TESTING PROPORTIONALITY IN DURATION MODELS WITH RESPECT TO CONTINUOUS COVARIATES *

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Abstract

Several omnibus tests of the proportional hazards assumption have been proposed in the duration models literature. In the two-sample case (*i.e.,* when the covariate is binary), tests have also been developed against non-parametrically specified ordered alternatives. This paper considers a natural extension of such monotone ordering to the case of continuous covariates, and develops tests for the proportional hazards assumption against such ordered alternatives. Small sample properties of the test are explored. The use of the test statistics, and use of histogram sieve estimators in the case where proportionality does not hold, are illustrated with application to data on strike durations.

Key words: Proportional Hazards model; Increasing hazard ratio; Increasing ratio of cumulative hazards; Ordered (monotone) alternatives; Two-sample problem; Histogram sieve estimators.

JEL Classification: C12, C14, C41.

1 Introduction

The proportional hazards (PH) model, and more specifically the Cox regression model (Cox, 1972) (a specific formulation of the PH model) has become almost universal in econometric applications with duration data.[1] This is because, the PH model (and the Cox regression model) provides a convenient way to evaluate the influence of one or several concomitant variables (covariates) on the probability of conclusion of duration spells. However, the PH specification substantially restricts interdependence between the explanatory variables and the duration in determining the hazard. In particular, the PH model restricts the coefficients of the regressors in the logarithm of the hazard function to be constant over the duration. This may not hold in many situations, or may even be unreasonable from the point of view of relevant economic theory (McCall, 1994). Further, such and other kinds of misspecifications often leads to misleading

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[1]The PH model and the Cox regression model have been used almost synonymously in the literature. For most of this paper, we use the term PH model, since the arguments, in general, apply to the PH model, and specifically also for the Cox regression model.

inferences about duration dependence, and potentially, to misleading inferences about the effects of included explanatory variables (Kiefer, 1988). Understandably, testing the PH model, both graphically and analytically, has been an area of active research.[2]

Most of these available tests are either omnibus tests (and hence, usually have low power), or are tests in which the PH model is embedded in a broader (semiparametric) model. As opposed to such broad alternatives, it is often of interest to explore whether the hazard rate for one level of the covariate increases in duration, relative to another level (*i.e.*, the hazard ratio increases/ decreases with duration), particularly when the covariate is discrete (two-sample or k-sample setup).[3] In the two-sample setup, Gill and Schumacher (1987), Lin (1991) and Deshpande and Sengupta (1995) have constructed analytical tests of the PH hypothesis against the alternative of 'increasing hazard ratio' [4], which is equivalent to convex ordering of the duration distribution in one sample with respect to the other. Under the same setup, Sengupta, Bhattacharjee and Rajeev (1998) have proposed a test of the PH model against the weaker alternative hypothesis of 'increasing ratio of cumulative hazards' (star ordering of the two samples). The above alternative hypotheses ('increasing hazard ratio' and 'increasing ratio of cumulative hazards') often provide an explanation for the phenomenon of 'crossing hazards' frequently observed in applications. In fact, it has now generally come to be accepted in the empirical literature that convex-ordering/ star-ordering of one sample with respect to another in the two-sample setup, or one cause of failure to another in the competing risks setup, as well as their duals (the concave-ordering/ negative-star-ordering hypotheses), are natural ordered alternatives to the proportional hazards model. Empirical evidence of such ordering are abundant in the literature on economic duration models and bio-medicine. The use of the above (two-sample) inferential procedures in econometric applications is limited by the fact that, it is often of importance to infer on the effects of one or more continuous covariates (Horowitz and Neumann, 1992).

On the other hand, monotone/ ordered departures are common and potentially meaningful alternatives to the PH model in the case of continuous covariates. For example, if the coefficient corresponding to a covariate under the Cox PH model is increasing with duration, the distribution of the duration conditional on a higher value of the covariate would be convex ordered with respect to the duration distribution conditional on a smaller covariate value. In Section 2, we build on the above idea and develop concepts of ordered alternatives to the PH model, with respect to continuous covariates. Tests of the PH assumption against such ordered alternatives are constructed and their asymptotic properties described in Section 3 (all proofs are in the Appendix), while their small sample properties are explored in Section 4. In Section 5, we illustrate the notions and tests proposed in the paper using an application to Kennan's (1985) data on strike durations. We also demonstrate how the histogram sieve estimator of Murphy and Sen (1991) can be used in the situation when proportionality is rejected, to derive credible and interpretable inference. Finally, Section 5 presents the concluding remarks.

[2]Andersen *et. al.* (1992) and Neumann (1994) provide partial reviews of the tests available in the literature, while Sengupta (1995) gives a review specifically of the graphical tests.

[3]This kind of situation could arise, for example, if the coefficient of the covariate is not constant over time, or is dependent on some other (possibly unobserved) covariate.

[4]Throughout this paper, the word 'increasing' would mean 'non-decreasing', and 'decreasing' would mean 'non-increasing'.

2 Partial ordering of duration models with continuous covariates

The concept of partial ordering of duration distributions (or lifetime distributions) is quite popular, especially in bio-medical applications, and are relevant to duration models in economics. The notions have also been used in applications in political science (see, for example, Box-Steffensmeier and Jones, 1997). The most popular of these available notions of partial ordering are convex ordering and star ordering (Kalashnikov and Rachev, 1986). These are defined as follows:

Let X and Y be two duration distributions with distribution functions F and G respectively, cumulative hazard functions Λ_F and Λ_G respectively, and hazard functions λ_F and λ_G respectively (whenever they exist).

Definition 1 X (F) is said to be convex ordered with respect to Y (G), denoted $X \prec_c Y$ or $F \prec_c G$, if $F \circ G^{-1}$ represents the distribution function of an increasing failure rate (IFR) distribution, or equivalently, $\Lambda_F \circ \Lambda_G^{-1}$ is a convex function on $[0, \infty)$.

Definition 2 X (F) is said to be star ordered with respect to Y (G), denoted $X \prec_* Y$ or $F \prec_* G$, if $F \circ G^{-1}$ represents the distribution function of an increasing failure rate average (IFRA) distribution, or equivalently, $\Lambda_F \circ \Lambda_G^{-1}$ is a star-shaped function on $[0, \infty)$.

Correspondingly, if $Y \prec_c X$ or $G \prec_c F$ ($Y \prec_* X$ or $G \prec_* F$), then X or F is said to be concave ordered (negative star ordered) with respect to Y or G respectively. Convex and concave ordering are stronger notions of ordering as compared to star-ordering and negative star-ordering respectively; in other words, $F \prec_c G \;\Rightarrow\; F \prec_* G$. Further, the function $\Lambda_F \circ \Lambda_G^{-1}$ is convex if and only if the ratio of hazard rates λ_F/λ_G is increasing (provided the ratio exists), and star-shaped if and only if Λ_F/Λ_G is increasing (Sengupta and Deshpande, 1994).

Convex-ordering and star-ordering are intuitive and meaningful departures from the PH model in two samples, and in the competing risks framework. Also, these ordered alternatives to the PH model can be conveniently studied in terms of monotonicity of ratios of hazards/ cumulative hazards (under the PH model, the hazard ratio and cumulative hazard ratio are both constant over duration). It has been observed in several econometric studies that the departure from the PH model is evident from the fact that the ratio of the hazard rates is not constant over the duration.[5] Such orderings can also often explain the phenomenon of 'crossing hazards' commonly observed in empirical applications.[6] As mentioned earlier, in the two sample setup, Gill and Schumacher (1987), Lin (1991) and Deshpande and Sengupta (1995) have developed tests of the PH model against the "increasing hazard ratio" alternative, which is equivalent to convex ordering of the life-time distribution in one sample with respect to the other, and Sengupta, Bhattacharjee and Rajeev (1998) have con-

[5]For example, Jayet and Moreau (1991), using French data on employment durations, found that the ratio of hazard function for individuals in the age groups 24–28 years to that for 37–40 years was increasing upto a duration of approximately 120 days, and then decreasing. However, the ratio of the cumulative hazards corresponding to these two groups was increasing over the duration scale (star ordering). Similarly, Sengupta and Bhattacharjee (1994), in their analysis of the unemployment duration data of Han and Hausman (1990), observed that the risk due to 'recall to old job' is convex ordered with respect to the competing risk of 'new job'. In other words, the ratio of the hazards due to the competing causes of termination of an unemployment spell, *viz.* 'recall to old job' and 'new job', is increasing in duration of the spell.

[6]This situation is apparent, for example, in Katz (1986), where the hazard functions for 'new job' and 'recall to old job' cross each other.

structed a test for the weaker alternative hypothesis of "increasing ratio of cumulative hazards" (star ordering of the two samples).

Following a suggestion in Fleming and Harrington (1991), we introduce the following definition which provides a natural extension of the above notions of partial ordering in the two-sample case to the continuous covariate case. Let T be a lifetime variable, X a continuous covariate and let $\lambda(t|x)$ denote the hazard rate of T, given $X = x$, at $T = t$.

Definition 3 The duartion random variable T is defined to be *increasing hazard ratio for continuous covariate (IHRCC)* with respect to the covariate X if, whenever $x_1 > x_2$, $\lambda(t|x_1)/\lambda(t|x_2) \uparrow t$ $(\equiv (T|X = x_1) \prec_c (T|X = x_2))$. Similarly, T is defined to be *increasing cumulative hazard ratio for continuous covariate (ICHRCC)* with respect to X if, whenever $x_1 > x_2$, $\Lambda(T|x_1)/\Lambda(t|x_2) \uparrow t$ $(\equiv (T|X = x_1) \prec_* (T|X = x_2))$. The duals *decreasing hazard ratio for continuous covariate (DHRCC)* and *decreasing cumulative hazard ratio for continuous covariate (DCHRCC)* are correspondingly defined.

Definition 3 gives a notion of positive ageing with respect to a continuous covariate. The higher the covariate, the faster the ageing of the individual – a situation which may be reasonably commonplace in empirical applications. For example, it has been observed that the impact of real wage changes varied with duration of strikes, and the variation may be in the nature of ordered departures (Metcalf, Wadsworth and Ingram (1992); Card and Olson, 1992).

Example 1 Let the DGP be defined by the hazard function $\lambda(t|x) = \lambda_0(t). \exp(\beta(t).x)$, where x is the covariate, and $\beta(.)$ is an increasing function of duration t. This model could be appropriate when the influence (or prognostic value) of the covariate is expected to be higher at higher durations. Then, if $x_1 > x_2$, $\lambda(t|x_1)/\lambda(t|x_2) = \exp(\beta(t).(x_1 - x_2))$ is increasing in t. In other words, the lifetime random variable T is $IHRCC$ with respect to the covariate X. Correspondingly, if $\beta(.)$ is an decreasing function of the duration, T would be $DHRCC$ with respect to X.

Example 2 Consider a changepoint duration model given by the cumulative hazard function $\Lambda(t|x) = \Lambda_0(t). \exp(I\,(t > t^*)\,.\beta x)$, where x is the covariate, $I(.)$ the indicator function, and t^* is a duration in the interior of the sample space. This is a model where the effect of the covariate begins as soon as the duration crosses a certain threshold t^*, and it lifts the distribution function upto a level where it would have been, if the effect of the covariate would have persisted over the entire past life of the duration variable. If $\beta > 0$, this model is $ICHRCC$, but not $IHRCC$.[7]

Example 3 Consider the hazard function $\lambda(t|x) = \lambda_0(t). \exp(\beta(t). |x - a|)$, where x is the covariate, a is a point on the covariate space, and $\beta(.)$ is an increasing function of duration t. This model is neither $IHRCC$ nor $DHRCC$; it is $IHRCC$ on one region of the covariate space $(x > a)$, and $DHRCC$ on another region $(x < a)$. As we shall see later, our tests of the PH model would also be able to detect this kind of departures from the PH model.

As the above examples illustrate, the notions of ordering introduced in *Definition 3* encompass a wide range of non-PH data generation processes. And, it appears likely that such ordered alternatives would be useful in many empirical applications. Even from a theoretical point of view, ordered duration models may often be expected in

[7]The distribution function here has a jump discontinuity, but one can easily construct examples where $ICHRCC$ holds, and the distribution function is absolutely continuous.

econometric applications. If, as in Example 1, instead of being constant over duration, the coefficient of a covariate is assumed to be increasing, the duration model would then be IHRCC with respect to that covariate. There may be a number of different explanations for such change in the coefficients to occur, including learning effect, shift in life-course position, maturational changes, and so on. For example, using British data, Atkinson *et. al.* (1984) and Narendranathan and Stewart (1993) find that unemployment benefits have different effects on the hazards from unemployments as the spell lengthens. Mortensen (1977) presents search theoretic arguments on why this should be so, based on maximum benifit periods, implying that the impact declines as the benefits come close to exhaustion. Similarly, the hypothesis of Mortensen (1986) that liquidity constraints generate a reservation wage that declines with time spent unemployed could give rise to ordered departures from proportionality.

All the above examples demonstrate that, many situations exist in practice, where the PH model appears to be a poor description of the duration generating procedure, as compared with ordered alternatives. In several cases, monotone departures from the PH model may be reasonable to expect, even from a theoretical point of view. Since, estimation of PH models with non-proportional hazards rates in the data lead to biased estimation, incorrect standard errors, and faulty inferences about the substantive impact of independent variables (Kalbfleisch and Prentice, 1980; Schemper, 1992; Horowitz and Neumann, 1992), one has to be careful about such departures while using the PH model for inference. This is the context in which construction of tests of the PH model against monotone alternatives with respect to continuous covariates assumes particular significance.

3 Construction of the test statistics

Several two-sample tests of the PH model against monotone alternatives exist in the literature. For a continuous covariate, a natural way of testing the PH assumption against the alternatives $IHRCC$ and $ICHRCC$ (and their duals) would be repeated applications of the corresponding tests in the two-sample setup (for example, Gill and Schumacher (1987) test (GS), and Sengupta, Bhattacharjee and Rajeev (1998) test (SBR), *etc.*). We, thus, propose a simple construction of our tests as follows. First, we (randomly) select a fixed number of pairs of distinct points on the covariate space, and construct the usual two-sample test statistics (T_{GS} and T_{SBR}) for each pair, based on counting processes conditional on these two distinct covariate points. We shall then construct our test statistic, by taking supremum/ infemum or average of these basic test statistics (suitably standardized) over these fixed number of pairs.[8]

For the alternative of 'increasing hazard ratio' (convexity) in two samples (having cumulative hazard functions $\Lambda_1(t)$ and $\Lambda_2(t)$), the test statistic proposed by Gill and Schumacher (1987) is

$$\begin{aligned}
T_{GS,std} &= \frac{T_{GS}}{\sqrt{\widehat{Var}\left[T_{GS}\right]}}, \text{ where} \\
T_{GS} &= T_{11}T_{22} - T_{12}T_{21}, \\
\widehat{Var}\left[T_{GS}\right] &= T_{21}T_{22}V_{11} - T_{21}T_{12}V_{12} - T_{11}T_{22}V_{21} + T_{11}T_{12}V_{22}, \\
T_{ij} &= \int_0^\tau L_i(t)d\widehat{\Lambda}_j(t), \qquad (i,j = 1,2),
\end{aligned}$$

[8]While supremum/ infemum have the interpretation of Rao's union-intersection principle, averages have the interpretation of invoking the central limit theorem.

$$V_{ij} = \int_0^\tau L_i(t)L_j(t)\{Y_1(t)Y_2(t)\}^{-1}d\left(N_1+N_2\right)(t), \qquad (i,j=1,2),$$

τ is a random stopping time (in particular, τ may be taken as the time at the final observation in the combined sample),

$L_j(t)$ $(j=1,2)$, are predictable processes,

$\widehat{\Lambda}_j(t)$ $(j=1,2)$, is the Nelson-Aalen estimator of the cumulative hazard function in the j^{th} sample,

$Y_j(t)$ $(j=1,2)$, is the number of individuals on test in sample j at time t,

and N_1, N_2 are the counting processes counting the number of failures in each sample.

Gill and Schumacher (1987) have shown that the unstandardised test statistic (T_{GS}) has mean zero under the null hypothesis (PH) and positive (negative) mean if the hazard ratio $\lambda_1(t)/\lambda_2(t)$ is monotonically increasing in t on $[0,\infty)$ and L_1 and L_2 are so chosen that $L_1(t)/L_2(t)$ is monotonically decreasing (increasing), and that its standard error would decrease to zero as sample size increases to ∞ under both the null and alternative hypotheses. Hence, while the standardized test statistic $T_{GS,std}$ would be asymptotically standard normal under the null hypothesis, it's mean would increase (decrease) to ∞ $(-\infty)$ under the alternative hypothesis.[9]

For testing $H_0 : PH$ vs. $H_1 : IHRCC$, we propose the following procedure. We fix $r>1$, and select $2r$ distinct points $\{x_{11}, x_{21}, \ldots, x_{r1}, x_{12}, x_{22}, \ldots, x_{r2}\}$ on the covariate space $\mathcal{X}$, such that $x_{l2} > x_{l1}, l = 1, \ldots, r$.

We then construct our test statistics T^*_{GS}, T^{**}_{GS} and $\overline{T}_{GS}$ based on the r statistics $T_{GS,std}(x_{l1}, x_{l2}), l = 1, \ldots, r$ (each testing convexity with respect to the pair of counting processes $N(t, x_{l1})$ and $N(t, x_{l2})$), where

$$\begin{aligned}
T_{GS,std}(x_{l1}, x_{l2}) &= \frac{T_{GS}(x_{l1}, x_{l2})}{\sqrt{\widehat{Var}\left[T_{GS}(x_{l1}, x_{l2})\right]}},\\
T_{GS}(x_{l1}, x_{l2}) &= T_{l11}T_{l22} - T_{l12}T_{l21},\\
\widehat{Var}\left[T_{GS}(x_{l1}, x_{l2})\right] &= T_{l21}T_{l22}V_{l11} - T_{l21}T_{l12}V_{l12} - T_{l11}T_{l22}V_{l21} + T_{l11}T_{l12}V_{l22},
\end{aligned}$$

$$\begin{aligned}
T_{lij} &= \int_0^\tau L_i(x_{l1}, x_{l2})(t)d\widehat{\Lambda}(t, x_{lj}), \qquad (i,j=1,2), \text{ and}\\
V_{lij} &= \int_0^\tau L_i(x_{l1}, x_{l2})(t)L_j(x_{l1}, x_{l2})(t)\frac{d\left[N(t,x_{l1}) + N(t,x_{l2})\right]}{Y(t,x_{l1})Y(t,x_{l2})}, (i,j=1,2).
\end{aligned}$$

[9]Typically, L_1 and L_2 may be chosen corresponding to the Gehan-Wilkoxon and log rank tests, whereby $L_1 = Y_1Y_2$ and $L_2 = Y_1Y_2(Y_1+Y_2)^{-1}$, so that $L_1(t)/L_2(t)$ is monotonically decreasing in t.

Then, our test statistics are:

$$T^*_{GS} = \max\left\{T_{GS,std}(x_{11},x_{12}), T_{GS,std}(x_{21},x_{22}), \ldots, T_{GS,std}(x_{r1},x_{r2})\right\},$$

$$T^{**}_{GS} = \min\left\{T_{GS,std}(x_{11},x_{12}), T_{GS,std}(x_{21},x_{22}), \ldots, T_{GS,std}(x_{r1},x_{r2})\right\},$$

$$\text{and } \overline{T}_{GS} = \frac{1}{r}\sum_{l=1}^{r} T_{GS,std}(x_{l1},x_{l2}).$$

For the choice of L_1 and L_2 mentioned above, this statistic will be close to zero under the null hypothesis. Under the alternative hypothesis $IHRCC$, $\overline{T}_{GS}$ and T^*_{GS} would increase to ∞ as sample size increases, while under $DHRCC$, $\overline{T}_{GS}$ and T^{**}_{GS} would decrease to $-\infty$.

The test statistic proposed by Sengupta, Bhattacharjee and Rajeev (1998) for testing the proportional hazards model against the 'increasing cumulative hazard ratio' (star-ordering) alternative is quite similar in structure to the test statistic of Gill and Schumacher (1987). The test statistic is given by

$$\begin{aligned}
T_{SBR,std} &= \frac{T_{SBR}}{\sqrt{\widehat{Var}\left[T_{SBR}\right]}}, \text{ where} \\
T_{SBR} &= S_{11}S_{22} - S_{12}S_{21}, \\
\widehat{Var}\left[T_{SBR}\right] &= S_{21}S_{22}W_{11} - S_{21}S_{12}W_{12} - S_{11}S_{22}W_{21} + S_{11}S_{12}W_{22}, \\
S_{ij} &= \int_0^{\tau} K_i(t)\widehat{\Lambda}_j(t)dt, \qquad (i,j=1,2), \\
W_{ij} &= \int_0^{\tau}\int_0^{\tau} K_i(t)K_j(s)W\left(\min(s,t)\right)dsdt, \qquad (i,j=1,2), \\
\tau & \text{ is a large positive number such that } \Lambda_j(\tau) < \infty, j=1,2, \\
& \text{(need not be a stopping time)}, \\
W(t) &= \int_0^{t} \left(Y_1(s)Y_2(s)\right)^{-1} d\left(N_1+N_2\right)(s) \\
\text{and} \quad K_j(t) & \quad (j=1,2), \text{ are right continuous functions with left limits} \\
& \text{(need not be predictable processes).}
\end{aligned}$$

This standardised test statistic is also asymptotically standard normal under the null hypothesis of proportional hazards, and asymptotically normal with mean increasing (decreasing) to ∞ $(-\infty)$ if the cumulative hazard ratio $\Lambda_1(t)/\Lambda_2(t)$ is monotonically increasing in t on $[0,\infty)$ and K_1 and K_2 are so chosen that $K_1(t)/K_2(t)$ is monotonically increasing (decreasing). As before, we construct our test statistics T^*_{SBR}, T^{**}_{SBR} and $\overline{T}_{SBR}$ based on the r statistics $T_{SBR,std}(x_{l1},x_{l2}), l=1,\ldots,r$ (each testing star-ordering with respect to the pair of counting processes $N(t,x_{l1})$ and $N(t,x_{l2})$). Thus, we have:

$$T^*_{SBR} = \max\left\{T_{SBR,std}(x_{11},x_{12}), T_{SBR,std}(x_{21},x_{22}), \ldots, T_{SBR,std}(x_{r1},x_{r2})\right\},$$

$$T^{**}_{SBR} = \min\left\{T_{SBR,std}(x_{11},x_{12}), T_{SBR,std}(x_{21},x_{22}), \ldots, T_{SBR,std}(x_{r1},x_{r2})\right\},$$

$$\text{and } \overline{T}_{SBR} = \frac{1}{r}\sum_{l=1}^{r} T_{SBR,std}(x_{l1},x_{l2}).$$

We now derive the large sample results for the proposed test statistics, using the counting process methods of *e.g.* Gill and Schumacher (1987) and Andersen *et. al.* (1992). It is also indicated how these results, in conjunction with extreme value theory as contained in *e.g.* Berman (1992), can be used to obtain convenient expressions for the relevant p-values of T^*_{GS}, T^{**}_{GS}, T^*_{SBR} and T^{**}_{SBR}.

Consider a counting processes $\{N(t,x) : t \in [0,\tau], x \in \mathcal{X}\}$, indexed on a continuous covariate x, with intensity processes $\{Y(t,x)\lambda(t,x)\}$ such that $\lambda(t,x) = \theta_x \lambda(t)$ for all t (under the null hypothesis of proportional hazards). Let, as before, L_1 and L_2 be two predictable processes, each indexed on a pair of distinct values of the continuous covariate x (*i.e.,* indexed on (x_1,x_2), $x_1 \neq x_2$, $x_1,x_2 \in \mathcal{X}$), and let τ be a stopping time. Then, from Gill and Schumacher (1987), it follows that, as $n \longrightarrow \infty$, $T_{GS,std}(x_1,x_2) \xrightarrow{D} N(0,1)$, $\forall x_1,x_2 \in \mathcal{X}$, $x_1 \neq x_2$. Similarly, if K_1 and K_2 are right continuous functions with left limits, which are each indexed on $\{(x_1,x_2),\ x_1 \neq x_2,\ x_1,x_2 \in \mathcal{X}\}$, and τ is a large positive time such that $\Lambda(\tau,x_i) < \infty$, $i = 1,2$, then, Sengupta, Bhattacharjee and Rajeev (1998) have showed that, as $n \longrightarrow \infty$, $T_{SBR,std}(x_1,x_2) \xrightarrow{D} N(0,1)$, $\forall x_1,x_2 \in \mathcal{X}$, $x_1 \neq x_2$.

Now, let $\{x_{11}, x_{21}, \ldots, x_{r1}, x_{12}, x_{22}, \ldots, x_{r2}\}$ be $2r$ (r is a fixed positive integer, $r > 1$) distinct points on the covariate space $\mathcal{X}$, such that $x_{l2} > x_{l1}, l = 1, \ldots, r$.

Assumption 1 For each l, $l = 1,2,\ldots,r$, let $L_1(x_{l1},x_{l2})(t)$ and $L_2(x_{l1},x_{l2})(t)$ be predictable processes (predictable with respect to t).

Assumption 2 Let τ be a random stopping time.[10]

Assumption 3 The sample paths of $L_i(x_{l1},x_{l2})$ and $Y(t,x_{li})^{-1}$ are almost surely bounded with respect to t, for $i = 1,2$ and $l = 1,\ldots,r$. Further, for each $l = 1,\ldots,r$, $L_i(x_{l1},x_{l2})$ $(i = 1,2)$ are both zero whenever $Y(t,x_{l1})$ or $Y(t,x_{l2})$ are.

Assumption 4 There exists a sequence $a^{(n)}$, $a^{(n)} \longrightarrow \infty$ as $n \longrightarrow \infty$, and fixed functions $y(t,x)$, $l_1(x_{l1},x_{l2})(t)$ and $l_2(x_{l1},x_{l2})(t)$, $l = 1,\ldots,r$ such that

$$\begin{array}{lll} sup_{t\in[0,\tau]} \left|Y(t,x)/a^{(n)} - y(t,x)\right| \xrightarrow{P} 0 & \text{as } n \longrightarrow \infty, & \forall x \in \mathcal{X} \\ sup_{t\in[0,\tau]} \left|L_i(x_{l1},x_{l2})(t) - l_i(x_{l1},x_{l2})(t)\right| \xrightarrow{P} 0 & \text{as } n \longrightarrow \infty, & i = 1,2; l = 1,\ldots,r \end{array}$$

where $|l_i(x_{l1},x_{l2})(.)|$ $(i = 1,2; l = 1,\ldots,r)$ are bounded on $[0,\tau]$, and $y^{-1}(.,x)$ is bounded on $[0,\tau]$, for each $x \in \mathcal{X}$.[11]

Let the test statistics T^*_{GS}, T^{**}_{GS} and $\overline{T}_{GS}$ be as defined earlier.

Theorem 1: Let Assumptions 1 to 4 hold. Then, under $H_0 : PH$, as $n \rightarrow \infty$,

(a) $P\left[T^*_{GS} \leq z\right] \longrightarrow [\Phi(z)]^r$,

(b) $P\left[T^{**}_{GS} \geq -z\right] \longrightarrow [\Phi(z)]^r$,

and

(c) $\sqrt{r}\ \overline{T}_{GS} \xrightarrow{D} N(0,1)$,

[10] In particular, τ may be taken as the time at the final observation of the counting process $\Sigma_{l=1}^{r}\Sigma_{j=1}^{2} N(t,x_{lj})$. In principle, one could also have different stopping times $\tau(x_{l1},x_{l2}), l = 1,\ldots,r$ for each of the r basic test statistics.

[11] The condition on probability limit of $Y(t,x)$ can be replaced by a set of weaker conditions (see, for example, Sengupta, Bhattacharjee and Rajeev, 1998).

where $\Phi(z)$ is the distribution function of a standard normal variate.
(Proof in Appendix.)

<u>*Corollary 1*</u>:

$$\begin{aligned} P\left[a_r\left\{T^*_{GS}-b_r\right\}\le z\right] &\longrightarrow \exp\left[-\exp(-z)\right] \text{ as } r\longrightarrow\infty \text{ and} \\ P\left[a_r\left\{T^{**}_{GS}+b_r\right\}\ge z\right] &\longrightarrow \exp\left[-\exp(z)\right] \text{ as } r\longrightarrow\infty, \\ \text{where } a_r &= (2\ln r)^{1/2}, \\ \text{and } b_r &= (2\ln r)^{1/2}-\frac{1}{2}(2\ln r)^{-1/2}(\ln\ln r+\ln 4\pi). \end{aligned}$$

(Proof in Appendix).

<u>*Assumption 5*</u> For each l, $l=1,2,\ldots,r$, let $K_1(x_{l1},x_{l2})(t)$ and $K_2(x_{l1},x_{l2})(t)$ be stochastic processes (with respect to t) with sample paths in $D[0,\infty)$ (*i.e.*, are right continuous and have left limits).

<u>*Assumption 6*</u> Let τ be a positive duration such that $\Lambda(t,x_{lj})<\infty,\ l=1,2,\ldots,r,\ j=1,2$.

<u>*Assumption 7*</u> There exists a sequence $a^{(n)}$, $a^{(n)}\longrightarrow\infty$ as $n\longrightarrow\infty$, and deterministic functions $y(t,x)$, $k_1(x_{l1},x_{l2})(t)$ and $k_2(x_{l1},x_{l2})(t)$, $l=1,\ldots,r$ such that

$$\begin{aligned} &sup_{t\in[0,\tau]}\left|Y(t,x)/a^{(n)}-y(t,x)\right| \xrightarrow{P} 0 && \text{as } n\longrightarrow\infty, \quad \forall x\in\mathcal{X} \\ &sup_{t\in[0,\tau]}\left|K_i(x_{l1},x_{l2})(t)-k_i(x_{l1},x_{l2})(t)\right| \xrightarrow{P} 0 && \text{as } n\longrightarrow\infty, \quad i=1,2; l=1,\ldots,r \end{aligned}$$

where $k_1(x_{l1},x_{l2})(t)$ and $k_2(x_{l1},x_{l2})(t)$, $l=1,\ldots,r$ are continuous functions with respect to t, and $y^{-1}(.,x)$ is bounded on $[0,\tau]$, for each $x\in\mathcal{X}$.
Let the test statistics T^*_{SBR}, T^{**}_{SBR} and $\overline{T}_{SBR}$ be as defined earlier.

<u>*Theorem 2*</u>: Let Assumptions 5 to 7 hold. Then, under $H_0: PH$, as $n\to\infty$,
(a) $P\left[T^*_{SBR}\le z\right]\longrightarrow[\Phi(z)]^r$,

(b) $P\left[T^{**}_{SBR}\ge -z\right]\longrightarrow[\Phi(z)]^r$,

and

(c) $\sqrt{r}\,\overline{T}_{SBR}\xrightarrow{D}N(0,1)$,

where $\Phi(z)$ is the distribution function of a standard normal variate.
(Proof in Appendix.)

<u>*Corollary 2*</u>:

$$\begin{aligned} P\left[a_r\left\{T^*_{SBR}-b_r\right\}\le z\right] &\longrightarrow \exp\left[-\exp(-z)\right] \text{ as } r\longrightarrow\infty \text{ and} \\ P\left[a_r\left\{T^{**}_{SBR}+b_r\right\}\ge z\right] &\longrightarrow \exp\left[-\exp(z)\right] \text{ as } r\longrightarrow\infty, \\ \text{where } a_r &= (2\ln r)^{1/2}, \\ \text{and } b_r &= (2\ln r)^{1/2}-\frac{1}{2}(2\ln r)^{-1/2}(\ln\ln r+\ln 4\pi). \end{aligned}$$

(Proof in Appendix.)

<u>*Remark 1*</u>: Restricting the statistics T^*_{GS}, T^{**}_{GS}, T^*_{SBR} and T^{**}_{SBR} to depend on a fixed number (r) of distinct pairs of points is a crucial step in the construction of the test statistics. This is because, the processes $T_{GS,std}(x_1, x_2)$ and $T_{SBR,std}(x_1, x_2)$ on the space $\{(x_1, x_2) : \; x_2 > x_1, \; x_1, x_2 \in \mathcal{X}\}$, being pointwise standard normal and independent (not independent increments), may not have well-defined limiting processes. And the supremum of these limiting processes would have a degenerate distribution, with all the mass at $+\infty$.

<u>*Remark 2*</u>: The significance of the Corollary is that it gives a simple way of calculating the p-values for the extremal test statistics T^*_{GS} and T^{**}_{GS} (and similarly, T^*_{SBR} and T^{**}_{SBR}), if r is reasonably large. Note that r is held fixed, and hence cannot increase to ∞, but then it can be fixed at a large enough value, so that the approximation can be fruitfully used.

Since the covariate under consideration is continuous, it is practically not feasible to construct the basic tests (GS, SBR *etc.*) based on two distinct fixed points on the covariate space, since the number of observations will be limited. In pursuance of usual practice in such situations, we recommend considering two "small" intervals/ neighborhoods around these chosen points, such that the hazard function within these intervals can be (at least approximately) regarded as constant (across covariate values). While the derived distributions are for counting processes pertaining to specified pairs of points in the covariate space, the tests would go through for small intervals around these points, provided the covariate values are so chosen that they are continuity points of the hazard function (for T^*_{GS}, T^{**}_{GS} and $\overline{T}_{GS}$, when the alternative is $IHRCC$ or its dual), or the cumulative hazard function (for T^*_{SBR}, T^{**}_{SBR} and $\overline{T}_{SBR}$, when the alternative hypotheses are $ICHRCC$ or its dual). However, in small samples, these intervals may be overlapping, and therefore independence of the basic test statistics may be violated. Our Monte Carlo simulations suggest that the average test statistics are particularly susceptible to this problem (they have a small sample variance larger than $1/r$), and we suggest making a standard error correction in such cases, by normalizing the average statistic using a jacknife or bootstrap estimate of the standard error. In this paper, we have used the Quenouille-Tukey jacknife variance estimator for this purpose (see Efron and Stein, 1981). This strengthens the performance of the test in small samples, and does not change our asymptotic results. We denote these adjusted test statistics by $\overline{T}_{GS,Adj}$ and $\overline{T}_{SBR,Adj}$ respectively.

The choice between the supremum/ infemum and average test statistics can be crucial in practice. The supremum/ infemum test statistics detect more complicated departures from the PH model, and thereby facilitate detailed investigation of monotonicity across the duration. If, as in Example 3, the relevant hazard / cumulative hazard ratio is monotonically increasing over one region of the covariate space, and decreasing over another (a not very unusual phenomenon), this approach would be useful in detecting such behaviour. We shall return to this aspect of the tests later in the paper, when we discuss small sample properties and empirical applications. On the other hand, as we shall see in the Monte Carlo simulations in the next section, the adjusted average statistics outperform the supremum/ infemum statistics in terms of power.

4 Simulation Results

The asymptotic distributions of the proposed test statistics have been derived in the previous Section. In this Section, we shall explore the finite sample performance of the tests for different specifications of the baseline hazard function and covariate dependence. The selected DGPs are broadly drawn on the models used in Horowitz (1999) and Martinussen, Scheike and Skovgaard (2000). In particular, we consider models of the form

$$\lambda(t,x) = \lambda_0(t).\exp\left[\beta(t,x)\right],$$

where $\lambda_0(t)$ and $\beta(t,x)$ are chosen to assume a variety of functional forms. The PH model holds if and only if $\beta(t,x)$ does not depend on t. If, for fixed x, $\beta(t,x)$ increases/ decreases in t, we have $IHRCC$/ $DHRCC$ alternatives.[12] If, on the other hand, $\beta(t,x)$ increases in t over some range of the covariate space, and decreases over another (as in Example 3), neither PH nor $IHRCC$/ $DHRCC$ hold. However, our tests based on supremum/ infemum would still be consistent for these kinds of alternatives to the null hypothesis of proportional hazards.

The Monte Carlo simulations are based on independent right-censored data from the following 8 DGPs, generated using the Gauss 386 random number generator, where the covariate X are i.i.d. $U(-1,1)$, and the censoring duration C are i.i.d. $U(0.2, 2.2)$.

Model	$\lambda_0(t)$	$\beta(t,x)$	Median cens.dur.	% cens.	Expected significance
DGP_{11}	2	0	0.36	16.4	None
DGP_{12}	2	x	0.30	19.2	None
DGP_{13}	2	$\ln(t).x$	0.25	15.8	$T^*_{GS}, \overline{T}_{GS,Adj}, T^*_{SBR}, \overline{T}_{SBR,Adj}$
DGP_{14}	2	$\ln(t).\lvert x\rvert$	0.52	26.9	$T^*_{GS}, T^{**}_{GS}, T^*_{SBR}, T^{**}_{SBR}$
DGP_{21}	$12t$	0	0.32	8.9	None
DGP_{22}	$12t$	x	0.32	9.6	None
DGP_{23}	$12t$	$\ln(t).x$	0.30	8.9	$T^*_{GS}, \overline{T}_{GS,Adj}, T^*_{SBR}, \overline{T}_{SBR,Adj}$
DGP_{24}	$12t$	$\ln(t).\lvert x\rvert$	0.42	13.8	$T^*_{GS}, T^{**}_{GS}, T^*_{SBR}, T^{**}_{SBR}$

Here, DGP_{11}, DGP_{12}, DGP_{21} and DGP_{22} belong to the null hypothesis of PH, and DGP_{13} and DGP_{23} belong to $IHRCC$ and $ICHRCC$. DGP_{14} and DGP_{24}, on the other hand, are $IHRCC$ and $ICHRCC$ over the range $x \in [0,1]$ and $DHRCC$ and $DCHRCC$ over the range $x \in [-1,0]$. Table 1 reports, for each of the above 8 DGPs, the observed rejection rates (in percentage) of each of the test statistics, at 5 per cent confidence level, for different sample sizes. The reported percentages of rejection are based on 200 Monte Carlo simulations in each case, and asymptotic distributions are used to compute the cut-offs. The test statistics are computed based on 100 random pairs of distinct points on the covariate space ($r = 100$) in each case. For the supremum and infemum test statistics, the one-sided cut-off for the relevant extreme value approximation has been used; for the average test statistic, a two-sided test has been used. The average test statistics have been standardized using the Quenouille-Tukey jacknife estimator of variance.

The results show that the tests have reasonable power in small samples, in most cases, excepting for DGP_{24}. This is not surprising, since this DGP is $IHRCC$ over one-half of the covariate space, and $DHRCC$ over the other half. Hence, when a pair of points are drawn at random from the covariate space, only a quarter of them may be expected to reflect the $IHRCC$ character of the hazard function, and another

[12] $ICHRCC$/ $DCHRCC$ alternatives also hold in this case.

quarter would reflect the $DHRCC$ nature. When we increased the sample size to 1500, the rejection percentages for T^{*}_{GS}, T^{**}_{GS}, T^{*}_{SBR} and T^{**}_{SBR} rose to 77, 68, 83 and 61 per cent respectively. Overall, the tests appear to be fairly powerful and robust in finite samples. The results also reflect the strength of the supremum/ infemum test statistics in their ability to detect non-monotonic departures from the PH model (DGP_{14} and DGP_{24}).

5 Empirical Application

In this section, we illustrate the tests proposed in this paper by way of an application to data on durations of strikes in the US (Kennan, 1985). Several authors have analysed these data, including Kennan (1985), Kiefer (1988), Horowitz and Neumann (1992), and Neumann (1994); a major focus of the analysis is on the effect of business cycles (measured by production index) on strike duration, and this production index represents the continuous covariate in our application. Given that, strike durations are also known to exhibit some seasonal effects (Neumann, 1994), we use only the data on 292 strikes beginning in the first half of each year.

As mentioned earlier, empirical investigations of Kennan's strike data by earlier authors suggest that the level of production index significantly affects strike duration (Kennan, 1985; Neumann, 1994). Higher values of the production index were observed to be associated with higher conditional probability of ending the strike, implying significant counter cyclical pattern of strike duration. However, the PH model specifies considerably more than merely the direction of impact of the covariate on the hazard function of strike duration. In order to graphically explore whether monotone departures from the PH model exist, we first constructed Lee-Pirie plots (Lee and Pirie, 1981) of cumulative hazard functions conditional on various randomly chosen pairs of covariate values. Many of these plots indicate an increasing ratio of the hazards, as evident from the convexity (or even marginal star-shapedness) of the plot[13] (as an illustration, see Figure 1, the Lee-Pirie plot conditional on covariate values -0.048 and 0.037), lending credence to *a priori* suspicion of monotone ordering of the $IHRCC$ type. Nexy, we applied our tests of the PH model on these data (Table 2). Each of the tests were based on 150 pairs of distinct covariate values. The results of the tests indicate confirmation of our *a priori* notion; the null hypothesis of PH model is rejected in favour of the alternative $IHRCC$ (and $ICHRCC$), with production index as the continuous covariate.

This implies that the impact of production index on the hazard rate is such that, the duration distribution conditional on a higher value of the covariate is convex-ordered with respect to that conditional on a lower production index. In other words, the impact of production index on the hazard of a spell of strike duration ending increases in the duration of the strike. This has significant consequences for an economist modelling strike durations, in that (i) the PH model does not hold, (ii) that there is a systematic monotone departure from the PH model in the sense of $IHRCC$, and (iii) that the theoretical model of strike durations applied to this situation must accommodate such covariate dependence.

Further, the supremum/ infemum test statistics provide additional information on the covariate pairs for which the basic test statistics assume extreme values, which may be useful in further investigating the nature of departures from proportionality.

[13]Empirical Lee-Pirie plots often display distortions at the far end of the duration spectrum, because of sampling fluctuations. Gill and Schumacher (1987) and Sengupta, Bhattacharjee and Rajeev (1998) have, among others discussed such distortions and proposed modifications to norm out the impact of these farthest observations.

For Kennan's strike data, for example, the significant test-statistic T^{*}_{GS} is attained for the covariate pair $\{-0.0478, 0.0371\}$. The test statistic T^{**}_{GS} (covariate pair 0.0371 and 0.0675) had a p-value of 5.4 per cent, which could imply weak evidence of concave-ordering towards the upper spectrum of the covariate space (as in Example 3). This evidence can also be assessed from Figure 2, which shows a contour diagram of the standardized test statistic (smoothed using the Epanechnikov kernel), on the covariate $\times$ covariate two-dimensional plane.

The evidence from the tests and Figure 2 may lead the researcher to enquire whether the impact of the covariate is indeed monotone, or instead increasing upto a point and then decreasing. To illustrate how the researcher may incorporate this kind of covariate dependence into an econometric model, we present parameter estimates for three different models in Table 3.[14] Model 1 is a simple Cox PH model, with production index as the continuous covariate. In Model 2, we allow the effect of the covariate to be time-varying, using the histogram-sieve estimator proposed by Murphy and Sen (1991). This can accomodate monotone departures from proportionality, in the nature of $IHRCC$. In Model 3, we allow the coefficient of the covariate to vary not only over the duration, but also for covariate values. Based on the results in Table 2, we allow, in Model 3, the coefficients to be different for covariate values below and over 0.0371. Model 3 is, thus, of the type DGP_{14}, DGP_{24} or Example 3. Here again, we use the estimators given by Murphy and Sen (1991) for inference.[15]

The time- and covariate-varying nature of the parameter estimates conform to our initial intuition based on the tests, regarding the nature of covariate dependence. For lower values of the covariate, the coefficient increases in duration, and decreases in duration for higher values.

The application, thus, illustrates the usefulness of our test statistics based on the supremum and infemum in deriving inference under non-monotonic structures of covariate dependence in economic duration models. Some other applications in which similar estimation methods have been used can be found in Bhattacharjee *et. al.* (2001) and Bhalotra and Bhattacharjee (2001). The former is an application to firm exits in the UK, and the latter to child mortality in India.

6 Conclusion

In this paper, we have introduced notions of monotone ordering of duration distributions with respect to continuous covariates and proposed tests of the PH model against monotone/ ordered departures. We have thus provided a framework wherein such monotone departures can be detected. As we have seen, such departures are common in the econometric applications, and empirical or theoretical work in duration models would require to have a framework flexible enough to accomodate such covariate dependence. Our tests demonstrate reasonable small sample properties, and are useful in applications. The empirical application to strike duration data demonstrates how our tests can be used, in conjunction with available estimation methods, to derive inference in cases of monotonic and non-monotonic covariate dependence.

The proposed tests are also suitable for testing proportionality in several dimensions (multiple continuous covariates). In this case, however, it would be necessary

[14]The model estimates presented in Table 3 are only of illustrative value, and intended to suggest how one might incorporate additional information about monotonic (or non-monotonic) departures from the PH model, with respect to a continuous covariate, in an econometric duration model.

[15]Alternatively, one may use the estimators proposed by Zucker and Karr (1990), or Martinussen, Scheike and Skovgaard (2000).

to extend the notion of monotone/ ordered alternatives appropriately. For example, one could define the duration T to be $IHRCC$ with respect to continuous covariates X and Z if, whenever $x_1 > x_2$ and $z_1 > z_2$, $\lambda(t|x_1, z_1)/\lambda(t|x_2, z_2) \uparrow t$. More generally, one may define T to be $IHRCC$ with respect to X and Z if, given some function $h(.,.)$, $\lambda(t|x_1, z_1)/\lambda(t|x_2, z_2) \uparrow t$ whenever $h(x_1, z_1) > h(x_2, z_2)$. The appropriate specification of the function $h(.,.)$ may depend on the particular application the researcher is interested in, and any prior knowledge about the nature of covariate dependence. Further, one may use contour plots, as in Figure 2, to suggest the appropriate choice of the function $h(.,.)$ that may be appropriate for a particular application.

The tests proposed here can also be adapted for analysing monotone departures in k-sample (discrete covariate) problems. In this case, an *a priori* partial order of the k samples can be derived either using the usual estimates of hazard ratio (or cumulative hazard ratio) proposed by Gill and Schumacher (1987) and Sengupta, Bhattacharjee and Rajeev (1998), or the tree-structured modeling approach (Ahn and Loh, 1994). One can then test for the PH model against ordered alternatives defined in this paper. Similarly, the testing procedures in this paper can also be easily adapted to the competing risks setup where there are more than 2 competing risks.

Some promising areas of future research emerge from the work in this paper. Firstly, research can be directed towards extension of the tests to the situation where unobserved heterogeneity is present. The notion of partial ordering introduced in this paper would be valid in the presence of unobserved heterogeneity, and one can construct tests using the estimator of cumulative hazard proposed in Horowitz (1999).

Second, estimation of semiparametric regression models under monotone dependence structures appears to be an area of considerable research potential. While we have suggested using time dependent coefficient estimates proposed by Murphy and Sen (1991), Zucker and Karr (1990) or Martinussen, Scheike and Skovgaard (2000), it may be more appropriate to develop estimates under appropriate monotonicity constraints suggested by the tests. Future research in these directions would be useful.

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Appendix

Proof of Theorem 1: It follows from Gill and Schumacher (1987) that, under PH, as $n \longrightarrow \infty$,

$$\begin{aligned} \left(a^{(n)}\right)^{1/2} T_{GS}\left(x_{l1}, x_{l2}\right) &\xrightarrow{D} N\left(0, \sigma^2_{GS,l}\right), \text{ and} \\ a^{(n)}\widehat{Var}\left[T_{GS}\left(x_{l1}, x_{l2}\right)\right] &\xrightarrow{\mathcal{P}} \sigma^2_{GS,l}\ , \end{aligned}$$

$$\begin{aligned} \text{where } \sigma^2_{GS,l} &= \int_0^\tau \left[\, \bar{l}_2\left(x_{l1}, x_{l2}\right) l_1\left(x_{l1}, x_{l2}\right)(t) - \bar{l}_1\left(x_{l1}, x_{l2}\right) l_2\left(x_{l1}, x_{l2}\right)(t)\right]^2 \\ &\qquad\qquad \theta_{x_{l1}}\theta_{x_{l2}}\left(\frac{d\Lambda(t, x_{l1})}{y(t, x_{l2})} + \frac{d\Lambda(t, x_{l2})}{y(t, x_{l1})}\right), \\ \text{and } \bar{l}_i\left(x_{l1}, x_{l2}\right) &= \int_0^\tau l_i\left(x_{l1}, x_{l2}\right)(t) d\Lambda(t, x_{li}), \qquad i = 1, 2. \end{aligned}$$

so that,

$$T_{GS,std}\left(x_{l1}, x_{l2}\right) = \frac{T_{GS}\left(x_{l1}, x_{l2}\right)}{\sqrt{\widehat{Var}\left[T_{GS}\left(x_{l1}, x_{l2}\right)\right]}} \xrightarrow{D} N(0, 1), \qquad l = 1, \ldots, r.$$

The proof of the Theorem would follow, if it further holds that $T_{GS,std}\left(x_{l1}, x_{l2}\right),\ l = 1, \ldots, r$ are asymptotically independent. In other words,

$$\begin{bmatrix} T_{GS,std}\left(x_{11}, x_{12}\right) \\ T_{GS,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{GS,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \mathbf{I_r}\right),$$

where $\mathbf{I_r}$ is the identity matrix of order r.
Let

$$Z_{lij}^{(n)} = \int_0^\tau L_i(x_{l1}, x_{l2})(t) d\left\{\widehat{\Lambda}(t, x_{lj}) - \Lambda(t, x_{lj})\right\}, \qquad (i,j = 1,2; l = 1,\ldots,r).$$

Then

$$\begin{aligned}\left(a^{(n)}\right)^{1/2} Z_{lij}^{(n)} &= \left(a^{(n)}\right)^{1/2} \int_0^\tau L_i(x_{l1}, x_{l2})(t) \frac{dN(t, x_{lj}) - Y(t, x_{lj}) d\Lambda(t, x_{lj})}{Y(t, x_{lj})} \\ &\xrightarrow{D} \int_0^\tau l_i(x_{l1}, x_{l2})(t) dM(t, x_{lj}),\end{aligned}$$

where $M(t, x_{lj}),\ l = 1,\ldots,r,\ j = 1,2$ are independent Gaussian processes with zero means, independent increments and variance functions

$$Var\left[M(t, x_{lj})\right] = \int_0^\tau \frac{d\Lambda\left(s, x_{lj}\right)}{y(s, x_{lj})}.$$

This follows from a version of Rebolledo's central limit theorem (see Andersen *et. al.*, 1992), which essentially states that the innovation martingales corresponding to components of a vector counting process are orthogonal, and the vector of these martingales asymptotically converge to a Gaussian martingale.
It follows, by a version of the δ-method proved in Gill and Schumacher (1987), that

$$\left(a^{(n)}\right)^{1/2} \begin{bmatrix} T_{GS,std}\left(x_{11}, x_{12}\right) \\ T_{GS,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{GS,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} \begin{bmatrix} \sum_{i,j=1}^2 \bar{l}^{\ 1ij} \int_0^\tau l_i(x_{11}, x_{12})(t) dM(t, x_{1j}) \\ \sum_{i,j=1}^2 \bar{l}^{\ 2ij} \int_0^\tau l_i(x_{21}, x_{22})(t) dM(t, x_{2j}) \\ \vdots \\ \sum_{i,j=1}^2 \bar{l}^{\ rij} \int_0^\tau l_i(x_{r1}, x_{r2})(t) dM(t, x_{rj}) \end{bmatrix}$$

where

$$\begin{aligned}\bar{l}^{\ lij} &= (-1)^{i+j} \bar{l}_{l,3-i,3-j} \\ \text{and } \bar{l}_{lij} &= \int_0^\tau l_i(x_{l1}, x_{l2})(t) d\Lambda(t, x_{lj}); \qquad l = 1,\ldots,r; i,j = 1,2.\end{aligned}$$

Now, under $H_0 : PH, \bar{l}_{lij} = \theta_{x_{lj}} \bar{l}_i\left(x_{l1}, x_{l2}\right)$, so that

$$\begin{aligned}\sum_{i,j=1}^2 \bar{l}^{lij} \int_0^\tau l_i(x_{l1}, x_{l2})(t) dM(t, x_{lj}) &= \int_0^\tau \left[\bar{l}_{l22} l_1(x_{l1}, x_{l2})(t) - \bar{l}_{l12} l_2(x_{l1}, x_{l2})(t)\right] dM(t, x_{l1}) \\ &+ \int_0^\tau \left[-\bar{l}_{l21} l_1(x_{l1}, x_{l2})(t) + \bar{l}_{l11} l_2(x_{l1}, x_{l2})(t)\right] dM(t, x_{l2}).\end{aligned}$$

It follows that

$$\begin{bmatrix} T_{GS}\left(x_{11}, x_{12}\right) \\ T_{GS}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{GS}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \sum\right),$$

where $\sum = diag\left((\sigma^2_{GS,l})\right), l = 1, \ldots, r$, with

$$\begin{aligned}\sigma^2_{GS,l} &= \int_0^\tau \left[\bar{l}_{l22}l_1(x_{l1}, x_{l2})(t) - \bar{l}_{l12}l_2(x_{l1}, x_{l2})(t)\right]^2 \frac{d\Lambda(t, x_{l1})}{y(t, x_{l1})} \\ &\quad + \int_0^\tau \left[-\bar{l}_{l21}l_1(x_{l1}, x_{l2})(t) + \bar{l}_{l11}l_2(x_{l1}, x_{l2})(t)\right]^2 \frac{d\Lambda(t, x_{l2})}{y(t, x_{l2})}.\end{aligned}$$

Further, following Gill and Schumacher (1987), it can be shown that $\sigma^2_{GS,l}$ can be consistently estimated by $\widehat{Var}\left[T_{GS}\left(x_{l1}, x_{l2}\right)\right]$. Hence, it follows that

$$\begin{bmatrix} T_{GS,std}\left(x_{11}, x_{12}\right) \\ T_{GS,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{GS,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \mathbf{I_r}\right),$$

where $\mathbf{I_r}$ is the identity matrix of order r.

Proofs of (a), (b) and (c) follow.

♢

<u>*Proof of Corollary 1*</u>: Proof follows from the well known result in extreme value theory regarding the asymptotic distribution of the maximum of a sample of iid $N(0, 1)$ variates (see, for example, Berman, 1992), and invoking the δ-method by noting that maxima and minima are continuous functions.

♢

<u>*Proof of Theorem 2*</u>: It follows from Sengupta, Bhattacharjee and Rajeev (1998) that, under H_0, as $n \longrightarrow \infty$,

$$\begin{aligned}\left(a^{(n)}\right)^{1/2} T_{SBR}\left(x_{l1}, x_{l2}\right) &\xrightarrow{D} N\left(0, \sigma^2_{SBR,l}\right), \text{ and} \\ a^{(n)}\widehat{Var}\left[T_{SBR}\left(x_{l1}, x_{l2}\right)\right] &\xrightarrow{\mathcal{P}} \sigma^2_{SBR,l},\end{aligned}$$

$$\begin{aligned}\text{where } \sigma^2_{SBR,l} &= \int_0^\tau \int_0^\tau \left[\, c(t)c(s)V\left(\min(s,t), x_{l1}\right) + d(t)d(s)V\left(\min(s,t), x_{l2}\right)\right] dsdt, \\ V\left(t, x_{lj}\right) &= \int_0^\tau \frac{d\Lambda\left(s, x_{lj}\right)}{y(s, x_{lj})}, \qquad j = 1, 2, \\ c(t) &= s_2\left(x_{l2}\right) k_1\left(x_{l1}, x_{l2}\right)(t) - s_1\left(x_{l2}\right) k_2\left(x_{l1}, x_{l2}\right)(t), \\ d(t) &= s_2\left(x_{l1}\right) k_1\left(x_{l1}, x_{l2}\right)(t) - s_1\left(x_{l1}\right) k_2\left(x_{l1}, x_{l2}\right)(t), \\ \text{and } s_i\left(x_{lj}\right) &= \int_0^\tau k_i\left(x_{l1}, x_{l2}\right)(s).\Lambda(s, x_{lj})ds, \qquad i = 1, 2, j = 1, 2.\end{aligned}$$

so that,

$$T_{SBR,std}\left(x_{l1}, x_{l2}\right) = \frac{T_{SBR}\left(x_{l1}, x_{l2}\right)}{\sqrt{\widehat{Var}\left[T_{SBR}\left(x_{l1}, x_{l2}\right)\right]}} \xrightarrow{D} N(0, 1), \qquad l = 1, \ldots, r.$$

Like Theorem 1, the proof will follow, if it further holds that

$$\begin{bmatrix} T_{SBR,std}\left(x_{11}, x_{12}\right) \\ T_{SBR,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{SBR,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \mathbf{I_r}\right),$$

where $\mathbf{I_r}$ is the identity matrix of order r.
The essential difference in the arguments required here to establish asymptotic distributions, from those in Theorem 1, lie in the fact that the integrals considered in Theorem 1 are transformations of stochastic integrals, while here, they are functions of ordinary Steiljes integrals of stochastic processes. Hence, we require different resuts to establish the asymptotic properties.
Let us define

$$Z_{lij}^{*(n)} = \int_0^\tau K_i(x_{l1}, x_{l2})(t) \left\{ \widehat{\Lambda}(t, x_{lj}) - \Lambda(t, x_{lj}) \right\} \, dt, \qquad (i, j = 1, 2; l = 1, \ldots, r).$$

Then, by Rebolledo's central limit theorem and Theorem 3.1 in Sengupta, Bhattacharjee and Rajeev (1998), we have, as $n \longrightarrow \infty$,

$$\left(a^{(n)}\right)^{1/2} Z_{lij}^{*(n)} \xrightarrow{D} \int_0^\tau k_i(x_{l1}, x_{l2})(t) \; M(t, x_{lj}) \; dt,$$

where $M(t, x_{lj}), l = 1, \ldots, r, j = 1, 2$ are independent Gaussian processes with zero means, independent increments and variance functions

$$Var\left[M(t, x_{lj})\right] = \int_0^\tau \frac{d\Lambda\left(s, x_{lj}\right)}{y(s, x_{lj})}.$$

Now, as in Theorem 1, invoking the δ-method of Gill and Schumacher (1987), it follows that

$$\left(a^{(n)}\right)^{1/2} \begin{bmatrix} T_{SBR,std}\left(x_{11}, x_{12}\right) \\ T_{SBR,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{SBR,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} \begin{bmatrix} \sum_{i,j=1}^2 \overline{k}^{\,1ij} \int_0^\tau k_i(x_{11}, x_{12})(t) \; M(t, x_{1j}) \; dt \\ \sum_{i,j=1}^2 \overline{k}^{\,2ij} \int_0^\tau k_i(x_{21}, x_{22})(t) \; M(t, x_{2j}) \; dt \\ \vdots \\ \sum_{i,j=1}^2 \overline{k}^{\,rij} \int_0^\tau k_i(x_{r1}, x_{r2})(t) \; M(t, x_{rj}) \; dt \end{bmatrix}$$

where

$$\begin{aligned} \overline{k}^{\,lij} &= (-1)^{i+j} \overline{k}_{l,3-i,3-j} \\ \text{and } \overline{k}_{lij} &= \int_0^\tau k_i(x_{l1}, x_{l2})(t) \Lambda(t, x_{lj}) dt; \qquad l = 1, \ldots, r; i, j = 1, 2, \end{aligned}$$

and under H_0,

$$\begin{aligned} \sum_{i,j=1}^2 \overline{k}^{\,lij} \int_0^\tau k_i(x_{l1}, x_{l2})(t) \; M(t, x_{lj}) \; dt \quad &= \int_0^\tau \left[\overline{k}_{l22} k_1(x_{l1}, x_{l2})(t) - \overline{k}_{l12} k_2(x_{l1}, x_{l2})(t) \right] M(t, x_{l1}) dt \\ &+ \int_0^\tau \left[-\overline{k}_{l21} k_1(x_{l1}, x_{l2})(t) + \overline{k}_{l11} k_2(x_{l1}, x_{l2})(t) \right] M(t, x_{l2}) dt. \end{aligned}$$

As in Theorem 1, it follows that

$$\begin{bmatrix} T_{SBR}\left(x_{11}, x_{12}\right) \\ T_{SBR}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{SBR}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \sum\right),$$

where $\sum = diag\left(\left(\sigma^2_{SBR,l}\right)\right), l = 1, \ldots, r$, and following Sengupta, Bhattacharjee and Rajeev (1998), it can be shown that $\sigma^2_{SBR,l}$ can be consistently estimated by $\widehat{Var}\left[T_{SBR}\left(x_{l1}, x_{l2}\right)\right]$. Hence, it follows that

$$\begin{bmatrix} T_{SBR,std}\left(x_{11}, x_{12}\right) \\ T_{SBR,std}\left(x_{21}, x_{22}\right) \\ \vdots \\ T_{SBR,std}\left(x_{r1}, x_{r2}\right) \end{bmatrix} \xrightarrow{D} N\left(\underline{\mathbf{0}}, \mathbf{I_r}\right),$$

where $\mathbf{I_r}$ is the identity matrix of order r.

Proofs of (a), (b) and (c) follow.

$\diamondsuit$

Proof of Corollary 2: Proof follows from extreme value theory and the δ-method, as in Corollary 1.

$\diamondsuit$

x

TABLE 1:
REJECTION RATES (%) AT THE 5% ASYMPTOTIC CONFIDENCE LEVEL

Model	Test statistic	Sample size			
		100	**200**	**500**	**1000**
DGP_{11}	T^{*}_{GS}	19.0	18.0	4.5	4.0
	T^{**}_{GS}	23.0	13.5	6.0	4.5
	$\overline{T}_{GS,Adj}$	4.0	4.5	11.0	7.0
	T^{*}_{SBR}	13.0	7.0	4.0	1.5
	T^{**}_{SBR}	13.0	10.0	6.5	3.5
	$\overline{T}_{SBR,Adj}$	5.0	2.0	9.0	5.0
DGP_{12}	T^{*}_{GS}	19.5	16.0	9.5	1.0
	T^{**}_{GS}	18.0	11.5	6.0	3.5
	$\overline{T}_{GS,Adj}$	12.0	12.5	15.5	4.5
	T^{*}_{SBR}	13.5	7.5	4.0	4.0
	T^{**}_{SBR}	17.0	5.0	5.5	3.0
	$\overline{T}_{SBR,Adj}$	5.5	16.5	13.5	10.0
DGP_{13}	T^{*}_{GS}	52.0	83.5	100.0	100.0
	T^{**}_{GS}	12.0	6.0	0.5	0.0
	$\overline{T}_{GS,Adj}$	37.5	100.0	100.0	100.0
	T^{*}_{SBR}	84.0	100.0	100.0	100.0
	T^{**}_{SBR}	4.5	0.0	0.0	0.5
	$\overline{T}_{SBR,Adj}$	41.5	100.0	100.0	100.0
DGP_{14}	T^{*}_{GS}	31.0	33.0	57.5	89.5
	T^{**}_{GS}	29.5	41.0	70.5	94.5
	$\overline{T}_{GS,Adj}$	15.5	12.0	7.5	10.0
	T^{*}_{SBR}	10.5	21.0	39.5	87.0
	T^{**}_{SBR}	21.0	33.0	72.0	97.5
	$\overline{T}_{SBR,Adj}$	9.5	13.5	9.5	8.5
DGP_{21}	T^{*}_{GS}	13.0	19.0	7.0	3.0
	T^{**}_{GS}	21.5	14.0	7.0	4.0
	$\overline{T}_{GS,Adj}$	5.5	5.5	3.5	2.0
	T^{*}_{SBR}	11.5	9.5	3.0	3.0
	T^{**}_{SBR}	13.0	6.5	4.0	3.0
	$\overline{T}_{SBR,Adj}$	15.0	3.5	5.0	5.0
DGP_{22}	T^{*}_{GS}	29.0	20.5	5.5	6.0
	T^{**}_{GS}	16.5	10.5	2.0	2.0
	$\overline{T}_{GS,Adj}$	5.5	8.0	3.0	4.5
	T^{*}_{SBR}	12.5	11.5	4.0	3.5
	T^{**}_{SBR}	12.0	8.5	4.0	3.0
	$\overline{T}_{SBR,Adj}$	3.0	7.5	3.5	8.0
DGP_{23}	T^{*}_{GS}	33.0	49.5	100.0	100.0
	T^{**}_{GS}	13.5	5.5	2.0	2.0
	$\overline{T}_{GS,Adj}$	76.0	92.0	100.0	100.0
	T^{*}_{SBR}	14.5	26.5	100.0	100.0
	T^{**}_{SBR}	4.0	2.0	0.0	0.0
	$\overline{T}_{SBR,Adj}$	87.5	98.5	100.0	100.0
DGP_{24}	T^{*}_{GS}	24.5	23.5	22.0	44.0
	T^{**}_{GS}	21.0	22.5	22.0	46.0
	$\overline{T}_{GS,Adj}$	0.0	10.5	11.0	5.5
	T^{*}_{SBR}	11.5	15.0	11.0	25.5
	T^{**}_{SBR}	14.0	18.5	28.5	56.5
	$\overline{T}_{SBR,Adj}$	1.0	10.0	14.0	4.5

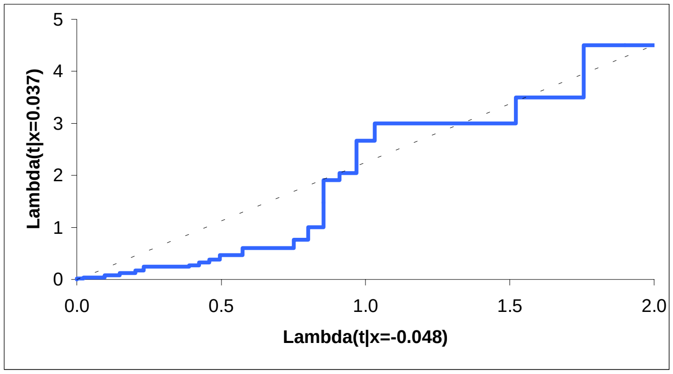

Figure 1:

Lee-Pirie Plot of $\widehat{\Lambda}(t|x = 0.037)$ **versus** $\widehat{\Lambda}(t|x = -0.048)$

TABLE 2:
Tests of the PH model: Strike Duration data

Test	**Test Statistic**	**P-Value** (%)
T^{*}_{GS}	3.619	3.0
T^{**}_{GS}	-3.426	5.4
$\overline{T}_{GS,Adj}$	4.093	0.0
T^{*}_{SBR}	3.415	5.6
T^{**}_{SBR}	-2.703	42.0
$\overline{T}_{SBR,Adj}$	3.808	0.0

TABLE 3:
MODEL ESTIMATES: STRIKE DURATION DATA

Model/ Parameter	**Coefficient**	**t-stat.**
MODEL 1		
Production Index, x	3.529	3.17
MODEL 2		
$x.I\left[t \in [0, 75)\right]$	5.179	3.90
$x.I\left[t \in [75, 150)\right]$	0.360	0.27
$x.I\left[t \in [150, \infty)\right]$	9.416	1.19
MODEL 3		
$x.I\left[x \in (-\infty, 0.037)\right].I\left[t \in [0, 75)\right]$	-1.178	-0.75
$x.I\left[x \in (-\infty, 0.037)\right].I\left[t \in [75, 150)\right]$	9.362	4.32
$x.I\left[x \in (-\infty, 0.037)\right].I\left[t \in [150, \infty)\right]$	45.266	3.43
$x.I\left[x \in [0.037, \infty)\right].I\left[t \in [0, 75)\right]$	10.173	4.96
$x.I\left[x \in [0.037, \infty)\right].I\left[t \in [75, 150)\right]$	-14.910	-5.96
$x.I\left[x \in [0.037, \infty)\right].I\left[t \in [150, \infty)\right]$	-27.619	-5.90

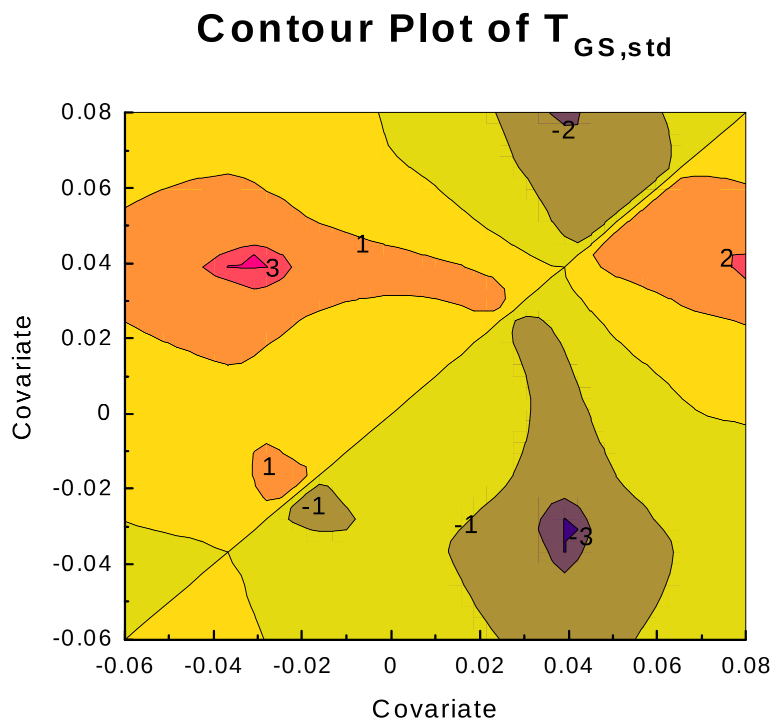
Contour Plot of T GS,std
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
Covariate
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
Covariate
1
3
-2
2
1
-1
-1
-3

Figure 2: