines of angle* θ, which are the flow lines obtained by adding $\theta\chi$ to the boundary data (i.e., a flow line of h of angle θ is an ordinary flow line of $h + \theta\chi$).

For $\kappa \in (0,4)$, we have $\kappa' = 16/\kappa \in (4,\infty)$. When $\rho_1', \rho_2' > -2$ we can obtain $\mathrm{SLE}_{\kappa'}(\rho_1';\rho_2')$ curves as flow lines in the same way as above; however, since $\chi(\kappa') = -\chi(\kappa)$, in order that we can couple SLE_κ and $\mathrm{SLE}_{\kappa'}$ with the same field, we perform a sign change and use $\chi = \chi(\kappa)$ for both, thus coupling $\mathrm{SLE}_{\kappa'}(\rho_1';\rho_2')$ with $-h$; the boundary conditions for $\mathrm{SLE}_{\kappa'}(\rho_1';\rho_2')$ then become

$$\mathfrak{h}_0(x) = \begin{cases} \lambda'(1+\rho_1') & x \in (0,\infty), \\ -\lambda'(1+\rho_2') & x \in (0,\infty). \end{cases}$$

The $\mathrm{SLE}_{\kappa'}(\rho_1'; \rho_2')$ is then referred to as the **counterflow line of h starting from 0**. Again one can define counterflow lines from other boundary points via conformal maps.

2.6 Distortion estimates for conformal maps

We now briefly recall some distortion estimates for conformal maps that will be useful in §5, beginning with the Koebe quarter theorem [Law05, Thm 3.16]:

Lemma 2.6.1 (Koebe quarter theorem)**.** *Let $D \subsetneq \mathbb{C}$ be a simply connected domain and let $f : \mathbb{D} \to D$ be a conformal map. Then $B(f(0), |f'(0)|/4) \subseteq D$.*

As a corollary of the Koebe quarter theorem one can deduce the following [Law05, Cor. 3.18]:

Lemma 2.6.2. *Let $f : D \to \widetilde{D}$ be a conformal map between domains $D, \widetilde{D} \subsetneq \mathbb{C}$. Fix $z \in D$ and let $\widetilde{z} = f(z)$. Then*

$$\frac{\operatorname{dist}(\widetilde{z}, \partial\widetilde{D})}{4 \operatorname{dist}(z, \partial D)} \le |f'(z)| \le \frac{4 \operatorname{dist}(\widetilde{z}, \partial\widetilde{D})}{\operatorname{dist}(z, \partial D)}.$$

Combining the Koebe quarter theorem with the growth theorem [Law05, Thm 3.21] for schlicht functions, one can obtain the following [Law05, Cor. 3.23]:

Lemma 2.6.3. *Let $f : D \to \widetilde{D}$ be a conformal map between domains $D, \widetilde{D} \subsetneq \mathbb{C}$. Fix $z \in D$ and let $\widetilde{z} = f(z)$. Then for all $r \in (0, 1)$ and all $|w - z| \le r \operatorname{dist}(z, \partial D)$,*

$$|f(w) - \widetilde{z}| \le \frac{4|w - z|}{1 - r^2} \frac{\operatorname{dist}(\widetilde{z}, \partial\widetilde{D})}{\operatorname{dist}(z, \partial D)} \le \frac{4r}{(1 - r)^2} \operatorname{dist}(\widetilde{z}, \partial\widetilde{D}).$$

2.7 Quasisymmetric embeddings and Assouad dimension

In preparation for Chapter 4, we discuss the notions of quasisymmetric embedding and Assouad dimension.

Definition 2.7.1. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ an injective function. Let $\Psi : (0, \infty) \to (0, \infty)$ be an increasing homeomorphism. Then f is **Ψ-quasisymmetric** (equivalently, a **Ψ-quasisymmetric embedding**) if for any three distinct points $x, y, z \in X$, we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \le \Psi\left(\frac{d_X(x, y)}{d_X(x, z)}\right). \tag{2.7.1}$$

We say f is **quasisymmetric** (equivalently, a **quasisymmetric embedding**) if there exists some Ψ for which f is Ψ-quasisymmetric.

Recall that *quasiconformal* maps between planar domains are intuitively those that send infinitesimal circles to infinitesimal ellipses of bounded eccentricity. More generally, with X, Y, f as above, and $K \geq 1$, f is K**-quasiconformal** if, for all $x \in X$,

$$\limsup_{r \downarrow 0} \frac{\sup\{d_Y(f(x), f(y)) : d_X(x,y) \leq r\}}{\inf\{d_Y(f(x), f(y)) : d_X(x,y) \geq r\}} \leq K.$$

For open subsets of $\mathbb{R}^n$ with $n \geq 2$, locally quasisymmetric embeddings (i.e. embeddings for which there exists Ψ such that each point has a neighbourhood on which the embedding is Ψ-quasisymmetric) are equivalent to locally quasiconformal embeddings [Väi81, Cor. 2.6]. Indeed, this equivalence holds quantitatively: for $n \geq 2$ and D any domain in $\mathbb{R}^n$, the following holds [Väi81, Thm 2.3]. For each Ψ, if $f : D \to \mathbb{R}^n$ is a locally Ψ-quasisymmetric embedding, then f is in fact K-quasiconformal for some $K \geq 1$ depending only on Ψ and n. Conversely [Väi81, Thm 2.4], for each $K \geq 1$, if $f : D \to \mathbb{R}^n$ is K-quasiconformal, and $x \in D$, $\alpha > 1$, $r > 0$ such that $B(x, \alpha r) \subseteq D$, then $f|_{B(x,r)}$ is Ψ-quasisymmetric for some Ψ depending only on K, n and α. Similar results hold for smooth connected Riemannian manifolds [AB21, Thm 2.6].

Definition 2.7.2. Let X be a metric space. For $E \subseteq X$, let $N_r(E)$ be the smallest possible cardinality of a set of open balls of radius r that cover E. Then the **Assouad dimension** $\dim_A X$ of X is defined by

$$\dim_A X := \inf\{\alpha \geq 0 : \exists C \in (0, \infty) \text{ s.t. } \forall 0 < r < R, \forall x \in X, N_r(B(x,R)) \leq C(R/r)^\alpha\}.$$

In [Tro21] an alternative definition of $\dim_A$ is used that only quantifies over $R < 1$. The Assouad dimension thus defined can be strictly smaller than the one defined in Definition 2.7.2 (for instance they assign 0 and 1 respectively to $\mathbb{Z}$), though they are equal when X is compact. Our results and proofs apply regardless of which definition is used, but we use Definition 2.7.2 since under this definition we have the equivalence ([Fra21, Thm 13.1.1]) that $\dim_A X < \infty$ if and only if X is a **doubling space**, i.e. there exists a finite constant K such that any open ball in X can be covered by at most K open balls of half its radius. (Under the other definition this equivalence fails; for example, under our definition the set of points in ℓ^2 with integer coordinates, which is not doubling, has infinite dimension, but under the other definition it has dimension zero.)

As observed by Coifman and Weiss [CW71, Ch. III, Lemma 1.1], a sufficient condition for a metric space X to be doubling is the existence of a *doubling measure*, that is a Borel measure μ on X for which there is a constant $D > 0$ such that, for all $x \in X$ and $r > 0$,

$$0 < \mu(B(x, 2r)) \leq D\mu(B(x, r)) < \infty.$$

A partial converse holds: whilst noting that $\mathbb{Q}$ is a doubling space for which there is no doubling measure (since the inequality would imply each point had measure zero), Assouad [Ass80] conjectured that every complete doubling space has a doubling measure, which was proven by Luukkainen and Saksman [LS98] building on Vol'berg and Konyagin's [VK87] proof for compact spaces. The Bishop–Gromov inequality ([BC01, §11.10, Corollary 3]; see also [Gro81, §2.1]) straightforwardly implies that, for any complete Riemannian manifold with non-negative Ricci curvature, the measure given by the volume form is doubling, and thus such manifolds are doubling spaces.

Chapter 3

Equivalence of metric gluing and conformal welding in γ-Liouville quantum gravity for $\gamma \in (0,2)$

This chapter is structured as follows. In §3.1 we show that the LQG metric corresponding to a free-boundary GFF on $\mathbb{H}$ extends continuously to a metric on $\overline{\mathbb{H}}$ that is locally Hölder continuous w.r.t. the Euclidean metric. In §3.2 we prove that the LQG metric on the boundary is locally Hölder continuous w.r.t. the LQG boundary measure, and that the Euclidean metric is locally Hölder continuous w.r.t. the LQG metric. In §3.3 we use an SLE/GFF coupling to establish a bound on the amount of LQG area within an LQG-metric neighbourhood of a boundary segment. Finally §3.4 contains the proofs of the main results.

3.1 γ-LQG metric boundary estimates for the free-boundary GFF on $\overline{\mathbb{H}}$

Throughout this section h will be a free-boundary GFF on $\mathbb{H}$, though not always with the same choice of additive constant. Indeed, although the statements of our results require the additive constant for h to be fixed in some way, it is easily seen that all the results of this section remain true regardless of how the constant is fixed, so we will not always specify a choice. In this section we show that the γ-LQG metric induced by h extends continuously to a metric on $\overline{\mathbb{H}} \times \overline{\mathbb{H}}$, and give some estimates for the regularity of this metric, showing that, almost surely, it is locally Hölder continuous with respect to the Euclidean metric on $\overline{\mathbb{H}}$.)

For a fixed $\gamma \in (0,2)$ we denote by $\mathfrak{d}_h, \mu_h, \nu_h$ respectively the γ-LQG metric, area measure and boundary length measure associated to h on $\mathbb{H}$. As noted previously, the γ-LQG metric was constructed in [GM21c] for the whole-plane GFF, but it is explained in [GM21c, Remark 1.5] how to adapt this to get the LQG metric on a proper domain $U \subset \mathbb{C}$ associated to $\mathring{h} + f$ where $\mathring{h}$ is a zero-boundary GFF on U and f is a continuous function on U; this is done as follows. If h^{wp} is a whole-plane GFF, then we can write $h^{\mathrm{wp}}|_U = \mathring{h} + \widehat{\mathfrak{h}}$ where $\mathring{h}$ is a zero-boundary GFF on U and $\widehat{\mathfrak{h}}$ is a random harmonic function (modulo additive constant) independent of $\mathring{h}$. Recall that $\mathring{h}$ and $\widehat{\mathfrak{h}}$ are the projections of h onto the spaces of functions that are, respectively, supported in U and harmonic in U. Note that fixing the additive constant for h corresponds to fixing that for $\widehat{\mathfrak{h}}$ but may or may not preserve the independence of $\mathring{h}$ and $\widehat{\mathfrak{h}}$; for instance, we can fix the constant by requiring $\widehat{\mathfrak{h}}(z) = 0$ for some choice of $z \in U$, in which case $\widehat{\mathfrak{h}}$ as a bona fide (random) function is still independent of $\mathring{h}$, or we can require that the average of h on some circle $\Gamma \subset U$ vanishes, in which case $\mathring{h}$ and $\widehat{\mathfrak{h}}$ are *not* independent, since their averages on Γ are required to sum to zero.

Having fixed the additive constant in some way – whether or not $\widehat{\mathfrak{h}}$ with the constant fixed is independent of $\mathring{h}$ – we can define $\mathfrak{d}_{\mathring{h}}$ on U as a Weyl scaling of the internal metric induced by $\mathfrak{d}_{h^{\mathrm{wp}}}$ on U, i.e.

$$\mathfrak{d}_{\mathring{h}}(\cdot,\cdot) = e^{-\xi\widehat{\mathfrak{h}}} \cdot \mathfrak{d}_{h^{\mathrm{wp}}}(\cdot,\cdot;U).$$

(This is well-defined since the definition of the internal metric only involves paths which stay in U, so it does not matter that $\widehat{\mathfrak{h}}$ does not extend continuously to the boundary.) Moreover one can define $\mathfrak{d}_{\mathring{h}+f} = e^{\xi f} \cdot \mathfrak{d}_{\mathring{h}}$ for f continuous on U. It is easy to see that $\mathfrak{d}_{\mathring{h}+f}$ thus defined is a metric on $\mathbb{H}$ that satisfies the axioms in [GM21c, §1.2] and conformal covariance. Observe also that $\mathfrak{d}_{\mathring{h}+f}$ induces the Euclidean topology on $\mathbb{H}$. Indeed, the internal metric $\mathfrak{d}_{h^{\mathrm{wp}}}(\cdot,\cdot;\mathbb{H})$ is at least as large as $\mathfrak{d}_{h^{\mathrm{wp}}}$, so since Euclidean open sets in $\mathbb{H}$ are open w.r.t. $\mathfrak{d}_{h^{\mathrm{wp}}}$ they must also be open w.r.t. the internal metric. Around each point $z \in \mathbb{H}$ the $\mathfrak{d}_{h^{\mathrm{wp}}}$-metric balls of sufficiently small radius must be contained in $\mathbb{H}$, so coincide with the $\mathfrak{d}_{h^{\mathrm{wp}}}(\cdot,\cdot;\mathbb{H})$-metric balls of the same radius. These thus contain Euclidean open discs, which shows that $\mathfrak{d}_{h^{\mathrm{wp}}}(\cdot,\cdot;\mathbb{H})$ induces the Euclidean topology. Since $-\mathfrak{h} + f$ is a continuous function on $\mathbb{H}$ (and thus locally bounded) the same has to be true for $\mathfrak{d}_{\mathring{h}+f}$.

Since we can write $h = \mathring{h} + \widetilde{\mathfrak{h}}$ for $\mathring{h}$ a zero-boundary GFF on $\mathbb{H}$ and $\widetilde{\mathfrak{h}}$ a (random) harmonic function on $\mathbb{H}$, we may define $\mathfrak{d}_h$ (as a function on $\mathbb{H} \times \mathbb{H}$) similarly. Recall that h and thus $\widetilde{\mathfrak{h}}$ are only defined modulo a global additive constant, so the above construction only defines $\mathfrak{d}_h$ modulo a multiplicative constant. Once the constant is fixed, it follows (as above for $\mathfrak{d}_{\mathring{h}+f}$) that this $\mathfrak{d}_h$ is a metric on $\mathbb{H}$ that induces the Euclidean topology on $\mathbb{H}$ and satisfies the axioms in [GM21c, §1.2] and conformal covariance. As noted, we will often be able to fix the constant somewhat arbitrarily. Note however that the same caveat applies as above: not every choice for fixing the constant makes the zero-boundary and harmonic parts of h independent.

We can extend the LQG metric to the boundary of $\mathbb{H}$ as follows. Firstly, we say that a path $P\colon [a,b] \to \overline{\mathbb{H}}$ is **admissible** if $P^{-1}(\partial\mathbb{H})$ is finite, and define the **$\widehat{d}_h$-length** of an admissible path P to be

$$\widehat{d}_h(P) := \sup\left\{\sum_{i=1}^{n} \mathfrak{d}_h(P(t_{i-1}), P(t_i)) :\ a \le t_0 < t_1 < \cdots < t_n \le b,\ P(t_i) \in \mathbb{H}\right\}.$$

$P^{-1}(\mathbb{H})$ can be written uniquely as a finite union of disjoint intervals I, each of which is open as a subset of $[a,b]$; it is straightforward to check that the length of P is the sum of the lengths of the $P|_{\overline{I}}$.

We now define the $\widehat{d}_h$-distance between two points of $\overline{\mathbb{H}}$ as the infimum of the lengths of admissible paths between them. To see that this definition actually does restrict to $\mathfrak{d}_h$ on $\mathbb{H}\times\mathbb{H}$, note that for $z,w \in \mathbb{H}$ we know that $\mathfrak{d}_h(z,w)$ is finite (indeed, one can find a path between z and w of finite $\mathfrak{d}_{\mathring{h}}$-length L that stays in some bounded open set U at positive distance from $\partial\mathbb{H}$, then we have $\mathfrak{d}_h(z,w) \le L \sup_U e^{\xi\widetilde{\mathfrak{h}}} < \infty$). Given $\varepsilon > 0$, we can then take a path P in $\overline{\mathbb{H}}$ with $\widehat{d}_h$-length in $[\widehat{d}_h(z,w), \widehat{d}_h(z,w)+\varepsilon)$, and thus find a subdivision of that path with

$$\sum_{i=1}^{n} \mathfrak{d}_h(P(t_{i-1}), P(t_i)) \le \widehat{d}_h(z,w) + \varepsilon.$$

We know that $\mathfrak{d}_h$ is almost surely a length metric, so for each i we can find a path from $P(t_{i-1})$ to $P(t_i)$ in $\mathbb{H}$ with $\mathfrak{d}_h$-length at most $\mathfrak{d}_h(P(t_{i-1}), P(t_i)) + \varepsilon/n$ and concatenate these to see that $\mathfrak{d}_h(z,w) \le \widehat{d}_h(z,w) + 2\varepsilon$. On the other hand clearly $\mathfrak{d}_h \ge \widehat{d}_h$ so the two must agree. We will henceforth use $\mathfrak{d}_h$ to refer to the function extended to all of $\overline{\mathbb{H}}$ (which we will show is a metric on $\overline{\mathbb{H}}$).

3.1.1 Joint Hölder continuity of the semicircle average

For a point $x \in \mathbb{R}$ and $\varepsilon > 0$, recall that $h_\varepsilon(x)$ denotes the average of h on the semicircular arc $\partial B(x,\varepsilon) \cap \mathbb{H}$, defined as $(h, \rho^+_{x,\varepsilon}) = -2\pi(h, \Delta^{-1}\rho^+_{x,\varepsilon})_\nabla$ where $\rho^+_{x,\varepsilon}$ is the uniform probability measure on $\partial B(x,\varepsilon)\cap\mathbb{H}$. Since h is only defined modulo additive constant, we have to fix the constant in order for the $h_\varepsilon(x)$ to be well defined – the results of this subsection (§3.1.1) hold however the constant is fixed, but for concreteness, we will state and prove them using the normalization $h_1(0)=0$. We can establish continuity for this semicircle average via the same Kolmogorov–Čentsov-type argument as in [HMP10, Prop. 2.1].

Lemma 3.1.1. *Let h be a free-boundary GFF on $\mathbb{H}$ with the additive constant fixed such that $h_1(0)=0$. There exist $\alpha,\beta>0$ such that, for any $U\subseteq\mathbb{R}$ bounded and open, $\zeta>1/\alpha$ and $\gamma\in(0,\beta/\alpha)$, there is a modification $\widetilde{X}$ of the process $X(z,r)=h_r(z)$ such that, for some random $M\in(0,\infty)$,*

$$|\widetilde{X}(z,r)-\widetilde{X}(w,s)|\le M\left(\log\frac{2}{r}\right)^{\zeta}\frac{|(z,r)-(w,s)|^{\gamma}}{r^{\frac{1+\beta}{\alpha}}}$$

whenever $z,w\in U$, $r,s\in(0,1]$ and $1/2\le r/s\le 2$. (This is unique in that any two such modifications are almost surely equal, by continuity.)

Proof. By the "modified Kolmogorov–Čentsov" result [HMP10, Lemma C.1] it suffices to show that there exist $\alpha,\beta,C>0$ such that for all $z,w\in U$ and $r,s\in(0,1]$ we have

$$\mathbb{E}[|h_r(z)-h_s(w)|^{\alpha}]\le C\left(\frac{|(z,r)-(w,s)|}{r\wedge s}\right)^{2+\beta}. \tag{3.1.1}$$

Thus we can show continuity for the semicircle average by bounding the absolute moments of $h_r(z)-h_s(w)$. In fact, since this is a centred Gaussian, we need only bound its second moment. We can do this by considering the Green's function for h, given by the Neumann Green's function in $\mathbb{H}$:

$$G(x,y)=-\log|x-\bar{y}|-\log|x-y|.$$

This G is the Green's function such that

$$-\Delta^{-1}\rho(\cdot)=\frac{1}{2\pi}\int_{\mathbb{H}}G(\cdot,y)\rho(y)\,dy.$$

Recall that $\rho_{x,\varepsilon}$ denotes the uniform probability measure on $\partial B(x,\varepsilon)$ and $\rho^+_{x,\varepsilon}$ denotes that on $\partial B(x,\varepsilon)\cap\mathbb{H}$. Since

$$\begin{aligned}\int G(z,y)\,\rho^+_{x,\varepsilon}(dy)&=\int(-\log|z-\bar{y}|-\log|z-y|)\,\rho^+_{x,\varepsilon}(dy)\\&=\int-2\log|z-y|\,\rho_{x,\varepsilon}(dy)=-2\log\max(|z-x|,\varepsilon),\end{aligned}$$

we find that

$$\begin{aligned}
&\mathbb{E}[(h_s(w) - h_r(z))(h_s(w) - h_r(z))] \\
&= \int [-2\log\max(|\zeta - w|, s) + 2\log\max(|\zeta - z|, r)]\,(\rho^+_{w,s} - \rho^+_{z,r})(d\zeta) \qquad (3.1.2)\\
&\le \int \left|-2\log\max(|\zeta - w|, s) + 2\log\max(|\zeta - z|, r)\right|\,(\rho^+_{w,s} + \rho^+_{z,r})(d\zeta) \\
&\le \int 2\,\frac{|z-w| + |r-s|}{r \wedge s}\,(\rho^+_{w,s} + \rho^+_{z,r})(d\zeta) = 4\,\frac{|z-w| + |r-s|}{r \wedge s}.
\end{aligned}$$

Here we used (as in the proof of [HMP10, Prop. 2.1]) that $|\log \frac{a}{b}| \le \frac{|a-b|}{a\wedge b}$ for $a, b > 0$, and that $|(a \vee b) - (c \vee d)| \le |a - c| \vee |b - d|$ for all a, b, c, d. Since $h_s(w) - h_r(z)$ is a centred Gaussian, we now have that for every $\alpha > 0$ there is C_α such that

$$\mathbb{E}[|h_s(w) - h_r(z)|^\alpha] \le C_\alpha \left(\frac{|(z,r) - (w,s)|}{r \wedge s}\right)^{\frac{\alpha}{2}},$$

concluding the proof. Observe also that when $w = z$ and $s > r$ the integral (3.1.2) becomes

$$\int 2\log\left(\frac{\max(|\zeta - z|, r)}{\max(|\zeta - z|, s)}\right)(\rho^+_{z,s}(d\zeta) - \rho^+_{z,r}(d\zeta));$$

the integral w.r.t. $\rho^+_{z,s}(d\zeta)$ vanishes whilst the integral w.r.t. $\rho^+_{z,r}(d\zeta)$ gives $2\log(s/r)$. A similar computation shows that the increments $h_s(z) - h_r(z)$ and $h_t(z) - h_u(z)$ have zero covariance when $r < s \le t < u$, which together with continuity implies that $\widetilde{X}(z, e^{-t}) - \widetilde{X}(z, 1)$ evolves as $\sqrt{2}$ times a two-sided standard Brownian motion. ■

Note that, since the boundary conditions are Neumann rather than Dirichlet, the semi-circle average process evolves as $\sqrt{2}$ times a Brownian motion, as opposed to circle averages which yield standard Brownian motion. This remains true for the free-boundary GFF; since we will need it later and the calculation is similar, we will now give a corresponding estimate for circle averages of the free-boundary GFF.

Lemma 3.1.2. *Let h be a free-boundary GFF with the additive constant fixed such that $h_1(0) = 0$. Let $K \subset \overline{\mathbb{H}}$ be compact. Then there exists a constant $C = C(K)$ such that, for all $w \in K$ and $s \in (0, 1]$ such that $B(w, s) \subset \overline{\mathbb{H}}$, we have*

$$\operatorname{var}[h^{\mathrm{circ}}_s(w) - h^{\mathrm{circ}}_1(i)] \le -\log s + C(K).$$

Proof. First we compute

$$\begin{aligned}\int G(z,y)\rho_{x,\varepsilon}(dy) &= \int (-\log|z-\bar{y}| - \log|z-y|)\rho_{x,\varepsilon}(dy)\\ &= -\log\max(|z-x|,\varepsilon) - \log\max(|z-\bar{x}|,\varepsilon).\end{aligned}$$

Thus, for $w, z \in \mathbb{H}$ and $r, s > 0$ such that $r \le \operatorname{Im} z$, $s \le \operatorname{Im} w$,

$$\begin{aligned}&\mathbb{E}[(h_s^{\mathrm{circ}}(w) - h_r^{\mathrm{circ}}(z))(h_s^{\mathrm{circ}}(w) - h_r^{\mathrm{circ}}(z))]\\ &= \int \Big[-\log\max(|\zeta - w|, s) - \log\max(|\zeta-\bar{w}|, s)\\ &\qquad + \log\max(|\zeta - z|, r) + \log\max(|\zeta - \bar{z}|, r)\Big] (\rho_{w,s} - \rho_{z,r})(dy)\\ &= \int [-\log\max(|\zeta - w|, s) + \log\max(|\zeta - z|, r)] (\rho_{w,s} - \rho_{z,r})(dy),\end{aligned}$$

where in the last line we used that $|\zeta - \bar{w}| \ge s$ and $|\zeta - \bar{z}| \ge r$ for $\zeta \in \mathbb{H}$, so the corresponding integrals w.r.t. $\rho_{w,s}$ and $\rho_{z,r}$ cancel.

Setting $z = i$, $r = 1$, note that the term $\log\max(|\zeta - i|, 1)$ vanishes on $\partial B(i, 1)$ and is bounded above (the bound depending only on K) on the closed Euclidean 1-neighbourhood of K, whereas $\log\max(|\zeta - w|, s)$ is equal to $\log s$ on $\partial B(w, s)$ and bounded above by some constant (depending only on K) on $\partial B(i, 1)$. The claimed result follows. ∎

3.1.2 Thick points on the boundary

We refer to $x \in \mathbb{R}$ as an **α-thick point** if

$$\lim_{r\to 0} \frac{h_r(x)}{\log(1/r)} = \alpha.$$

Our aim in this subsection is to show that boundary points have maximum thickness 2. This matches the maximum thickness in all of $\mathbb{H}$ for the zero-boundary GFF, as calculated in [HMP10]. This is because $\partial\mathbb{H}$ has Euclidean dimension half that of $\mathbb{H}$, but the semicircle averages centred at boundary points for the free-boundary GFF behave like $\sqrt{2}$ times the circle averages of the zero-boundary GFF; these two effects cancel each other out.

Lemma 3.1.3. *Let h be a free-boundary GFF with the additive constant fixed such that $h_1(0) = 0$.*

Almost surely, for every $x \in \mathbb{R}$ we have

$$\limsup_{r\to 0} \frac{h_r(x)}{\log(1/r)} \le 2. \tag{3.1.3}$$

The proof is an application of the argument that proves [HMP10, Lemma 3.1].

Proof. Fix $a > 2$, and choose ε such that $0 < \varepsilon < \frac{1}{2} \wedge (\frac{a^2}{8} - \frac{1}{2})$. For each $n \in \mathbb{N}$ let $r_n = n^{-1/\varepsilon}$. By setting $U = (-1,1)$, $\zeta = \frac{1}{2}$, $\alpha = \frac{16}{\varepsilon}$, $\beta = \frac{8}{\varepsilon} - 2$, $\gamma = \frac{1}{2} - \frac{3\varepsilon}{16}$ in Lemma 3.1.1 there is a random $M \in \mathbb{R}$ such that

$$|h_{s'}(y) - h_s(x)| \le M\left(\log\frac{2}{s'}\right)^{\frac{1}{2}} \frac{|(y,s') - (x,s)|^{\frac{1}{2}-\frac{3\varepsilon}{16}}}{s'^{\frac{1}{2}-\frac{\varepsilon}{16}}}, \tag{3.1.4}$$

whenever $x, y \in (-1,1)$, $s, s' \in (0,1]$ with $1/2 \le s'/s \le 2$. Thus, for all $x \in (0,1)$ and $n > (2^\varepsilon - 1)^{-1}$ (so that $r_n/r_{n+1} \in (1,2)$), and all $\log\frac{1}{r_n} < t \le \log\frac{1}{r_{n+1}}$, we have

$$\begin{aligned}
|h_{e^{-t}}(x) - h_{r_n}(x)| &\le M\left(\log\frac{2}{r_n}\right)^{\frac{1}{2}} \frac{(r_n - r_{n+1})^{\frac{1}{2}-\frac{3\varepsilon}{16}}}{r_n^{\frac{1}{2}-\frac{\varepsilon}{16}}} \\
&= M\left(\log 2 + \frac{1}{\varepsilon}\log n\right)^{\frac{1}{2}} \left(1 - \left(\frac{n}{n+1}\right)^{\frac{1}{\varepsilon}}\right)^{\frac{1}{2}-\frac{3\varepsilon}{16}} n^{\frac{1}{8}} \\
&\le 2M\varepsilon^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} \left(1 - \left(\frac{n}{n+1}\right)^{\frac{1}{\varepsilon}}\right)^{\frac{1}{2}-\frac{3\varepsilon}{16}} n^{\frac{1}{8}} \\
&\le 2M\varepsilon^{-1+\frac{3\varepsilon}{16}} n^{-\frac{3}{8}+\frac{3\varepsilon}{16}} (\log n)^{\frac{1}{2}},
\end{aligned}$$

where in the last step we used that $(1 - 1/(n\varepsilon))^\varepsilon < (1 - 1/((n+1)\varepsilon))^\varepsilon < 1 - 1/(n+1) = n/(n+1)$, so that $1 - (n/(n+1))^{1/ve} \le 1/(n\varepsilon)$. This shows

$$\limsup_{r\to 0} \frac{h_r(x)}{\log(1/r)} > a \Leftrightarrow \limsup_{n\to\infty} \frac{h_{r_n}(x)}{\log(1/r_n)} > a. \tag{3.1.5}$$

Given $x \in (0,1)$ we can find $k \in \mathbb{N}$ such that $kr_n^{1+\varepsilon} \in (0,1)$ with $|x - kr_n^{1+\varepsilon}| < r_n^{1+\varepsilon}$. Then (3.1.4) gives

$$|h_{r_n}(x) - h_{r_n}(kr_n^{1+\varepsilon})| < M\left(\log\frac{2}{r_n}\right)^{\frac{1}{2}} r_n^{(1+\varepsilon)\frac{\varepsilon}{8}} = M\left(\log\frac{2}{r_n}\right)^{\frac{1}{2}} n^{-\frac{1}{8}(1+\varepsilon)}.$$

So if the right-hand side of (3.1.5) holds for some $x \in (0,1)$, then for some $\delta > 0$ there are infinitely many n for which $h_{r_n}(x) > (a+\delta)\log\frac{1}{r_n}$, and thus infinitely many n for which some $k \in \{1, 2, \ldots, \lfloor r_n^{-(1+\varepsilon)} \rfloor\}$ makes the event $A_{n,k} = \{h_{r_n}(kr_n^{1+\varepsilon}) > a \log \frac{1}{r_n}\}$ hold. We can now apply a Gaussian tail bound. Indeed, for each x, $(h_{e^{-t}}(x) - h_1(x))_t$ is $\sqrt{2}$ times a standard Brownian motion, independent of $h_1(x) = h_1(x) - h_1(0)$ (recall our choice of additive constant) whose variance is at most some constant C for $x \in (0, 1)$, as follows from (3.1.1). Thus $h_{r_n}(kr_n^{1+\varepsilon}) \sim \mathcal{N}(0, c + 2 \log \frac{1}{r_n})$ for some $c \leq C$. Thus for any $\eta > 0$, if n is sufficiently large we have

$$\mathbb{P}[A_{n,k}] = \mathbb{P}\left[Z > \frac{a \log \frac{1}{r_n}}{\sqrt{c + 2 \log \frac{1}{r_n}}}\right] < \mathbb{P}\left[Z > \frac{(a-\eta) \log \frac{1}{r_n}}{\sqrt{2 \log \frac{1}{r_n}}}\right] < r_n^{\frac{(a-\eta)^2}{4}},$$

where $Z \sim \mathcal{N}(0, 1)$. So by a union bound we have, for n sufficiently large depending on η,

$$\mathbb{P}\left[\bigcup_{k=1}^{\lfloor r_n^{-(1+\varepsilon)} \rfloor} A_{n,k}\right] \leq r_n^{\frac{(a-\eta)^2}{4} - 1 - \varepsilon} = n^{1 + \frac{1}{\varepsilon}\left(1 - \frac{(a-\eta)^2}{4}\right)}.$$

Since $1 + \frac{1}{\varepsilon}\left(1 - \frac{a^2}{4}\right) < -1$, we can choose η so that $1 + \frac{1}{\varepsilon}\left(1 - \frac{(a-\eta)^2}{4}\right) < -1$, in which case the Borel–Cantelli lemma gives

$$\mathbb{P}\left[\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{\lfloor r_n^{-(1+\varepsilon)} \rfloor} A_{n,k}\right] = 0.$$

Thus almost surely there is no $x \in (0, 1)$ for which

$$\limsup_{r \to 0} \frac{h_r(x)}{\log(1/r)} > a.$$

By translation invariance we can conclude that there is no such $x \in \mathbb{R}$; all that changes is the bound c on the variance of $h_1(x)$, but we will still locally have boundedness. Since $a > 2$ was arbitrary we are done. ■

Remark 3.1.4. *In the proof of Lemma 3.1.3, we find that almost surely there is some N such that $\bigcup_k A_{n,k}$ never happens for $n \geq N$, which gives a uniform bound on $h_r(x)/\log(1/r)$ for $x \in (0, 1)$ and $r \leq r_N$. Using this, and translation invariance, we can deduce the stronger statement that, almost surely, for every $K \subset \mathbb{R}$ compact we have*

$$\limsup_{r \to 0} \sup_{x \in K} \frac{h_r(x)}{\log(1/r)} \leq 2. \tag{3.1.6}$$

Indeed, since we have

$$\mathbb{P}\left[\bigcup_{n=m}^{\infty}\bigcup_{k=1}^{\lfloor r_n^{-(1+\varepsilon)}\rfloor} A_{n,k}\right] = O\left(m^{2+\frac{1}{\varepsilon}\left(1-\frac{(a-\eta)^2}{4}\right)}\right),$$

it in fact holds that, as $r \to 0$,

$$\mathbb{P}\left[\sup_{s\le r}\sup_{x\in K}\frac{h_s(x)}{\log(1/s)} > a\right] = O\left(r^{\frac{(a-\eta)^2}{4}-1-2\varepsilon}\right), \tag{3.1.7}$$

where since η and ε can be made arbitrarily small the exponent is arbitrary subject to being less than $a^2/4 - 1$.

3.1.3 Controlling pointwise distance via semicircle averages

We give an analogue of part of [DFG+20, Prop. 3.14].

Proposition 3.1.5. *Fix $a \in \mathbb{R}$ and $r > 0$. Let h be a free-boundary GFF on $\mathbb{H}$ and let $K \subset \mathbb{H} \cap \partial B(0,1)$ be a closed arc consisting of more than a single point. Then we have*

$$\mathfrak{d}_h(a, a + rK) \le C \int_{\log(1/r)}^{\infty} e^{\xi(h_{e^{-t}}(a)-Qt)}\,dt \tag{3.1.8}$$

with superpolynomially high probability as $C \to \infty$, at a rate which is uniform in a and r.

Proof. Without loss of generality fix the additive constant for h such that $h_1(a) = 0$ (note that the statement does not depend on the normalization, since adding a constant c to the field scales both sides by $e^{\xi c}$).

Couple h with h^{wp}, a whole-plane GFF with the constant fixed so that $h_1^{\mathrm{wp}}(a) = 0$, such that $h^{\mathrm{wp}}|_{\mathbb{H}} = \mathring{h} + \widehat{\mathfrak{h}}$ and $h = \mathring{h} + \widetilde{\mathfrak{h}}$, where $\mathring{h}$ is a zero-boundary GFF on $\mathbb{H}$ and $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$ are independent random functions harmonic in $\mathbb{H}$. (Note that $h_1(a)$ is a semicircle average whereas $h_1^{\mathrm{wp}}(a)$ is a circle average; note also that although we could choose the normalizations differently to make $\mathring{h}$ independent of $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$, we do not do so in this proof as we will not require this independence.)

Let $\mathfrak{h} = \widetilde{\mathfrak{h}} - \widehat{\mathfrak{h}}$; then $\mathfrak{d}_h = e^{\xi\mathfrak{h}} \cdot \mathfrak{d}_{h^{\mathrm{wp}}|_{\mathbb{H}}}$, by Weyl scaling. We will prove the result by obtaining an upper bound on $\mathfrak{d}_h(a + e^{1-n}K, a + e^{-1-n}K)$ for each $n \in \mathbb{N}$. Let U be a bounded connected open set containing $eK \cup K \cup e^{-1}K$ and at positive distance from $\partial\mathbb{H}$. We can then apply the result [DFG+20, Prop. 3.1][1] to find that, with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in n, we have

$$\mathfrak{d}_{h^{\mathrm{wp}}}(a + e^{1-n}K, a + e^{-1-n}K; a + e^{-n}U) \leq Ae^{\xi(h^{\mathrm{wp}}_{e^{-n}}(a) - Qn)}. \tag{3.1.9}$$

(Recall that, for an open set $V \subset \mathbb{C}$, $\mathfrak{d}_{h^{\mathrm{wp}}}(\cdot, \cdot; V)$ is the internal metric on V induced by $\mathfrak{d}_{h^{\mathrm{wp}}}$.) Since $a + U$ is at positive distance from $\partial\mathbb{H}$, $\mathfrak{h}$ is almost surely bounded on $a + U$. Thus the variables

$$\{\mathfrak{h}(z) - [h_1(a) - h^{\mathrm{wp}}_1(a)] : z \in a + U\}$$

form an almost surely bounded Gaussian process, so by the Borell–TIS inequality (see [AT07, Thm 2.1.1]) the supremum has a Gaussian tail: there exist $c_1, c_2 > 0$ such that for all M sufficiently large we have

$$\mathbb{P}\left[\sup_{z \in a+U} (\mathfrak{h}(z) - [h_1(a) - h^{\mathrm{wp}}_1(a)]) > M\right] < c_1 e^{-c_2 M^2}.$$

Setting $u > 0$ to be the Euclidean distance between U and $\partial\mathbb{H}$ and writing $h^{\mathrm{circ}}_r(z)$ for the average of h on a circle $\partial B(z, r) \subset \overline{\mathbb{H}}$, we can write

$$\sup_{z \in a+U} (\mathfrak{h}(z) - [h_1(a) - h^{\mathrm{wp}}_1(a)]) = \sup_{z \in a+U} ((h^{\mathrm{circ}}_u(z) - h_1(a)) - (h^{\mathrm{wp}}_u(z) - h^{\mathrm{wp}}_1(a))),$$

where both differences on the right hand side are independent of how the additive constants for h and h^{wp} are fixed, since they only depend on the fields h and h^{wp} when integrated against mean-zero test functions). Thus, by scale invariance, with the same c_1, c_2 we have for each n

$$\mathbb{P}\left[\sup_{z \in a+e^{-n}U} (\mathfrak{h}(z) - [h_{e^{-n}}(a) - h^{\mathrm{wp}}_{e^{-n}}(a)]) > M\right] < c_1 e^{-c_2 M^2}.$$

It follows that

$$\sup_{z \in a+e^{-n}U} \mathfrak{h}(z) - [h_{e^{-n}}(a) - h^{\mathrm{wp}}_{e^{-n}}(a)] \leq \log A \tag{3.1.10}$$

[1] Many of the results in [DFG+20] involve constants $\mathfrak{c}_r$ for each $r > 0$, which describe the scaling of LQG distances. In [DFG+20, Thm 1.5] a "tightness" result is obtained for the $\mathfrak{c}_r$ in lieu of actual scale invariance, which was established later in [GM21c]; in this work, we will thus use the subsequent result (see [GM21c, Thm 1.8]) that we can take $\mathfrak{c}_r = r^{\xi Q}$.

with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in n. Since

$$\begin{aligned}&\mathfrak{d}_h(a + e^{1-n}K, a + e^{-1-n}K; a + e^{-n}U)\\&\le e^{\xi(\sup_{z\in a+e^{-n}U}\mathfrak{h}(z))}\mathfrak{d}_{h^{\mathrm{wp}}}(a + e^{1-n}K, a + e^{-1-n}K; a + e^{-n}U)\end{aligned}$$

and $h_{e^{-n}}(a) = h^{\mathrm{wp}}_{e^{-n}}(a) + \mathfrak{h}_{e^{-n}}(a)$, we find that

$$\mathfrak{d}_h(a + e^{1-n}K, a + e^{-1-n}K; a + e^{-n}U) \le Ae^{\xi(h_{e^{-n}}(a)-Qn)} \tag{3.1.11}$$

on the intersection of the events of (3.1.9) with A replaced by $A^{1/2}$ and (3.1.10) with A replaced by $A^{1/(2\xi)}$; the probability of this event is superpolynomially high as $A \to \infty$.

By replacing U by a suitable bounded connected open neighbourhood $\widetilde{U}$ of K, again with positive distance to $\partial\mathbb{H}$, and using compact subsets of $\widetilde{U}$ on either side of K, a similar argument shows that for each n there is a path γ_n in $e^{-n}\widetilde{U}$ whose intersection with U disconnects $e^{1-n}K$ and $e^{-1-n}K$ in U such that

$$\mathrm{length}(\gamma_n; \mathfrak{d}_h) \le Ae^{\xi(h_{e^{-n}}(a)-Qn)} \tag{3.1.12}$$

with superpolynomially high probability as $A \to \infty$, uniformly in n. This provides the adaptation of [DFG+20, Prop. 3.1] that we need – namely, fixing $\zeta > 0$ small, as $C \to \infty$ the probability is superpolynomially high that (3.1.11) holds, and there is a path γ_n such that (3.1.12) holds, with $A = C$ whenever $n \le C^{1/\zeta}$ and with $A = n^\zeta$ whenever $n > C^{1/\zeta}$. Stringing together the paths γ_n with paths of near-minimal length connecting $a + e^{1-n}K$ and $a + e^{-1-n}K$ for each n, we find that

$$\mathfrak{d}_h(a, a + rK) \le Cr^{\xi Q}\sum_{n=0}^{\lfloor C^{1/\zeta}\rfloor} e^{\xi h_{re^{-n}}(0)-\xi Qn} + \sum_{\lfloor C^{1/\zeta}\rfloor+1}^{\infty} n^\zeta e^{\xi h_{re^{-n}}(0)-\xi Qn}. \tag{3.1.13}$$

We now have to bound the right-hand side by the integral in (3.1.8). The argument for this, using Gaussian tail bounds, is exactly the same as in Steps 2–3 of the proof of [DFG+20, Prop. 3.14].[2] We thus conclude that (3.1.8) holds with superpolynomially high probability. (Uniformity in a follows by translation invariance for $\mathfrak{d}_h$ and the fact that the result, and in

[2]The proof there gives their result with an added factor $\psi(t) = o(t)$ in the exponent, which arises in the proof from the fact that exact scale invariance was not then known, but ψ can be taken to be identically zero in light of the relation $\mathfrak{c}_r = r^{\xi Q}$.

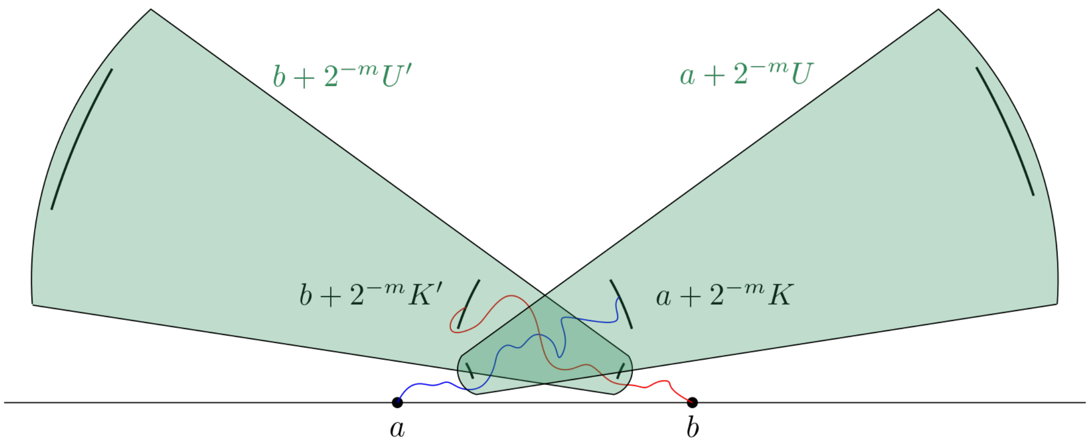

Figure 3.1: An illustration of the arcs K, K' and their neighbourhoods U, U' from the proof of Prop. 3.1.6.

particular the probability (3.1.8), does not depend on the choice of constant for h.) ■

Given points a, b with n maximal such that $|b - a| < 2^{1-n}$, arcs K, K', and open sets U and U' at positive distance to $\partial\mathbb{H}$ and such that $eK \cup K \cup e^{-1}K \subset U$, $eK' \cup K' \cup e^{-1}K' \subset U'$, we can apply Prop. 3.1.5 to find paths from a to $a + 2^{-n}K$ and from b to $b + 2^{-n}K'$ that respectively stay in $a + \bigcup_{m \geq n} e^{-m}U$, $a + \bigcup_{m \geq n} e^{-m}U'$ and whose lengths are respectively bounded by

$$C \int_{\log(|b-a|^{-1})}^{\infty} e^{\xi(h_{e^{-t}}(a) - Qt)}\, dt, \quad C \int_{\log(|b-a|^{-1})}^{\infty} e^{\xi(h_{e^{-t}}(b) - Qt)}\, dt,$$

with superpolynomially high probability as $C \to \infty$. By judiciously choosing K, K' and the open sets U, U' (see Figure 3.1), we can arrange that the path from a to $a + 2^{-n}K$ and that from b to $b + 2^{-n}K'$ cross each other, provided $2^{1-n} > |b - a|$, giving an analogue of part of [DFG+20, Prop. 3.15]:

Proposition 3.1.6. *For h a free-boundary GFF on $\mathbb{H}$ and $a, b \in \mathbb{R}$ with $0 < |b - a| \leq 1$, we have*

$$\mathfrak{d}_h(a, b) \leq C \int_{\log(|b-a|^{-1})}^{\infty} \left[e^{\xi(h_{e^{-t}}(a) - Qt)} + e^{\xi(h_{e^{-t}}(b) - Qt)} \right] dt \tag{3.1.14}$$

with superpolynomially high probability as $C \to \infty$.

Remark 3.1.7. *Since the choices of K, K', U, U' depend only on $2^{1-n}/|b - a|$, if we assume that $|b - a| = 2^{-n}$ for some $n \in \mathbb{N}$ then we get that the rate at which the probability decays is uniform in the choice of a and b (and n).*

We now want to use Prop. 3.1.6 to find sequences of points in $\mathbb{H}$ that converge to a point in $\mathbb{R}$ w.r.t. both $\mathfrak{d}_h$ and the Euclidean metric; eventually we will use these to show that, almost surely, both metrics induce the same topology.

Lemma 3.1.8. *Let h be a free-boundary GFF on $\mathbb{H}$ and fix $a \in \mathbb{R}$. Almost surely, for every closed arc $K \subset \mathbb{H} \cap \partial B(0,1))$ consisting of more than one point, we have*

$$\mathfrak{d}_h(a, a + 2^{-n}K) \to 0$$

as $n \to \infty$. Thus, this convergence almost surely holds simultaneously for all such K and all dyadic rationals a.

Proof. Since every such K contains an arc with endpoints at rational angles, we can assume K is fixed. Fix also $\varepsilon \in (0, Q-2)$. Applying Prop. 3.1.5 and the Borel–Cantelli lemma for all $n \in \mathbb{N}$ with $r = 2^{-n}$ and $C = 2^{\xi\varepsilon n}$, it almost surely holds that for n large enough, we have

$$\mathfrak{d}_h(a, a + 2^{-n}K) \le 2^{\xi\varepsilon n} \int_{n \log 2}^{\infty} e^{\xi(h_{e^{-t}}(a) - Qt)}\, dt \le \int_{n \log 2}^{\infty} e^{\xi(h_{e^{-t}}(a) - (Q-\varepsilon)t)}\, dt.$$

Moreover, for $u \in (0, Q - \varepsilon - 2)$, by (3.1.6) it almost surely holds that the integrand of the rightmost integral is bounded by e^{-ut} for n large enough, so the rightmost integral almost surely tends to 0 as $n \to \infty$, as required. ■

3.1.4 Local Hölder continuity w.r.t. the Euclidean metric

We now prove that $\mathfrak{d}_h$ is almost surely locally Hölder continuous w.r.t. the Euclidean metric on $\overline{\mathbb{H}}$.

Proposition 3.1.9. *Let h be a free-boundary GFF on $\mathbb{H}$ with some choice of additive constant. Almost surely, for each $u \in (0, \xi(Q-2))$ and each compact $K \subset \overline{\mathbb{H}}$ there exists $C > 0$ finite such that whenever $z, w \in K$, we have*

$$\mathfrak{d}_h(z, w) \le C|z - w|^{\xi(Q-2)-u}. \tag{3.1.15}$$

Proof. By scale invariance it suffices to consider $K = [0,1]^2$. We will use the domain Markov property to couple h with a zero-boundary field $\mathring{h}$ and a harmonic correction $\widetilde{\mathfrak{h}} := h - \mathring{h}$ given respectively by the projections of h onto the spaces of functions, respectively, supported in and harmonic on $\mathbb{H}$. Unlike in the proof of Prop. 3.1.5, this time we will need $\mathring{h}$ and $\widetilde{\mathfrak{h}}$ to be independent, so we will fix the additive constant for h so that $\widetilde{\mathfrak{h}}(i) = 0$. (Note that the claimed result does not depend on the choice of normalization.) Fix S to be the rectangle $[0,1] \times [\frac{1}{2}, 1]$, and U to be a neighbourhood of S at positive distance from $\partial\mathbb{H}$. By [DFG+20, Prop. 3.9] we know that, for h^{wp} a whole-plane GFF with the additive constant fixed such that $h_1^{\mathrm{wp}}(0) = 0$, and for any $p < 4d_\gamma/\gamma^2$,

$$\mathbb{E}[(\sup_{z,w\in S} \mathfrak{d}_{h^{\mathrm{wp}}}(z,w;U))^p] = \mathbb{E}[(e^{-\xi h_1^{\mathrm{wp}}(0)} \sup_{z,w\in S} \mathfrak{d}_{h^{\mathrm{wp}}}(z,w;U))^p] < \infty. \tag{3.1.16}$$

Now consider the coupling of h^{wp} with $\mathring{h}$ and $\widehat{\mathfrak{h}}$ from Prop. 3.1.5 (we will not need independence here). Using that $\sup_S \widehat{\mathfrak{h}}$ has a Gaussian tail by Borell–TIS and thus $\mathbb{E}[\sup_S e^{q\xi\widehat{\mathfrak{h}}}] < \infty$ for all $q > 0$, we get that

$$\mathbb{E}[(\sup_{z,w\in S} \mathfrak{d}_{\mathring{h}}(z,w;U))^p] < \infty \tag{3.1.17}$$

for each $p < 4d_\gamma/\gamma^2$ (by applying (3.1.16) for a slightly larger value of p and using Hölder's inequality). Now define for each $n \in \mathbb{N}$, $1 \le j \le 2^{n+1} - 1$, $1 \le k \le 2^n$ the sets

$$S_{n,j,k} = 2^{-n}\left(S + k - 1 + \frac{1}{2}(j-1)i\right) = [(k-1)2^{-n}, k2^{-n}] \times [j2^{-(n+1)}, (j+1)2^{-(n+1)}],$$

$$U_{n,j,k} = 2^{-n}\left(U + k - 1 + \frac{1}{2}(j-1)i\right).$$

Note that for each n the $S_{n,j,k}$ divide the rectangle $[0,1] \times [2^{-(n+1)}, 1]$ into rectangles of dimensions $2^{-n} \times 2^{-(n+1)}$. Then by scaling and translation invariance, for each $p < 4d_\gamma/\gamma^2$, there exists $M_p < \infty$ such that

$$\mathbb{E}[(2^{n\xi Q} \sup_{z,w\in S_{n,j,k}} \mathfrak{d}_{\mathring{h}}(z,w;U_{n,j,k}))^p] \le M_p, \tag{3.1.18}$$

Moreover, we know that

$$\sup_{z,w\in S_{n,j,k}} \mathfrak{d}_h(z,w;U_{n,j,k}) \le \sup_{z,w\in S_{n,j,k}} \mathfrak{d}_{\mathring{h}}(z,w;U_{n,j,k}) \cdot \sup_{z\in U_{n,j,k}} e^{\xi\widetilde{\mathfrak{h}}(z)}.$$

Since $\mathring{h}$ and $\widetilde{\mathfrak{h}}$ are independent, we have

$$\begin{aligned}\mathbb{E}&\left[\left(2^{n\xi Q}\sup_{z,w\in S_{n,j,k}}\mathfrak{d}_h(z,w;U_{n,j,k})\right)^p\right]\\&\le\mathbb{E}\left[\left(2^{n\xi Q}\sup_{z,w\in S_{n,j,k}}\mathfrak{d}_{\mathring{h}}(z,w;U_{n,j,k})\right)^p\right]\mathbb{E}\left[\sup_{z\in U_{n,j,k}}e^{p\xi\widetilde{\mathfrak{h}}(z)}\right].\end{aligned}\tag{3.1.19}$$

Now note that $\sup_{z\in U}\xi(\widetilde{\mathfrak{h}}(z)=\sup_{z\in U}\xi(\widetilde{\mathfrak{h}}(z)-\widetilde{\mathfrak{h}}(i))$ has a Gaussian tail by Borell–TIS, since U is at positive distance from $\partial\mathbb{H}$. By scaling and translation invariance, we can conclude that there are σ, c for which, for all $t>0$ and all n, j, k,

$$\mathbb{P}\left[\sup_{z\in U_{n,j,k}}\xi(\widetilde{\mathfrak{h}}(z)-\widetilde{\mathfrak{h}}(2^{-n}(k-1+\frac{1}{2}(j+1)i)))\ge t\right]\le ce^{-t^2/2\sigma^2}.$$

This means that for every $p>0$, there is a constant $K_p<\infty$ such that, for all n, j, k,

$$\mathbb{E}\left[\sup_{z\in U_{n,j,k}}e^{p\xi(\widetilde{\mathfrak{h}}(z)-\widetilde{\mathfrak{h}}(2^{-n}(k-1+\frac{1}{2}(j+1)i)))}\right]\le K_p.\tag{3.1.20}$$

Note that $\widetilde{\mathfrak{h}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))$ is a Gaussian variable; we proceed to bound its variance. Denoting by $\widetilde{\mathfrak{h}}^{\mathrm{circ}}_r(z)$ the average of $\widetilde{\mathfrak{h}}$ on the circle $\partial B(z,r)$, by harmonicity we can write $\widetilde{\mathfrak{h}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))=\widetilde{\mathfrak{h}}^{\mathrm{circ}}_{2^{-n}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))$, using that $2^{-n}(k-1+\frac{1}{2}(j+1)i)$ is at distance at least 2^{-n} from the boundary. We now use that $\widetilde{\mathfrak{h}}$ is an orthogonal projection of h, so that there is a constant $c>0$ not depending on n, j, k such that (by Lemma 3.1.2)

$$\begin{aligned}\operatorname{var}\widetilde{\mathfrak{h}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))&=\operatorname{var}[\widetilde{\mathfrak{h}}^{\mathrm{circ}}_{2^{-n}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))-\widetilde{\mathfrak{h}}^{\mathrm{circ}}_1(i)]\\&\le\operatorname{var}[h^{\mathrm{circ}}_{2^{-n}}(2^{-n}(k-1+\frac{1}{2}(j+1)i))-h^{\mathrm{circ}}_1(i)]\le n\log 2+c.\end{aligned}\tag{3.1.21}$$

(We could also compute this variance exactly using a similar argument to [GMS18, Lemma 2.9].)

Now fix $q' > q > 0$; we compute

$$\begin{aligned}\mathbb{E}\left[\sup_{z\in U_{n,j,k}} e^{q\xi\mathfrak{h}(z)}\right] &\le (K_{\frac{qq'}{q'-q}})^{\frac{q'-q}{q'}}\left(\mathbb{E}\left[e^{\xi q'\mathfrak{h}(2^{-n}(k-1+\frac{1}{2}(j+1)i))}\right]\right)^{q/q'} && ((3.1.20)\text{ and Hölder})\\ &\le (K_{\frac{qq'}{q'-q}})^{\frac{q'-q}{q'}} e^{\xi^2 qq'(n\log 2+c)/2} && ((3.1.21)\text{ and Gaussian m.g.f.})\\ &= C(q,q')2^{n\xi^2 qq'/2}.\end{aligned}$$

Using these bounds (with $\delta > 0$, $q = p$, $q' = p + \delta/p$) in (3.1.19), we get that for $p < 4d_\gamma/\gamma^2$, we have

$$\mathbb{E}\left[\left(2^{n\xi Q}\sup_{z,w\in S_{nj,,k}}\mathfrak{d}_h(z,w;U_{n,k})\right)^p\right] \le \widetilde{C}(p,\delta)2^{n\xi^2(p^2+\delta)/2}$$

and thus, for any $s \in \mathbb{R}$,

$$\mathbb{P}\left[2^{n\xi Q}\sup_{z,w\in S_{n,j,k}}\mathfrak{d}_h(z,w;U_{n,j,k}) \ge 2^{ns}\right] \le 2^{-nps}\widetilde{C}(p,\delta)2^{n\xi^2(p^2+\delta)/2}. \tag{3.1.22}$$

If $s > 2\xi$ is sufficiently close to 2ξ, setting $p = s/\xi^2$ we have $p < 4d_\gamma/\gamma^2$ since $\gamma > 2$ implies $2/\xi > 4/(\xi\gamma) = 4d_\gamma/\gamma^2$, so the right-hand side becomes $\widetilde{C}(p,\delta)2^{n(-s^2/(2\xi^2)+\xi^2\delta/2)}$, and if δ is sufficiently small we have $-s^2/(2\xi^2) + \xi^2\delta/2 < -2$. Thus, using Borel–Cantelli and setting $u = s - 2\xi$, we conclude that, almost surely, for every $u > 0$, there exists n_0 such that for all $n \ge n_0$, $1 \le j \le 2^{n+1} - 1$ and $1 \le k \le 2^n$, we have

$$\sup_{z,w\in S_{n,j,k}}\mathfrak{d}_h(z,w;U_{n,j,k}) < 2^{-n(\xi(Q-2)-u)}. \tag{3.1.23}$$

Fixing $u > 0$ and taking n as above, if $m \ge n_0$, $b \in (0, 2^{-m}]$, $a \in [0, 1]$, we can concatenate near-minimal paths connecting $a + 2^{-n}i$ to $a + 2^{-(n+1)i}$ in $U_{n,1,\lceil 2^n a\rceil}$ for each $n \ge m + 1$ with a near-minimal path connecting $a + 2^{-(m+1)}i$ to $a + bi$ in $U_{m,1,\lceil 2^m a\rceil}$, to find that

$$\mathfrak{d}_h(a, a + bi) \le 2^{(1-m)(\xi(Q-2)-u)}. \tag{3.1.24}$$

Indeed, by the same token it follows that, whenever $a \in [0, 1]$ and $b \le 2^{-n_0}$, the $\mathfrak{d}_h$-diameter of the vertical line segment $[a, a + bi]$ from a to $a + bi$ satisfies

$$\operatorname{diam}([a, a + bi]; \mathfrak{d}_h) \le (4b)^{\xi(Q-2)-u}. \tag{3.1.25}$$

Now consider general w, z in K. If $|w - z| \le 2\min\{\operatorname{Im} w, \operatorname{Im} z\}$, then with n such that $2^{-n} \le |w - z| < 2^{-(n-1)}$, we have $\min\{\operatorname{Im} w, \operatorname{Im} z\} \ge 2^{-(n+1)}$, so we can find j_1, j_2, k_1, k_2 such that $z \in S_{n,j_1,k_1}$ and $w \in S_{n,j_2,k_2}$. Moreover, $|w - z| < 2^{-(n-1)}$ implies that $|j_1 - j_2| \le 5$ and $|k_1 - k_2| \le 3$, which means that if $n \ge n_0$, then by applying (3.1.23) to a set of rectangles of the form $S_{n,j,k}$ connecting S_{n,j_1,k_1} and S_{n,j_2,k_2} we find that

$$\mathfrak{d}_h(z, w) \le 7 \cdot 2^{-n(\xi(Q-2)-u)} \le 7|w - z|^{\xi(Q-2)-u}. \tag{3.1.26}$$

On the other hand, suppose $|w - z| > 2\min\{\operatorname{Im} w, \operatorname{Im} z\}$. Since we also have $2|w - z| > 2(\max\{\operatorname{Im} w, \operatorname{Im} z\} - \min\{\operatorname{Im} w, \operatorname{Im} z\})$, adding these inequalities yields $\max\{\operatorname{Im} w, \operatorname{Im} z\} < \frac{3}{2}|w - z|$. Moreover, with n such that $2^{-n} \le |w - z| < 2^{-(n-1)}$, we can find k_1 and k_2 such that $\operatorname{Re} w + |w - z|i \in S_{n-1,1,k_1}$, $\operatorname{Re} z + |w - z|i \in S_{n-1,1,k_2}$ and (since $|\operatorname{Re} w - \operatorname{Re} z| \le |w - z| < 2^{-(n-1)}$) $|k_1 - k_2| \le 1$. Thus

$$\begin{aligned}
\mathfrak{d}_h(w, z) \le\ & \mathfrak{d}_h(w, \operatorname{Re} w + |w - z|i) + \mathfrak{d}_h(\operatorname{Re} w + |w - z|i, \operatorname{Re} z + |w - z|i) \\
& + \mathfrak{d}_h(z, \operatorname{Re} z + |w - z|i) \\
\le\ & \operatorname{diam}([\operatorname{Re} w, \operatorname{Re} w + \tfrac{3}{2}|w - z|i]; \mathfrak{d}_h) + \operatorname{diam}(S_{n-1,1,k_1}; \mathfrak{d}_h(\cdot,\cdot; U_{n-1,1,k_1})) \\
& + \operatorname{diam}(S_{n-1,1,k_2}; \mathfrak{d}_h(\cdot,\cdot; U_{n-1,1,k_2})) + \operatorname{diam}([\operatorname{Re} z, \operatorname{Re} z + \tfrac{3}{2}|w - z|i]; \mathfrak{d}_h).
\end{aligned}$$

Assuming now that $n - 2 \ge n_0$ we can use (3.1.25) to bound the first and fourth terms each by $(6|w - z|)^{\xi(Q-2)-u}$ and use (3.1.23) to bound the second and third terms each by $(2|w - z|)^{\xi(Q-2)-u}$. Along with (3.1.26), we have just shown that (3.1.15) holds with $C = \max\{7, 2(2^{\xi(Q-2)-u} + 6^{\xi(Q-2)-u})\}$ provided $|w - z| < 2^{-(n_0+1)}$. Since K clearly has finite $\mathfrak{d}_h$-diameter (e.g., combine (3.1.25) with $b = 2^{-n_0}$ and (3.1.23) for all j and k with $n = n_0$), the result for general w and z in K follows by possibly increasing the constant C.

Note that our exponent matches the one in [DFG+20, Prop. 3.18] for the zero-boundary GFF in the bulk, which is proved there to be optimal in the sense that $\mathfrak{d}_{h^{\text{wp}}}$ is almost surely not locally $(\xi(Q - 2) + u)$-Hölder continuous w.r.t. the Euclidean metric on any bounded open set for any $u > 0$. Since $\mathfrak{h}$ is continuous away from the boundary it is easy to see that the same holds for $\mathfrak{d}_h$. We obtain the same optimal exponent here because, as we have already seen, for the free-boundary GFF the maximum thickness at the boundary is the same as that in the bulk. ∎

Remark 3.1.10. *Note that the above argument provides near-minimal paths that do not intersect $\partial\mathbb{H}$ except possibly at their endpoints. In particular, it follows that with h, u, K, C as in the statement of Prop. 3.1.9, we have*

$$\sup_{z,w\in K\cap\mathbb{H}} \inf\left\{\mathrm{length}(P;\mathfrak{d}_h)\,\middle|\,P : z \rightsquigarrow w, P^{-1}(\partial\mathbb{H}) = \varnothing\right\} \le C(\mathrm{diam}\,K)^{\xi(Q-2)-u}. \tag{3.1.27}$$

3.1.5 Positive definiteness

The aim of this subsection is to prove the following:

Proposition 3.1.11. *If h is a free-boundary GFF on $\mathbb{H}$ with some choice of additive constant, then the function $\mathfrak{d}_h$ is almost surely a metric on $\overline{\mathbb{H}}$; in particular it is almost surely positive definite.*

Note that, since finiteness follows for instance from Lemma 3.1.8, we now have only to establish positive definiteness. Firstly we show positive definiteness at the boundary:

Lemma 3.1.12. *If h is a free-boundary GFF on $\mathbb{H}$ with some choice of additive constant, then almost surely for all $a, b \in \partial\mathbb{H}$ we have $\mathfrak{d}_h(a, b) > 0$.*

Proof. We want to show that, almost surely, $\mathfrak{d}_h(a, b) > 0$ whenever $a, b \in \partial\mathbb{H}$ are distinct. Firstly we can consider the analogous problem for the quantum wedge. Recall that part of [She16a, Thm 1.8] states that, if a $(\gamma - 2/\gamma)$-quantum wedge $(\mathbb{H}, h, 0, \infty)$ (equivalently, a wedge of weight 4) is decorated by an independent SLE_{γ^2} η in $\mathbb{H}$ from 0 to ∞, then the surfaces parametrized by the left and right components $(W_1, h|_{W_1}, 0, \infty)$ and $(W_2, h|_{W_2}, 0, \infty)$ are independent γ-quantum wedges (equivalently, wedges of weight 2).

If we take any two distinct points on $\eta \setminus \{0\}$, we know that they are at positive $\mathfrak{d}_h$-distance w.r.t. h in $\mathbb{H}$, since they are away from the boundary $\partial\mathbb{H}$ (since η does not hit $\partial\mathbb{H} \setminus \{0\}$ by [RS05, Thm 6.1]). The distance w.r.t. $\mathfrak{d}_h(\cdot, \cdot; W_i)$ cannot be less than that w.r.t. h, which means that any two distinct points on the right-hand (resp. left-hand) side of the boundary of W_1 (resp. W_2) are at positive LQG distance w.r.t. h in W_1 (resp. W_2). (By conformal covariance, this remains true regardless of the embedding of these wedges.) This suffices to establish positive definiteness of $\mathfrak{d}_h$ on $(0, \infty)$ for the γ-wedge. Note also that if we consider the canonical (circle-average) embedding of the wedge given by W_1 into $\mathbb{H}$, then fix a particular compact set $K \subset \overline{\mathbb{H}}$ not containing 0, we can find $L \subset \overline{\mathbb{H}}$ not containing 0 such that $K \subset \mathrm{int}\,L$ (i.e. the relative interior of L within $\overline{\mathbb{H}}$); then almost surely the LQG metric distance between K and ∂L in $\overline{\mathbb{H}}$ is positive, so that the positive definiteness of the LQG metric on $K \cap \partial\mathbb{H}$ is determined by the field values inside int L.

Given the result for the γ-wedge, we can deduce it for the free-boundary GFF on $\mathbb{H}$ by an absolute continuity argument, using the radial–lateral decomposition. Indeed, recalling the construction of the circle average embedding from §2.2.1, when we parametrize by $\mathcal{S}$, the field average on $\{t\} \times [0, \pi]$ for the γ-wedge can be expressed, for a standard two-sided Brownian motion B considered modulo vertical translation, as $B_{2(t+\tau)} + (Q-\gamma)(t+\tau)$ where τ is the last time at which $B_{2t} + (Q - \gamma)t$ hits 0. Translating horizontally by τ gives a field whose average on $\{t\} \times [0, \pi]$ is given by $B_{2t} + (Q - \gamma)t$, and whose lateral part (i.e., the part with mean zero on vertical line segments) has the same distribution as that of the wedge's lateral part (since it is independent of the radial part, with scale-invariant distribution). By conformal covariance, we again have positive definiteness for the LQG distance defined with respect to this field (when we map back to $\mathbb{H}$, on $(0, \infty)$).

Finally the process $B_{2t} + (Q - \gamma)t$, with t restricted to any compact subset of $\mathbb{R}$, is mutually absolutely continuous with the law of B_{2t}. So, considering the radial–lateral decomposition, the result for the wedge implies that for the free-boundary GFF (at least away from 0, but translation invariance then covers the case when a or b is 0). ■

In the light of Lemma 3.1.12 it is straightforward to complete the proof of Prop. 3.1.11.

Proof of Prop. 3.1.11. It remains only to rule out the possibility that there exist some $a \in \mathbb{R}$, $z \in \mathbb{H}$ at $\mathfrak{d}_h$-distance zero from each other. If we had $a \in \mathbb{R}$ and $\mathfrak{d}_h(a, z) = 0$ for some $z \neq a$, then by Lemma 3.1.12 we necessarily have $z \in \mathbb{H}$. Taking $z_n \to a$ w.r.t. the Euclidean metric, by Prop. 3.1.9 we also have $z_n \to a$ w.r.t. $\mathfrak{d}_h$. Thus $z_n \to z$ w.r.t. $\mathfrak{d}_h$, but (since $\operatorname{Im} z_n \to 0$) not w.r.t. the Euclidean metric. This contradicts the fact (noted in the discussion at the start of §3.1) that $\mathfrak{d}_h$ induces the Euclidean topology on $\mathbb{H}$, concluding the proof. ■

3.2 Further Hölder continuity estimates for γ-LQG metrics

In this section we will show that, almost surely, $\mathfrak{d}_h$ is in fact locally *bi*-Hölder continuous w.r.t. the Euclidean metric on $\overline{\mathbb{H}}$ (so in particular $\mathfrak{d}_h$ induces the Euclidean topology on $\overline{\mathbb{H}}$). First, however, we will show that the $\mathfrak{d}_h$-distance between two points of $\partial\mathbb{H}$ is almost surely bi-Hölder continuous w.r.t. the ν_h-measure of the interval between them. Throughout this subsection, h will always be a free-boundary GFF on $\mathbb{H}$ with some choice of additive constant (in this subsection it will never be necessary to specify the choice).

3.2.1 Upper bound on distance in terms of boundary measure

For what follows we will need to adapt the version of [DS11, Lemma 4.6] for the boundary measure ν_h.

Lemma 3.2.1. *There exist $c_1, c_2 > 0$ such that, for all $a \in \mathbb{R}$, $\varepsilon > 0$ and $\eta \leq 0$, we have*

$$\mathbb{P}[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}\sup_{\delta \leq \varepsilon}(h_\delta(a) - Q\log\frac{1}{\delta})}] \leq c_1 e^{-c_2\eta^2}.$$

Proof. Note that this result does not depend on the choice of additive constant for h, since adding a constant c scales both sides of the inequality by $e^{\gamma c/2}$. The original [DS11, Lemma 4.6] gives the almost sure lower tail bound

$$\mathbb{P}\left[\nu_h([a-\varepsilon, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}(h_\varepsilon(a) - Q\log\frac{1}{\varepsilon})} \middle| (h_{\varepsilon'}(a) : \varepsilon' \geq \varepsilon)\right] \leq C_1 e^{-C_2\eta^2},$$

for all $\eta \leq 0$, with $C_1, C_2 > 0$ deterministic constants independent of a, ε, η. (Actually in [DS11] the statement and proof are given just for an analogue involving the area measure μ_h instead of the boundary measure ν_h, but the proof is similar for the boundary measure – see [DS11, §6.3].) We will need a lower tail bound for the conditional law of $\nu_h([a, a+\varepsilon])$ given $(h_{\varepsilon'}(a) : \varepsilon' \geq \varepsilon)$, rather than for that of $\nu_h([a-\varepsilon, a+\varepsilon])$, but this in fact follows from the proofs of [DS11, Lemmas 4.5–4.6], which (when reformulated for the boundary measure) proceed by partitioning $[a-\varepsilon, a+\varepsilon]$ into $[a-\varepsilon, a]$ and $[a, a+\varepsilon]$ and thus actually obtain that, almost surely (possibly changing C_1, C_2):

$$\mathbb{P}\left[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}(h_\varepsilon(a) - Q\log\frac{1}{\varepsilon})} \middle| (h_{\varepsilon'}(a) : \varepsilon' \geq \varepsilon)\right] \leq C_1 e^{-C_2\eta^2}, \tag{3.2.1}$$

Observe also that if $\delta < \varepsilon$, since $[a, a+\delta] \subset [a, a+\varepsilon]$ we have, almost surely,

$$\mathbb{P}\left[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}(h_\delta(a) - Q\log\frac{1}{\delta})} \middle| (h_{\varepsilon'}(a) : \varepsilon' \geq \delta)\right] \leq C_1 e^{-C_2\eta^2}.$$

Moreover, if T is a stopping time for the process $(h_{e^{-t}}(a))_t$ such that $e^{-T} \leq \varepsilon$, then, by continuity of this process, the usual argument considering the discrete stopping times $T_n :=$

$2^{-n}\lceil 2^n T \rceil$ yields that

$$\mathbb{P}\left[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}(h_{e^{-T}}(a) - QT)} \middle| (h_{e^{-t}}(a) : t \leq T)\right] \leq C_1 e^{-C_2\eta^2}$$

almost surely on the event $\{T < \infty\}$. Note that, if we set $Y_t = h_{e^{-t}}(a) - Qt$, then $Z_{t,a} := \sup_{s\geq t} Y_s - Y_t$ is the maximum over all time of a standard Brownian motion with drift $-Q/2$, and is thus an $\mathrm{Exp}(Q)$ random variable. (This standard result can be obtained by using the Girsanov theorem to deduce the law of the maximum over the time interval $[0, T]$ from that for a Brownian motion without drift and then sending $T \to \infty$ – see [Pri14, Prop. 10.4].) Thus $\mathbb{P}[Z_{t,a} \leq \log 2] =: q$ is positive and independent of t and a. Now, let $T^M := \inf\{t \geq \log\frac{1}{\varepsilon} : Y_t \geq M\}$. Conditioning on $\{\sup_{t\geq \log\frac{1}{\varepsilon}} Y_t \geq M\}$ (equivalently, on $\{T^M < \infty\}$), the above applied to the stopping time T^M gives that

$$\mathbb{P}\left[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}M} \middle| \sup_{t\geq \log\frac{1}{\varepsilon}} Y_t \geq M\right] \leq C_1 e^{-C_2\eta^2}.$$

But since we also have

$$\mathbb{P}\left[\sup_{t\geq \log\frac{1}{\varepsilon}} Y_t \leq M + \log 2 \middle| \sup_{t\geq \log\frac{1}{\varepsilon}} Y_t \geq M\right] = \mathbb{P}\left[Z_{T^M,a} \leq \log 2 \middle| T^M < \infty\right] = q$$

(by the strong Markov property), by conditioning on which of the disjoint intervals of the form $[(n-1)\log 2, n\log 2)$ contains $\sup_{t\geq \log\frac{1}{\varepsilon}} Y_t$ we have

$$\mathbb{P}\left[\nu_h([a, a+\varepsilon]) < e^{\eta + \frac{\gamma}{2}(\sup_{t\geq \log\frac{1}{\varepsilon}} Y_t - \log 2)}\right] \leq C_1 q^{-1} e^{-C_2\eta^2},$$

from which the result follows. ■

Lemma 3.2.2. *Fix $\alpha \in (0, 2/d_\gamma)$. Then there exist a constant $\varepsilon > 0$ and a random integer N such that whenever $m \in \mathbb{N}$, $m \geq N$, $a \in [0,1] \cap 2^{-m}\mathbb{Z}$, we have*

$$\mathfrak{d}_h(a, a + 2^{-m}) \leq \nu_h([a, a+2^{-m}])^\alpha \cdot 2^{-m\varepsilon}. \tag{3.2.2}$$

Moreover, $\varepsilon > 0$ can be chosen so that the minimal such N satisfies $\mathbb{P}[N \geq n] = O(2^{-n\beta})$ for every $\beta \in (0, Q^2/4 - 1)$.

Proof. Recall $d_\gamma = \gamma/\xi$. Fix $\zeta > \gamma/(2\xi)$. Then to prove the result for $\alpha = \zeta^{-1}$, it suffices to show that for each fixed a and m, the complement of the event in (3.2.2) has probability $O(2^{-\lambda m})$ (uniformly for $a \in [0,1] \cap 2^{-m}\mathbb{Z}$) for some $\lambda > 1$ (then we can apply Borel–Cantelli to prove the result with $\beta = \lambda - 1$). Put $b = a + 2^{-m}$. By Prop. 3.1.6 (and the subsequent Remark 3.1.7), for any $\varepsilon > 0$, it holds that, for every n, we have

$$\begin{aligned}
&\mathbb{P}[\nu_h([a,b]) \le |a-b|^{-\zeta\varepsilon}\mathfrak{d}_h(a,b)^\zeta] \\
&\le \mathbb{P}\left[\nu_h([a,b]) \le |a-b|^{-2\zeta\varepsilon}\left(\int_{\log(|a-b|^{-1})}^\infty e^{\xi(h_{e^{-t}}(a)-Qt)}\,dt\right)^\zeta\right] \\
&\quad + \mathbb{P}\left[\nu_h([a,b]) \le |a-b|^{-2\zeta\varepsilon}\left(\int_{\log(|a-b|^{-1})}^\infty e^{\xi(h_{e^{-t}}(b)-Qt)}\,dt\right)^\zeta\right] \\
&\quad + O(|a-b|^n).
\end{aligned}$$

Consider the first of the probabilities on the RHS. By [DMS21, Lemma A.5], if we let the integral be I and the supremum of its integrand be M, then for any $p > 0$ there is some $c_p < \infty$ for which we have

$$\mathbb{E}[I^p | M] \le c_p M^p.$$

(Actually [DMS21, Lemma A.5] is stated for an integral with lower limit 0 and a Brownian motion started from 0, so to obtain our statement we can, say, use the lemma to bound the expectation conditional on both $h_{|a-b|}(a)$ and $\sup_{u\ge 0} e^{\xi(h_{e^{-u}|a-b|}(a)-h_{|a-b|}(a)-Qu)}$, and use that these have finite moments of all positive orders.) Dividing and taking expectations we find that I/M has finite moments of all positive orders, so that by Markov's inequality, for any $k > 0$ we have that I is bounded by $|a-b|^{-k}$ times M except on an event of superpolynomially decaying probability (in $|a-b|^{-1}$). Thus we have, for every n:

$$\begin{aligned}
&\mathbb{P}\left[\nu_h([a,b]) \le |a-b|^{-2\zeta\varepsilon}\left(\int_{\log(|a-b|^{-1})}^{\infty} e^{\xi(h_{e^{-t}}(a)-Qt)}\,dt\right)^{\zeta}\right] \\
&\le \mathbb{P}\left[\nu_h([a,b]) \le |a-b|^{-2\zeta\varepsilon}\exp\left(\zeta\xi \sup_{t\ge\log(|a-b|^{-1})}(h_{e^{-t}}(a)-Qt)\right)\right] + O(|a-b|^n) \\
&\le \mathbb{P}\left[\nu_h([a,b]) \le |a-b|^{\zeta\varepsilon}\exp\left(\frac{\gamma}{2}\sup_{t\ge\log(|a-b|^{-1})}(h_{e^{-t}}(a)-Qt)\right)\right] + O(|a-b|^n) \\
&\quad + \mathbb{P}\left[\zeta\xi \sup_{t\ge\log(|a-b|^{-1})}(h_{e^{-t}}(a)-Qt) \ge 3\zeta\varepsilon\log|a-b| + \frac{\gamma}{2}\sup_{t\ge\log(|a-b|^{-1})}(h_{e^{-t}}(a)-Qt)\right].
\end{aligned} \tag{3.2.3}$$

Note that the first probability on the right-hand side of (3.2.3) decays superpolynomially in $|a-b|$ by Lemma 3.2.1, whereas the last probability is equal to

$$\mathbb{P}\left[\sup_{t\ge\log(|a-b|^{-1})}(h_{e^{-t}}(a)-Qt) \ge \frac{3\zeta\varepsilon}{\zeta\xi-\gamma/2}\log|a-b|\right]. \tag{3.2.4}$$

It thus suffices to show that the union of the event in (3.2.4) over all $m \ge n$, $a \in [0,1]\cap 2^{-m}\mathbb{Z}$, $b = a + 2^{-m}$ has probability $O(2^{-n\beta})$ for some $\beta > 0$. Setting $\delta = \frac{3\zeta\varepsilon}{\zeta\xi-\gamma/2}$, this union is contained in the event

$$\left\{\sup_{r\le 2^{-n}}\sup_{x\in[0,1]}\frac{h_r(x)}{\log(1/r)} \ge Q-\delta\right\},$$

which by (3.1.6) has probability $O(2^{-n((Q-\delta)^2/4-1-u)})$ for every $u > 0$. Since ε and therefore δ can be made arbitrarily small, this completes the proof for arbitrary $\beta \in (0, Q^2/4-1)$ (note that the result is non-trivial since $Q > 2$). ■

By dyadically partitioning each boundary interval we can now establish local Hölder continuity for $\mathfrak{d}_h$ w.r.t. ν_h.

Proposition 3.2.3. *Fix $\alpha \in (0, 2/d_\gamma)$. Then almost surely there exists $C' \in (0,\infty)$ such that whenever a, b are in $[0,1]$, we have*

$$\mathfrak{d}_h(a,b) \le C'\nu_h([a,b])^{\alpha}.$$

Moreover $\mathbb{P}[C' \ge x]$ decays at worst polynomially in x.

Proof. With N as in Lemma 3.2.2, note that for $m < N$, for $a \in [0,1] \cap 2^{-m}\mathbb{Z}$, we have

$$\begin{aligned}\mathfrak{d}_h(a, a+2^{-m}) &\leq \sum_{i=1}^{2^{N-m}} \mathfrak{d}_h(a+(i-1)2^{-N}, a+i2^{-N}) \\ &\leq 2^{-N\varepsilon} \sum_{i=1}^{2^{N-m}} \nu_h([a+(i-1)2^{-N}, a+i2^{-N}])^\alpha \\ &\leq 2^{-N\varepsilon} 2^{(N-m)(1-\alpha)} \nu_h([a, a+2^{-m}])^\alpha,\end{aligned}$$

using the power mean inequality (since $\alpha < 2/d_\gamma < 1$). It follows that there is a random constant C such that whenever $m \in \mathbb{N}$, $a \in [0,1] \cap 2^{-m}\mathbb{Z}$, we have

$$\mathfrak{d}_h(a, a+2^{-m}) \leq C \nu_h([a, a+2^{-m}])^\alpha \cdot 2^{-m\varepsilon},$$

and that, assuming $\varepsilon < 1-\alpha$,

$$\mathbb{P}[C \geq 2^{n(1-\alpha-\varepsilon)}] \leq \mathbb{P}[N \geq n] = O(2^{-n\beta}),$$

i.e., C has polynomial decay. We now argue as in the proof of the Kolmogorov criterion: if a and b are dyadic rationals in $[0,1]$ we can partition $[a,b]$ as $a = a_0 < a_1 < \cdots < a_l = b$ where $[a_{i-1}, a_i] = [2^{-m_i} n_i, 2^{-m_i}(n_i+1)]$ for non-negative integer m_i, n_i such that no three of the m_i are equal. Then we have

$$\frac{\mathfrak{d}_h(a,b)}{\nu_h([a,b])^\alpha} \leq \sum_{i=1}^{l} \frac{\mathfrak{d}_h(a_{i-1}, a_i)}{\nu_h([a_{i-1}, a_i])^\alpha} \leq 2C \sum_{n=0}^{\infty} 2^{-n\varepsilon} =: C'. \tag{3.2.5}$$

The same argument works for a, b arbitrary, using a countably infinite partition $a = a_0 < a_1 < a_2 < \cdots$ with $a_n \uparrow b$. In order to obtain the analogue of the first inequality in (3.2.5) we thus have to justify that

$$\mathfrak{d}_h(a,b) \leq \sum_{i=1}^{\infty} \mathfrak{d}_h(a_{i-1}, a_i). \tag{3.2.6}$$

This follows by using the triangle inequality to obtain

$$\mathfrak{d}_h(a,b) \leq \sum_{i=1}^{n} \mathfrak{d}_h(a_{i-1}, a_i) + \mathfrak{d}_h(a_n, b)$$

and noting that $\mathfrak{d}_h(a_n, b) \to 0$ by Prop. 3.1.9. ∎

3.2.2 Local reverse Hölder continuity

For the metric gluing proof we will need a Hölder exponent for the Euclidean metric w.r.t. $\mathfrak{d}_h$. We know this exists away from $\partial\mathbb{H}$, since this is true for the whole-plane GFF by the results of [DFG+20], and the harmonic correction is well-behaved away from the boundary. However, we can obtain Hölder continuity even at the boundary. We can establish such a reverse Hölder inequality by an argument based on [DFG+20, Lemma 3.22]. First we will need the following lemma:

Lemma 3.2.4. *Let $\alpha < Q$, and let h be either the circle average embedding into $\mathbb{H}$ of an α-quantum wedge, or equal to $h^F - \alpha \log|\cdot|$ where h^F is a free-boundary GFF on $\mathbb{H}$ with $h_1^F(0) = 0$. Then, almost surely, every $\mathfrak{d}_h$-bounded subset of $\overline{\mathbb{H}}$ is also Euclidean-bounded.*

Proof. First observe that since $\mathfrak{d}_h$ is locally bounded (indeed, locally Hölder continuous) w.r.t. the Euclidean topology, as $a \downarrow 0$,

$$\mathbb{P}\left[\mathfrak{d}_h(\partial B(0,r) \cap \overline{\mathbb{H}}, \partial B(0,2r) \cap \overline{\mathbb{H}}) \geq a r^{\xi Q} e^{\xi h_r(0)}\right] \to 1,$$

and that by conformal covariance this probability does not depend on r. Since in (2.1.1), α_n and $\sigma(h|_{\mathbb{H}\setminus B(0,r)})$ are independent if $\operatorname{supp}\varphi_n \subset B(0,r)$, the tail σ-algebra $\bigcap_{r>0}\sigma(h|_{\mathbb{H}\setminus B(0,r)})$ is trivial, and from this and the fact that $\mathfrak{d}_h$ is locally determined by h it follows that for a large enough there are almost surely infinitely many $k \in \mathbb{N}$ for which

$$\mathfrak{d}_h(\partial B(0,2^k) \cap \overline{\mathbb{H}}, \partial B(0,2^{k+1}) \cap \overline{\mathbb{H}}) \geq a 2^{k\xi Q} e^{\xi h_{2^k}(0)}.$$

Since $e^{t\xi Q} e^{\xi h_{e^t}(0)} \to \infty$ as $r \to \infty$ (for $t \geq 0$ it has the law of the exponential of a Brownian motion with positive drift $Q - \alpha$, albeit started at the last time this process hits 0 in the case of a wedge), it follows that for any compact $K \subset \overline{\mathbb{H}}$, $\mathfrak{d}_h(K, \partial B(0,r)) \to \infty$ as $r \to \infty$. ■

Proposition 3.2.5. *Let h be a free-boundary GFF on $\mathbb{H}$ with some choice of additive constant. Almost surely, for each $u > 0$ and $K \subset \overline{\mathbb{H}}$ compact there exists $C \in (0,\infty)$ such that whenever a, $b \in \mathbb{R}$ with a, $a + bi \in K$, we have*

$$\mathfrak{d}_h(a, a+bi) \geq C^{-1} b^{\xi(Q+2)+u}.$$

Proof. Since we know that $\operatorname{diam}(K; \mathfrak{d}_h)$ is finite, by Lemma 3.2.4 we can find a (random)

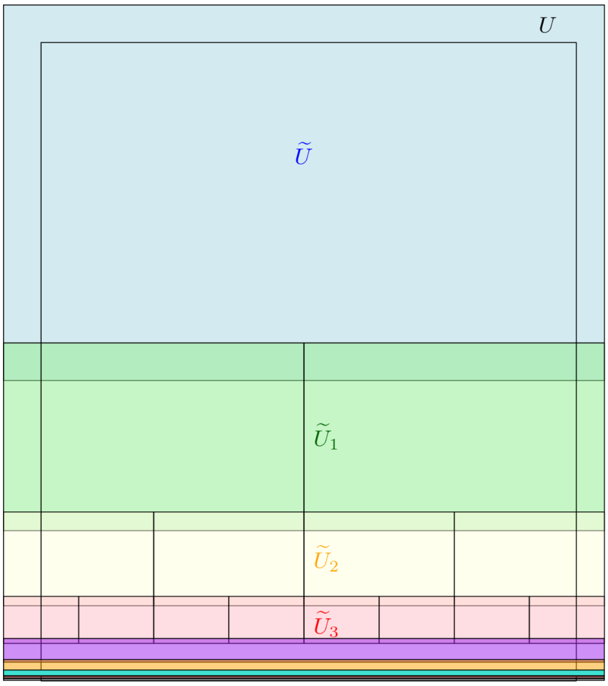

Figure 3.2: The sets $\widetilde{U}, \widetilde{U}_n$ used in the proof of Prop. 3.2.5.

$K' \supseteq K$ compact so that $\mathfrak{d}_h(K, \partial K' \setminus \partial\mathbb{H}) > \operatorname{diam}(K; \mathfrak{d}_h)$, so that $\mathfrak{d}_h(z,w) = \mathfrak{d}_h(z,w; \operatorname{int} K')$ for each $z, w \in K$. Thus it suffices to prove the result for $\mathfrak{d}_h(a, a+bi; U)$ and each $a, a+bi \in U$ for each bounded U open in $\overline{\mathbb{H}}$.

Choose U to be an axis-parallel rectangle containing 0 with dyadic rational vertices. Let $\widetilde{U}$ be another such rectangle containing $U \cap (\mathbb{H}+i)$ so that the lower vertices of $\widetilde{U}$ have imaginary part greater than $1/2$ and the upper vertices have imaginary part greater than 2, so that there exists $\widetilde{U}_n$ a union of 2^n horizontal translates of $2^{-n}\widetilde{U}$ covering

$$(U \cap (\mathbb{H} + 2^{-n}i)) \setminus (\mathbb{H} + 2^{-(n-1)}i)).$$

Define h^{wp} and $\mathfrak{h}$ by coupling with h as in Prop. 3.1.5. By [DFG+20, Lemma 3.22] (and the Borel–Cantelli lemma applied to $\varepsilon = 2^{-n}$) we know that the Euclidean metric on K is $(\chi')^{-1}$-Hölder continuous w.r.t. $\mathfrak{d}_{h^{\mathrm{wp}}}$ for each $\chi' > \xi(Q+2)$. By Borell–TIS and a union bound over the 2^n translates of $2^{-n}\widetilde{U}$, we find that there are c_1, c_2 for which, for each $t \geq 0$,

$$\mathbb{P}\left[\inf_{z \in \widetilde{U}_n} e^{\xi(\mathfrak{h}(z) - \mathfrak{h}(2^{-n}i))} \leq e^{-t}\right] \leq 2^n c_1 e^{-c_2 t^2}.$$

Note that setting $t = n\varepsilon \log 2$ makes this summable. As before, we can get a similar tail bound for $\mathfrak{h}(2^{-n}i) - \mathfrak{h}(i)$; combining all of these we find that (with $\chi = \chi' - 2\varepsilon$) we get that for every $\chi > \xi(Q+2)$ there is almost surely a finite constant $C > 0$ for which we have

$$\mathfrak{d}_h(\mathbb{R}+2^{-(n-1)}i, \mathbb{R}+2^{-n}i; U) \geq C2^{-n\chi} \tag{3.2.7}$$

for all n, from which the result follows. ■

We now deduce estimates for LQG areas of Euclidean balls from the results of [RV10].

Proposition 3.2.6. *Fix $K \subset \mathbb{H}$ compact; let h be a free-boundary GFF on $\mathbb{H}$ with the constant fixed such that $h_1(0) = 0$. Then whenever $\zeta_1 > \gamma(Q+2) > \gamma(Q-2) > \zeta_2 > 0$ there almost surely exists a random $\varepsilon_0 > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $z \in K$, if $B(z,\varepsilon)$ is the Euclidean ball of radius ε around z then*

$$\varepsilon^{\zeta_1} \leq \mu_h(B(z,\varepsilon) \cap \overline{\mathbb{H}}) \leq \varepsilon^{\zeta_2}.$$

Moreover, define

$$s_+ = \frac{4\gamma^2 + 2\sqrt{2}\gamma\sqrt{(2+\gamma^2)(8+\gamma^2)}}{(4+\gamma^2)^2}.$$

If the condition on K is weakened to $K \subset \overline{\mathbb{H}}$ compact (so that K is allowed to intersect $\partial\mathbb{H}$), then provided $\zeta_2 \in (0, \gamma(Q-2)(1-s_+))$ there still almost surely exists a random $\varepsilon_0 > 0$ such that the upper bound holds for all $z \in K$, $\varepsilon \in (0, \varepsilon_0)$.

Proof. We will first assume that K is at positive distance from $\partial\mathbb{H}$, using the fact [RV10, Prop. 3.5, Prop. 3.6] that the $\mu_{h^{\mathrm{wp}}}$-area of a fixed Euclidean ball has finite p^{th} moments for all $p \in (-\infty, 4/\gamma^2)$. Fix the additive constant for h^{wp} so that $h_1^{\mathrm{wp}}(0) = 0$, and for each $p \in (-\infty, 4/\gamma^2)$ define $C_p := \mathbb{E}[\mu_{h^{\mathrm{wp}}}(B(0,1))^p]$. Then by (1.1.2), we have

$$\mu_{h^{\mathrm{wp}}}(B(0,\varepsilon)) \overset{(d)}{=} e^{\gamma h_\varepsilon^{\mathrm{wp}}(0)}\varepsilon^{\gamma Q}\mu_{h^{\mathrm{wp}}}(B(0,1)). \tag{3.2.8}$$

If $p > 0$, fixing $q, q' > p$ with $1/q + 1/q' = 1/p$ and using that $h_\varepsilon^{\mathrm{wp}}(0) \sim N(0, \log(1/\varepsilon))$, we obtain by Hölder's inequality that

$$\mathbb{E}[\mu_{h^{\mathrm{wp}}}(B(0,\varepsilon))^p] \leq C_q^{p/q}\varepsilon^{\gamma Q p - \gamma^2 p q'/2}. \tag{3.2.9}$$

Likewise, since the centred Gaussian variables $h_\varepsilon^{\mathrm{wp}}(z) - h_1^{\mathrm{wp}}(0)$ have variance $\log(1/\varepsilon) + O(1)$ uniformly in ε and $z \in K$, we obtain the same result up to a multiplicative constant for

$\mu_{h^{\mathrm{wp}}}(B(z,\varepsilon))$ and conclude by taking q' sufficiently close to p that, for each $p, u > 0$, we have

$$\mathbb{E}[\mu_{h^{\mathrm{wp}}}(B(z,\varepsilon))^p] \le C(p,u)\varepsilon^{\gamma p(Q-\gamma p/2)-u}$$

for each $\varepsilon > 0$ and each $z \in K$, where $C(p,u)$ depends neither on ε nor on z.

Fixing a neighbourhood U of K still at positive distance from $\partial\mathbb{H}$, since $\sup_U(h - h^{\mathrm{wp}})$ has a Gaussian tail, a further application of Hölder's inequality gives the same result for h (except with a different $C(p,u)$ and only over choices of ε and z such that $B(z,\varepsilon) \subset U$). Setting $p = 2/\gamma < 4/\gamma^2$, we obtain the exponent $2Q - 2 - u$. Now we can cover K by $O(\varepsilon^{-2})$ balls of radius ε such that each ball B_ε satisfies

$$\mathbb{P}[\mu_h(B_\varepsilon) > \varepsilon^{\zeta_2}] \le \varepsilon^{-p\zeta_2}\mathbb{E}[\mu_h(B_\varepsilon)^p] = O(\varepsilon^{2Q-2-2\zeta_2/\gamma-u}).$$

This exponent is greater than 2 whenever $\zeta_2 < \gamma(Q-2)$ and u is chosen small enough, so applying Borel–Cantelli to covers C_n with $\varepsilon = 2^{-n}$ for each n, and noting that each ball of radius ε centred in K and such that $B(z, 2\varepsilon)$ is contained in U can be covered by an absolute constant number of balls in $C_{\lfloor \log_2(1/\varepsilon)\rfloor}$, we obtain the second inequality whenever $\zeta_2 < \gamma(Q-2)$. (Note that $Q - 2 > 0$.)

The same argument for $p = -2/\gamma$, considering $\inf_U(h - h^{\mathrm{wp}})$ instead, produces the bound

$$\mathbb{P}[\mu_h(B_\varepsilon) < \varepsilon^{\zeta_1}] = O(\varepsilon^{2\zeta_1/\gamma-2Q-2-u}),$$

giving summability whenever $\zeta_1 > \gamma(Q+2)$ and hence the first inequality.

We will again employ the result [She16a, Thm 1.8] that a $(\gamma - 2/\gamma)$-quantum wedge (a wedge of weight 4) is cut by an independent SLE_{γ^2} (call it η) into two independent γ-quantum wedges (of weight 2). (It suffices to prove the result for a wedge, by mutual absolute continuity.) Parametrize the original wedge by $(\mathbb{H}, h, 0, \infty)$, so that if η is an independent SLE_{γ^2} from 0 to ∞ and W^-, W^+ are respectively the left and right sides of η, then $(W^-, h|_{W^-}, 0, \infty)$ and $(W^+, h|_{W^+}, 0, \infty)$ are independent γ-quantum wedges.

It follows from [RS05, Thm 5.2] that the components of the complement of an SLE_κ in a smooth bounded domain for $\kappa \in (0,4)$ are almost surely Hölder domains. Therefore (e.g. by using a Möbius map to transfer from $\mathbb{H}$ to the unit disc) we have a conformal map $\varphi\colon \mathbb{H} \to W$ that is almost surely *locally* Hölder continuous (away from a single point on $\partial\mathbb{H}$)

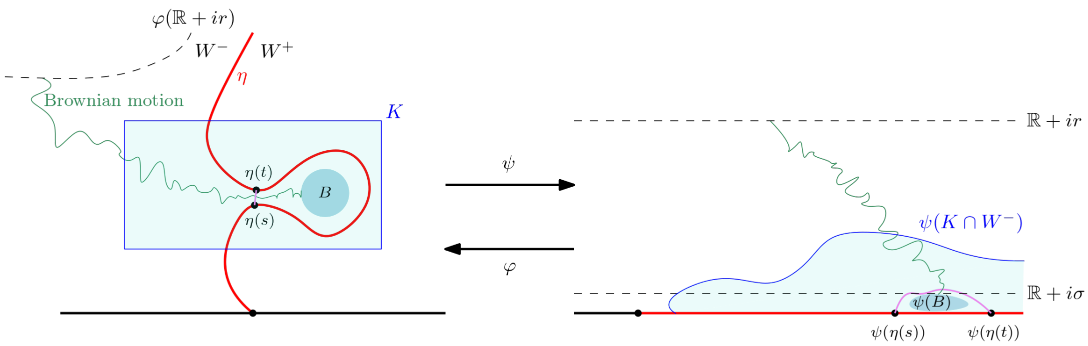

Figure 3.3: We prove a bound on the narrowness of bottlenecks in SLE_κ curves for $\kappa \in (0,4)$. If $\operatorname{diam}\eta([s,t]) \gg |\eta(s)-\eta(t)|$, we have a large ball B surrounded by the union of $\eta([s,t])$ and the line segment $[\eta(s),\eta(t)]$. Since a Brownian motion started on $\varphi(\mathbb{R}+ir)$ is unlikely to hit B before exiting W^-, a Brownian motion started on $\mathbb{R}+ir$ is unlikely to hit $\psi(B)$ before exiting $\mathbb{H}$, making $\operatorname{diam}\psi(B)$ small. This is impossible since the conformal coordinate change preserves quantum areas, which are bounded above and below by polynomials in Euclidean diameter, so the diameter of $\psi(B)$ is no smaller than a certain power of the diameter of B.

for $W = W^-$ or W^+. Moreover [GMS18, Cor. 1.8] gives that this holds with any exponent $\alpha < 1-s_+$ (and [GMS18, Remark 1.2] we have $s_+ < 1$). So even if K intersects $\partial\mathbb{H}$, if $0 \notin K$ we can use φ to map K Hölder-continuously to a subset of W away from $\partial\mathbb{H}$, then deduce the upper bound for K from that for $\varphi(K)$ and $\mu_{\widetilde{h}}$, where $\widetilde{h} = h\circ\varphi^{-1} + Q\log(|\varphi'|^{-1})$. ■

We now prove Hölder continuity for the Euclidean metric w.r.t. $\mathfrak{d}_h$. We begin by establishing this at the boundary:

Proposition 3.2.7. *Let h be a free-boundary GFF on $\mathbb{H}$ with the additive constant fixed in some way. There exists $\widetilde{\beta} > 0$ such that the following holds. Almost surely, for every $u \in (0,\widetilde{\beta})$ and each fixed compact interval $I \subset \mathbb{R}$ there is a finite constant $C > 0$ such that $|x-y| \le C\mathfrak{d}_h(x,y)^{(\widetilde{\beta}-u)}$ for all $x,y \in I$.*

In order to prove this we will begin by proving Prop. 1.3.8, showing that SLE_κ curves for $\kappa < 4$ cannot bottleneck too much.

Proof of Prop. 1.3.8. Let h, η and $W^\pm$ be as in the proof of Prop. 3.2.6 (so η cuts the wedge $(\mathbb{H},h,0,\infty)$ of weight 4 into independent wedges of weight 2 parametrized by $W^\pm$). We will need the result of [MMQ21, §4.2] that for $\kappa < 8$, chordal SLE_κ curves in $\mathbb{H}$ from 0 to ∞ almost surely satisfy the following *non-tracing hypothesis*: for any compact rectangle $K \subset \mathbb{H}$ and any $\widetilde{\alpha} > \widetilde{\xi} > 1$ there exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$, and any t such that $\eta(t) \in K$, there exists a point y with the following properties:

- $B(y,\delta^{\widetilde{\alpha}}) \subseteq B(\eta(t),\delta)\setminus\eta$, and $B(y,2\delta^{\widetilde{\alpha}})$ intersects η;

- if O is the connected component of y in $B(\eta(t), \delta) \setminus \eta$, and $a \in \partial O \setminus \eta(t;\delta)$, then every path in $O \cup \{a\}$ from y to a exits the ball $B(y, \delta^{\tilde{\xi}})$. (Here $\eta(t;\delta))$ is defined as the SLE segment $\eta([\sigma, \tau])$, where σ and τ are respectively the last time before t, and the first time after t, that η hits $B(\eta(t), \delta)$.)

Note that the proof in that paper can be used to find such points on either side of η, and that the hypothesis does not depend on the parametrization of η; indeed, the parametrization will not matter for what follows, so we will choose to parametrize our SLE curves by capacity.

Let W be either W^- or W^+, fix compact axis-parallel rectangles $K', K \subset \mathbb{H}$ such that $K' \subset \operatorname{int} K$, and fix a conformal map $\psi : W \to \mathbb{H}$ fixing 0 and ∞, with inverse φ. Fix r such that $\operatorname{Im} z \le r/2$ for each $z \in \psi(K \cap W)$. Given $0 < s < t$ such that $\eta(s), \eta(t) \in K$, let $[\eta(s), \eta(t)]$ be the straight line segment from $\eta(s)$ to $\eta(t)$, and let $\ell = |\eta(s) - \eta(t)|$. Also let $\mathbb{P}_z$ be the law of a complex Brownian motion started at z. Then if B is a closed ball in $\mathbb{H}$ contained in K' and B is disconnected from $\varphi(\mathbb{R} + ir)$ by the union of η and $[\eta(s), \eta(t)]$, the Beurling estimate gives that there exists $c > 0$ such that, for each $z \in B$, $a \in \mathbb{R}$,

$$\mathbb{P}_z[\text{hit}\, \varphi(\mathbb{R} + ir)\, \text{before}\, \eta \cup \mathbb{R}] \le \frac{c\,\ell^{1/2}}{(\operatorname{dist}(K', \partial K))^{1/2}},$$
$$\mathbb{P}_{\varphi(a+ir)}[\text{hit}\, B\, \text{before}\, \eta \cup \mathbb{R}] \le \frac{c\,\ell^{1/2}}{(\operatorname{dist}(K', \partial K))^{1/2}}.$$

By conformal invariance of Brownian motion, applying ψ gives

$$\mathbb{P}_{\psi(z)}[\text{hit}\, \mathbb{R} + ir\, \text{before}\, \mathbb{R}] \le \frac{c\,\ell^{1/2}}{(\operatorname{dist}(K', \partial K))^{1/2}},$$
$$\mathbb{P}_{a+ir}[\text{hit}\, \psi(B)\, \text{before}\, \mathbb{R}] \le \frac{c\,\ell^{1/2}}{(\operatorname{dist}(K', \partial K))^{1/2}}. \tag{3.2.10}$$

Let $\sigma = \sup_{z \in \psi(B)} \operatorname{Im} z$. Then by compactness we can choose $a \in \mathbb{R}$ so that $a + i\sigma \in \psi(B)$. Now by gambler's ruin we have

$$\mathbb{P}_{a+i\sigma}[\text{hit}\, \mathbb{R} + ir\, \text{before}\, \mathbb{R}] = \frac{\sigma}{r} \tag{3.2.11}$$

and therefore $\sigma/r \le \frac{c\,\ell^{1/2}}{(\operatorname{dist}(K', \partial K))^{1/2}}$.

Suppose $z, w \in \psi(B)$ with $|\operatorname{Re}(z-w)| = \rho > 0$. Without loss of generality suppose $\operatorname{Im} z > \operatorname{Im} w$. Set $a = \operatorname{Re}(\frac{z+w}{2})$. Then set L_1, L_2, L_3 to be horizontal line segments of respective lengths $\rho/2$, $2\rho/3$, $2\rho/3$ and centres $a + i \operatorname{Im} z$, a, $a + i \operatorname{Im} z$. Let R be the unique rectangle that has L_2 and L_3 as opposite sides. If a Brownian motion from $a+ir$ exits $\mathbb{H}+i\operatorname{Im} z$ through L_1 and after hitting L_1 makes an excursion across R from L_3 to L_2 without hitting the vertical sides of R or hitting $(\mathbb{R}+i\operatorname{Im} z)\setminus L_3$, then it must hit $\psi(B)$ before $\mathbb{R}$. Since we already have $\operatorname{Im} z, \operatorname{Im} w = O(\ell^{1/2})$, the probability from $a+ir$ of exiting $\mathbb{H}+i\operatorname{Im} z$ through L_1 is $\frac{2}{\pi}\arctan\frac{\rho}{4(r-\operatorname{Im} z)}$ (using the Poisson kernel in $\mathbb{H}$), whereas assuming $\rho \geq \ell^{1/2}$ (and thus $\rho = \Omega(\sigma) = \Omega(\operatorname{Im} z)$) the probability from u of the latter event can be bounded below by a constant uniformly for $u \in L_1$, so by the second inequality in (3.2.10) we must have $\rho = O(\ell^{1/2})$. We have therefore shown that the (Euclidean) diameter of $\psi(B)$ is $O(\ell^{1/2})$.

Now, given $s<t$ and $\widetilde{\alpha}>1$, we can apply the non-self-tracing hypothesis to the ball

$$B\left(\eta(u), \frac{1}{4}\operatorname{diam}\eta([s,t])\right)$$

for some $s<u<t$, so that the connected component of $\eta(u)$ in that ball is a subsegment of $\eta([s,t])$. This gives us a ball B of radius $\left(\frac{1}{4}\operatorname{diam}\eta([s,t])\right)^{\widetilde{\alpha}}$ which is disconnected from ∞ by the union of η and the straight line segment from $\eta(s)$ to $\eta(t)$ (recalling that we can choose the ball to be on the appropriate side of η).

We can now compare $\operatorname{diam}\eta([s,t])$ to ℓ using Prop. 3.2.6, which implies that for ζ_1 and ζ_2 as in the statement of that proposition, if B is the ball above, we have $\mu_h(B) = \Omega((\operatorname{diam}\eta([s,t]))^{\widetilde{\alpha}\zeta_1})$, but if $\widetilde{h} = h\circ\varphi + Q\log|\varphi'|$ then $\widetilde{h}$ itself has the law of a quantum wedge, so we have $\mu_h(B) = \mu_{\widetilde{h}}(\psi(B)) = O(\ell^{\zeta_2/2})$. So $\operatorname{diam}\eta([s,t]) = O(\ell^{\zeta_2/(2\widetilde{\alpha}\zeta_1)})$, as required. ■

We will now prove Prop. 3.2.7 from the results of [DFG+20] giving Hölder continuity away from the boundary.

Proof of Prop. 3.2.7. Continuing in the setting of the proof of Prop. 1.3.8, observe that another use of the Poisson kernel in $\mathbb{H}$ gives that, for $a = \frac{1}{2}\operatorname{Re}(\psi(\eta(s))+\psi(\eta(t)))$,

$$\mathbb{P}_{a+ir}[\text{exit}\mathbb{H}\text{through}[\psi(\eta(s)),\psi(\eta(t))]] = \Theta(|\psi(\eta(s))-\psi(\eta(t))|).$$

But the LHS is equal to

$$\mathbb{P}_{\varphi(a+ir)}[\text{exit}W\text{through}\eta([s,t])] = O((\operatorname{diam}\eta([s,t]))^{1/2})$$

(by the Beurling estimate). Combining this with the diameter estimate we find

$$|\psi(\eta(s)) - \psi(\eta(t))| = O(\ell^{\zeta_2/(4\widetilde{\alpha}\zeta_1)}) = O(|\eta(s) - \eta(t)|^{\zeta_2/(4\alpha\zeta_1)}).$$

In other words, ψ is locally Hölder continuous on η. Note that for a fixed compact set $K \subset \mathbb{H}$ at positive distance from $\partial\mathbb{H}$, we have χ^{-1}-Hölder continuity of the Euclidean metric w.r.t. $\mathfrak{d}_h$ on K for any $\chi > \xi(Q+2)$ (this follows from [DFG+20, Prop. 3.18] for the whole-plane GFF h^{wp} and the almost sure finiteness of $\sup_K \mathfrak{h}$). So if $\eta(s), \eta(t) \in K$ we have

$$|\psi(\eta(s)) - \psi(\eta(t))| = O(|\eta(s) - \eta(t)|^{\zeta_2/(4\widetilde{\alpha}\zeta_1)}) = O(\mathfrak{d}_h(\eta(s), \eta(t))^{\zeta_2/(4\widetilde{\alpha}\chi\zeta_1)}).$$

Observing finally that $\mathfrak{d}_h(\eta(s), \eta(t)) \le \mathfrak{d}_h(\eta(s), \eta(t); W) = \mathfrak{d}_{\widetilde{h}}(\psi(\eta(s)), \psi(\eta(t)))$ gives the desired Prop. 3.2.7 with $\widetilde{\beta} = \zeta_2/(4\widetilde{\alpha}\chi\zeta_1)$. Observe that since we require $\widetilde{\alpha} > 1$, $\chi > \xi(Q+2)$, $\zeta_1 > \gamma(Q+2) > \gamma(Q-2)(1-s_+) > \zeta_2$, we obtain that the result holds in the range

$$0 < \widetilde{\beta} < \frac{(Q-2)(1-s_+)}{4\xi(Q+2)^2}.$$

■

Proof of Prop. 1.3.7. It suffices to prove the left-hand inequality, since the right-hand inequality is given by Prop. 3.1.9. Let $\chi > \xi(Q+2)$ and $\sigma < \xi(Q-2)$ be arbitrary. Fix some $\beta < \widetilde{\beta}$, i.e. $\beta < (Q-2)(1-s_+)/(4\xi(Q+2)^2)$. To recap what we have proven so far, suppose we are on the intersection of the almost sure events of Prop. 3.2.5 and Prop. 3.2.7. Then fixing $K \subset \overline{\mathbb{H}}$ compact, there is some finite $C > 0$ on which we have

$$|(a_1 + b_1 i) - (a_2 + b_2 i)| \le C\mathfrak{d}_h(a_1 + b_1 i, a_2 + b_2 i)^\beta \tag{3.2.12}$$

provided $a_1 + b_1 i, a_2 + b_2 i \in K$ and *either* $b_1 = b_2 = 0$ *or* $b_1 = 0, a_1 = a_2$. (Note that we can use Prop. 3.2.5 because $\beta < (\xi(Q+2))^{-1}$.)

In order to remove this second condition and thus deduce a Hölder exponent for the Euclidean metric w.r.t. $\mathfrak{d}_h$ on a compact set $K \subseteq \overline{\mathbb{H}}$, we will split into cases. Fix $\rho > 1/(\beta\sigma)$. Note that $1/(\beta\sigma)$ can be made arbitrarily close to $4(Q+2)^2/((Q-2)^2(1-s_+)$ and thus can be chosen to force $\rho > 1$. Suppose $\operatorname{Im} w \ge \operatorname{Im} z$ and $\operatorname{Im} w > 0$. We will justify that, almost surely, there exist finite constants $C_1, C_2, C_3, C_4 > 0$ such that:

1. Firstly, if $\operatorname{Im} w > |w-z|^\rho$ then

$$\mathfrak{d}_h(w, \mathbb{R} + (i/4)\operatorname{Im} w) \geq C_1 |w-z|^{\chi\rho}; \tag{3.2.13}$$

2. Secondly, if $4\operatorname{Im} z > \operatorname{Im} w$ and $\operatorname{Im} w > |w-z|^\rho$ then

$$\mathfrak{d}_h(w, z; \mathbb{H} + (i/4)\operatorname{Im} w) \geq C_2 |w-z|^{\chi}. \tag{3.2.14}$$

3. Finally, if $\operatorname{Im} w \leq |w-z|^\rho$ and $|w-z| \leq 4^{1/(1-\rho)}$ then:

$$\mathfrak{d}_h(\operatorname{Re} w, \operatorname{Re} z) \geq C_3 |w-z|^{1/\beta}; \tag{3.2.15}$$

$$\mathfrak{d}_h(w, \operatorname{Re} w) \leq C_4 |w-z|^{\sigma\rho}; \tag{3.2.16}$$

$$\mathfrak{d}_h(z, \operatorname{Re} z) \leq C_4 |w-z|^{\sigma\rho}. \tag{3.2.17}$$

Fixing the compact set K, we can choose U a bounded axis-parallel rectangle open in $\overline{\mathbb{H}}$ and containing K with the property that $\operatorname{dist}(z, \partial U \setminus \mathbb{R}) \geq 1$ for all $z \in K$. By the local Hölder continuity of $\mathfrak{d}_h$ w.r.t. the Euclidean metric, this ensures that $\mathfrak{d}_h(z, \partial U \setminus \mathbb{R})$ will almost surely be bounded below by some $C_K > 0$ uniformly in $z \in K$. We now bound $\mathfrak{d}_h(w, \mathbb{R} + \frac{i}{4}\operatorname{Im} w)$ from below by the minimum of C_K and the internal metric distance $\mathfrak{d}_h(w, \mathbb{R} + \frac{i}{4}\operatorname{Im} w; U)$. By (3.2.7), there exists an almost surely finite constant C' such that for all $w \in K$, we have

$$\mathfrak{d}_h(w, \mathbb{R} + (i/4)\operatorname{Im} w; U) \geq C'(\operatorname{Im} w)^{\chi},$$

which together with the assumption $\operatorname{Im} w > |w-z|^\rho$ establishes (3.2.13).

By [DFG+20, Prop. 3.18], for $\varepsilon > 0$ sufficiently small the Euclidean metric is almost surely $(\chi - \varepsilon)^{-1}$-Hölder continuous on K w.r.t. $\mathfrak{d}_{h^{\mathrm{wp}}}$ (say, with the additive constant fixed so that $h_1^{\mathrm{wp}}(0) = 0$), and thus also w.r.t. the larger internal metrics $\mathfrak{d}_{h^{\mathrm{wp}}}(\cdot, \cdot; \mathbb{H} + yi)$ for each $y > 0$, with the same Hölder constant. Thus there almost surely exists $C_5 > 0$ such that

$$\mathfrak{d}_{h^{\mathrm{wp}}}(w, z; \mathbb{H} + (i/4)\operatorname{Im} w) \geq C_5 |w-z|^{\chi - \varepsilon}.$$

This implies that

$$\mathfrak{d}_h(w,z;\mathbb{H}+(i/4)\operatorname{Im} w) \geq \min\{C_K, C_5|w-z|^{\chi-\varepsilon} \cdot \inf_{U\cap(\mathbb{H}+(i/4)\operatorname{Im} w)} e^{\xi(h-h^{\mathrm{wp}})}\}.$$

By the proof of Prop. 3.2.5, if h^{wp} is coupled with h so that the difference $h^{\mathrm{wp}}-h$ is a harmonic function, we almost surely have

$$\sup_{U\cap(\mathbb{H}+yi)} e^{\xi(h^{\mathrm{wp}}-h)} = O(y^{-\varepsilon/\rho}) \tag{3.2.18}$$

for each $\varepsilon > 0$, while the same holds for $e^{\xi(h-h^{\mathrm{wp}})}$.In other words, $\inf_{U\cap(\mathbb{H}+yi)} e^{\xi(h-h^{\mathrm{wp}})} = \Omega(y^{\varepsilon/\rho})$, and since $\operatorname{Im} w > |w-z|^\rho$ we have $(\operatorname{Im} w)^{\varepsilon/\rho} > |w-z|^\varepsilon$, which establishes (3.2.14).

Now turn to the case $\operatorname{Im} w \leq |w-z|^\rho$. Since the assumption $|w-z| \leq 4^{1/(1-\rho)}$ gives that $|w-z|^\rho \leq \frac{1}{4}|w-z|$, we have $|\operatorname{Re} w - \operatorname{Re} z| \geq \frac{1}{2}|w-z|$, and thus

$$\mathfrak{d}_h(\operatorname{Re} w, \operatorname{Re} z) \geq C^{-1/\beta}|\operatorname{Re} w - \operatorname{Re} z|^{1/\beta} \geq (2C)^{-1/\beta}|w-z|^{1/\beta}.$$

We thus obtain (3.2.15) with $C_3 = (2C)^{-1/\beta}$.

The existence of $C_4 > 0$ finite satisfying (3.2.16) and (3.2.17) follows from Prop. 3.1.9 together with the assumption that $\operatorname{Im} z \leq \operatorname{Im} w \leq |w-z|^\rho$.

Having justified the estimates (3.2.13)–(3.2.17) we finish the proof. If $\operatorname{Im} w > |w-z|^\rho$ then either $4\operatorname{Im} z \leq \operatorname{Im} w$, in which case it follows from (3.2.13) that we have

$$\mathfrak{d}_h(w,z) \geq \mathfrak{d}_h(w,\mathbb{R}+(i/4)\operatorname{Im} w) \geq C_1|w-z|^{\chi\rho},$$

or $4\operatorname{Im} z > \operatorname{Im} w$. In this latter case, since $\rho > 1$ and $\operatorname{Im} w > |w-z|^\rho$, it follows from (3.2.13) and (3.2.14) that there almost surely exists $C_6 > 0$ such that

$$\mathfrak{d}_h(w,z) \geq \min\{\mathfrak{d}_h(w,\mathbb{R}+(i/4)\operatorname{Im} w), \mathfrak{d}_h(w,z;\mathbb{H}+(i/4)\operatorname{Im} w)\} \geq C_5|w-z|^{\chi\rho}.$$

But if $\operatorname{Im} w \leq |w-z|^\rho$ then, since $\mathfrak{d}_h(\operatorname{Re} w, \operatorname{Re} z) \leq \mathfrak{d}_h(\operatorname{Re} w, w) + \mathfrak{d}_h(w,z) + \mathfrak{d}_h(z, \operatorname{Re} z)$ by the triangle inequality, it follows from (3.2.15), (3.2.16) and (3.2.17) that

$$\mathfrak{d}_h(w,z) \geq C_3|w-z|^{1/\beta} - 2C_4|w-z|^{\sigma\rho}.$$

Since $\sigma\rho > 1/\beta$, the last three displays imply the left-hand inequality of the proposition for any α_1 subject to

$$\alpha_1^{-1} < \frac{(Q-2)^2(1-s_+)}{4\xi(Q+2)^3} \tag{3.2.19}$$

at least when $|w-z| \le 4^{1/(1-\rho)}$, but we can deduce it for general w, z by considering points $w = w_0, w_1, \ldots, w_k = z$ along a path of finite $\mathfrak{d}_h$-length from w to z such that $|w_i - w_{i-1}| \le 4^{1/(1-\rho)}$ and using a Kolmogorov criterion-type argument as in (3.2.5). ■

Note that local bi-Hölder continuity on $\overline{\mathbb{H}}$ implies that $\mathfrak{d}_h$ induces the Euclidean topology on $\overline{\mathbb{H}}$ and not just on $\mathbb{H}$, which completes the proof of Prop. 1.3.6. Finally, we show the existence of geodesics.

Proposition 3.2.8. *Let h be a free-boundary GFF on $\mathbb{H}$ minus $\alpha \log|\cdot|$ for some $\alpha < Q$, with the additive constant fixed such that $h_1(0) = 0$. Then it is almost surely the case that for any z, $w \in \overline{\mathbb{H}}$ there exists a $\mathfrak{d}_h$-geodesic between z and w, which does not hit ∞.*

Proof. Since $\mathfrak{d}_h$-bounded subsets are also Euclidean-bounded by Lemma 3.2.4, and the two metrics induce the same topology, the Heine–Borel theorem gives that $(\overline{\mathbb{H}}, \mathfrak{d}_h)$ is almost surely a *boundedly compact* space (i.e., closed bounded sets are compact), which implies that there exists a geodesic between any two points (provided they are connected by a rectifiable curve, which we know holds for any two points since $\mathfrak{d}_h$ is a length metric). This is a standard result in metric geometry [BBI01, Cor. 2.5.20] proven by taking an infimizing sequence of paths which by bounded compactness can be assumed to lie in a compact set, extracting a uniformly converging subsequence by an Arzelà–Ascoli-type result, and applying lower semicontinuity of length to conclude that the limit is a geodesic. ■

3.3 Bound on γ-LQG area near the boundary

Our aim in the entirety of this section is to prove the following lower bound on the μ_h-area near a boundary segment. We will achieve this via the result [DMS21, Thm 1.2] that an independent SLE-type curve cuts a quantum wedge into two independent wedges, but here we will use several curves to cut out many independent surfaces that each have a positive chance to accumulate a large μ_h-area within a $\mathfrak{d}_h$-neighbourhood of our boundary segment. These surfaces can be described as contiguous portions of a space-filling $\mathrm{SLE}_{16/\gamma^2}$-type curve similar to the one that generates the "topological mating" in [DMS21, §8], but we will not need that description here.

Proposition 3.3.1. *Let h be a free-boundary GFF with the constant fixed so that $h_1(0) = 0$. For $\delta > 0$ and $a, b \in \mathbb{R}$ with $a < b$, define $\mathcal{B}_\delta([a, b])$ to be the set of points at $\mathfrak{d}_h$-distance $< \delta$ from the interval $[a, b]$. Fix $I \subset \mathbb{R}$ a compact interval and $u > 0$. Then there almost surely exists $M > 0$ such that for each $[a, b] \subset I$ and $\delta \in (0, 1)$ such that $\nu_h([a, b]) \geq 4\delta^{d_\gamma/2-u}$, we have*

$$\mu_h(\mathcal{B}_\delta([a, b])) \geq M\delta^{\frac{d_\gamma}{2}}\nu_h([a, b]). \tag{3.3.1}$$

Proof. As before, using the radial–lateral decomposition and mutual absolute continuity, it is enough to prove this for a quantum wedge. In particular, we will consider a γ-wedge $(\mathbb{H}, h, 0, \infty)$ (i.e., a wedge of weight 2, which is thick since $2 > \gamma^2/2$ for $\gamma \in (0, 2)$), and use the result [She16a, Prop. 1.7] that the law of such a wedge is invariant under translating one marked point by a fixed amount of ν_h-length. More precisely, if $(\mathbb{H}, h, 0, \infty)$ is a γ-wedge and we fix $L > 0$ and let $y > 0$ be defined by $\nu_h([0, y]) = L$, then the surface given by recentring the wedge such that y becomes the origin (which can be described either by $(\mathbb{H}, h, y, \infty)$ or by $(\mathbb{H}, h(\cdot + y), 0, \infty)$) is itself a γ-wedge.

By the conformal welding/cutting result [DMS21, Thm 1.2], an $\mathrm{SLE}_{\gamma^2}(-1; -1)$ from 0 to ∞ independent of h cuts the wedge $(\mathbb{H}, h, 0, \infty)$ into two independent wedges of weight 1; by shift invariance, for any $L > 0$, the same is true for an independent $\mathrm{SLE}_{\gamma^2}(-1; -1)$ from a_L to ∞, where a_L is defined as the point in $(0, \infty)$ for which $\nu_h([0, a_L]) = L$.

We can couple $\mathrm{SLE}_{\gamma^2}(-1; -1)$ curves η_x from each $x \in \mathbb{R}$ to ∞ (or at least from each x in a countable dense subset of $\mathbb{R}$) using the imaginary geometry results from [MS16a]. Indeed, by [MS16a, Thm 1.1], the flow line of a zero-boundary GFF $\mathring{h}$ on $\mathbb{H}$ started at $x \in \mathbb{R}$ is an $\mathrm{SLE}_{\gamma^2}(-1; -1)$ curve from x to ∞, so we can simultaneously generate η_r for different values of $r > 0$ by sampling such a GFF $\mathring{h}$ independently of h. By [MS16a, Thm 1.5(ii)], almost surely, whenever any two such curves η_c, $\eta_{c'}$ intersect, they merge immediately upon intersecting and never subsequently separate. Moreover, by [MS16a, Lemma 7.7], if K is the set formed by the initial portions of two such curves $\eta_c, \eta_{c'}$ run until they intersect, then the subsequent merged curve stays in the unbounded component of $\mathbb{H} \setminus K$.

Note that for $c < c'$ the curves $\eta_c, \eta_{c'}$ will merge almost surely. Indeed, if $-1 < \gamma^2/2 - 2$, i.e. $\gamma > \sqrt{2}$, then η_c hits $(0, \infty)$ almost surely, and by scale invariance η_c will then almost surely hit arbitrarily large $x > 0$. Thus η_c swallows c' and then the transience of $\eta_{c'}$ implies that the two curves merge. On the other hand, when $\gamma \leq \sqrt{2}$, η_c almost surely does not hit $\partial\mathbb{H}$. In this case one can map the unbounded region to the right of η_c back to $\mathbb{H}$ via

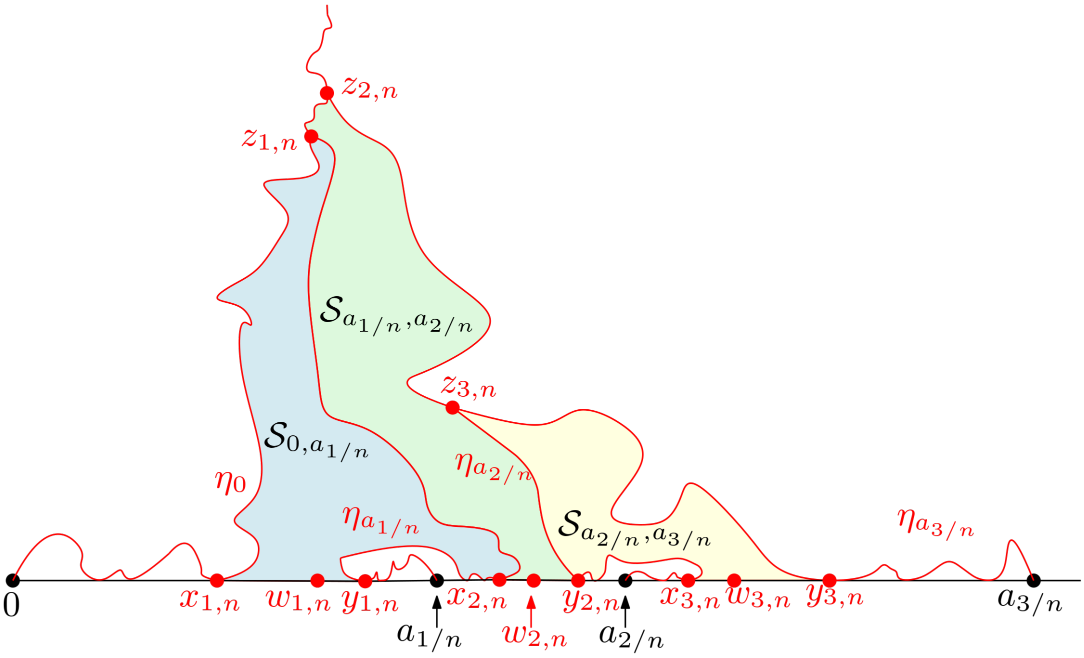

Figure 3.4: We show a lower bound on the μ_h-area near a boundary segment by using coupled $\mathrm{SLE}_{\gamma^2}(-1;-1)$ curves to cut a wedge into independent surfaces each of which have a positive chance of accumulating some positive amount of μ_h-area within a small $\mathfrak{d}_h$-distance of the boundary.

a conformal map ϕ; since η_c is a flow line, the field on $\mathbb{H}$ given by $\mathring{h} \circ \phi^{-1} - \chi \arg(\phi^{-1})'$ (the appropriate imaginary geometry coordinate change formula for $\phi(\eta_{c'})$ to be a flow line in $\mathbb{H}$) has boundary conditions $\lambda = \pi/\gamma$ on $(-\infty, \phi(c))$ and 0 on $(\phi(c), \infty)$. This means that, by [MS16a, Thm 1.1], $\phi(\eta_{c'})$ is an $\mathrm{SLE}_{\gamma^2}(\underline{\rho})$ from $\phi(c')$ with two left-hand force points of weight -1 at $\phi(c)$ and $\phi(c')^-$ and a right-hand force point of weight -1 at $\phi(c')^+$. Since the weights of the force points on the left sum to -2, the curve $\phi(\eta_{c'})$ must collide with a left-hand force point, meaning that it merges with the left-hand boundary segment $\phi(\eta_c)$ – indeed the denominator $V^{2,L} - W$ in [MS16a, (1.11)] evolves until hitting 0 as a Bessel process of dimension 1, i.e. a Brownian motion, and thus will hit 0 almost surely.

Given $c < c'$ denote by $\mathcal{S}_{a_c,a_{c'}}$ the quantum surface described by the restriction of the field h to the unique connected component $S_{a_c,a_{c'}}$ of $\mathbb{H} \setminus (\eta_{a_c} \cup \eta_{a_{c'}})$ which is to the right of η_{a_c} and to the left of $\eta_{a_{c'}}$ and whose boundary contains non-trivial segments of both η_{a_c} and $\eta_{a_{c'}}$. Almost surely, $\eta_{a_c}, \eta_{a_{c'}}$ do not intersect on $\mathbb{R}$. Indeed, since it is an $\mathrm{SLE}_{\gamma^2}(-1;-1)$ from a_c to ∞ and given a_c is conditionally independent of h, η_{a_c} almost surely does not hit $a_{c'}$. On this event, $\eta_{a_{c'}}$ is an $\mathrm{SLE}_{\gamma^2}(-1;-1)$ from $a_{c'}$ to ∞, and since given $a_{c'}$ this curve is conditionally independent of h, $\eta_{a_{c'}}$ almost surely does not hit the unique point on $\mathbb{R} \cap \eta_{a_c}$ that is on the boundary of the unbounded component of $\mathbb{H} \setminus \eta_{a_c}$ to the right of η_{a_c}). Thus, it is almost surely the case that the intersection of $\partial S_{a_c,a_{c'}}$ with $\mathbb{R}$ is a bounded interval and that, for each interior point x of this interval, there exists $r > 0$ such that $S_{a_c,a_{c'}}$ contains the Euclidean semi-disc $\overline{B}(x,r) \cap \overline{\mathbb{H}}$. Moreover, since the law of h on each deterministic open set not containing 0 is absolutely continuous w.r.t. that of a free-boundary Gaussian free field, it almost surely holds that for each such x and r, the smaller semi-disc $\overline{B}(x,r/2) \cap \overline{\mathbb{H}}$ has finite diameter w.r.t. the internal metric $\mathfrak{d}_h(\cdot,\cdot; B(x,r) \cap \overline{\mathbb{H}})$ (since we can find rationals q_1, q_2, r_1 and r_2 for which $\overline{B}(x,r/2) \subseteq \overline{B}(q_1,r_1) \subset B(q_2,r_2) \subseteq \overline{B}(x,r)$, and it is almost surely the case that, for all q_1, q_2, r_1, r_2 such that $\overline{B}(q_1,r_1) \subset B(q_2,r_2)$, $\overline{B}(q_1,r_1) \cap \overline{\mathbb{H}}$ has finite diameter w.r.t. $\mathfrak{d}_h(\cdot,\cdot; B(q_2,r_2) \cap \overline{\mathbb{H}})$).

Notice $\mathcal{S}_{a_{c_1},a_{c_2}}$ and $\mathcal{S}_{a_{c_3},a_{c_4}}$ are independent as quantum surfaces (i.e., modulo embedding) when $c_1 < c_2 \le c_3 < c_4$. Indeed, we know from the conformal welding result [DMS21, Thm 1.2] that the surfaces given by the restrictions of h to the regions to the left and right of η_{c_2} are independent; the same holds for $\mathring{h}$ since η_{c_2} is a flow line. Moreover, h and $\mathring{h}$ are independent of each other. These independences together imply that $\mathcal{S}_{a_{c_1},a_{c_2}}$ and $\mathcal{S}_{a_{c_3},a_{c_4}}$ are independent.

For each $k,n \in \mathbb{N}$, we can consider the surfaces $\mathcal{S}_{a_{(k-1)/n},a_{k/n}}$, with three marked points $x_{k,n}$, $y_{k,n}$, $z_{k,n}$ given by, respectively, the last point on $\mathbb{R} \cap S_{a_{(k-1)/n},a_{k/n}}$ that $\eta_{a_{(k-1)/n}}$ hits before merging with $\eta_{a_{k/n}}$, the last point on $\mathbb{R} \cap S_{a_{(k-1)/n},a_{k/n}}$ that $\eta_{a_{k/n}}$ hits before merging with $\eta_{a_{(k-1)/n}}$, and the point in $\mathbb{H}$ where the two curves merge. As explained above these surfaces are independent. By shift invariance, these surfaces are identically distributed when considered as triply marked surfaces modulo embedding.

Consider a point $w_{k,n}$ in the interval $(x_{k,n}, y_{k,n})$ (which has positive length almost surely, since $\eta_{a_{(k-1)/n}}$ and $\eta_{a_{k/n}}$ do not merge on $\mathbb{R}$); for concreteness we may set $w_{k,n}$ to be the unique point in the interval such that $\nu_h([x_{k,n}, w_{k,n}]) = \nu_h([w_{k,n}, y_{k,n}])$. As explained earlier, we can almost surely find $r > 0$ such that $\overline{B}(w_{k,n}, r) \cap \overline{\mathbb{H}}$ is contained in $S_{a_{(k-1)/n},a_{k/n}}$, and that $\overline{B}(w_{k,n}, r/2) \cap \overline{\mathbb{H}}$ has finite diameter w.r.t. the internal metric $\mathfrak{d}_h(\cdot,\cdot; B(w_{k,n}, r) \cap \overline{\mathbb{H}})$. Thus, the set $\mathcal{B}_{k,n}^{n^{-2/d_\gamma}}$ consisting of the intersection of $\operatorname{int} S_{a_{(k-1)/n},a_{k/n}}$ with the open ball of radius n^{-2/d_γ} centred on $w_{k,n}$ w.r.t. the internal metric $\mathfrak{d}_h(\cdot,\cdot; \operatorname{int} S_{a_{(k-1)/n},a_{k/n}})$ is non-empty and open w.r.t. the Euclidean topology, and thus has positive μ_h-measure almost surely. Thus, for every $p \in (0,1)$, there exists $c > 0$ such that

$$p_c := \mathbb{P}[\mu_h(\mathcal{B}_{k,n}^{n^{-2/d_\gamma}}) \ge c n^{-2}] > p. \tag{3.3.2}$$

Indeed as $c \to 0$ this probability tends to 1. Observe that by shift invariance p_c does not depend on k.

Adding a constant C to the field h scales ν_h-lengths by $e^{\gamma C/2}$ and μ_h-areas by $e^{\gamma C}$, as well as scaling $\mathfrak{d}_h$-distances by $e^{\xi C}$. By [DMS21, Prop. 4.7(i)], the circle-average embedding of $h + C$ into $\mathbb{H}$ has the same law as that of h, so if we add a constant C to the field h on $\mathbb{H}$ and then rescale appropriately to achieve the circle-average embedding, the resulting surface has the same law as $(\mathbb{H}, h, 0, \infty)$. The rescaling factor is independent of $\mathring{h}$, which itself has a scale-invariant law, so if we also apply the rescaling to the field $\mathring{h}$ and η then the joint law is invariant. This shows that, as a triply marked quantum surface, the law of η_{a_0,a_t} is the same as the law of η_{a_0,a_1} but with ν_h-lengths scaled by t, μ_h-areas scaled by t^2 and $\mathfrak{d}_h$-lengths scaled by t^{2/d_γ}. This implies that the probability p_c in (3.3.2) does not depend on n.

For $c > 0$ and $k, n \in \mathbb{N}$, define the event

$$A_{c,k,n} = \left\{\mu_h(\mathcal{B}_{k,n}^{n^{-2/d_\gamma}}) \geq cn^{-2}\right\}.$$

If $c > 0$ is chosen so that $p_c > 1/2$, then by a standard binomial tail estimate (see, for example, [MQ20, Lemma 2.6]), there exists $C_0(p_c) > 0$ for each $N, n_0 \in \mathbb{N}$, we have

$$\mathbb{P}\left[\sum_{n=n_0}^{n_0+N-1} \mathbf{1}_{A_{c,k,n}} \leq N/2\right] \leq e^{-C_0(p_c)N}. \tag{3.3.3}$$

Moreover $C_0(p_c) \to \infty$ as $p_c \to 1$, i.e. as $c \to 0$. Thus, since the $\mathcal{B}_{k,n}^{n^{-2/d_\gamma}}$ are disjoint for different k, if $t, s \in \mathbb{R}$ are such that $t - s \geq 1$ (so that $\lfloor t - s\rfloor \geq \frac{1}{2}(t - s)$), we have

$$\mathbb{P}\left[\mu_h(\mathcal{B}_{n^{-2/d_\gamma}}([a_{s/n}, a_{t/n}])) \leq \frac{1}{4}(t-s)cn^{-2} \leq \frac{1}{2}\lfloor t - s\rfloor cn^{-2}\right] \leq e^{-C_0(p_c)\lfloor t-s\rfloor} \leq e^{-\frac{1}{2}C_0(p_c)(t-s)}.$$

For $T, v > 0$ fixed, this probability converges when summed over all choices of $n = 2^m$, $s = j2^{mv}$, $t = (j + 1)2^{mv}$ with m, j non-negative integers such that $(j + 1)2^{-m(1-v)} \leq T$. Indeed, the sum is bounded by $\sum_m 2^{m(1-v)}Te^{-\frac{1}{2}C_0(p_c)2^{mv}}$ which converges superpolynomially fast in 2^m. Thus we find that, with superpolynomially high probability in 2^{m_0} as $m_0 \to \infty$, whenever $m \geq m_0$ and $j \geq 0$ is an integer such that $(j + 1)2^{-m+mv} \leq T$, we have

$$\mu_h(\mathcal{B}_{2^{-2m/d_\gamma}}([a_{j2^{-m+mv}}, a_{(j+1)2^{-m+mv}}])) > \frac{c}{4}2^{-2m+mv}. \tag{3.3.4}$$

Furthermore, by disjointness of the $\mathcal{B}_{k,n}^{n^{-2/d_\gamma}}$, on the event considered above (and thus still with superpolynomially high probability in 2^{m_0} as $m_0 \to \infty$) it holds that whenever $j < k$ are non-negative integers with $k2^{-m+mv} \le T$, we have

$$\mu_h(\mathcal{B}_{2^{-2m/d_\gamma}}([a_{j2^{-m+mv}}, a_{k2^{-m+mv}}])) > \frac{c}{4}2^{-2m+mv}(k-j). \tag{3.3.5}$$

On this event, for each $m \ge m_0$ (3.3.1) holds for each subinterval $[a, b]$ of $[a_0, a_T]$ of ν_h-length at least $2^{-m+mv+2}$ with $\delta = 2^{-2m/d_\gamma}$, with $u = d_\gamma v/2$ and with $M = \frac{c}{16}$, since we can find a subinterval of $[a, b]$ of the form $[a_{j2^{-m+mv}}, a_{k2^{-m+mv}}]$ whose ν_h-measure is at least 1/4 that of $[a, b]$. This gives an overall constant (i.e., one holding for all $\delta \in (0, 1)$) of $M = c2^{-m_0}/16$ (using the right-hand side of (3.3.1) with $\delta = 2^{-2m_0/d_\gamma}$ as the lower bound for all larger δ) and holds with probability bounded by $T\sum_{m\ge m_0} 2^{m(1-v)}e^{-\frac{1}{2}C_0(p_c)2^{mv}}$. Since $\nu_h(I)$ has a finite first moment, for any $\alpha > 0$ we can set $T = 2^{m_0\alpha}$ and observe that the probability that we can take $M = c2^{-m_0}/16$ is bounded by $2^{m_0\alpha}\sum_{m\ge m_0} 2^{m(1-v)}e^{-\frac{1}{2}C_0(p_c)2^{mv}}$ plus the probability that $\nu_h(I)$ is greater than T, which is $O(2^{-m_0\alpha})$. Since α is arbitrary, this gives superpolynomial decay of the constant M. ∎

3.4 Proofs of main results

3.4.1 Proof of Theorem 1.3.2

We now prove Theorem 1.3.2, the extension of [GM19, Thm 1.5] to the γ-LQG metric for all $\gamma \in (0, 2)$. Suppose we are in the setup of Theorem 1.3.2. That is, fix $\gamma \in (0, 2)$ and $\mathfrak{w}^-$, $\mathfrak{w}^+ > 0$, and let $(\mathbb{H}, h, 0, \infty)$ be a quantum wedge of weight $\mathfrak{w} := \mathfrak{w}^- + \mathfrak{w}^+$ if $\mathfrak{w} \ge \gamma^2/2$ (so that the wedge is thick), or a single bead of a wedge of weight $\mathfrak{w}$, with specified γ-LQG area $\mathfrak{a}$ and γ-LQG boundary lengths $\mathfrak{l}^-$, $\mathfrak{l}^+ > 0$, if $\mathfrak{w} < \gamma^2/2$ (corresponding to a thin wedge). Let η be an independent $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}^- - 2; \mathfrak{w}^+ - 2)$ from 0 to ∞ which we will parametrize by ν_h-length as measured on either side of the curve (recall that these two boundary length measures agree by [DMS21, Thm 1.4]). As in [GM19], we define $V_\rho = \{z \in \mathbb{C} : |z| < \rho, \operatorname{Im} z > \rho^{-1}\}$ for $\rho > 1$. For $z \in \mathbb{H}$ and $r > 0$, write $B_r(z; \mathfrak{d}_h)$ for the open $\mathfrak{d}_h$-metric ball of radius r centred at z.

We will replicate the argument of [GM19, §4], establishing analogues of the lemmas in that section, beginning with an analogue of [GM19, Lemma 4.1]:

Lemma 3.4.1. *In the setting of Theorem 1.3.2, let $R > 1$ and let z_1 and z_2 be independent samples from $\mu_h|_{V_R}$, normalized to be a probability measure. Almost surely, there exists a $\mathfrak{d}_h$-geodesic from z_1 to z_2 that does not hit 0 or ∞.*

Proof. First note that [GM19, Lemmas 4.2, 4.3] hold for general γ just as in the $\gamma = \sqrt{8/3}$ case, since their proofs just rely on the locality and Weyl scaling properties of the γ-LQG metric, along with (in the case of [GM19, Lemma 4.3]) calculations for the Gaussian free field ([MS21a, Lemma 5.4]) that do not depend on γ. This establishes that no $\mathfrak{d}_h$-geodesic between points of $\mathbb{H}$ hits 0. For our analogue of [GM19, Lemma 4.1], we also need to know that for quantum typical points z_1, z_2 (i.e. if z_1 and z_2 are sampled independently according to μ_h) there almost surely exists a $\mathfrak{d}_h$-geodesic. For a thick wedge, existence of geodesics that do not intersect ∞ follows from Prop. 3.2.8 plus absolute continuity with the free-boundary GFF plus a log singularity. For beads of thin wedges, since $(\mathbb{H}, h, \infty, 0) \overset{(d)}{=} (\mathbb{H}, h, 0, \infty)$, the analogue of [GM19, Lemma 4.3] gives that paths of near-minimal $\mathfrak{d}_h$-length between z_1 and z_2 must stay in a set that is Euclidean-bounded and thus $\mathfrak{d}_h$-compact (since $\mathfrak{d}_h$ still induces the Euclidean topology away from 0, by absolute continuity w.r.t. the free-boundary GFF away from 0), and thus we can still deduce the existence of a geodesic between z_1 and z_2 by the argument of [BBI01, Cor. 2.5.20]. ∎

We will not address the question of whether geodesics are unique here, since we do not need uniqueness for our results.

We now proceed to state and prove analogues of [GM19, Lemmas 4.5–4.9]. We begin by using the estimates established in the previous sections to prove that a global regularity event G_C holds with high probability, which is analogous to [GM19, Lemma 4.5]. The remaining lemmas (the analogues of [GM19, Lemmas 4.6–4.9]) will follow from this one in essentially the same way as in [GM19, §4], though we will give the proofs here since there are minor differences, since conditions (iii) and (iv) in Lemma 3.4.2 are slightly weaker than those in [GM19, Lemma 4.5], and we have not ruled out the possibility of geodesics hitting the boundary. Given these lemmas, the remainder of the proof of Theorem 1.3.2 will be identical to the argument in [GM19, Thm 1.5].

Lemma 3.4.2. *In the setting of Theorem 1.3.2, there exists $\beta > 0$ such that, for all $u \in (0,1)$, $\rho > 2$, $p \in (0,1)$, there is $C > \rho$ such that $\mathbb{P}[G_C] \geq 1 - p$, where G_C is the event that all the following hold:*

(i) *For each* $z \in V_\rho$ *and* $0 < \delta \leq 1$ *such that* $B_\delta(z; \mathfrak{d}_h) \cap \mathbb{R} = \emptyset$, *we have* $\mu_h(B_\delta(z; \mathfrak{d}_h)) \leq C\delta^{d_\gamma - u}$.

(ii) *For each* $U \in \mathcal{U}^- \cup \mathcal{U}^+$ *with* $U \cap V_\rho \neq \emptyset$, *each* $z \in \overline{U} \cap V_\rho$, *and each* $0 < \delta \leq 1$, *we have* $\mu_h(B_\delta(z; \mathfrak{d}_{h|_U})) \geq C^{-1}\delta^{d_\gamma + u}$.

(iii) *For each* $U \in \mathcal{U}^- \cup \mathcal{U}^+$ *with* $U \cap V_\rho \neq \emptyset$, *and each* $x, y \in \partial U \cap V_\rho$, *we have*

$$\mathfrak{d}_{h|_U}(x, y) \leq C\nu_h([x, y]_{\partial U})^{(2/d_\gamma) - u}.$$

(iv) *For each* $U \in \mathcal{U}^- \cup \mathcal{U}^+$ *with* $U \cap V_\rho \neq \emptyset$, *each* $0 < \delta < 1$, *and each* $x, y \in \partial U \cap V_\rho$ *with* $\nu_h([x, y]_{\partial U}) \geq 4\delta^{d_\gamma/2 - u}$, *we have*

$$\mu_h(B_\delta([x, y]_{\partial U}; \mathfrak{d}_{h|_U})) \geq C^{-1}\delta^{(d_\gamma/2)}\nu_h([x, y]_{\partial U}).$$

(v) *For each* $z \in V_\rho$ *and* $0 < \delta \leq 1$, *we have* $B_\delta(z; \mathfrak{d}_h) \subseteq B(z, C\delta^\beta)$.

(vi) *For each* $t > s > 0$ *such that* $\eta(s) \in V_{\rho/2}$ *and* $|t - s| \leq C^{-1}$, *we have* $\eta(t) \in V_\rho$.

Proof. Note first that it suffices to show that for each item, there almost surely exists some $C \in (\rho, \infty)$ for which that item holds, since this forces $\mathbb{P}[G_C] \to 1$ as $C \to \infty$.

With this in mind, item (i) follows from [AFS20, Thm 1.1]. Indeed, that result gives us that, for h^{wp} a whole-plane GFF normalized so that the circle average $h_1^{\mathrm{wp}}(0) = 0$, K a compact set and $\varepsilon > 0$, we almost surely have

$$\sup_{s \in (0,1)} \sup_{z \in K} \frac{\mu_{h^{\mathrm{wp}}}(B_s(z; \mathfrak{d}_{h^{\mathrm{wp}}}))}{s^{d_\gamma - \varepsilon}} < \infty \quad \text{and} \quad \inf_{s \in (0,1)} \inf_{z \in K} \frac{\mu_{h^{\mathrm{wp}}}(B_s(z; \mathfrak{d}_{h^{\mathrm{wp}}}))}{s^{d_\gamma + \varepsilon}} > 0. \tag{3.4.1}$$

Recall that we can couple a free-boundary GFF h^{F} on $\mathbb{H}$, normalized so that the semicircle average $h_1(0)$ is zero, with h^{wp}, so that $\mathfrak{h} = h - h^{\mathrm{wp}}$ is a random harmonic function. We thus find that (3.4.1) holds with h^{F} in place of h^{wp} provided $K \subset \mathbb{H}$ is at positive Euclidean distance from $\mathbb{R}$ (and thus positive $\mathfrak{d}_{h^{\mathrm{F}}}$-distance, so that we need only consider $s < \mathfrak{d}_{h^{\mathrm{F}}}(K, \mathbb{R})$), and we can then deduce the same for h either a thick quantum wedge or a bead of a thin quantum wedge (in the latter case with specified area and boundary lengths) by local absolute continuity, which implies that there almost surely exists $C < \infty$ for which item (i) holds.

Item (v) follows from Prop. 1.3.7 (for a free-boundary GFF, then for a wedge or bead thereof by absolute continuity). Just as in [GM19], item (vi) follows from the continuity and transience of SLE from 0 to ∞ with force points, proved in [MS16a, Thm 1.3] (although the parametrization by quantum length depends on γ, observe that if (vi) holds for one parametrization then it holds for any other parametrization, though not necessarily with the same C). We now turn to items (ii)–(iv), which are required to hold for each of the surfaces U cut out by η that intersect V_ρ. We can reduce to considering finitely many such surfaces: exactly as explained in the first part of the proof of [GM19, Lemma 4.5], it suffices to show that for each $U \in \mathcal{U}^- \cup \mathcal{U}^+$ intersecting V_ρ, there almost surely exists $C \in (\rho, \infty)$ such that items (ii)–(iv) hold for U. We will map to $\mathbb{H}$ and use absolute continuity arguments; in particular for each U we will consider the surface $\phi_U(U)$, where we define x_U (resp. y_U) as the first (resp. last) point on ∂U to be hit by η (with $y_U = \infty$ when U is a thick wedge) and set ϕ_U to be the unique conformal map $U \to \mathbb{H}$ sending x_U to 0 and y_U to ∞ with the property that the covariantly transformed field $h_U := h \circ \phi_U^{-1} + Q \log |(\phi_U^{-1})'|$ satisfies

$$\mu_{h_U}(\mathbb{D} \cap \mathbb{H}) = \begin{cases} 1 & \mu_h(U) = \infty \\ \mu_h(U)/2 & \mu_h(U) < \infty \end{cases}.$$

As in the proof of [GM19, Lemma 4.5], we can find $\widetilde{\rho}$ such that $\phi_U(U \cap V_{2\rho}) \subset V_{\widetilde{\rho}}$ with high probability (since the marked points x_U and y_U must be in $\mathbb{R} \cup \infty$). For the free-boundary GFF, item (ii) follows since (3.4.1) holds with $K = \phi_U(\overline{U \cap V_\rho})$, whereas items (iii) and (iv) follow from Prop. 3.2.3 and Prop. 3.3.1 respectively, so it suffices to observe that the restriction of h_U to $V_{\widetilde{\rho}}$ is absolutely continuous w.r.t. the same restriction of h^F. ■

We now proceed as in [GM19]. Our version of [GM19, Lemma 4.6] is as follows:

Lemma 3.4.3. *In the setting of Theorem 1.3.2, for each $v \in (0,1)$ there exists $u_0 = u_0(v) \in (0,1)$ such that whenever $0 < u \le u_0$, $\rho > 2$, $C > 1$, and $G_C = G_C(u, \rho)$ is the event of Lemma 3.4.2, there exists $\varepsilon_0 = \varepsilon_0(C, u, v, \rho) > 0$ such that the following holds almost surely on G_C. If $0 < a < b \le a + \varepsilon_0 < \infty$ and $\eta([a,b]) \cap V_{\rho/2} \neq \emptyset$, then we have*

$$\operatorname{diam}(\eta([a,b]); \mathfrak{d}_h) \ge 7(b-a)^{2(1+v)/d_\gamma}.$$

Proof. The proof is essentially the same as that of [GM19, Lemma 4.6]. Fixing v, C, u, ρ, by condition (v) in Lemma 3.4.2 we can choose $\varepsilon_0 \in (0,1)$ such that whenever $z \in V_{\rho/2}$, we have $B_{8\varepsilon_0^{2(1+v)/d_\gamma}}(z;\mathfrak{d}_h) \subseteq V_\rho$. In particular this ball does not intersect $\mathbb{R}$.

Now suppose G_C occurs and fix $0 < a < b < a+\varepsilon_0 < \infty$ and $z \in \eta([a,b]) \cap V_{\rho/2}$. Setting $\delta = (b-a)^{2(1+v)/d_\gamma}$, if we assume the statement of the lemma is false we have $\eta([a,b]) \subseteq B_{7\delta}(z;\mathfrak{d}_h) \subseteq V_\rho$. Noting that V_ρ does not intersect $\mathbb{R}$, we can find $U \in \mathcal{U}^-$ such that $\eta([a,b]) \subseteq \partial U$. Now, since $b-a \le \varepsilon_0 < 1$ and $d_\gamma > 2$, we have $b-a = \delta^{d_\gamma/(2+2v)} \ge 4\delta^{d_\gamma/2-u}$ provided $u < \frac{d_\gamma}{2}(1-(1+v)^{-1})$ (i.e. provided u is sufficiently small depending on v) and ε_0 is sufficiently small depending on u and v, so by condition (iv) in Lemma 3.4.2, we have

$$\mu_h(B_\delta(\eta([a,b]);\mathfrak{d}_h)) \ge \mu_h(B_\delta(\eta([a,b]);\mathfrak{d}_{h|_U})) \ge C^{-1}(b-a)^{2+v}.$$

Condition (i) in Lemma 3.4.2 gives us

$$\mu_h(B_{8\delta}(z;\mathfrak{d}_h)) \le 8^{d_\gamma-u}C(b-a)^{2(1+v)(1-u/d_\gamma)}.$$

If u is sufficiently small (depending only on v), then if ε_0 is small enough depending on u and C we can ensure that, whenever $b-a \le \varepsilon_0$,

$$8^{d_\gamma-u}C(b-a)^{2(1+v)(1-u/d_\gamma)} < C^{-1}(b-a)^{2+v}.$$

Thus $B_\delta(\eta([a,b]);\mathfrak{d}_h) \not\subseteq B_{8\delta}(z;\mathfrak{d}_h)$, so $\eta([a,b])$ cannot be contained in $B_{7\delta}(z;\mathfrak{d}_h)$. ∎

Next we give a version of [GM19, Lemma 4.7], which bounds the number of segments of η of a fixed quantum length that can intersect a $\mathfrak{d}_h$-metric ball.

Lemma 3.4.4. *In the setting of Theorem 1.3.2, for each $v \in (0,1)$, there exists $u_0 = u_0(v) \in (0,1)$ such that, whenever $\rho > 2$, $C > 1$, $0 < u \le u_0$, there exists $\delta_0 = \delta_0(C,u,v,\rho) > 0$ such that, almost surely on $G_C = G_C(u,\rho)$, for each $z \in V_{\rho/2}$ and $\delta \in (0,\delta_0]$, the number of $k \in \mathbb{N}$ for which $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$ intersects $B_{\delta^{1+v}}(z;\mathfrak{d}_h)$ is at most δ^{-v}.*

Proof. Assume G_C occurs; then for δ small enough depending on C and ρ and $z \in V_{\rho/2}$, we have (using condition (v) in Lemma 3.4.2) that $B_{3\delta^{1+v}}(z;\mathfrak{d}_h) \subseteq V_\rho$. By Lemma 3.4.3, if u is small enough depending on v and δ is small enough depending on C, u, v, ρ, we have that for all $z \in V_{\rho/2}$ and all $k \in \mathbb{N}$, $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}]) \not\subseteq B_{2\delta^{1+v}}(z;\mathfrak{d}_h)$. Assume that δ and u are chosen so that the above conditions hold. Let K be the set of $k \in \mathbb{N}$ for which the segment $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$ intersects $B_{\delta^{1+v}}(z;\mathfrak{d}_h)$; we now know that each $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$ also intersects $\mathbb{H} \setminus B_{2\delta^{1+v}}(z;\mathfrak{d}_h)$. Let $\mathcal{V}$ be the set of connected components of $\mathbb{H} \setminus B_{\delta^{1+v}}(z;\mathfrak{d}_h)$, and for each $V \in \mathcal{V}$ let O_V be the set of those connected components of $V \setminus \eta$ which intersect $V \setminus B_{2\delta^{1+v}}(z;\mathfrak{d}_h)$. A topological argument given in Step 1 of the proof of [GM19, Lemma 4.7] shows that we have $|K| \leq 2 + 2\sum_{\mathcal{V}} |O_V|$. This argument relies only on the facts that η is continuous and transient and does not hit itself and that $\mathfrak{d}_h$ induces the Euclidean topology, so it applies here unchanged.

Fixing $V \in \mathcal{V}$, $O \in O_V$, by the definition of O_V and the fact that $B_{3\delta^{1+v}}(z;\mathfrak{d}_h)$ does not intersect $\mathbb{R}$, there exists $w_O \in \partial O \cap \eta$ satisfying

$$\mathfrak{d}_h(w_O, \partial B_{\delta^{1+v}}(z;\mathfrak{d}_h)) = \mathfrak{d}_h(w_O, \partial B_{2\delta^{1+v}}(z;\mathfrak{d}_h)) = \frac{1}{2}\delta^{1+v}.$$

Let U_O be the connected component of $\mathbb{H}\setminus\eta$ containing O (so $U_O \in \mathcal{U}^- \cup \mathcal{U}^+$), and let $B_O = B_{\frac{1}{2}\delta^{1+v}}(w_O;\mathfrak{d}_{h|_{U_O}})$. Then by construction, η does not cross B_O, and $B_O \subseteq B_{\frac{1}{2}\delta^{1+v}}(w_O;\mathfrak{d}_h) \subseteq B_{2\delta^{1+v}}(z;\mathfrak{d}_h) \setminus B_{\delta^{1+v}}(z;\mathfrak{d}_h)$. In particular $B_O \subseteq O$, which implies that B_O and $B_{O'}$ are disjoint when O and O' are distinct elements of $\bigcup_{V\in\mathcal{V}} O_V$.

Since $B_{3\delta^{1+v}}(z;\mathfrak{d}_h) \subseteq V_\rho$, each U_O intersects V_ρ, so by condition (ii) in Lemma 3.4.2, for each $O \in \bigcup_{V\in\mathcal{V}} O_V$ we have $\mu_h(B_O) \geq C^{-1}(\frac{1}{2}\delta)^{(d_\gamma+u)(1+v)}$. We thus find that

$$C^{-1}(\delta/2)^{(d_\gamma+u)(1+v)} \sum_{V\in\mathcal{V}} |O_V| \leq \mu_h(B_{2\delta^{1+v}}(z;\mathfrak{d}_h)) \leq C(2\delta)^{(d_\gamma-u)(1+v)},$$

where the second inequality is by condition (i) in Lemma 3.4.2. This, combined with the earlier fact that $|K| \leq 2 + 2\sum_{\mathcal{V}} |O_V|$, gives us a bound on $|K|$ of a universal constant times $C^2\delta^{-2u(1+v)}$, which after possibly shrinking u and δ is enough to prove the lemma. ∎

Next we adapt [GM19, Lemma 4.8]:

Lemma 3.4.5. *In the setting of Theorem 1.3.2, let $v \in (0,1)$ and let $u_0 = u_0(v)$ be as in Lemma 3.4.4. Let $u \in (0,u_0]$, $\rho > 2$ and $C > 1$, and let $G_C = G_C(u,\rho)$. Let $z_1, z_2 \in V_\rho$ and let γ_{z_1,z_2} be a $\mathfrak{d}_h$-geodesic from z_1 to z_2 contained in $V_{\rho/2}$, all chosen in a manner that is independent from η. For $\delta \in (0,1)$ let $K^\delta_{z_1,z_2}$ be the set of $k \in \mathbb{N}$ for which γ_{z_1,z_2} intersects $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$. Then there is an exponent $\alpha > 0$ depending only on γ, $\mathfrak{w}^-$, $\mathfrak{w}^+$ and the exponent β in Lemma 3.4.2, and a deterministic constant $M = M(C,u,v,\rho)$, such that for* *each $\delta \in (0,1)$ we have*

$$\mathbb{E}\left[|K^{\delta}_{z_1,z_2}| \cdot \mathbf{1}_{G_C} \,\middle|\, h, \gamma_{z_1,z_2}\right] \le M\delta^{-1-2v+\alpha(1+v)}\mathfrak{d}_h(z_1,z_2). \tag{3.4.2}$$

Proof. It suffices to prove (3.4.2) for $\delta \le \delta_0$ where $\delta_0 = \delta_0(C,u,v,\rho)$ is as in Lemma 3.4.4, since it can then be extended to $\delta \in (0,1)$ by (deterministically) increasing M. Fixing z_1, z_2 as in the statement, let $N := \lfloor \delta^{-(1+v)}\mathfrak{d}_h(z_1,z_2)\rfloor + 1$. For $j \in \{1,\dots,N-1\}$ let $t_j = j\delta^{1+v}$ and let $t_N = \mathfrak{d}_h(z_1,z_2)$. Now define $\mathcal{V}_j = B_{\delta^{1+v}}(\gamma_{z_1,z_2}(t_j);\mathfrak{d}_h)$, where we parametrize the path $\gamma_{z_1,z_2}\colon [0,t_N] \to V_{\rho/2}$ by $\mathfrak{d}_h$-distance, so that the $\mathcal{V}_j$ cover γ_{z_1,z_2}. Let $J^{\delta}_{z_1,z_2}$ be the number of j in $\{1,\dots,N\}$ for which $\mathcal{V}_j$ intersects η. Lemma 3.4.4 gives that, on G_C, if $\delta \in (0,\delta_0]$ then for each j there are at most δ^{-v} elements of $K^{\delta}_{z_1,z_2}$ for which $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$ intersects $\mathcal{V}_j$. This shows that

$$|K^{\delta}_{z_1,z_2}| \le \delta^{-v}|J^{\delta}_{z_1,z_2}|.$$

So it suffices to show that

$$\mathbb{E}\left[|J^{\delta}_{z_1,z_2}| \cdot \mathbf{1}_{G_C} \,\middle|\, h, \gamma_{z_1,z_2}\right] \le M\mathfrak{d}_h(z_1,z_2)\delta^{-1-v+\alpha(1+v)},$$

for appropriately chosen α and M. On G_C, condition (v) in Lemma 3.4.2 ensures that $\mathcal{V}_j$ is contained in the Euclidean ball $\widetilde{\mathcal{V}}_j := B(\gamma_{z_1,z_2}(t_j), C\delta^{\beta(1+v)})$. There exists α_0 depending only on γ, $\mathfrak{w}^-$ and $\mathfrak{w}^+$ such that for each $w \in V_{\rho/2}$ and $\varepsilon > 0$ we have

$$\mathbb{P}[\eta \cap B(w,\varepsilon) \neq \varnothing] \le c(\rho, C, \gamma, \mathfrak{w}^-, \mathfrak{w}^+)\varepsilon^{\alpha_0}$$

(this is [GM19, Lemma B.1]), and since (h,γ_{z_1,z_2}) is independent of (the trace of) η but determines $\widetilde{\mathcal{V}}_j$, this probability bound applies here to give

$$\mathbb{P}[\eta \cap \widetilde{\mathcal{V}}_j \neq \varnothing] \le c(\rho, C, \gamma, \mathfrak{w}^-, \mathfrak{w}^+)\delta^{\alpha_0\beta(1+v)}.$$

Summing over $1 \le j \le N$ we get the result with $\alpha = \alpha_0\beta$ and $M = c(\rho, C, \gamma, \mathfrak{w}^-, \mathfrak{w}^+)$. ■

We now adapt [GM19, Lemma 4.9], which states that $\mathfrak{d}_h$ is equal to the metric gluing at quantum typical points.

Lemma 3.4.6. *In the setting of Theorem 1.3.2, let $\widetilde{d}_h$ be the quotient metric on $\mathbb{H}$ obtained by the metric gluing of $(U, \mathfrak{d}_{h|_U})$. Fix $R > 1$ and sample z_1, z_2 independently from the probability* *measure obtained by normalizing $\mu_h|_{V_R}$. Then almost surely we have $\mathfrak{d}_h(z_1, z_2) = \widetilde{d}_h(z_1, z_2)$.*

Proof. Let $v \in (0, \alpha/100)$ where α is as in Lemma 3.4.5 and let $u \in (0, u_0]$ where u_0 is as in Lemma 3.4.4. Also fix $p \in (0, 1)$ and $\varepsilon > 0$. Choose $\rho > 2$ such that the event

$$E_\rho := \{\mathfrak{d}_h(z_1, z_2; V_{\rho/2}) - \varepsilon < \mathfrak{d}_h(z_1, z_2) \leq \rho\}$$

has probability at least $1 - p/5$. We can do this because, by definition, $\mathfrak{d}_h(z_1, z_2)$ is the infimum of the $\mathfrak{d}_h$-lengths of paths between them that only intersect $\mathbb{R}$ finitely often, and by Remark 3.1.10 we can replace a small segment of such a path near each intersection point with a path that stays in $\mathbb{H}$ with arbitrarily close $\mathfrak{d}_h$-length. Since we also know by Lemma 3.4.1 that near-minimal paths from z_1 to z_2 cannot hit ∞, it follows that $\mathbb{P}[E_\rho] \to 1$ as $\rho \to \infty$.

Having chosen ρ and u, choose $C = C(\rho, u)$ so that $G_C = G_C(u, \rho)$ has probability at least $1 - p/5$. Work now on the event $E_\rho \cap G_C$. By [Gwy21, Thm 1.7] there can only be finitely many geodesics from z_1 to z_2 w.r.t. the internal metric $\mathfrak{d}_h(\cdot, \cdot; V_{\rho/2})$ (which must also be geodesics from z_1 to z_2 w.r.t. $\mathfrak{d}_h$); let γ_{z_1,z_2} be the leftmost of these (i.e., when started from z_1, the path γ_{z_1,z_2} stays to the left of all other $\mathfrak{d}_h(\cdot, \cdot; V_{\rho/2})$-geodesics from z_1 to z_2). By Lemma 3.4.5 we have

$$\mathbb{E}\left[|K^\delta_{z_1,z_2}| \cdot \mathbf{1}_{G_C \cap E_\rho} \,\middle|\, h, \gamma_{z_1,z_2}\right] \leq M\delta^{-1-2v+\alpha(1+v)}\rho,$$

and by taking a further expectation this bound also holds for $\mathbb{E}\left[|K^\delta_{z_1,z_2}| \cdot \mathbf{1}_{G_C \cap E_\rho}\right]$. So by Markov's inequality, there exists $\delta_0 = \delta_0(u, v, C, \rho) > 0$ such that when $\delta \leq \delta_0$, it holds with probability at least $1 - p/2$ that $E_\rho \cap G_C$ occurs and

$$|K^\delta_{z_1,z_2}| \leq \delta^{-1-3v+\alpha(1+v)} \leq \delta^{-1+\alpha/2} \tag{3.4.3}$$

(the second inequality holds because $v < \alpha/100$). Now fix $\delta \in (0, \delta_0]$ and assume that $E_\rho \cap G_C$ occurs and (3.4.3) holds. We need to show that $\widetilde{d}_h(z_1, z_2) \leq \mathfrak{d}_h(z_1, z_2)$ (note that the reverse inequality is clear by locality of the LQG metric, as pointed out in the discussion after the statement of Theorem 1.3.2). To this end we construct a path from z_1 to z_2 by concatenating finitely many paths each of which is contained in some $\overline{U}$, for $U \in \mathcal{U}^- \cup \mathcal{U}^+$.

By condition (vi) in Lemma 3.4.2, as long as $\delta \leq C^{-2/d_\gamma}$ (which we can guarantee by possibly shrinking δ_0), we have $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}]) \subseteq V_\rho$ for each $k \in K^\delta_{z_1,z_2}$, and thus these segments are disjoint from $\mathbb{R}$, so that we may choose $U_k \in \mathcal{U}^-$ such that U_k intersects V_ρ and such that $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}]) \subseteq \partial U_k$. Let τ_k and σ_k be respectively the first and last times γ_{z_1,z_2} hits $\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}])$. Let $\widetilde{\gamma}_k$ be a $\mathfrak{d}_{h|_{U_k}}$-geodesic from $\gamma_{z_1,z_2}(\tau_k)$ to $\gamma_{z_1,z_2}(\sigma_k)$. By condition (iii) in Lemma 3.4.2, almost surely on G_C we have $\operatorname{diam}(\eta([(k-1)\delta^{d_\gamma/2}, k\delta^{d_\gamma/2}]); \mathfrak{d}_{h|_{U_k}}) \leq C\delta^{1-ud_\gamma/2}$, and thus

$$\operatorname{length}(\widetilde{\gamma}_k; \mathfrak{d}_{h|_{U_k}}) \leq C\delta^{1-ud_\gamma/2} \quad \forall k \in K^\delta_{z_1,z_2}. \tag{3.4.4}$$

Pick $k_1 \in K^\delta_{z_1,z_2}$ with τ_{k_1} minimal, and inductively define $k_2, \ldots, k_{|K^\delta_{z_1,z_2}|}$ such that τ_{k_j} is the smallest τ_k with $k \in K^\delta_{z_1,z_2}$ for which $\tau_k \geq \sigma_{k_{j-1}}$, if this exists; if there is no such τ_k let $k_j = \infty$. Let J be the smallest $j \in \mathbb{N}$ for which $k_j = \infty$. Let $\mathring{\gamma}_1 = \gamma_{z_1,z_2}|_{[0,\tau_{k_1}]}$, let $\mathring{\gamma}_J = \gamma_{z_1,z_2}|_{[\sigma_J, \mathfrak{d}_h(z_1,z_2)]}$ and let $\mathring{\gamma}_j = \gamma_{z_1,z_2}|_{[\sigma_{j-1},\tau_j]}$ for $2 \leq j \leq J-1$. Then for $1 \leq j \leq J$, the curve $\mathring{\gamma}_j$ does not hit η except at its endpoints, so that we can find $\mathring{U}_j \in \mathcal{U}^- \cup \mathcal{U}^+$ such that $\mathring{\gamma}_j \subseteq \overline{\mathring{U}_j}$, and by locality we have $\operatorname{length}(\mathring{\gamma}_j; \mathfrak{d}_{h|_{\mathring{U}_j}}) = \operatorname{length}(\mathring{\gamma}_j; \mathfrak{d}_h)$ for each j. We now concatenate the curves $\mathring{\gamma}_1, \widetilde{\gamma}_{k_1}, \mathring{\gamma}_2, \widetilde{\gamma}_{k_2}, \ldots, \mathring{\gamma}_{J-1}, \widetilde{\gamma}_{k_{J-1}}, \mathring{\gamma}_J$, to get a path $\widetilde{\gamma}$ from z_1 to z_2 such that

$$\begin{aligned}
\widetilde{d}_h(z_1; z_2) &\leq \sum_{j=1}^{J-1} \operatorname{length}(\widetilde{\gamma}_{k_j}; \mathfrak{d}_{h|_{U_{k_j}}}) + \sum_{j=1}^{J} \operatorname{length}(\mathring{\gamma}_j; \mathfrak{d}_{h|_{\mathring{U}_j}}) \\
&\leq \sum_{j=1}^{J-1} \operatorname{length}(\widetilde{\gamma}_{k_j}; \mathfrak{d}_{h|_{U_{k_j}}}) + \mathfrak{d}_h(z_1, z_2) + \varepsilon \\
&\leq C\delta^{\alpha/2 - ud_\gamma/2} + \mathfrak{d}_h(z_1, z_2) + \varepsilon,
\end{aligned}$$

where the last inequality comes from (3.4.3) to bound J by $\delta^{-1+\alpha/2}$ and (3.4.4) to bound the length of each $\widetilde{\gamma}_{k_j}$. By possibly shrinking u_0 we can ensure that $ud_\gamma < \alpha$, so that sending $\delta \to 0$ gives $\widetilde{d}_h(z_1, z_2) \leq \mathfrak{d}_h(z_1, z_2) + \varepsilon$ as required. Since p and ε can be made arbitrarily small, we are done. ∎

The last step to prove Theorem 1.3.2 is the same as in the proof of [GM19, Thm 1.5]. (Essentially, we now have that $\widetilde{d}_h$ and $\mathfrak{d}_h$ agree on a set of μ_h-full measure, which is dense since open sets have positive μ_h-measure, so we can conclude quickly by an approximation argument.)

3.4.2 Proofs of Theorems 1.3.3, 1.3.4 and 1.3.5

We now turn to the proofs of Theorems 1.3.3, 1.3.4 and 1.3.5. In fact, as with [GM19, Thm 1.6], in the case that $\mathfrak{w} \geq \gamma^2/2$ (so that $(U, h|_U)$ is a thick wedge), the proof of Theorem 1.3.3 is essentially the same as that of the previous theorem, so we just need to treat the case where cutting along η gives a thin wedge. The reason this case is more difficult is that we have to approximate geodesics with paths that avoid the points at which η intersects itself. However, we can still deduce this case from the previous results.

Proof of Theorem 1.3.3 in the case $\mathfrak{w} \in (0, \gamma^2/2)$. Fix $z \in \mathbb{C} \setminus \{0\}$ and $0 < r < s < s' < |z|$. Let τ_1 be the first time that η hits $\partial B(z, r)$ and let σ_1 be the first time after τ_1 that η hits $\partial B(z, s)$. Having defined τ_j, σ_j, let τ_{j+1} be the first time after σ_j that η hits $\partial B(z, r)$ and let σ_{j+1} be the first time after τ_{j+1} that η hits $\partial B(z, s)$. We will show that for each j it is almost surely the case that the internal metric $\mathfrak{d}_h(\cdot, \cdot; B(z, r))$ agrees with the metric gluing of the components of $B(z, r) \setminus \eta|_{[0,\sigma_j]}$ along $\eta|_{[0,\sigma_j]}$. This suffices to prove the theorem, since then the result almost surely holds for all j, all $z \in \mathbb{Q}^2 \setminus \{0\}$ and all $0 < r < s < |z|$ rational, so that we can split any path not hitting 0 into finitely many pieces each contained in a ball $B(z, r)$ for which the result holds. Then the length of each such piece is the same according to $\mathfrak{d}_h(\cdot, \cdot; B(z, r))$ and the metric gluing across η (which, since η is transient by [MS17, Thm 1.12], is the same as the metric gluing along $\eta|_{[0,\sigma_j]}$ for j sufficiently large).

We proceed by induction on j; first we consider the case $j = 1$. The conditional law of $\eta|_{[\tau_1,\sigma_1]}$ given $\eta|_{[0,\tau_1]}$ is that of a radial $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ in the unbounded component $\widetilde{D}_1$ of $\mathbb{C} \setminus \eta|_{[0,\tau_1]}$, started from $\eta(\tau_1)$, targeted at ∞ and stopped at time σ_1, and thus has the same law (up to time change) as a chordal $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ in $\widetilde{D}_1$ from $\eta(\tau_1)$ targeted at ∞ and stopped upon hitting $\partial B(z, s)$ [SW05, Thm 3]. Moreover, if we define the domain D_1 to be the component of $B(z, s') \setminus \eta|_{[0,\tau_1]}$ containing $\eta([\tau_1, \sigma_1])$ (note that this component is determined by $\eta|_{[0,\tau_1]}$), this latter law is mutually absolutely continuous with that of a chordal $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ in $\widetilde{D}_1$ from $\eta(\tau_1)$ targeted at ∞ and stopped upon hitting $\partial B(z, s)$, and indeed the Radon–Nikodym derivatives between the two laws are bounded by [MW17, Lemma 2.8]. Therefore, if we now fix (in some way which is measurable w.r.t. $\eta|_{[0,\tau_1]}$) a conformal map $\psi_1 \colon D_1 \to \mathbb{H}$ such that $\psi_1(\eta(\tau_1)) = 0$, then the law of $\psi_1 \circ \eta([\tau_1, \sigma_1])$ is absolutely continuous up to time change w.r.t. the law of a chordal $\mathrm{SLE}_{\gamma^2}(\mathfrak{w} - 2)$ from 0 to ∞ in $\mathbb{H}$ stopped upon exiting $\psi_1(B(z, s))$, and the Radon–Nikodym derivatives between

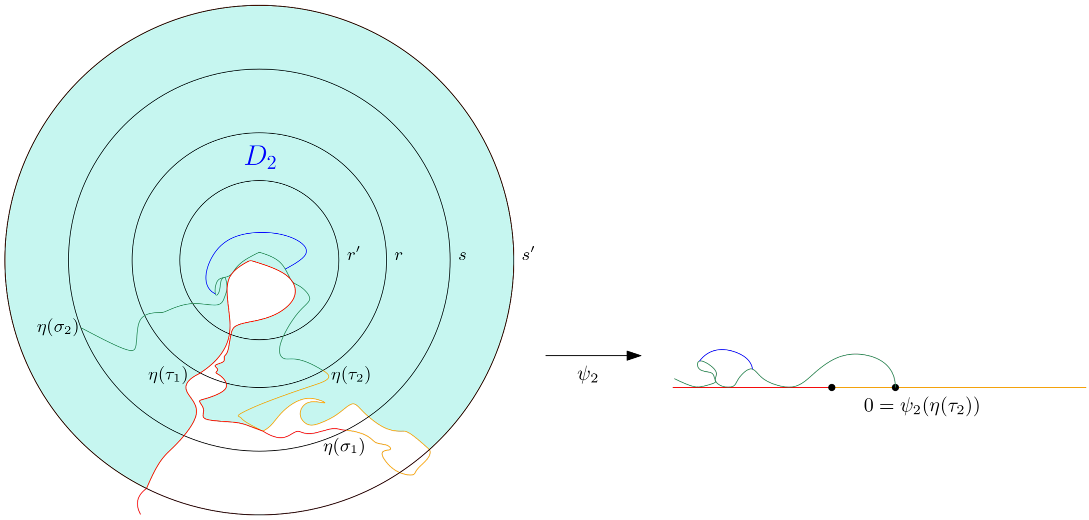

Figure 3.5: In order to deduce Theorem 1.3.3 for thin wedges from Theorem 1.3.2, we draw the whole-plane $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}-2)$ curve η up to a stopping time then map a domain bounded by η and a circular arc to $\mathbb{H}$. The law of the image of the remaining part of η up to a later stopping time is absolutely continuous w.r.t. that of a chordal $\mathrm{SLE}_\kappa(\mathfrak{w}-2)$, so this puts us in the setting of Theorem 1.3.2.

the two laws are bounded independently of the choice of ψ_1.

Letting $h^{\psi_1} = h \circ \psi_1^{-1} + Q \log|(\psi_1^{-1})'|$ be the covariantly transformed field on $\mathbb{H}$, whenever $0 < r' < r$, the law of the pair $(\psi_1(B(z,r')), h^{\psi_1}|_{(\psi_1(B(z,r')))})$ is absolutely continuous w.r.t. that of $(\psi_1(B(z,r')), h^F|_{(\psi_1(B(z,r')))})$ where h^F is a free-boundary GFF on $\mathbb{H}$ (say, normalized so that $h_1^F(0) = 0$). This follows since $\psi_1(B(z,r'))$ is at positive Euclidean distance from $\partial\mathbb{H}$ and the laws of the two GFF variants are mutually absolutely continuous away from the boundary (which can be seen by coupling them so that their difference is a random harmonic function and using the Girsanov theorem to express the Radon–Nikodym derivative in terms of this harmonic function). We can thus apply the proof of Theorem 1.3.2 to $(\mathbb{H}, h^F, 0, \infty)$ (nothing changes, since the required GFF estimates in Lemma 3.4.2 are proved for h^F anyway) and, by absolute continuity, deduce the conclusion of that theorem for h^{ψ_1}. That is to say, almost surely, for each rational $r' \in (0, r)$, the length of any path in $B(z,r')$ is the same w.r.t. $\mathfrak{d}_h(\cdot,\cdot; B(z,r))$ and the metric gluing along $\eta|_{[0,\sigma_1]}$. This completes the base case.

Suppose that the result holds for $j \geq 1$; we prove that it holds also for $j+1$. By the induction hypothesis, it holds almost surely that if w_1, w_2 are any two distinct points in $B(z,r)$, then for each $\varepsilon > 0$ there is a path P in $B(z,r)$ which crosses $\eta|_{[0,\sigma_j]}$ only finitely many times whose $\mathfrak{d}_h(\cdot,\cdot; B(z,r))$-length is at most $\mathfrak{d}_h(w_1, w_2; B(z,r)) + \varepsilon$. We thus aim to show that it is almost surely the case that each path $\widetilde{P}$ in $B(z,r)$ which does not intersect $\eta|_{[0,\sigma_j]}$ except possibly at the endpoints of $\widetilde{P}$ has the same length w.r.t. $\mathfrak{d}_h(\cdot,\cdot; B(z,r))$ as w.r.t. the metric gluing along $\eta|_{[0,\sigma_{j+1}]}$. This implies that if w_1 and w_2 are quantum typical points (i.e., sampled independently from $h|_{B(z,r)}$ normalized to be a probability measure), then w_1 and w_2 have the same distance w.r.t. $\mathfrak{d}_h(\cdot,\cdot;B(z,r))$ and the gluing along $\eta|_{[0,\sigma_{j+1}]}$ (since we can choose an almost-minimal path P as above between w_1 and w_2 and split into subpaths $\widetilde{P}$ with the same length according to each of the two metrics). We can then conclude that these two metrics on $B(z,r)$ are equal using the same argument as at the end of the proof of [GM21c, Thm 1.5].

Analogously to the base case, let D_{j+1} be the component of $B(z,s')\setminus\eta|_{[0,\tau_{j+1}]}$ containing $\eta([\tau_{j+1},\sigma_{j+1}])$ and, in some way which is measurable w.r.t. $\eta|_{[0,\tau_{j+1}]}$, fix a conformal map $\psi_{j+1}\colon D_{j+1}\to\mathbb{H}$ such that $\psi_{j+1}(\eta(\tau_{j+1}))=0$ and let $h^{\psi_{j+1}}=h\circ\psi_{j+1}^{-1}+Q\log|(\psi_{j+1}^{-1})'|$ be the covariantly transformed field on $\mathbb{H}$.

As before, the conditional law of $\eta|_{[\tau_{j+1},\sigma_{j+1}]}$ given $\eta|_{[0,\tau_{j+1}]}$ is (up to time change) that of a chordal $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}-2)$ in the unbounded component $\widetilde{D}_{j+1}$ of $\mathbb{C}\setminus\eta|_{[0,\tau_{j+1}]}$, started from $\eta(\tau_{j+1})$ and stopped upon hitting $\partial B(z,s)$, and thus the law of $\psi_{j+1}\circ\eta([\tau_{j+1},\sigma_{j+1}])$ is absolutely continuous up to time change w.r.t. that of a chordal $\mathrm{SLE}_{\gamma^2}(\mathfrak{w}-2)$ from 0 to ∞ in $\mathbb{H}$ stopped upon exiting $\psi_{j+1}(B(z,s))$.

Moreover, for each $\delta>0$, $r'\in(0,r)$, the law of the pair

$$(\psi_{j+1}(B(z,r')\setminus B(\eta([0,\sigma_j]),\delta)),h^{\psi_{j+1}}|_{\psi_{j+1}(B(z,r'))})$$

is absolutely continuous w.r.t. that of $(\psi_{j+1}(B(z,r')\setminus B(\eta([0,\sigma_j]),\delta)),h^F|_{(\psi_{j+1}(B(z,r')))})$, since the set $\psi_{j+1}(B(z,r')\setminus B(\eta([0,\sigma_j]),\delta))$ will have positive Euclidean distance from $\partial\mathbb{H}$. We can thus argue as in the case $j=1$ that, almost surely, for any $\delta>0$ and $r'\in(0,r)$, any path in $B(z,r')\setminus B(\eta([0,\sigma_j]),\delta)$ has the same length w.r.t. $\mathfrak{d}_h(\cdot,\cdot;B(z,r))$ and the gluing along $\eta|_{[0,\sigma_{j+1}]}$. This suffices to complete the inductive step, since w.r.t. either metric we can find the length of any path in $B(z,r')$ intersecting $\eta|_{[0,\sigma_{j+1}]}$ only at its endpoints by considering the amount of length it accumulates in $B(z,r')\setminus B(\eta([0,\sigma_j]),\delta)$ and sending δ to 0. ∎

Theorem 1.3.4 follows by the same method as in [GM19]. The left boundary η_L of $\eta'((-\infty,0])$ is an $\mathrm{SLE}_{\gamma^2}(2-\gamma^2)$ by [DMS21, Footnote 4] and [MS17, Thm 1.1]. Now [DMS21, Thm 1.5] gives that $(\mathbb{C}\setminus\eta_L,h|_{\mathbb{C}\setminus\eta_L},0,\infty)$ is a wedge of weight $4-\gamma^2$. We apply Theorem 1.3.3. By [MS17, Thm 1.11], the conditional law of the right boundary η_R of $\eta'((-\infty,0])$ given η_L is a $\mathrm{SLE}_{\gamma^2}(-\gamma^2/2;-\gamma^2/2)$. Thus η_R cuts the wedge into two wedges of weight $2-\gamma^2/2$ and now we deduce Theorem 1.3.4 by applying Theorem 1.3.2. Finally, Theorem 1.3.5 follows by the same absolute continuity argument (between quantum spheres and γ-quantum cones) as in [GM19].

Chapter 4

Liouville quantum gravity metrics are not doubling

This chapter is devoted to the proof of Theorem 1.3.9, that LQG metrics are not doubling and thus cannot be quasisymmetrically embedded into finite-dimensional Euclidean spaces.

4.1 Non-doubling metric spaces

We begin by giving an alternate characterization of non-doubling metric spaces (equivalently, those with infinite Assouad dimension) that we will verify for the LQG metric in order to rule out embeddability into $\mathbb{R}^n$. Namely, we observe that having infinite Assouad dimension is equivalent to containing arbitrarily large finite sets of points that are all approximately equidistant from each other, a characterization that does not seem to have appeared in previous literature.

Definition 4.1.1. Let (X, d) be a metric space. Given $N \in \mathbb{N}$ and $K > 1$, we say that distinct points $x_1, \ldots, x_N \in X$ form an (N, K)**-clique** if

$$\max_{1 \le i < j \le N} d(x_i, x_j) \le K \min_{1 \le i < j \le N} d(x_i, x_j).$$

For $K > 1$ we say (X, d) is K**-cliquey** if it contains an (N, K)-clique for each $N \in \mathbb{N}$.

Instead of considering (N, K)-cliques, [Tro21] considers "approximate N-stars" in which the N points of a clique are also roughly equidistant from a central point that is closer to each outer point than the outer points are to each other. The proofs in both that paper and this one actually find approximate N-stars, but for our purposes the more simply defined (N, K)-

cliques suffice, since quasisymmetric images of K-cliquey spaces must have infinite Assouad dimension:

Proposition 4.1.2. *Let (X, d_X) be a K-cliquey metric space for some $K > 1$ and $f : (X, d_X) \to (Y, d_Y)$ a quasisymmetric mapping. Then $\dim_A Y = \infty$.*

Proof. Suppose (X, d_X) is K-cliquey and $f : (X, d_X) \to (Y, d_Y)$ is Ψ-quasisymmetric. Suppose also that $\dim_A Y < \infty$, so that there exist $\alpha, C \in (0, \infty)$ for which $N_r(B(y, R)) \leq C(R/r)^\alpha$ whenever $0 < r < R$ and $y \in Y$.

Choose $N > 4^\alpha C(\Psi(K)^2 + 1)^\alpha$ and let $x_1, \dots, x_N$ form an (N, K)-clique in X. Now by (2.7.1), for $1 \leq i, j, k \leq N$ distinct we have

$$\frac{d_Y(f(x_i), f(x_j))}{d_Y(f(x_i), f(x_k))} \leq \Psi\left(\frac{d_X(x_i, x_j)}{d_X(x_i, x_k)}\right) \leq \Psi(K) \leq \Psi(K)^2 + 1, \tag{4.1.1}$$

since $x_1, \dots, x_N$ form an (N, K)-clique and Ψ is increasing. Applying the above twice, for $1 \leq i, j, k, l \leq N$ distinct we have

$$\frac{d_Y(f(x_i), f(x_j))}{d_Y(f(x_k), f(x_l))} = \frac{d_Y(f(x_i), f(x_j))}{d_Y(f(x_i), f(x_k))} \cdot \frac{d_Y(f(x_i), f(x_k))}{d_Y(f(x_k), f(x_l))} \leq \Psi(K)^2 \leq \Psi(K)^2 + 1. \tag{4.1.2}$$

(4.1.1) and (4.1.2) together imply that $f(x_1), \dots, f(x_N)$ form an $(N, \Psi(K)^2 + 1)$-clique. Now set $r = \frac{1}{2} \min_{1 \leq i < j \leq N} d_Y(f(x_i), f(x_j))$ and $R = 2 \max_{1 \leq i < j \leq N} d_Y(f(x_i), f(x_j))$. Then $B(f(x_1), R)$ contains all the $f(x_i)$ but no open ball of radius r can contain more than one of the $f(x_i)$, so $N_r(B(f(x_1), R)) \geq N > 4^\alpha C(\Psi(K)^2 + 1)^\alpha$. But $R/r \leq 4(\Psi(K)^2 + 1)$ since the $f(x_i)$ form a $(\Psi(K)^2 + 1)$-clique, so this contradicts $N_r(B(y_1, R)) \leq C(R/r)^\alpha$ and we must have $\dim_A Y = \infty$. ∎

In fact, being K-cliquey is equivalent to not being doubling (cf. [Tro21, Prop. 2.7]):

Proposition 4.1.3. *Let (X, d) be a metric space. The following are equivalent:*

(i) $\dim_A(X) = \infty$;

(ii) *X is not a doubling space;*

(iii) *X is K-cliquey for some $K > 1$;*

(iv) *X is K-cliquey for every $K > 1$.*

Proof. **(i) ⇒ (ii):** Contrapositively, if X is a doubling space, then it is straightforward to show that $\dim_{\mathrm{A}}(X) < \infty$ by iterating the operation of covering a ball with a fixed number of balls of half its radius (see [Fra21, Thm 13.1.1]).

(ii) ⇒ (iii): If X is not doubling, then given any $N \in \mathbb{N}$ we can find $x \in X$ and $R > 0$ such that $B(x,R)$ cannot be covered by less than N balls of radius $R/2$. Let $x_1 = x$ and construct $x_2, \ldots, x_N$ inductively so that $x_k \in B(x,R) \setminus \bigcup_{i=1}^{k-1} B(x_i, R/2)$ for $k = 2, \ldots, N$ (possible by choice of x and R). Now for $1 \le i < j \le N$ we have

$$R/2 \le d(x_i, x_j) \le d(x_i, x) + d(x, x_j) < 2R,$$

so the x_i form an $(N,4)$-clique. Thus X is 4-cliquey.

(iii) ⇒ (iv): If $K > 1$ and X is K-cliquey, then for any N we can find an $(R(N), K)$-clique $x_1, x_2, \ldots, x_{R(N)}$ in X, where $R(N)$ is the Nth Ramsey number, and by definition of $R(N)$ such a clique must contain N points whose pairwise distances are either all in $[\min_{i<j} d(x_i,x_j), K^{1/2} \min_{i<j} d(x_i,x_j)]$ or all in $(K^{1/2} \min_{i<j} d(x_i,x_j), \max_{i<j} d(x_i,x_j)]$, in either case forming an $(N, K^{1/2})$-clique. Iterating this argument, we find that X is $K^{1/4}$-cliquey, $K^{1/8}$-cliquey, and so on.

(iv) ⇒ (i): Apply Prop. 4.1.2 to the identity on X. ∎

Remark 4.1.4. *From Prop. 4.1.2 and Prop. 4.1.3 we deduce the well-known result that quasisymmetric images of non-doubling spaces are not doubling, and conversely (since the inverse of a Ψ-quasisymmetric bijection is $1/\Psi(1/\cdot)$-quasisymmetric) that quasisymmetric images of doubling spaces are doubling.*

We briefly observe another property that contrasts spaces of infinite and finite Assouad dimension. Given a metric space (X, d_X) and $\beta \in (0,1)$ one can define the β-**snowflaking** d_X^β of d_X as the metric on X given by $d_X^\beta(x,y) = d_X(x,y)^\beta$. The *Assouad embedding theorem* [Ass83, Prop. 2.6] states that for each $\alpha \in (0,\infty)$ and $\beta \in (0,1)$ there exists $n = n(\alpha,\beta) \in \mathbb{N}$ such that, if (X,d) is a metric space such that $\dim_{\mathrm{A}}(X) = \alpha$, then there is a bi-Lipschitz embedding of (X, d^β) into $\mathbb{R}^n$. (Naor and Neiman [NN12] later proved that one can choose $n = n(\alpha)$ such that $\mathbb{R}^n$ admits bi-Lipschitz embeddings of (X, d^β) for all $\beta \in (1/2, 1)$ and all X with $\dim_{\mathrm{A}}(X) = \alpha$.) For spaces of infinite Assouad dimension, however, snowflaking does not facilitate bi-Lipschitz embeddings into $\mathbb{R}^n$:

Remark 4.1.5. *Note that, for $\beta \in (0,1)$, if (X, d_X) is K-cliquey then (X, d_X^β) is K^β-cliquey;*

thus, if $\dim_A X = \infty$ *then the* β*-snowflaking of* X *cannot be embedded quasisymmetrically into any doubling space (cf. [Tro21, Thm 2.6]).*

4.2 Proof of Theorem 1.3.9

We will begin by proving the result for the whole-plane GFF and deduce it for other variants via local absolute continuity. The main task is to show that for N, δ fixed, a fixed closed disc contains an $(N, 1+\delta)$-clique with positive probability (by scale and translation invariance, this probability will not depend on the disc). The basic idea for this is to consider a polygonal star with N arms and add bump functions to the field in order to force geodesics between the arms to stay within the star, recalling that the law of the modified field will be mutually absolutely continuous with that of the original field. The near-independence of the field in disjoint regions then allows us to translate positive probability for a fixed disc into an almost sure result: a Markovian exploration of the domain (we will use the annulus exploration from [GM20, Lemma 3.1]) will almost surely find a disc containing an $(N, 1+\delta)$-clique. Since this holds for every N, we have that γ-LQG metric spaces are $(1+\delta)$-cliquey, so by Prop. 4.1.2 their quasisymmetric images must have infinite Assouad dimension, which as mentioned is equivalent to not being doubling.

Fix $N \geq 2$, $z_0 \in \mathbb{C}$, $r > 0$, $\delta \in (0,1)$ and $\varepsilon \in (0, 1/14)$. Let h be a whole-plane GFF, normalized so that (say) the circle average $h_1(0)$ is zero. Now set $z_k = z_0 + 6re^{2\pi ik/N}$, $z'_k = z_0 + 7re^{2\pi ik/N}$ and $w_k = z_0 + re^{\pi i(2k+1)/N}$ for $k = 1, \dots, N$, and let K_N be the compact set consisting of the polygon whose sides are the line segments joining

$$(z'_1, w_1), (w_1, z'_2), (z'_2, w_2), (w_2, z'_3), \dots, (z'_N, w_N), (w_N, z'_1)$$

together with this polygon's interior. For $\beta \in (0,1)$ let $K_N^\beta = z_0 + (1-\beta)(K_N - z_0)$. Fix $\zeta(\varepsilon) > 0$ such that the Euclidean $2\zeta(\varepsilon)$-neighbourhood of $K_N^{\varepsilon/2}$ is contained in $K_N^{\varepsilon/4}$. Define the event

$$A_C^1(h) = \left\{ \begin{array}{c} \inf\left\{ \mathfrak{d}_h(z,w) \,\middle|\, z, w \in B(z_0, 7r) \setminus K_N^{\varepsilon/2}, |z-w| \geq \zeta(\varepsilon) \right\} > 1/C; \\ \mathfrak{d}_h(\partial B(z_0, 2r), \partial B(z_0, 5r); \operatorname{int} K_N^\varepsilon) \leq C \end{array} \right\}.$$

If $\widetilde{h}$ is another field, we define $A_C^1(\widetilde{h})$ to be the event given by replacing h by $\widetilde{h}$ throughout in the definition of $A_C^1(h)$. (We will later tacitly use further definitions of this kind.)

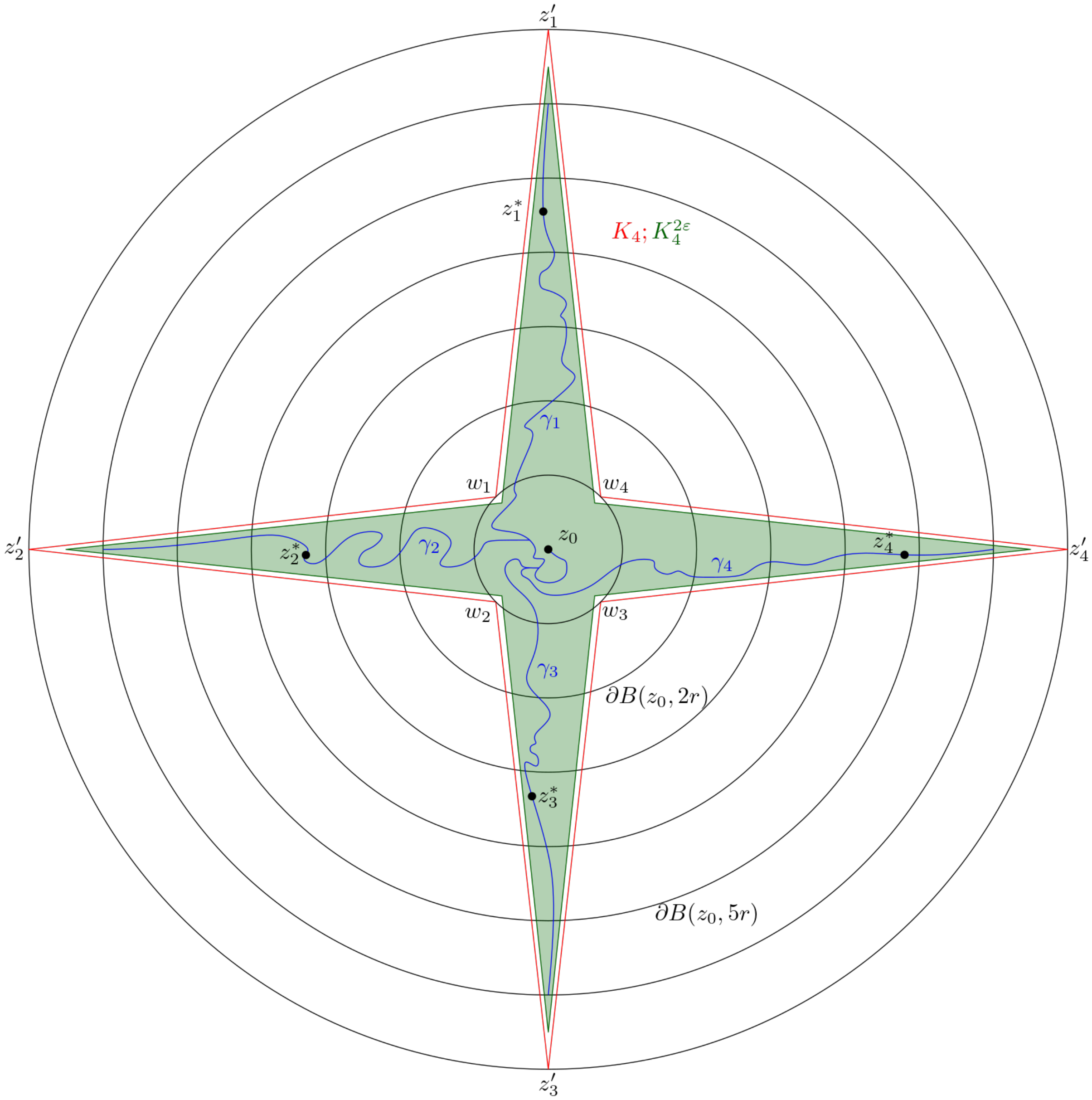

Figure 4.1: The points z_i^* are chosen to be equidistant from $\partial B(z_0, 2r)$; we then arrange that geodesics between them stay within $K_N \cup \overline{B}(z_0, 2r)$ and that the diameter of $\overline{B}(z_0, 2r)$ is small, making the z_i^* almost equidistant from each other.

We check that $\mathbb{P}[A^1_C(h)] \to 1$ as $C \to \infty$. It suffices to observe that, almost surely,

$$0 < \mathfrak{d}_h(\partial B(z_0, 2r), \partial B(z_0, 5r)) < \mathfrak{d}_h(\partial B(z_0, 2r), \partial B(z_0, 5r); \operatorname{int} K^\varepsilon_N) < \infty,$$

whilst

$$\inf\left\{\mathfrak{d}_h(z, w) \,\middle|\, z, w \in B(z_0, 7r) \setminus K^{\varepsilon/2}_N, |z - w| \geq \zeta(\varepsilon)\right\} > 0,$$

since if not we could find $z_{(n)}, w_{(n)} \in B(z_0, 7r) \setminus K^{\varepsilon/2}_N$ for each $n \in \mathbb{N}$, with $|z_{(n)} - w_{(n)}| \geq \zeta(\varepsilon)$ and $\mathfrak{d}_h(z_{(n)}, w_{(n)}) \to 0$ as $n \to \infty$, and by Bolzano–Weierstrass and continuity of $\mathfrak{d}_h$ w.r.t. the Euclidean metric, extract subsequences converging to z and w (w.r.t. both the Euclidean metric and $\mathfrak{d}_h$) with $\mathfrak{d}_h(z, w) = 0$ but $|z - w| \geq \zeta(\varepsilon)$, a contradiction. (For the subcritical case, we could also use the local Hölder continuity of the Euclidean metric w.r.t. $\mathfrak{d}_h$ as proven in [DFG+20, Prop. 3.18]; neither of the critical LQG metric and the Euclidean metric is locally Hölder continuous w.r.t. the other, but we could instead use the polylogarithmic modulus of continuity established in [DG21, Prop. 1.8].)

Since $\mathbb{P}[A^1_C(h)] \to 1$ as $C \to \infty$, we can choose $C_1 > 0$ such that $\mathbb{P}[A^1_{C_1}(h)] > 0$. Let ψ be a bump function supported in $B(z_0, 8r) \setminus K^\varepsilon_N$ such that $\psi \equiv 1$ on $B(z_0, 7r) \setminus K^{\varepsilon/2}_N$. Let M be such that $e^{\xi M} > 2C_1^2$.

For $\eta > 0$, let $E_\eta(h)$ be the event that

$$\begin{aligned}\inf\left\{\mathfrak{d}_h(z, w; \mathbb{A}_{2r,7r}(z_0) \setminus K^{\varepsilon/2}_N) \,\middle|\, z, w \in \mathbb{A}_{2r,7r}(z_0) \setminus K^{\varepsilon/2}_N, |z - w| \geq \zeta(\varepsilon)\right\} \\ \geq 2\left(\mathfrak{d}_h(\partial B(z_0, 2r), \partial B(z_0, 5r)) + \eta\right).\end{aligned}$$

If we choose M depending on C_1 as above, and choose $\eta < e^{\xi M}/(2C_1) - C_1$, then by Weyl scaling we have $A^1_{C_1}(h) \subseteq E_\eta(h + M\psi)$. Thus, since h and $h + M\psi$ have mutually absolutely continuous laws and $\mathbb{P}[A^1_{C_1}(h)] > 0$, we can fix $\eta_1 > 0$ so that $\mathbb{P}[E_{\eta_1}(h)] > 0$.

Observe that, almost surely, $\sup_{z \in \mathbb{A}_{(2-u)r,2r}(z_0)} \mathfrak{d}_h(z, \partial B(z_0, (2 - u)r)) \to 0$ as $u \downarrow 0$, since otherwise we could find $v > 0$ and sequences $z_{(n)}$, $u_{(n)}$ such that $u_{(n)} \downarrow 0$, $z_{(n)} \in \mathbb{A}_{(2-u_{(n)})r,2r}(z_0)$ and $\mathfrak{d}_h(z_{(n)}, \partial B(z_0, (2 - u_{(n)})r)) \geq v$, then extract a convergent subsequence whose limit $z \in \partial B(z_0, 2r)$ must have $\mathfrak{d}_h$-distance $\geq v$ from $B(z_0, 2r)$, a contradiction (indeed, given any $u \in (0, 2)$, once n is large enough that $u_{(n)} \leq u$ it must hold that $\mathfrak{d}_h(z_{(n)}, \overline{B}(z_0, (2 - u)r)) \geq v$ and, taking the subsequential limit, $\mathfrak{d}_h(z, \overline{B}(z_0, (2 - u)r)) \geq v$). This convergence holds almost surely and thus also in probability, so given any $t > 0$,

$p \in (0,1)$, we can fix $u > 0$ such that

$$\mathbb{P}\left[\sup_{z \in \mathbb{A}_{(2-u)r,2r}(z_0)} \mathfrak{d}_h(z, \partial B(z_0,(2-u)r)) < t/3\right] > p,$$

then fix a bump function σ supported in $B(z_0, 2r)$ such that $\sigma \equiv 1$ on $B(z_0,(2-u/2)r)$. Since $\mathbb{P}[\operatorname{diam}(\overline{B}(z_0,(2-u)r); \mathfrak{d}_h(\cdot,\cdot; B(z_0,(2-u/2)r))) \le C] \to 1$ as $C \to \infty$, given any $p \in (0,1)$ we can fix $C_2 = C_2(p)$ so that

$$\mathbb{P}[F_{C_2,t,u}(h)] := \mathbb{P}\left[\begin{array}{c}\sup_{z \in \mathbb{A}_{(2-u)r,2r}(z_0)} \mathfrak{d}_h(z, \partial B(z_0,(2-u)r)) < t/3; \\ \operatorname{diam}(\overline{B}(z_0,(2-u)r); \mathfrak{d}_h(\cdot,\cdot; B(z_0,(2-u/2)r))) \le C_2\end{array}\right] > p.$$

Now for M' large enough depending on C_2, on the event above we have

$$\operatorname{diam}(\overline{B}(z_0,(2-u)r); \mathfrak{d}_{h-M'\sigma}) \le t/3,$$

whereas $\mathfrak{d}_{h-M'\sigma} \le \mathfrak{d}_h$ pointwise, so

$$\sup_{z \in \mathbb{A}_{(2-u)r,2r}(z_0)} \mathfrak{d}_{h-M'\sigma}(z, \partial B(z_0,(2-u)r)) < t/3,$$

which forces $\operatorname{diam}(\overline{B}(z_0,2r); \mathfrak{d}_{h-M'\sigma}) < t$.

Since $\mathfrak{d}_h(\partial B(z_0,2r), \partial B(z_0,5r))$ is almost surely positive, we can choose t sufficiently small and p sufficiently close to 1, then fix u and C_2 appropriately, such that with positive probability both $E_{\eta_1}(h) \cap F_{C_2,t,u}(h)$ holds and $\mathfrak{d}_h(\partial B(z_0,2r), \partial B(z_0,5r)) > t/(2\delta)$ (since the probability of the latter event tends to 1 as $t \to 0$). On this event, with M' chosen depending on C_2 as above, $\operatorname{diam}(\overline{B}(z_0,2r); \mathfrak{d}_{h-M'\sigma}) < t$, $E_{\eta_1}(h - M'\sigma)$ holds, and $\mathfrak{d}_{h-M'\sigma}(\partial B(z_0,2r), \partial B(z_0,5r)) > t/(2\delta)$. Indeed, the latter two events only depend on the field outside $B(z_0,2r)$ so are invariant under replacing h by $h - M'\sigma$. Since h and $h - M'\sigma$ have mutually absolutely continuous laws, we may conclude that with positive probability, $\operatorname{diam}(\overline{B}(z_0,2r); \mathfrak{d}_h) < t$, $E_{\eta_1}(h)$ holds, and $\mathfrak{d}_h(\partial B(z_0,2r), \partial B(z_0,5r)) > t/(2\delta)$.

Since $\varepsilon < 1/14$, we have $z_i \in \operatorname{int} K_N^{2\varepsilon}$ for each i, $0 \le i \le N$, so we can almost surely find paths $\gamma_i = \gamma_i(h|_{B(z_0,8r)}) \subset \operatorname{int} K_N^{2\varepsilon}$ from z_0 to z_i for $1 \le i \le N$ with finite $\mathfrak{d}_h$-length (e.g., by [DFG+20, Prop. 3.9]), which we can fix in some manner that is measurable w.r.t. $h|_{B(z_0,8r)}$ considered modulo additive constant. (For instance, the proof of [MQ20, Thm 1.2] still works for the internal metric of $\mathfrak{d}_h$ on a domain $U \subset \mathbb{C}$, so we could take γ_i to be the almost surely unique $\mathfrak{d}_h(\cdot,\cdot,\operatorname{int} K_N^{\varepsilon+1/14})$-geodesic from z_0 to z_i.)

For $1 \le i \le N$, explore γ_i from z_0 towards z_i and let z_i^* be the first point of $\gamma_i \setminus \overline{B}(z_0,2r)$ reached by this exploration such that

$$\mathfrak{d}_h(z_i^*, \partial B(z_0,2r)) = \mathfrak{d}_h(\partial B(z_0,2r), \partial B(z_0,5r))$$

(such a point exists by continuity of $\mathfrak{d}_h(\cdot, \partial B(z_0,2r))$ along γ_i).

We argue that, on the event

$$G_{z_0,8r}(h) := E_{\eta_1}(h) \cap \{\operatorname{diam}(\overline{B}(z_0,2r);\mathfrak{d}_h) < 2\delta\mathfrak{d}_h(\partial B(z_0,2r),\partial B(z_0,5r))\},$$

which we have just shown to have positive probability, the z_i^* form an $(N,1+\delta)$-clique. On $E_{\eta_1}(h)$, for $1 \le i < j \le N$ we have

$$\mathfrak{d}_h(z_i^*,z_j^*) \ge \mathfrak{d}_h(z_i^*,\partial B(z_0,2r)) + \mathfrak{d}_h(z_j^*,\partial B(z_0,2r)) = 2\mathfrak{d}_h(\partial B(z_0,2r),\partial B(z_0,5r)).$$

Indeed, this lower bound certainly holds for any path from z_i^* to z_j^* that intersects $\overline{B}(z_0,2r)$; however, since $\overline{B}(z_0,2r)$ disconnects the prongs of the star $K_N^{\varepsilon/4}$, any path from z_i^* to z_j^* that does *not* enter $\overline{B}(z_0,2r)$ must have a subpath contained in $\mathbb{A}_{2r,7r}(z_0) \setminus K_N^{\varepsilon/2}$ of Euclidean diameter at least $\zeta(\varepsilon)$, which on $E_{\eta_1}(h)$ must have $\mathfrak{d}_h$-length $> 2\mathfrak{d}_h(\partial B(z_0,2r),\partial B(z_0,5r))$.

Finally, on the event that $\operatorname{diam}(\overline{B}(z_0,2r);\mathfrak{d}_h) < 2\delta\mathfrak{d}_h(\partial B(z_0,2r),\partial B(z_0,5r))$, we have

$$\begin{aligned}\mathfrak{d}_h(z_i^*,z_j^*) &\le \mathfrak{d}_h(z_i^*,\partial B(z_0,2r)) + \mathfrak{d}_h(z_j^*,\partial B(z_0,2r)) + \operatorname{diam}(\overline{B}(z_0,2r);\mathfrak{d}_h)\\ &< 2(1+\delta)\mathfrak{d}_h(\partial B(z_0,2r),\partial B(z_0,5r)).\end{aligned}$$

Therefore the z_i^* form an $(N,1+\delta)$-clique. Thus, on the event $G_{z_0,8r}(h)$ which we have just shown to have positive probability, there exist points in $B(z_0,8r)$ that form an $(N,1+\delta)$-clique w.r.t. $\mathfrak{d}_h$. Note that, since $G_{z_0,8r}(h)$ only depends on ratios between distances and thus is determined by the field modulo additive constant, the scale and translation invariance properties of h imply that the analogous event $G_{z,r'}(h)$ with z_0 and r replaced respectively by z and r' (and the necessary changes made in the definitions of γ_i, z_i^*, $E_{\eta_1}(h)$) has the same probability for any $z \in \mathbb{C}$ and any $r' > 0$. Moreover, since $G_{z,r'}(h)$ is determined by $h|_{B(z,8r')}$, it is in fact determined by $(h - h_R(z'))|_{B(z,8r')}$ whenever $\overline{B}(z, 8r') \subset B(z', R)$.

We can now consider a sequence of nested concentric annuli within which we have near-independence of the field, meaning that if we take a closed disc $\overline{B}(z^{(k)}, 8r^{(k)})$ within each annulus then at least one of the events $G_{z^{(k)},r^{(k)}}(h)$ holds. Indeed we are in the setting of [GM20, Lemma 3.1], which implies that, say, for the annuli $(\mathbb{A}_{2^{-2k-1},2^{-2k}}(0))_{k\in\mathbb{N}}$ and $z^{(k)} = 3 \cdot 2^{-2k-2}$, $8r^{(k)} = 2^{-2k-3}$, a positive proportion of the events $\{G_{z^{(k)},8r^{(k)}}(h)\}_{k=1}^{K}$ hold with probability exponentially high in K. In particular, it is almost surely the case that at least one of the events $G_{z^{(k)},8r^{(k)}}(h)$ holds.

Since we have now shown that an $(N, 1 + \delta)$-clique almost surely exists for all N within a fixed closed disc, the surface $(\mathbb{C}, \mathfrak{d}_h)$ is almost surely $(1 + \delta)$-cliquey and thus cannot be embedded quasisymmetrically into any doubling space. The fact that this argument finds all the $(N, 1+\delta)$-cliques within the same disc also means that the local mutual absolute continuity of GFF variants gives the same result for other LQG surfaces, and thus we conclude the proof of Theorem 1.3.9.

Chapter 5

Upper bound on the conformal covariance exponent for the CLE chemical distance metric

The purpose of this chapter is to prove Theorem 1.3.12. As well as considering simple CLE_κ coupled with two-sided whole-plane SLE_κ for $\kappa \in (8/3, 4)$, in the non-simple regime we will denote the parameter by $\kappa' \in (4, 8)$ and refer to $\mathrm{CLE}_{\kappa'}$. We will couple such $\mathrm{CLE}_{\kappa'}$ with two-sided whole-plane SLE_κ where $\kappa = 16/\kappa' \in (2, 4)$, and make the definitions $\lambda = \pi/\sqrt{\kappa}$, $\lambda' = \pi/\sqrt{\kappa'}$. As explained in the preliminaries, in both the regimes $\kappa \in (8/3, 4)$ and $\kappa' \in (4, 8)$ we define $\chi = 2/\sqrt{\kappa} - \kappa/\sqrt{2}$.

5.1 BCLE/GFF couplings

We begin by explaining that the iterative BCLEs used to construct a CLE can be coupled with the GFF via imaginary geometry.

Lemma 5.1.1. *Fix $\kappa \in (8/3, 4)$. There exists a coupling of a CLE_κ and a GFF in $\mathbb{H}$ such that the loops of the CLE_κ are all traced by flow lines of the same angle modulo 2π. More precisely, using the standard GFF boundary conditions for such a coupling, they all have angle $\frac{\pi}{2}$ modulo 2π.*

Proof. Recall the iterative $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)/\mathrm{BCLE}^{\circlearrowright}_{\kappa}(-\kappa/2)$ construction of CLE_κ, where $\kappa' = 16/\kappa$. We first construct a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$. Note that the boundary conditions to couple this with a GFF on $\mathbb{H}$ are given by

$$f(x) = \begin{cases} \lambda' & x \in (-\infty, 0), \\ \lambda' - 2\pi\chi & x \in (0, \infty), \end{cases}$$

see [MSW17, Table 1]. Inside each of the true (clockwise) loops, the boundary data is as follows. If we map the interior of the loop to $\mathbb{H}$ with the first (equivalently last) point on its boundary visited by the BCLE exploration sent to 0, then the boundary data is given by

$$f_1(x) = \begin{cases} -\lambda' = -\lambda + \frac{\pi}{2}\chi & x \in (-\infty, 0), \\ -\lambda' - 2\pi\chi = -\lambda - \frac{3\pi}{2}\chi & x \in (0, \infty), \end{cases}$$

see [MSW17, Table 2]. Next, we note that to couple a $\mathrm{BCLE}_\kappa^{\circlearrowleft}(-\kappa/2)$ on $\mathbb{H}$ with a GFF h, we need the boundary data to be

$$f_2(x) = \begin{cases} \lambda(1-\kappa/2) + 2\pi\chi & x \in (-\infty, 0), \\ \lambda(1-\kappa/2) & x \in (0, \infty), \end{cases}$$

see [MSW17, Table 1]. We note that $f_2(x) = f_1(x) + \frac{5\pi}{2}\chi$. Thus, the loops of the CLE_κ that have been discovered so far (that is, the true, counterclockwise, loops of the $\mathrm{BCLE}_\kappa^{\circlearrowleft}(-\kappa/2)$) are flow lines at angle $+\frac{5\pi}{2}$.

The iteration then starts over in each region bounded by a false (counterclockwise) loop of the first $\mathrm{BCLE}_{\kappa'}^{\circlearrowleft}(0)$ or a false (clockwise) loop of the $\mathrm{BCLE}_\kappa^{\circlearrowleft}(-\kappa/2)$. We shall see that the loops of the second iteration of $\mathrm{BCLE}_{\kappa'}(0)$ will all have angle $\pm 2\pi$, and hence the result follows inductively.

When mapping the interior of the false (counterclockwise) loops of the first $\mathrm{BCLE}_{\kappa'}^{\circlearrowleft}(0)$ to $\mathbb{H}$ with the first (equivalently last) point on its boundary visited by the BCLE exploration sent to 0, then the resulting boundary data is

$$\widetilde{f}(x) = \begin{cases} \lambda' + 2\pi\chi = \lambda + \frac{3\pi}{2}\chi & x \in (-\infty, 0), \\ \lambda' = \lambda - \frac{\pi}{2}\chi & x \in (0, \infty), \end{cases}$$

and the boundary data needed to couple a new $\mathrm{BCLE}_{\kappa'}^{\circlearrowleft}(0)$ is $f(x) = \widetilde{f}(x) - 2\pi\chi$, thus the angle of the loops in the new $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$ is -2π.

Finally, if we map the interior of the false (clockwise) $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(-\kappa/2)$ loops back to $\mathbb{H}$ with the first (equivalently last) point on its boundary visited by the BCLE exploration sent to 0, we get boundary data

$$\widehat{f}(x) = \begin{cases} \lambda - \frac{5\pi}{2}\chi & x \in (-\infty, 0), \\ \lambda - \frac{9\pi}{2}\chi & x \in (0, \infty). \end{cases}$$

This is because, by [MSW17, Table 2], we get boundary data

$$f_3(x) = \begin{cases} \lambda & x \in (-\infty, 0), \\ \lambda - 2\pi\chi & x \in (0, \infty), \end{cases}$$

for the field of which the $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(-\kappa/2)$ loops are flow lines, which as we have seen is $h+\frac{5\pi}{2}\chi$. So the boundary data for h is $\widehat{f} = f_3 - \frac{5\pi}{2}\chi$. When coupling a new $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$, we need boundary data $\widehat{f}(x) + 2\pi\chi$. Consequently, the angle of those loops is -2π, and hence the result follows. ■

This calculation is easier in the case of $\mathrm{CLE}_{\kappa'}$ for $\kappa' \in (4, 8)$ since the BCLE construction is simpler. In this case we start with a $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$, and then in each of its boundary-intersecting false (counterclockwise) loops we sample a new $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$, and so on. The corresponding result is as follows:

Lemma 5.1.2. *Fix $\kappa' \in (4, 8)$. There exists a coupling of a $\mathrm{CLE}_{\kappa'}$ and a GFF in $\mathbb{H}$ such that the loops of the $\mathrm{CLE}_{\kappa'}$ are all traced by counterflow lines of the same angle modulo 2π. Indeed, using the standard GFF boundary conditions for such a coupling, they all have angle 0 modulo 2π.*

Proof. As before, to couple the initial $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(0)$ with a GFF, the boundary data (as can be seen in [MSW17, Table 1]) is

$$f(x) = \begin{cases} \lambda' & x \in (-\infty, 0), \\ \lambda' - 2\pi\chi & x \in (0, \infty). \end{cases}$$

If we map a complementary component which is bounded by a false (counterclockwise) loop (under all the true loops that have been drawn so far) back to $\mathbb{H}$ with the first (resp.

equivalently last) point on its boundary sent to 0, then the boundary data (see [MSW17, Table 2]) is

$$\widetilde{f}(x) = \begin{cases} \lambda' + 2\pi\chi & x \in (-\infty, 0), \\ \lambda' & x \in (0, \infty), \end{cases}$$

i.e. $\widetilde{f} = f + 2\pi\chi$. ∎

5.2 Scaling and translation for two-sided whole-plane SLE

When $\kappa \in (8/3, 4)$ we will define $d := 1 + \kappa/8$. Given κ, we work with a fixed collection of CLE_κ metric probability measures (μ_D) as in Assumption 1.3.10. Our argument will consider a two-sided whole-plane SLE_κ curve η from ∞ to ∞ through 0, parametrized via the natural parametrization (recall that this is a multiple of the d-dimensional Minkowski content), with $\eta(0) = 0$. We will then construct a CLE_κ Γ in the domain D_η, defined to be the component of $\mathbb{C} \setminus \eta$ on the left-hand side of η, with associated CLE metric d_Γ. (In other words, the joint law of (η, Γ, d_Γ) is determined by the properties that the marginal law of η is that of a two-sided whole-plane SLE_κ curve η from ∞ to ∞ through 0 and that, given η, the conditional law of (Γ, d_Γ) is the joint law μ_{D_η} of a CLE_κ in D_η and a CLE metric associated to this CLE_κ.)

In the other case of Theorem 1.3.12 we will instead denote the parameter of the CLE by $\kappa' \in (4, 8)$, and make the definition $\kappa := 16/\kappa'$. We will again define $d := 1 + \kappa/8$, so that as before Theorem 1.3.12 states that $\alpha < d$, and this time fix a collection (μ_D) of $\mathrm{CLE}_{\kappa'}$ metric probability measures. Again, we define η to be a two-sided whole-plane SLE_κ curve from ∞ to ∞ through 0, parametrized via the natural parametrization, with $\eta(0) = 0$, and D_η to be the component of $\mathbb{C} \setminus \eta$ on the left-hand side of η. In this case we define (Γ, d_Γ) so that their conditional joint law given η is μ_{D_η}, which this time gives a $\mathrm{CLE}_{\kappa'}$ in D_η along with an associated CLE metric. Recall that we are making Assumption 1.3.11, which says that $\sup_{t<1} d_\Gamma(\eta(0), \eta(t))$ is integrable in both cases.

Recall also that by [Zha21, Cor. 4.7], for each $r > 0$ the scaling map

$$S_r : \eta(\cdot) \mapsto r^{-1/d}\eta(r\cdot)$$

and the translation map

$$T_r : \eta(\cdot) \mapsto \eta(\cdot + r) - \eta(r)$$

are measure-preserving (w.r.t. the law of η).

Write $S_r(\Gamma)$ for the process in $D_{S_r(\eta)} := r^{-1/d} D_\eta$ obtained by scaling the loops of Γ by $r^{-1/d}$. Note that because the scaling map from D_η to $D_{S_r(\eta)}$ is conformal, $S_r(\Gamma)$ is a CLE$_\kappa$ in $D_{S_r(\eta)}$, and by conformal covariance, if we define

$$d_{S_r(\Gamma)}(r^{-1/d}\cdot, r^{-1/d}\cdot) = d_\Gamma(\cdot,\cdot) r^{-\alpha/d}$$

then the law of $(S_r(\Gamma), d_{S_r(\Gamma)})$ given $S_r(\eta)$ is $\mu_{D_{S_r(\eta)}}$. Because S_r is measure-preserving, the map

$$(\eta, \Gamma, d_\Gamma(\cdot,\cdot)) \mapsto (S_r(\eta), S_r(\Gamma), d_{S_r(\Gamma)}(r^{-1/d}\cdot, r^{-1/d}\cdot))$$

is measure-preserving w.r.t. the joint law of (η, Γ, d_Γ). Note that our first-moment assumption now gives that $\sup_{t<r} d_\Gamma(\eta(0), \eta(t))$ is integrable for each $r > 0$.

Write $T_r(\Gamma)$ for the process obtained by translating the loops of Γ by $-\eta(r)$. Then, again by conformal covariance, if we define

$$d_{T_r(\Gamma)}(\cdot - \eta(r), \cdot - \eta(r)) = d_\Gamma(\cdot,\cdot)$$

then the law of $(T_r(\Gamma), d_{T_r(\Gamma)})$ given $T_r(\eta)$ is $\mu_{D_{T_r(\eta)}}$, so that the map

$$(\eta, \Gamma, d_\Gamma(\cdot,\cdot)) \mapsto (T_r(\eta), T_r(\Gamma), d_{T_r(\Gamma)}(\cdot - \eta(r), \cdot - \eta(r)))$$

is measure-preserving w.r.t. the joint law of (η, Γ, d_Γ).

Lemma 5.2.1. *In the setting of Theorem 1.3.12 we have $\alpha \le d$.*

Proof. Assume $\alpha > d$ and fix r such that r^{-1} is an integer greater than 2. Scaling by S_r and using conformal covariance and that S_r is measure-preserving, we find that $d_\Gamma(\eta(0), \eta(r^{-1}))$ has the same law as

$$d_{S_r(\Gamma)}(S_r(\eta)(0), S_r(\eta)(r^{-1})) = d_{S_r(\Gamma)}(r^{-1/d}\eta(0), r^{-1/d}\eta(1)) = d_\Gamma(\eta(0), \eta(1)) r^{-\alpha/d}.$$

Therefore we have

$$\mathbb{E}[d_\Gamma(\eta(0),\eta(r^{-1}))] = \mathbb{E}[d_\Gamma(\eta(0),\eta(1))]r^{-\alpha/d}.$$

Note that these expectations are finite by our first-moment assumption. However, since T_1 is measure-preserving, we know that

$$\mathbb{E}[d_\Gamma(\eta(0),\eta(r^{-1}))] \le \sum_{k=1}^{r^{-1}} \mathbb{E}[d_\Gamma(\eta(k-1),\eta(k))] = r^{-1}\mathbb{E}[d_\Gamma(\eta(0),\eta(1))].$$

This is a contradiction, since $r^{-1} > 1$ so $r^{-1} < r^{-\alpha/d}$. ■

It remains to rule out $\alpha = d$. For this our plan is as follows. We will first show that η and Γ can be coupled with a whole-plane GFF h in such a way that h determines both η and Γ. We will use this coupling to prove that the scaling S_r is ergodic w.r.t. the joint law of η and Γ, then use that ergodicity to argue that, on the assumption $\alpha = d$, conditional expectations of d_Γ-distances between points on η given *both* η and Γ are determined by Γ alone. To derive a contradiction from this, we will use the GFF coupling again to argue that one can resample a segment of η to make it longer (in terms of natural parametrization) but without increasing the conditional expectation given η and Γ of the d_Γ-distance between the segment's endpoints.

5.3 Coupling CLE with two-sided whole-plane SLE

We will show that we can couple (η,Γ) with h, a whole-plane GFF modulo $2\pi(\chi+\zeta)$, so that h determines (η,Γ).

The first step is to use [MS17, Thm 1.4] to couple the two-sided whole-plane SLE$_\kappa$ η with a whole-plane GFF. Once we have coupled η with the GFF, we will show that the boundary conditions for the GFF in D_η are the appropriate ones to construct Γ from its counterflow lines.

We begin with the case $\kappa \in (8/3,4)$. Fix $\zeta = \sqrt{\kappa}/2$ and let h be a whole-plane GFF modulo $2\pi(\chi+\zeta)$. Using [MS17, Thm 1.4], the flow line of angle 0 from 0 corresponding to the field $h - \zeta\arg(\cdot)$ is a whole-plane SLE$_\kappa(2-\kappa+2\pi\zeta/\lambda)$, provided $\zeta > -\chi := \sqrt{\kappa}/2 - 2/\sqrt{\kappa}$. Since $\zeta = \sqrt{\kappa}/2$ and $\kappa > 0$, the inequality is satisfied and we get a whole-plane SLE$_\kappa(2)$. Thus, conditional on $\eta|_{[0,\infty)}$, the boundary conditions for the field $h - \zeta\arg(\cdot)$ are the so-called *flow* *line boundary conditions*. Specifically, if φ is a conformal map from $\mathbb{C} \setminus \eta([0,\infty))$ to $\mathbb{H}$ that swaps 0 and ∞, then the boundary conditions of the field $h - \zeta \arg(\cdot)$ on $\mathbb{C} \setminus \eta([0,\infty))$ are the same as those of $f \circ \varphi - \chi \arg \varphi'$ (as fields modulo $2\pi(\chi + \zeta)$), where f is the harmonic function on $\mathbb{H}$ with boundary conditions

$$f(x) = \begin{cases} -\lambda & x \in (-\infty, 0), \\ \lambda & x \in (0, \infty). \end{cases} \tag{5.3.1}$$

This means that [MS16a, Thm 1.1] the flow line of angle 0 from ∞ to 0 corresponding to the field $h - \zeta \arg(\cdot)$ in the remaining domain $\mathbb{C} \setminus \eta([0,\infty))$ is a chordal SLE_κ in that domain from ∞ to 0. Indeed, this flow line corresponds to the flow line of angle 0 from 0 to ∞ in $\mathbb{H}$ with the boundary conditions in (5.3.1).

We can thus construct $\eta|_{(-\infty,0]}$ to be this 0-angle flow line. Again working in $\mathbb{H}$ with boundary conditions (5.3.1), if we map the part of $\mathbb{H}$ to the left of the flow line to $\mathbb{H}$ via a conformal map ψ that fixes 0 and ∞, and consider the field $(h - \zeta \arg) \circ \psi^{-1} - \chi \arg(\psi^{-1})'$ on $\mathbb{H}$, then the boundary conditions on $\partial\mathbb{H}$ are constantly $-\lambda$. If we add $\lambda + \lambda' - \pi\chi$ to this field, where $\kappa' = 16/\kappa$ and $\lambda' = \pi/\sqrt{\kappa'}$, we get $\lambda' - \pi\chi$ on the boundary, which is the appropriate boundary data for the counterflow line from ∞ to 0 to be an $\mathrm{SLE}_{\kappa'}(\kappa' - 6)$ process with the force point at ∞^+, i.e. on the counterclockwise side of ∞. (This can be seen by considering the boundary data for this process in a rectangle $[-r, r] \times [0, 1]$, which would have boundary data $\lambda' - 2\pi\chi$ on $(-r, 0) \times \{1\}$ and λ' on $(0, r) \times \{1\}$ – see [MS17, §4.1] – and thus constantly $\lambda' - \pi\chi$ on $(-r, r) \times \{0\}$, then applying a conformal map looking like the identity at 0 to send i to ∞.)

By translation invariance, it follows that for each $x \in \mathbb{R}$ the counterflow line from ∞ to x is an $\mathrm{SLE}_{\kappa'}(\kappa' - 6)$ process from ∞ to x with the force point at ∞^+, and thus that the process given by the collection of counterflow lines from ∞ targeted at a countable dense set of boundary points is an $\mathrm{SLE}_{\kappa'}(\kappa' - 6)$ branching tree with the force point on the left-hand side of the image of $\eta|_{(-\infty,0]}$. The set of boundary-touching loops of this branching tree is by definition a $\mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(0)$. At this point, one can for instance apply (2.5.2) to swap 0 and ∞, thus bringing us into the setting of the proof of Lemma 5.1.1, which shows that the boundary conditions are the appropriate ones to construct $\mathrm{BCLE}^{\circlearrowleft}_{\kappa}(-\kappa/2)$ within the loops of the $\mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(0)$ and thus continue the iterative BCLE construction of a CLE_κ Γ. So we have coupled (η, Γ) with the field h.

The case $\kappa' \in (4,8)$ is simpler, since we are done once we have generated the branching tree. Indeed, we again set $\zeta = \sqrt{\kappa}/2$ (remember $\kappa = 16/\kappa'$). As before we can couple η with $h - \zeta \arg(\cdot)$, yielding the appropriate boundary conditions for the process given by the collection of counterflow lines from ∞ targeted at a countable dense set of boundary points to be an $\mathrm{SLE}_{\kappa'}(\kappa'-6)$ branching tree with the force point on the left-hand side of the image of $\eta|_{(-\infty,0]}$. We can thus define a $\mathrm{CLE}_{\kappa'}$ Γ via the branching tree construction.

Proof of Prop. 1.3.13. The idea to complete the proof of Prop. 1.3.13 is that the σ-algebras generated by local restrictions of the field become trivial in the limit as the domains to which we restrict shrink.

Indeed, by [HS18, Lemma 2.2], $\bigcap_{r>0} \mathcal{F}_r$ is trivial. Now, if $r > 1$ and A is an S_r-invariant (equivalently, $S_{r^{-1}}$-invariant) event measurable w.r.t. (η, Γ) (and thus w.r.t. h), then the martingale convergence theorem implies that, almost surely,

$$\mathbb{P}[A|\mathcal{F}_{r^{n/d}}] \to \mathbf{1}_A \quad \text{as} \quad n \to \infty.$$

However, invariance under $S_{r^{2n}}$ gives

$$\mathbb{P}[A|\mathcal{F}_{r^{n/d}}] \overset{(d)}{=} \mathbb{P}[A|\mathcal{F}_{r^{-n/d}}]$$

almost surely, whereas backward martingale convergence gives

$$\mathbb{P}[A|\mathcal{F}_{r^{-n/d}}] \to \mathbb{P}\left[A \,\middle|\, \bigcap_{t>0} \mathcal{F}_t\right] = \mathbb{P}[A]$$

almost surely and thus also weakly. Hence $\mathbb{P}[A|\mathcal{F}_{r^{n/d}}] \to \mathbb{P}[A]$ weakly, but since we know that $\mathbb{P}[A|\mathcal{F}_{r^{n/d}}] \to \mathbf{1}_A$ almost surely we must have $\mathbf{1}_A = \mathbb{P}[A]$ almost surely and thus $\mathbb{P}[A] \in \{0,1\}$.

We have thus established that S_r is ergodic w.r.t. the joint law of (η, Γ), in particular establishing Prop. 1.3.13 (ergodicity w.r.t. the marginal law of η) – note that in order to couple just the two-sided whole-plane SLE_κ with the GFF, we only needed [MS17, Thm 1.4] which holds for any whole-plane $\mathrm{SLE}_\kappa(\rho)$ curve with $\kappa \in (0,4)$ (the inequality $\zeta > -\chi$ is equivalent to the condition $\rho > -2$ for the whole-plane $\mathrm{SLE}_\kappa(\rho)$ process to be generated by a continuous curve, and here we have $\rho = 0$), and thus we have indeed established Prop. 1.3.13

for the entire stated range $\kappa \in (0, 4)$, not just the range $\kappa \in (8/3, 4)$ that we need for our present purpose. ■

5.4 Concluding the proof: boundary distances are not determined by the SLE

Since by the triangle inequality the process $\{\mathbb{E}[d_\Gamma(\eta(0), \eta(t))|\eta, \Gamma] : t \geq 0\}$ is subadditive, Kingman's subadditive ergodic theorem applied to the family $\{T_r : r > 0\}$ shows that there is a random variable X such that, almost surely,

$$\frac{\mathbb{E}[d_\Gamma(\eta(0), \eta(t))|\eta, \Gamma]}{t} \to X$$

as $t \to \infty$. (In order to apply the continuous-time version [Kin73, Thm 4] of Kingman's theorem, we use the assumption that $\sup_{t<1} d_\Gamma(\eta(0), \eta(t))$ is integrable.)

We now have

$$\begin{aligned} X \circ S_r &= \lim_{t \to \infty} \frac{\mathbb{E}[d_{S_r(\Gamma)}(S_r(\eta)(0), S_r(\eta)(t))|\eta, \Gamma]}{t} \\ &= \lim_{t \to \infty} \frac{\mathbb{E}[d_\Gamma(\eta(0), \eta(rt))|\eta, \Gamma] r^{-\alpha/d}}{t} = r^{1-\alpha/d} X. \end{aligned}$$

Thus if $\alpha = d$, then X is invariant under the scaling map S_r for all $r > 0$. Note that $X < \infty$ almost surely since, for instance, Kingman's theorem also gives convergence in L^1 and we have assumed that $\sup_{t<1} d_\Gamma(\eta(0), \eta(t))$ is integrable. Thus, by ergodicity of S_r, X is almost surely equal to a finite constant $c \geq 0$. It follows that with probability 1 we have $\mathbb{E}[d(\eta(0), \eta(t))|\eta, \Gamma] = ct$ for all $t \geq 0$; this is because, by scale invariance, the probability of the event

$$\left\{ \sup_{s \geq t} \left| \frac{\mathbb{E}[d_\Gamma(\eta(0), \eta(s))|\eta, \Gamma]}{s} - c \right| > \varepsilon \right\}$$

for fixed $\varepsilon > 0$ does not depend on t, but these events are nested and their intersection over all t has zero probability, so each of the events themselves has zero probability.

If $c = 0$, we obtain a contradiction to the assumption in the statement of Thm 1.3.12 that the CLE$_\kappa$ metric does not vanish on the boundary of D.

Lemma 5.4.1. *In the setup above, let $\sigma := \inf\{t \in \mathbb{R} : \eta(t) \in \partial B(0, 1)\}$. Then there exists a coupling of η and Γ with a curve γ from $\eta(0)$ to $\eta(\sigma)$ that is either a CPI in the carpet of the*

CLE_κ Γ (when $\kappa \in (8/3, 4)$) or a strand of an exploration of the $\mathrm{CLE}_{\kappa'}$ Γ (when $\kappa' \in (4, 8)$) such that, conditional on $\eta|_{[0,\infty)}$, $\eta|_{(-\infty,\sigma]}$ and γ, the law of the unexplored portion of η is that of an $\mathrm{SLE}_\kappa(\kappa - 4)$ in the region to the right of γ between the two endpoints $\eta(0) = 0$ and $\eta(\sigma)$ of the explored parts of η.

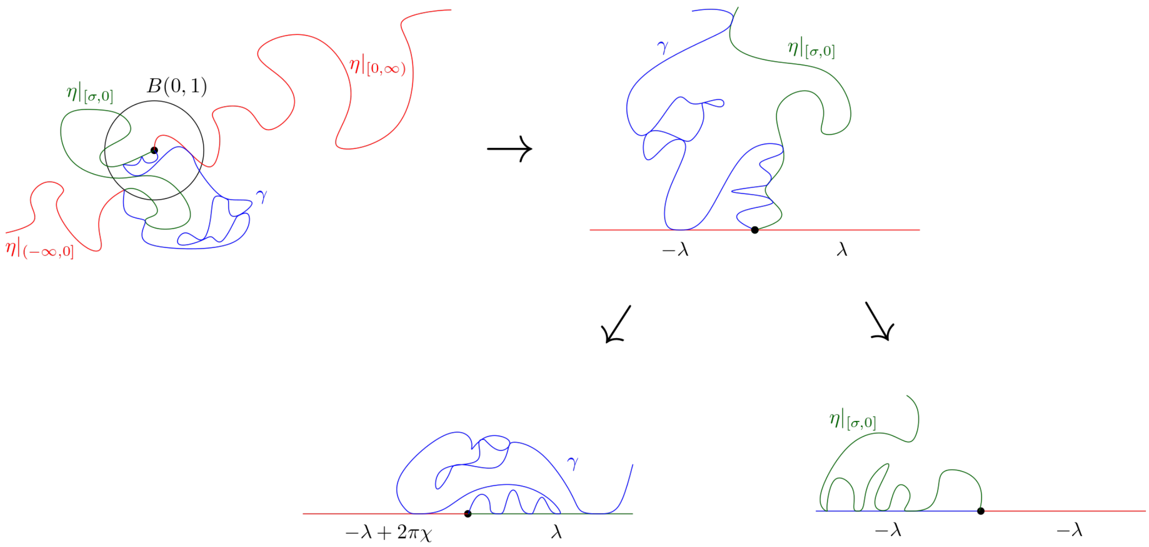

Figure 5.1: In the proof of Lemma 5.4.1, we use GFF couplings to draw a CLE exploration γ and part of the domain boundary η in either order; the resulting boundary conditions tell us the conditional laws of each curve given the other.

Proof. If $\kappa < 4$, then we can construct $\eta|_{[0,\infty)}$ as a whole-plane $\mathrm{SLE}_\kappa(2)$, then (by reversibility of SLE_κ for $\kappa \le 4$ [Zha08]) construct part of $\eta|_{(-\infty,0]}$ as a chordal SLE_κ from ∞ to 0 in the remaining domain, but stop this process when it first hits $\partial B(0,1)$. We can then conformally map the domain we have obtained to $\mathbb{H}$, sending $\eta(\sigma)$ to 0 and 0 to ∞, so that the remaining portion of η is a chordal SLE from 0 to ∞. We can thus construct the remaining portion of η as a flow line of angle 0 from 0 of a GFF h with boundary conditions $-\lambda$ on the negative real axis and λ on the positive real axis.

If we then map the region to the left of this flow line to $\mathbb{H}$, fixing 0 and ∞, the boundary conditions of the transformed field become λ on the boundary, so as before, if we add $\beta = \lambda + \lambda' - \pi\chi$ to the field, the counterflow line from ∞ to 0 is an $\mathrm{SLE}_{\kappa'}(\kappa' - 6)$ process, from which we can construct a CLE_κ in D_η. By [MS16a, Thm 1.4], its right-hand boundary γ, which by [MSW17, §4] is a CPI in the CLE_κ, is therefore a flow line of $h + \beta$ with angle $-\pi/2$, or equivalently a flow line of h with angle $\beta/\chi - \pi/2$.

If we instead drew this flow line first, then conformally mapped the region to its right back to $\mathbb{H}$, fixing 0 and ∞, the boundary conditions would therefore become λ on the positive real axis and $\lambda' - \beta + \pi\chi = -\lambda + 2\pi\chi$ on the negative real axis. With these boundary conditions, the flow line of angle 0 from 0 has the law of an $\mathrm{SLE}_\kappa(\kappa-4)$ from 0 to ∞ with the force point at 0^- [MS16a, Thm 1.1]. We have thus found that the conditional law of the remaining part of η given the parts we have already sampled *and* the CPI γ is that of an $\mathrm{SLE}_\kappa(\kappa-4)$.

If $\kappa' > 4$ we can use the same construction to get $\eta|_{[0,\infty)}$, $\eta|_{(-\infty,\sigma]}$ and h; then instead of considering just the counterflow line of $h+\beta$ from ∞ to 0, we can consider the $\mathrm{SLE}_{\kappa'}(\kappa'-6)$ branching tree given by the collection of counterflow lines from ∞ targeted at a countable dense set of boundary points in order to define Γ. This time, the right-hand boundary γ of the counterflow line from ∞ to 0 will be a strand of an exploration of the $\mathrm{CLE}_{\kappa'}$ Γ, but since the construction of η and γ was the same as in the $\kappa < 4$ case, we get the same result on the conditional law of the remaining portion of η. ∎

Now $d_\Gamma(0,\eta(\sigma))$ is bounded by the distance between 0 and $\eta(\sigma)$ w.r.t. the internal metric induced by d_Γ on the region to the left of γ. This will have finite conditional expectation w.r.t. (η,Γ) almost surely by Assumption 1.3.11, since γ is a bounded portion of an SLE_κ-type curve and $c < \infty$. Moreover, since γ is either a CPI or a strand of a CLE exploration, this expectation is almost surely the same as that w.r.t. just $\eta|_{(-\infty,\sigma]}$, $\eta|_{[0,\infty)}$, γ and the loops of Γ on the left-hand side of γ. In order to prove Theorem 1.3.12, it thus suffices to show that, conditional on $\eta|_{(-\infty,\sigma]}$, $\eta|_{[0,\infty)}$ and γ, the natural length of $\eta|_{[\sigma,0]}$ can be arbitrarily large with positive probability.

Proof of Theorem 1.3.12. Consider a conformal map sending the domain to the right of γ to $\mathbb{H}$, with 0 and $\eta(\sigma)$ respectively mapping to ∞ and 0. Since the derivative of this map will be bounded on a fixed compact set away from the boundary, it is enough to show that the image of $\eta|_{[\sigma,0]}$ (which is just an $\mathrm{SLE}_\kappa(\kappa-4)$ from 0 to ∞) can attain arbitrarily high natural length inside a fixed ball away from the boundary with positive probability.

Fix $\delta > 0$. We show that the natural length attained inside $B(i,1/2)$ by an $\mathrm{SLE}_\kappa(\kappa-4)$ $\widetilde{\eta}$ from 0 to ∞ in $\mathbb{H}$ can be $\Omega(\delta^{-1+\kappa/8})$ with positive probability; since this goes to ∞ as $\delta \to 0$, this will suffice.

Fix an ordering on a set of $N = \Omega(\delta^{-2})$ balls of radius δ inside $B(i,1/2)$ at positive distance from each other, say $B_1, B_2, \ldots$, with centres $z_1, z_2, \ldots$. With positive probability, for all i, $\widetilde{\eta}$

will enter $\overline{B}_{z_i}(3\delta/4)$ for the first time (say at time σ_i) before entering any B_j for $j > i$ or the negative real axis. (This is because an initial segment of $\widetilde{\eta}$ has positive probability to stay close to a given smooth simple curve in $\mathbb{H}$ – see [MW17, Lemma 2.3].) We can realize $\widetilde{\eta}$ as the 0-angle flow line from 0 of a GFF h in $\mathbb{H}$ with boundary conditions $-\lambda + 2\pi\chi$ on $(-\infty, 0)$ and λ on $(0, \infty)$.

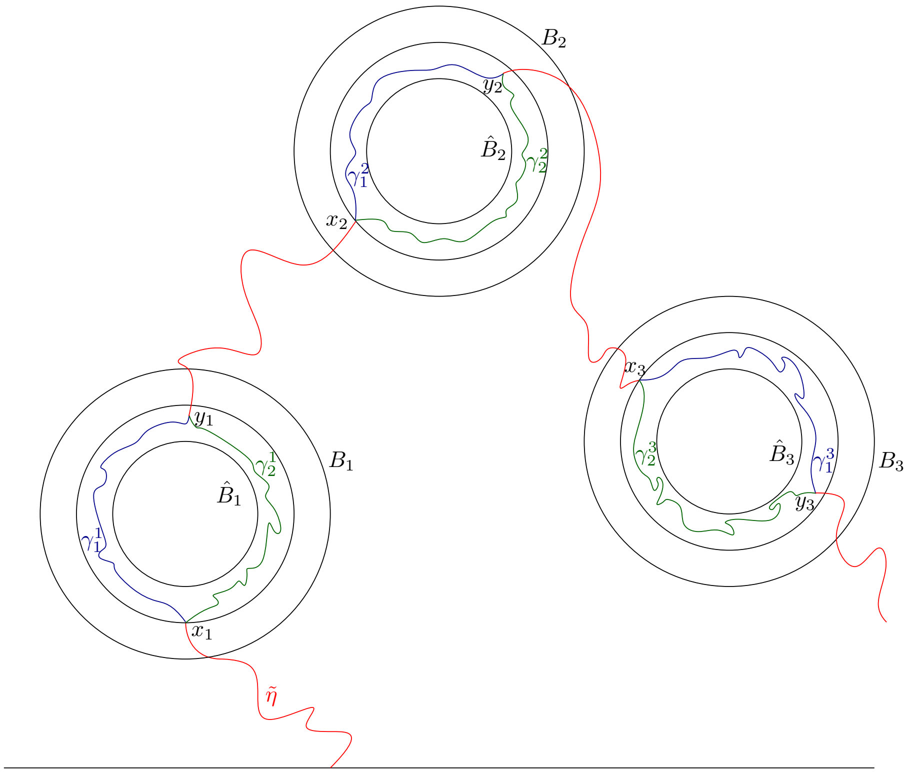

Figure 5.2: We illustrate the argument that concludes the proof that $\alpha < d$, in which we use auxiliary flow lines to generate domains in which we can resample the field to attain arbitrarily high natural length for η.

Now take the conformal map $\psi_i \colon \mathbb{H} \setminus \widetilde{\eta}([0, \sigma_i]) \to \mathbb{H}$ sending $x_i := \widetilde{\eta}(\sigma_i)$ to 0 and looking like the identity at ∞. Let h_i be the field $h \circ \psi_i^{-1} - \chi \arg(\psi_i^{-1})'$ on $\mathbb{H}$, which has the appropriate boundary conditions for the flow line of angle 0 from 0 to be an $\mathrm{SLE}_\kappa(\kappa - 4)$ from 0 to ∞. We can then sample the flow lines in $\mathbb{H}$ of h_i from 0 of angles θ_1 and θ_2, where

$$\theta_2 - \pi\kappa/(4-\kappa) < \theta_1 < 0 < \theta_2 < 2\pi \min\{1, (\kappa-2)/(4-\kappa)\},$$

and the θ_i are small enough for those flow lines to be simple curves, and let their images under ψ_i^{-1} be γ_1^i and γ_2^i respectively. (Note that these are the flow lines of h from x_i of angles θ_1 and θ_2 respectively.) The values of θ_1 and θ_2 are chosen so that, by [MS16a, Thm 1.5], there is a positive probability that γ_1^i and γ_2^i bounce off each other. Indeed, letting $\widehat{B}_i = B(z_i, \delta/2)$, we will show that there is a positive probability that for all i, when run until bouncing off each other for the first time, γ_1^i and γ_2^i disconnect $\partial\widehat{B}_i$ and ∂B_i, without hitting either circle. On the event that this happens for each i, let D_i be the connected component of $\widehat{B}_i$ in the complement of the traces of γ_1^i and γ_2^i run until bouncing off each other, say at y_i. Our aim is to define an exploration discovering the D_i in which, with positive probability, all the D_i have "nice" geometries; we will then be able to explain why, when we resample the field inside each D_i, $\widetilde{\eta}$ has a positive probability of attaining an appropriately large natural length.

We begin by sampling $\widetilde{\eta}$ until time σ_1. If we fix a deterministic simple curve Γ_1 from 0 to a point on $\partial B(z_i, 3\delta/4)$ that does not hit $\partial\mathbb{H}$ except at the beginning, and some $\varepsilon > 0$ such that $\varepsilon \ll \delta$, then (by slightly extending the curve into $B(z_i, 3\delta/4)$) [MW17, Lemma 2.3] guarantees that with positive probability $\widetilde{\eta}$ remains in the ε-neighbourhood of Γ_1 until time σ_1. Likewise, if we now apply ψ_1, we can fix another curve Γ_1^1 depending only on ψ_1 (say, one whose image under ψ_1^{-1} is a semicircular arc of $\partial B(z_i, 3\delta/4)$) and then with positive probability γ_1^1 will get within ε of the far end of Γ_1^1 (say, at time σ_1^1 for the natural parametrization in $\mathbb{H} \setminus \widetilde{\eta}([0, \sigma_i])$) before leaving the ε-neighbourhood of Γ_1^1.

Now by [MW17, Lemma 2.5], γ_2^1 (which, conditionally on the exploration so far along with the entirety of γ_1^1, has the law of an $\mathrm{SLE}_\kappa((\kappa-4)(1+\theta_2/(2\pi)); (\theta_2-\theta_1)\chi/\lambda-2)$ process, as can be checked by mapping the region to the left of γ_1^1 back to $\mathbb{H}$ – note that both weights are greater than -2) has positive probability (conditional on γ_1^1 up until time σ_1^1) of staying within ε of the opposite semicircular arc of $\partial B(z_i, 3\delta/4)$, call it Γ_2^1, until hitting γ_1^1.

Since the conditions on θ_1 and θ_2 are such that $\widetilde{\eta}$ does not cross γ_1^i or γ_2^i, $\widetilde{\eta}$ must exit $\overline{D_i}$ at y_i; let $\widetilde{\sigma}_i$ be the first time after σ_i that $\widetilde{\eta}$ hits y_i. We now apply a conformal map from the domain $\mathbb{H} \setminus (D_1 \cup \widetilde{\eta}([0, \widetilde{\sigma}_1]))$ to $\mathbb{H}$ and repeat this argument (again, noting that all the curves we discover are $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ processes with $\rho^L, \rho^R > -2$ conditionally on what has already been explored) to discover the other D_i, so that, defining Γ_1^i and Γ_2^i analogously, there is a positive probability that γ_1^i and γ_2^i respectively stay within ε of them; we then work on the positive-probability event E_δ that this happens for all i.

We can condition on the portions of $\widetilde{\eta}$ up to time σ_N not contained in any D_i and on each γ_1^i and γ_2^i run until they hit each other at y_i. Then we can resample h inside each D_i. By [MS16a, Lemma 7.1], the law of $\widetilde{\eta}|_{[\sigma_i, \widetilde{\sigma}_i]}$ will be that of an $\mathrm{SLE}_\kappa(\theta_2\chi/\lambda - 2; -\theta_1\chi/\lambda - 2)$ in D_i from x_i to y_i. Moreover, by locality, these portions will be conditionally independent for each i. We now just have to argue that there exists $c > 0$ independent of δ and i such that, for each i, the segment $\widetilde{\eta}|_{[\sigma_i,\widetilde{\sigma_i}]}$ has natural length at least $c\delta^{1+\kappa/8}$ with positive probability.

If we fix conformal maps $\varphi_i\colon \mathbb{D} \to D_i$ mapping 0 to z_i, then since $B(z_i, \delta/2) \subseteq D_i$ we can apply Lemma 2.6.2 to find that $|\varphi_i'(z)| \geq \delta/16$ whenever $\varphi_i(z) \in B(z_i, \delta/4)$, and that $|\varphi_i'(0)| \leq 4\delta$. Moreover, by the Koebe quarter theorem, we have that $\varphi_i^{-1}(B(z_i, \delta/4))$ contains $B(0, |\varphi_i'(0)|^{-1}\delta/16) \supseteq B(0, 1/64)$.

Note also that on E_δ we can bound $|\varphi_i^{-1}(x_i) - \varphi_i^{-1}(y_i)|$ away from 0: it suffices to consider a Brownian motion B started from z_i and bound both the probabilities that B exits D_i through γ_1^i and through γ_2^i away from 0, but this can be done because γ_1^i and γ_2^i are close to Γ_1^i and Γ_2^i respectively.

Since $\varphi_i^{-1}(\widetilde{\eta}|_{[\sigma_i,\widetilde{\sigma_i}]})$ is an $\mathrm{SLE}_\kappa(\theta_2\chi/\lambda-2; -\theta_1\chi/\lambda-2)$ from $\varphi_i^{-1}(x_i)$ to $\varphi_i^{-1}(y_i)$ in $\mathbb{D}$, it has a positive chance of hitting $B(0, 1/64)$ and accumulating a macroscopic amount of length, say $\ell > 0$, inside this small ball. Certainly such an $\ell > 0$ exists for an $\mathrm{SLE}_\kappa(\theta_2\chi/\lambda-2; -\theta_1\chi/\lambda-2)$ between antipodal points. On E_δ, since $|\varphi_i^{-1}(x_i) - \varphi_i^{-1}(y_i)|$ is bounded away from 0, we can uniformly bound the derivative on $B(0, 1/64)$ of a conformal automorphism of the disc sending $\varphi_i^{-1}(x_i)$ and $\varphi_i^{-1}(y_i)$ to antipodal points, and therefore (by giving up a constant) we can choose ℓ independently of where $\varphi_i^{-1}(x_i)$ and $\varphi_i^{-1}(y_i)$ are.

We know that $|\varphi_i'(z)| \geq \delta/16$ for $z \in \varphi_i^{-1}(B(z_i, \delta/4))$, so conformal covariance implies that the natural length attained by $\widetilde{\eta}|_{[\sigma_i,\widetilde{\sigma_i}]}$ is at least $\ell(\delta/16)^{1+\kappa/8}$, and we have proven the result with $c = 16^{-(1+\kappa/8)}\ell$. ■

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