# Applying the POWHEG method to top pair production and decays at the ILC 

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#### Abstract

We study the effects of gluon radiation in top pair production and their decays for $e^{+} e^{-}$annihilation at the ILC. To achieve this we apply the POWHEG method and interface our results to the Monte Carlo event generator Herwig++. We consider a center-of-mass energy of $\sqrt{s}=500 \mathrm{GeV}$ and compare decay correlations and bottom quark and anti-quark distributions before hadronization.


Keywords: QCD, NLO Computations, Phenomenological Models, e+-e- Experiments.

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## 1. Introduction

High energy polarized $e^{+} e^{-}$colliders will be essential instruments in the search for the fundamental constituents of matter and their interactions. One such collider being designed is the International Linear Collider (ILC) which is expected to run at centre-of-mass energies $\geq 500 \mathrm{GeV}$. At these energies there will be a significant proportion of top quark pairs produced from the annihilation process so the ILC provides an impressive tool to carry out detailed studies of top quark physics. The top quark radiates gluons both in its production phase and its decay phase, thus it is useful to see how the leading order experimental analysis will be affected by these QCD corrections. There have been numerous studies of top quark production and their decays both at leading order and next-to leading order in QCD. Details can be found in [1-8] and many more besides. In this paper we consider the process at next-to-leading order in the production and semi-leptonic decays of the top pairs using the POWHEG method [9,10] in conjunction with the Monte Carlo event generator Herwig++ [11]. The POWHEG method has been successfully applied to $Z$ pair production [12], heavy flavour production [13], $e^{+} e^{-}$annihilation into hadrons and

Drell-Yan vector boson production $[14,15]$. We work in the narrow width approximation and hence do not include interference between the production and decay emissions which are negligible in this limit $[16,17]$. We also take account of the beam polarization and spin correlations of the top pairs. Finally, we present plots of some relevant distributions.

## 2. Hardest emission generation: Production

The order- $\alpha_{s}$ differential cross section for the process $e^{+} e^{-} \rightarrow V \rightarrow t \bar{t} g$ where $V$ represents a vector current, can be written as

$$
\begin{equation*}
R(x, y)=\sigma_{V} W_{V}(x, y)=\frac{\sigma_{V}}{v} \frac{2 \alpha_{s}}{3 \pi}\left[\frac{(x+2 \rho)^{2}+(y+2 \rho)^{2}+\zeta_{V}}{(1+2 \rho)(1-x)(1-y)}-\frac{2 \rho}{(1-x)^{2}}-\frac{2 \rho}{(1-y)^{2}}\right] \tag{2.1}
\end{equation*}
$$

where $\sigma_{V}$ is the Born cross section for heavy quark production, $W_{V}=R / \sigma_{V}, \rho=m_{\mathrm{t}}{ }^{2} / \mathrm{s}$ where $m_{\mathrm{t}}$ is the mass of the top quark and $s$ is the square of the center of mass energy, $\zeta_{V}=-8 \rho(1+2 \rho), v=\sqrt{1-4 \rho}$ and $x, y$ are the energy fractions of $t$ and $\bar{t}$ respectively.

In the case of the axial current contribution $e^{+} e^{-} \rightarrow A \rightarrow t \bar{t} g$, we have

$$
\begin{equation*}
R(x, y)=\sigma_{A} W_{A}(x, y)=\frac{\sigma_{A}}{v} \frac{2 \alpha_{s}}{3 \pi}\left[\frac{(x+2 \rho)^{2}+(y+2 \rho)^{2}+\zeta_{A}}{(1-4 \rho)(1-x)(1-y)}-\frac{2 \rho}{(1-x)^{2}}-\frac{2 \rho}{(1-y)^{2}}\right] \tag{2.2}
\end{equation*}
$$

where $\sigma_{A}$ is the Born cross section for heavy quark production by the axial current, $W_{A}=$ $R / \sigma_{A}$ and $\zeta_{A}=2 \rho\left[\left(3+x_{g}\right)^{2}-19+4 \rho\right]$ where $x_{g}=2-x-y$.

Because of the top mass, the phase space for gluon emission is reduced and the collinear divergences present in the massless quark cross-section are regularized here. However, the infra-red divergence as the gluon momentum goes to zero is still present. We can write down the cross-section for the hardest emission as

$$
\begin{equation*}
d \sigma=\sum \bar{B}(v) d \Phi_{v}\left[\Delta_{R}^{(N L O)}(0)+\Delta_{R}^{(N L O)}\left(p_{T}\right) \frac{R(v, r)}{B(v)} d \Phi_{r}\right] \tag{2.3}
\end{equation*}
$$

where $B(v)$ is the Born cross section and $v$ represents the Born variables, $r$ represents the radiation variables and $d \Phi_{v}$ and $d \Phi_{r}$ are the Born and real emission phase spaces respectively. $\Delta_{R}^{N L O}\left(p_{T}\right)$ is the modified Sudakov form factor for the hardest emission with transverse momentum $p_{T}$, as indicated by the Heaviside function in the exponent of (2.4),

$$
\begin{equation*}
\Delta_{R}^{N L O}\left(p_{T}\right)=\exp \left[-\int d \Phi r \frac{R(v, r)}{B(v)} \Theta\left(k_{T}(v, r)-p_{T}\right)\right] . \tag{2.4}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
\bar{B}(v)=B(v)+V(v)+\int(R(v, r)-C(v, r)) d \Phi_{r} . \tag{2.5}
\end{equation*}
$$

$\bar{B}(v)$ is the sum of the Born, $B(v)$, virtual, $V(v)$ and real, $R(v, r)$ terms, (with some counter-terms, $C(v, r))$. It overcomes the problem of negative weights since in the region where $\bar{B}(v)$ is negative, the NLO negative terms must have overcome the Born term and
hence perturbation theory must have failed. It is used to generate the variables of the Born subprocess to which the real-emission contributions factorize in the collinear limit.

Now explicitly for $e^{+} e^{-} \rightarrow t \bar{t} g$,

$$
\begin{equation*}
\left.\Delta_{R}^{N L O}\left(p_{T}\right)=\exp \left[-\int d x d y W(x, y) \Theta\left(k_{T}(x, y)\right)-p_{T}\right)\right] \tag{2.6}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{T}(x, y)=\sqrt{s \frac{(1-x)(1-y)(x+y-1)-\rho(2-x-y)^{2}}{\max (x, y)^{2}-4 \rho}} \tag{2.7}
\end{equation*}
$$

is the transverse momentum of the hardest emitted gluon relative to the splitting axis, as illustrated in Figure 1 below.


Figure 1: Transverse momentum, $k_{T}$.

### 2.1 Generation of radiation variables, $x$ and $y$

The radiation variables, $x$ and $y$ are to be generated according to the probability distribution

$$
\begin{equation*}
\Delta^{W}\left(k_{T}\right) W(x, y) d x d y \tag{2.8}
\end{equation*}
$$

where for $e^{+} e^{-}$annihilation via a vector and axial current, $W(x, y)$ and $\Delta^{W}\left(k_{T}\right)$ are defined in (2.1),(2.2) and (2.6).

In the region where $x>y$, let us define the dimensionless variable, $\kappa$ as

$$
\begin{equation*}
\kappa=\frac{k_{T}^{2}}{s}=\frac{(1-x)(1-y)(x+y-1)-\rho(2-x-y)^{2}}{x^{2}-4 \rho} . \tag{2.9}
\end{equation*}
$$

There are two solutions for $y$ for each value of $x$ and $\kappa$.

$$
\begin{equation*}
y_{1,2}=\frac{x^{2}-3 x-2 \rho x+2+4 \rho \pm \sqrt{\left(x^{2}-4 \rho\right)\left(4 \kappa(x-1-\rho)+(x-1)^{2}\right)}}{2(1+\rho-x)} . \tag{2.10}
\end{equation*}
$$

Exchanging the $y$ variable for $\kappa$, we find

$$
\begin{align*}
\int W(x, y) \Theta\left(k_{T}(x, y)-p_{T}\right) d x d y & =\int_{x_{\min }}^{x_{\max }} d x \int_{\kappa}^{\kappa_{\max }} d \kappa \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{d y}{d \kappa} W\left(x, y_{1,2}\right) \\
& =\int d x \int d \kappa \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{\sqrt{x^{2}-4 \rho}}{\sqrt{\left(4 \kappa(x-1-\rho)-(x-1)^{2}\right)}} W\left(x, y_{1,2}\right) \tag{2.11}
\end{align*}
$$

Now the integrand, $W^{\prime}=\frac{\sqrt{x^{2}-4 \rho}}{\sqrt{\left(4 \kappa(x-1-\rho)-(x-1)^{2}\right)}} W\left(x, y_{1,2}\right)$ in (2.11) yields a complicated integral so we look for an upper bound on $W^{\prime}$ which we denote as $V^{\prime}=\frac{d y}{d \kappa} V\left(x, y_{1,2}\right)$ to simplify the integration. We then generate the radiation variables as outlined below:

1. Set $p_{\max }=k_{T_{\text {max }}}$.
2. For a random number, $n$ between 0 and 1 , solve the equation below for $p_{T}$

$$
\begin{equation*}
n=\frac{\Delta^{V}\left(p_{T}\right)}{\Delta^{V}\left(p_{\max }\right)} \tag{2.12}
\end{equation*}
$$

where $\Delta^{V}\left(p_{T}\right)=\exp \left[-\int d x d \kappa V^{\prime}(x, \kappa)\right]$
3. Generate the variables $x$ and $y$ according to the distribution

$$
\begin{equation*}
V(x, y) \delta\left(k_{T}(x, y)-p_{T}\right) \tag{2.13}
\end{equation*}
$$

4. Accept the generated value of $p_{T}$ with probability $W / V$. If the event is rejected set $p_{\max }=p_{T}$ and go to 2).

Using our knowledge of the form of the integrand in the massless quark case [18], we guess that $V^{\prime}$ should take the form,

$$
\begin{equation*}
V^{\prime}(x, \kappa)=N_{\kappa} \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{4}{(1-x+\gamma(\kappa, x)) \gamma(\kappa, x)} \tag{2.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma(\kappa, x)=\sqrt{(1-x)\left(1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}\right)} . \tag{2.15}
\end{equation*}
$$

and $N_{\kappa}$ is a normalisation factor which depends on $\kappa$, which has to be tuned to ensure that $V^{\prime}$ is an upper bound of $W^{\prime}$. Both $V^{\prime}$ and $W^{\prime}$ have the same divergent behaviour at $x_{\text {max }}=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}$.

Table 1 shows the $N_{\kappa}$ values for different ranges of $\kappa$ and the two solutions for $y$ used for both axial and vector currents. The lower limit on $\kappa$ was set by choosing $k_{T}=\Lambda_{\mathrm{QCD}}=0.2$ GeV thus setting a lower bound on the transverse momentum.

We now consider the specific case where $m_{\mathrm{t}}=175 \mathrm{GeV}$ and $\sqrt{s}=500 \mathrm{GeV}$ i.e. $\rho=0.1225$. For $\kappa \leq 0.024$, there are two $y$ solutions in the region of phase space where $x>y$. This is illustrated in Figure 2 below for $\kappa=0.01$. The red line denotes the phase

| Range of $\kappa$ | $N_{\kappa}\left(y_{1}\right)$ | $N_{\kappa}\left(y_{2}\right)$ |
| :--- | :--- | :--- |
| $0.024-0.03$ | 0.4 | 0.4 |
| $0.015-0.024$ | 0.7 | 0.7 |
| $0.005-0.015$ | 1.2 | 1.1 |
| $0.0005-0.005$ | 4.0 | 2.6 |
| $0.0001-0.0005$ | 9.0 | 6.0 |
| $0.00005-0.0001$ | 13.0 | 7.0 |
| $0.00003-0.00005$ | 17.0 | 10.0 |
| $0.00000016-0.00003$ | 45.0 | 35.0 |

Table 1: $N_{\kappa}$ for different values of $\kappa$ for both axial and vector currents


Figure 2: Phase space and $y$ solutions for $\kappa=0.01$ in the region $x>y$.
space for gluon emissions. The two solutions lie on either side of the dashed line

$$
\begin{equation*}
y=\frac{(2-x)(1-x+2 \rho)}{2(1-x+\rho)} \tag{2.16}
\end{equation*}
$$

and are equal when

$$
\begin{equation*}
x=x_{\max }=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa} \tag{2.17}
\end{equation*}
$$

which lies on the dotted line. At $\kappa=0.024$, the branches meet along the line $y=x$ and there is only one solution for $y$ in the region (the lower branch). So for $\kappa>0.024$, only one $y$ solution exists for $x>y$. This is illustrated in Figure 3 for $\kappa=0.028$. In addition there are no $y$ solutions for $\kappa>0.03$ for $x>y$. Also note that for $x<x^{u}$, there is only one solution for $y$.


Figure 3: Phase space and $y$ solutions for $\kappa=0.028$ in the region $x>y$.

In the region where there are two solutions, the integral (with $V^{\prime}$ in place of $W^{\prime}$ ) in (2.11) is performed along both branches independently and summed. For the upper branch, $x$ runs from $x^{u}$ to $x_{\max }=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}$ while for the lower branch, $x$ runs from $x^{l}$ to $x_{\text {max }}$ where if we define

$$
\begin{align*}
x_{a} & =39 \kappa-1+12 \rho-168 \rho \kappa-48 \rho^{2}+\kappa^{3}+15 \kappa^{2}+12 \rho \kappa^{2}+48 \rho^{2} \kappa+64 \rho^{3}, \\
x_{b} & =6\left(-33 \kappa^{2}+3 \kappa-3 \kappa^{3}+288 \rho^{2} \kappa-768 \rho^{3} \kappa-48 \rho \kappa+168 \rho \kappa^{3}-204 \rho^{2} \kappa^{2}+300 \rho \kappa^{2}\right. \\
& \left.+12 \rho \kappa^{4}+768 \rho^{4} \kappa+144 \rho^{2} \kappa^{3}+486 \rho^{3} \kappa^{2}\right)^{\frac{1}{2}}, \\
x_{c} & =\sqrt{x_{a}^{2}+x_{b}{ }^{2}}, \\
x_{d} & =\tan ^{-1}\left(\frac{x_{b}}{x_{a}}\right), \\
x_{e} & =-\frac{1}{12} x_{c}^{\frac{1}{3}} \cos \left(\frac{x_{d}}{3}\right), \\
x_{f} & =\frac{\left(-1-\kappa^{2}-10 \kappa+8 \rho-2 \rho \kappa-16 \rho^{2}\right) \cos \left(\frac{x_{d}}{3}\right)}{12 x_{c}^{\frac{1}{3}}}, \\
x_{g} & =\frac{\kappa+5+4 \rho}{6}, \\
x_{h} & =\frac{\sqrt{3}}{12} \sin \left(\frac{x_{d}}{3}\right)\left(x_{c}^{\frac{1}{3}}+\frac{1+\kappa^{2}+10 \kappa-8 \rho+2 \rho \kappa+16 \rho^{2}}{x_{c}^{\frac{1}{3}}}\right), \tag{2.18}
\end{align*}
$$

we can write $x^{u}$ and $x^{l}$ as

$$
\begin{align*}
x^{u} & =x_{e}+x_{f}+x_{g}+x_{h}, \\
x^{l} & =x_{e}+x_{f}+x_{g}-x_{h} ; \tag{2.19}
\end{align*}
$$

In the region where there is only one solution for $y, x$ runs from $x^{l}$ to $x^{u}$.
The $\kappa$ integration can then be performed numerically. Having performed the integration, values for $\kappa$ and hence $k_{T}$ are then generated according to steps 1 and 2 in Section 2.1. The variables $x$ and $y$ are then to be distributed according to $W^{\prime}(x, y) \delta\left(k_{T}(x, y)-p_{T}\right)$. This is the subject of the next section.

### 2.2 Distributing $x$ and $y$ according to $W(x, y)$

To generate $x$ and $y$ values with a distribution proportional to $V(x, y) \delta\left(k_{T}(x, y)-p_{T}\right)$, where from (2.11), $V(x, y)$ is the $\frac{V^{\prime}}{d y / d \kappa}$, we can use the $\delta$ function to eliminate the $y$ variable by computing

$$
\begin{equation*}
D(x)=\int d y \delta\left(k_{T}-p_{T}\right) V(x, y)=\left.\frac{V(x, y)}{\frac{\partial k_{T}}{\partial y}}\right|_{y=\bar{y}} \tag{2.20}
\end{equation*}
$$

where $\bar{y}$ is such that $k_{T}(x, \bar{y})=p_{T}$. Note that $\frac{\partial k_{T}}{\partial y}$ is the same for both $y$ solutions. We then generate $x$ values with a probability distribution proportional to $D$ with hit-and-miss techniques as described below. All events generated have uniform weights.

1. Randomly sample $x, N_{x}$ times (we used $N_{x}=10^{5}$ ) in the range [ $x_{\text {min }}: x_{\max }$ ] for the selected value of $\kappa$.
2. For each value of $x$, evaluate $\bar{D}=D\left(x, y_{1}\right)+D\left(x, y_{2}\right)$ if there are 2 solutions for the selected $\kappa$ and $\bar{D}=D\left(x, y_{2}\right)$ if there is only one solution. Also, if $\kappa<0.024$ and $x<x^{u}$ (see Figure 2), there is only one $y$ solution so evaluate $\bar{D}=D\left(x, y_{2}\right)$.
3. Find the maximum value $\bar{D}_{\max }$ of $\bar{D}$ for the selected value of $\kappa$.
4. Next, select a value for $x$ in the allowed range and evaluate $\bar{D}$.
5. If $\bar{D}>r \bar{D}_{\text {max }}$ (where $r$ is a random number between 0 and 1 ), accept the event, otherwise go to 4.) and generate a new value for $x$.
6. If for the chosen value of $x$, there are two solutions for $y$, select a value for $y$ in the ratio $D\left(x, y_{1}\right): D\left(x, y_{2}\right)$.
7. Compare $V^{\prime}(x, y)$ with the true integrand, $W^{\prime}(x, y)$. If the event fails this veto, set $\kappa_{\text {max }}=\kappa$ and regenerate a new $\kappa$ value as discussed in Section 2.1.

NB: For the region $y>x$, exchange $x$ and $y$ in the above discussion. In this way, the smooth phase space distribution in Figure 5 below was obtained for the hardest emission events for an axial current. The plot show 2,500 of these events.


Figure 4: Phase space and distribution of hardest emissions for axial (left) and vector(right) currents with $\rho=0.1225$.


Figure 5: Phase space and distribution of hardest emissions for an axial current with $\rho=0.0625$ (left) and $\rho=0.01361$ (right).

The procedure was repeated for $\sqrt{s}=700 \mathrm{GeV}, \rho=0.0625$ and $\sqrt{s}=1500 \mathrm{GeV}$, $\rho=0.01361$ and the corresponding plots are shown below. As can be seen, the method is stable as $\rho \rightarrow 0$. This is not surprising because the upper bound function $V(x, \kappa)$ in $(2.14)$ is stable as $\rho \rightarrow 0$ and tends to $W^{\prime}$, the true value of the integrand in this limit.

## 3. Hardest emission generation: Decays

In addition to gluon emission in top production, we also studied the emission in its decay,

$$
\begin{equation*}
t\left(p_{1}\right) \rightarrow W^{+}\left(w_{1}\right) b\left(r_{1}\right) g(k) . \tag{3.1}
\end{equation*}
$$

The procedure for generating the hardest emission in this case follows the same lines as discussed in Section 2. We parameterize the phase space for the decay in terms of variables $x$ and $y$ defined as

$$
\begin{align*}
& y=\frac{2 w_{1} \cdot p_{1}}{m_{\mathrm{t}}^{2}}-a \\
& x=\frac{2 k \cdot p_{1}}{m_{\mathrm{t}}^{2}} \tag{3.2}
\end{align*}
$$

where $a=m_{w}^{2} / m_{\mathrm{t}}^{2}$ with $m_{w}$ and $m_{\mathrm{t}}$ the masses of the $W$ boson and top quark respectively. $(y+a) / 2$ and $x / 2$ are the energy fractions of the $W$ boson and gluon in the top frame. Therefore the corresponding energy fraction of the $b$ quark in this frame is given by

$$
\begin{equation*}
\frac{x_{b}}{2}=\frac{2-y-a-x}{2} . \tag{3.3}
\end{equation*}
$$

In this paper, we neglect the $b$ mass and work in the narrow-width approximation so that the top quarks and $W$ boson are on-shell. The $t \rightarrow W b g$ differential decay rate is given by:

$$
\begin{equation*}
\frac{1}{\Gamma_{0}} \frac{d^{2} \Gamma}{d x d y}=\frac{\alpha_{S}}{\pi} \frac{C_{F}}{(1-y) x^{2}}\left[x-\frac{(1-y)(1-x)+x^{2}}{1-a}+x \frac{(y+x-1)^{2}}{2(1-a)^{2}}+\frac{2 a(1-y) x^{2}}{(1-a)^{2}(1+2 a)}\right], \tag{3.4}
\end{equation*}
$$

where $\Gamma_{0}$ is the leading order decay rate. The phase space limits for the decay are:

$$
\begin{array}{r}
\frac{a x}{1-x}+(1-x)<y<1, \\
0<x<1-a . \tag{3.5}
\end{array}
$$

Working in the rest frame of the top quark where the parton shower is formulated in Herwig++, we identify the splitting axis corresponding to the original $b-W$ boson axis and therefore the relative transverse momentum for gluon emission is:

$$
\begin{equation*}
k_{T}(x, y)=m_{\mathrm{t}} \sqrt{\frac{(1-y)\left(y+x(2-y-a)-x^{2}-1\right)}{(y+a)^{2}-4 a}} . \tag{3.6}
\end{equation*}
$$

Now defining a dimensionless variable $\kappa=\frac{k_{T}^{2}}{m_{\mathrm{t}}^{2}}$, we find that in analogy to the production case, there are 2 solutions for $y$ for each value of $x$ and $\kappa$.
$y_{1,2}=\frac{x^{2}+a x+2-3 x-2 a \kappa \pm \sqrt{\left(x^{2}-4 \kappa(1+a)\right)(x-1)^{2}+4 a \kappa(4 \kappa+1-a)+x^{2}(a+2 x-2)}}{2(\kappa+1-x)}$.
These solutions may be identified with either initial state gluon emission from the top quark $\left(y_{2}\right)$ or final state radiation from the bottom quark $\left(y_{1}\right)$. A plot of the phase space


Figure 6: Phase space(solid), $y^{\prime}$ (dot-dash) and solutions $y_{1}$ (dots) and $y_{2}$ (dashes).
and the 2 solutions for $\kappa=0.01$ is shown in Figure 6. We then construct the modified Sudakov form factor for the generation of the hardest emission. The exponent of the form factor is given by

$$
\begin{equation*}
\int W(x, y) \Theta\left(k_{T}(x, y)-p_{T}\right) d x d y=\int_{x_{\min }}^{x_{\max }} d x \int_{\kappa}^{\kappa_{\max }} d \kappa \frac{\alpha_{S}\left(\kappa m_{\mathrm{t}}^{2}\right) C_{F}}{\pi} \frac{d y}{d \kappa} W(x, \kappa), \tag{3.8}
\end{equation*}
$$

where $W(x, \kappa)$ is the differential cross-section (3.4) and $\frac{d y}{d \kappa}$ is the Jacobian for the change of variables from $y$ to $\kappa$. Note that $\kappa=\kappa_{\max }=\frac{(1-\sqrt{a})^{2}}{4}$ when the $W$ boson is at rest and $x=1-\sqrt{a}, y=2 \sqrt{a}$. For a given $\kappa, x_{\min }$ and $x_{\max }$ are also given by

$$
\begin{align*}
& x_{\min }=2 \sqrt{\kappa} \\
& x_{\max }=1-a-2 \sqrt{\kappa a} \tag{3.9}
\end{align*}
$$

To make the integral simpler, we again look for an upper bound $V^{\prime}(x, \kappa)$ on the integrand as we did for the production case. To do this we replace the Jacobian with the simpler expression,

$$
\begin{equation*}
\frac{d y^{\prime}}{d \kappa}=\frac{d}{d \kappa}\left(\frac{x^{2}-3 x-2 \kappa a+2+x a}{2(\kappa+1-x)}\right)=\frac{-a}{\kappa+1-x}-\frac{x^{2}-3 x-2 \kappa a+2+x a}{2(\kappa+1-x)^{2}}, \tag{3.10}
\end{equation*}
$$

where $y^{\prime}$ lies in between $y_{1}$ and $y_{2}$ and is indicated in Figure 6. We also overestimate the differential cross-section by replacing (3.4) with

$$
\begin{equation*}
U(x, y)=N_{\kappa} \frac{\alpha_{S} C_{F}}{\pi} \frac{1-a}{2 x^{2}\left(1-y^{\prime}\right)}, \tag{3.11}
\end{equation*}
$$

where $N_{\kappa}$ is a normalisation factor dependent on $\kappa$ and is chosen such that $V^{\prime}=U \frac{d y^{\prime}}{d \kappa}$ is greater than the integrand in (3.8). The $N_{\kappa}$ values are given in Table 2 for the 2 solutions.

| Range of $\kappa$ | $N_{\kappa}\left(y_{1}\right) \times 10^{4}$ | $N_{\kappa}\left(y_{2}\right) \times 10^{4}$ |
| :--- | :--- | :--- |
| $0.01-0.0737$ | 0.005 | 0.006 |
| $0.005-0.01$ | 0.0175 | 0.02 |
| $0.001-0.005$ | 0.03 | 0.045 |
| $0.0001-0.001$ | 0.08 | 0.12 |
| $0.00005-0.0001$ | 0.2 | 0.2 |
| $0.000025-0.00005$ | 0.3 | 0.2 |
| $0.0000075-0.000025$ | 1.0 | 0.9 |
| $0.000005-0.0000075$ | 2.0 | 0.9 |
| $0.0000025-0.000005$ | 3.0 | 0.9 |
| $0.0000013-0.0000025$ | 6.0 | 0.9 |

Table 2: $N_{\kappa}$ for different values of $\kappa$


Figure 7: Phase space distribution of POWHEG events

The lower limit on $\kappa$ is $1.3 \times 10^{-6}$ and was set by choosing $k_{T}=\Lambda_{\mathrm{QCD}}=0.2 \mathrm{GeV}$ thus setting a lower bound on the transverse momentum. We then generate the values of $\kappa$ and distribute $x$ and $y$ according to the true differential (3.4) using vetoes as described for the production case in Sections 2. Figure 7 shows the phase space distribution obtained.

## 4. Spin Correlations and the distribution of Born variables

In [19], it was observed that the lepton matrix element for the production process

$$
\begin{align*}
e^{+}(p)+e^{-}(q) & \rightarrow t\left(p_{1}\right)+\bar{t}\left(p_{2}\right)+g\left(p_{3}\right) \rightarrow W^{+}\left(w_{1}\right)+b\left(r_{1}\right)+W^{-}\left(w_{2}\right)+b\left(r_{2}\right)+g\left(p_{3}\right) \\
& \rightarrow l^{+}\left(k_{1}\right)+\nu\left(x_{1}\right)+b\left(r_{1}\right)+l^{-}\left(k_{2}\right)+\nu\left(x_{2}\right)+\bar{b}\left(r_{2}\right)+g\left(p_{3}\right) \tag{4.1}
\end{align*}
$$

is bounded from above in the narrow width approximation by the undecayed matrix element obtained by eliminating the decay products i.e. $W^{+}, W^{-}, b, \bar{b}$ and putting the parent particles i.e. $t, \bar{t}$ on-shell, multiplied by a process dependent constant. We can then use the undecayed matrix elements to perform computer-intensive tasks such as event generation and finally, by using the hit-and-miss method, replace the parent particles with their decay products. This procedure is outlined below:

1. Evaluate the undecayed matrix elements which are proportional to the upper bounds on the lepton matrix elements. Generate hard events using the POWHEG method described above with the top and anti-top quarks in the final state.
2. For each event, generate the decay products and their four-momenta according to the phase space.
3. Evaluate the leptonic decay matrix element for each event. If the decay matrix element divided by the corresponding upper bound is less than a random number $r$ between 0 and 1 , throw away the decay momenta and return to step 2 .
4. Otherwise, replace the top and anti-top momenta with the decay momenta and shower the event.

### 4.1 Undecayed matrix elements

At the ILC, the electron and positron beams will be polarized i.e. either $e_{L}^{-} e_{R}^{+}$or $e_{R}^{-} e_{L}^{+}$ where the subscripts $L$ and $R$ represent the left-handed and right-handed helicity states respectively. The corresponding undecayed matrix element for $e_{L}^{-} e_{R}^{+}$annihilation is:

$$
\begin{align*}
& \tilde{M}\left(e_{L}^{-}(p) e_{R}^{+}(q) \rightarrow t_{s_{t}}\left(p_{1}\right) \bar{t}_{s_{\bar{t}}}\left(p_{2}\right) g\left(p_{3}\right)\right)= \\
& {\left[v(\bar{q}) \gamma_{\mu L} u(p)\right] \bar{u}\left(p_{1}, s_{t}\right)\left[\frac{1}{2 p_{2} \cdot p_{3}}\left(\frac{a_{L L}}{2} \gamma_{L}^{\mu}+\frac{a_{L R}}{2} \gamma_{R}^{\mu}\right)\left(-\hat{p_{2}}-\hat{p_{3}}+m_{\mathrm{t}}\right) \gamma_{\nu}\right.} \\
& \left.+\frac{1}{2 p_{1} \cdot p_{3}} \gamma_{\nu}\left(\hat{p_{1}}+\hat{p_{3}}+m_{\mathrm{t}}\right)\left(\frac{a_{L L}}{2} \gamma_{L}^{\mu}+\frac{a_{L R}}{2} \gamma_{R}^{\mu}\right)\right] T^{a} v\left(p_{2}, s_{\bar{t}}\right) \epsilon_{a}^{\nu}\left(p_{3}\right) \tag{4.2}
\end{align*}
$$

where $\epsilon$ is the polarization vector of the gluon, $T^{a}$ is the colour matrix, $\hat{p}=p^{\mu} \gamma_{\mu}$ and $\gamma_{R / L}^{\mu}=\gamma^{\mu}\left(1 \pm \gamma_{5}\right) / 2$. For $e_{R}^{-} e_{L}^{+}$annihilation, interchange $L, R$ in the above equation. $s_{t}, s_{\bar{t}}$ are the spin vectors of the top and anti-top quarks respectively and satisfy the relations:

$$
\begin{align*}
s_{t} \cdot p_{1} & =0 \\
s_{\bar{t}} \cdot p_{2} & =0 \\
s_{t} \cdot s_{t} & =-1 \\
s_{\bar{t}} \cdot s_{\bar{t}} & =-1 \tag{4.3}
\end{align*}
$$

The massive spinors $u(p, s), v(p, s)$ are given in terms of the massless spinors $u(p), v(p)$ by

$$
\begin{align*}
& u(p, \uparrow)=\frac{1+\gamma_{5} \hat{s}}{2} u(p) \\
& u(p, \downarrow)=\frac{1-\gamma_{5} \hat{s}}{2} u(p) \\
& v(p, \uparrow)=\frac{1+\gamma_{5} \hat{s}}{2} v(p) \\
& v(p, \downarrow)=\frac{1-\gamma_{5} \hat{s}}{2} v(p) \tag{4.4}
\end{align*}
$$

The coupling constants $a_{I J}$ are given by

$$
\begin{equation*}
a_{I J}=\frac{e^{2} g}{s}\left[-Q_{t}+Q_{e}^{I} Q_{t}^{J} \frac{1}{\sin ^{2} \theta_{W}} \frac{s}{s-M_{Z}^{2}+i M_{Z} \Gamma_{Z}}\right] \tag{4.5}
\end{equation*}
$$

where $M_{Z}$ is the $Z$ boson mass, $\Gamma_{Z}$ is the width of the $Z$ boson, $\theta_{W}$ is the Weinberg angle, $Q_{t}$ is the electric charge of the top in units of the electric charge $e, g=\sqrt{4 \pi \alpha_{S}}$ and $s$ is the center of mass energy squared. The couplings to the $Z$ boson are given by

$$
\begin{align*}
Q_{e}^{L} & =\frac{2 \sin ^{2} \theta_{W}-1}{2 \cos \theta_{W}} \\
Q_{e}^{R} & =\frac{\sin ^{2} \theta_{W}}{\cos \theta_{W}} \\
Q_{t}^{L} & =\frac{3-4 \sin ^{2} \theta_{W}}{6 \cos \theta_{W}} \\
Q_{t}^{R} & =-\frac{2 \sin ^{2} \theta_{W}}{3 \cos \theta_{W}} \tag{4.6}
\end{align*}
$$

In Section 2.2, we distributed our events according to the vector and axial vector current matrix elements separately using the POWHEG method. To obtain a full unpolarized distribution we can select events from either current distribution according to their contributions to the full cross-section given below.

$$
\begin{align*}
\sigma & =3 \beta(1+2 \rho)\left(1+c_{1} \frac{\alpha_{S}}{\pi}\right) \sigma_{V V}+3 \beta^{3}\left(1+d_{1} \frac{\alpha_{S}}{\pi}\right) \sigma_{A A} \\
\sigma_{V V} & =\frac{4 \pi \alpha_{\mathrm{em}}^{2}}{s}\left[Q_{t}^{2}-2 Q_{t} V_{e} V_{t} \chi_{1}(s)+\left(A_{e}^{2}+V_{e}^{2}\right) V_{t}^{2} \chi_{2}(s)\right] \\
\sigma_{A A} & =\frac{4 \pi \alpha_{e m}^{2}}{s}\left[\left(A_{e}^{2}+V_{e}^{2}\right) A_{t}^{2} \chi_{2}(s)\right], \tag{4.7}
\end{align*}
$$

where $\beta=\sqrt{1-\rho}, \alpha_{\mathrm{em}}$ is the electromagnetic coupling, $A_{t}, A_{e}$ and $V_{t}, V_{e}$ are the axial and vector coupling constants of the top $t$ and electron $e$ to the $Z$ boson and $c_{1}=3.5$ and $d_{1}=2.25$ are the QCD correction coefficients defined at $m_{t}=175 \mathrm{GeV}$ and $\sqrt{s}=500 \mathrm{GeV}$
i.e. $\rho=0.1225[20] . \chi_{1}(s)$ and $\chi_{2}(s)$ are given by

$$
\begin{align*}
\chi_{1}(s) & =\kappa \frac{s\left(s-M_{Z}^{2}\right)}{\left(s-M_{Z}^{2}\right)^{2}+\Gamma_{Z}^{2} M_{Z}^{2}} \\
\chi_{2}(s) & =\kappa^{2} \frac{s^{2}}{\left(s-M_{Z}^{2}\right)^{2}+\Gamma_{Z}^{2} M_{Z}^{2}} \\
\kappa & =\frac{\sqrt{2} G_{F} M_{Z}^{2}}{16 \pi \alpha_{\mathrm{em}}} \tag{4.8}
\end{align*}
$$

where $G_{F}$ is the Fermi constant and $M_{Z}$ and $\Gamma_{Z}$ are the mass and decay width of the $Z$ boson respectively.

Explicit expressions for the Born, virtual and real polarization dependent squared matrix elements for the production process are given in [5] in terms of the energy fractions $x, y$ of the top and anti-top quarks and the polar angle and azimuthal angles orienting the $t \bar{t} g$ plane relative to the $e^{+} e^{-}$beam axis. For each initial polarization, we then assign final-state polarizations to each event in proportion to the squared matrix elements and distribute the polar and azimuthal angles of the top/anti-top pairs accordingly using wellknown Monte Carlo techniques.

### 4.2 Decay matrix elements

Next, we investigate the decays of the top and anti-top pair. The leptonic matrix elements for the process in (4.1) are dependent on the spins of the top and anti-top quark. This dependence can be written in the form of a decay density matrix. The decay density matrix $\rho_{\lambda, \lambda^{\prime}}$, for an on-shell top quark is given by

$$
\begin{aligned}
& \rho_{\lambda, \lambda^{\prime}}=\frac{4 g_{w}^{4} V_{t b}^{2}}{\left(w_{1}^{2}-m_{w}^{2}\right)^{2}+\left(m_{w} \Gamma_{W}\right)^{2}} \times \\
& {\left[\begin{array}{cl}
\left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)-\left(s_{t} \cdot k_{1}\right)\left(r_{1} \cdot x_{1}\right) m_{\mathrm{t}} & -\left(k_{1} \cdot n\right)\left(x_{1} \cdot r_{1}\right) m_{\mathrm{t}}-i \epsilon\left(p_{1}, k_{1}, s_{t}, n\right)\left(x_{1} \cdot r_{1}\right) \\
-\left(k_{1} \cdot n\right)\left(x_{1} \cdot r_{1}\right) m_{\mathrm{t}}+i \epsilon\left(p_{1}, k_{1}, s_{t}, n\right)\left(x_{1} \cdot r_{1}\right) & \left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)+\left(s_{t} \cdot k_{1}\right)\left(r_{1} \cdot x_{1}\right) m_{\mathrm{t}}
\end{array}\right]}
\end{aligned}
$$

where $\lambda, \lambda^{\prime}$ are spin labels, $s_{t}$ is the top spin vector, $n$ is a spacelike vector perpendicular to $s_{t}$ and $p_{1}$ and $m_{w}, \Gamma_{W}$ are the mass and width of the $W$ boson respectively. In this paper we work in the helicity basis for which the top quark spin is defined along its direction of motion. A similar matrix can be derived for $\bar{t}$ decay. The spin-specific decay matrix elements are therefore of the form:

$$
\begin{equation*}
S_{\lambda_{t} \lambda_{\bar{t}} \lambda_{t}^{\prime} \lambda_{\bar{t}}^{\prime}}=\tilde{M}_{\lambda_{t} \lambda_{\bar{t}}} \rho_{\lambda_{t} \lambda_{t}^{\prime}} \rho_{\lambda_{\bar{t}} \lambda_{\bar{t}}^{\prime}} \tilde{M}_{\lambda_{t}^{\prime} \lambda_{\bar{t}}^{\prime}} \tag{4.9}
\end{equation*}
$$

where $\lambda_{t}, \lambda_{\bar{t}}$ are spin labels for the top and anti-top respectively and $\tilde{M}$ is the matrix element for the undecayed process introduced in Section 4.1. By diagonalizing the density matrix, we can obtain the largest possible value of the matrix elements and hence the upper bound. An explicit computation gives this upper bound $\left|M_{u b}^{t}\right|^{2}$ on the top decay as [19],

$$
\begin{equation*}
\left|M_{u b}^{t}\right|^{2}=\frac{4 g_{w}^{4}\left|V_{t b}\right|^{2}\left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)}{\left[\left(w_{1}^{2}-m_{w}^{2}\right)^{2}+\left(m_{w} \Gamma_{W}\right)^{2}\right]\left[\left(p_{1}^{2}-m_{\mathrm{t}}^{2}\right)^{2}-\left(m_{\mathrm{t}} \Gamma_{t}\right)^{2}\right]}|\tilde{M}|^{2} \tag{4.10}
\end{equation*}
$$

where $|\tilde{M}|^{2}$ is the undecayed matrix element for unpolarized $t \bar{t} g$ production. A similar expression $\left|M_{u b}^{\bar{t}}\right|^{2}$ can be obtained for the decay of the top anti-quark by interchanging the labels 1 and 2 and $t$ and $\bar{t}$ in (4.10). Hence the full upper bound can be written as:

$$
\begin{equation*}
\left|M_{u b}^{t \bar{t}}\right|^{2}=\frac{\left|M_{u b}^{t}\right|^{2}\left|M_{u b}^{\bar{t}}\right|^{2}}{|\tilde{M}|^{2}} \tag{4.11}
\end{equation*}
$$

Having obtained the decay matrix elements and their upper bounds, we then proceed to generate events with leptons in the final state as outlined at the beginning of this section.

For the POWHEG decays, we apply the same method where in this case the undecayed matrix elements $|\tilde{M}|$ are the leading order matrix elements for the process

$$
\begin{equation*}
e^{+}+e^{-} \rightarrow t+\bar{t} . \tag{4.12}
\end{equation*}
$$

We then use the next-to-leading order decay matrix for which the helicity amplitudes can be found in [5]. These are given in terms of the polar and azimuthal angles of the decay w.r.t the top/anti-top axis and we distribute them as described for the production process in Section 4.1. Note that in this case, we generate two decay gluons, one each from the top and anti-top quark.

In addition, we also consider POWHEG radiations in both the production and decay process by independently generating the emission and distributing the Born variables of the production process first and then generating the emission and distributing the Born variables of the decay process to yield three gluons in the final state.

## 5. Decay NLO lepton spectra comparisons

Extensive studies have been carried out on the lepton angular and energy distributions from the semi-leptonic decays of polarized top and anti-top quarks at next-to-leading order in $\alpha_{S}[1,3]$.

$$
\begin{align*}
& t \rightarrow W^{+}+b+g \rightarrow e^{+}+\nu_{e}+b+g \\
& \bar{t} \rightarrow W^{-}+\bar{b}+g \rightarrow e^{-}+\overline{\nu_{e}}+\bar{b}+g . \tag{5.1}
\end{align*}
$$

In the top rest frame, we define $\theta$ as the angle between the spin 3 -vectors $\mathbf{s}_{\mathbf{t}}, \mathbf{s}_{\bar{t}}$ of the decaying quark and the lepton. We also defined the scaled energies $x_{l, n}$ of the charged lepton and the neutrino respectively as

$$
\begin{align*}
x_{l} & =\frac{2 E_{l}}{m_{\mathrm{t}}} \\
x_{n} & =\frac{2 E_{n}}{m_{\mathrm{t}}} \tag{5.2}
\end{align*}
$$

where $E_{l}$ and $E_{n}$ are the energies of the charged lepton and neutrino in the top rest frame. In these variables, the NLO double differential distribution of the charged lepton
and neutrino in the decay of a heavy top or anti-top quark with polarization $S$ has been shown to be of the form

$$
\begin{align*}
\frac{d \Gamma^{l, n}}{d x_{l, n} d \cos \theta} & =\frac{G_{F}^{2} m_{\mathrm{t}}^{5}}{32 \pi^{3}}\left[F_{0}^{l, n}\left(x_{l, n}, a\right)+S \cos \theta J_{0}^{l, n}\left(x_{l, n}, a\right)\right. \\
& \left.-\frac{2 \alpha_{S}}{3 \pi}\left(F_{1}^{l, n}\left(x_{l, n}, a\right)+S \cos \theta J_{1}^{l, n}\left(x_{l, n}, a\right)\right)\right] \tag{5.3}
\end{align*}
$$

in the narrow width limit for the decay of the $W$ boson. Expressions for $F_{0,1}^{l, n}$ and $J_{0,1}^{l, n}$ can be found in [3]. Integrating over $\cos \theta$ gives us the differential energy distribution,

$$
\begin{equation*}
\frac{d \Gamma^{l, n}}{d x_{l, n}}=\frac{G_{F}^{2} m_{\mathrm{t}}^{5}}{16 \pi^{3}}\left[F_{0}^{l, n}\left(x_{l, n}, y\right)-\frac{2 \alpha_{S}}{3 \pi} F_{1}^{l, n}\left(x_{l, n}, y\right)\right] \tag{5.4}
\end{equation*}
$$

We compared this theoretical prediction with the distribution obtained from the POWHEG method before interfacing with the Herwig++ parton shower. The best fit distributions shown in Figure 8 were obtained by setting $\alpha_{S}$ to 0.1 in (5.4).


Figure 8: Scaled energy fractions of the charged lepton (left) and neutrino (right) from top decay. Black $($ solid $)=$ Theory, Blue (dashes $)=$ Decay.

## 6. Truncated Shower

The POWHEG method requires the addition of a 'truncated shower' before the hardest gluon emission in order to simulate the soft radiation distribution [21]. Due to angular ordering, the 'truncated' radiation is emitted at a wider angle than the angle of the hardest emission but at a lower $p_{T}$. This means the 'truncated' radiation does not appreciably degrade
the energy entering the hardest emission and justifies our decision to generate the hardest emission first.

In [18], there is a description of a method to generate a truncated shower of at most one gluon for the case of light quark production from $e^{+} e^{-}$annihilation. In this section, we extend the discussion to top pair production. Below is an outline of how the 'truncated shower' was generated. We will consider the case in which at most one extra gluon is emitted by the top or anti-top before the hardest emission. The outline closely follows the Herwig++ parton shower evolution method described in [22, 23] where the evolution variables $z$, the momentum fractions, and $\tilde{q}$, the evolution scale, determine the kinematics of the shower.
i) Having generated the $p_{T}$ of the hardest emission as discussed in Section 2 and the energy fractions $x$ and $y$, calculate the light-cone momentum fractions $z$ and $1-z$ of the partons involved in the hardest emission. We will assume henceforth that $x>y$ and that $y$ is the energy fraction of the quark, i.e. the quark is involved in the hardest emission. Then

$$
\begin{equation*}
z=\frac{\alpha_{b}}{\alpha_{b}+\alpha_{g}} \tag{6.1}
\end{equation*}
$$

where if we define

$$
\begin{align*}
b & =\frac{m_{\mathrm{t}}^{2}}{s} \\
\lambda & =\sqrt{(1-4 b)} \tag{6.2}
\end{align*}
$$

we have

$$
\begin{align*}
& \alpha_{b}=\frac{x(1+\lambda)+\sqrt{x^{2}(1+\lambda)^{2}-8(b+\kappa)(1+\lambda-2 b)}}{2(1+\lambda-2 b)} \\
& \alpha_{c}=\frac{y(1+\lambda)-\sqrt{y^{2}(1+\lambda)^{2}-8 b(1+\lambda-2 b)}}{2(1+\lambda-2 b)} \\
& \alpha_{g}=\frac{2}{1+\lambda}-\alpha_{b}-\alpha_{c} \tag{6.3}
\end{align*}
$$

ii) Next generate the light-cone momentum fraction $z_{t}$ of the 'truncated' radiation within the range

$$
\begin{equation*}
\frac{m_{\mathrm{t}}}{\tilde{q}_{i}}<z_{t}<1-\frac{Q_{g}}{\tilde{q}_{i}} \tag{6.4}
\end{equation*}
$$

and distributed according to the massive splitting function, $P_{Q Q}=C_{F}\left[\frac{1+z_{t}^{2}}{1-z_{t}}-\frac{2 m_{\mathrm{t}}^{2}}{z_{t}\left(1-z_{t}\right) \tilde{q}^{2}}\right]$. $\tilde{q}_{i}$ is the initial evolution scale, i.e. $\sqrt{s}=500 \mathrm{GeV}$, and $Q_{g}$ is a cutoff introduced to regularize soft gluon singularities in the splitting functions. In this report, a $Q_{g}$ value of 0.75 GeV was used. $z_{t}$ is the momentum fraction of the quark after emitting the 'truncated' gluon with momentum fraction $1-z_{t}$.
iii) Determine the scale $\tilde{q}_{h}$ of the hardest emission from

$$
\begin{equation*}
\tilde{q}_{h}=\sqrt{\frac{p_{T}{ }^{2}}{z^{2}(1-z)^{2}}+\frac{m_{\mathrm{t}}^{2}}{z^{2}}+\frac{Q_{g}{ }^{2}}{z(1-z)^{2}}} \tag{6.5}
\end{equation*}
$$

iv) Starting from an initial scale $\tilde{q}_{i}$, the probability of there being an emission next at the scale $\tilde{q}$ is given by

$$
\begin{equation*}
S\left(\tilde{q}_{i}, \tilde{q}\right)=\frac{\Delta\left(\tilde{q}_{c}, \tilde{q}_{i}\right)}{\Delta\left(\tilde{q}_{c}, \tilde{q}\right)} \tag{6.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta\left(\tilde{q}_{c}, \tilde{q}\right)=\exp \left[-\int_{\tilde{q}_{c}}^{\tilde{q}} \frac{d \tilde{q}^{2}}{\tilde{q}^{2}} \int d z \frac{\alpha_{s}}{2 \pi} P_{Q Q} \Theta\left(0<p_{T}^{t}<p_{T}\right)\right] . \tag{6.7}
\end{equation*}
$$

$\tilde{q}_{c}$ is the lower cutoff of the parton shower which was set to the default value of 0.631 GeV in this report, $\alpha_{s}$ is the running coupling constant evaluated at $z(1-z) \tilde{q}, P_{Q Q}$ is the $Q \rightarrow Q g$ splitting function and $p_{T}$ is the transverse momentum of the hardest emission. The Heaviside function ensures that the transverse momentum, $p_{T}^{t}$ of the truncated emission is real and is less than $p_{T}$. To evaluate the integral in (6.7), we overestimate the integrands and apply vetoes with weights as described in [22]. With $r$ a random number between 0 and 1 , we then solve the equation

$$
\begin{equation*}
S\left(\tilde{q}_{i}, \tilde{q}\right)=r \tag{6.8}
\end{equation*}
$$

for $\tilde{q}$. If $\tilde{q}>\tilde{q}_{h}$, the event has a 'truncated' emission. If $\tilde{q}<\tilde{q}_{h}$, there is no 'truncated' emission and the event is showered from the scale of the hardest emission.
v) If there is a 'truncated' emission, the next step is to determine the transverse momentum $p_{T}^{t}$ of the emission. This is given by

$$
\begin{equation*}
p_{T}^{t}=\sqrt{\left(1-z_{t}\right)^{2}\left(z_{t}^{2} \tilde{q}^{2}-m_{\mathrm{t}}^{2}\right)-z_{t}{Q_{g}}^{2}} . \tag{6.9}
\end{equation*}
$$

If $p_{T}^{t^{2}}<0$ or $p_{T}^{t}>p_{T}$ go to ii).
vi) We now have values for $z_{t}$, the momentum fraction of the quark after the first emission, $p_{T}^{t}$, the transverse momentum of the first emission, $z$, the momentum fraction of the hardest emission and $p_{T}$, the transverse momentum of the hardest emission. We can then reconstruct the momenta of the partons as described in [22]. The orientation of the quark, antiquark and hardest emission with respect to the beam axis is determined as explained there for the hard matrix element correction.

In this paper, we consider only truncated emissions in the production process, not in the decay.

## 7. Parton shower distributions

Next we interface the generated events with the Herwig++ 2.2.0 [24] parton shower and veto the hardest emissions in the production and decay of the top and anti-top pairs. In this section we will consider collisions at $\sqrt{s}=500 \mathrm{GeV}$ and only include the truncated shower for the production emissions. We considered four cases:

1. Leading Order (LO): No POWHEG emissions.
2. Production (Pr): Only POWHEG emissions in the production are allowed including the truncated shower.
3. Decay (Dc): Only POWHEG emissions in the decays of the top/anti-top pairs are allowed.
4. Production + Decay ( PrDc ): Both production and decay emissions are allowed.

The following distributions were investigated in the lab frame for the two different $e^{+} e^{-}$ initial polarizations:
i) The angle between the lepton from the decay of the top anti-quark and the top quark are presented in Figure 9.
ii) The angle between the lepton and anti-lepton from the decays of the top pairs are presented in Figure 10.
iii) The energy distributions of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 11 and 12.
iv) The transverse momenta w.r.t the beam axis of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 13 and 14.
v) The longitudinal momenta (along the beam axis) of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 15 and 16.


Figure 9: Angle between the lepton from the decay of the top anti-quark and the top quark.


Figure 10: Angle between the lepton and anti-lepton from the decays of the top pairs.


Figure 11: Energy of the b-quark before hadronization.


Figure 12: Energy of the b anti-quark before hadronization.


Figure 13: Transverse momentum of the b quark before hadronization.


Figure 14: Transverse momentum of the b anti-quark before hadronization.


Figure 15: Longitudinal momentum of the b quark before hadronization.


Figure 16: Longitudinal momentum of the b anti-quark before hadronization.

At leading order, the leptonic correlations in Figures 9 and 10 are as expected with higher correlations seen for $e_{R}^{-} e_{L}^{+}$annihilation than for $e_{L}^{-} e_{R}^{+}$annihilation. At next-toleading order, it can be observed that the POWHEG emissions do not change the shapes of the distributions much except for a slight broadening of the peaks.

Also at leading order, the distributions in Figures 11-16 have the expected shapes with the $b$ quarks and anti-quarks having softer (harder) energy, longitudinal momentum and transverse momentum spectra for $e_{R}^{-} e_{L}^{+}\left(e_{L}^{-} e_{R}^{+}\right)$annihilation.

Now comparing the POWHEG production and decay distributions for the $b$ quark in Figures 11-16, we observe that the decay emissions soften the spectra more than the production emissions and therefore these have the greater effect in the production + decay distributions. This is expected since the scale range available for the production emissions $\approx \log \left(\sqrt{s} / m_{\mathrm{t}}\right)$ is less than the range available for the decay emissions $\approx \log \left(m_{\mathrm{t}} / m_{\mathrm{b}}\right)$.

## 8. Conclusions

Using the Monte Carlo event generator Herwig++, we have successfully applied the POWHEG method to investigate angular correlation distributions at next-to-leading order in top pair production and decays at ILC energies. In all distributions studied, the POWHEG emissions have the effect of broadening the peaks of the leading order predictions slightly. We also compared momentum distributions of the $b$ quarks and anti-quarks before hadronization and observe that the decay emissions soften the spectra more at next-to-leading order as expected.

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# Applying the POWHEG method to top pair production and decays at the ILC 

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#### Abstract

We study the effects of gluon radiation in top pair production and their decays for $e^{+} e^{-}$annihilation at the ILC. To achieve this we apply the POWHEG method and interface our results to the Monte Carlo event generator Herwig++. We consider a center of mass energy of $\sqrt{s}=500 \mathrm{GeV}$ and compare decay correlations and bottom quark and anti-quark distributions before hadronization.


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## 1. Introduction

High energy polarized $e^{+} e^{-}$colliders will be essential instruments in the search for the fundamental constituents of matter and their interactions. One such collider being designed is the International Linear Collider (ILC) which is expected to run at centre-of-mass energies $\geq 500 \mathrm{GeV}$. At these energies there will be a significant proportion of top quark pairs produced from the annihilation process so the ILC provides an impressive tool to carry out detailed studies of top quark physics. The top quark radiates gluons both in its production phase and its decay phase, thus it is useful to see how the leading order experimental analysis will be affected by these QCD corrections. There have been numerous studies of top quark production and their decays both at leading order and next-to leading order in QCD. Details can be found in [1-8] and many more besides. In this paper we consider the process at next-to-leading order in the production and semi-leptonic decays of the top pairs using the POWHEG method $[9,10]$ in conjunction with the Monte Carlo event generator Herwig++ [11]. The POWHEG method has been successfully applied to $Z$ pair production [12], heavy flavour production [13], $e^{+} e^{-}$annihilation into hadrons and Drell-Yan vector boson
production $[14,15]$. We work in the narrow width approximation and hence do not include interference between the production and decay emissions which are negligible in this limit $[16,17]$. We also take account of the beam polarization and spin correlations of the top pairs. Finally, we present plots of some relevant distributions.

## 2. Hardest emission generation: Production

The order- $\alpha_{s}$ differential cross section for the process $e^{+} e^{-} \rightarrow V \rightarrow t \bar{t} g$ where $V$ represents a vector current, can be written as

$$
\begin{equation*}
R(x, y)=\sigma_{V} W_{V}(x, y)=\frac{\sigma_{V}}{v} \frac{2 \alpha_{s}}{3 \pi}\left[\frac{(x+2 \rho)^{2}+(y+2 \rho)^{2}+\zeta_{V}}{(1+2 \rho)(1-x)(1-y)}-\frac{2 \rho}{(1-x)^{2}}-\frac{2 \rho}{(1-y)^{2}}\right] \tag{2.1}
\end{equation*}
$$

where $\sigma_{V}$ is the Born cross section for heavy quark production, $W_{V}=R / \sigma_{V}, \rho=m_{\mathrm{t}}{ }^{2} / \mathrm{s}$ where $m_{\mathrm{t}}$ is the mass of the top quark and $s$ is the square of the center of mass energy, $\zeta_{V}=-8 \rho(1+2 \rho), v=\sqrt{1-4 \rho}$ and $x, y$ are the energy fractions of $t$ and $\bar{t}$ respectively.

In the case of the axial current contribution $e^{+} e^{-} \rightarrow A \rightarrow t \bar{t} g$, we have

$$
\begin{equation*}
R(x, y)=\sigma_{A} W_{A}(x, y)=\frac{\sigma_{A}}{v} \frac{2 \alpha_{s}}{3 \pi}\left[\frac{(x+2 \rho)^{2}+(y+2 \rho)^{2}+\zeta_{A}}{(1-4 \rho)(1-x)(1-y)}-\frac{2 \rho}{(1-x)^{2}}-\frac{2 \rho}{(1-y)^{2}}\right] \tag{2.2}
\end{equation*}
$$

where $\sigma_{A}$ is the Born cross section for heavy quark production by the axial current, $W_{A}=$ $R / \sigma_{A}$ and $\zeta_{A}=2 \rho\left[\left(3+x_{g}\right)^{2}-19+4 \rho\right]$ where $x_{g}=2-x-y$.

Because of the top mass, the phase space for gluon emission is reduced and the collinear divergences present in the massless quark cross-section are regularized here. However, the infra-red divergence as the gluon momentum goes to zero is still present. We can write down the cross-section for the hardest emission as

$$
\begin{equation*}
d \sigma=\sum \bar{B}(v) d \Phi_{v}\left[\Delta_{R}^{(N L O)}(0)+\Delta_{R}^{(N L O)}\left(p_{T}\right) \frac{R(v, r)}{B(v)} d \Phi_{r}\right] \tag{2.3}
\end{equation*}
$$

where $B(v)$ is the Born cross section and $v$ represents the Born variables, $r$ represents the radiation variables and $d \Phi_{v}$ and $d \Phi_{r}$ are the Born and real emission phase spaces respectively. $\Delta_{R}^{N L O}\left(p_{T}\right)$ is the modified Sudakov form factor for the hardest emission with transverse momentum $p_{T}$, as indicated by the Heaviside function in the exponent of (2.4),

$$
\begin{equation*}
\Delta_{R}^{N L O}\left(p_{T}\right)=\exp \left[-\int d \Phi r \frac{R(v, r)}{B(v)} \Theta\left(k_{T}(v, r)-p_{T}\right)\right] . \tag{2.4}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
\bar{B}(v)=B(v)+V(v)+\int(R(v, r)-C(v, r)) d \Phi_{r} \tag{2.5}
\end{equation*}
$$

$\bar{B}(v)$ is the sum of the Born, $B(v)$, virtual, $V(v)$ and real, $R(v, r)$ terms, (with some counter-terms, $C(v, r)$ ). It overcomes the problem of negative weights since in the region where $\bar{B}(v)$ is negative, the NLO negative terms must have overcome the Born term and
hence perturbation theory must have failed. It is used to generate the variables of the Born subprocess to which the real-emission contributions factorize in the collinear limit.

Now explicitly for $e^{+} e^{-} \rightarrow t \bar{t} g$,

$$
\begin{equation*}
\left.\Delta_{R}^{N L O}\left(p_{T}\right)=\exp \left[-\int d x d y W(x, y) \Theta\left(k_{T}(x, y)\right)-p_{T}\right)\right] \tag{2.6}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{T}(x, y)=\sqrt{s \frac{(1-x)(1-y)(x+y-1)-\rho(2-x-y)^{2}}{\max (x, y)^{2}-4 \rho}} \tag{2.7}
\end{equation*}
$$

is the transverse momentum of the hardest emitted gluon relative to the splitting axis, as illustrated in Figure 1 below.


Figure 1: Transverse momentum, $k_{T}$.

### 2.1 Generation of radiation variables, $x$ and $y$

The radiation variables, $x$ and $y$ are to be generated according to the probability distribution

$$
\begin{equation*}
\Delta^{W}\left(k_{T}\right) W(x, y) d x d y \tag{2.8}
\end{equation*}
$$

where for $e^{+} e^{-}$annihilation via a vector and axial current, $W(x, y)$ and $\Delta^{W}\left(k_{T}\right)$ are defined in (2.1),(2.2) and (2.6).

In the region where $x>y$, let us define the dimensionless variable, $\kappa$ as

$$
\begin{equation*}
\kappa=\frac{k_{T}^{2}}{s}=\frac{(1-x)(1-y)(x+y-1)-\rho(2-x-y)^{2}}{x^{2}-4 \rho} . \tag{2.9}
\end{equation*}
$$

There are two solutions for $y$ for each value of $x$ and $\kappa$.

$$
\begin{equation*}
y_{1,2}=\frac{x^{2}-3 x-2 \rho x+2+4 \rho \pm \sqrt{\left(x^{2}-4 \rho\right)\left(4 \kappa(x-1-\rho)+(x-1)^{2}\right)}}{2(1+\rho-x)} . \tag{2.10}
\end{equation*}
$$

Exchanging the $y$ variable for $\kappa$, we find

$$
\begin{align*}
\int W(x, y) \Theta\left(k_{T}(x, y)-p_{T}\right) d x d y & =\int_{x_{\min }}^{x_{\max }} d x \int_{\kappa}^{\kappa_{\max }} d \kappa \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{d y}{d \kappa} W\left(x, y_{1,2}\right) \\
& =\int d x \int d \kappa \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{\sqrt{x^{2}-4 \rho}}{\sqrt{\left(4 \kappa(x-1-\rho)-(x-1)^{2}\right)}} W\left(x, y_{1,2}\right) \tag{2.11}
\end{align*}
$$

Now the integrand, $W^{\prime}=\frac{\sqrt{x^{2}-4 \rho}}{\sqrt{\left(4 \kappa(x-1-\rho)-(x-1)^{2}\right)}} W\left(x, y_{1,2}\right)$ in (2.11) yields a complicated integral so we look for an upper bound on $W^{\prime}$ which we denote as $V^{\prime}=\frac{d y}{d \kappa} V\left(x, y_{1,2}\right)$ to simplify the integration. We then generate the radiation variables as outlined below:

1. Set $p_{\text {max }}=k_{T \text { max }}$.
2. For a random number, $n$ between 0 and 1 , solve the equation below for $p_{T}$

$$
\begin{equation*}
n=\frac{\Delta^{V}\left(p_{T}\right)}{\Delta^{V}\left(p_{\max }\right)} \tag{2.12}
\end{equation*}
$$

where $\Delta^{V}\left(p_{T}\right)=\exp \left[-\int d x d \kappa V^{\prime}(x, \kappa)\right]$
3. Generate the variables $x$ and $y$ according to the distribution

$$
\begin{equation*}
V(x, y) \delta\left(k_{T}(x, y)-p_{T}\right) \tag{2.13}
\end{equation*}
$$

4. Accept the generated value of $p_{T}$ with probability $W / V$. If the event is rejected set $p_{\text {max }}=p_{T}$ and go to 2).

Using our knowledge of the form of the integrand in the massless quark case [18], we guess that $V^{\prime}$ should take the form,

$$
\begin{equation*}
V^{\prime}(x, \kappa)=N_{\kappa} \frac{2 \alpha_{s}(\kappa s)}{3 \pi} \frac{4}{(1-x+\gamma(\kappa, x)) \gamma(\kappa, x)} \tag{2.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma(\kappa, x)=\sqrt{(1-x)\left(1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}\right)} . \tag{2.15}
\end{equation*}
$$

and $N_{\kappa}$ is a normalisation factor which depends on $\kappa$, which has to be tuned to ensure that $V^{\prime}$ is an upper bound of $W^{\prime}$. Both $V^{\prime}$ and $W^{\prime}$ have the same divergent behaviour at $x_{\text {max }}=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}$.

Table 1 shows the $N_{\kappa}$ values for different ranges of $\kappa$ and the two solutions for $y$ used for both axial and vector currents. The lower limit on $\kappa$ was set by choosing $k_{T}=\Lambda_{\mathrm{QCD}}=0.2$ GeV thus setting a lower bound on the transverse momentum.

We now consider the specific case where $m_{\mathrm{t}}=175 \mathrm{GeV}$ and $\sqrt{s}=500 \mathrm{GeV}$ i.e. $\rho=0.1225$. For $\kappa \leq 0.024$, there are two $y$ solutions in the region of phase space where $x>y$. This is illustrated in Figure 2 below for $\kappa=0.01$. The red line denotes the phase

| Range of $\kappa$ | $N_{\kappa}\left(y_{1}\right)$ | $N_{\kappa}\left(y_{2}\right)$ |
| :--- | :--- | :--- |
| $0.024-0.03$ | 0.4 | 0.4 |
| $0.015-0.024$ | 0.7 | 0.7 |
| $0.005-0.015$ | 1.2 | 1.1 |
| $0.0005-0.005$ | 4.0 | 2.6 |
| $0.0001-0.0005$ | 9.0 | 6.0 |
| $0.00005-0.0001$ | 13.0 | 7.0 |
| $0.00003-0.00005$ | 17.0 | 10.0 |
| $0.00000016-0.00003$ | 45.0 | 35.0 |

Table 1: $N_{\kappa}$ for different values of $\kappa$ for both axial and vector currents


Figure 2: Phase space and $y$ solutions for $\kappa=0.01$ in the region $x>y$.
space for gluon emissions. The two solutions lie on either side of the dashed line

$$
\begin{equation*}
y=\frac{(2-x)(1-x)+2 \rho)}{2(1-x+\rho)} \tag{2.16}
\end{equation*}
$$

and are equal when

$$
\begin{equation*}
x=x_{\max }=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa} \tag{2.17}
\end{equation*}
$$

which lies on the dotted line. At $\kappa=0.024$, the branches meet along the line $y=x$ and there is only one solution for $y$ in the region (the lower branch). So for $\kappa>0.024$, only one $y$ solution exists for $x>y$. This is illustrated in Figure 3 for $\kappa=0.028$. In addition there are no $y$ solutions for $\kappa>0.03$ for $x>y$. Also note that for $x<x^{u}$, there is only one solution for $y$.


Figure 3: Phase space and $y$ solutions for $\kappa=0.028$ in the region $x>y$.

In the region where there are two solutions, the integral (with $V^{\prime}$ in place of $W^{\prime}$ ) in (2.11) is performed along both branches independently and summed. For the upper branch, $x$ runs from $x^{u}$ to $x_{\max }=1-2 \kappa-2 \sqrt{\kappa^{2}+\rho \kappa}$ while for the lower branch, $x$ runs from $x^{l}$ to $x_{\text {max }}$ where if we define

$$
\begin{align*}
x_{a} & =39 \kappa-1+12 \rho-168 \rho \kappa-48 \rho^{2}+\kappa^{3}+15 \kappa^{2}+12 \rho \kappa^{2}+48 \rho^{2} \kappa+64 \rho^{3}, \\
x_{b} & =6\left(-33 \kappa^{2}+3 \kappa-3 \kappa^{3}+288 \rho^{2} \kappa-768 \rho^{3} \kappa-48 \rho \kappa+168 \rho \kappa^{3}-204 \rho^{2} \kappa^{2}+300 \rho \kappa^{2}\right. \\
& \left.+12 \rho \kappa^{4}+768 \rho^{4} \kappa+144 \rho^{2} \kappa^{3}+486 \rho^{3} \kappa^{2}\right)^{\frac{1}{2}}, \\
x_{c} & =\sqrt{x_{a}^{2}+x_{b}^{2}}, \\
x_{d} & =\tan ^{-1}\left(\frac{x_{b}}{x_{a}}\right), \\
x_{e} & =-\frac{1}{12} x_{c}^{\frac{1}{3}} \cos \left(\frac{x_{d}}{3}\right), \\
x_{f} & =\frac{\left(-1-\kappa^{2}-10 \kappa+8 \rho-2 \rho \kappa-16 \rho^{2}\right) \cos \left(\frac{x_{d}}{3}\right)}{12 x_{c}^{\frac{1}{3}}}, \\
x_{g} & =\frac{\kappa+5+4 \rho}{6}, \\
x_{h} & =\frac{\sqrt{3}}{12} \sin \left(\frac{x_{d}}{3}\right)\left(x_{c}^{\frac{1}{3}}+\frac{1+\kappa^{2}+10 \kappa-8 \rho+2 \rho \kappa+16 \rho^{2}}{x_{c}^{\frac{1}{3}}}\right), \tag{2.18}
\end{align*}
$$

we can write $x^{u}$ and $x^{l}$ as

$$
\begin{align*}
x^{u} & =x_{e}+x_{f}+x_{g}+x_{h}, \\
x^{l} & =x_{e}+x_{f}+x_{g}-x_{h} ; \tag{2.19}
\end{align*}
$$

In the region where there is only one solution for $y, x$ runs from $x^{l}$ to $x^{u}$.
The $\kappa$ integration can then be performed numerically. Having performed the integration, values for $\kappa$ and hence $k_{T}$ are then generated according to steps 1 and 2 in Section 2.1. The variables $x$ and $y$ are then to be distributed according to $W^{\prime}(x, y) \delta\left(k_{T}(x, y)-p_{T}\right)$. This is the subject of the next section.

### 2.2 Distributing $x$ and $y$ according to $W(x, y)$

To generate $x$ and $y$ values with a distribution proportional to $V(x, y) \delta\left(k_{T}(x, y)-p_{T}\right)$, where from $(2.11), V(x, y)$ is the $\frac{V^{\prime}}{d y / d \kappa}$, we can use the $\delta$ function to eliminate the $y$ variable by computing

$$
\begin{equation*}
D(x)=\int d y \delta\left(k_{T}-p_{T}\right) V(x, y)=\left.\frac{V(x, y)}{\frac{\partial k_{T}}{\partial y}}\right|_{y=\bar{y}} \tag{2.20}
\end{equation*}
$$

where $\bar{y}$ is such that $k_{T}(x, \bar{y})=p_{T}$. Note that $\frac{\partial k_{T}}{\partial y}$ is the same for both $y$ solutions. We then generate $x$ values with a probability distribution proportional to $D$ with hit-and-miss techniques as described below. All events generated have uniform weights.

1. Randomly sample $x, N_{x}$ times (we used $N_{x}=10^{5}$ ) in the range $\left[x_{\min }: x_{\max }\right]$ for the selected value of $\kappa$.
2. For each value of $x$, evaluate $\bar{D}=D\left(x, y_{1}\right)+D\left(x, y_{2}\right)$ if there are 2 solutions for the selected $\kappa$ and $\bar{D}=D\left(x, y_{2}\right)$ if there is only one solution. Also, if $\kappa<0.024$ and $x<x^{u}$ (see Figure 2), there is only one $y$ solution so evaluate $\bar{D}=D\left(x, y_{2}\right)$.
3. Find the maximum value $\bar{D}_{\max }$ of $\bar{D}$ for the selected value of $\kappa$.
4. Next, select a value for $x$ in the allowed range and evaluate $\bar{D}$.
5. If $\bar{D}>r \bar{D}_{\max }$ (where $r$ is a random number between 0 and 1 ), accept the event, otherwise go to 4.) and generate a new value for $x$.
6. If for the chosen value of $x$, there are two solutions for $y$, select a value for $y$ in the ratio $D\left(x, y_{1}\right): D\left(x, y_{2}\right)$.
7. Compare $V^{\prime}(x, y)$ with the true integrand, $W^{\prime}(x, y)$. If the event fails this veto, set $\kappa_{\text {max }}=\kappa$ and regenerate a new $\kappa$ value as discussed in Section 2.1.

NB: For the region $y>x$, exchange $x$ and $y$ in the above discussion. In this way, the smooth phase space distribution in Figure 5 below was obtained for the hardest emission events for an axial current. The plot show 2,500 of these events.


Figure 4: Phase space and distribution of hardest emissions for axial (left) and vector(right) currents with $\rho=0.1225$.


Figure 5: Phase space and distribution of hardest emissions for an axial current with $\rho=0.0625$ (left) and $\rho=0.01361$ (right).

The procedure was repeated for $\sqrt{s}=700 \mathrm{GeV}, \rho=0.0625$ and $\sqrt{s}=1500 \mathrm{GeV}$, $\rho=0.01361$ and the corresponding plots are shown below. As can be seen, the method is stable as $\rho \rightarrow 0$. This is not surprising because the upper bound function $V(x, \kappa)$ in (2.14) is stable as $\rho \rightarrow 0$ and tends to $W^{\prime}$, the true value of the integrand in this limit.

## 3. Hardest emission generation: Decays

In addition to gluon emission in top production, we also studied the emission in its decay,

$$
\begin{equation*}
t\left(p_{1}\right) \rightarrow W^{+}\left(w_{1}\right) b\left(r_{1}\right) g(k) . \tag{3.1}
\end{equation*}
$$

The procedure for generating the hardest emission in this case follows the same lines as discussed in Section 2. We parameterize the phase space for the decay in terms of variables $x$ and $y$ defined as

$$
\begin{array}{r}
y=\frac{2 w_{1} \cdot p_{1}}{m_{\mathrm{t}}^{2}}-a \\
x=\frac{2 k \cdot p_{1}}{m_{\mathrm{t}}^{2}} \tag{3.2}
\end{array}
$$

where $a=m_{w}^{2} / m_{\mathrm{t}}^{2}$ with $m_{w}$ and $m_{\mathrm{t}}$ the masses of the $W$ boson and top quark respectively. $(y+a) / 2$ and $x / 2$ are the energy fractions of the $W$ boson and gluon in the top frame. Therefore the corresponding energy fraction of the $b$ quark in this frame is given by

$$
\begin{equation*}
\frac{x_{b}}{2}=\frac{2-y-a-x}{2} . \tag{3.3}
\end{equation*}
$$

In this paper, we neglect the $b$ mass and work in the narrow-width approximation so that the top quarks and $W$ boson are on-shell. The $t \rightarrow W b g$ differential decay rate is given by:

$$
\begin{equation*}
\frac{1}{\Gamma_{0}} \frac{d^{2} \Gamma}{d x d y}=\frac{\alpha_{S}}{\pi} \frac{C_{F}}{(1-y) x^{2}}\left[x-\frac{(1-y)(1-x)+x^{2}}{1-a}+x \frac{(y+x-1)^{2}}{2(1-a)^{2}}+\frac{2 a(1-y) x^{2}}{(1-a)^{2}(1+2 a)}\right], \tag{3.4}
\end{equation*}
$$

where $\Gamma_{0}$ is the leading order decay rate. The phase space limits for the decay are:

$$
\begin{array}{r}
\frac{a x}{1-x}+(1-x)<y<1, \\
0<x<1-a . \tag{3.5}
\end{array}
$$

Working in the rest frame of the top quark where the parton shower is formulated in Herwig++, we identify the splitting axis corresponding to the original $b-W$ boson axis and therefore the relative transverse momentum for gluon emission is:

$$
\begin{equation*}
k_{T}(x, y)=m_{\mathrm{t}} \sqrt{\frac{(1-y)\left(y+x(2-y-a)-x^{2}-1\right)}{(y+a)^{2}-4 a}} \tag{3.6}
\end{equation*}
$$

Now defining a dimensionless variable $\kappa=\frac{k_{r}^{2}}{m_{t}^{2}}$, we find that in analogy to the production case, there are 2 solutions for $y$ for each value of $x$ and $\kappa$.
$y_{1,2}=\frac{x^{2}+a x+2-3 x-2 a \kappa \pm \sqrt{\left(x^{2}-4 \kappa(1+a)\right)(x-1)^{2}+4 a \kappa(4 \kappa+1-a)+x^{2}(a+2 x-2)}}{2(\kappa+1-x)}$.
These solutions may be identified with either initial state gluon emission from the top quark $\left(y_{2}\right)$ or final state radiation from the bottom quark $\left(y_{1}\right)$. A plot of the phase space


Figure 6: Phase space(solid), $y^{\prime}$ (dot-dash) and solutions $y_{1}$ (dots) and $y_{2}$ (dashes).
and the 2 solutions for $\kappa=0.01$ is shown in Figure 6. We then construct the modified Sudakov form factor for the generation of the hardest emission. The exponent of the form factor is given by

$$
\begin{equation*}
\int W(x, y) \Theta\left(k_{T}(x, y)-p_{T}\right) d x d y=\int_{x_{\min }}^{x_{\max }} d x \int_{\kappa}^{\kappa_{\max }} d \kappa \frac{\alpha_{S}\left(\kappa m_{\mathrm{t}}^{2}\right) C_{F}}{\pi} \frac{d y}{d \kappa} W(x, \kappa), \tag{3.8}
\end{equation*}
$$

where $W(x, \kappa)$ is the differential cross-section (3.4) and $\frac{d y}{d \kappa}$ is the Jacobian for the change of variables from $y$ to $\kappa$. Note that $\kappa=\kappa_{\max }=\frac{(1-\sqrt{a})^{2}}{4}$ when the $W$ boson is at rest and $x=1-\sqrt{a}, y=2 \sqrt{a}$. For a given $\kappa, x_{\min }$ and $x_{\max }$ are also given by

$$
\begin{array}{r}
x_{\min }=2 \sqrt{\kappa} \\
x_{\max }=1-a-2 \sqrt{\kappa a} . \tag{3.9}
\end{array}
$$

To make the integral simpler, we again look for an upper bound $V^{\prime}(x, \kappa)$ on the integrand as we did for the production case. To do this we replace the Jacobian with the simpler expression,

$$
\begin{equation*}
\frac{d y^{\prime}}{d \kappa}=\frac{d}{d \kappa}\left(\frac{x^{2}-3 x-2 \kappa a+2+x a}{2(\kappa+1-x)}\right)=\frac{-a}{\kappa+1-x}-\frac{x^{2}-3 x-2 \kappa a+2+x a}{2(\kappa+1-x)^{2}}, \tag{3.10}
\end{equation*}
$$

where $y^{\prime}$ lies in between $y_{1}$ and $y_{2}$ and is indicated in Figure 6. We also overestimate the differential cross-section by replacing (3.4) with

$$
\begin{equation*}
U(x, y)=N_{\kappa} \frac{\alpha_{S} C_{F}}{\pi} \frac{1-a}{2 x^{2}\left(1-y^{\prime}\right)}, \tag{3.11}
\end{equation*}
$$

where $N_{\kappa}$ is a normalisation factor dependent on $\kappa$ and is chosen such that $V^{\prime}=U \frac{d y^{\prime}}{d \kappa}$ is greater than the integrand in (3.8). The $N_{\kappa}$ values are given in Table 2 for the 2 solutions.

| Range of $\kappa$ | $N_{\kappa}\left(y_{1}\right) \times 10^{4}$ | $N_{\kappa}\left(y_{2}\right) \times 10^{4}$ |
| :--- | :--- | :--- |
| $0.01-0.0737$ | 0.005 | 0.006 |
| $0.005-0.01$ | 0.0175 | 0.02 |
| $0.001-0.005$ | 0.03 | 0.045 |
| $0.0001-0.001$ | 0.08 | 0.12 |
| $0.00005-0.0001$ | 0.2 | 0.2 |
| $0.000025-0.00005$ | 0.3 | 0.2 |
| $0.0000075-0.000025$ | 1.0 | 0.9 |
| $0.000005-0.0000075$ | 2.0 | 0.9 |
| $0.0000025-0.000005$ | 3.0 | 0.9 |
| $0.0000013-0.0000025$ | 6.0 | 0.9 |

Table 2: $N_{\kappa}$ for different values of $\kappa$


Figure 7: Phase space distribution of POWHEG events

The lower limit on $\kappa$ is $1.3 \times 10^{-6}$ and was set by choosing $k_{T}=\Lambda_{\mathrm{QCD}}=0.2 \mathrm{GeV}$ thus setting a lower bound on the transverse momentum. We then generate the values of $\kappa$ and distribute $x$ and $y$ according to the true differential (3.4) using vetoes as described for the production case in Sections 2. Figure 7 shows the phase space distribution obtained.

## 4. Spin Correlations and the distribution of Born variables

In [19], it was observed that the lepton matrix element for the production process

$$
\begin{align*}
e^{+}(p)+e^{-}(q) & \rightarrow t\left(p_{1}\right)+\bar{t}\left(p_{2}\right)+g\left(p_{3}\right) \rightarrow W^{+}\left(w_{1}\right)+b\left(r_{1}\right)+W^{-}\left(w_{2}\right)+b\left(r_{2}\right)+g\left(p_{3}\right) \\
& \rightarrow l^{+}\left(k_{1}\right)+\nu\left(x_{1}\right)+b\left(r_{1}\right)+l^{-}\left(k_{2}\right)+\nu\left(x_{2}\right)+\bar{b}\left(r_{2}\right)+g\left(p_{3}\right) \tag{4.1}
\end{align*}
$$

is bounded from above in the narrow width approximation by the undecayed matrix element obtained by eliminating the decay products i.e. $W^{+}, W^{-}, b, \bar{b}$ and putting the parent particles i.e. $t, \bar{t}$ on-shell, multiplied by a process dependent constant. We can then use the undecayed matrix elements to perform computer-intensive tasks such as event generation and finally, by using the hit-and-miss method, replace the parent particles with their decay products. This procedure is outlined below:

1. Evaluate the undecayed matrix elements which are proportional to the upper bounds on the lepton matrix elements. Generate hard events using the POWHEG method described above with the top and anti-top quarks in the final state.
2. For each event, generate the decay products and their four-momenta according to the phase space.
3. Evaluate the leptonic decay matrix element for each event. If the decay matrix element divided by the corresponding upper bound is less than a random number $r$ between 0 and 1 , throw away the decay momenta and return to step 2 .
4. Otherwise, replace the top and anti-top momenta with the decay momenta and shower the event.

### 4.1 Undecayed matrix elements

At the ILC, the electron and positron beams will be polarized i.e. either $e_{L}^{-} e_{R}^{+}$or $e_{R}^{-} e_{L}^{+}$ where the subscripts $L$ and $R$ represent the left-handed and right-handed helicity states respectively. The corresponding undecayed matrix element for $e_{L}^{-} e_{R}^{+}$annihilation is:

$$
\begin{align*}
& \tilde{M}\left(e_{L}^{-}(p) e_{R}^{+}(q) \rightarrow t_{s_{t}}\left(p_{1}\right) \bar{t}_{s_{\bar{t}}}\left(p_{2}\right) g\left(p_{3}\right)\right)= \\
& {\left[v(\bar{q}) \gamma_{\mu L} u(p)\right] \bar{u}\left(p_{1}, s_{t}\right)\left[\frac{1}{2 p_{2} \cdot p_{3}}\left(\frac{a_{L L}}{2} \gamma_{L}^{\mu}+\frac{a_{L R}}{2} \gamma_{R}^{\mu}\right)\left(-\hat{p_{2}}-\hat{p_{3}}+m_{\mathrm{t}}\right) \gamma_{\nu}\right.} \\
& \left.+\frac{1}{2 p_{1} \cdot p_{3}} \gamma_{\nu}\left(\hat{p_{1}}+\hat{p_{3}}+m_{\mathrm{t}}\right)\left(\frac{a_{L L}}{2} \gamma_{L}^{\mu}+\frac{a_{L R}}{2} \gamma_{R}^{\mu}\right)\right] T^{a} v\left(p_{2}, s_{\bar{t}}\right) \epsilon_{a}^{\nu}\left(p_{3}\right) \tag{4.2}
\end{align*}
$$

where $\epsilon$ is the polarization vector of the gluon, $T^{a}$ is the colour matrix, $\hat{p}=p^{\mu} \gamma_{\mu}$ and $\gamma_{R / L}^{\mu}=\gamma^{\mu}\left(1 \pm \gamma_{5}\right) / 2$. For $e_{R}^{-} e_{L}^{+}$annihilation, interchange $L, R$ in the above equation. $s_{t}, s_{\bar{t}}$ are the spin vectors of the top and anti-top quarks respectively and satisfy the relations:

$$
\begin{align*}
s_{t} \cdot p_{1} & =0 \\
s_{\bar{t}} \cdot p_{2} & =0 \\
s_{t} \cdot s_{t} & =-1 \\
s_{\bar{t}} \cdot s_{\bar{t}} & =-1 \tag{4.3}
\end{align*}
$$

The massive spinors $u(p, s), v(p, s)$ are given in terms of the massless spinors $u(p), v(p)$ by

$$
\begin{align*}
& u(p, \uparrow)=\frac{1+\gamma_{5} \hat{s}}{2} u(p) \\
& u(p, \downarrow)=\frac{1-\gamma_{5} \hat{s}}{2} u(p) \\
& v(p, \uparrow)=\frac{1+\gamma_{5} \hat{s}}{2} v(p) \\
& v(p, \downarrow)=\frac{1-\gamma_{5} \hat{s}}{2} v(p) \tag{4.4}
\end{align*}
$$

The coupling constants $a_{I J}$ are given by

$$
\begin{equation*}
a_{I J}=\frac{e^{2} g}{s}\left[-Q_{t}+Q_{e}^{I} Q_{t}^{J} \frac{1}{\sin ^{2} \theta_{W}} \frac{s}{s-M_{Z}^{2}+i M_{Z} \Gamma_{Z}}\right] \tag{4.5}
\end{equation*}
$$

where $M_{Z}$ is the $Z$ boson mass, $\Gamma_{Z}$ is the width of the $Z$ boson, $\theta_{W}$ is the Weinberg angle, $Q_{t}$ is the electric charge of the top in units of the electric charge $e, g=\sqrt{4 \pi \alpha_{S}}$ and $s$ is the center of mass energy squared. The couplings to the $Z$ boson are given by

$$
\begin{align*}
Q_{e}^{L} & =\frac{2 \sin ^{2} \theta_{W}-1}{2 \cos \theta_{W}} \\
Q_{e}^{R} & =\frac{\sin ^{2} \theta_{W}}{\cos \theta_{W}} \\
Q_{t}^{L} & =\frac{3-4 \sin ^{2} \theta_{W}}{6 \cos \theta_{W}} \\
Q_{t}^{R} & =-\frac{2 \sin ^{2} \theta_{W}}{3 \cos \theta_{W}} \tag{4.6}
\end{align*}
$$

In Section 2.2, we distributed our events according to the vector and axial vector current matrix elements separately using the POWHEG method. To obtain a full unpolarized distribution we can select events from either current distribution according to their contributions to the full cross-section given below.

$$
\begin{align*}
\sigma & =3 \beta(1+2 \rho)\left(1+c_{1} \frac{\alpha_{S}}{\pi}\right) \sigma_{V V}+3 \beta^{3}\left(1+d_{1} \frac{\alpha_{S}}{\pi}\right) \sigma_{A A} \\
\sigma_{V V} & =\frac{4 \pi \alpha_{\mathrm{em}}^{2}}{s}\left[Q_{t}^{2}-2 Q_{t} V_{e} V_{t} \chi_{1}(s)+\left(A_{e}^{2}+V_{e}^{2}\right) V_{t}^{2} \chi_{2}(s)\right] \\
\sigma_{A A} & =\frac{4 \pi \alpha_{e m}^{2}}{s}\left[\left(A_{e}^{2}+V_{e}^{2}\right) A_{t}^{2} \chi_{2}(s)\right], \tag{4.7}
\end{align*}
$$

where $\beta=\sqrt{1-\rho}, \alpha_{\mathrm{em}}$ is the electromagnetic coupling, $A_{t}, A_{e}$ and $V_{t}, V_{e}$ are the axial and vector coupling constants of the top $t$ and electron $e$ to the $Z$ boson and $c_{1}=3.5$ and $d_{1}=2.25$ are the QCD correction coefficients defined at $m_{t}=175 \mathrm{GeV}$ and $\sqrt{s}=500 \mathrm{GeV}$
i.e. $\rho=0.1225[20] \cdot \chi_{1}(s)$ and $\chi_{2}(s)$ are given by

$$
\begin{align*}
& \chi_{1}(s)=\kappa \frac{s\left(s-M_{Z}^{2}\right)}{\left(s-M_{Z}^{2}\right)^{2}+\Gamma_{Z}^{2} M_{Z}^{2}} \\
& \chi_{2}(s)= \\
&=\kappa^{2} \frac{s^{2}}{\left(s-M_{Z}^{2}\right)^{2}+\Gamma_{Z}^{2} M_{Z}^{2}}  \tag{4.8}\\
& \kappa=\frac{\sqrt{2} G_{F} M_{Z}^{2}}{16 \pi \alpha_{\mathrm{em}}}
\end{align*}
$$

where $G_{F}$ is the Fermi constant and $M_{Z}$ and $\Gamma_{Z}$ are the mass and decay width of the $Z$ boson respectively.

Explicit expressions for the Born, virtual and real polarization dependent squared matrix elements for the production process are given in [5] in terms of the energy fractions $x, y$ of the top and anti-top quarks and the polar angle and azimuthal angles orienting the $t \bar{t} g$ plane relative to the $e^{+} e^{-}$beam axis. For each initial polarization, we then assign final-state polarizations to each event in proportion to the squared matrix elements and distribute the polar and azimuthal angles of the top/anti-top pairs accordingly using wellknown Monte Carlo techniques.

### 4.2 Decay matrix elements

Next, we investigate the decays of the top and anti-top pair. The leptonic matrix elements for the process in (4.1) are dependent on the spins of the top and anti-top quark. This dependence can be written in the form of a decay density matrix. The decay density matrix $\rho_{\lambda, \lambda^{\prime}}$, for an on-shell top quark is given by

$$
\begin{aligned}
& \rho_{\lambda, \lambda^{\prime}}=\frac{4 g_{w}^{4} V_{t b}^{2}}{\left(w_{1}^{2}-m_{w}^{2}\right)^{2}+\left(m_{w} \Gamma_{W}\right)^{2}} \times \\
& {\left[\begin{array}{cl}
\left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)-\left(s_{t} \cdot k_{1}\right)\left(r_{1} \cdot x_{1}\right) m_{\mathrm{t}} & -\left(k_{1} \cdot n\right)\left(x_{1} \cdot r_{1}\right) m_{\mathrm{t}}-i \epsilon\left(p_{1}, k_{1}, s_{t}, n\right)\left(x_{1} \cdot r_{1}\right) \\
-\left(k_{1} \cdot n\right)\left(x_{1} \cdot r_{1}\right) m_{\mathrm{t}}+i \epsilon\left(p_{1}, k_{1}, s_{t}, n\right)\left(x_{1} \cdot r_{1}\right)\left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)+\left(s_{t} \cdot k_{1}\right)\left(r_{1} \cdot x_{1}\right) m_{\mathrm{t}}
\end{array}\right]}
\end{aligned}
$$

where $\lambda, \lambda^{\prime}$ are spin labels, $s_{t}$ is the top spin vector, $n$ is a spacelike vector perpendicular to $s_{t}$ and $p_{1}$ and $m_{w}, \Gamma_{W}$ are the mass and width of the $W$ boson respectively. In this paper we work in the helicity basis for which the top quark spin is defined along its direction of motion. A similar matrix can be derived for $\bar{t}$ decay. The spin-specific decay matrix elements are therefore of the form:

$$
\begin{equation*}
S_{\lambda_{t} \lambda_{\bar{t}} \lambda_{t}^{\prime} \lambda_{\bar{t}}^{\prime}}=\tilde{M}_{\lambda_{t} \lambda_{\bar{t}}} \rho_{\lambda_{t} \lambda_{\bar{t}}^{\prime}} \rho_{\lambda_{\bar{t}} \lambda_{\bar{t}}} \tilde{M}_{\lambda_{t}^{\prime} \lambda_{\bar{t}}^{\prime}}^{\prime} \tag{4.9}
\end{equation*}
$$

where $\lambda_{t}, \lambda_{\bar{t}}$ are spin labels for the top and anti-top respectively and $\tilde{M}$ is the matrix element for the undecayed process introduced in Section 4.1. By diagonalizing the density matrix, we can obtain the largest possible value of the matrix elements and hence the upper bound. An explicit computation gives this upper bound $\left|M_{u b}^{t}\right|^{2}$ on the top decay as [19],

$$
\begin{equation*}
\left|M_{u b}^{t}\right|^{2}=\frac{4 g_{w}^{4}\left|V_{t b}\right|^{2}\left(r_{1} \cdot x_{1}\right)\left(p_{1} \cdot k_{1}\right)}{\left[\left(w_{1}^{2}-m_{w}^{2}\right)^{2}+\left(m_{w} \Gamma_{W}\right)^{2}\right]\left[\left(p_{1}^{2}-m_{\mathrm{t}}^{2}\right)^{2}-\left(m_{\mathrm{t}} \Gamma_{t}\right)^{2}\right]}|\tilde{M}|^{2} \tag{4.10}
\end{equation*}
$$

where $|\tilde{M}|^{2}$ is the undecayed matrix element for unpolarized $t \bar{t} g$ production. A similar expression $\left|M_{u b}^{\bar{t}}\right|^{2}$ can be obtained for the decay of the top anti-quark by interchanging the labels 1 and 2 and $t$ and $\bar{t}$ in (4.10). Hence the full upper bound can be written as:

$$
\begin{equation*}
\left|M_{u b}^{t \bar{t}}\right|^{2}=\frac{\left|M_{u b}^{t}\right|^{2}\left|M_{u b}^{\bar{t}}\right|^{2}}{|\tilde{M}|^{2}} \tag{4.11}
\end{equation*}
$$

Having obtained the decay matrix elements and their upper bounds, we then proceed to generate events with leptons in the final state as outlined at the beginning of this section.

For the POWHEG decays, we apply the same method where in this case the undecayed matrix elements $|\tilde{M}|$ are the leading order matrix elements for the process

$$
\begin{equation*}
e^{+}+e^{-} \rightarrow t+\bar{t} \tag{4.12}
\end{equation*}
$$

We then use the next-to-leading order decay matrix for which the helicity amplitudes can be found in [5]. These are given in terms of the polar and azimuthal angles of the decay w.r.t the top/anti-top axis and we distribute them as described for the production process in Section 4.1. Note that in this case, we generate two decay gluons, one each from the top and anti-top quark.

In addition, we also consider POWHEG radiations in both the production and decay process by independently generating the emission and distributing the Born variables of the production process first and then generating the emission and distributing the Born variables of the decay process to yield three gluons in the final state.

## 5. Decay NLO lepton spectra comparisons

Extensive studies have been carried out on the lepton angular and energy distributions from the semi-leptonic decays of polarized top and anti-top quarks at next-to-leading order in $\alpha_{S}[1,3]$.

$$
\begin{gather*}
t \rightarrow W^{+}+b+g \rightarrow e^{+}+\nu_{e}+b+g \\
\bar{t} \rightarrow W^{-}+\bar{b}+g \rightarrow e^{-}+\overline{\nu_{e}}+\bar{b}+g . \tag{5.1}
\end{gather*}
$$

In the top rest frame, we define $\theta$ as the angle between the spin 3 -vectors $\mathbf{s}_{\mathbf{t}}, \mathbf{s}_{\bar{t}}$ of the decaying quark and the lepton. We also defined the scaled energies $x^{l, n}$ of the charged lepton and the neutrino respectively as

$$
\begin{align*}
x_{l} & =\frac{2 E_{l}}{m_{\mathrm{t}}} \\
x_{n} & =\frac{2 E_{n}}{m_{\mathrm{t}}} \tag{5.2}
\end{align*}
$$

where $E_{l}$ and $E_{n}$ are the energies of the charged lepton and neutrino in the top rest frame. In these variables, the NLO double differential distribution of the charged lepton
and neutrino in the decay of a heavy top or anti-top quark with polarization $S$ has been shown to be of the form

$$
\begin{align*}
\frac{d \Gamma^{l, n}}{d x^{l, n} d \cos \theta} & =\frac{G_{F}^{2} m_{\mathrm{t}}^{5}}{32 \pi^{3}}\left[F_{0}^{l, n}\left(x^{l, n}, a\right)+S \cos \theta J_{0}^{l, n}\left(x^{l, n}, a\right)\right. \\
& \left.-\frac{2 \alpha_{S}}{3 \pi}\left(F_{1}^{l, n}\left(x^{l, n}, a\right)+S \cos \theta J_{1}^{l, n}\left(x^{l, n}, a\right)\right)\right] \tag{5.3}
\end{align*}
$$

in the narrow width limit for the decay of the $W$ boson. Expressions for $F_{0,1}^{l, n}$ and $J_{0,1}^{l, n}$ can be found in [3]. Integrating over $\cos \theta$ gives us the differential energy distribution,

$$
\begin{equation*}
\frac{d \Gamma^{l, n}}{d x^{l, n}}=\frac{G_{F}^{2} m_{\mathrm{t}}^{5}}{16 \pi^{3}}\left[F_{0}^{l, n}\left(x^{l, n}, y\right)-\frac{2 \alpha_{S}}{3 \pi} F_{1}^{l, n}\left(x^{l, n}, y\right)\right] \tag{5.4}
\end{equation*}
$$

We compared this theoretical prediction with the distribution obtained from the POWHEG method before interfacing with the Herwig++ parton shower. The best fit distributions shown in Figure 8 were obtained by setting $\alpha_{S}$ to 0.1 in (5.4).


Figure 8: Scaled energy fractions of the charged lepton (left) and neutrino (right) from top decay. Black $($ solid $)=$ Theory, Blue $($ dashes $)=$ Decay.

## 6. Truncated Shower

The POWHEG method requires the addition of a 'truncated shower' before the hardest gluon emission in order to simulate the soft radiation distribution [21]. Due to angular ordering, the 'truncated' radiation is emitted at a wider angle than the angle of the hardest emission but at a lower $p_{T}$. This means the 'truncated' radiation does not appreciably degrade the energy entering the hardest emission and justifies our decision to generate the hardest emission first.

In [18], there is a description of a method to generate a truncated shower of at most one gluon for the case of light quark production from $e^{+} e^{-}$annihilation. In this section, we extend the discussion to top pair production. Below is an outline of how the 'truncated shower' was generated. We will consider the case in which at most one extra gluon is emitted by the top or anti-top before the hardest emission. The outline closely follows the Herwig++ parton shower evolution method described in [22,23] where the evolution variables $z$, the momentum fractions, and $\tilde{q}$, the evolution scale, determine the kinematics of the shower.
i) Having generated the $p_{T}$ of the hardest emission as discussed in Section 2 and the energy fractions $x$ and $y$, calculate the light-cone momentum fractions $z$ and $1-z$ of the partons involved in the hardest emission. We will assume henceforth that $x>y$ and that $y$ is the energy fraction of the quark, i.e. the quark is involved in the hardest emission. Then

$$
\begin{equation*}
z=\frac{\alpha_{b}}{\alpha_{b}+\alpha_{g}} \tag{6.1}
\end{equation*}
$$

where if we define

$$
\begin{align*}
b & =\frac{m_{\mathrm{t}}^{2}}{s} \\
\lambda & =\sqrt{(1-4 b)} \tag{6.2}
\end{align*}
$$

we have

$$
\begin{align*}
& \alpha_{b}=\frac{x(1+\lambda)+\sqrt{x^{2}(1+\lambda)^{2}-8(b+\kappa)(1+\lambda-2 b)}}{2(1+\lambda-2 b)} \\
& \alpha_{c}=\frac{y(1+\lambda)-\sqrt{y^{2}(1+\lambda)^{2}-8 b(1+\lambda-2 b)}}{2(1+\lambda-2 b)} \\
& \alpha_{g}=\frac{2}{1+\lambda}-\alpha_{b}-\alpha_{c} \tag{6.3}
\end{align*}
$$

ii) Next generate the light-cone momentum fraction $z_{t}$ of the 'truncated' radiation within the range

$$
\begin{equation*}
\frac{m_{\mathrm{t}}}{\tilde{q}_{i}}<z_{t}<1-\frac{Q_{g}}{\tilde{q}_{i}} \tag{6.5}
\end{equation*}
$$

and distributed according to the massive splitting function, $P_{Q Q}=C_{F}\left[\frac{1+z_{t}^{2}}{1-z_{t}}-\frac{2 m_{t}^{2}}{z_{t}\left(1-z_{t}\right) \tilde{q}^{2}}\right]$. $\tilde{q}_{i}$ is the initial evolution scale, i.e. $\sqrt{s}=500 \mathrm{GeV}$, and $Q_{g}$ is a cutoff introduced to regularize soft gluon singularities in the splitting functions. In this report, a $Q_{g}$ value of 0.75 GeV was used. $z_{t}$ is the momentum fraction of the quark after emitting the 'truncated' gluon with momentum fraction $1-z_{t}$.
iii) Determine the scale $\tilde{q}_{h}$ of the hardest emission from

$$
\begin{equation*}
\tilde{q}_{h}=\sqrt{\frac{p_{T}{ }^{2}}{z^{2}(1-z)^{2}}+\frac{m_{\mathrm{t}}^{2}}{z^{2}}+\frac{Q_{g}{ }^{2}}{z(1-z)^{2}}} \tag{6.6}
\end{equation*}
$$

iv) Starting from an initial scale $\tilde{q}_{i}$, the probability of there being an emission next at the scale $\tilde{q}$ is given by

$$
\begin{equation*}
S\left(\tilde{q}_{i}, \tilde{q}\right)=\frac{\Delta\left(\tilde{q}_{c}, \tilde{q}_{i}\right)}{\Delta\left(\tilde{q}_{c}, \tilde{q}\right)} \tag{6.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta\left(\tilde{q}_{c}, \tilde{q}\right)=\exp \left[-\int_{\tilde{q}_{c}}^{\tilde{q}} \frac{d \tilde{q}^{2}}{\tilde{q}^{2}} \int d z \frac{\alpha_{s}}{2 \pi} P_{Q Q} \Theta\left(0<p_{T}^{t}<p_{T}\right)\right] \tag{6.8}
\end{equation*}
$$

$\tilde{q}_{c}$ is the lower cutoff of the parton shower which was set to the default value of 0.631 GeV in this report, $\alpha_{s}$ is the running coupling constant evaluated at $z(1-z) \tilde{q}, P_{Q Q}$ is the $Q \rightarrow Q g$ splitting function and $p_{T}$ is the transverse momentum of the hardest emission. The Heaviside function ensures that the transverse momentum, $p_{T}^{t}$ of the truncated emission is real and is less than $p_{T}$. To evaluate the integral in (6.8), we overestimate the integrands and apply vetoes with weights as described in [22]. With $r$ a random number between 0 and 1 , we then solve the equation

$$
\begin{equation*}
S\left(\tilde{q}_{i}, \tilde{q}\right)=r \tag{6.9}
\end{equation*}
$$

for $\tilde{q}$. If $\tilde{q}>\tilde{q}_{h}$, the event has a 'truncated' emission. If $\tilde{q}<\tilde{q}_{h}$, there is no 'truncated' emission and the event is showered from the scale of the hardest emission.
v) If there is a 'truncated' emission, the next step is to determine the transverse momentum $p_{T}^{t}$ of the emission. This is given by

$$
\begin{equation*}
p_{T}^{t}=\sqrt{\left(1-z_{t}\right)^{2}\left(z_{t}^{2} \tilde{q}^{2}-m_{\mathrm{t}}^{2}\right)-z_{t}{Q_{g}}^{2}} \tag{6.10}
\end{equation*}
$$

If $p_{T}^{t^{2}}<0$ or $p_{T}^{t}>p_{T}$ go to ii).
vi) We now have values for $z_{t}$, the momentum fraction of the quark after the first emission, $p_{T}^{t}$, the transverse momentum of the first emission, $z$, the momentum fraction of the hardest emission and $p_{T}$, the transverse momentum of the hardest emission. We can then reconstruct the momenta of the partons as described in [22]. The orientation of the quark, antiquark and hardest emission with respect to the beam axis is determined as explained there for the hard matrix element correction.

In this paper, we consider only truncated emissions in the production process, not in the decay.

## 7. Parton shower distributions

Next we interface the generated events with the Herwig++ 2.2 .0 [24] parton shower and veto the hardest emissions in the production and decay of the top and anti-top pairs. In this section we will consider collisions at $\sqrt{s}=500 \mathrm{GeV}$ and only include the truncated shower for the production emissions. We considered four cases:

1. Leading Order (LO): No POWHEG emissions.
2. Production ( Pr ): Only POWHEG emissions in the production are allowed including the truncated shower.
3. Decay (Dc): Only POWHEG emissions in the decays of the top/anti-top pairs are allowed.
4. Production + Decay (PrDc): Both production and decay emissions are allowed.

The following distributions were investigated in the lab frame for the two different $e^{+} e^{-}$ initial polarizations:
i) The angle between the lepton from the decay of the top anti-quark and the top quark are presented in Figure 9.
ii) The angle between the lepton and anti-lepton from the decays of the top pairs are presented in Figure 10.
iii) The energy distributions of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 11 and 12.
iv) The transverse momenta w.r.t the beam axis of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 13 and 14.
v) The longitudinal momenta (along the beam axis) of the $b$ quark and $b$ anti-quark before hadronization are presented in Figures 15 and 16.


Figure 9: Angle between the lepton from the decay of the top anti-quark and the top quark.


Figure 10: Angle between the lepton and anti-lepton from the decays of the top pairs.


Figure 11: Energy of the b-quark before hadronization.


Figure 12: Energy of the b anti-quark before hadronization.


Figure 13: Transverse momentum of the b quark before hadronization.


Figure 14: Transverse momentum of the $b$ anti-quark before hadronization.


Figure 15: Longitudinal momentum of the b quark before hadronization.


Figure 16: Longitudinal momentum of the $b$ anti-quark before hadronization.

At leading order, the leptonic correlations in Figures 9 and 10 are as expected with higher correlations seen for $e_{R}^{-} e_{L}^{+}$annihilation than for $e_{L}^{-} e_{R}^{+}$annihilation. At next-toleading order, it can be observed that the POWHEG emissions do not change the shapes of the distributions much except for a slight broadening of the peaks.

Also at leading order, the distributions in Figures 11-16 have the expected shapes with the $b$ quarks and anti-quarks having softer (harder) energy, longitudinal momentum and transverse momentum spectra for $e_{R}^{-} e_{L}^{+}\left(e_{L}^{-} e_{R}^{+}\right)$annihilation.

Now comparing the POWHEG production and decay distributions for the $b$ quark in Figures 11-16, we observe that the decay emissions soften the spectra more than the production emissions and therefore these have the greater effect in the production + decay distributions. This is expected since the scale range available for the production emissions $\approx \log \left(\sqrt{s} / m_{\mathrm{t}}\right)$ is less than the range available for the decay emissions $\approx \log \left(m_{\mathrm{t}} / m_{\mathrm{b}}\right)$.

## 8. Conclusions

Using the Monte Carlo event generator Herwig++, we have successfully applied the POWHEG method to investigate angular correlation distributions at next-to-leading order in top pair production and decays at ILC energies. In all distributions studied, the POWHEG emissions have the effect of broadening the peaks of the leading order predictions slightly. We also compared momentum distributions of the $b$ quarks and anti-quarks before hadronization and observe that the decay emissions soften the spectra more at next-to-leading order as expected.

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