Initial and Boundary Value Problems in Two and Three Dimensions

Konstantinos Kalimeris

Trinity College, Cambridge.

A Dissertation submitted for
the degree of Doctor of Philosophy
at the University of Cambridge

30 August 2009

First and foremost, I would like to express my gratitude to my supervisor Thanasis Fokas. His intuition, insight and precision have always been a great inspiration in my attempt to become a researcher.

My research and studies in the University of Cambridge were supported by a student scholarship from the "John S. Latsis Ileians Scholarships Foundation". Also, my tuition fees were paid by EPSRC and I was receiving a part-cost bursary from "George & Marie Vergottis Cambridge Bursaries". I hope they are all satisfied by the outcome of their investments.

Also I would like to sincerely thank George Dassios for the knowledge and advice he shared with me, both mathematical and non-mathematical, but mostly for his wholehearted support in both my undergraduate and graduate studentship.

At this point I wish to extend my thanks to the members of "the Fokas group": First of all, Euan Spence and Michail Dimakos for the interesting (and sometimes enthusiastic) conversations we had regarding many mathematical ideas, speculations and facts. Furthermore, I would like to thank Dionyssios Mantzavinos, Anthony Ashton, and Jonatan Lenells. I have enjoyed all our discussions.

Also, thanks should go to Stefanos and Stergios who seem to always have an elegant answer to my mathematical questions.

Special thanks must be attributed to the "Kastritsi people". Especially to Psilos and Bogias for the periods of relaxation and Totis, Andrew and Nikolas for the times of great fun. Kokos and Ilias were the companions in my life who were willing to cope with any issue that seemed to be a dead-end and conclude with something more than a hope. More special thanks are attributed to Constantinos for the nice time we had in all the places we have been and the greatest thanks go to Anatoli for supporting me in any aspect of my life, ignoring the potential cost.

Finally, I would like to thank my parents and my sister for their contribution in my becoming an independent and responsible person.

Declaration

This dissertation is based on research done at the Department of Applied Mathematics and Theoretical Physics from October 2005 to June 2009.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Konstantinos Kalimeris

Cambridge,

30 August 2009

Abstract

This thesis: (a) presents the solution of several boundary value problems (BVPs) for the Laplace and the modified Helmholtz equations in the interior of an equilateral triangle; (b) presents the solution of the heat equation in the interior of an equilateral triangle; (c) computes the eigenvalues and eigenfunctions of the Laplace operator in the interior of an equilateral triangle for a variety of boundary conditions; (d) discusses the solution of several BVPs for the non-linear Schrödinger equation on the half line.

In 1967 the Inverse Scattering Transform method was introduced; this method can be used for the solution of the initial value problem of certain integrable equations including the celebrated Korteweg-de Vries and nonlinear Schrödinger equations. The extension of this method from initial value problems to BVPs was achieved by Fokas in 1997, when a unified method for solving BVPs for both integrable nonlinear PDEs, as well as linear PDEs was introduced. This thesis applies "the Fokas method" to obtain the results mentioned earlier.

For linear PDEs, the new method yields a novel integral representation of the solution in the spectral (transform) space; this representation is not yet effective because it contains certain unknown boundary values. However, the new method also yields a relation, known as "the global relation", which couples the unknown boundary values and the given boundary conditions. By manipulating the global relation and the integral representation, it is possible to eliminate the unknown boundary values and hence to obtain an effective solution involving only the given boundary conditions. This approach is used to solve several BVPs for elliptic equations in two dimensions, as well as the heat equation in the interior of an equilateral triangle.

The implementation of this approach: (a) provides an alternative way for obtaining classical solutions; (b) for problems that can be solved by classical methods, it yields

novel alternative integral representations which have both analytical and computational advantages over the classical solutions; (c) yields solutions of BVPs that apparently cannot be solved by classical methods.

In addition, a novel analysis of the global relation for the Helmholtz equation provides a method for computing the eigenvalues and the eigenfunctions of the Laplace operator in the interior of an equilateral triangle for a variety of boundary conditions.

Finally, for the nonlinear Schrödinger on the half line, although the global relation is in general rather complicated, it is still possible to obtain explicit results for certain boundary conditions, known as "linearizable boundary conditions". Several such explicit results are obtained and their significance regarding the asymptotic behavior of the solution is discussed.

Contents

Chapter 1

Introduction.

1.1 The problems.

In this thesis the following PDEs are discussed:

(1) The second order linear elliptic PDEs in two spatial dimensions

$$q_{xx}(x,y) + q_{yy}(x,y) + 4\lambda q(x,y) = 0 \quad (x,y) \in D, \tag{1.1.1}$$

where λ is a *complex* constant and D is some 2 dimensional domain with piecewise smooth boundary. For $\lambda = 0$ this is the Laplace equation, $\lambda > 0$ the Helmholtz equation, $\lambda < 0$ the modified Helmholtz equation and otherwise the "generalized Helmholtz" equation.

(2) The heat equation, which is a second order linear evolution PDE, in three dimensions

$$q_t - q_{x_1x_1} - q_{x_2x_2} = f, \quad (x_1, x_2) \in D, \ 0 < t < T, \tag{1.1.2}$$

where $f(x_1, x_2, t)$ is a given function and D is some 2 dimensional domain with piecewise smooth boundary.

(3) The nonlinear Schrödinger (NLS) equation on the half line

$$iq_t + q_{xx} - 2\lambda|q|^2 q = 0, \ \lambda = \pm 1, \quad 0 < x < \infty, \ t > 0. \tag{1.1.3}$$

For the first class of equations several classes of Boundary Value problems (BVP) are solved explicitly, when D is an equilateral triangle. Moreover, the Helmholtz equation($\lambda > 0$) provides the eigenvalues and eigenfunction of the associated Laplace operator($\lambda = 0$). The boundary value problems analyzed have the following boundary conditions

- Dirichlet: $q(x, y) = \text{known}, \quad (x, y) \in \partial D$
- Neumman: $\frac{\partial q}{\partial N}(x, y) = \text{known}, \quad (x, y) \in \partial D$
- Robin: $\frac{\partial q}{\partial N}(x, y) - \chi q(x, y) = \text{known}, \ \chi \text{ constant}, \quad (x, y) \in \partial D$
- oblique Robin: $\sin\delta \frac{\partial q}{\partial N}(x, y) + \cos\delta \frac{\partial q}{\partial T}(x, y) - \chi q(x, y) = \text{known}, \quad \delta, \ \chi \text{ constants}, \ (x, y) \in \partial D$
- Poincaré: $\sin\delta_j \frac{\partial q}{\partial N}(x, y) + \cos\delta_j \frac{\partial q}{\partial T}(x, y) - \chi_j q(x, y) = \text{known}, \quad \delta_j, \ \chi_j \text{ constants}, \ (x, y) \in \partial D,$

where $\frac{\partial q}{\partial N} = \nabla q \cdot N$, N is the unit outward-pointing normal vector to D, $\frac{\partial q}{\partial T} = \nabla q \cdot T$, T is the unit tangent vector to ∂D; the terminology "oblique Robin" can be justified by rewriting the relevant condition as

$$(\sin\delta, \cos\delta) \cdot \left(\frac{\partial q}{\partial T}(x, y), \frac{\partial q}{\partial N}(x, y) \right) - \chi q(x, y) = 0, \quad (x, y) \in \partial D$$

thus it involves the derivative of q in the direction making an angle δ with the tangent vector on the boundary, i.e. with every side of the equilateral triangle; the Poincaré condition describes the case when there exist different oblique Robin conditions in each piece of the piecewise smooth boundary, i.e. in each side of the equilateral triangle, see Figure 1.1.

Similar considerations are valid for the Initial Boundary Value problems (IBVP) for the heat equation in the equilateral triangle; in this case we mainly analyze the Dirichlet problem, i.e.

$$q(x_1, x_2, t) = \text{known}, \quad (x_1, x_2, t) \in \partial\mathcal{T}, \ \text{where } \mathcal{T} = \{x_1, x_2 \in D, \ 0 < t < T\}.$$

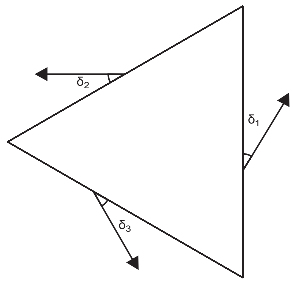

Figure 1.1:

For the NLS equation, the following types of boundary conditions, the so-called "linearizable", are discussed:

$$q(0,t) = 0; \quad q_x(0,t) = 0; \quad q_x(0,t) - \chi q(0,t) = 0, \quad \chi \in \mathbb{R}^*. \tag{1.1.4}$$

Furthermore, we will analyze three classes of Initial Boundary Value problems (IBVP); these problems involve one of the boundary conditions (1.1.4), as well as initial conditions characterized by the following functions: (a) a soliton evaluated at $t = 0$; (b) a function describing a hump; and (c) an exponential function.

All these problems, i.e. BV and IBV for both linear and integrable nonlinear PDEs are analyzed by the unified method, called the "Fokas method" introduced in [1], in 1997; it was further developed by several authors, see for example [2], [3], [4], [5], [6], [7], [8] and the monograph [9].

1.2 Classical theory and techniques.

In this section we review briefly the classical theory for solving (1.1.1) for $\lambda \in \mathbb{R}$. We discuss only the techniques which can be applied to the boundary value problems considered in the thesis. [10] provides an excellent survey of both these techniques and many other exact and approximate methods for solving boundary value problems for linear PDEs.

1.2.1 Green's integral representation.

Green's theorem gives an integral representation of the solution of (1.1.1), involving the fundamental solution (sometimes known as the free space Green's function) and both the known and unknown boundary values. We note that a drawback for both Helmholtz and modified Helmholtz in 2-d is that the fundamental solution is given as a special function.

In order to formulate, for instance, an integral representation of the solution of (1.1.1) for the Dirichlet problem one should first determine the Green's function for the corresponding domain, i.e.,

$$\begin{aligned} (\Delta_y + 4\lambda)G(y,x) &= \delta(y-x), \quad y \in \Omega \\ G(y,x) &= 0, \quad y \in \partial\Omega. \end{aligned} \tag{1.2.1}$$

Alternatively, if the eigenvalues and eigenfunctions of the Laplacian are known in Ω then the problem is solved since the Greens function can be constructed as an *infinite* sum of the eigenfunctions.

1.2.2 Separation of variables.

Start with a given boundary value problem in a separable domain (one where $\Omega = \{a_1 \le x_1 \le b_1\} \times \{a_2 \le x_2 \le b_2\}$ where x_j are the co-ordinates under which the differential operator is separable). This method involves the separation of the PDE into two ODEs and the derivation of the associated completeness relation (i.e. transform pair) depending on the boundary conditions for one of the ODEs. Then the solution of the boundary value problem is given as a superposition of eigenfunctions of this ODE.

Some of the main limitations of this method for solving boundary value problems are the following:

- It fails for BVPs with non-separable boundary conditions (for example, those which include a derivative at an angle to the boundary).
- The appropriate transform depends on the boundary conditions and so the process must be repeated for different boundary conditions.
- The solution is not uniformly convergent on the whole boundary of the domain (since it is given as a superposition of eigenfunctions of one of the ODEs).

In the author's opinion the best references on separation of variables are: [11] volume 1 chapter 4 (spectral analysis of differential operators), [12] paragraph 5.1 (separable coordinates), [13] chapter 4 (spectral analysis), chapter 5 (transforms and switching between the alternative representations), [14] chapter 7 (spectral analysis) chapter 8 paragraph 8.1.3 (transform methods), [15] paragraphs 4.4, 5.7, 5.8 (transform methods), [16] and [17].

1.2.3 The method of images/reflections.

This technique can be used to find either the Green's function or the eigenfunctions and eigenvalues. The domains on which this technique works are the half plane, the infinite strip, the semi-infinite strip, the wedge of angle π/n, $n \in Z^+$, the rectangle and three types of triangles (the equilateral, the right isosceles and the 30-60-90 right triangle).

This applies to Dirichlet and Neumann boundary conditions, as well as some mixed boundary conditions where Dirichlet conditions are posed on part of the boundary and Neumann conditions on the rest (the mixed boundary conditions which are allowed for each domain are detailed in [18]). For all the domains except for the half plane and wedge, an infinite number of images is required, and so the Green's function is given as an infinite sum. The extension of the method to Robin and oblique Robin boundary conditions in the upper half plane is given in [19] and [20]. The Green's function is given as the source, plus one image, plus an semi-infinite line of images. Robin and oblique Robin boundary conditions in a wedge of angle $\pi/n,\ n \in Z^+$ are considered in [21]. For the Robin problem the Greens function is given as a source point, plus infinite lines of images, plus infinite regions of images. The oblique Robin problem can only be solved if n is odd and under some restrictions on the angle of derivative in the boundary conditions (this is to ensure no images lie inside the domain).

For the four bounded domains mentioned above, the method of images can be used to find their eigenfunctions and eigenvalues under Dirichlet, or Neumann, or some mixed Dirichlet-Neumann boundary conditions (the same ones for which the Green's function can be found) by reflecting to one of

- the whole space [22], [23]
- a parallelogram [24],
- a rectangle [25],

where one can use separation of variables in cartesian co-ordinates, then reflecting back. This reflection technique does *not* work for Robin or more complex boundary conditions.

Some references that have interesting results concerning the method of images in polar co-ordinates are [26] and [27].

1.2.4 Conformal mapping.

The Laplace equation has the unique property that the Dirichlet and Neumann problems can be solved using conformal mapping, in particular Schwarz-Christoffel mapping. When the mapping function is given explicitly, this gives an integral representation of the solution. However, this is *not* the case for the equilateral triangle in section 2.2, where inversion of special functions is involved.

The other classical techniques, and the Fokas method, become competitive when more general boundary conditions, such as Robin, are prescribed, which cannot be solved by conformal mapping. Similar advantages of the Fokas method appear in the modified Helmholtz and Helmholtz equations.

1.3 The Fokas method.

The Fokas method has the following basic ingredients:

(1) the **global relation**, which is an algebraic equation that involves certain transforms of all initial and boundary values; the existence of these transforms justifies the terminology "global" relation.

(2) the **integral representation** of the solution, given in terms of the global form of all the initial and boundary values.

Firstly, we will illustrate how the Fokas method works for linear PDEs:

- Given a PDE, construct a *scalar* differential form which is closed iff the PDE is satisfied.
- From this differential form define two compatible linear eigenvalue equations with *scalar* eigenfunctions, which are called a Lax pair.
- On the one hand, by employing Green's theorem, this differential form yields the global relation, which is an *algebraic* equation coupling the relevant spectral functions.
- On the other hand, the simultaneous spectral analysis of both parts of the Lax pair yields a *scalar* Riemann-Hilbert problem, which consequently yields the relevant integral representation of the solution in terms of the spectral functions.

Finally, the explicit solution of the associated problem is derived through the elimination of the unknown boundary values in the integral representation, by using appropriately the global relation.

The situation in the nonlinear PDEs is conceptually similar, but more complicated. Now, we construct a *matrix* differential form, which yields a Lax pair containing matrix eigenfunctions. This implies that the spectral functions are not given explicitly by the relevant initial and boundary values(they are given as the solutions of linear integral equations of the Volterra type). Furthermore, the integral representation of the solution is given through a matrix Riemann-Hilbert problem which cannot be solved in closed form(its solution is characterized by a linear integral equation of Fredholm type). However, there exist certain class of boundary conditions, called "linearizable", for which the unknown spectral functions can be obtained through the algebraic manipulation of the global relation.

1.4 Achievements of the thesis.

Boundary value problems for $q_{\bar{z}} = 0$ and the Modified Helmholtz equation were solved in [28], [29], [30] and [31]. Solutions in terms of infinite series have been derived for several problems of the Laplace, Helmholtz and modified Helmholtz equations in the interior of an equilateral triangle in [32] and for the Laplace equation in the interior of a right isosceles triangle in [33], employing the Fokas method; this is to be contrasted to other techniques based on the eigenvalues of the relevant operators that yield the solution as a bi-infinite series.

The eigenvalues of the Laplace operator for the Dirichlet, Neumann and Robin problems in the interior of an equilateral triangle were first obtained by Lamé in 1833 [34]. Completeness for the associated expansions for the Dirichlet and Neumann problems was obtained in [23], [24], [35], [25] using group theoretic techniques. Completeness for the associated expansion for the Robin problem was achieved in [36] using a homotopy argument. These results have been rederived by several authors, see for example [37]-[38].

The classical problem of the heat equation is solved in several ways in separable domains, but for non-separable has been mainly related with the results obtained for the modified Helmholtz equation, through the Laplace transform. Moreover, the Fokas method was extended to evolution PDEs in two spatial dimensions in [39] and [40].

The integral representations of the initial-boundary value problems on the half line, applied on the NLS, the sine-Gordon(sG) and the Korteweg-de Vries(KdV), were derived in [3] and [2]. Furthermore, the linearizable boundary conditions were obtained for each one of the equations. These results were reviewed in [9].

Considering these problems, the main achievements of this thesis are:

- The solutions of the same problems with those considered in [32], for the Laplace and modified Helmholtz equations in the interior of an equilateral triangle(non-separable domain), are now given as an integral(as opposed to an infinite sum in [32], [33], and a bi-infinite sum classically). Furthermore, a novel approach has been introduced which employs the global relation at the same time that the contours of the integral representation are being deformed. As a result, the integrands of the relevant integrals are exponentially decaying functions; this has analytical and numerical advantages.

- A specific choice for the contours of integration in the integral representation and Cauchy's theorem, yields the solution in terms of an infinite series of the relevant residues, which provides a relationship between the discrete and the continuous spectrum of these problems.

- The integral representation of the generalized Helmholtz equation in the interior of a convex polygon is given for the first time; this is also the case for the solution of the Dirichlet problem in the interior of an equilateral triangle. These results are interesting, in particular taking into consideration the relation of this equation with certain evolution PDEs in higher dimensions.

- Regarding the eigenvalues of the Laplace operator a simple, unified approach for rederiving the previous results is presented. Furthermore the eigenvalues for the oblique Robin and certain Poincaré problems are derived for the first time. The method introduced here is based on the analysis of the global relation, see [7]. In addition, combining these results with the integral representation of the solution of the Helmholtz equation, yields the corresponding eigenfunctions.

- The solution of the heat equation in an equilateral triangle is expressed as an integral in the complex Fourier space, i.e. the complex k_1 and k_2 planes, involving appropriate integral transforms of the known boundary conditions. Moreover, the solution is expressed in terms of an integral whose integrand decays exponentially as $|k| \to \infty$. Hence, it is possible to evaluate this integral numerically in an efficient and straightforward manner.

- The distribution of zeros of the spectral functions of the linearizable boundary value problems for the NLS yields the explicit asymptotic behavior of the solution. In particular, it yields the number of solitons generated from the given initial and boundary conditions.

1.5 Structure of the thesis.

Chapter 2: Linear Elliptic Equations in an Equilateral Triangle.

- We solve:
  - ◇ Laplace equation in an equilateral triangle for symmetric Dirichlet (the *same* function is prescribed in all three sides), as well as arbitrary Dirichlet boundary conditions.
  - ◇ modified Helmholtz equation in an equilateral triangle for symmetric Dirichlet and Poincaré boundary conditions.

    - ◇ generalized Helmholtz equation in an equilateral triangle for symmetric Dirichlet boundary conditions.

- Particular cases of the Poincaré problem yield the solution of other problems, e.g. oblique Robin, Robin and Neumann.

- Common characteristics appear in the solution of all the above problems.

- The solution is given in terms of integrals that have exponentially decaying integrands on the contours of integration.

Chapter 3: Eigenvalues for the Laplace operator in the interior of an equilateral triangle.

- We find explicitly the eigenvalues of the Laplace operator for the Dirichlet and the Neumann problems in the equilateral triangle.

- We derive explicit formulae for the computation of the eigenvalues of the Laplace operator for the Robin, the oblique Robin and certain Poincaré problems in the equilateral triangle.

- The formulae for Poincaré problem, yield the relevant eigenvalues of all other problems, via particular limits.

- We find the eigenfunctions of the Laplace operator for the Dirichlet problem and also indicate how the eigenfunctions for all other problems can be computed.

Chapter 4: The heat equation in the interior of an equilateral triangle.

- We solve the heat equation in the interior of an equilateral triangle for symmetric Dirichlet and arbitrary Dirichlet boundary conditions. this is achieved by employing similar techniques with those used in Chapter 2.

- The solution is given in terms of integrals that have exponentially decaying integrands on the contours of integration.

Chapter 5: Explicit soliton asymptotics for the nonlinear Schrödinger equation on the half-line.

- A review of the Fokas method is given, in connection with initial and boundary value problems for nonlinear integrable PDEs on the half line; emphasis is placed in the NLS.
- The linearizable boundary conditions, for which the unknown spectral functions are computed via algebraic manipulation of the global relation, are derived; furthermore, for this class of boundary conditions three initial-boundary value problems are analyzed. These problems are characterized by the following initial conditions:
  - ◇ a soliton evaluated at $t = 0$;
  - ◇ a function describing a hump;
  - ◇ an exponential function.
- The analysis of the spectral functions yields effective asymptotic results using the Deift-Zhou techniques for the asymptotic analysis of the relevant Riemann-Hilbert problem, see [41].

Chapter 2

Linear elliptic equations in an equilateral triangle.

Below, we describe the solutions of some boundary value problems for the basic elliptic equations using the Fokas method, introduced in [1]. For linear PDEs, this method involves the following steps(see [9]):

(1) Given a PDE, construct a differential form which is closed iff the PDE is satisfied.

(2) From this differential form define two compatible linear eigenvalue equations which, in analogy with the theory of nonlinear integrable PDEs, are called a **Lax pair**.

(3) Employing Green's theorem in this differential form yields a relation between certain functions $\hat{q}_j(k)$, called the spectral functions; these functions are certain integrals of the values of q and of its derivatives on the boundary of the domain. From now on we will refer to this relation as the **"global relation"**.

(4) Perform the simultaneous spectral analysis of the Lax pair, which yields an integral representation of the solution $q(z, \bar{z})$ in terms of the spectral functions $\hat{q}(k)$.

(5) Given appropriate boundary conditions, use the invariants of the global relation to eliminate the unknown boundary values appearing in the integral formula obtained in (4).

The implementation of the approach presented here has certain novel features. In particular, it constructs the solution in terms of integrals which involve integrands that have strong decay as $|k| \to \infty$. This is to be contrasted with earlier investigations (see [32]) where the solution was expressed in terms of a combination of an infinite series and integrals with oscillating kernels.

2.1 The problems.

We implement this approach to the Laplace, modified Helmholtz and "generalized Helmholtz" equations for some boundary value problems in the interior of an equilateral triangle.

(a) Fundamental Domain

Let $D \subset \mathbb{C}$ be the interior of the equilateral triangle depicted in Figure 2.1 and defined by its three vertices (z_1, z_2, z_3) ,

$$z_1 = \frac{l}{\sqrt{3}} e^{\frac{-i\pi}{3}},\ z_2 = \bar{z}_1,\ z_3 = -\frac{l}{\sqrt{3}}, \tag{2.1.1}$$

where l is the length of the side.

The sides $(z_1, z_2),\ (z_2, z_3),\ (z_3, z_1)$ will be referred as sides (1),(2),(3).

The complex variable z, on each of the sides (1),(2),(3), satisfies the following relations:

$$\frac{dz}{ds}^{(1)}(s) = i, \quad \frac{dz}{ds}^{(2)}(s) = ia, \quad \frac{dz}{ds}^{(3)}(s) = i\bar{a}, \qquad a = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

where s denotes the arclength. Integrating the above equations and using the boundary conditions

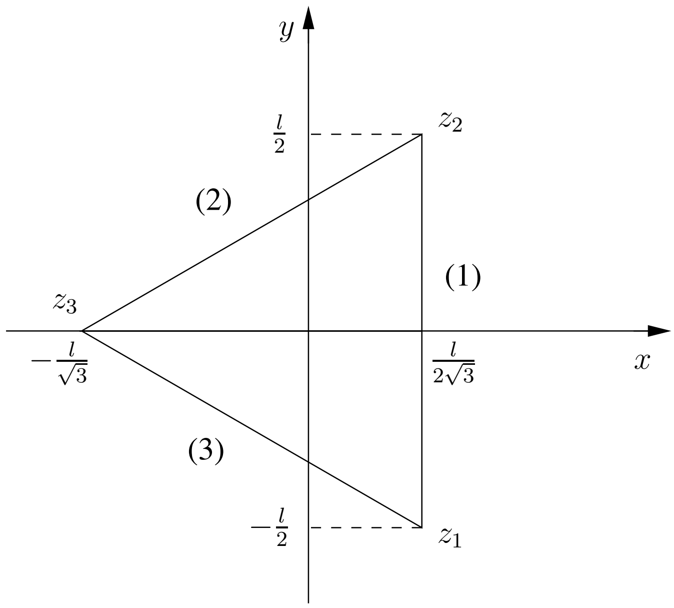

Figure 2.1: *The Equilateral Triangle.*

$$z^{(1)}\left(-\frac{l}{2}\right) = z_1,\ z^{(2)}\left(-\frac{l}{2}\right) = z_2,\ z^{(3)}\left(-\frac{l}{2}\right) = z_3,$$

we find the following expressions parametrizing each of the three sides:

$$\begin{aligned} z^{(1)}(s) &= \frac{l}{2\sqrt{3}} + is, \quad z^{(2)}(s) = \left(\frac{l}{2\sqrt{3}} + is\right) a, \\ z^{(3)}(s) &= \left(\frac{l}{2\sqrt{3}} + is\right) \bar{a}, \quad -\frac{l}{2} < s < \frac{l}{2}. \end{aligned} \tag{2.1.2}$$

(b) Formulation of the problems

The equations investigated in this chapter are given by (1.1.1), where D denotes the interior of the equilateral triangle. Using the transformation $\lambda = \beta^2\gamma$, with $\beta \geq 0$, $\gamma \in \mathbb{C}$ and $|\gamma| = 1$, we obtain the following form of (1.1.1):

$$q_{xx} + q_{yy} + 4\gamma\beta^2 q = 0, \quad (x, y) \in D. \tag{2.1.3}$$

The cases $\{\beta = 0\}$, $\{\beta > 0, \gamma = -1\}$ and $\{\beta > 0, \gamma \neq 1\}$ correspond to the Laplace, the modified Helmholtz and the generalized Helmholtz equations respectively.

- The problems analyzed in the first section of this chapter are:

  (i) The Symmetric Dirichlet problem for the Laplace equation, i.e. the case with the boundary conditions

$$q^{(j)}(s) = g(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3. \tag{2.1.4}$$

  (ii) The Dirichlet problem for the Laplace equation, i.e. the case with the boundary conditions

$$q^{(j)}(s) = g_j(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3. \tag{2.1.5}$$

- The problems analyzed in the second section of this chapter are:

  (i) The Symmetric Dirichlet problem for the modified Helmholtz equation, i.e. the case with the boundary conditions

$$q^{(j)}(s) = d(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3. \tag{2.1.6}$$

  (ii) The Poincaré problem for the modified Helmholtz equation, i.e. the case with the boundary conditions

$$\sin\delta_j q_N^{(j)}(s) + \cos\delta_j \frac{d}{ds} q^{(j)}(s) - \chi_j q^{(j)}(s) = g_j(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3, \tag{2.1.7}$$

  where δ_1 is a real constant so that $\sin\delta_1 \neq 0$, δ_2 and δ_3 satisfy $\sin\delta_2 \neq 0$ and $\sin\delta_3 \neq 0$ and are given in terms of δ_1 by the expressions

$$\delta_2 = \delta_1 + \frac{n\pi}{3}, \quad \delta_3 = \delta_1 + \frac{m\pi}{3}, \quad m, n \in \mathbb{Z}, \tag{2.1.8}$$

  whereas the real constants χ_j , $j = 1, 2, 3$ satisfy the relations

$$[\chi_2(3\beta^2 - \chi_2^2) + e^{in\pi}\chi_1(3\beta^2 - \chi_1^2)] \sin 3\delta_1 = 0 \tag{2.1.9}$$

and

$$[\chi_3(3\beta^2-\chi_3^2)+e^{im\pi}\chi_1(3\beta^2-\chi_1^2)]\sin 3\delta_1=0. \qquad (2.1.10)$$

Note that the assumption $\sin\delta_j \neq 0$ is without loss of generality since if $\sin\delta_j = 0$ then after integration the boundary condition can be rewritten as $\frac{d}{ds}q^{(j)}(s) = d_j(s)$, which becomes the Dirichlet problem.

- The problem analyzed in the third section of this chapter is the Symmetric Dirichlet problem for the generalized Helmholtz equation.

It is assumed that the functions $g_j(s)$ have sufficient smoothness and that they are compatible at the vertices of the triangle.

Recall the following identities:

(a) If

$$z = x+iy, \quad \bar{z} = x-iy, \quad (x,y)\in\mathbb{R}^2,$$

then

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \qquad (2.1.11)$$

(b) If a side of a polygon is parametrical by s, then

$$q_z dz = \frac{1}{2}(\dot{q}+iq_N)ds, \quad q_{\bar{z}}d\bar{z} = \frac{1}{2}(\dot{q}-iq_N)ds, \qquad (2.1.12)$$

where $\dot{q}$ is the derivative along the side, i.e. $\dot{q} = dq(z(s))/ds$ and q_N is the derivative normal to the side in the outward direction.

Under the transformation (2.1.11) equation (2.1.3) can be written in this form

$$q_{z\bar{z}} + \gamma\beta^2 q = 0, \quad \text{where } z = x+iy. \qquad (2.1.13)$$

2.2 The Laplace Equation.

The substitution $\beta = 0$ in (2.1.13) yields the following form of the Laplace equation

$$q_{z\bar{z}} = 0. \tag{2.2.1}$$

Hence, since $(q_z)_{\bar{z}} = 0$, it follows that q is harmonic iff q_z is an analytic function on z. This implies that it is easier to obtain an integral representation for q_z instead of q. In this respect we note that q satisfies the Laplace equation iff the following differential form is closed,

$$W(z,k) = e^{-ikz} q_z dz, \ \ k \in \mathbb{C}. \tag{2.2.2}$$

In what follows, we will use the spectral analysis of the differential form

$$d\left[e^{-ikz}\mu(z,k)\right] = e^{-ikz} q_z dz, \ \ k \in \mathbb{C}, \tag{2.2.3}$$

to obtain an integral representation for q_z in the interior of a convex polygon Ω. Furthermore the following global relations are valid

$$\int_{\partial\Omega} e^{-ikz} q_z dz = 0, \quad \int_{\partial\Omega} e^{ik\bar{z}} q_{\bar{z}} d\bar{z} = 0, \ \ k \in \mathbb{C}. \tag{2.2.4}$$

If q is real then the second equation comes from the Schwarz conjugate of the first of the equations (2.2.4). If q is complex, the second of the equations (2.2.4) is a consequence of the differential form

$$\overline{W}(z,\bar{k}) = e^{ik\bar{z}} q_{\bar{z}} d\bar{z}, \ \ k \in \mathbb{C}, \tag{2.2.5}$$

which is also closed iff q satisfies the Laplace equation.

The following theorem, which can be found slightly different in [9] and [42], gives the formulae for the global relation and the integral representation for the Laplace's equations in the interior of a convex polygon.

Theorem 2.1. *Let Ω be the interior of a convex closed polygon in the complex z-plane, with corners $z_1, \ldots, z_n, z_{n+1} \equiv z_1$. Assume that there exists a solution $q(z, \bar{z})$ of the Laplace equation, i.e. of equation (2.2.1), valid on Ω and suppose that this solution has sufficient smoothness on the boundary of the polygon.*

Then q_z can be expressed in the form

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{l_j} e^{ikz} \hat{q}_j(k) dk, \tag{2.2.6}$$

where $\{\hat{q}_j(k)\}_1^n$ are defined by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz} q_z dz, \;\; k \in \mathbb{C}, \;\; j = 1, \ldots, n \tag{2.2.7}$$

and $\{l_j\}_1^n$ are the rays in the complex k-plane

$$l_j = \{k \in \mathbb{C} : argk = -arg(z_{j+1} - z_j)\}, \;\; j = 1, \ldots, n \tag{2.2.8}$$

oriented from zero to infinity.

Furthermore, the following global relations are valid

$$\sum_{j=1}^{n} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{q}_j(k) = 0, k \in \mathbb{C}, \tag{2.2.9}$$

where $\{\tilde{q}_j(k)\}_1^n$ are defined by

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{ik\bar{z}} q_{\bar{z}} d\bar{z}, \;\; k \in \mathbb{C}, \;\; j = 1, \ldots, n. \tag{2.2.10}$$

Proof. Integrating equation (2.2.3) we find

$$\mu_j(z,k) = \int_{z_j}^{z} e^{ik(z-\zeta)} q_\zeta d\zeta, \ z \in \Omega, \ j = 1, \dots, n. \tag{2.2.11}$$

The term $\exp[ik(z-\zeta)]$ is bounded as $k \to \infty$ for

$$0 \le \arg k + \arg(z-\zeta) \le \pi. \tag{2.2.12}$$

If z is inside the polygon and ζ is on a curve from z to z_j, see Figure 2.2, then

$$\arg(z_{j+1} - z_j) \le \arg(z-\zeta) \le \arg(z_{j-1} - z_j), \quad j = 1, \dots, n.$$

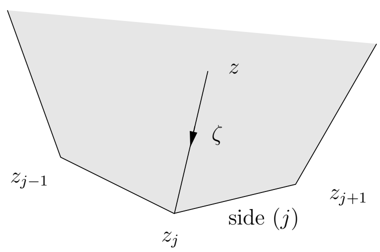

Figure 2.2: *Part of the convex polygon.*

Hence, the inequalities (2.2.12) are satisfied provided that

$$-\arg(z_{j+1} - z_j) \le \arg k \le \pi - \arg(z_{j-1} - z_j).$$

Hence, the function μ_j is an entire function of k which is bounded as $k \to \infty$ in the sector Σ_j defined by

$$\Sigma_j = \{k \in \mathbb{C}, \ \ \arg k \in [-\arg(z_{j+1} - z_j), \pi - \arg(z_{j-1} - z_j)]\}, \quad j = 1, \dots, n. \tag{2.2.13}$$

The angle of the sector Σ_j, which we denote by ψ_j, equals

$$\psi_j = \pi - \arg(z_{j-1} - z_j) + \arg(z_{j+1} - z_j) = \pi - \phi_j, \tag{2.2.14}$$

where ϕ_j is the angle at the corner z_j. Hence

$$\sum_{j=1}^{n} \psi_j = n\pi - \sum_{j=1}^{n} \phi_j = n\pi - \pi(n-2) = 2\pi, \tag{2.2.15}$$

thus the sectors $\{\Sigma_j\}_1^n$ precisely cover the complex k-plane. Hence, the function

$$\mu = \mu_j, \quad z \in \Omega, \quad k \in \Sigma_j, \quad j = 1, \ldots, n, \tag{2.2.16}$$

defines a sectionally analytic function in the complex k-plane.

For the solution of the inverse problem, we note that integration by parts implies that $\mu_j = O(1/k)$ as $k \to \infty$ in Σ_j, i.e.

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.2.17}$$

Furthermore, by subtracting equation (2.2.11) and the analogous equation for μ_{j+1} we find

$$\mu_j - \mu_{j+1} = e^{ikz}\hat{q}_j(k), \quad z \in \Omega, \quad k \in l_j, \quad j = 1, \ldots, n, \tag{2.2.18}$$

where $\{\hat{q}_j(k)\}_1^n$ are defined by equation (2.2.7) and l_j is the ray of overlap of the sectors Σ_j and Σ_{j+1}. Using the identity

$$\pi - \arg(z_j - z_{j+1}) = -\arg(z_{j+1} - z_j) \pmod{2\pi}, \tag{2.2.19}$$

it follows that l_j is defined by equation (2.2.8). Furthermore, Σ_j is to the left of Σ_{j+1}, see Figure 2.3.

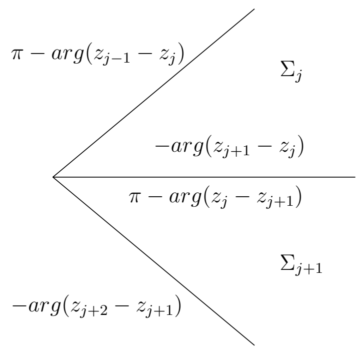

Figure 2.3: *The sectors Σ_j and Σ_{j+1}.*

The solution of the RH problem defined by equations (2.2.16) - (2.2.18) is given by

$$\mu = \frac{1}{2i\pi}\sum_{j=1}^{n}\int_{l_j} e^{ilz}\hat{q}_j(l)\frac{dl}{l-k}, \quad z\in\Omega, \quad k\in\mathbb{C}\backslash\{l_j\}_1^n. \tag{2.2.20}$$

Substituting this expression in equation (2.2.3), i.e. in the equation

$$\mu_z - ik\mu = q_z,$$

we find equation (2.2.6).

Using the definitions of $\{\hat{q}_j\}_1^n$ and of $\{\tilde{q}_j\}_1^n$, i.e. equations (2.2.7) and (2.2.10) respectively, equations (2.2.4) yield the two global relations (2.2.9).

$\square$

Substituting equations (2.1.12) in the definition of the function $\hat{q}_j(k)$ and $\tilde{q}_j(k)$ we find the following expressions

$$\hat{q}_j(k) = \frac{1}{2}\int_{z_j}^{z_{j+1}} e^{-ikz}\left(iq_N^{(j)} + \dot{q}^{(j)}\right)ds, \ k\in\mathbb{C}, \tag{2.2.21}$$

and

$$\tilde{q}_j(k) = \frac{1}{2}\int_{z_j}^{z_{j+1}} e^{ikz}\left(-iq_N^{(j)} + \dot{q}^{(j)}\right)ds, \ k\in\mathbb{C}, \tag{2.2.22}$$

where the index (j) denotes the value of the corresponding functions on side (j). Observe that the solution (2.2.6) is given in terms of the spectral functions $\hat{q}$ which involve both $\dot{q}$ and q_n on the boundary, i.e. both known and unknown functions. In what follows the unknown functions will be eliminated from the integral representation of the solution, by using appropriately the global relations.

2.2.1 Symmetric Dirichlet Problem.

The problem analyzed here is the Symmetric Dirichlet problem for the Laplace equation in the Equilateral Triangle($\Omega \equiv D$), i.e. the case with the boundary conditions

$$q^{(j)}(s) = g(s),\ s \in \left[-\frac{l}{2}, \frac{l}{2}\right],\ j = 1, 2, 3.$$

For convenience we define

$$d(s) = \dot{q}(s),\ s \in \left[-\frac{l}{2}, \frac{l}{2}\right],\ j = 1, 2, 3.$$

It is also assumed that the function $d(s)$ has sufficient smoothness and that it is compatible at the vertices of the triangle, i.e. $d(\frac{l}{2}) = d(-\frac{l}{2})$.

Applying the parametrization of the fundamental domain given in equations (2.1.2), on equations (2.2.21) and (2.2.22), we obtain the following expressions for the spectral functions $\{\hat{q}_j(k)\}_1^3$ and $\{\tilde{q}_j(k)\}_1^3$:

$$\begin{gathered} \hat{q}_1(k) = \hat{q}(k), \quad \hat{q}_2(k) = \hat{q}(ak), \quad \hat{q}_3(k) = \hat{q}(\bar{a}k), \\ \text{with} \\ \hat{q}(k) = E(-ik)[iU(k) + D(k)] \end{gathered} \tag{2.2.23}$$

and

$$\begin{gathered} \tilde{q}_1(k) = \tilde{q}(k), \quad \tilde{q}_2(k) = \tilde{q}(\bar{a}k), \quad \tilde{q}_3(k) = \tilde{q}(ak), \\ \text{with} \\ \tilde{q}(k) = E(ik)[-iU(k) + D(k)], \end{gathered} \tag{2.2.24}$$

where

$$E(k) = e^{k\frac{l}{2\sqrt{3}}},\quad D(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} d(s) ds,\quad U(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} q_N(s) ds,\ k \in \mathbb{C}.$$

The function $D(k)$ is known, whereas the unknown function $U(k)$ contains the unknown Neumann boundary value q_N.

It turns out that, using algebraic manipulations of the global relations and appropriate contour deformations of the $\{l_j\}_1^3$, it is possible to eliminate the unknown functions $U(k)$,

$U(ak)$, $U(\bar{a}k)$ from the representation of the solution at (2.2.6). In this way we will obtain the following integral representation:

$$\begin{aligned}\frac{\partial q(z)}{\partial z} &= \frac{1}{2\pi}\int_{l_1} A(k,z,\bar{z})E(-ik)\left[D(k)+\frac{G(k)}{\Delta(ak)}\right]dk \\ &+\frac{1}{2\pi}\int_{l_1'} A(k,z,\bar{z})E^2(iak)\frac{G(k)}{\Delta(ak)\Delta(k)}dk\end{aligned} \tag{2.2.25}$$

where $l_1=\{k\in\mathbb{C}:\arg k=-\frac{\pi}{2}\}$, l_1' is the ray with $-\frac{\pi}{2}\le\arg k\le-\frac{\pi}{6}$(see Figure 2.4) and

$$A(k,z,\bar{z})=e^{ikz}+\bar{a}e^{i\bar{a}kz}+ae^{iakz}, \tag{2.2.26a}$$

$$G(k)=\Delta^+(ak)D(k)+2D(\bar{a}k)+\Delta^+(k)D(ak), \tag{2.2.26b}$$

$$\Delta(k)=e(k)-e(-k),\ \Delta^+(k)=e(k)+e(-k),\ e(k)=e^{k\frac{l}{2}}. \tag{2.2.26c}$$

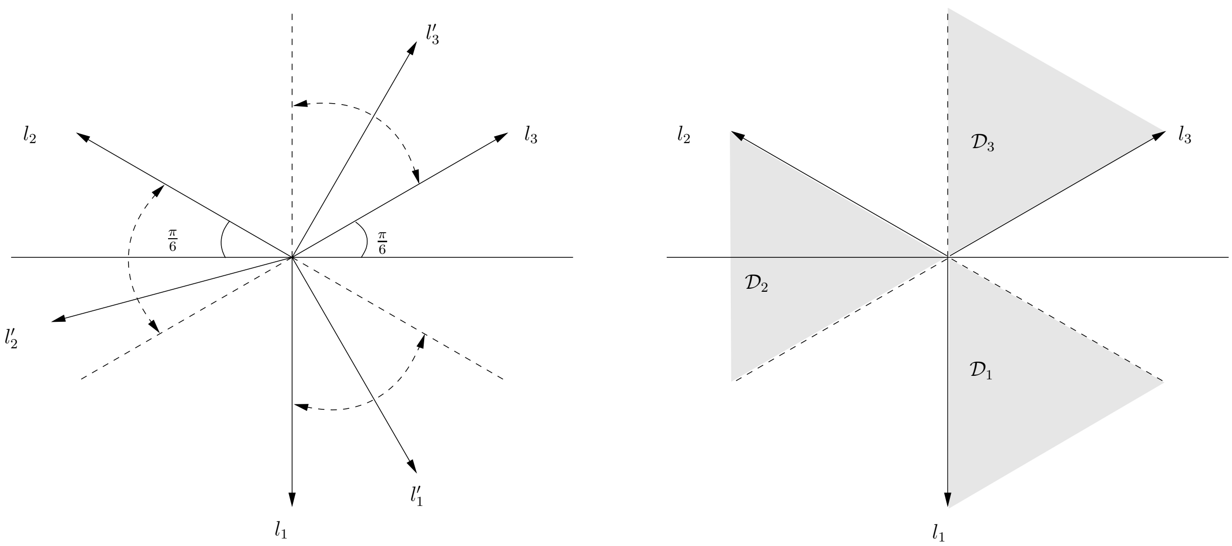

Figure 2.4: *The contours l_j and l_j'.*

Using the Global Relations

Applying (2.2.23) in the first of the global relations (2.2.9) and multiplying by $E(i\bar{a}k)$ we obtain the equation

$$e(-ak)U(k) + e(k)U(ak) + U(\bar{a}k) = iJ(k), \ k \in \mathbb{C}, \tag{2.2.27}$$

where

$$J(k) = e(-ak)D(k) + e(k)D(ak) + D(\bar{a}k).$$

Applying (2.2.24) in the second of the global relations (2.2.9) and multiplying by $E(-i\bar{a}k)$ we obtain the equation

$$e(ak)U(k) + e(-k)U(ak) + U(\bar{a}k) = -ie(-k)\overline{J(\bar{k})}, \ k \in \mathbb{C}, \tag{2.2.28}$$

where $\overline{J(\bar{k})}$ denotes the function obtained from $J(k)$ by taking the complex conjugate of all the terms in $J(\overline{k})$ except $d(s)$. In this respect, note that if $d(s)$ is a real function, then equation (2.2.28) can be obtained by taking the Schwarz conjugate of (2.2.27) and multiplying by $e(-k)$.

Subtracting equations (2.2.27) and (2.2.28) we find the following equation which is valid for all $k \in \mathbb{C}$,

$$\Delta(ak)U(k) = \Delta(k)U(ak) - iG(k), \tag{2.2.29}$$

where $G(k) = J(k) + e(-k)\overline{J(\bar{k})}$.

Substituting $U(k)$ in the expression of $\hat{q}(k)$ in (2.2.23) we find

$$\hat{q}(k) = E(-ik)D(k) + \frac{E(-ik)G(k)}{\Delta(k)} + i[E^2(i\bar{a}k) - E^2(iak)]\frac{U(ak)}{\Delta(ak)}. \tag{2.2.30}$$

The functions $\hat{q}_2(k)$ and $\hat{q}_3(k)$ can be obtained from (2.2.30) by replacing k with ak and $\bar{a}k$ respectively.

In what follows we will show that the contribution of the unknown functions $U(ak)$, $U(k)$ and $U(\bar{a}k)$ can be computed in terms of the given boundary conditions, using the following basic facts.

Basic facts

1. The zeros of the functions $\Delta(k)$, $\Delta(ak)$, $\Delta(\bar{a}k)$ occur on the following lines respectively in the complex k-plane
$$i\mathbb{R}, \quad e^{\frac{5i\pi}{6}}\mathbb{R}, \quad e^{\frac{i\pi}{6}}\mathbb{R}.$$
Indeed,
$$\Delta(k) = 0 \Leftrightarrow \sinh\left(k\frac{l}{2}\right) = 0.$$
Hence, the zeros of $\Delta(k)$ occur on the imaginary axis and then the zeros of $\Delta(ak)$ and $\Delta(\bar{a}k)$ can be obtained by appropriate rotations.

2. The functions
$$e^{ikz}E^2(iak), \quad e^{ikz}E^2(ik), \quad e^{ikz}E^2(i\bar{a}k)$$
are bounded and analytic for all $z \in D$, for $\arg k$ in
$$\left[-\frac{\pi}{2}, \frac{\pi}{6}\right], \quad \left[\frac{\pi}{6}, \frac{5\pi}{6}\right], \quad \left[\frac{5\pi}{6}, \frac{3\pi}{2}\right],$$
respectively, as shown in Figure 2.5.

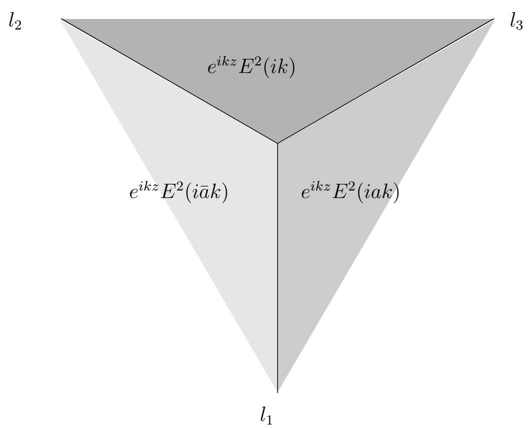

Figure 2.5: *The domains of boundedness and analyticity.*

Indeed, let us consider the first exponential $e^{ikz}E^2(iak) = e^{ik(z-z_1)}$. If $z \in D$ then
$$\frac{\pi}{2} \leq \arg(z - z_1) \leq \frac{5\pi}{6},$$

thus if

$$-\frac{\pi}{2} \leq \arg k \leq \frac{\pi}{6},$$

it follows that

$$0 \leq \arg[k(z - z_1)] \leq \pi.$$

Hence the exponential $e^{i\beta k(z-z_1)}$ is bounded. Similarly for the other two exponentials.

3. The functions $\frac{U(k)}{\Delta(k)}$, $\frac{U(ak)}{\Delta(ak)}$ and $\frac{U(\bar{a}k)}{\Delta(\bar{a}k)}$ are bounded and analytic in $\mathbb{C}$ apart from the above lines where $\Delta(k)$, $\Delta(ak)$ and $\Delta(\bar{a}k)$ have zeros.

   Indeed, regarding $\frac{U(k)}{\Delta(k)}$ observe that $\Delta(k)$ is dominated by $e(k)$ for $Rek > 0$ and by $e(-k)$ for $Rek < 0$, hence

$$\frac{U(k)}{\Delta(k)} \sim \begin{cases} e(-k)U(k), & Rek > 0 \\ -e(k)U(k), & Rek < 0. \end{cases}$$

   Furthermore $e(-k)U(k)$ involves $e^{k(s-\frac{l}{2})}$ which is bounded for $Rek \geq 0$ and $e(k)U(k)$ involves $e^{k(s+\frac{l}{2})}$ which is bounded for $Rek \leq 0$.

The unknown $U(ak)$ in the expression for $\hat{q}(k)$ at (2.2.30), yields the contribution $C_1(z)$ to the solution q given in (2.2.6),

$$C_1 = \frac{i}{2\pi} \int_{l_1} e^{ikz}[E^2(i\bar{a}k) - E^2(iak)] \frac{U(ak)}{\Delta(ak)} dk.$$

The integral of the second term in the rhs of C_1 can be deformed from l_1 to l_1', where l_1' is a ray with $-\frac{\pi}{2} \leq \arg k \leq -\frac{\pi}{6}$.

Hence,

$$C_1 = \frac{i}{2\pi} \int_{l_1} e^{ikz} E^2(i\bar{a}k) \frac{U(ak)}{\Delta(ak)} dk - \frac{i}{2\pi} \int_{l_1'} e^{ikz} E^2(iak) \frac{U(ak)}{\Delta(ak)} dk.$$

In the second integral of the rhs of this equation we replace $U(ak)$ by using (2.2.29), i.e.,

$$\Delta(ak)U(k) = \Delta(k)U(ak) - iG(k).$$

Hence

$$C_1 = \frac{i}{2\pi}\int_{l_1} e^{ikz}E^2(i\bar{a}k)\frac{U(ak)}{\Delta(ak)}dk - \frac{i}{2\pi}\int_{l_1'} e^{ikz}E^2(iak)\frac{U(k)}{\Delta(k)}dk$$
$$+\frac{1}{2\pi}\int_{l_1'} e^{ikz}E^2(iak)\frac{G(k)}{\Delta(k)\Delta(ak)}dk. \qquad (2.2.31)$$

In summary the term $\hat{q}(k)$ gives rise to the contribution $F_1 + \widetilde{U}_1$, where $\widetilde{U}_1$ denotes the first two terms of the rhs of (2.2.31) and F_1 is defined by

$$F_1 = \frac{1}{2\pi}\int_{l_1} e^{ikz}\left[E(-ik)D(k) + \frac{E(-ik)G(k)}{\Delta(ak)}\right]dk$$
$$+\frac{1}{2\pi}\int_{l_1'} e^{ikz}E^2(iak)\frac{G(k)}{\Delta(k)\Delta(ak)}dk. \qquad (2.2.32)$$

The contributions to the solution of $\hat{q}_2$ and $\hat{q}_3$ can be obtained from $F_1 + \widetilde{U}_1$ with the aid of the substitutions

$$l_1 \to l_2 \to l_3, \quad l_1' \to l_2' \to l_3', \quad k \to ak \to \bar{a}k. \qquad (2.2.33)$$

The contribution of $\widetilde{U}_j, j = 1, 2, 3$ vanish due to analyticity. Indeed, the integrands

$$e^{ikz}E^2(i\bar{a}k)\frac{U(ak)}{\Delta(ak)}, \; e^{ikz}E^2(ik)\frac{U(\bar{a}k)}{\Delta(\bar{a}k)}, \; e^{ikz}E^2(iak)\frac{U(k)}{\Delta(k)} \qquad (2.2.34)$$

occur in $l_1 \cup l_2'$, $l_2 \cup l_3'$, $l_3 \cup l_1'$, and in the corresponding domains the above functions are bounded and analytic.

Hence,

$$q = F_1 + F_2 + F_3, \qquad (2.2.35)$$

where F_2 and F_3 are obtained from F_1 using the substitutions (2.2.33). In order to derive the integral representation (2.2.25), we make the change of variables $k \to \bar{a}k$ on the integrals in F_2 and the change of variables $k \to ak$ on the integrals in F_3. In particular, regarding F_2 this leads to the following changes:

1. The differential dk becomes $\bar{a}\, dk$.
2. The rays l_2 and l_2' become l_1 and l_1' respectively.
3. The exponential e^{ikz} becomes $e^{i\bar{a}kz}$.
4. The remaining integrand is equal to the corresponding integrand in F_1.

Similar changes occur in F_3.

The integrands appearing in the integrals along l_1 and l_1' defined in equation (2.2.25) contain terms which decay exponentially. Regarding the integral along l_1', observe that $\frac{G(k)}{\Delta(k)\Delta(ak)}$ is bounded in the domain of deformation of l_1' (see Figure 2.4). The function $A(k, z, \bar{z})E^2(iak)$ is also exponentially decaying. In particular, the first term of the function $e^{ikz}E^2(iak)$ is an exponential whose exponent has negative real part in the domain $\mathcal{D}_1$ (see Figure 2.4). Similar considerations are also valid for the other two remaining terms of the function $A(k, z, \bar{z})E^2(iak)$. Hence we conclude that the integrand of the integral along l_1' is an exponentially decaying function.

Regarding the integral along l_1, observe that $D(k)$ and $\frac{G(k)}{\Delta(ak)}$ are bounded for $k \in l_1$ and since the function $e^{ikz}E(-ik)$ is an exponential whose exponent has negative real part when $k \in l_1$, we conclude that the integrand of the integral along l_1 is an exponentially decaying function.

The above facts can be explicitly verified in the following example.

Example 2.1. *Set* $l = \pi$ *and* $d(s) = \cos s$ *(i.e.* $q_j(s) = \sin s,\ j = 1, 2, 3$*).*

In this case,

$$D(k) = \frac{1}{1+k^2}\cosh\left(k\frac{\pi}{2}\right)$$

and

$$G(k) = \left[\frac{2}{1+k^2} + \frac{2}{1+(ak)^2}\right]\cosh\left(k\frac{\pi}{2}\right)\cosh\left(ak\frac{\pi}{2}\right) + \frac{2}{1+(\bar{a}k)^2}\cosh\left(\bar{a}k\frac{\pi}{2}\right), \quad (2.2.36)$$

where we have used that $\Delta^+(k) = 2\cosh\left(k\frac{\pi}{2}\right)$.

In order to check the convergence of all the integrals appearing in the representation (2.2.25) observe the following:

- *For the first integral along* l_1, $Re(k) = 0$ *and* $Im(k) < 0$. *Hence, as* $k \to \infty$*:*
  1. $e^{ikz}E(-ik) \sim e^{ik\left(Re(z)-\frac{\pi}{2\sqrt{3}}\right)} \sim e^{-Im(k)\left(x-\frac{\pi}{2\sqrt{3}}\right)}, \ x < \frac{\pi}{2\sqrt{3}}$;
  2. $D(k) \sim \frac{1}{k^2}$;
  3. $\frac{G(k)}{\Delta(ak)} \sim \frac{1}{k^2}$.
- *For the second integral, along* l_1', $\arg k \in (-\frac{\pi}{2}, -\frac{\pi}{6})$. *Hence, as* $k \to \infty$*:*
  1. $e^{ikz}E^2(iak) \sim exp\left[\left(x - \frac{\pi}{2\sqrt{3}}\right)\cos\left(\phi + \frac{\pi}{2}\right) - \left(y + \frac{\pi}{2}\right)\sin\left(\phi + \frac{\pi}{2}\right)\right]$, *where* $x < \frac{\pi}{2\sqrt{3}}$ *and* $y > -\frac{\pi}{2}$ *and* $\phi = \arg k$. *Hence the exponent is negative when* $\arg k \in \left(-\frac{\pi}{2}, -\frac{\pi}{6}\right)$ *i.e. in the domain of* l_1' *deformation;*
  2. $\frac{G(k)}{\Delta(k)\Delta(ak)} \sim \frac{1}{k^2}$.

Similar arguments are valid for the other two terms of $A(k, z, \bar{z})$.

2.2.2 The General Dirichlet Problem.

We now consider the solution of the arbitrary Dirichlet problem, i.e. of the problem with the boundary conditions

$$\dot{q}^{(j)}(s) = d_j(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3,$$

where the function $d_j(s)$ have sufficient smoothness and are compatible at the vertices of the triangle, i.e. $d_j(\frac{l}{2}) = d_{j+1}(-\frac{l}{2}), \ j = 1, 2, 3, \ \ d_4(s) = d_1(s)$.

The solution of this problem can be obtained in two different ways:

(i) In the first approach we use the solution of the symmetric Dirichlet problem, as well as the fact that the arbitrary Dirichlet problem can be decomposed into three problems, which are solved in a way similar to the symmetric Dirichlet problem.

(ii) In the second approach we use the invariants of the global relations and follow the general methodology used for the symmetric Dirichlet problem.

The second approach is more complicated, however, it has the advantage that it can be used to solve other problems that do not admit the decomposition mentioned in (i) above. Such problems are:

(a) the Poincaré problem defined in equation (2.1.7);

(b) the oblique Robin problem defined in equation (2.1.7), with

$$\delta_j = \delta, \quad \text{and} \quad \chi_j = \chi, \;\; j = 1, 2, 3;$$

(c) the Robin problem defined in equation (2.1.7), with

$$\delta_j = \frac{\pi}{2}, \quad \text{and} \quad \chi_j = \chi \neq 0, \;\; j = 1, 2, 3.$$

The First Approach

The general Dirichlet problem can be decomposed into the following three problems:

1. Let q satisfy the symmetric Dirichlet problem for (2.2.1) in the domain D defined in (2.1.1), i.e.

$$\dot{q}^{(j)}(s) = f(s), \;\; s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \;\; j = 1, 2, 3,$$

where $f(s)$ is sufficiently smooth and compatible at the corners of the triangle i.e. $f\left(\frac{l}{2}\right) = f\left(-\frac{l}{2}\right)$.

2. Let q satisfy (2.2.1) in the domain D defined in (2.1.1), with the following Dirichlet boundary conditions

$$\dot{q}^{(1)}(s) = g(s), \;\; \dot{q}^{(2)}(s) = a g(s), \;\; \dot{q}^{(3)}(s) = \bar{a} g(s), \;\; s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \;\; j = 1, 2, 3,$$

where $g(s)$ is sufficiently smooth and compatible at the corners of the triangle i.e. $g\left(\frac{l}{2}\right) = ag\left(-\frac{l}{2}\right)$.

3. Let q satisfy (2.2.1) in the domain D defined in (2.1.1), with the following Dirichlet boundary conditions

$$\dot{q}^{(1)}(s) = h(s),\ \dot{q}^{(2)}(s) = \bar{a}h(s),\ \dot{q}^{(3)}(s) = ah(s),\ s \in \left[-\frac{l}{2}, \frac{l}{2}\right],\ j = 1, 2, 3,$$

where $h(s)$ is sufficiently smooth and compatible at the corners of the triangle i.e. $h\left(\frac{l}{2}\right) = \bar{a}h\left(-\frac{l}{2}\right)$.

A general Dirichlet boundary value problem can be written as the sum of above three boundary value problems. Indeed, suppose that the following Dirichlet condition is valid

$$\dot{q}^{(j)}(s) = d_j(s),\ s \in \left[-\frac{l}{2}, \frac{l}{2}\right],\ j = 1, 2, 3.$$

The matrix of the following 3×3 algebraic system is non-singular:

$$\begin{pmatrix} d_1(s) \\ d_2(s) \\ d_3(s) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix} \begin{pmatrix} f(s) \\ g(s) \\ h(s) \end{pmatrix},\quad Det[\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix}] = i3\sqrt{3}. \tag{2.2.37}$$

Due to uniqueness, the solution of the general Dirichlet problem is given by the *sum* of the three problems defined earlier.

The solution of the problems (2) and (3) above can be obtained in a way similar to the symmetric case. Indeed, let us consider problem (2), where the Dirichlet conditions are given by

$$\dot{q}^{(1)}(s) = d(s),\ \dot{q}^{(2)}(s) = ad(s),\ \dot{q}^{(3)}(s) = \bar{a}d(s),\ s \in \left[-\frac{l}{2}, \frac{l}{2}\right],\ j = 1, 2, 3.$$

As mentioned earlier the function $d(s)$ has sufficient smoothness and is compatible at the vertices of the triangle, i.e. $d(\frac{l}{2}) = a\ d(-\frac{l}{2})$.

Applying the parametrization of the fundamental domain on equations (2.2.21) and (2.2.22)we obtain the following expressions for the spectral functions $\{\hat{q}_j(k)\}_1^3$ and $\{\tilde{q}_j(k)\}_1^3$:

$$\begin{aligned} &\hat{q}_1(k) = \hat{q}(k), \quad \hat{q}_2(k) = a\ \hat{q}(ak), \quad \hat{q}_3(k) = \bar{a}\ \hat{q}(\bar{a}k), \\ &\text{with } \hat{q}(k) = E(-ik)[iU(k) + D(k)] \end{aligned} \tag{2.2.38}$$

and

$$\begin{aligned} &\tilde{q}_1(k) = \tilde{q}(k), \quad \tilde{q}_2(k) = \bar{a}\ \tilde{q}(\bar{a}k), \quad \tilde{q}_3(k) = a\ \tilde{q}(ak), \\ &\text{with } \tilde{q}(k) = E(ik)[-iU(k) + D(k)] \end{aligned} \tag{2.2.39}$$

where

$$E(k) = e^{k\frac{l}{2\sqrt{3}}}, \quad D(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} d(s)ds, \quad U(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} q_N(s)ds, \ k \in \mathbb{C}. \tag{2.2.40}$$

The function $D(k)$ is known, whereas the unknown function $U(k)$ contains the unknown Neumann boundary value q_N. The solution of this problem can now be obtained by adopting the methodology of the symmetric case and also making the following substitutions:

$$\begin{aligned} &U(k) \to U(k), \quad U(ak) \to aU(ak), \quad U(\bar{a}k) \to \bar{a}U(\bar{a}k) \\ &D(k) \to D(k), \quad D(ak) \to aD(ak), \quad D(\bar{a}k) \to \bar{a}D(\bar{a}k) \end{aligned} \tag{2.2.41}$$

Thus, the solution is given by

$$\begin{aligned} \frac{\partial q(z)}{\partial z} = &\frac{1}{2\pi}\int_{l_1} B(k,z,\bar{z})E(-ik)D(k)dk \\ &+ \frac{1}{2\pi}\int_{l_1} A(k,z,\bar{z})E(-ik)\frac{G(k)}{\Delta(ak)}dk \\ &+ \frac{1}{2\pi}\int_{l_1'} A(k,z,\bar{z})E^2(iak)\frac{G(k)}{\Delta(ak)\Delta(k)}dk \end{aligned} \tag{2.2.42}$$

where

$$A(k,z,\bar{z}) = e^{ikz} + \bar{a}e^{i\bar{a}kz} + ae^{iakz}, \tag{2.2.43a}$$

$$B(k,z,\bar{z}) = e^{ikz} + e^{i\bar{a}kz} + e^{iakz}, \tag{2.2.43b}$$

$$G(k) = \Delta^+(ak)D(k) + 2\bar{a}D(\bar{a}k) + a\Delta^+(k)D(ak), \tag{2.2.43c}$$

$$\Delta(k) = e(k) - e(-k), \ \Delta^+(k) = e(k) + e(-k), \ e(k) = e^{k\frac{l}{2}}, \tag{2.2.43d}$$

with $D(k)$ given in (2.2.40).

The Second Approach

In what follows we illustrate a direct way to find the solution of the Dirichlet problem in the interior of the equilateral triangle, Figure 2.1. Applying the parametrization of the fundamental domain, given in equations (2.1.2), on equations (2.2.21) and (2.2.22)we obtain the following expressions for the spectral functions $\{\hat{q}_j(k)\}_1^3$ and $\{\tilde{q}_j(k)\}_1^3$:

$$\begin{aligned} &\hat{q}_1(k) = E(-ik)[iU_1(k) + D_1(k)],\ \hat{q}_2(k) = E(-iak)[iU_2(ak) + D_2(ak)], \\ &\hat{q}_3(k) = E(-i\bar{a}k)[iU_3(\bar{a}k) + D_3(\bar{a}k)], \end{aligned} \tag{2.2.44}$$

where

$$E(k) = e^{k\frac{l}{2\sqrt{3}}},\quad D_j(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} d_j(s)ds,\quad U_j(k) = \frac{1}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks} q_N^{(j)}(s)ds,\ k \in \mathbb{C}. \tag{2.2.45}$$

Using algebraic manipulations of the global relations and appropriate contour deformation, it is possible to eliminate the unknown functions $U_1(k)$, $U_2(ak)$, $U_3(\bar{a}k)$ from the representation of the solution at (2.2.6) and thus, obtain the representation

$$\begin{aligned} \frac{\partial q(z)}{\partial z} &= \frac{1}{2\pi}\int_{l_1} e^{ikz}E(-ik)\left[D_1(k) + i\frac{\Gamma_1(ak)}{\Delta(ak)}\right]dk + \frac{i}{2\pi}\int_{l_1'} e^{ikz}E^2(iak)\frac{\Gamma_{123}(ak)}{\Delta(k)\Delta(ak)}dk \\ &+ \frac{1}{2\pi}\int_{l_2} e^{ikz}E(-iak)\left[D_2(ak) + i\frac{\Gamma_2(\bar{a}k)}{\Delta(\bar{a}k)}\right]dk + \frac{i}{2\pi}\int_{l_2'} e^{ikz}E^2(i\bar{a}k)\frac{\Gamma_{231}(\bar{a}k)}{\Delta(ak)\Delta(\bar{a}k)}dk \\ &+ \frac{1}{2\pi}\int_{l_3} e^{ikz}E(-i\bar{a}k)\left[D_3(\bar{a}k) + i\frac{\Gamma_3(k)}{\Delta(k)}\right]dk + \frac{i}{2\pi}\int_{l_3'} e^{ikz}E^2(ik)\frac{\Gamma_{312}(k)}{\Delta(\bar{a}k)\Delta(k)}dk, \end{aligned} \tag{2.2.46}$$

where $\{l_j\}_1^3, \{l_j'\}_1^3$ are depicted in Figure 2.4 ,

$$\Delta(k) = e^3(k) - e^{-3}(k),\quad e(k) = e^{k\frac{l}{2}}, \tag{2.2.47}$$

$$\Gamma_{lmn}(k) = E^{-2\sqrt{3}}(k)\Gamma_l(k) + \Gamma_m(k) + E^{2\sqrt{3}}(k)\Gamma_n(k), \tag{2.2.48}$$

$$\begin{aligned}
\Gamma_3(k) &= [E^3(iak)+E^{-3}(iak)]e(-k)D_1(k)\\
&+[E^3(iak)+E^{-3}(iak)]e(k)D_2(k)\\
&+[E^3(i\bar{a}k)+E^{-3}(i\bar{a}k)]D_3(k) \qquad (2.2.49)\\
&+2e^2(k)D_1(ak)+2e^2(-k)D_2(ak)+2D_3(ak)\\
&+2e^2(-k)D_1(\bar{a}k)+2e^2(k)D_2(\bar{a}k)+[e^3(k)+e^{-3}(k)]D_3(\bar{a}k),
\end{aligned}$$

$\Gamma_1(k)$ is obtained by making the rotations $3\to 1$, $1\to 2$, $2\to 3$ on the subscripts of $\Gamma_3(k)$ and $\Gamma_2(k)$ is obtained by making the rotations $3\to 2$, $2\to 1$, $1\to 3$ on the subscripts of $\Gamma_3(k)$.

Using the Global Relations.

Applying (2.2.44) in the first of the global relations (2.2.9) we obtain the following equation

$$E(-ik)U_1(k)+E(-iak)U_2(ak)+E(-i\bar{a}k)U_3(\bar{a}k)=iF_1(k),\ \ k\in\mathbb{C}, \qquad (2.2.50)$$

where

$$F_1(k)=E(-ik)D_1(k)+E(-iak)D_2(ak)+E(-i\bar{a}k)D_3(\bar{a}k).$$

Furthermore, applying (2.2.44) in the second of the global relations we obtain the following equation

$$E(ik)U_1(k)+E(i\bar{a}k)U_2(\bar{a}k)+E(iak)U_3(ak)=-iF_2(k),\ \ k\in\mathbb{C}, \qquad (2.2.51)$$

where

$$F_2(k)=E(ik)D_1(k)+E(i\bar{a}k)D_2(\bar{a}k)+E(iak)D_3(ak).$$

Applying the transformations $k\to ak$ and $k\to\bar{a}k$ in both (2.2.50) and (2.2.51) we find an algebraic system of *6 equations* which involves the following *9 unknown functions*:

$$\left\{U_1(k),\ U_1(ak),\ U_1(\bar{a}k), U_2(k),\ U_2(ak),\ U_2(\bar{a}k), U_3(k),\ U_3(ak),\ U_3(\bar{a}k)\right\}.$$

Hence, we can solve this system for one of the unknown functions in terms of three other unknown functions and some known function. Hence, solving this system for $U_3(\bar{a}k)$ in terms of $\{U_1(k), U_2(k), U_3(k)\}$ we obtain the following relation(see [32])

$$\begin{aligned}\Delta(k)U_3(\bar{a}k) = &[E^3(iak) - E^{-3}(iak)]e(-k)U_1(k)\\ &+ [E^3(iak) - E^{-3}(iak)]e(k)U_2(k)\\ &+ [E^3(i\bar{a}k) - E^{-3}(i\bar{a}k)]U_3(k) + \Gamma_3(k),\end{aligned} \tag{2.2.52}$$

where $\Delta(k)$ is defined in (2.2.47) and $\{\Gamma_j(k)\}_1^3$ involve the known functions $\{F_1(k), F_2(k), F_1(ak), F_2(ak), F_1(\bar{a}k), F_2(\bar{a}k)\}$ which are defined in equation (2.2.49). Solving again the system of equation for $U_1(k)$ in terms of $\{U_1(ak), U_2(ak), U_3(ak)\}$, or by simply making the substitution $k \to ak$ and the rotations $3 \to 1$, $1 \to 2$, $2 \to 3$ on the subscripts of (2.2.52) we obtain the following relation

$$\begin{aligned}\Delta(ak)U_1(k) = &[E^3(ik) - E^{-3}(ik)]U_1(ak)\\ &+ [E^3(i\bar{a}k) - E^{-3}(i\bar{a}k)]e(-ak)U_2(ak)\\ &+ [E^3(i\bar{a}k) - E^{-3}(i\bar{a}k)]e(ak)U_3(ak) + \Gamma_1(ak).\end{aligned} \tag{2.2.53}$$

Following the same pattern, we obtain the expression of $U_2(ak)$ in terms of $\{U_1(\bar{a}k), U_2(\bar{a}k), U_3(\bar{a}k)\}$, by substituting $k \to \bar{a}k$ and the rotations $2 \to 1$, $1 \to 3$, $3 \to 2$ on the subscripts of the equation (2.2.52):

$$\begin{aligned}\Delta(\bar{a}k)U_2(ak) = &[E^3(ik) - E^{-3}(ik)]e(\bar{a}k)U_1(\bar{a}k)\\ &+ [E^3(iak) - E^{-3}(iak)]U_2(\bar{a}k)\\ &+ [E^3(ik) - E^{-3}(ik)]e(-\bar{a}k)U_3(\bar{a}k) + \Gamma_2(\bar{a}k).\end{aligned} \tag{2.2.54}$$

Rotating the subscripts of (2.2.53) we obtain also the expressions of $U_2(k)$ and $U_3(k)$ in terms of $\{U_1(ak), U_2(ak), U_3(ak)\}$. These expressions yield the following identity

$$\begin{aligned}&\frac{E^{-2\sqrt{3}}(k)U_1(k) + U_2(k) + E^{2\sqrt{3}}(k)U_3(k)}{\Delta(k)} =\\ &\frac{E^{-2\sqrt{3}}(ak)U_1(ak) + U_2(ak) + E^{2\sqrt{3}}(ak)U_2(ak)}{\Delta(ak)} + \frac{\Gamma_{123}(ak)}{\Delta(k)\Delta(ak)},\end{aligned} \tag{2.2.55}$$

where $k \in \mathbb{C}/\{k; \Delta(k) = 0 \cup \Delta(ak) = 0\}$ and $\Gamma_{lmn}(k)$ are the known functions defined in (2.2.48). Furthermore, employing the substitution $k \to \bar{a}k$ and the rotations $1 \to 3$,

$3 \to 2$, $2 \to 1$ on the subscripts of (2.2.55), we find

$$\frac{U_1(\bar{a}k) + E^{2\sqrt{3}}(\bar{a}k)U_2(\bar{a}k) + E^{-2\sqrt{3}}(\bar{a}k)U_3(\bar{a}k)}{\Delta(\bar{a}k)} = \frac{U_1(k) + E^{2\sqrt{3}}(k)U_2(k) + E^{-2\sqrt{3}}(k)U_3(k)}{\Delta(k)} + \frac{\Gamma_{312}(k)}{\Delta(k)\Delta(\bar{a}k)}. \tag{2.2.56}$$

Similarly the substitution $k \to ak$ and the rotations $1 \to 2$, $2 \to 3$, $3 \to 1$ on the subscripts of (2.2.55) yield

$$\frac{E^{2\sqrt{3}}(ak)U_1(ak) + E^{-2\sqrt{3}}(ak)U_2(ak) + U_3(ak)}{\Delta(ak)} = \frac{E^{2\sqrt{3}}(\bar{a}k)U_1(\bar{a}k) + E^{-2\sqrt{3}}(\bar{a}k)U_2(\bar{a}k) + U_2(\bar{a}k)}{\Delta(\bar{a}k)} + \frac{\Gamma_{231}(\bar{a}k)}{\Delta(ak)\Delta(\bar{a}k)}. \tag{2.2.57}$$

Replacing in $\hat{q}_1(k)$ given in (2.2.44) the term $U_1(k)$ with the expression given in (2.2.53) we find:

$$\begin{aligned}\hat{q}_1(k) &= E(-ik)D_1(k) + iE(-ik)\frac{\Gamma_1(ak)}{\Delta(ak)} \\ &+ \frac{i}{\Delta(ak)}\left\{\left[E^2(i\bar{a}k)E^{2\sqrt{3}}(ak) - E^2(iak)E^{-2\sqrt{3}}(ak)\right]U_1(ak)\right. \\ &+ \left[E^2(i\bar{a}k)E^{-2\sqrt{3}}(ak) - E^2(iak)\right]U_2(ak) \\ &+ \left.\left[E^2(i\bar{a}k) - E^2(iak)E^{2\sqrt{3}}(ak)\right]U_3(ak)\right\},\end{aligned} \tag{2.2.58}$$

where we have used that $a = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\bar{a} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $E(k) = e^{k\frac{l}{2\sqrt{3}}}$ and $e(k) = e^{k\frac{l}{2}}$. Equation (2.2.58) can be rewritten in the following form

$$\begin{aligned}\hat{q}_1(k) &= E(-ik)D_1(k) + iE(-ik)\frac{\Gamma_1(ak)}{\Delta(ak)} \\ &+ i\frac{E^2(i\bar{a}k)}{\Delta(ak)}\left[E^{2\sqrt{3}}(ak)U_1(ak) + E^{-2\sqrt{3}}(ak)U_2(ak) + U_3(ak)\right] \\ &- i\frac{E^2(iak)}{\Delta(ak)}\left[E^{-2\sqrt{3}}(ak)U_1(ak) + U_2(ak) + E^{2\sqrt{3}}(ak)U_3(ak)\right].\end{aligned} \tag{2.2.59}$$

Hence, the unknown functions $\{U_j(ak)\}_1^3$ in (2.2.59) yield the following contribution to the solution

$$\begin{aligned}C_1(z) &= \frac{i}{2\pi}\int_{l_1} e^{ikz}E^2(i\bar{a}k)\frac{1}{\Delta(ak)}\left[E^{2\sqrt{3}}(ak)U_1(ak) + E^{-2\sqrt{3}}(ak)U_2(ak) + U_3(ak)\right]dk \\ &- \frac{i}{2\pi}\int_{l_1} e^{ikz}E^2(iak)\frac{1}{\Delta(ak)}\left[E^{-2\sqrt{3}}(ak)U_1(ak) + U_2(ak) + E^{2\sqrt{3}}(ak)U_3(ak)\right]dk.\end{aligned} \tag{2.2.60}$$

Arguments of boundedness and analyticity similar to the "basic facts" used in the symmetric case, allow us to deform the second integral of (2.2.60) from l_1 to l_1', where l_1' is a ray with $-\frac{\pi}{2} < \arg k < -\frac{\pi}{6}$. Indeed, the first two elements are exactly the same. For the third, observe that the terms

$$\left\{ E^{-2\sqrt{3}} \frac{U_j(k)}{\Delta(k)}, \quad \frac{U_j(k)}{\Delta(k)}, \quad E^{2\sqrt{3}} \frac{U_j(k)}{\Delta(k)} \right\}_{j=1}^{3}$$

are bounded and analytic in $\mathbb{C}$ apart from the lines where the zeros of $\Delta(k)$ occur. Indeed, observe that $\Delta(k)$ is dominated by $e^3(k)$ for $Rek > 0$ and by $e^{-3}(k)$ for $Rek < 0$, hence

$$\frac{U_j(k)}{\Delta(k)} \sim \begin{cases} e^{-3}(k)U(k), & Rek > 0 \\ -e^{3}(k)U(k), & Rek < 0. \end{cases}$$

Furthermore

$$E^{-2\sqrt{3}} e^{-3}(k) U_1(k), \quad e^{-3}(k) U_2(k) \quad \text{and} \quad E^{2\sqrt{3}} e^{-3}(k) U_3(k)$$

involve

$$e^{k\left(s-\frac{5l}{2}\right)}, \quad e^{k\left(s-\frac{3l}{2}\right)} \quad \text{and} \quad e^{k\left(s-\frac{l}{2}\right)},$$

respectively, which are bounded for $Rek \geq 0$. Also,

$$E^{-2\sqrt{3}} e^{3}(k) U_1(k), \quad e^{3}(k) U_2(k) \quad \text{and} \quad E^{2\sqrt{3}} e^{3}(k) U_3(k)$$

involve

$$e^{k\left(s+\frac{l}{2}\right)}, \quad e^{k\left(s+\frac{3l}{2}\right)} \quad \text{and} \quad e^{k\left(s+\frac{5l}{2}\right)},$$

respectively, which are bounded for $Rek \leq 0$. Applying now equation (2.2.55) to the deformed integral, we obtain the following expression:

$$\begin{aligned} C_1(z) = &\frac{i}{2\pi} \int_{l_1} e^{ikz} E^2(i\bar{a}k) \frac{1}{\Delta(ak)} \left[E^{2\sqrt{3}}(ak) U_1(ak) + E^{-2\sqrt{3}}(ak) U_2(ak) + U_3(ak) \right] dk \\ &- \frac{i}{2\pi} \int_{l_1'} e^{ikz} E^2(iak) \frac{1}{\Delta(k)} \left[E^{-2\sqrt{3}}(k) U_1(k) + U_2(k) + E^{2\sqrt{3}}(k) U_3(k) \right] dk \\ &+ \frac{i}{2\pi} \int_{l_1'} e^{ikz} E^2(iak) \frac{\Gamma_{123}(ak)}{\Delta(k)\Delta(ak)} dk. \end{aligned} \tag{2.2.61}$$

In summary, the term $\hat{q}_1(k)$ gives rise to the contribution $\widetilde{F}_1 + \widetilde{U}_1$, where $\widetilde{U}_1$ denotes the first two terms of the rhs of (2.2.61) and $\widetilde{F}_1$ is defined by

$$\begin{aligned}\widetilde{F}_1(z) &= \frac{1}{2\pi}\int_{l_1} e^{ikz}E(-ik)\left[D_1(k) + i\frac{\Gamma_1(ak)}{\Delta(ak)}\right]dk \\ &+ \frac{i}{2\pi}\int_{l_1'} e^{ikz}E^2(iak)\frac{\Gamma_{123}(ak)}{\Delta(k)\Delta(ak)}dk.\end{aligned} \tag{2.2.62}$$

The contributions of $\hat{q}_2(ak)$ and $\hat{q}_3(\bar{a}k)$ to the solution, i.e. $\widetilde{F}_2(z)+\widetilde{U}_2(z)$ and $\widetilde{F}_3(z)+\widetilde{U}_3(z)$, respectively, are obtained in a similar way. Using equations (2.2.54) and (2.2.57) we find that

$$\begin{aligned}\widetilde{F}_2(z) &= \frac{1}{2\pi}\int_{l_2} e^{ikz}E(-iak)\left[D_2(ak) + i\frac{\Gamma_2(\bar{a}k)}{\Delta(\bar{a}k)}\right]dk \\ &+ \frac{i}{2\pi}\int_{l_2'} e^{ikz}E^2(i\bar{a}k)\frac{\Gamma_{231}(\bar{a}k)}{\Delta(ak)\Delta(\bar{a}k)}dk,\end{aligned} \tag{2.2.63}$$

where l_2' is a ray with $\frac{5\pi}{6} < \arg k < \frac{7\pi}{6}$. This result is also obtained from $\widetilde{F}_1(z)$ by making the substitution $k \to ak$ on the arguments and the rotations $1 \to 2$, $2 \to 3$, $3 \to 1$ on the subscripts of the functions of the integrand.

Using (2.2.52) and (2.2.56) we find that

$$\begin{aligned}\widetilde{F}_3(z) &= \frac{1}{2\pi}\int_{l_3} e^{ikz}E(-i\bar{a}k)\left[D_3(\bar{a}k) + i\frac{\Gamma_3(k)}{\Delta(k)}\right]dk \\ &+ \frac{i}{2\pi}\int_{l_3'} e^{ikz}E^2(ik)\frac{\Gamma_{312}(k)}{\Delta(\bar{a}k)\Delta(k)}dk,\end{aligned} \tag{2.2.64}$$

where l_3' is a ray with $\frac{\pi}{6} < \arg k < \frac{\pi}{2}$. This result is also obtained from $\widetilde{F}_1(z)$ by making the substitution $k \to \bar{a}k$ on the arguments and the rotations $1 \to 3$, $3 \to 2$, $2 \to 1$ on the subscripts of the functions of the integrand.

The contribution of $\{\widetilde{U}_j\}_1^3$ vanishes due to analyticity. Indeed, following similar arguments with those used to obtain $\widetilde{F}_2$ and $\widetilde{F}_3$, we find that

$$\begin{aligned}\widetilde{U}_2(z) &= \frac{i}{2\pi}\int_{l_2} e^{ikz}E^2(ik)\frac{1}{\Delta(\bar{a}k)}\left[U_1(\bar{a}k) + E^{2\sqrt{3}}(\bar{a}k)U_2(\bar{a}k) + E^{-2\sqrt{3}}(\bar{a}k)U_3(\bar{a}k)\right]dk \\ &- \frac{i}{2\pi}\int_{l_2'} e^{ikz}E^2(i\bar{a}k)\frac{1}{\Delta(ak)}\left[E^{2\sqrt{3}}(ak)U_1(ak) + E^{-2\sqrt{3}}(ak)U_2(ak) + U_3(ak)\right]dk\end{aligned} \tag{2.2.65}$$

and

$$\begin{aligned}\widetilde{U}_3(z) = &\frac{i}{2\pi}\int_{l_3} e^{ikz}E^2(iak)\frac{1}{\Delta(k)}\left[E^{-2\sqrt{3}}(k)U_1(k) + U_2(k) + E^{2\sqrt{3}}(k)U_3(k)\right] dk \\ &- \frac{i}{2\pi}\int_{l_3'} e^{ikz}E^2(ik)\frac{1}{\Delta(\bar{a}k)}\left[U_1(\bar{a}k) + E^{2\sqrt{3}}(\bar{a}k)U_2(\bar{a}k) + E^{-2\sqrt{3}}(\bar{a}k)U_3(\bar{a}k)\right] dk.\end{aligned} \tag{2.2.66}$$

The integrands which occur in $l_1 \cup l_2'$, $l_2 \cup l_3'$, $l_3 \cup l_1'$, are bounded and analytic in the corresponding domains, see Figure 2.4.

Hence, the solution is given by

$$q(z) = \widetilde{F}_1(z) + \widetilde{F}_2(z) + \widetilde{F}_3(z), \tag{2.2.67}$$

where $\widetilde{F}_1(z)$, $\widetilde{F}_2(z)$ and $\widetilde{F}_3(z)$ are given in (2.2.62), (2.2.63) and (2.2.64), respectively. Equation (2.2.67) yields the solution of the Dirichlet problem given in (2.2.46).

2.3 The Modified Helmholtz Equation.

In this section we discuss the modified Helmholtz equation, which is equation (2.1.13) with the choices $\{\beta > 0, \gamma = -1\}$, i.e.

$$q_{z\bar{z}} - \beta^2 q = 0, \quad \text{where } z = x + iy. \tag{2.3.1}$$

In order to formulate the general solution for the modified Helmholtz equation in the interior of a convex polygon Ω, we will state a theorem analogous to Theorem 2.1. A similar procedure is followed in [9]. In this respect, we construct the following differential form

$$W(z,\bar{z},k) = e^{-i\beta\left(kz - \frac{\bar{z}}{k}\right)}\left[(q_z + ik\beta q)\, dz - \left(q_{\bar{z}} + \frac{\beta}{ik}q\right) d\bar{z}\right], \; k \in \mathbb{C}, \tag{2.3.2}$$

which is closed iff the modified Helmholtz equation is satisfied, i.e.,

$$dW = 2e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}[q_{z\bar{z}} - \beta^2 q]d\bar{z} \wedge dz, \;\; k \in \mathbb{C}. \tag{2.3.3}$$

Note that we can obtain this differential form from the formal adjoint of the equation (2.3.1). Indeed, the formal adjoint $\tilde{q}$ also satisfies the modified Helmholtz equation

$$\tilde{q}_{z\bar{z}} - \beta^2 \tilde{q} = 0. \tag{2.3.4}$$

Multiplying equation (2.3.1) by $\tilde{q}$, equation (2.3.4) by q and subtracting, we find

$$\tilde{q}q_{z\bar{z}} - q\tilde{q}_{z\bar{z}} = 0, \tag{2.3.5}$$

or equivalently

$$\frac{\partial}{\partial z}\left(\tilde{q}q_{\bar{z}} - \tilde{q}_{\bar{z}}q\right) + \frac{\partial}{\partial \bar{z}}\left(q\tilde{q}_z - q_z\tilde{q}\right) = 0. \tag{2.3.6}$$

This implies that the differential form

$$\widetilde{W}(z, \bar{z}, k) = -\left(q\tilde{q}_z - q_z\tilde{q}\right) dz + \left(\tilde{q}q_{\bar{z}} - \tilde{q}_{\bar{z}}q\right) d\bar{z}, \tag{2.3.7}$$

is closed. Using that $e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}$ is a special solution of equation (2.3.4) we obtain the differential form $W(z, \bar{z}, k)$ defined in (2.3.2).

In what follows, we will use the spectral analysis of the differential form

$$d\left[e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\mu(z, k)\right] = W(z, \bar{z}, k), \;\; k \in \mathbb{C}, \tag{2.3.8}$$

to obtain an integral representation for q in Ω. Also, the following global relation, due to Green's theorem, is valid

$$\int_{\partial\Omega} W(z, \bar{z}, k) = 0, \quad k \in \mathbb{C}. \tag{2.3.9}$$

If q is real, another independent global relation can be obtained from equation (2.3.9), via Schwarz conjugation, i.e. by replacing $W(z, \bar{z}, k)$ with $\overline{W}(z, \bar{z}, \bar{k})$ in (2.3.9). This yields

$$\int_{\partial\Omega} e^{i\beta\left(k\bar{z}-\frac{z}{k}\right)}\left[\left(q_{\bar{z}} - ik\beta q\right) d\bar{z} - \left(q_z - \frac{\beta}{ik}q\right) dz\right] = 0, \quad k \in \mathbb{C}. \tag{2.3.10}$$

Actually (2.3.10) is valid even if q is not real. Indeed, replacing in equation (2.3.2) k by $\frac{1}{k}$ it follows that $W(z,\bar{z},\frac{1}{k})$ is closed iff equation (2.3.1) is satisfied; then, Green's theorem for the closed differential form $W(z,\bar{z},\frac{1}{k})$ yields equation (2.3.10).

The following theorem, which can be also found in [9] and [43], gives the formulae for the global relation and the integral representation for the modified Helmholtz's equations in the interior of a convex polygon.

Theorem 2.2. *Let Ω be the interior of a convex closed polygon in the complex z-plane, with corners $z_1,\ldots,z_n,z_{n+1}\equiv z_1$. Assume that there exists a solution $q(z,\bar{z})$ of the modified Helmholtz equation, i.e. of equation (2.3.1), valid on Ω and suppose that this solution has sufficient smoothness all the way to the boundary of the polygon.*

Then q can be expressed in the form

$$q(z,\bar{z})=\frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\hat{q}_j(k)\frac{dk}{k}, \tag{2.3.11}$$

where $\{\hat{q}_j(k)\}_1^n$ are defined by

$$\hat{q}_j(k)=\int_{z_j}^{z_{j+1}}e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[(q_z+ik\beta q)\,dz-\left(q_{\bar{z}}+\frac{\beta}{ik}q\right)d\bar{z}\right],\ \ k\in\mathbb{C}, \tag{2.3.12}$$

and $\{l_j\}_1^n$ are the rays in the complex k-plane

$$l_j=\{k\in\mathbb{C}:argk=-arg(z_{j+1}-z_j)\},\ \ j=1,\ldots,n \tag{2.3.13}$$

oriented from zero to infinity.

Furthermore, the following global relations are valid

$$\sum_{j=1}^{n}\hat{q}_j(k)=0,\quad \sum_{j=1}^{n}\tilde{q}_j(k)=0,k\in\mathbb{C}, \tag{2.3.14}$$

where $\{\tilde{q}_j(k)\}_1^n$ *are defined by*

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta\left(k\bar{z}-\frac{z}{k}\right)} \left[(q_{\bar{z}} - ik\beta q)\, d\bar{z} - \left(q_z - \frac{\beta}{ik} q \right) dz \right]. \tag{2.3.15}$$

Proof. We will follow the same steps as in the proof of Theorem 2.1, i.e. we will perform the spectral analysis of the differential form (2.3.8), with W defined by equation (2.3.2).

Integrating equation (2.3.8) we find that for $z \in \Omega$

$$\mu_j(z,\bar{z},k) = \int_{z_j}^{z} e^{i\beta[k(z-\zeta)-\frac{1}{k}(\bar{z}-\bar{\zeta})]} \left[(q_\zeta + ik\beta q) d\zeta - \left(q_{\bar{\zeta}} + \frac{\beta}{ik} q \right) d\bar{\zeta} \right]. \tag{2.3.16}$$

This is an entire function of k which is bounded as $k \to \infty$ and $k \to 0$ in the sector Σ_j of the complex k-plane defined by (2.2.13). Indeed, equation (2.3.16) involves the two exponentials

$$e^{i\beta k(z-\zeta)}, \quad e^{-\frac{i\beta}{k}(\bar{z}-\bar{\zeta})} = e^{-\frac{i\beta\bar{k}}{|k|^2}(\bar{z}-\bar{\zeta})}.$$

The real part of these two exponentials have the same sign, thus the exponentials have identical domains of boundedness as k and $1/k$ tend to infinity.

The differential form (2.3.8) is equivalent to the following Lax pair,

$$\mu_z - i\beta k\mu = q_z + i\beta kq, \quad \mu_{\bar{z}} + \frac{i\beta}{k}\mu = -\left(q_{\bar{z}} + \frac{\beta}{ik} q \right). \tag{2.3.17}$$

The first of these equations suggests that

$$\mu = -q + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.3.18}$$

This can be verified using equation (2.3.16) with $k \in \Sigma_j$ and integration by parts. Also subtracting equation (2.3.16) and the analogous equation for μ_{j+1} we find

$$\mu_j - \mu_{j+1} = e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)} \hat{q}_j(k), \quad k \in l_j, \tag{2.3.19}$$

where $\{\hat{q}_j\}_1^n$ are defined by equation (2.3.12).

The solution of the RH problem defined by (2.3.18) and (2.3.19) is given for all $k \in \mathbb{C} \backslash \left\{ \bigcup \{l_j\}_1^n \right\}$ by

$$\mu = -q + \frac{1}{2i\pi} \sum_{j=1}^{n} \int_{l_j} e^{i\beta\left(lz-\frac{\bar{z}}{l}\right)} \hat{q}_j(l) \frac{dl}{l-k}, \quad z \in \Omega. \tag{2.3.20}$$

Substituting this expression in the second of equations (2.3.17) we find equation (2.3.11).

Using in equations (2.3.9) and (2.3.10) the definitions of $\{\hat{q}_j\}_1^n$ and $\{\tilde{q}_j\}_1^n$ (i.e. equations (2.3.12) and (2.3.15)), we find the global relations (2.3.14).

□

Using the identities (2.1.12), which expresses $q_z dz$ and $q_{\bar{z}} d\bar{z}$ in terms of $\dot{q}$ and q_N, in the equations (2.3.12) and (2.3.15), the expression for $\hat{q}_j$ and $\tilde{q}_j$ become

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz - \frac{\bar{z}}{k}\right)} \left[iq_N + i\beta\left(\frac{1}{k}\frac{d\bar{z}}{ds} + k\frac{dz}{ds}\right) q\right] ds, \; k \in \mathbb{C} \tag{2.3.21}$$

and

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta\left(k\bar{z} - \frac{z}{k}\right)} \left[iq_N + i\beta\left(k\frac{d\bar{z}}{ds} + \frac{1}{k}\frac{dz}{ds}\right) q\right] ds, \; k \in \mathbb{C}, \tag{2.3.22}$$

respectively.

2.3.1 The Symmetric Dirichlet Problem.

In what follows we will solve the symmetric Dirichlet problem for the modified Helmholtz equation in the interior of the equilateral triangle D, i.e. we will solve the problem with the boundary conditions

$$q^{(j)}(s) = d(s), \; s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \; j = 1, 2, 3,$$

where the function $d(s)$ has sufficient smoothness and is compatible at the vertices of the triangle, i.e. $d(\frac{l}{2}) = d(-\frac{l}{2})$.

It turns out that the analysis of Laplace and modified Helmholtz equations is very similar. Indeed, applying the parametrization of the fundamental domain on the general solution (2.3.11) we obtain

$$\hat{q}_1(k) = \hat{q}(k), \quad \hat{q}_2(k) = \hat{q}(ak), \quad \hat{q}_3(k) = \hat{q}(\bar{a}k),$$

with

$$\hat{q}(k) = E(-ik)[iU(k) + D(k)], \tag{2.3.23}$$

where

$$E(k) = e^{\beta\left(k+\frac{1}{k}\right)\frac{l}{2\sqrt{3}}}, \quad D(k) = \beta\left(\frac{1}{k} - k\right)\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k+\frac{1}{k}\right)s} d(s)ds,$$

$$U(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k+\frac{1}{k}\right)s} q_N(s)ds, \; k \in \mathbb{C}.$$

Hence, we obtain the following integral representation:

$$\begin{aligned} q(z,\bar{z}) = & \frac{1}{4i\pi}\int_{l_1} A(k,z,\bar{z})E(-ik)\left(D(k) + \frac{G(k)}{\Delta(ak)}\right)\frac{dk}{k} \\ & + \frac{1}{4i\pi}\int_{l_1'} A(k,z,\bar{z})E^2(iak)\frac{G(k)}{\Delta(ak)\Delta(k)}\frac{dk}{k} \end{aligned} \tag{2.3.24}$$

where

$$A(k,z,\bar{z}) = e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} + e^{i\beta\left(\bar{a}kz - \frac{\bar{z}}{\bar{a}k}\right)} + e^{i\beta\left(akz - \frac{\bar{z}}{ak}\right)}, \tag{2.3.25.1}$$

$$G(k) = \Delta^+(ak)D(k) + 2D(\bar{a}k) + \Delta^+(k)D(ak), \tag{2.3.25.2}$$

$$\Delta(k) = e(k) - e(-k), \; \Delta^+(k) = e(k) + e(-k), \; e(k) = e^{\beta\left(k+\frac{1}{k}\right)\frac{l}{2}}. \tag{2.3.25.3}$$

Following, now, step by step the analysis of the symmetric Dirichlet problem for the Laplace equation we derive the solution (2.3.24).

Remark 2.1. *Notice that the three "basic facts" used for the Laplace equation remain true, but slightly more complicated to prove. In particular,*

1. *The zeros of $\Delta(k)$ occur when $k + \frac{1}{k} \in e^{-i\frac{\pi}{2}}\mathbb{R} \Rightarrow k \in e^{-i\frac{\pi}{2}}\mathbb{R}$.*
   Therefore by rotation

$$\Delta(ak) = 0 \Rightarrow k \in e^{\frac{i5\pi}{6}}\mathbb{R} \quad \textit{and} \quad \Delta(\bar{a}k) = 0 \Rightarrow k \in e^{\frac{i\pi}{6}}\mathbb{R}.$$

2. *The functions*

$$e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}E^2(iak), \quad e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}E^2(ik), \quad e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}E^2(i\bar{a}k), \tag{2.3.26}$$

with z in the interior of the triangle, are bounded as $k \to 0$ and $k \to \infty$, for $\arg k$ *in*

$$\left[-\frac{\pi}{2}, \frac{\pi}{6}\right], \quad \left[\frac{\pi}{6}, \frac{5\pi}{6}\right], \quad \left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$$

respectively, see Figure 2.5. Indeed, let us consider the first exponential in (2.3.26). *Using $z_1 = -l\alpha/\sqrt{3}$, this exponential can be written as*

$$e^{i\beta k(z-z_1)+\frac{\beta(\bar{z}-\bar{z}_1)}{ik}}.$$

If z is in the interior of the triangle then

$$\frac{\pi}{2} \leq \arg(z - z_1) \leq \frac{5\pi}{6},$$

thus, if

$$-\frac{\pi}{2} \leq \arg k \leq \frac{\pi}{6},$$

it follows that

$$0 \leq \arg\left[k(z - z_1)\right] \leq \pi.$$

Hence, the exponentials

$$e^{i\beta k(z-z_1)} \quad \text{and} \quad e^{\frac{\beta(\bar{z}-\bar{z}_1)}{ik}}$$

are bounded as $|k| \to \infty$ and $|k| \to 0$ respectively. The analogous results for the second and third exponentials in (2.3.26) *can be obtained in a similar way.*

3. *The functions $\frac{U(k)}{\Delta(k)}$, $\frac{U(ak)}{\Delta(ak)}$ and $\frac{U(\bar{a}k)}{\Delta(\bar{a}k)}$ are bounded and analytic in $\mathbb{C}$ except for k on the lines where the functions $\Delta(k)$, $\Delta(ak)$ and $\Delta(\bar{a}k)$ have zeros.*

   Indeed, regarding $\frac{U(k)}{\Delta(k)}$ observe that $\Delta(k)$ is dominated by $e(k)$ for $Rek > 0$ and by $e(-k)$ for $Rek < 0$, hence

$$\frac{U(k)}{\Delta(k)} \sim \begin{cases} e(-k)U(k), & Rek > 0 \\ -e(k)U(k), & Rek < 0. \end{cases}$$

   Furthermore $e(-k)U(k)$ involves $e^{\left(k+\frac{1}{k}\right)(s-\frac{l}{2})}$ which is bounded for $Rek \geq 0$ and $e(k)U(k)$ involves $e^{\left(k+\frac{1}{k}\right)(s+\frac{l}{2})}$ which is bounded for $Rek \leq 0$.

Example 2.2. *Set $l = \pi$ and $d(s) = \cos s$.*

Hence,

$$D(k) = 2\beta \frac{\frac{1}{k} - k}{1 + \beta^2 \left(k + \frac{1}{k}\right)^2} \cosh\left[\beta\left(k + \frac{1}{k}\right)\frac{\pi}{2}\right]$$

and

$$G(k)=4\beta\left[\frac{\frac{1}{k}-k}{1+\beta^2\left(k+\frac{1}{k}\right)^2}+\frac{\frac{1}{ak}-ak}{1+\beta^2\left(ak+\frac{1}{ak}\right)^2}\right]\cosh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right]\cosh\left[\beta\left(ak+\frac{1}{ak}\right)\frac{\pi}{2}\right]$$
$$+4\beta\frac{\frac{1}{\bar{a}k}-\bar{a}k}{1+\beta^2\left(\bar{a}k+\frac{1}{\bar{a}k}\right)^2}\cosh\left[\beta\left(\bar{a}k+\frac{1}{\bar{a}k}\right)\frac{\pi}{2}\right], \tag{2.3.27}$$

where we have used that $\Delta^+(k)=2\cosh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right]$.

In order to verify the convergence of the integrals in the representation (2.3.24) observe the following:

- *For the first integral along* l_1, $Re(k)=0$ *and* $Im(k)<0$, *hence as* $k\to 0$ *and* $k\to\infty$*:*
  1. $e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E(-ik)\sim e^{\beta\left(ik+\frac{1}{ik}\right)\left(Re(z)-\frac{\pi}{2\sqrt{3}}\right)}\sim e^{-\beta\left(t+\frac{1}{t}\right)\left(x-\frac{\pi}{2\sqrt{3}}\right)},\ t<0,\ x<\frac{\pi}{2\sqrt{3}}$;
  2. $D(k)\sim\frac{1}{k+\frac{1}{k}}$;
  3. $\frac{G(k)}{\Delta(ak)}\sim\frac{1}{k+\frac{1}{k}}$.
- *For the second integral along* l_1', $\arg k\in(-\frac{\pi}{2},-\frac{\pi}{6})$, *hence as* $k\to 0$ *and* $k\to\infty$*:*
  1. $e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E^2(iak)\sim exp\left[\left(x-\frac{\pi}{2\sqrt{3}}\right)\cos\left(\phi+\frac{\pi}{2}\right)-\left(y+\frac{\pi}{2}\right)\sin\left(\phi+\frac{\pi}{2}\right)\right]$, *where* $x<\frac{\pi}{2\sqrt{3}}$, $y>-\frac{\pi}{2}$ *and* $\phi=\arg k$. *Thus the associated exponent has negative real part when* $\arg k\in\left(-\frac{\pi}{2},-\frac{\pi}{6}\right)$ *i.e. in the domain of the deformation of* l_1';
  2. $\frac{G(k)}{\Delta(k)\Delta(ak)}\sim\frac{1}{k+\frac{1}{k}}$.

Similar arguments are valid for the other two terms of $A(k,z,\bar{z})$.

Example 2.3. *Set* $l=2L$, $d_1(s)=\cos(n\frac{2\pi}{L}s)$ *and* $d_2(s)=\sin\left(n\frac{2\pi}{L}s\right)$, $n\in\mathbb{Z}$.

After some calculations we find

$$D_1(k)=2\frac{\beta\left(\frac{1}{k^2}-k^2\right)}{\beta^2\left(k+\frac{1}{k}\right)^2+\left(\frac{2n\pi}{L}\right)^2}\sinh\left[\beta\left(k+\frac{1}{k}\right)L\right]$$

and

$$D_2(k)=2\frac{\frac{2n\pi}{L}\left(\frac{1}{k}-k\right)}{\beta^2\left(k+\frac{1}{k}\right)^2+\left(\frac{2n\pi}{L}\right)^2}\sinh\left[\beta\left(k+\frac{1}{k}\right)L\right],$$

$$\Delta^{+}(k) = 2\cosh\left[\beta\left(k+\frac{1}{k}\right)L\right], \quad \Delta(k) = 2\sinh\left[\beta\left(k+\frac{1}{k}\right)L\right].$$

The convergence of the integrals in the representation of the solution can be verified easily.

2.3.2 The Poincaré Problem.

Replacing in the definition of $\hat{q}_j$, in (2.3.21), the term $q_N^{(j)}$ with

$$\frac{1}{\sin\delta_j}\left[f_j - \cos\delta_j \frac{dq^{(j)}}{ds} + \chi_j q^{(j)}\right],$$

and integrating by parts the term involving $\frac{dq^{(j)}}{ds}$, we find the following:

$$\begin{aligned}
\hat{q}_1(k) &= iE(-ik)\left[H_1(k)\Psi_1(k) + F_1(k) + C_1(k)\right],\\
\hat{q}_2(k) &= iE(-iak)\left[H_2(ak)\Psi_2(ak) + F_2(ak) + C_2(ak)\right]\\
\hat{q}_3(k) &= iE(-i\bar{a}k)\left[H_3(\bar{a}k)\Psi_3(\bar{a}k) + F_3(\bar{a}k) + C_3(\bar{a}k)\right],
\end{aligned} \tag{2.3.28}$$

where

$$H_j(k) = i\beta\left(ke^{i\delta_j} + \frac{1}{ke^{i\delta_j}}\right) + \chi_j, \tag{2.3.29}$$

the function $F_j(k)$ is known, $\Psi_j(k)$ involves the unknown Dirichlet boundary values

$$\begin{aligned}
F_j(k) &= \frac{1}{\sin\delta_j}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k+\frac{1}{k}\right)s} f_j(s)ds,\\
\Psi_j(k) &= \frac{1}{\sin\delta_j}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k+\frac{1}{k}\right)s} q^{(j)}(s)ds
\end{aligned} \tag{2.3.30}$$

and the function $C_j(k)$ involves the values of $q(x,y)$ at the vertices,

$$C_j(k) = \frac{\cos\delta_j}{\sin\delta_j}\left[e(-k)q^{(j)}\left(-\frac{l}{2}\right) - e(k)q^{(j)}\left(\frac{l}{2}\right)\right]. \tag{2.3.31}$$

Applying equations (2.3.28) in the first of the global relations (2.3.14) we find that

$$E(-ik)H_1(k)\Psi_1(k) + E(-iak)H_2(ak)\Psi_2(ak) + E(-i\bar{a}k)H_3(\bar{a}k)\Psi_3(\bar{a}k) = F(k) + C(k), \tag{2.3.32}$$

where $F(k)$ is the known function

$$F(k) = E(-ik)F_1(k) + E(-iak)F_2(ak) + E(-i\bar{a}k)F_3(\bar{a}k)$$

and $C(k)$ depends on $\{C_j(k)\}_1^3$ and also involve the values of $q(x,y)$ at the vertices:

$$\begin{aligned} C(k) = {} & \frac{\cos\delta_1}{\sin\delta_1} E(-ik) e^{\left(k+\frac{\lambda}{k}\right)s} q_1(s)\Big|_{-\frac{l}{2}}^{\frac{l}{2}} + \frac{\cos\delta_2}{\sin\delta_2} E(-iwk) e^{\left(wk+\frac{\lambda}{wk}\right)s} q_2(s)\Big|_{-\frac{l}{2}}^{\frac{l}{2}} \\ & + \frac{\cos\delta_3}{\sin\delta_3} E(-i\bar{w}k) e^{\left(\bar{w}k+\frac{\lambda}{\bar{w}k}\right)s} q_3(s)\Big|_{-\frac{l}{2}}^{\frac{l}{2}}. \end{aligned} \tag{2.3.33}$$

Here, we will sketch the method used to solve this problem, following precisely the same steps used for the general Dirichlet problem in the Laplace equation. First, we formulate the global relation and then formulate a system of *6 equations* involving the *9 unknowns*

$$\{\Psi_j(k),\ \Psi_j(ak),\ \Psi_j(\bar{a}k)\}_{j=1}^3 .$$

Then we find the analogue of equation (2.2.52) which is an expression of $\Psi_3(\bar{a}k)$ in terms of $\{\Psi_j(k)\}_1^3$ (see [32]):

$$D^3(k)H_3(\bar{a}k)\Psi_3(\bar{a}k) = \sum_{j=1}^{3} \Gamma_j^3(k)H_j(k)\Psi_j(k) + T^3(k) + C^3(k) \tag{2.3.34}$$

$$
\begin{aligned}
D^3(k) &= \frac{P_1(\bar{a}k)}{P_2(ak)P_3(ak)}\left(e^{-3}(k) - e^3(k)\frac{P_1(ak)P_2(ak)P_3(ak)}{P_1(\bar{a}k)P_2(\bar{a}k)P_3(\bar{a}k)}\right),\\
\Gamma_1^3(k) &= \frac{1}{P_1(k)}e(-\bar{a}k) - e^2(-k)e(\bar{a}k)\frac{P_1(\bar{a}k)}{P_2(ak)P_3(ak)},\\
\Gamma_2^3(k) &= e^2(k)e(-\bar{a}k)\frac{P_1(ak)}{P_3(k)P_3(\bar{a}k)} - e(\bar{a}k)\frac{1}{P_2(ak)},\\
\Gamma_3^3(k) &= e^2(-k)e(-\bar{a}k)\frac{P_1(\bar{a}k)}{P_2(k)P_2(ak)} - e^2(k)e(\bar{a}k)\frac{1}{P_3(\bar{a}k)},
\end{aligned}
\tag{2.3.35}
$$

with

$$
P_j(k) = \frac{H_j(k)}{\overline{\overline{H_j(\bar{k})}}}
\tag{2.3.36}
$$

and $\{T^j(k)\}_1^3$ involves the known functions $\{F(k), \overline{F}(\bar{k}), F(ak), \overline{F}(a\bar{k}), F(\bar{a}k), \overline{F}(\bar{a}\bar{k})\}$ and $\{C^j(k)\}_1^3$ involves $\{C(k), \overline{C}(\bar{k}), C(ak), \overline{C}(a\bar{k}), C(\bar{a}k), \overline{C}(\bar{a}\bar{k})\}$.

Making the substitution $k \to ak$ and the rotations $3 \to 1$, $1 \to 2$, $2 \to 3$ on the subscripts of (2.3.34), we find the analogue of equation (2.2.53) for the Poincaré problem, namely, we find the equation

$$
D^1(ak)H_1(k)\Psi_1(k) = \sum_{j=1}^{3}\Gamma_j^1(ak)H_j(ak)\Psi_j(ak) + T^1(ak) + C^1(ak),
\tag{2.3.37}
$$

where $D^1(k)$ and $\{\Gamma_j^1(k)\}_1^3$ are obtained by making the rotations $3 \to 1$, $1 \to 2$, $2 \to 3$ on the subscripts of $D^3(k)$ and $\{\Gamma_j^3(k)\}_1^3$ in (2.3.35). Similarly, making the substitution $k \to \bar{a}k$ and the rotations $3 \to 2$, $2 \to 1$, $1 \to 3$ on the subscripts of (2.3.34) we find the analogue of equation (2.2.54) for the Poincaré problem, namely, we find the equation

$$
D^2(\bar{a}k)H_2(ak)\Psi_2(ak) = \sum_{j=1}^{3}\Gamma_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k) + T^2(\bar{a}k) + C^2(\bar{a}k),
\tag{2.3.38}
$$

where $D^2(k)$ and $\{\Gamma_j^2(k)\}_1^3$ are obtained by making the rotations $3 \to 2$, $2 \to 1$, $1 \to 3$ on the subscripts of $D^3(k)$ and $\{\Gamma_j^3(k)\}_1^3$ in (2.3.35).

Replacing in the expression $\hat{q}_1(k)$, defined by (2.3.28), the term $H_1(k)\Psi_1(k)$ with the expression given in (2.3.37) we find

$$\begin{aligned}\hat{q}_1(k) &= iE(-ik)\left[F_1(k)+C_1(k)\right] + i\frac{E(-ik)}{D^1(ak)}\left[T^1(ak)+C^1(ak)\right]\\ &+ i\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak) - i\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}A_j^1(ak)H_j(ak)\Psi_j(ak),\end{aligned}\tag{2.3.39}$$

where we have used the notation $\Gamma_j^i(k) = \Theta_j^i(k) - A_j^i(k)$. Hence, the contribution of $\hat{q}_1(k)$ to the solution is

$$\begin{aligned}\widetilde{C}_1(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E(-ik)\left(F_1(k)+C_1(k)+\frac{T^1(ak)+C^1(ak)}{D^1(ak)}\right)\frac{dk}{k}\\ &+\frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak)\frac{dk}{k}\\ &-\frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}A_j^1(ak)H_j(ak)\Psi_j(ak)\frac{dk}{k}.\end{aligned}\tag{2.3.40}$$

Considerations of boundedness and analyticity, allow us to deform the integral of the last term from l_1 to l_1', where l_1' is a ray with $-\frac{\pi}{2} \le \arg k \le -\frac{\pi}{6}$.

Remark 2.2. *The analysis of the zeros of $D^1(ak)$ is now slightly more complicated. Indeed, if $D^1(k) = 0$ then k does not necessarily belong to the imaginary axis. However, the following relations are valid*

$$\lim_{|k|\to 0}|P(k)| = 1, \qquad \lim_{|k|\to\infty}|P(k)| = 1.$$

The definition of $D^3(k)$ in (2.3.35) implies that

$$D^3(k) = 0 \Rightarrow e^6(k) = \frac{P_1(\bar{a}k)P_2(\bar{a}k)P_3(\bar{a}k)}{P_1(ak)P_2(ak)P_3(ak)}.$$

The last equation yields

$$\lim_{|k|\to 0}|e(k)| = 1, \qquad \lim_{|k|\to\infty}|e(k)| = 1.$$

Hence, the roots of $D^3(k)$ are on the imaginary axis as $|k|\to 0$ and $|k|\to\infty$. The same is true for $\{D^j(k)\}_1^3$. Thus,

$$D^1(ak) = 0 \Rightarrow k \in e^{i\frac{5\pi}{6}}\mathbb{R}, \quad |k|\to 0, |k|\to\infty.$$

The last integral in (2.3.40)*, in addition to the deformed integral, also yields a* finite *sum of unknown functions evaluated at some zeros of* $D^1(ak)$*, which can be computed in terms of known functions by equation* (2.3.37)*. However, choosing* l_1' *to be an appropriate curve, oriented from 0 to infinity, in the domain* $D_1 = \{k \in \mathbb{C} : -\frac{\pi}{2} \le \arg k \le -\frac{\pi}{6}\}$*, it is possible to avoid all these poles.*

Similarly, replacing in the expression of $\hat{q}_2(k)$, defined in (2.3.28) the term $H_2(ak)\Psi_2(ak)$ with the expression given in (2.3.38) we find

$$\begin{aligned}\hat{q}_2(k) = iE(-iak)\left[F_2(ak) + C_2(ak)\right] + i\frac{E(-iak)}{D^2(\bar{a}k)}\left[T^2(\bar{a}k) + C^2(\bar{a}k)\right]\\ + i\frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}\Theta_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k) - i\frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}A_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k).\end{aligned} \tag{2.3.41}$$

Hence, the contribution of $\hat{q}_2(k)$ to the solution is

$$\begin{aligned}\widetilde{C}_2(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_2} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}E(-iak)\left[F_2(ak) + C_2(ak) + \frac{T^2(\bar{a}k) + C^2(\bar{a}k)}{D^2(\bar{a}k)}\right]\frac{dk}{k}\\ &+ \frac{1}{4\pi}\int_{l_2} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}\frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}\Theta_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k)\frac{dk}{k}\\ &- \frac{1}{4\pi}\int_{l_2'} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}\frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}A_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k)\frac{dk}{k} + S_2(k_n),\end{aligned} \tag{2.3.42}$$

where l_2' is a ray with $\frac{5\pi}{6} < \arg k < \frac{7\pi}{6}$ and $S_2(k_n)$ is a finite sum of known functions. In analogy with the earlier results we expect that the following relation is valid

$$\begin{aligned}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak) = \frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}A_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k)\\ + E^2(i\bar{a}k)\frac{T^{231}(\bar{a}k) + C^{231}(\bar{a}k)}{D^1(ak)D^2(\bar{a}k)},\end{aligned} \tag{2.3.43}$$

where $T^{231}(k)$ and $C^{231}(k)$ are functions of $\{T^j\}_1^3$ and $\{C^j\}_1^3$, respectively. In order to verify this equation, we compute $\Psi_1(ak)$ and $\Psi_3(ak)$ by making the rotations $2 \to 1$, $1 \to 3$, $3 \to 2$ and $2 \to 3$, $3 \to 1$, $1 \to 1$ in equation (2.3.38), respectively. Similarly, we compute $\Psi_1(\bar{a}k)$ and $\Psi_2(\bar{a}k)$ from equation (2.3.34). Hence, computing the rhs and the lhs side of equation (2.3.43) we conclude that this equation is indeed valid iff the following condition is valid

$$P_1(k)P_1(ak)P_1(\bar{a}k) = P_2(k)P_2(ak)P_2(\bar{a}k) = P_3(k)P_3(ak)P_3(\bar{a}k).$$

Employing in this expression the definitions of $P_j(k)$ and $H_j(k)$ given in (2.3.36) and in (2.3.29) respectively, we find the conditions (2.1.8)-(2.1.10). Furthermore,

$$T^{lmn}(k) = e(ak)\sum_{j=1}^{3}\Theta_j^n(\bar{a}k)T^j(k) \quad \text{and} \quad C^{lmn}(k) = e(ak)\sum_{j=1}^{3}\Theta_j^n(\bar{a}k)C^j(k).$$

Equation (2.3.43) is the analogue to equation (2.2.57). Thus, we also obtain the analogue of the equations (2.2.55) and (2.2.56):

$$\begin{aligned}\frac{E(-i\bar{a}k)}{D^3(k)}\sum_{j=1}^{3}\Theta_j^3(k)H_j(k)\Psi_j(k) = \frac{E(-ik)}{D^1(ak)}&\sum_{j=1}^{3}A_j^1(ak)H_j(ak)\Psi_j(ak)\\ &+E^2(iak)\frac{T^{123}(ak)+C^{123}(ak)}{D^3(k)D^1(ak)}\end{aligned} \tag{2.3.44}$$

and

$$\begin{aligned}\frac{E(-iak)}{D^2(\bar{a}k)}\sum_{j=1}^{3}\Theta_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k) = \frac{E(-i\bar{a}k)}{D^3(k)}&\sum_{j=1}^{3}A_j^3(k)H_j(k)\Psi_j(k)\\ &+E^2(ik)\frac{T^{312}(k)+C^{312}(k)}{D^2(\bar{a}k)D^3(k)}.\end{aligned} \tag{2.3.45}$$

Employing equations (2.3.39) and (2.3.44) in the integral representation (2.3.11) we conclude that the contribution of $\hat{q}_1(k)$ to the solution is

$$\begin{aligned}\widetilde{C}_1(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E(-ik)\left(F_1(k)+C_1(k)+\frac{T^1(ak)+C^1(ak)}{D^1(ak)}\right)\frac{dk}{k}\\ &+\frac{1}{4\pi}\int_{l_1'} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E^2(iak)\frac{T^{123}(ak)+C^{123}(ak)}{D^3(k)D^1(ak)}\frac{dk}{k} \quad + \quad S_1(k_n)\\ &+\frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak)\frac{dk}{k}\\ &-\frac{1}{4\pi}\int_{l_1'} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-i\bar{a}k)}{D^3(k)}\sum_{j=1}^{3}\Theta_j^3(k)H_j(k)\Psi_j(k)\frac{dk}{k}.\end{aligned} \tag{2.3.46}$$

The solution is given by

$$q(z)=\widetilde{F}_1(z)+\widetilde{F}_2(z)+\widetilde{F}_3(z), \tag{2.3.47}$$

where $\widetilde{F}_1(z)$ is given by the first three terms of equation (2.3.46); $\widetilde{F}_2(z)$ is obtained from $\widetilde{F}_1(z)$ by substituting the arguments of the functions of the integrand with $k\to ak$ and using the rotations $1\to 2$, $2\to 3$, $3\to 1$; $\widetilde{F}_3(z)$ is obtained from $\widetilde{F}_1(z)$ by substituting the arguments of the functions of the integrand with $k\to \bar{a}k$ and using the rotations $1\to 3$, $3\to 2$, $2\to 1$.

Indeed, define $\widetilde{\Psi}_1$ to be equal to the last two terms of equation (2.3.46), i.e.

$$\begin{aligned}\widetilde{\Psi}_1(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_1} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak)\frac{dk}{k}\\ &-\frac{1}{4\pi}\int_{l_1'} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-i\bar{a}k)}{D^3(k)}\sum_{j=1}^{3}\Theta_j^3(k)H_j(k)\Psi_j(k)\frac{dk}{k}.\end{aligned} \tag{2.3.48}$$

Thus, the contribution of $\hat{q}_2(k)$ and $\hat{q}_3(k)$ are

$$\begin{aligned}\widetilde{\Psi}_2(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_2} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-iak)}{D^1(\bar{a}k)}\sum_{j=1}^{3}\Theta_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k)\frac{dk}{k}\\ &-\frac{1}{4\pi}\int_{l_2'} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-ik)}{D^1(ak)}\sum_{j=1}^{3}\Theta_j^1(ak)H_j(ak)\Psi_j(ak)\frac{dk}{k}\end{aligned} \tag{2.3.49}$$

and

$$\begin{aligned}\widetilde{\Psi}_3(z,\bar{z}) =& \frac{1}{4\pi}\int_{l_3} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-i\bar{a}k)}{D^3(k)}\sum_{j=1}^{3}\Theta_j^3(k)H_j(k)\Psi_j(k)\frac{dk}{k}\\ &-\frac{1}{4\pi}\int_{l_3'} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\frac{E(-iak)}{D^2(k)}\sum_{j=1}^{3}\Theta_j^2(\bar{a}k)H_j(\bar{a}k)\Psi_j(\bar{a}k)\frac{dk}{k}.\end{aligned} \tag{2.3.50}$$

The contribution of $\{\widetilde{\Psi}_j\}_1^3$ to the solution vanishes, because the integrands which occur in $l_1\cup l_2',\ l_2\cup l_3',\ l_3\cup l_1'$, are bounded and analytic in the corresponding domains, see Figure 2.5.

Remark 2.3. *The corner term $C(k)$ which involves the values of the function q at the vertices, given in* (2.3.33)*, vanishes iff*

$$\cot\delta_1 = \cot\delta_2 = \cot\delta_3 \quad \Leftrightarrow \{\sin(\delta_1-\delta_2)=0,\ \sin(\delta_2-\delta_3)=0\}, \tag{2.3.51}$$

since $q_1\left(\frac{l}{2}\right)=q_2\left(-\frac{l}{2}\right),\ q_2\left(\frac{l}{2}\right)=q_3\left(-\frac{l}{2}\right),\ q_3\left(\frac{l}{2}\right)=q_1\left(-\frac{l}{2}\right)$. In this case, $C^j(k)=0,\ j=1,2,3,$ in the integral representation of the solution (2.3.47).

If equation (2.3.51) *is not valid then we can find the solution via the integral representation of the solution* (2.3.47) *in terms of $C^j(k)$, i.e. in terms of $q_j\left(\frac{l}{2}\right)$, and then evaluate the solution at these points. Hence, we can determine the associated values by solving a 3×3 system of linear equations.*

Remark 2.4. *Having solved the Poincaré problem, we can then immediately obtain the solutions of the Neumann, Robin and oblique Robin problems via appropriate limits. Indeed, we can solve:*

- *the Neumann problem, by putting $\delta_j=\frac{\pi}{2}, \chi_j=0,\ j=1,2,3;$*
- *the Robin problem, by putting $\delta_j=\frac{\pi}{2},$ and $\chi_j=\chi\neq 0,\ j=1,2,3;$*
- *the oblique Robin problem, by putting $\delta_j=\delta,$ and $\chi_j=\chi,\ j=1,2,3.$*

In all these three cases, observe that the corner terms always vanish, i.e. $\cot\delta_1=\cot\delta_2=\cot\delta_3$. Furthermore, for the Neumann and Robin problem the poles of the integrands have the same distribution as in the Dirichlet case. Indeed, $P_j(k)=1$. Hence, the definition of $D^3(k)$ in (2.3.35) *implies that $D^j(k)=e^{-3}(k)-e^3(k),\ j=1,2,3;$ this coincides with the definition of $\Delta(k)$ in* (2.2.47).

2.4 The "Generalized Helmholtz" Equation.

In this section we discuss the "generalized Helmholtz" equation (2.1.13), i.e.

$$q_{z\bar{z}} + \gamma\beta^2 q = 0, \tag{2.4.1}$$

where $\beta > 0$ and $\gamma \in \mathbb{C}$ with $\{\ |\gamma| = 1,\ \gamma \neq 1\}$.

In order to formulate the general solution of the generalized Helmholtz equation in the interior of a convex polygon Ω, we will state the analogue to Theorem 2.1. In this respect, we consider the following differential form

$$W(z,\bar{z},k) = e^{-i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}\left[(q_z + ik\beta q)\,dz - \left(q_{\bar{z}} + \gamma\frac{i\beta}{k}q\right)d\bar{z}\right],\ k \in \mathbb{C}, \tag{2.4.2}$$

which is closed iff the generalized Helmholtz equation is satisfied, i.e.,

$$dW = 2e^{-i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}[q_{z\bar{z}} + \gamma\beta^2 q]d\bar{z}\wedge dz,\ k \in \mathbb{C}. \tag{2.4.3}$$

In what follows, the spectral analysis of the differential form

$$d\left[e^{-i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}\mu(z,k)\right] = W(z,\bar{z},k),\ k \in \mathbb{C} \tag{2.4.4}$$

yields an integral representation for q in Ω. Also, the following global relation is valid

$$\int_{\partial\Omega} W(z,\bar{z},k) = 0, \quad k \in \mathbb{C}. \tag{2.4.5}$$

Another independent global relation can be obtained from equation (2.4.2) by replacing k with $-\frac{\gamma}{k}$. It follows that $W(z,\bar{z},-\frac{\gamma}{k})$ is closed iff equation (2.4.1) is satisfied; Green's theorem for the closed differential form $W(z,\bar{z},-\frac{\gamma}{k})$ yields the global relation

$$\int_{\partial\Omega} e^{i\beta\left(\frac{\gamma}{k}z+k\bar{z}\right)}\left[(q_{\bar{z}} - ik\beta q)\,d\bar{z} - \left(q_z - \frac{i\beta\gamma}{k}q\right)dz\right] = 0, \quad k \in \mathbb{C}. \tag{2.4.6}$$

Theorem 2.3. *Let Ω be the interior of a convex closed polygon in the complex z-plane, with corners $z_1, \ldots, z_n, z_{n+1} \equiv z_1$. Assume that there exists a solution $q(z, \bar{z})$ of the generalized Helmholtz equation, i.e. of equation (2.4.1), valid on Ω and suppose that this solution has sufficient smoothness all the way to the boundary of the polygon.*

Then q can be expressed in the form

$$q(z, \bar{z}) = \frac{1}{4\pi i} \sum_{j=1}^{n} \int_{l_j} e^{i\beta\left(kz + \gamma\frac{\bar{z}}{k}\right)} \hat{q}_j(k) \frac{dk}{k}, \tag{2.4.7}$$

where $\{\hat{q}_j(k)\}_1^n$ are defined by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz + \gamma\frac{\bar{z}}{k}\right)} \left[(q_z + ik\beta q)\, dz - \left(q_{\bar{z}} + \gamma \frac{i\beta}{k} q \right) d\bar{z} \right], \; k \in \mathbb{C}, \tag{2.4.8}$$

and $\{l_j\}_1^n$ are the contours in the complex k-plane, oriented from zero to infinity:

$$l_j = \left\{ k \in \mathbb{C} : \tan(\phi + \phi_j) = \frac{\sin\theta}{\cos\theta - |k|^2} \right\}, \; j = 1, \ldots, n, \tag{2.4.9}$$

where $\theta = \arg\gamma$, $\phi = \arg k$ and $\phi_j = \arg\{z_{j+1} - z_j\}$; see Figure 2.6.

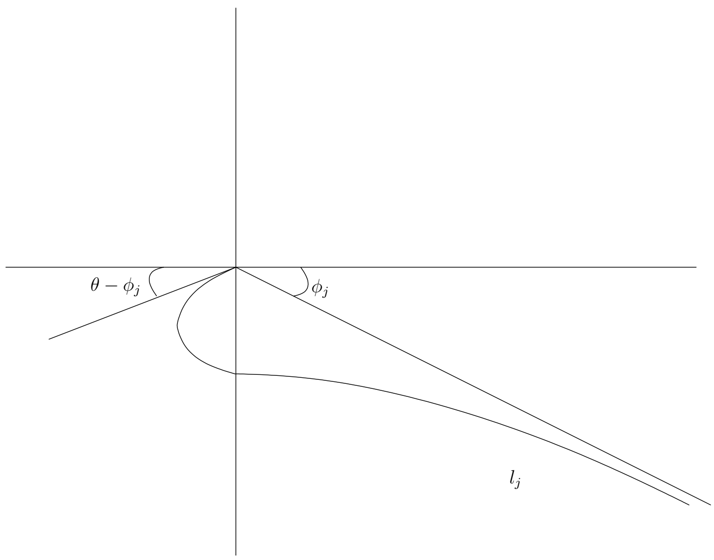

Figure 2.6: The l_j contour.

Furthermore, the following global relations are valid

$$\sum_{j=1}^{n} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{q}_j(k) = 0, k \in \mathbb{C}, \tag{2.4.10}$$

where $\{\tilde{q}_j(k)\}_1^n$ *are defined by*

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta\left(\frac{\gamma}{k}z+k\bar{z}\right)} \left[(q_{\bar{z}} - ik\beta q)\, d\bar{z} - \left(q_z - \frac{i\beta\gamma}{k}q\right) dz\right]. \tag{2.4.11}$$

Proof. We will follow the same steps as in the proof of Theorem 2.1, i.e. we will perform the spectral analysis of the differential form (2.4.4), with W defined by equation (2.4.2).

Integrating equation (2.4.4) we find that for $z \in \Omega$

$$\mu_j(z, \bar{z}, k) = \int_{z_j}^{z} e^{i\beta[k(z-\zeta)+\frac{\gamma}{k}(\bar{z}-\bar{\zeta})]} \left[(q_\zeta + ik\beta q)d\zeta - \left(q_{\bar{\zeta}} + \gamma\frac{i\beta}{k}q\right) d\bar{\zeta}\right]. \tag{2.4.12}$$

This equation involves the following exponential

$$e^{i\beta\left[k(z-\zeta)+\frac{\gamma}{|k|^2}\bar{k}(\bar{z}-\bar{\zeta})\right]}.$$

The real part of this exponentials is bounded as k and $1/k$ tend to infinity, in the domains Σ_j where $\partial\Sigma_j = l_{j-1} \cup \{-l_j\}$, with l_j defined in (2.4.9). Indeed, this exponential is bounded iff

$$Im\left\{k(z-\zeta) + \frac{\gamma}{k}(\bar{z}-\bar{\zeta})\right\} \geq 0. \tag{2.4.13}$$

Introducing the notations $\theta = \arg\gamma$, $\phi = \arg k$ and $\Phi = \arg\{z-\zeta\}$, equation (2.4.13) yields

$$(|k|^2 - \cos\theta)\sin(\phi+\Phi) + \sin\theta\cos(\phi+\Phi) \geq 0, \tag{2.4.14}$$

or, equivalently

$$\sin(\phi+\Phi+\Theta) \geq 0, \quad \tan\Theta = \frac{\sin\theta}{|k|^2 - \cos\theta}, \tag{2.4.15}$$

which yields

$$0 \leq \phi + \Phi + \Theta \leq \pi. \tag{2.4.16}$$

If z is inside the polygon and ζ is on a curve from z to z_j, see Figure 2.2, then

$$\arg(z_{j+1} - z_j) \leq \arg(z-\zeta) \leq \arg(z_{j-1} - z_j), \quad j = 1, \ldots, n. \tag{2.4.17}$$

Using the identity

$$\pi - \arg(z_j - z_{j+1}) = -\arg(z_{j+1} - z_j) = -\phi_j \pmod{2\pi}, \tag{2.4.18}$$

equation (2.4.17) becomes

$$\phi_j \le \Phi \le \phi_{j-1} + \pi, \quad j = 1, \ldots, n. \tag{2.4.19}$$

Hence, the inequalities (2.4.16) are satisfied provided that

$$-\phi_j \le \phi + \Theta \le -\phi_{j-1}, \quad j = 1, \ldots, n.$$

Thus, the boundaries of the domain $\{\Sigma_j\}_1^n$ are defined by

$$\phi + \phi_j + \Theta = 0 \;\text{ and }\; \phi + \phi_{j-1} + \Theta = 0, \quad j = 1, \ldots, n,$$

or, equivalently

$$\tan(\phi + \phi_j) = -\tan\Theta \;\text{ and }\; \tan(\phi + \phi_{j-1}) = -\tan\Theta, \quad j = 1, \ldots, n. \tag{2.4.20}$$

Applying in this equation the definition of Θ given in (2.4.15), we obtain that

$$\partial\Sigma_j = l_{j-1} \cup \{-l_j\}, \quad j = 1, \ldots, n,$$

where $\{l_j\}_1^n$ is given in (2.4.9).

The differential form (2.4.4) is equivalent to the following Lax pair,

$$\mu_z - i\beta k\mu = q_z + i\beta kq, \quad \mu_{\bar z} - \frac{i\gamma\beta}{k}\mu = -\left(q_{\bar z} + \frac{i\gamma\beta}{k}q\right). \tag{2.4.21}$$

The first of these equations suggests that

$$\mu = -q + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.4.22}$$

This can be verified using equation (2.4.12) with $k \in \Sigma_j$ and integration by parts. Also subtracting equation (2.4.12) and the analogous equation for μ_{j+1} we find

$$\mu_j - \mu_{j+1} = e^{i\beta\left(kz + \gamma\frac{\bar z}{k}\right)}\hat q_j(k), \quad k \in l_j, \tag{2.4.23}$$

where $\{\hat q_j\}_1^n$ are defined by equation (2.4.8).

The solution of the RH problem defined by (2.4.22) and (2.4.23) is given for all $k \in \mathbb{C}\backslash\{\bigcup\{l_j\}_1^n\}$ by

$$\mu = -q + \frac{1}{2i\pi}\sum_{j=1}^{n}\int_{l_j} e^{i\beta\left(lz+\gamma\frac{\bar{z}}{l}\right)}\hat{q}_j(l)\frac{dl}{l-k}, \quad z \in \Omega. \tag{2.4.24}$$

Substituting this expression in the second of equations (2.4.21) we find equation (2.4.7).

Using in equations (2.4.5) and (2.4.6) the definitions of $\{\hat{q}_j\}_1^n$ and $\{\tilde{q}_j\}_1^n$ (i.e. equations (2.4.8) and (2.4.11)), we find the global relations (2.4.10).

□

Remark 2.5. *For the behavior of l_j observe the following:*

As $|k| \to \infty \Rightarrow \tan(\phi+\phi_j) = 0 \Rightarrow \phi = -\phi_j$. *Hence, these curves asymptote at infinity to the curve $\{l_j\}_1^3$ defined for the modified Helmholtz in equation* (2.3.13).

As $|k| \to 0 \Rightarrow \tan(\phi+\phi_j) = \tan\theta \Rightarrow \phi = \theta - \phi_j$. *Hence, these curves have as tangent lines at 0 the curves $\{k \in \mathbb{C} : \arg k = \theta - \phi_j\}$, see Figure 2.6.*

Remark 2.6. *Substituting $\gamma = -1$ in the formulae of Theorem 2.3 yields precisely the relations appearing in Theorem 2.2 for the modified Helmholtz. Furthermore, in this case $\theta = \pi$, which yields* $\tan(\phi+\phi_j) = 0$, $j = 1,\ldots,n$. *Thus* $\phi = -\phi_j$, $j = 1,\ldots,n$, *which is the definition of $\{l_j\}_1^n$ given in* (2.3.13).

Using in equations (2.4.8) and (2.4.11) the identities (2.1.12), which expresses $q_z dz$ and $q_{\bar{z}} d\bar{z}$ in terms of $\dot{q}$ and q_N, the expressions for $\hat{q}_j$ and $\tilde{q}_j$ become

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}\left[iq_N + i\beta\left(-\frac{\gamma}{k}\frac{d\bar{z}}{ds} + k\frac{dz}{ds}\right)q\right]ds, \ k \in \mathbb{C} \tag{2.4.25}$$

and

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta\left(\frac{\gamma}{k}z+k\bar{z}\right)}\left[iq_N + i\beta\left(k\frac{d\bar{z}}{ds} - \frac{\gamma}{k}\frac{dz}{ds}\right)q\right]ds, \ k \in \mathbb{C}, \tag{2.4.26}$$

respectively.

2.4.1 The Symmetric Dirichlet problem in the Equilateral Triangle.

Here, we will solve of the symmetric Dirichlet problem for the generalized Helmholtz equation in the interior of the equilateral triangle D, i.e. we will solve the problem with boundary conditions

$$q^{(j)}(s) = d(s), \ s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \ j = 1, 2, 3,$$

where the function $d(s)$ has sufficient smoothness and is compatible at the vertices of the triangle, i.e. $d(\frac{l}{2}) = d(-\frac{l}{2})$.

The analysis of the generalized Helmholtz equation is identical to the analysis of the Laplace and the modified Helmholtz. Indeed, applying the parametrization of the fundamental domain on the general solution (2.4.7) we obtain that:

$$\hat{q}_1(k) = \hat{q}(k), \quad \hat{q}_2(k) = \hat{q}(ak), \quad \hat{q}_3(k) = \hat{q}(\bar{a}k),$$

with

$$\hat{q}(k) = E(-ik)[iU(k) + D(k)], \tag{2.4.27}$$

where

$$E(k) = e^{\beta\left(k-\frac{\gamma}{k}\right)\frac{l}{2\sqrt{3}}}, \quad D(k) = -\beta\left(\frac{\gamma}{k} + k\right)\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k-\frac{\gamma}{k}\right)s} d(s)ds,$$

$$U(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k-\frac{\gamma}{k}\right)s} q_N(s)ds, \ k \in \mathbb{C}.$$

Hence, we obtain the following integral representation:

$$\begin{aligned} q(z,\bar{z}) = &\frac{1}{4i\pi}\int_{l_1} A(k,z,\bar{z})E(-ik)\left(D(k) + \frac{G(k)}{\Delta(ak)}\right)\frac{dk}{k} \\ &+ \frac{1}{4i\pi}\int_{l_1'} A(k,z,\bar{z})E^2(iak)\frac{G(k)}{\Delta(ak)\Delta(k)}\frac{dk}{k}, \end{aligned} \tag{2.4.28}$$

where,

$$l_1 = \left\{ k \in \mathbb{C} : \tan(\phi + \frac{\pi}{2}) = \frac{\sin\theta}{\cos\theta - |k|^2} \right\}, \tag{2.4.29a}$$

$$l_2 = \left\{ k \in \mathbb{C} : \tan(\phi - \frac{5\pi}{6}) = \frac{\sin\theta}{\cos\theta - |k|^2} \right\}, \tag{2.4.29b}$$

$$l_3 = \left\{ k \in \mathbb{C} : \tan(\phi - \frac{\pi}{6}) = \frac{\sin\theta}{\cos\theta - |k|^2} \right\}, \tag{2.4.29c}$$

$$l_1' \in \hat{D}_1, \text{ with } \partial\hat{D}_1 = l_1 \cup -\tilde{l}_2 \text{ and } \tilde{l}_j = \{k \in \mathbb{C} : -k \in l_j\}, \; j = 1, 2, 3, \tag{2.4.29d}$$

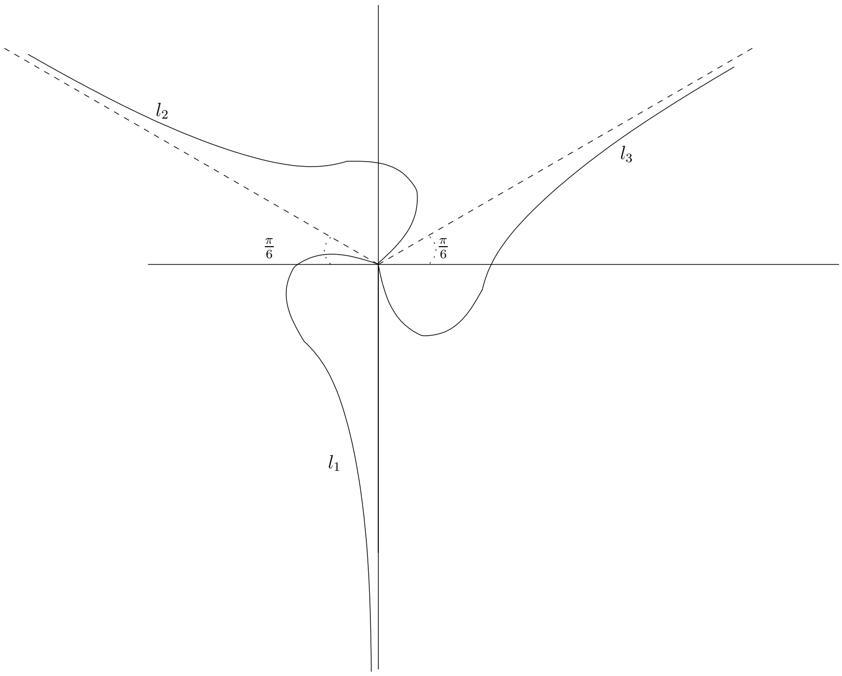

Figure 2.7: *The curves $\{l_j\}_1^3$ for the equilateral triangle.*

$$A(k, z, \bar{z}) = e^{i\beta\left(kz + \gamma\frac{\bar{z}}{k}\right)} + e^{i\beta\left(\bar{a}kz + \gamma\frac{\bar{z}}{\bar{a}k}\right)} + e^{i\beta\left(akz + \gamma\frac{\bar{z}}{ak}\right)}, \tag{2.4.30a}$$

$$G(k) = \Delta^+(ak)D(k) + 2D(\bar{a}k) + \Delta^+(k)D(ak), \tag{2.4.30b}$$

$$\Delta(k) = e(k) - e(-k), \; \Delta^+(k) = e(k) + e(-k), \; e(k) = e^{\beta\left(k - \frac{\gamma}{k}\right)\frac{l}{2}}. \tag{2.4.30c}$$

Following, now, step by step the analysis of the symmetric Dirichlet problem for the Laplace equation, we can derive the solution (2.4.28).

Remark 2.7. *The three "basic facts" used for the Laplace equation remain true, but it is now slightly more complicated to prove them. In particular:*

1. *The zeros of* $\Delta(k)$ *occur when* $k - \frac{\gamma}{k} \in e^{-i\frac{\pi}{2}}\mathbb{R} \Rightarrow k \in\ l_1 \cup \tilde{l}_1$, *where* $\tilde{l}_j = \{k \in \mathbb{C} : -k \in l_j\}$, $j = 1, 2, 3$. *Therefore by rotation,*

$$\Delta(ak) = 0 \Rightarrow k \in\ l_2 \cup \tilde{l}_2 \quad \text{and} \quad \Delta(\bar{a}k) = 0 \Rightarrow k \in\ l_3 \cup \tilde{l}_3.$$

2. *The functions*

$$e^{i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}E^2(iak), \quad e^{i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}E^2(i\bar{a}k), \quad e^{i\beta\left(kz+\gamma\frac{\bar{z}}{k}\right)}E^2(ik), \tag{2.4.31}$$

with z *in the interior of the triangle, are bounded as* $k \to 0$ *and* $k \to \infty$, *for* $k \in\ D_1,\ D_2,\ D_3$, *respectively, where the boundaries of these domains are defined respectively by*

$$\partial D_1 = \{-l_3\} \cup l_1, \quad \partial D_2 = \{-l_1\} \cup l_2, \quad \partial D_3 = \{-l_2\} \cup l_3,$$

see Figure 2.8. Indeed, let us consider the first exponential in (2.4.31).

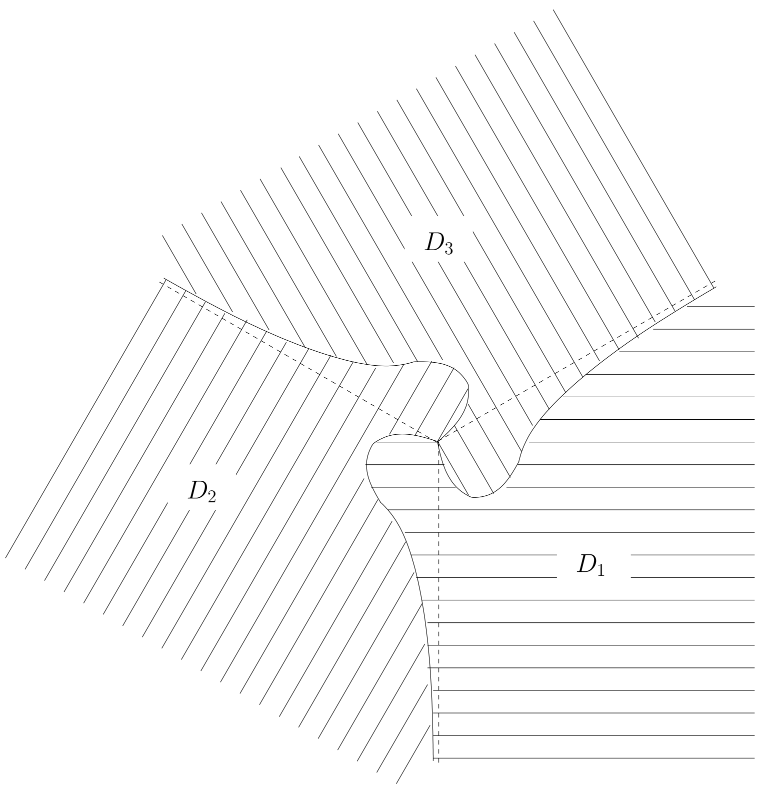

Figure 2.8: *The domains of boundedness and analyticity* $\{D_j\}_1^3$.

Using $z_1 = -la\sqrt{3}$*, this exponential can be written as*

$$e^{i\beta k(z-z_1)+\frac{i\beta\gamma(\bar{z}-\bar{z}_1)}{k}}. \tag{2.4.32}$$

If z *is in the interior of the triangle then*

$$\frac{\pi}{2} \leq \arg(z - z_1) \leq \frac{5\pi}{6}.$$

Using the notations $\phi = \arg k$*,* $\theta = \arg \gamma$ *and* $\psi = \arg\{z - z_1\}$*, we find that the exponential given in* (2.4.32) *is bounded iff*

$$(|k|^2 - \cos\theta)\sin(\phi + \psi) + \sin\theta\cos(\phi + \psi) \geq 0.$$

Using exactly the same analysis used for the inequality (2.4.14)*, we now find that the relevant exponential is bounded, when* $k \in D_1$*. The analogous results for the second and third exponentials in* (2.3.26) *can be obtained in a similar way.*

3. *In exactly the same way as in the Laplace equation we can prove that* $\frac{U(k)}{\Delta(k)}$ *is bounded and analytic in* $\mathbb{C} - \{l_1 \cup \tilde{l}_1\}$.

Chapter 3

Eigenvalues for the Laplace operator in the interior of an equilateral triangle.

An important role in the Fokas method is played by the *global relation*, which for linear PDEs is an equation in the spectral (or Fourier) space coupling the given boundary data with the unknown boundary values [7]-[9].

3.1 Formulation of the problems.

Let $q(z, \bar{z})$ satisfy the Helmholtz equation in the interior of an equilateral triangle, namely

$$\frac{\partial^2 q}{\partial z \partial \bar{z}} - \lambda q = 0, \ z \in D, \lambda < 0, \tag{3.1.1}$$

where D denotes the interior of the equilateral triangle defined in Chapter 2, but for convenience we will make the substitution $l = 2L$. Hence, D has vertices at $\{z_j\}_1^3$, where

$$z_1 = \frac{2L}{\sqrt{3}} e^{-i\frac{\pi}{3}}, \ z_2 = \frac{2L}{\sqrt{3}} e^{i\frac{\pi}{3}}, \ z_3 = -\frac{2L}{\sqrt{3}}, \quad L \text{ positive constant.} \tag{3.1.2}$$

This triangle is depicted in Figure 3.1; the sides $(z_1, z_2), (z_2, z_3), (z_3, z_1)$ will be referred to as side $(1), (2), (3)$ respectively.

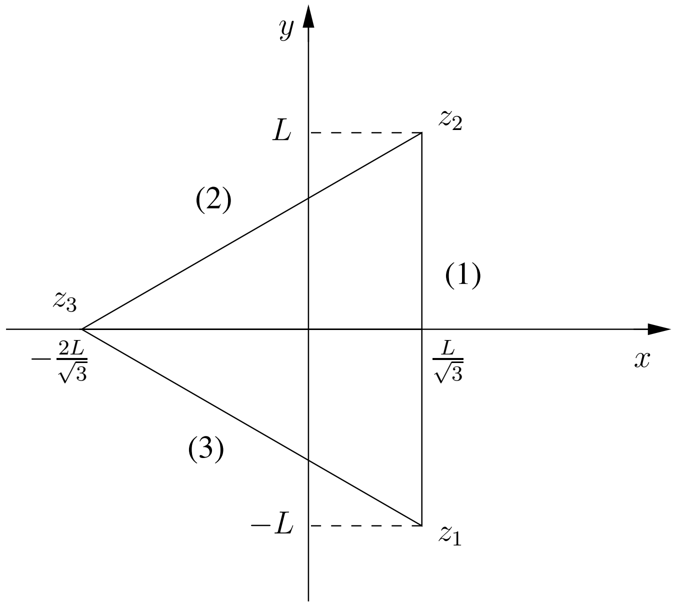

Figure 3.1: *The Equilateral Triangle.*

Eigenvalues

The method followed in Chapter 2 suggests the definition of the following differential form:

$$W = \left[e^{-ikz-\frac{\lambda}{ik}\bar{z}}(q_z + ikq)\right]dz - \left[e^{-ikz-\frac{\lambda}{ik}\bar{z}}\left(q_{\bar{z}} + \frac{\lambda}{ik}q\right)\right]d\bar{z},\ k \in \mathbb{C}, \tag{3.1.3}$$

which is closed iff equation (3.1.1) is satisfied. Indeed,

$$dW = 2e^{-ikz-\frac{\lambda}{ik}\bar{z}}(q_{z\bar{z}} - \lambda q).$$

Hence, the complex form of Green's theorem yields the following global relation:

$$\int_{\partial D}\left\{\left[e^{-ikz-\frac{\lambda}{ik}\bar{z}}(q_z + ikq)\right]dz - \left[e^{-ikz-\frac{\lambda}{ik}\bar{z}}\left(q_{\bar{z}} + \frac{\lambda}{ik}q\right)\right]d\bar{z}\right\} = 0. \tag{3.1.4}$$

Each of the sides of the triangle can be parametrized as follows:

$$\begin{aligned} z^{(1)}(s) &= \frac{L}{\sqrt{3}} + is, \quad z^{(2)}(s) = \left(\frac{L}{\sqrt{3}} + is\right) w, \\ z^{(3)}(s) &= \left(\frac{L}{\sqrt{3}} + is\right) \bar{w}, \quad -L < s < L, \end{aligned} \tag{3.1.5}$$

where

$$w = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \tag{3.1.6}$$

Hence, equation (3.1.4) becomes

$$\sum_{j=1}^{3} \hat{q}_j(w^{j-1}k) = 0, \tag{3.1.7a}$$

$$\hat{q}_j(k) = e^{\left(-ik+\frac{\lambda}{-ik}\right)\frac{L}{\sqrt{3}}} \int_{-L}^{L} e^{\left(k+\frac{\lambda}{k}\right)s} \left[iq_N^{(j)} - (k - \frac{\lambda}{k})q^{(j)} \right] ds, \; j = 1, 2, 3, \; k \in \mathbb{C}. \tag{3.1.7b}$$

The Dirichlet eigenvalues correspond to the case of $\left\{q^{(j)}\right\}_1^3 = 0$; in this case the global relation (3.1.7a) becomes a single homogeneous equation involving the three unknown functions $\left\{q_N^{(j)}\right\}_1^3$, where q_N denotes the derivative in the outward normal direction. For the Dirichlet problem, it will be shown in section 3.2 that there exist nontrivial functions $\left\{q_N^{(j)}\right\}_1^3$ satisfying this equation provided that

$$\lambda = -(m^2 + mn + n^2)\frac{\pi^2}{9L^2} \; , \; m, n \in \mathbb{Z}. \tag{3.1.8}$$

Similarly, in the case of the Neumann problem $\left\{q_N^{(j)}\right\}_1^3 = 0$, and thus the global relation (3.1.7a) becomes a single homogeneous equations involving the three unknown functions $\left\{q^{(j)}\right\}_1^3$. Moreover, in section 3.3 for the same values of λ, this equation is satisfied by nontrivial functions. These results are the rederivation of the eigenvalues found firstly by Lamé in [34] and later by several authors, see for example [37] and [38].

The Robin, oblique Robin and Poincaré problems correspond to the following boundary conditions:

$$q_N^{(j)} - \chi q^{(j)} = 0, \tag{3.1.9}$$

$$\sin\delta q_N^{(j)} + \cos\delta \frac{dq^{(j)}}{ds} - \chi q^{(j)} = 0, \tag{3.1.10}$$

$$\sin\delta_j q_N^{(j)} + \cos\delta_j \frac{dq^{(j)}}{ds} - \chi_j q^{(j)} = 0, \ \ j = 1, 2, 3, \tag{3.1.11}$$

where $\chi,\ \delta,\ \{\chi_j\}_1^3,\ \{\delta_j\}_1^3$ are real constants and

$$\sin\delta \neq 0, \ \ \sin\delta_j \neq 0. \tag{3.1.12}$$

The Poincaré condition can be rewritten in the form

$$(\sin\delta_j, \cos\delta_j) \cdot \left(\frac{\partial q^{(j)}}{\partial T}, \frac{\partial q^{(j)}}{\partial N}\right) - \chi_j q^{(j)} = 0, \quad j = 1, 2, 3,$$

thus it involves the derivative of q in the direction making an angle δ_j with every side of the triangle, see Figure 3.2.

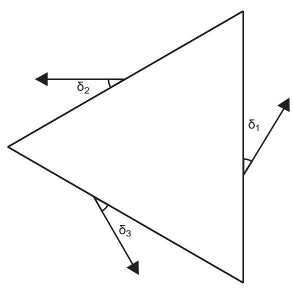

Figure 3.2:

It will be shown in section 3.6 that the method introduced here is still capable of obtaining the associated eigenvalues, provided that the constants $\{\delta_j, \chi_j\}$ satisfy the following constraints:

$$\begin{aligned} &\delta_1 = \delta_2 + \nu\pi, \quad \delta_2 = \delta_3 + \mu\pi, \qquad \nu, \mu = 0, 1, \\ &(\sin 3\delta_1)\left[\left(\chi_1^2 - 3\lambda\right)\chi_1 - (-1)^\nu\left(\chi_2^2 - 3\lambda\right)\chi_2\right] = 0, \\ &(\sin 3\delta_2)\left[\left(\chi_2^2 - 3\lambda\right)\chi_2 - (-1)^\mu\left(\chi_3^2 - 3\lambda\right)\chi_3\right] = 0. \end{aligned} \tag{3.1.13}$$

Eigenfunctions

For linear PDEs the method introduced in [1] and [43] yields an integral representation of the solution which involves certain integrals in the spectral (Fourier) space. In the case of equation (3.1.1) the relevant integral representation is given by

$$q(z,\bar{z}) = \sum_{j=1}^{3} \int_{L_j} e^{ikz+\frac{\lambda}{ik}\bar{z}} \hat{q}_j(w^{j-1}k)\frac{dk}{k} \tag{3.1.14}$$

where $\{\hat{q}_j(k)\}_1^3$ are defined in (3.1.7b) and the contours $\{L_j\}_1^3$, depicted in Figure 3.3, and defined as follows:

$$\begin{aligned}
L_1 =&\Bigg\{k\in\mathbb{C}, \quad \{|k|\le\sqrt{-\lambda}\}\cap\{\arg k=\frac{\pi}{2}\}, \quad \{|k|\ge\sqrt{-\lambda}\}\cap\{\arg k=\frac{3\pi}{2}\},\\
&\quad\{|k|=\sqrt{-\lambda}\}\cap\Bigg\{\{\frac{\pi}{6}>\arg k>\frac{-\pi}{6}\}\cup\{\frac{5\pi}{6}<\arg k<\frac{7\pi}{6}\}\Bigg\}\Bigg\}\\
L_2 =&\Bigg\{k\in\mathbb{C}, \quad \{|k|\le\sqrt{-\lambda}\}\cap\{\arg k=-\frac{\pi}{6}\}, \quad \{|k|\ge\sqrt{-\lambda}\}\cap\{\arg k=\frac{5\pi}{6}\},\\
&\quad\{|k|=\sqrt{-\lambda}\}\cap\Bigg\{\{\frac{3\pi}{2}>\arg k>\frac{7\pi}{6}\}\cup\{\frac{\pi}{6}<\arg k<\frac{\pi}{2}\}\Bigg\}\Bigg\}\\
L_3 =&\Bigg\{k\in\mathbb{C}, \quad \{|k|\le\sqrt{-\lambda}\}\cap\{\arg k=\frac{7\pi}{6}\}, \quad \{|k|\ge\sqrt{-\lambda}\}\cap\{\arg k=\frac{\pi}{6}\},\\
&\quad\{|k|=\sqrt{-\lambda}\}\cap\Bigg\{\{\frac{5\pi}{6}>\arg k>\frac{\pi}{2}\}\cup\{\frac{3\pi}{2}<\arg k<\frac{11\pi}{6}\}\Bigg\}\Bigg\}.
\end{aligned} \tag{3.1.15}$$

Indeed, using arguments similar to those used in the proof of Theorem 2.2 in Chapter 2, we formulate a Riemman-Hilbert problem on the sectors $\{\Sigma_j\}_1^3$ of the k-plane where the exponentials

$$e^{ik(z-\zeta)+\frac{\lambda}{ik}(\bar{z}-\bar{\zeta})}$$

are bounded. This implies that $Re\left\{ik(z-\zeta)+\frac{\lambda}{ik}(\bar{z}-\bar{\zeta})\right\}\le 0$, and equivalently

$$(|k|^2+\lambda)\sin(\phi+\theta)\ge 0,$$

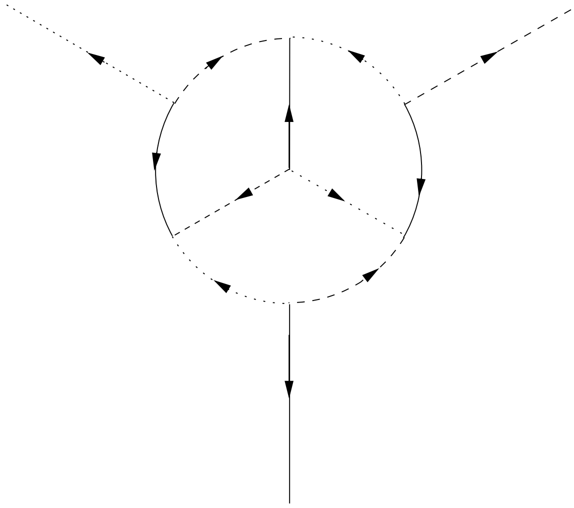

Figure 3.3: *The contours L_j are depicted as follows: L_1——— , $L_2 \cdots\cdots$, L_3 − − − −.*

where $\phi = \arg k$ and $\theta = \arg(z - \zeta)$. Hence,

$$\text{if } |k| \geq \sqrt{-\lambda}, \text{ then } \sin(\phi + \theta) \geq 0$$

and

$$\text{if } |k| \leq \sqrt{-\lambda}, \text{ then } \sin(\phi + \theta) \leq 0.$$

Following the analysis of the proof of Theorem 2.1 these inequalities imply the definitions of the sectors $\{\Sigma_j\}_1^3$ depicted in Figure 3.4. The contours $\{L_j\}_1^3$ are defined as the following intersections $\Sigma_j \cap \Sigma_{j+1}$, $j = 1, 2, 3$.

For the Dirichlet problem $\left\{q^{(j)}\right\}_1^3 = 0$ and $\left\{q_N^{(j)}\right\}_1^3$ are obtained in section 3.2. Hence, the functions $\{\hat{q}_j(k)\}_1^3$ appearing in (3.1.7b) can be computed explicitly and equation (3.1.14) expresses q in terms of integrals involving explicit integrands. By employing Cauchy's theorem it is straightforward to compute the relevant integrals and hence q can be found explicitly. Similar considerations are valid for the other boundary value problems.

The eigenvalues for the Dirichlet, Robin, oblique Robin and Poincaré problems are computed in sections 3.2-3.6, respectively. It is shown in section 3.7 that the formulae defining the eigenvalues of the Poincaré problem yield, via appropriate limits, the corresponding formulae for the oblique Robin, Robin, Neumman and Dirichlet problems. The associated eigenfunctions for the Dirichlet problem are computed in section 3.8.

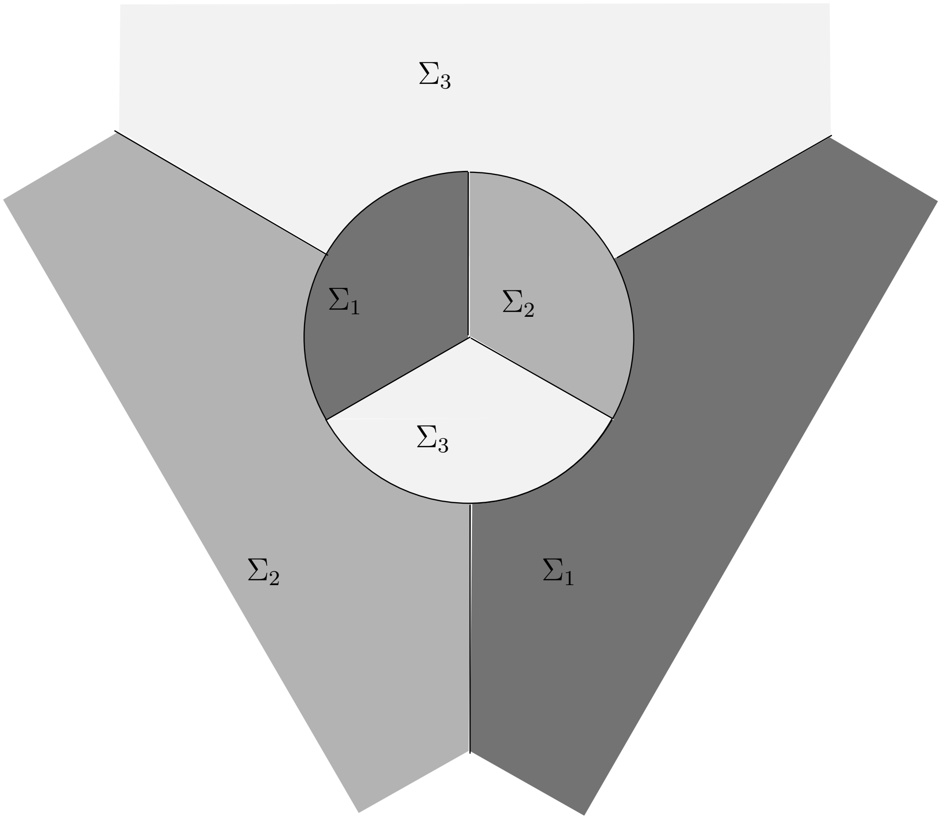

Figure 3.4: *The sectors Σ_j.*

3.2 The Dirichlet Problem.

In this case the global relation (3.1.7a) becomes

$$E(-ik)N_1(k) + E(-iwk)N_2(wk) + E(-i\bar{w}k)N_3(\bar{w}k) = 0, \;\; E(k) = e^{\left(k+\frac{\lambda}{k}\right)\frac{L}{\sqrt{3}}}, \qquad (3.2.1)$$

where $k \in \mathbb{C}$ and the unknown functions $\{N_j\}_1^3$ are defined by

$$N_j(k) = \int_{-L}^{L} e^{\left(k+\frac{\lambda}{k}\right)s} u_j(s)ds, \quad j = 1, 2, 3, \;\; k \in \mathbb{C}, \qquad (3.2.2)$$

with $\{u_j\}_1^3$ denoting the unknown Neumann boundary values.

Proposition 3.1. *Let each of the unknown Neumann boundary values be expressed as a sum of three exponentials, namely*

$$u_1(s) = \sum_{l=1}^{3} \alpha_l e^{ia_l s}, \;\; u_2(s) = \sum_{l=1}^{3} \beta_l e^{ib_l s}, \;\; u_3(s) = \sum_{l=1}^{3} \gamma_l e^{ic_l s}, \qquad (3.2.3)$$

where $\{a_l, b_l, c_l\}_1^3$ *are real constants and* $\{\alpha_l, \beta_l, \gamma_l\}_1^3$ *are complex constants. Then, the global relation* (3.2.1) *implies the following results:*

The constants a_1, a_2, a_3 *are given by the equations*

$$a_1 = (m+2n)\frac{\pi}{3L}, \quad a_2 = (m-n)\frac{\pi}{3L}, \ m,n \in \mathbb{Z}, \quad a_3 = -a_1 - a_2; \tag{3.2.4a}$$

the constants $\{b_l\}_1^3$ *and* $\{c_l\}_1^3$ *can be expressed in terms of* a_1, a_2 *by the equations*

$$b_1 = -a_1 - a_2, \quad b_2 = a_1, \quad b_3 = a_2, \tag{3.2.4b}$$

and

$$c_1 = a_2, \quad c_2 = -a_1 - a_2, \quad c_3 = a_1; \tag{3.2.4c}$$

the value of λ *is given by*

$$-3\lambda = a_1^2 + a_2^2 + a_1 a_2, \tag{3.2.4d}$$

hence (3.2.4a) *implies that the possible eigenvalues are given by*

$$\lambda = -(m^2 + mn + n^2)\frac{\pi^2}{9L^2}\ , \ m,n \in \mathbb{Z}; \tag{3.2.4e}$$

the constants $\{\alpha_l\}_2^3$, $\{\beta_l\}_1^3$ *and* $\{\gamma_l\}_1^3$ *can be expressed in terms of* α_1 *by the equations*

$$\alpha_2 = -(-1)^n \frac{m+n}{m}\alpha_1, \quad \alpha_3 = (-1)^{n+m}\frac{n}{m}\alpha_1, \tag{3.2.4f}$$

$$\beta_1 = e^{-i\frac{m-n}{3}\pi}\frac{n}{m}\alpha_1, \quad \beta_2 = (-1)^m e^{-i\frac{m+2n}{3}\pi}\alpha_1,$$
$$\beta_3 = -(-1)^{n+m}e^{-i\frac{m-n}{3}\pi}\frac{m+n}{m}\alpha_1, \tag{3.2.4g}$$

$$\gamma_1 = -e^{i\frac{2m+n}{3}\pi}\frac{m+n}{m}\alpha_1, \quad \gamma_2 = (-1)^n e^{i\frac{2n+m}{3}\pi}\frac{n}{m}\alpha_1,$$
$$\gamma_3 = (-1)^{n+m}e^{i\frac{m-n}{3}\pi}\alpha_1. \tag{3.2.4h}$$

Proof. In order to compute a typical term appearing in N_j we integrate $\alpha_l \exp[ia_l s] \exp\left[\left(k + \frac{\lambda}{k}\right)s\right]$ with respect to s from $-L$ to L; this yields

$$\frac{\alpha_l}{k + \frac{\lambda}{k} + ia_l}\left[e^{ia_l L}e^{\left(k+\frac{\lambda}{k}\right)L} - e^{-ia_l L}e^{-\left(k+\frac{\lambda}{k}\right)L}\right].$$

Multiplying this expression by $E(-ik)$, making use of the relation $1+i/\sqrt{3} = 2\exp\left[i\pi/6\right]/\sqrt{3}$, and summing the resulting expression over l we find the equation

$$E(-ik)N_1(k) = \sum_{l=1}^{3} \frac{\alpha_l}{k + \frac{\lambda}{k} + ia_l}$$
$$\left\{ e^{ia_l L} \exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}}\right)\right] - e^{-ia_l L} \exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{5i\pi}{6}} + \frac{\lambda}{ke^{-\frac{5i\pi}{6}}}\right)\right]\right\}. \tag{3.2.5}$$

Let k_l and $-\bar{k}_l$ denote the two roots of $k^2 + ika_l + \lambda = 0$, i.e.,

$$k_l = \frac{1}{2}[-ia_l + A_l], \ A_l = \sqrt{-a_l^2 - 4\lambda}, \ l = 1, 2, 3. \tag{3.2.6}$$

Then

$$\frac{\alpha_l}{k + \frac{\lambda}{k} + ia_l} = \frac{\alpha_l}{A_l}\left[\frac{k_l}{k - k_l} + \frac{\bar{k}_l}{k + \bar{k}_l}\right]. \tag{3.2.7}$$

Thus the first terms in the global relation (3.2.1) yields the RHS of (3.2.5) with $\alpha_l / \left(k + \frac{\lambda}{k} + ia_l\right)$ replaced by the RHS of (3.2.7).

The second and third terms can be obtained from the first term using the substitutions $k \to wk$ and $k \to \bar{w}k$ respectively. Thus the second and third terms involve the following exponentials:

$$\exp\left[\frac{2L}{\sqrt{3}}\left(ke^{\frac{i\pi}{2}} + \frac{\lambda}{ke^{\frac{i\pi}{2}}}\right)\right], \ \exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}}\right)\right] \tag{3.2.8a}$$

and

$$\exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{5i\pi}{6}} + \frac{\lambda}{ke^{-\frac{5i\pi}{6}}}\right)\right], \ \exp\left[\frac{2L}{\sqrt{3}}\left(ke^{\frac{i\pi}{2}} + \frac{\lambda}{ke^{\frac{i\pi}{2}}}\right)\right], \tag{3.2.8b}$$

respectively. Furthermore,

$$\frac{\beta_l}{wk + \frac{\lambda}{wk} + ib_l} = \frac{\beta_l}{B_l}\left[\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l}\right] \tag{3.2.9a}$$

and

$$\frac{\gamma_l}{\bar{w}k + \frac{\lambda}{\bar{w}k} + ic_l} = \frac{\gamma_l}{C_l}\left[\frac{w\mu_l}{k - w\mu_l} + \frac{w\bar{\mu}_l}{k + w\bar{\mu}_l}\right], \tag{3.2.9b}$$

where $(\lambda_l, -\bar{\lambda}_l)$ and $(\mu_l, -\bar{\mu}_l)$ denote the roots of

$$k^2 + ikb_l + \lambda = 0, \quad k^2 + ikc_l + \lambda = 0,$$

i.e.,

$$\lambda_l = \frac{1}{2}[-ib_l + B_l], \ B_l = \sqrt{-b_l^2 - 4\lambda}, \ l = 1, 2, 3 \tag{3.2.10}$$

and

$$\mu_l = \frac{1}{2}[-ic_l + C_l],\ C_l = \sqrt{-c_l^2 - 4\lambda},\ l = 1,2,3. \tag{3.2.11}$$

Thus, the second term in the global relation gives rise to a term similar to (3.2.5), but involving the exponentials (3.2.8a) and the expressions in (3.2.9a), whereas the third term in the global relation gives rise to a term similar to (3.2.5), but involving the exponentials (3.2.8b) and the expressions in (3.2.9b). Hence, using the fact that the coefficients of the three exponentials

$$\exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}}\right)\right],\ \exp\left[\frac{2L}{\sqrt{3}}\left(ke^{-\frac{5i\pi}{6}} + \frac{\lambda}{ke^{-\frac{5i\pi}{6}}}\right)\right],$$
$$\exp\left[\frac{2L}{\sqrt{3}}\left(ke^{\frac{i\pi}{2}} + \frac{\lambda}{ke^{\frac{i\pi}{2}}}\right)\right],$$

must vanish, the global relation yields the following set of three equations, each of which is valid for all $k \in \mathbb{C}$:

$$\sum_{l=1}^{3} \frac{\alpha_l e^{ia_l L}}{A_l}\left[\frac{k_l}{k - k_l} + \frac{\bar{k}_l}{k + \bar{k}_l}\right] = \sum_{l=1}^{3} \frac{\beta_l e^{-ib_l L}}{B_l}\left[\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l}\right], \tag{3.2.12}$$

$$\sum_{l=1}^{3} \frac{\beta_l e^{ib_l L}}{B_l}\left[\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l}\right] = \sum_{l=1}^{3} \frac{\gamma_l e^{-ic_l L}}{C_l}\left[\frac{w\mu_l}{k - w\mu_l} + \frac{w\bar{\mu}_l}{k + w\bar{\mu}_l}\right], \tag{3.2.13}$$

$$\sum_{l=1}^{3} \frac{\gamma_l e^{ic_l L}}{C_l}\left[\frac{w\mu_l}{k - w\mu_l} + \frac{w\bar{\mu}_l}{k + w\bar{\mu}_l}\right] = \sum_{l=1}^{3} \frac{\alpha_l e^{-ia_l L}}{A_l}\left[\frac{k_l}{k - k_l} + \frac{\bar{k}_l}{k + \bar{k}_l}\right]. \tag{3.2.14}$$

Equations (3.2.4) can be obtained by solving equations (3.2.12)-(3.2.14); the relevant analysis consists of the following four steps.

The first set of Poles

Equations (3.2.12)-(3.2.14) imply the following relations between the associated poles:

$$k_l = \bar{w}\lambda_l = w\mu_l, \quad l = 1,2,3. \tag{3.2.15}$$

Using these equations and the definitions of $\{k_l, \lambda_l, \mu_l\}_1^3$ it is possible to characterize $\{b_l, B_l\}_1^3$ and $\{c_l, C_l\}_1^3$ in terms of $\{a_l, A_l\}_1^3$:

$$b_l = -\frac{a_l}{2} - \frac{\sqrt{3}}{2}A_l, \quad B_l = \frac{\sqrt{3}}{2}a_l - \frac{A_l}{2}, \quad l = 1,2,3 \tag{3.2.16}$$

and

$$c_l = -\frac{a_l}{2} + \frac{\sqrt{3}}{2}A_l, \quad C_l = -\frac{\sqrt{3}}{2}a_l - \frac{A_l}{2}, \quad l = 1, 2, 3. \tag{3.2.17}$$

Indeed, the first set of equations (3.2.15) yields $\lambda_l = wk_l$, which using equations (3.2.6) and (3.2.10) becomes

$$-ib_l + B_l = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(-ia_l + A_l).$$

The real and the imaginary parts of this equation yields equations (3.2.16). Similarly, the equation $\mu_l = \bar{w}k_l$ yields (3.2.17).

The second set of Poles

For the second set of poles of equations (3.2.12)-(3.2.14), without loss of generality, we make the following associations:

$$\bar{k}_l = \bar{w}\bar{\lambda}_{l-1}, \; \bar{w}\bar{\lambda}_l = w\bar{\mu}_{l-1}, \; w\bar{\mu}_l = \bar{k}_{l-1}, \quad l = 1, 2, 3, \tag{3.2.18}$$

where

$$k_0 = k_3, \; \lambda_0 = \lambda_3, \; \mu_0 = \mu_3. \tag{3.2.19}$$

The equation $k_l = w\lambda_{l-1}$ together with the equation $\lambda_{l-1} = wk_{l-1}$ (see the first of equations in (3.2.15)) yields

$$k_l = w\lambda_{l-1} = wwk_{l-1} = \bar{w}k_{l-1}.$$

Similarly,

$$\lambda_l = w\mu_{l-1} = ww\lambda_{l-1} = \bar{w}\lambda_{l-1}$$

and

$$\mu_l = wk_{l-1} = ww\mu_{l-1} = \bar{w}\mu_{l-1}.$$

Hence,

$$k_l = \bar{w}k_{l-1}, \; \lambda_l = \bar{w}\lambda_{l-1}, \; \mu_l = \bar{w}\mu_{l-1}, \quad l = 1, 2, 3. \tag{3.2.20}$$

Equations (3.2.16), (3.2.17) and (3.2.20) imply equations (3.2.4b)-(3.2.4d). Indeed, the real and imaginary parts of the equation $k_2 = \bar{w}k_1$ yield

$$\sqrt{3}A_1 = a_1 + 2a_2, \quad A_2 = -\frac{\sqrt{3}}{2}a_1 - \frac{A_1}{2}. \tag{3.2.21}$$

Taking the square of the first of these equations and using the definition of A_1 given in (3.2.6), we find (3.2.4d). Furthermore, multiplying the second of the equations (3.2.21) by $\sqrt{3}$ and replacing in the resulting expression $\sqrt{3}A_1$ by the RHS of the first of the equations (3.2.21), we find

$$\sqrt{3}A_2 = -2a_1 - a_2. \tag{3.2.22}$$

The equation

$$k_3 = \bar{w}k_2 = \bar{w}\bar{w}k_1 = wk_1,$$

yields

$$-ia_3 + A_3 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(-ia_1 + A_1).$$

Using the first of equations (3.2.21) to replace A_1, the real and imaginary part of this equation yield the last of equations (3.2.4a), as well as the equation

$$\sqrt{3}A_3 = a_1 - a_2. \tag{3.2.23}$$

The first of equations (3.2.16), where A_1 is given by (3.2.21), yields $b_1 = -a_1 - a_2,\ b_2 = a_1,\ b_3 = a_2$, whereas the first of equations (3.2.17) yields the analogous equations in (3.2.4c), which express $\{c_l\}_1^3$ in terms of a_1 and a_2. Similarly, the second of equations (3.2.16) and (3.2.17) yield

$$B_1 = A_3,\ B_2 = A_1,\ B_3 = A_2, \quad C_1 = A_2,\ C_2 = A_3,\ C_3 = A_1, \tag{3.2.24}$$

where $\{A_j\}_1^3$ are expressed in terms of a_1 and a_2 by the first of equations (3.2.21) and by equations (3.2.22) and (3.2.23).

The first set of Residues

Employing equations (3.2.15) in (3.2.12)-(3.2.14) we find the following residue conditions:

$$\frac{\alpha_l}{A_l}e^{ia_lL} = \frac{\beta_l}{B_l}e^{-ib_lL},\ \frac{\beta_l}{B_l}e^{ib_lL} = \frac{\gamma_l}{C_l}e^{-ic_lL},\ \frac{\gamma_l}{C_l}e^{ic_lL} = \frac{\alpha_l}{A_l}e^{-ia_lL},\ l = 1,2,3. \tag{3.2.25}$$

The first two equations yield

$$\frac{\beta_l}{B_l} = \frac{\alpha_l}{A_l}e^{i(a_l+b_l)L},\ \frac{\gamma_l}{C_l} = \frac{\alpha_l}{A_l}e^{i(a_l+2b_l+c_l)L},\ l = 1,2,3. \tag{3.2.26}$$

The third equation in (3.2.25) is satisfied identically. Indeed, replacing in this equation $\frac{\gamma_l}{C_l}$ by the RHS of (3.2.26) we find an identity; in this respect we note that equations (3.2.4b) and (3.2.4c) imply the relations

$$a_l + b_l + c_l = 0,\ l = 1,2,3. \tag{3.2.27}$$

The second set of Residues

Employing equations (3.2.18) in (3.2.12)-(3.2.14) we obtain the following residue conditions:

$$\begin{gathered}\frac{\alpha_l}{A_l}e^{ia_lL}=\frac{\beta_{l-1}}{B_{l-1}}e^{-ib_{l-1}L},\quad \frac{\beta_l}{B_l}e^{ib_lL}=\frac{\gamma_{l-1}}{C_{l-1}}e^{-ic_{l-1}L},\\ \frac{\gamma_l}{C_l}e^{ic_lL}=\frac{\alpha_{l-1}}{A_{l-1}}e^{-ia_{l-1}L},\quad l=1,2,3,\end{gathered}\tag{3.2.28}$$

where

$$\alpha_0=\alpha_3,\ \beta_0=\beta_3,\ \gamma_0=\gamma_3,\ A_0=A_3,\ B_0=B_3,\ C_0=C_3.\tag{3.2.29}$$

Expressing in equations (3.2.28) the ratios $\frac{\beta_l}{B_l}$ and $\frac{\gamma_l}{C_l}$ in terms of $\frac{\alpha_l}{A_l}$ via equations (3.2.26), equations (3.2.28) yield the following relations:

$$\begin{gathered}\frac{\alpha_l}{A_l}=\frac{\alpha_{l-1}}{A_{l-1}}e^{i(a_{l-1}-a_l)L},\quad \frac{\alpha_l}{A_l}=\frac{\alpha_{l-1}}{A_{l-1}}e^{i(a_{l-1}-a_l)L}e^{2i(b_{l-1}-b_l)L},\\ \frac{\alpha_l}{A_l}=\frac{\alpha_{l-1}}{A_{l-1}}e^{i(a_l-a_{l-1})L},\quad l=1,2,3.\end{gathered}\tag{3.2.30}$$

These equations imply

$$e^{i(a_{l-1}-a_l)L}=e^{i(a_{l-1}-a_l)L}e^{2i(b_{l-1}-b_l)L}=e^{i(a_l-a_{l-1})L},\ l=1,2,3,$$

or

$$e^{2i(a_l-a_{l-1})L}=1,\quad e^{2i(b_l-b_{l-1})L}=1,\ l=1,2,3.\tag{3.2.31}$$

Equations in (3.2.4b), which express $\{b_l\}_1^3$ in terms of a_1 and a_2, can be rewritten in the form $b_l=a_{l-1},\ l=1,2,3,$ thus the second set of equations (3.2.31) follows from the first set, which yields

$$e^{2i(a_2-a_1)L}=1,\ e^{2i(2a_2+a_1)L}=1.$$

Hence,

$$a_1-a_2=\frac{n\pi}{L},\quad 2a_2+a_1=\frac{m\pi}{L},\quad m,n\in\mathbb{Z}.\tag{3.2.32}$$

These equations imply (3.2.4a). Furthermore equations (3.2.21)-(3.2.23) yield

$$A_1=m\frac{\pi}{L\sqrt{3}},\ A_2=-(m+n)\frac{\pi}{L\sqrt{3}},\ A_3=n\frac{\pi}{L\sqrt{3}},\quad m,n\in\mathbb{Z}.$$

Hence, equations (3.2.30) imply equations (3.2.4f). Moreover, (3.2.26) with equations (3.2.24) imply equations (3.2.4g) and (3.2.4h). □

Remark 3.1. *We assumed that the unknown Neumman functions are the sum of three exponentials. However, the same analysis is valid for any finite sum of exponentials. Indeed, let these functions defined by a sum of four exponentials, then equations* (3.2.15) *and* (3.2.20) *imply that*

$$k_4 = \bar{w}k_3 = \bar{w}^2\lambda_3 = \bar{w}^3\lambda_2 = \lambda_2 = \bar{w}\mu_2 = \bar{w}^2\mu_1 = w\mu_1 = k_1.$$

Hence, $a_4 = a_1$*; similarly* $b_4 = b_1$ *and* $c_4 = c_1$*. Thus,*

$$a_i = a_j,\ b_i = b_j,\ c_i = c_j \quad \textit{iff} \quad i \equiv j(mod 3).$$

3.3 The Neumann Problem.

In this case the global relation (3.1.7a) becomes

$$\left(k - \frac{\lambda}{k}\right) E(-ik)D_1(k) + \left(wk - \frac{\lambda}{wk}\right) E(-iwk)D_2(wk) \\ + \left(\bar{w}k - \frac{\lambda}{\bar{w}k}\right) E(-i\bar{w}k)D_3(\bar{w}k) = 0,\ k \in \mathbb{C}, \tag{3.3.1}$$

where the unknown functions $\{D_j\}_1^3$ are defined by

$$D_j(k) = \int_{-L}^{L} e^{\left(k+\frac{\lambda}{k}\right)s} d_j(s)ds, \quad j = 1, 2, 3,\ k \in \mathbb{C} \tag{3.3.2}$$

and $\{d_j\}_1^3$ denote the unknown boundary values.

Proposition 3.2. *Let each of the unknowns Dirichlet boundary values be expressed as a sum of three exponentials, namely*

$$d_1(s) = \sum_{l=1}^{3} \alpha_l e^{ia_l s},\ d_2(s) = \sum_{l=1}^{3} \beta_l e^{ib_l s},\ d_3(s) = \sum_{l=1}^{3} \gamma_l e^{ic_l s}, \tag{3.3.3}$$

where $\{a_l, b_l, c_l\}_1^3$ *are real constants and* $\{\alpha_l, \beta_l, \gamma_l\}_1^3$ *are complex constants. Then, the global relation* (3.3.1) *implies the relations* (3.2.4a)-(3.2.4e)*, as well as the following relations:*

$$\alpha_2 = (-1)^n \alpha_1, \quad \alpha_3 = (-1)^{n+m}\alpha_1, \tag{3.3.4a}$$

$$\beta_1 = e^{-i\frac{m-n}{3}\pi}\alpha_1, \quad \beta_2 = (-1)^m e^{-i\frac{m+2n}{3}\pi}\alpha_1,$$

$$\beta_3 = (-1)^{n+m} e^{-i\frac{m-n}{3}\pi}\alpha_1, \tag{3.3.4b}$$

$$\gamma_1 = e^{i\frac{2m+n}{3}\pi}\alpha_1, \quad \gamma_2 = (-1)^n e^{i\frac{2n+m}{3}\pi}\alpha_1,$$

$$\gamma_3 = (-1)^{n+m} e^{i\frac{m-n}{3}\pi}\alpha_1. \tag{3.3.4c}$$

Proof. Proceeding as in section 3.2 and noting that

$$\frac{k - \frac{\lambda}{k}}{k + \frac{\lambda}{k} + ia_l} = 1 + \frac{k_l}{k - k_l} - \frac{\bar{k}_l}{k + \bar{k}_l},$$

in analogy with equations (3.2.12)-(3.2.14), we now find the following set of equations, which are valid for all $k \in \mathbb{C}$:

$$\sum_{l=1}^{3} \alpha_l e^{ia_l L}\left[1 + \frac{k_l}{k - k_l} - \frac{\bar{k}_l}{k + \bar{k}_l}\right] = \sum_{l=1}^{3} \beta_l e^{-ib_l L}\left[1 + \frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} - \frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l}\right], \tag{3.3.5}$$

$$\sum_{l=1}^{3} \beta_l e^{ib_l L}\left[1 + \frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} - \frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l}\right] = \sum_{l=1}^{3} \gamma_l e^{-ic_l L}\left[1 + \frac{w\mu_l}{k - w\mu_l} - \frac{w\bar{\mu}_l}{k + w\bar{\mu}_l}\right], \tag{3.3.6}$$

$$\sum_{l=1}^{3} \gamma_l e^{ic_l L}\left[1 + \frac{w\mu_l}{k - w\mu_l} - \frac{w\bar{\mu}_l}{k + w\bar{\mu}_l}\right] = \sum_{l=1}^{3} \alpha_l e^{-ia_l L}\left[1 + \frac{k_l}{k - k_l} - \frac{\bar{k}_l}{k + \bar{k}_l}\right]. \tag{3.3.7}$$

The analysis of the first and second steps associated with equations (3.3.5)-(3.3.7), i.e. the analysis of the first and second set of the relevant poles, is *identical* with the analysis of the corresponding steps of section 3.2. The only difference is that

$$\frac{\alpha_l}{A_l}, \ \frac{\beta_l}{B_l}, \ \frac{\gamma_l}{C_l} \ \longrightarrow \ \alpha_l, \ \beta_l, \ \gamma_l.$$

Hence, in analogy with equations (3.2.26) and (3.2.30), we now find the following equations:

$$\beta_l = \alpha_l e^{i(a_l + b_l)L}, \ \gamma_l = \alpha_l e^{i(a_l + 2b_l + c_l)L}, \ l = 1, 2, 3 \tag{3.3.8}$$

and

$$\begin{aligned} \alpha_l e^{ia_l L} = \alpha_{l-1} e^{ia_{l-1}L}, \quad \alpha_l e^{i(a_l - 2b_l)L} = \alpha_{l-1} e^{i(a_{l-1}-2b_{l-1})L}, \\ \alpha_l e^{-ia_l L} = \alpha_{l-1} e^{-ia_{l-1}L}, \quad l = 1, 2, 3. \end{aligned} \tag{3.3.9}$$

Equations (3.3.9) imply equations (3.2.31) and hence equations (3.2.4a). Furthermore, equations (3.3.8) imply equations (3.3.4a)-(3.3.4c).

It is important to note that the conditions obtained from the terms in (3.3.5)-(3.3.7) which are of order $O(1)$ in k, are identical with the first residue conditions, i.e. equations (3.3.8), thus these conditions do *not* impose additional constraints.

□

3.4 The Robin Problem.

Replacing in the expression $iq_N - (k - \frac{\lambda}{k} q)$ appearing in equation (3.1.7b), q_N with χq, it follows that the global relation becomes

$$\begin{aligned} &\left(k - i\chi - \frac{\lambda}{k}\right) E(-ik) D_1(k) + \left(wk - i\chi - \frac{\lambda}{wk}\right) E(-iwk) D_2(wk) \\ &+ \left(\bar{w}k - i\chi - \frac{\lambda}{\bar{w}k}\right) E(-i\bar{w}k) D_3(\bar{w}k) = 0, \ k \in \mathbb{C}, \end{aligned} \tag{3.4.1}$$

where χ is a real constant and the unknown functions $\{D_j\}_1^3$ are defined in (3.3.2).

Proposition 3.3. *Let each of the unknown Dirichlet boundary values be expressed as the sum of the three exponentials appearing in equations* (3.3.3). *Then, the global relation* (3.4.1) *implies relations* (3.2.4b)-(3.2.4d), *where* a_1 *and* a_2 *satisfy the following relations:*

$$\begin{aligned} e^{i\left(a_2 - \frac{N\pi}{3L}\right)} \sin\left[\left(a_2 - \frac{N\pi}{3L}\right) L\right] &= \frac{6i\sqrt{3}a_2\chi}{a_1^2 + a_1 a_2 - 2a_2^2 - 3i\sqrt{3}a_2\chi + 3\chi^2}, \\ e^{i\left(a_1 - \frac{N\pi}{3L}\right)} \sin\left[\left(a_1 - \frac{N\pi}{3L}\right) L\right] &= \frac{6i\sqrt{3}a_1\chi}{a_2^2 + a_1 a_2 - 2a_1^2 - 3i\sqrt{3}a_1\chi + 3\chi^2}, \quad N \in \mathbb{Z}. \end{aligned} \tag{3.4.2}$$

Furthermore,

$$\alpha_2 = \frac{\left(1+\frac{i\chi}{A_1}\right)\left(1-\frac{i\chi}{A_3}\right)}{\left(1-\frac{i\chi}{A_2}\right)\left(1+\frac{i\chi}{A_3}\right)} e^{i(a_2-a_1)L}\alpha_1,$$

$$\alpha_3 = \frac{\left(1-\frac{i\chi}{A_1}\right)\left(1+\frac{i\chi}{A_2}\right)}{\left(1+\frac{i\chi}{A_3}\right)\left(1-\frac{i\chi}{A_2}\right)} e^{i(a_3-a_1)L}\alpha_1, \tag{3.4.3a}$$

$$\beta_l = \frac{1-\frac{i\chi}{A_l}}{1-\frac{i\chi}{A_{l-1}}} e^{-ia_{l+1}L}\alpha_l, \quad \gamma_l = \frac{1-\frac{i\chi}{A_l}}{1-\frac{i\chi}{A_{l+1}}} e^{-ia_{l-1}L}\alpha_l, \quad l=1,2,3, \tag{3.4.3b}$$

where $\{A_l\}_1^3$ *are defined in terms of* a_1 *and* a_2 *by*

$$\sqrt{3}A_1 = a_1 + 2a_2, \;\; \sqrt{3}A_2 = -2a_1 - a_2, \;\; \sqrt{3}A_3 = a_1 - a_2. \tag{3.4.4}$$

Proof. The definitions (3.2.6) imply the following relations:

$$k^2 + ia_lk + \lambda = (k-k_l)(k+\bar{k}_l), \;\; ia_l = \bar{k}_l - k_l, \;\; \lambda = -|k_l|^2, \;\; k_l + \bar{k}_l = A_l. \tag{3.4.5}$$

Using these equations in the identities,

$$\frac{k - i\chi - \frac{\lambda}{k}}{k + ia_l + \frac{\lambda}{k}} = \frac{k^2 - i\chi k - \lambda}{k^2 + ia_lk + \lambda} = 1 - \frac{i(a_l+\chi)k + 2\lambda}{k^2 + ia_lk + \lambda},$$

we find

$$\frac{k - i\chi - \frac{\lambda}{k}}{k + ia_l + \frac{\lambda}{k}} = \left(1 - \frac{i\chi}{A_l}\right)\left(\frac{k_l}{k-k_l} + \frac{1}{2}\right) - \left(1 + \frac{i\chi}{A_l}\right)\left(\frac{\bar{k}_l}{k-\bar{k}_l} - \frac{1}{2}\right).$$

Hence, in analogy with equations (3.2.12)-(3.2.14), now the global relation implies that the following equations are valid for all $k \in \mathbb{C}$:

$$\begin{aligned}\sum_{l=1}^{3}\alpha_l e^{ia_lL}&\left[\left(1-\frac{i\chi}{A_l}\right)\left(\frac{k_l}{k-k_l}+\frac{1}{2}\right) - \left(1+\frac{i\chi}{A_l}\right)\left(\frac{\bar{k}_l}{k+\bar{k}_l}-\frac{1}{2}\right)\right] \\ = \sum_{l=1}^{3}\beta_l e^{-ib_lL}&\left[\left(1-\frac{i\chi}{B_l}\right)\left(\frac{\bar{w}\lambda_l}{k-\bar{w}\lambda_l}+\frac{1}{2}\right) - \left(1+\frac{i\chi}{B_l}\right)\left(\frac{\bar{w}\bar{\lambda}_l}{k+\bar{w}\bar{\lambda}_l}-\frac{1}{2}\right)\right],\end{aligned} \tag{3.4.6}$$

$$\begin{aligned}\sum_{l=1}^{3}\beta_l e^{ib_lL}&\left[\left(1-\frac{i\chi}{B_l}\right)\left(\frac{\bar{w}\lambda_l}{k-\bar{w}\lambda_l}+\frac{1}{2}\right) - \left(1+\frac{i\chi}{B_l}\right)\left(\frac{\bar{w}\bar{\lambda}_l}{k+\bar{w}\bar{\lambda}_l}-\frac{1}{2}\right)\right] \\ = \sum_{l=1}^{3}\gamma_l e^{-ic_lL}&\left[\left(1-\frac{i\chi}{B_l}\right)\left(\frac{w\mu_l}{k-w\mu_l}+\frac{1}{2}\right) - \left(1+\frac{i\chi}{B_l}\right)\left(\frac{w\bar{\mu}_l}{k+w\bar{\mu}_l}-\frac{1}{2}\right)\right],\end{aligned} \tag{3.4.7}$$

$$\begin{aligned}&\sum_{l=1}^{3}\gamma_l e^{ic_lL}\left[\left(1-\frac{i\chi}{B_l}\right)\left(\frac{w\mu_l}{k-w\mu_l}+\frac{1}{2}\right)-\left(1+\frac{i\chi}{B_l}\right)\left(\frac{w\bar{\mu}_l}{k+w\bar{\mu}_l}-\frac{1}{2}\right)\right]\\&=\sum_{l=1}^{3}\alpha_l e^{-ia_lL}\left[\left(1-\frac{i\chi}{A_l}\right)\left(\frac{k_l}{k-k_l}+\frac{1}{2}\right)-\left(1+\frac{i\chi}{A_l}\right)\left(\frac{\bar{k}_l}{k+\bar{k}_l}-\frac{1}{2}\right)\right].\end{aligned}\tag{3.4.8}$$

The analysis of the first and the second set of the relevant poles is identical with the analysis presented in section 3.2 and it yields equations (3.2.4b)- (3.2.4d). We next consider the relevant residues.

The first set of Residues

In analogy with equations (3.2.25) we now have the following equations:

$$\begin{aligned}\alpha_l e^{ia_lL}\left(1-\frac{i\chi}{A_l}\right)&=\beta_l e^{-ib_lL}\left(1-\frac{i\chi}{B_l}\right),\\ \beta_l e^{ib_lL}\left(1-\frac{i\chi}{B_l}\right)&=\gamma_l e^{-ic_lL}\left(1-\frac{i\chi}{C_l}\right),\quad l=1,2,3,\end{aligned}\tag{3.4.9}$$

$$\gamma_l e^{ic_lL}\left(1-\frac{i\chi}{C_l}\right)=\alpha_l e^{-ia_lL}\left(1-\frac{i\chi}{A_l}\right),\quad l=1,2,3.\tag{3.4.10}$$

Equations (3.4.9) yield

$$\begin{aligned}\beta_l\left(1-\frac{i\chi}{B_l}\right)&=\alpha_l e^{i(a_l+b_l)L}\left(1-\frac{i\chi}{A_l}\right),\\ \gamma_l\left(1-\frac{i\chi}{C_l}\right)&=\alpha_l e^{i(a_l+2b_l+c_l)L}\left(1-\frac{i\chi}{A_l}\right),\quad l=1,2,3\end{aligned}\tag{3.4.11}$$

and then, in view of (3.2.27), equation (3.4.10) is satisfied identically. Applying in (3.4.11) the definitions of $\{A_l,B_l,C_l\}_1^3$ given in (3.2.21)-(3.2.24) as well as equation (3.2.27), we obtain equations (3.4.3b).

The second set of Residues

In analogy with equations (3.2.28), we now have the following equations:

$$\begin{aligned}\alpha_l e^{ia_lL}\left(1+\frac{i\chi}{A_l}\right)&=\beta_{l-1}e^{-ib_{l-1}L}\left(1+\frac{i\chi}{B_{l-1}}\right),\\ \beta_l e^{ib_lL}\left(1+\frac{i\chi}{B_l}\right)&=\gamma_{l-1}e^{-ic_{l-1}L}\left(1+\frac{i\chi}{C_{l-1}}\right),\\ \gamma_l e^{ic_lL}\left(1+\frac{i\chi}{C_l}\right)&=\alpha_{l-1}e^{-ia_{l-1}L}\left(1+\frac{i\chi}{A_{l-1}}\right),\quad l=1,2,3,\end{aligned}\tag{3.4.12}$$

where equations (3.2.29) are still valid.

Expressing in equations (3.4.12) β_l and γ_l in terms of α_l via equations (3.4.11), equations (3.4.12) yield the following relations:

$$\begin{aligned}
\alpha_l &= \alpha_{l-1}\frac{\left(1-\frac{i\chi}{A_{l-1}}\right)}{\left(1+\frac{i\chi}{A_l}\right)}\frac{\left(1+\frac{i\chi}{B_{l-1}}\right)}{\left(1-\frac{i\chi}{B_{l-1}}\right)}e^{i(a_{l-1}-a_l)L},\\
\alpha_l &= \alpha_{l-1}\frac{\left(1-\frac{i\chi}{A_{l-1}}\right)}{\left(1-\frac{i\chi}{A_l}\right)}\frac{\left(1-\frac{i\chi}{B_l}\right)}{\left(1+\frac{i\chi}{B_l}\right)}\frac{\left(1+\frac{i\chi}{C_{l-1}}\right)}{\left(1-\frac{i\chi}{C_{l-1}}\right)}e^{i(a_{l-1}-a_l)L}e^{2i(b_{l-1}-b_l)L},\\
\alpha_l &= \alpha_{l-1}\frac{\left(1+\frac{i\chi}{A_{l-1}}\right)}{\left(1-\frac{i\chi}{A_l}\right)}\frac{\left(1-\frac{i\chi}{C_l}\right)}{\left(1+\frac{i\chi}{C_l}\right)}e^{i(a_l-a_{l-1})L},\quad l=1,2,3.
\end{aligned}\tag{3.4.13}$$

The last of the equations (3.4.13) imply (3.4.3a). Furthermore, equations (3.4.13) imply the equations

$$e^{2i(a_l-a_{l-1})L}=\frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{B}_{l-1}\widetilde{C}_l},\quad e^{2i(b_l-b_{l-1})L}=\frac{\widetilde{B}_{l-1}\widetilde{B}_l}{\widetilde{C}_{l-1}\widetilde{A}_l},\quad l=1,2,3,\tag{3.4.14}$$

where $\{\widetilde{A}_l,\widetilde{B}_l,\widetilde{C}_l\}_1^3$ are defined as follows:

$$\widetilde{A}_l=\frac{1-\frac{i\chi}{A_l}}{1+\frac{i\chi}{A_l}},\quad \widetilde{B}_l=\frac{1-\frac{i\chi}{B_l}}{1+\frac{i\chi}{B_l}},\quad \widetilde{C}_l=\frac{1-\frac{i\chi}{C_l}}{1+\frac{i\chi}{C_l}},\quad l=1,2,3.\tag{3.4.15}$$

Equations (3.2.4b) and (3.2.4c), which express $\{b_l\}_1^3$ and $\{c_l\}_1^3$ in terms of a_1 and a_2, can be rewritten in the form

$$b_l=a_{l-1},\quad c_l=a_{l+1},\quad l=1,2,3,\qquad a_0=a_3,\ a_4=a_1.\tag{3.4.16}$$

Hence,

$$\widetilde{B}_l=\widetilde{A}_{l-1},\quad \widetilde{C}_l=\widetilde{A}_{l+1},\quad l=1,2,3,\qquad \widetilde{A}_0=\widetilde{A}_3,\ \widetilde{A}_4=\widetilde{A}_1.\tag{3.4.17}$$

Thus, equations (3.4.14) can be rewritten in the form

$$e^{2i(a_l-a_{l-1})L}=\frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{A}_{l+1}^2},\quad e^{2i(a_{l-1}-a_{l-2})L}=\frac{\widetilde{A}_{l-2}\widetilde{A}_{l-1}}{\widetilde{A}_l^2},\quad l=1,2,3,\tag{3.4.18}$$

where we have used the identity $\widetilde{A}_{l-2}=\widetilde{A}_{l+1},\ l=1,2,3.$

The second set of equations (3.4.18) is identical with the first set of equations (3.4.18)(after replacing l with $l+1$). The first set of equations (3.4.18) yields:

$$e^{2i(a_1-a_3)L} = \frac{\widetilde{A}_1\widetilde{A}_3}{\widetilde{A}_2^2},\ e^{2i(a_2-a_1)L} = \frac{\widetilde{A}_2\widetilde{A}_1}{\widetilde{A}_3^2},\ e^{2i(a_3-a_2)L} = \frac{\widetilde{A}_3\widetilde{A}_2}{\widetilde{A}_1^2}. \tag{3.4.19}$$

The third of the above equations is equivalent to the product of the first two equations; the former equations, using $-a_3 = a_1 + a_2$, become

$$e^{2i(a_2-a_1)L} = \frac{\widetilde{A}_1\widetilde{A}_2}{\widetilde{A}_3^2}, \quad e^{2i(2a_1+a_2)L} = \frac{\widetilde{A}_1\widetilde{A}_3}{\widetilde{A}_2^2}. \tag{3.4.20}$$

These equations are equivalent to

$$e^{2ia_2L}e^{-2i\frac{N\pi}{3}} = \frac{\widetilde{A}_1}{\widetilde{A}_3}, \quad e^{2ia_1L}e^{-2i\frac{N\pi}{3}} = \frac{\widetilde{A}_3}{\widetilde{A}_2}, \quad N \in \mathbb{Z}. \tag{3.4.21}$$

Using the definitions of $\{\widetilde{A}_j\}_1^3$, see equations (3.2.21)-(3.2.23), equations (3.4.21) yield equations (3.4.2).

□

3.5 The Oblique Robin Problem.

Replacing in the global relation (3.1.7b), the term q_N with

$$-\frac{1}{\sin\delta}\left[\cos\delta\frac{dq}{ds} - \chi q\right],$$

and integrating by parts the term involving $\frac{dq}{ds}$, we find the equation

$$\hat{q}_j(k) = E(-ik)\left\{\frac{i}{\sin\delta}\left(ke^{i\delta} + \frac{\lambda}{ke^{i\delta}} + \chi\right)D_j(k) - \frac{i\cos\delta}{\sin\delta}e^{\left(k+\frac{\lambda}{k}\right)s}q_j(s)\Big|_{-L}^{L}\right\}.$$

We assume that the boundary terms vanish, i.e.,

$$\begin{aligned} &E(-ik)e^{\left(k+\frac{\lambda}{k}\right)s}q_1(s)\Big|_{-L}^{L} + E(-iwk)e^{\left(wk+\frac{\lambda}{wk}\right)s}q_2(s)\Big|_{-L}^{L} \\ &+ E(-i\bar{w}k)e^{\left(\bar{w}k+\frac{\lambda}{\bar{w}k}\right)s}q_3(s)\Big|_{-L}^{L} = 0. \end{aligned} \tag{3.5.1}$$

This is indeed the case provided that

$$q_1(L) = q_2(-L),\ q_2(L) = q_3(-L),\ q_3(L) = q_1(-L).$$

Then, the global relation becomes

$$\begin{aligned}&\left(ke^{i\delta} + \chi + \frac{\lambda}{ke^{i\delta}}\right) E(-ik)D_1(k) + \left(wke^{i\delta} + \chi + \frac{\lambda}{wke^{i\delta}}\right) E(-iwk)D_2(wk)\\ &+ \left(\bar{w}ke^{i\delta} + \chi + \frac{\lambda}{\bar{w}ke^{i\delta}}\right) E(-i\bar{w}k)D_3(\bar{w}k) = 0,\ k \in \mathbb{C},\end{aligned} \tag{3.5.2}$$

where χ, δ are real constants and $\sin\delta \neq 0$.

Proposition 3.4. *Let each of the unknown Dirichlet boundary values be expressed as the sum of the three exponentials appearing in equations* (3.3.3)*. Then, the global relation* (3.5.2) *implies relations* (3.2.4b)*,* (3.2.4c) *and* (3.2.4d)*, where a_1 and a_2 satisfy the following relations:*

$$\begin{aligned}&e^{i\left(a_2-\frac{N\pi}{3L}\right)L}\sin\left[\left(a_2 - \frac{N\pi}{3L}\right)L\right]\\ &\quad= \frac{3\sqrt{3}a_2\chi\sin\delta + i\sqrt{3}(a_1^2 + a_1a_2 + a_2^2)\sin 2\delta}{(i\chi + a_1\cos\delta + A_1\sin\delta)(i\chi - (a_1+a_2)\cos\delta + (A_1+A_2)\sin\delta)},\\ &e^{i\left(a_1-\frac{N\pi}{3L}\right)L}\sin\left[\left(a_1 - \frac{N\pi}{3L}\right)L\right] =\\ &\quad= \frac{3\sqrt{3}a_1\chi\sin\delta + i\sqrt{3}(a_1^2 + a_1a_2 + a_2^2)\sin 2\delta}{(i\chi + a_2\cos\delta - A_2\sin\delta)(i\chi - (a_1+a_2)\cos\delta - (A_1+A_2)\sin\delta)},\quad N \in \mathbb{Z}.\end{aligned} \tag{3.5.3}$$

where $\{A_l\}_1^3$ are defined in terms of a_1 and a_2 by (3.4.4)*. Furthermore,*

$$\alpha_2 = \frac{\left(\frac{a_1\cos\delta+i\chi}{A_1} + \sin\delta\right)\left(\frac{a_3\cos\delta+i\chi}{A_3} - \sin\delta\right)}{\left(\frac{a_2\cos\delta+i\chi}{A_2} - \sin\delta\right)\left(\frac{a_3\cos\delta+i\chi}{A_3} + \sin\delta\right)}e^{i(a_2-a_1)L}\alpha_1,$$

$$\alpha_3 = \frac{\left(\frac{a_1\cos\delta+i\chi}{A_1} - \sin\delta\right)\left(\frac{a_2\cos\delta+i\chi}{A_2} + \sin\delta\right)}{\left(\frac{a_3\cos\delta+i\chi}{A_3} + \sin\delta\right)\left(\frac{a_2\cos\delta+i\chi}{A_2} - \sin\delta\right)}e^{i(a_3-a_1)L}\alpha_1, \tag{3.5.4a}$$

$$\beta_l = \frac{\frac{a_l\cos\delta+i\chi}{A_l} - \sin\delta}{\frac{a_{l-1}\cos\delta+i\chi}{A_{l-1}} - \sin\delta}e^{-ia_{l+1}L}\alpha_l,$$

$$\gamma_l = \frac{\frac{a_l\cos\delta+i\chi}{A_l} - \sin\delta}{\frac{a_{l+1}\cos\delta+i\chi}{A_{l+1}} - \sin\delta}e^{-ia_{l-1}L}\alpha_l,\quad l = 1,2,3. \tag{3.5.4b}$$

Proof. Using the identity

$$\frac{ke^{i\delta} + \chi + \frac{\lambda}{ke^{i\delta}}}{k + ia_l + \frac{\lambda}{k}} = e^{i\delta} - \frac{k(ia_l e^{i\delta} - \chi) + \lambda(e^{i\delta} - e^{-i\delta})}{k^2 + ia_l k + \lambda},$$

we find that

$$\begin{aligned} i\frac{e^{i\delta}k + \chi + \frac{\lambda}{e^{i\delta}k}}{k + ia_l + \frac{\lambda}{k}} = &\left(\frac{a_l\cos\delta + i\chi}{A_l} - \sin\delta\right)\left(\frac{k_l}{k - k_l} + \frac{e^{i\delta}}{2i\sin\delta}\right) \\ &+ \left(\frac{a_l\cos\delta + i\chi}{A_l} + \sin\delta\right)\left(\frac{\bar{k}_l}{k + \bar{k}_l} - \frac{e^{i\delta}}{2i\sin\delta}\right). \end{aligned} \tag{3.5.5}$$

Hence, in analogy with equations (3.2.12)-(3.2.14), we now have the following equations valid for all $k \in \mathbb{C}$:

$$\begin{aligned} &\sum_{l=1}^{3} \alpha_l e^{ia_l L}\left[\left(\frac{a_l\cos\delta + i\chi}{A_l} - \sin\delta\right)\left(\frac{k_l}{k - k_l} + \frac{e^{i\delta}}{2i\sin\delta}\right)\right. \\ &\qquad \left. + \left(\frac{a_l\cos\delta + i\chi}{A_l} + \sin\delta\right)\left(\frac{\bar{k}_l}{k + \bar{k}_l} - \frac{e^{i\delta}}{2i\sin\delta}\right)\right] \\ &= \sum_{l=1}^{3} \beta_l e^{-ib_l L}\left[\left(\frac{b_l\cos\delta + i\chi}{B_l} - \sin\delta\right)\left(\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{e^{i\delta}}{2i\sin\delta}\right)\right. \\ &\qquad \left. + \left(\frac{b_l\cos\delta + i\chi}{B_l} + \sin\delta\right)\left(\frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l} - \frac{e^{i\delta}}{2i\sin\delta}\right)\right], \end{aligned} \tag{3.5.6}$$

$$\begin{aligned} &\sum_{l=1}^{3} \beta_l e^{ib_l L}\left[\left(\frac{b_l\cos\delta + i\chi}{B_l} - \sin\delta\right)\left(\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{e^{i\delta}}{2i\sin\delta}\right)\right. \\ &\qquad \left. + \left(\frac{b_l\cos\delta + i\chi}{B_l} + \sin\delta\right)\left(\frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l} - \frac{e^{i\delta}}{2i\sin\delta}\right)\right] \\ &= \sum_{l=1}^{3} \gamma_l e^{-ic_l L}\left[\left(\frac{c_l\cos\delta + i\chi}{C_l} - \sin\delta\right)\left(\frac{w\mu_l}{k - w\mu_l} + \frac{e^{i\delta}}{2i\sin\delta}\right)\right. \\ &\qquad \left. + \left(\frac{c_l\cos\delta + i\chi}{C_l} + \sin\delta\right)\left(\frac{w\bar{\mu}_l}{k + w\bar{\mu}_l} - \frac{e^{i\delta}}{2i\sin\delta}\right)\right] \end{aligned} \tag{3.5.7}$$

$$\begin{aligned}
&\sum_{l=1}^{3}\gamma_l e^{ic_lL}\left[\left(\frac{c_l\cos\delta+i\chi}{C_l}-\sin\delta\right)\left(\frac{w\mu_l}{k-w\mu_l}+\frac{e^{i\delta}}{2i\sin\delta}\right)\right.\\
&\qquad\left.+\left(\frac{c_l\cos\delta+i\chi}{C_l}+\sin\delta\right)\left(\frac{w\bar{\mu}_l}{k+w\bar{\mu}_l}-\frac{e^{i\delta}}{2i\sin\delta}\right)\right]\\
&=\sum_{l=1}^{3}\alpha_l e^{-ia_lL}\left[\left(\frac{a_l\cos\delta+i\chi}{A_l}-\sin\delta\right)\left(\frac{k_l}{k-k_l}+\frac{e^{i\delta}}{2i\sin\delta}\right)\right.\\
&\qquad\left.+\left(\frac{a_l\cos\delta+i\chi}{A_l}+\sin\delta\right)\left(\frac{\bar{k}_l}{k+\bar{k}_l}-\frac{e^{i\delta}}{2i\sin\delta}\right)\right].
\end{aligned}\tag{3.5.8}$$

The analysis of the first and the second set of the relevant poles is identical with the analysis presented in section 3.2 and it yields equations (3.2.4b)-(3.2.4d).

The analysis of the first and the second set of the relevant residues is identical with the analysis presented in section 3.4, provided that we make the following substitutions:

$$\begin{aligned}
1-\frac{i\chi}{A_l}&\longrightarrow\frac{a_l\cos\delta+i\chi}{A_l}-\sin\delta, \quad 1+\frac{i\chi}{A_l}\longrightarrow\frac{a_l\cos\delta+i\chi}{A_l}+\sin\delta,\\
1-\frac{i\chi}{B_l}&\longrightarrow\frac{b_l\cos\delta+i\chi}{B_l}-\sin\delta, \quad 1+\frac{i\chi}{B_l}\longrightarrow\frac{b_l\cos\delta+i\chi}{B_l}+\sin\delta,\\
1-\frac{i\chi}{C_l}&\longrightarrow\frac{c_l\cos\delta+i\chi}{C_l}-\sin\delta, \quad 1+\frac{i\chi}{C_l}\longrightarrow\frac{c_l\cos\delta+i\chi}{C_l}+\sin\delta.
\end{aligned}\tag{3.5.9}$$

Hence, equation (3.4.14) gives rise to a similar equation, where now $\{\widetilde{A}_l,\widetilde{B}_l,\widetilde{C}_l\}_1^3$ are defined as follows:

$$\begin{aligned}
\widetilde{A}_l&=\frac{\frac{a_l\cos\delta+i\chi}{A_l}-\sin\delta}{\frac{a_l\cos\delta+i\chi}{A_l}+\sin\delta}, \quad \widetilde{B}_l=\frac{\frac{b_l\cos\delta+i\chi}{B_l}-\sin\delta}{\frac{b_l\cos\delta+i\chi}{B_l}+\sin\delta},\\
\widetilde{C}_l&=\frac{\frac{c_l\cos\delta+i\chi}{C_l}-\sin\delta}{\frac{c_l\cos\delta+i\chi}{C_l}+\sin\delta}, \quad l=1,2,3.
\end{aligned}\tag{3.5.10}$$

Following the same steps used in section 3.4, we derive the analogue of equation (3.4.20), i.e.

$$e^{2ia_2L}e^{-2i\frac{N\pi}{3}}=\frac{\widetilde{A}_1}{\widetilde{A}_3}, \quad e^{2ia_1L}e^{-2i\frac{N\pi}{3}}=\frac{\widetilde{A}_3}{\widetilde{A}_2}, \qquad N\in\mathbb{Z}, \tag{3.5.11}$$

where $\{\widetilde{A}_l\}_1^3$ are replaced by the relevant definitions in (3.5.10). Hence, equations (3.5.3) (the analogue of equations (3.4.2)) are satisfied. Furthermore, equations (3.5.4a) and

(3.5.4b) are also satisfied; these equations follow from equations (3.4.3a) and (3.4.3b), respectively, after replacing $\{\widetilde{A}_l\}_1^3$ with the relevant expressions in (3.5.10).

□

3.6 The Poincaré Problem.

Replacing in the global relation (3.1.7b), the term q_N with

$$-\frac{1}{\sin\delta_j}\left[\cos\delta_j\frac{dq}{ds}-\chi_j q\right],$$

and integrating by parts the term involving $\frac{dq}{ds}$, we find

$$\hat{q}(k)=E(-ik)\left\{\frac{i}{\sin\delta_j}\left(ke^{i\delta_j}+\frac{\lambda}{ke^{i\delta_j}}+\chi_j\right)D_j(k)-\frac{i\cos\delta_j}{\sin\delta_j}e^{\left(k+\frac{\lambda}{k}\right)s}q_j(s)\Big|_{-L}^{L}\right\}.$$

We assume that the boundary terms vanish i.e.

$$\begin{aligned}&\frac{\cos\delta_1}{\sin\delta_1}E(-ik)e^{\left(k+\frac{\lambda}{k}\right)s}q_1(s)\Big|_{-L}^{L}+\frac{\cos\delta_2}{\sin\delta_2}E(-iwk)e^{\left(wk+\frac{\lambda}{wk}\right)s}q_2(s)\Big|_{-L}^{L}\\&+\frac{\cos\delta_3}{\sin\delta_3}E(-i\bar{w}k)e^{\left(\bar{w}k+\frac{\lambda}{\bar{w}k}\right)s}q_3(s)\Big|_{-L}^{L}=0.\end{aligned}\tag{3.6.1}$$

This is indeed the case provide that

$$\cot\delta_1=\cot\delta_2=\cot\delta_3\ \text{ and }\ q_1(L)=q_2(-L),\ q_2(L)=q_3(-L),\ q_3(L)=q_1(-L).$$

Then the global relation becomes

$$\begin{aligned}&\left(ke^{i\delta_1}+\chi_1+\frac{\lambda}{ke^{i\delta_1}}\right)E(-ik)D_1(k)+\left(wke^{i\delta_2}+\chi_2+\frac{\lambda}{wke^{i\delta_2}}\right)E(-iwk)D_2(wk)\\&+\left(\bar{w}ke^{i\delta_3}+\chi_3+\frac{\lambda}{\bar{w}ke^{i\delta_3}}\right)E(-i\bar{w}k)D_3(\bar{w}k)=0,\ k\in\mathbb{C},\end{aligned}\tag{3.6.2}$$

where $\{\chi_j\}_1^3,\{\delta_j\}_1^3$ are real constants and $\sin\delta_j\neq0,\ j=1,2,3$.

Proposition 3.5. *Let each of the unknown Dirichlet boundary values be expressed as the sum of the three exponentials appearing in equations* (3.3.3)*. Then, the global relation* (3.6.2) *implies relations* (3.2.4b)-(3.2.4d)*, where a_1 and a_2 satisfy the following relations:*

$$\begin{aligned}
e^{6ia_2L} &= \prod_{j=1}^{3} \frac{(i\chi_j + a_1\cos\delta_j - A_1\sin\delta_j)(i\chi_j - (a_1+a_2)\cos\delta_j - (A_1+A_2)\sin\delta_j)}{(i\chi_j + a_1\cos\delta_j + A_1\sin\delta_j)(i\chi_j - (a_1+a_2)\cos\delta_j + (A_1+A_2)\sin\delta_j)},\\
e^{6ia_1L} &= \prod_{j=1}^{3} \frac{(i\chi_j + a_2\cos\delta_j + A_2\sin\delta_j)(i\chi_j - (a_1+a_2)\cos\delta_j + (A_1+A_2)\sin\delta_j)}{(i\chi_j + a_2\cos\delta_j - A_2\sin\delta_j)(i\chi_j - (a_1+a_2)\cos\delta_j - (A_1+A_2)\sin\delta_j)},
\end{aligned} \tag{3.6.3}$$

where $\{A_l\}_1^3$ are defined in terms of a_1 and a_2 by (3.4.4)*. Furthermore,*

$$\begin{aligned}
\alpha_2 &= \frac{\left(\frac{a_1\cos\delta_1+i\chi_1}{A_1}+\sin\delta_1\right)\left(\frac{a_3\cos\delta_3+i\chi_3}{A_3}-\sin\delta_3\right)}{\left(\frac{a_2\cos\delta_1+i\chi_1}{A_2}-\sin\delta_1\right)\left(\frac{a_3\cos\delta_3+i\chi_3}{A_3}+\sin\delta_3\right)} e^{i(a_2-a_1)L}\alpha_1,\\
\alpha_3 &= \frac{\left(\frac{a_1\cos\delta_1+i\chi_1}{A_1}+\sin\delta_1\right)\left(\frac{a_3\cos\delta_3+i\chi_3}{A_3}-\sin\delta_3\right)}{\left(\frac{a_2\cos\delta_1+i\chi_1}{A_2}-\sin\delta_1\right)\left(\frac{a_3\cos\delta_3+i\chi_3}{A_3}+\sin\delta_3\right)}\\
&\quad\frac{\left(\frac{a_2\cos\delta_1+i\chi_1}{A_2}+\sin\delta_1\right)\left(\frac{a_1\cos\delta_3+i\chi_3}{A_1}-\sin\delta_3\right)}{\left(\frac{a_3\cos\delta_1+i\chi_1}{A_3}-\sin\delta_1\right)\left(\frac{a_1\cos\delta_3+i\chi_3}{A_1}+\sin\delta_3\right)} e^{i(a_3-a_1)L}\alpha_1,
\end{aligned} \tag{3.6.4a}$$

$$\begin{aligned}
\beta_l &= \frac{\frac{a_l\cos\delta_1+i\chi_1}{A_l}-\sin\delta_1}{\frac{a_{l-1}\cos\delta_2+i\chi_2}{A_{l-1}}-\sin\delta_2} e^{-ia_{l+1}L}\alpha_l,\\
\gamma_l &= \frac{\frac{a_l\cos\delta_1+i\chi_1}{A_l}-\sin\delta_1}{\frac{a_{l+1}\cos\delta_3+i\chi_3}{A_{l+1}}-\sin\delta_3} e^{-ia_{l-1}L}\alpha_l, \quad l=1,2,3.
\end{aligned} \tag{3.6.4b}$$

Proof. Using the identity

$$\frac{ke^{i\delta_1}+\chi_1+\frac{\lambda}{ke^{i\delta_1}}}{k+ia_l+\frac{\lambda}{k}} = e^{i\delta_1} - \frac{k(ia_le^{i\delta_1}-\chi_1)+\lambda(e^{i\delta_1}-e^{-i\delta_1})}{k^2+ia_lk+\lambda}$$

we find that

$$\begin{aligned}
i\frac{e^{i\delta_1}k+\chi_1+\frac{\lambda}{e^{i\delta_1}k}}{k+ia_l+\frac{\lambda}{k}} &= \left(\frac{a_l\cos\delta_1+i\chi_1}{A_l}-\sin\delta_1\right)\left(\frac{k_l}{k-k_l}+\frac{e^{i\delta_1}}{2i\sin\delta_1}\right)\\
&\quad+\left(\frac{a_l\cos\delta_1+i\chi_1}{A_l}+\sin\delta_1\right)\left(\frac{\bar{k}_l}{k+\bar{k}_l}-\frac{e^{i\delta_1}}{2i\sin\delta_1}\right).
\end{aligned} \tag{3.6.5}$$

Hence, in analogy with equations (3.2.12)-(3.2.14), we now have the following equations valid for all $k \in \mathbb{C}$:

$$\begin{aligned}
&\sum_{l=1}^{3} \alpha_l e^{ia_l L} \left[\left(\frac{a_l \cos\delta_1 + i\chi_1}{A_l} - \sin\delta_1 \right) \left(\frac{k_l}{k - k_l} + \frac{e^{i\delta_1}}{2i\sin\delta_1} \right) \right. \\
&\qquad \left. + \left(\frac{a_l \cos\delta_1 + i\chi_1}{A_l} + \sin\delta_1 \right) \left(\frac{\bar{k}_l}{k + \bar{k}_l} - \frac{e^{i\delta_1}}{2i\sin\delta_1} \right) \right] \\
&= \sum_{l=1}^{3} \beta_l e^{-ib_l L} \left[\left(\frac{b_l \cos\delta_2 + i\chi_2}{B_l} - \sin\delta_2 \right) \left(\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{e^{i\delta_2}}{2i\sin\delta_2} \right) \right. \\
&\qquad \left. + \left(\frac{b_l \cos\delta_2 + i\chi_2}{B_l} + \sin\delta_2 \right) \left(\frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l} - \frac{e^{i\delta_2}}{2i\sin\delta_2} \right) \right],
\end{aligned} \tag{3.6.6}$$

$$\begin{aligned}
&\sum_{l=1}^{3} \beta_l e^{ib_l L} \left[\left(\frac{b_l \cos\delta_2 + i\chi_2}{B_l} - \sin\delta_2 \right) \left(\frac{\bar{w}\lambda_l}{k - \bar{w}\lambda_l} + \frac{e^{i\delta_2}}{2i\sin\delta_2} \right) \right. \\
&\qquad \left. + \left(\frac{b_l \cos\delta_2 + i\chi_2}{B_l} + \sin\delta_2 \right) \left(\frac{\bar{w}\bar{\lambda}_l}{k + \bar{w}\bar{\lambda}_l} - \frac{e^{i\delta}}{2i\sin\delta_2} \right) \right] \\
&= \sum_{l=1}^{3} \gamma_l e^{-ic_l L} \left[\left(\frac{c_l \cos\delta_3 + i\chi_3}{C_l} - \sin\delta_3 \right) \left(\frac{w\mu_l}{k - w\mu_l} + \frac{e^{i\delta_3}}{2i\sin\delta_3} \right) \right. \\
&\qquad \left. + \left(\frac{c_l \cos\delta_3 + i\chi_3}{C_l} + \sin\delta_3 \right) \left(\frac{w\bar{\mu}_l}{k + w\bar{\mu}_l} - \frac{e^{i\delta_3}}{2i\sin\delta_3} \right) \right],
\end{aligned} \tag{3.6.7}$$

$$\begin{aligned}
&\sum_{l=1}^{3} \gamma_l e^{ic_l L} \left[\left(\frac{c_l \cos\delta_3 + i\chi_3}{C_l} - \sin\delta_3 \right) \left(\frac{w\mu_l}{k - w\mu_l} + \frac{e^{i\delta_3}}{2i\sin\delta_3} \right) + \right. \\
&\qquad \left. \left(\frac{c_l \cos\delta_3 + i\chi_3}{C_l} + \sin\delta_3 \right) \left(\frac{w\bar{\mu}_l}{k + w\bar{\mu}_l} - \frac{e^{i\delta_3}}{2i\sin\delta_3} \right) \right] \\
&= \sum_{l=1}^{3} \alpha_l e^{-ia_l L} \left[\left(\frac{a_l \cos\delta_1 + i\chi_1}{A_l} - \sin\delta_1 \right) \left(\frac{k_l}{k - k_l} + \frac{e^{i\delta_1}}{2i\sin\delta_1} \right) + \right. \\
&\qquad \left. \left(\frac{a_l \cos\delta_1 + i\chi_1}{A_l} + \sin\delta_1 \right) \left(\frac{\bar{k}_l}{k + \bar{k}_l} - \frac{e^{i\delta_1}}{2i\sin\delta_1} \right) \right].
\end{aligned} \tag{3.6.8}$$

The analysis of the first and the second set of the relevant poles is similar with the analysis presented in section 3.2; it yields equations (3.2.4d), (3.2.4b) and (3.2.4c), as well as the following additional conditions:

$$\frac{e^{i\delta_1}}{2i\sin\delta_1} = \frac{e^{i\delta_2}}{2i\sin\delta_2} = \frac{e^{i\delta_2}}{2i\sin\delta_2} \Leftrightarrow \cot\delta_1 = \cot\delta_2 = \cot\delta_3. \tag{3.6.9}$$

The analysis of the first and the second set of the relevant residues is similar with the analysis presented in section 3.4, provided that we make the following substitutions:

$$\begin{aligned}
1-\frac{i\chi}{A_l} &\longrightarrow \frac{a_l\cos\delta_1+i\chi_1}{A_l}-\sin\delta_1, \;\; 1+\frac{i\chi}{A_l}\longrightarrow \frac{a_l\cos\delta_1+i\chi_1}{A_l}+\sin\delta_1\\
1-\frac{i\chi}{B_l} &\longrightarrow \frac{b_l\cos\delta_2+i\chi_2}{B_l}-\sin\delta_2, \;\; 1+\frac{i\chi}{B_l}\longrightarrow \frac{b_l\cos\delta_2+i\chi_2}{B_l}+\sin\delta_2\\
1-\frac{i\chi}{C_l} &\longrightarrow \frac{c_l\cos\delta_3+i\chi_3}{C_l}-\sin\delta_3, \;\; 1+\frac{i\chi}{C_l}\longrightarrow \frac{c_l\cos\delta_3+i\chi_3}{C_l}+\sin\delta_3.
\end{aligned} \tag{3.6.10}$$

Hence, the equations (3.6.4a) and (3.6.4b) are satisfied, which follow from the equations (3.5.4a) and (3.5.4b) with the aid of the above substitutions.

Furthermore, the analysis presented in section 4, employing the substitutions (3.6.10), implies that equation (3.4.14) is valid, i.e.

$$e^{2i(a_l-a_{l-1})L}=\frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{B}_{l-1}\widetilde{C}_l},\quad e^{2i(b_l-b_{l-1})L}=\frac{\widetilde{B}_{l-1}\widetilde{B}_l}{\widetilde{C}_{l-1}\widetilde{A}_l},\quad l=1,2,3, \tag{3.6.11}$$

where now $\{\widetilde{A}_l,\widetilde{B}_l,\widetilde{C}_l\}_1^3$ are defined as follows:

$$\begin{aligned}
\widetilde{A}_l &= \frac{\frac{a_l\cos\delta_1+i\chi_1}{A_l}-\sin\delta_1}{\frac{a_l\cos\delta_1+i\chi_1}{A_l}+\sin\delta_1}, \quad \widetilde{B}_l=\frac{\frac{b_l\cos\delta_2+i\chi_2}{B_l}-\sin\delta_2}{\frac{b_l\cos\delta_2+i\chi_2}{B_l}+\sin\delta_2},\\
\widetilde{C}_l &= \frac{\frac{c_l\cos\delta_3+i\chi_3}{C_l}-\sin\delta_3}{\frac{c_l\cos\delta_3+i\chi_3}{C_l}+\sin\delta_3}, \quad l=1,2,3.
\end{aligned} \tag{3.6.12}$$

Using equations (3.4.16) in (3.6.11) we find

$$e^{2i(a_l-a_{l-1})L}=\frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{B}_{l-1}\widetilde{C}_l},\quad e^{2i(a_{l-1}-a_{l-2})L}=\frac{\widetilde{B}_{l-1}\widetilde{B}_l}{\widetilde{C}_{l-1}\widetilde{A}_l},\quad l=1,2,3. \tag{3.6.13}$$

The second set of equations (3.6.13) is identical with the first set of equation (3.6.13)(after replacing l with $l+1$), provided that

$$\frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{B}_{l-1}\widetilde{C}_l}=\frac{\widetilde{B}_l\widetilde{B}_{l+1}}{\widetilde{C}_l\widetilde{A}_{l+1}},\quad l=1,2,3. \tag{3.6.14}$$

Furthermore, the condition that $\prod_{l=1}^{3} e^{i(a_l-a_{l-1})L} = 1$ gives rise to the additional constraint

$$\prod_{l=1}^{3} \frac{\widetilde{A}_{l-1}\widetilde{A}_l}{\widetilde{B}_{l-1}\widetilde{C}_l} = 1. \tag{3.6.15}$$

Equations (3.6.14) and (3.6.15) are equivalent with the following conditions

$$\prod_{l=1}^{3} \widetilde{A}_l = \prod_{l=1}^{3} \widetilde{B}_l = \prod_{l=1}^{3} \widetilde{C}_l. \tag{3.6.16}$$

Note that the analogue of equations (3.6.16) is identically satisfied for the Oblique Robin problem.

Using equations (3.2.21) - (3.2.24), we write $\{A_l\}_1^3$ and $\{B_l\}_1^3$ in the equation $\prod_{l=1}^{3} \widetilde{A}_l = \prod_{l=1}^{3} \widetilde{B}_l$, in terms of a_1 and a_2; this yields the following relation:

$$\sin 3\delta_1 \left(\chi_1^2 - 3\lambda\right) \chi_1 - \sin 3\delta_2 \left(\chi_2^2 - 3\lambda\right) \chi_2 + i \prod_{j=1}^{3} a_j \ \sin\left(3\delta_1 - 3\delta_2\right) = 0, \tag{3.6.17}$$

where we have used the fact that $a_1^2 + a_1 a_2 + a_2^2 = -3\lambda$. Similarly, from the condition $\prod_{l=1}^{3} \widetilde{B}_l = \prod_{l=1}^{3} \widetilde{C}_l$, we obtain

$$\sin 3\delta_2 \left(\chi_2^2 - 3\lambda\right) \chi_2 - \sin 3\delta_3 \left(\chi_3^2 - 3\lambda\right) \chi_3 + i \prod_{j=1}^{3} a_j \ \sin\left(3\delta_2 - 3\delta_3\right) = 0. \tag{3.6.18}$$

Equations (3.6.17) and (3.6.18) imply the following conditions

$$\begin{gathered} \sin\left(3\delta_1 - 3\delta_2\right) = 0, \ \sin\left(3\delta_2 - 3\delta_3\right) = 0, \\ \left(\sin 3\delta_1\right) \left(\chi_1^2 - 3\lambda\right) \chi_1 - \left(\sin 3\delta_2\right) \left(\chi_2^2 - 3\lambda\right) \chi_2 = 0, \\ \left(\sin 3\delta_2\right) \left(\chi_2^2 - 3\lambda\right) \chi_2 - \left(\sin 3\delta_3\right) \left(\chi_3^2 - 3\lambda\right) \chi_3 = 0. \end{gathered} \tag{3.6.19}$$

Furthermore, by employing in (3.6.19) equations (3.6.9) we obtain conditions (3.1.13). Using these conditions, it follows that equations (3.6.13) are equivalent with the following conditions:

$$e^{2i(a_2-a_1)L} = \frac{\widetilde{A}_1\widetilde{A}_2}{\widetilde{B}_1\widetilde{C}_2}, \quad e^{2i(2a_1+a_2)L} = \frac{\widetilde{A}_3\widetilde{A}_1}{\widetilde{B}_3\widetilde{C}_1}, \tag{3.6.20}$$

where $\{\widetilde{A}_l, \widetilde{B}_l, \widetilde{C}_l\}_1^3$ are defined in equation (3.6.12). Using equations (3.2.24) we rewrite equations (3.6.20) in the following form:

$$e^{2i(a_2-a_1)L} = \frac{\Psi_{1,1}\Psi_{1,2}}{\Psi_{2,3}\Psi_{3,3}}, \quad e^{2i(2a_1+a_2)L} = \frac{\Psi_{1,3}\Psi_{1,1}}{\Psi_{2,2}\Psi_{3,2}}, \tag{3.6.21}$$

where

$$\Psi_{j,l} = \frac{\frac{a_l\cos\delta_j + i\chi_j}{A_l} - \sin\delta_j}{\frac{a_l\cos\delta_j + i\chi_j}{A_l} + \sin\delta_j}, \quad j = 1,2,3, \ l = 1,2,3. \tag{3.6.22}$$

Hence,

$$e^{6ia_2L} = \prod_{j=1}^{3} \frac{\Psi_{j,1}}{\Psi_{j,3}}, \quad e^{6ia_1L} = \prod_{j=1}^{3} \frac{\Psi_{j,3}}{\Psi_{j,2}}. \tag{3.6.23}$$

Replacing in equations (3.6.23) the functions $\Psi_{j,l}, \ j,l = 1,2,3$, by the expressions given in (3.6.22) we obtain equations (3.6.3).

$\square$

3.7 The oblique Robin, Robin, Neumman and Dirichlet eigenvalues as particular limits of the Poincaré eigenvalues.

From Poincaré to oblique Robin

Making in equations (3.6.3) the substitutions

$$\delta_j = \delta, \quad \chi_j = \chi, \ j = 1,2,3,$$

we find that

$$e^{6ia_2L} = \left(\frac{\widetilde{A}_1}{\widetilde{A}_3}\right)^3, \quad e^{6ia_1L} = \left(\frac{\widetilde{A}_3}{\widetilde{A}_2}\right)^3, \tag{3.7.1}$$

where $\{\widetilde{A}_l\}_1^3$ are defined in (3.5.10). Furthermore, equations (3.7.1) are equivalent to equations (3.5.11), which yield equations (3.5.3).

From oblique Robin to Robin

Replacing in equations (3.5.3) δ by $\frac{\pi}{2}$ and writing $\{A_l\}_1^3$ in terms of a_1 and a_2, as defined in (3.4.4), we find equations (3.4.2).

From Robin to Neumman and Dirichlet

Inserting in equations (3.4.2) either $\chi = \infty$ (Dirichlet condition), or $\chi = 0$ (Neumman condition) we find the following equations:

$$\sin\left[\left(a_2 - \frac{N\pi}{3L}\right)L\right] = 0, \quad \sin\left[\left(a_1 - \frac{N\pi}{3L}\right)L\right] = 0, \quad n \in \mathbb{Z}, \tag{3.7.2}$$

which yield

$$a_1 = \frac{(N+3M_1)\pi}{3L}, \quad a_2 = \frac{(N+3M_2)\pi}{3L}, \qquad N, M_1, M_2 \in \mathbb{Z}. \tag{3.7.3}$$

Making the substitutions

$$N \to m - n,\ M_1 \to -n \text{ and } M_2 \to 0, \qquad m, n \in \mathbb{Z},$$

we find equations (3.2.4a). Hence, the corresponding eigenvalues are given by equation (3.2.4e).

3.8 Eigenfunctions.

For the Dirichlet problem the integral representation of the solution given in (3.1.14) becomes

$$q(x,y) = \frac{1}{4i\pi}\sum_{j=1}^{3}\int_{L_j} P(k,z)E(-iw^{j-1}k)N_j(w^{j-1}k)\frac{dk}{k}, \tag{3.8.1}$$

where $P(k,z) = e^{ikz+\frac{\lambda}{ik}\bar{z}}$ and the contours L_j are depicted in Figure 1.3.

Proposition 3.6. *Assume that $q(x,y)$ satisfies equation* (3.1.1), *where q vanishes on the boundary of the equilateral triangle, i.e. q is an eigenfunction of the Dirichlet problem. This function is given by*

$$\begin{aligned} q_{m,n}(x,y) =& e^{i\frac{m+2n}{3}\frac{\pi}{L}(y-L)} \sin\left[\frac{m\pi}{L\sqrt{3}}\left(x+\frac{2L}{\sqrt{3}}\right)\right] \\ & - e^{i\frac{m-n}{3}\frac{\pi}{L}(y-L)} \sin\left[\frac{(m+n)\pi}{L\sqrt{3}}\left(x+\frac{2L}{\sqrt{3}}\right)\right] \\ & + e^{i\frac{-2m-n}{3}\frac{\pi}{L}(y-L)} \sin\left[\frac{n\pi}{L\sqrt{3}}\left(x+\frac{2L}{\sqrt{3}}\right)\right]. \end{aligned} \tag{3.8.2}$$

Proof. Equations (3.2.30) suggest the definitions

$$\frac{\alpha_l}{A_l}e^{ia_lL} = \mathcal{A}, \ \frac{\alpha_l}{A_l}e^{ia_lL}e^{2ib_lL} = \mathcal{B}, \ \frac{\alpha_l}{A_l}e^{-ia_lL} = \mathcal{C}, \quad l = 1,2,3. \tag{3.8.3}$$

Also, equations (3.2.4b), (3.2.4c) and (3.2.4a) yield

$$\mathcal{B} = w^{m-n}\mathcal{A}, \quad \mathcal{C} = w^{n-m}\mathcal{A}. \tag{3.8.4}$$

Hence using equations (3.2.12)-(3.2.14) together with equations (3.8.3), we can rewrite the spectral functions $N_j(k)$ as follows:

$$\begin{aligned} E(-ik)N_1(k) &= [\mathcal{A}e_1(k) - \mathcal{C}e_3(k)]F(k) \\ E(-iwk)N_2(wk) &= [\mathcal{B}e_2(k) - \mathcal{A}e_1(k)]F(k) \\ E(-i\bar{w}k)N_3(\bar{w}k) &= [\mathcal{C}e_3(k) - \mathcal{B}e_2(k)]F(k), \end{aligned} \tag{3.8.5}$$

where

$$e_j(k) = \exp\left[\frac{2L}{\sqrt{3}}\left(w^{j-1}ke^{i\frac{\pi}{6}} + \frac{\lambda}{w^{j-1}ke^{i\frac{\pi}{6}}}\right)\right]$$

and

$$F(k) = \sum_{j=1}^{3}\left(\frac{k_j}{k-k_j} + \frac{\bar{k}_j}{k+\bar{k}_j}\right).$$

Using equations (3.8.5) the integral representation (3.8.1) can be rewritten as follows:

$$\begin{aligned} q(x,y) = & \mathcal{A}\int_{\partial\tilde{D}_1} P(k,z)e_1(k)F(k)\frac{dk}{k} + \mathcal{B}\int_{\partial\tilde{D}_2} P(k,z)e_2(k)F(k)\frac{dk}{k} \\ & + \mathcal{C}\int_{\partial\tilde{D}_3} P(k,z)e_3(k)F(k)\frac{dk}{k} + \mathcal{A}\int_{\partial\hat{D}_1} P(k,z)e_1(k)F(k)\frac{dk}{k} \\ & + \mathcal{B}\int_{\partial\hat{D}_2} P(k,z)e_2(k)F(k)\frac{dk}{k} + \mathcal{C}\int_{\partial\hat{D}_3} P(k,z)e_3(k)F(k)\frac{dk}{k}, \end{aligned} \tag{3.8.6}$$

where the domains $\{\widetilde{D}_j\}_1^3$ and $\{\hat{D}_j\}_1^3$ are depicted in Figure 3.5.

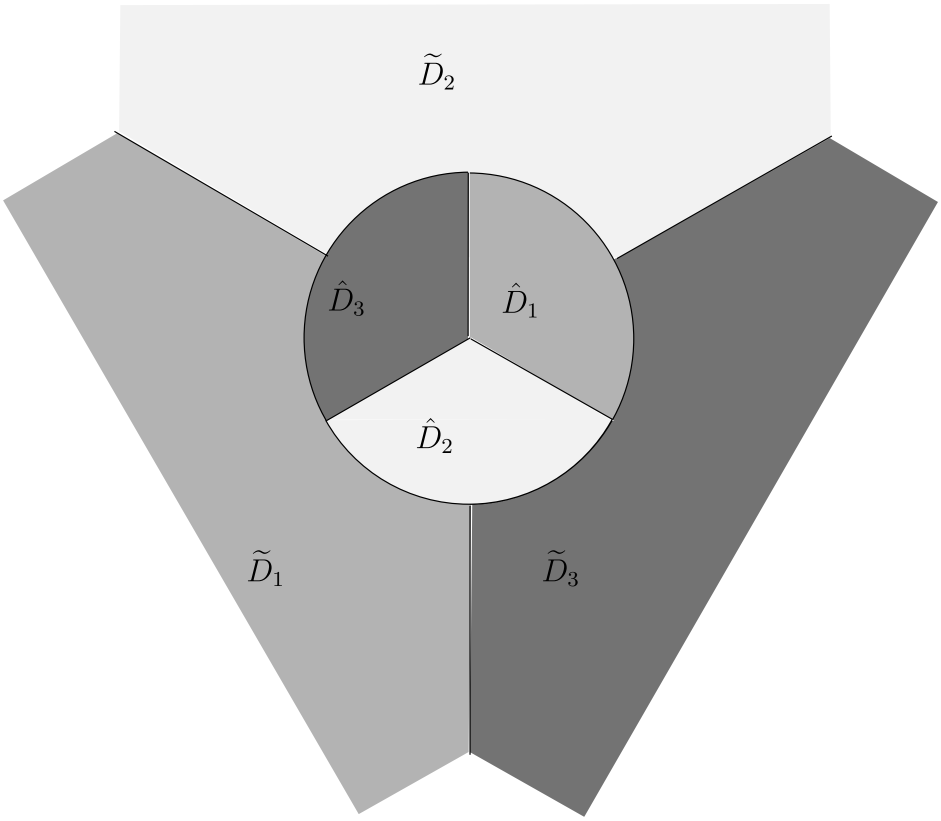

Figure 3.5: *The $\hat{D}_j$ and $\widetilde{D}_j$ domains.*

Using Cauchy theorem and appropriate arguments for boundedness and analyticity we find that the only contribution of the expression in (3.8.6) to the solution $q(x,t)$ is due to the poles of the function $F(k)$. The poles $\{k_j\}_1^3$ and $\{-\bar{k}_j\}_1^3$ satisfy

$$|k_j| = \sqrt{-\lambda}, \;\; j = 1, 2, 3.$$

Hence, without loss of generality, we can choose a point k_1 on the circle with radius $\sqrt{-\lambda}$ and then the position of the other five poles are fixed from the relation $k_j = \bar{w}k_{j-1}, \;\; j = 1, 2, 3$ (see Figure 3.6).

Computing the residues in the equation (3.8.6) we obtain the following contribution:

$$\begin{aligned} & P(k_1, z)[\mathcal{A}e_1(k_1) + \mathcal{B}e_2(k_1)] - P(-\bar{k}_1, z)[\mathcal{B}e_2(-\bar{k}_1) + \mathcal{C}e_3(-\bar{k}_1)] \\ & + P(k_2, z)[\mathcal{B}e_2(k_2) + \mathcal{C}e_3(k_2)] - P(-\bar{k}_2, z)[\mathcal{A}e_1(-\bar{k}_2) + \mathcal{B}e_2(-\bar{k}_2)] \\ & + P(k_3, z)[\mathcal{C}e_3(k_3) + \mathcal{A}e_1(k_3)] - P(-\bar{k}_3, z)[\mathcal{C}e_3(-\bar{k}_3) + \mathcal{A}e_1(-\bar{k}_3)]. \end{aligned} \tag{3.8.7}$$

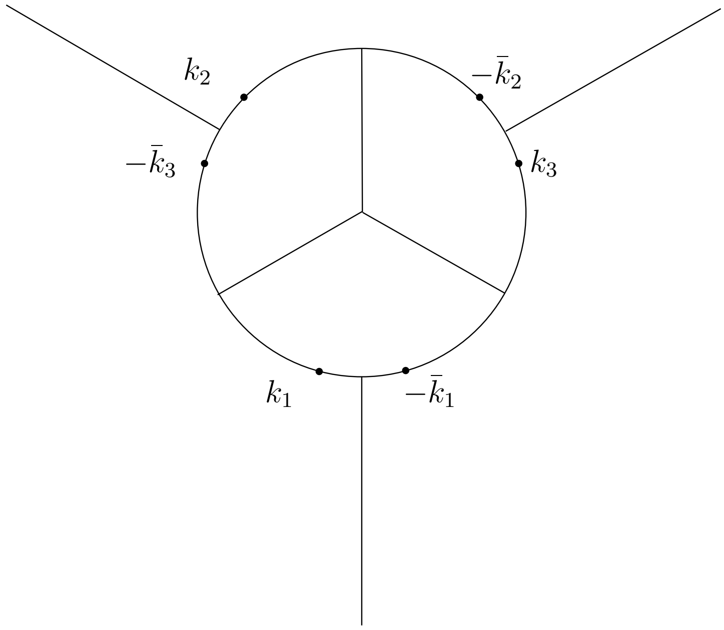

Figure 3.6: *The poles k_j.*

In order to simplify these expression we will use the identities

$$P(k_j, z) = e^{ia_j y} e^{iA_j x} \text{ and } P(-\bar{k}_j, z) = e^{ia_j y} e^{-iA_j x},$$

as well as relations (3.8.4), the expressions of $\{A_j\}_1^3$ in terms of a_1 and a_2 given in equations (3.2.21)-(3.2.23), and the definitions of $\{a_j\}_1^3$ in equations (3.2.4b), (3.2.4c) and (3.2.4a). Then, equation (3.8.7) yields the expression

$$\sum_{j=1}^{3} e^{ia_j y} e^{iA_j x} e^{-ia_j L} e^{2iA_j \frac{L}{\sqrt{3}}} - \sum_{j=1}^{3} e^{ia_j y} e^{-iA_j x} e^{-ia_j L} e^{-2iA_j \frac{L}{\sqrt{3}}}, \tag{3.8.8}$$

where we have put $\mathcal{A} = (-1)^m$. Therefore, the function $q(x, y)$ is given by the following expression:

$$q(x, y) = \sum_{j=1}^{3} e^{ia_j (y-L)} \sin \left[A_j \left(x + \frac{2L}{\sqrt{3}} \right) \right], \tag{3.8.9}$$

where

$$a_1 = (m + 2n)\frac{\pi}{3L},\ a_2 = (m - n)\frac{\pi}{3L},\ a_3 = -(2m + n)\frac{\pi}{3L},\ m, n \in \mathbb{Z}$$

and

$$A_1 = m\frac{\pi}{L\sqrt{3}},\ A_2 = -(m + n)\frac{\pi}{L\sqrt{3}},\ A_3 = n\frac{\pi}{L\sqrt{3}},\ m, n \in \mathbb{Z}.$$

$\square$

The eigenfunctions of the other problems can be computed in similar way. In this respect we note that the relation among $\mathcal{A}, \mathcal{B}, \mathcal{C}$ as well as the expression for $F(k)$ are different for each different problem, however the relevant poles can always be expressed in terms of a_1 and a_2.

Regarding the important question of the completeness of the associated eigenfunctions, we note that a novel approach for establishing completeness was introduced in [9], Chapter 4. This approach involves the following: Solve the heat equation in the interior of an equilateral triangle with given initial condition $q_0(x, y)$ and with homogeneous Dirichlet boundary conditions. The evaluation of this solution at $t = 0$ provides a *complete* spectral representation of the *arbitrary* function $q_0(x, y)$. The solution of the Dirichlet problem for the heat equation in the interior of an equilateral triangle is presented in the following Chapter (other types of boundary value problems can be analyzed in a similar way).

As an independent approach, mentioned in the Introduction, completeness for the associated expansions for the Dirichlet and Neumann problems was obtained in [23], [24], [35], [25] using group theoretic techniques and for the Robin problem was achieved in [36] using a homotopy argument.

Chapter 4

The heat equation in the interior of an equilateral triangle.

The "Fokas method" was developed further for linear PDEs in [39], [4], [5], [44] and the monograph [9]. This method was extended to evolution PDEs in two spatial dimensions in [39] and [40]. Here, we implement the new method to the heat equation in the interior of an equilateral triangle.

The new transform method involves the following steps:

1. Given a PDE and a domain, derive an *integral representation* in the Fourier space for the solution of this PDE, in terms of appropriate integral transforms of the boundary values. Furthermore, derive the *global relation*, which is an algebraic equation coupling the integral transforms of the boundary values.

For the heat equation in the interior of an equilateral triangle this step is implemented in Proposition 4.1, see (4.1.12) and (4.1.15), respectively.

2. Given appropriate boundary conditions, by employing the global relation, the equations obtained from the global relation via certain invariant transforms, and Cauchy's

theorem, eliminate from the integral representation the integral transforms of the unknown boundary values.

For the symmetric Dirichlet problem, i.e. for the case that the same function is prescribed as a Dirichlet boundary condition on every side of the triangle, this step is implemented in Proposition 4.2, see equation (4.1.25).

In more details, step 2 involves the following:

(i) For the Dirichlet problem, the integral representation, in addition to the transforms of the known Dirichlet data, it also involves the transforms of the unknown Neumann data in each side, which are denoted by $\{U^{(j)}\}_{j=1}^{3}$. The global relation and the equation obtained from the global relation via certain invariant transform, see (4.1.19) and (4.1.20), are two equations, see (4.1.30) and (4.1.32), involving the functions $\{U^{(j)}\}_{j=1}^{3}$, and $\hat{q}(\pm k_1, k_2, t)$, where $\hat{q}(k_1, k_2, t)$ denotes the Fourier transform of the solution $q(x_1, x_2, t)$. Eliminating $U^{(3)}$, we can express $U^{(1)}$ in terms of $U^{(2)}$ and $\hat{q}(\pm k_1, k_2, t)$.

(ii) Replacing in the integral representation of q the expression $U^{(1)}$ found in (i), we find that the contribution of the term $U^{(1)}$ involves integrals containing $U^{(2)}$ and $\hat{q}(\pm k_1, k_2, t)$. The latter integrals vanish because of analyticity, whereas the former integrals, using appropriate contour deformations, give rise to two different integrals involving $U^{(2)}$, see (4.1.42).

(iii) For one of the above integrals, we use again the relation found in (i) and we express $U^{(2)}$ in terms of $U^{(1)}$ and $\hat{q}(\pm k_1, k_2, t)$, see (4.1.43). Taking into consideration that the contribution of $\hat{q}(\pm k_1, k_2, t)$ again vanishes because of analyticity, we find that the contribution of $U^{(1)}$ to q yields an integral involving $U^{(1)}$ and an integral involving $U^{(2)}$, see (4.1.46).

(iv) We can compute the analogous contributions in the integral representation of q of the terms involving $U^{(2)}$ and $U^{(3)}$ in a similar way. It turns out, that the integrals involving the unknown terms $\{U^{(j)}\}_{j=1}^{3}$ cancel and hence q can be expressed in terms of the transforms of the given Dirichlet data.

It is obvious that there exist clear analogies between the method used here for the derivation of the solution of the symmetric Dirichlet problem for the heat equation in the interior of an equilateral triangle and the method used in Chapter 2, for the derivation of the solution for some elliptic PDEs in the same domain. This is of course expected since the Laplace and Helmholtz equation are related in several ways with the heat equation. For example, the relation between the eigenvalues of the Laplace operator and the solution of the heat equation was discussed at the end of Chapter 3.

An illustrative example of the main result of this Chapter is presented in section 4.2, see equations (4.2.1)-(4.2.4). It should be noted that the new method yields integrals in the Fourier space involving integrands which decay exponentially as $|k| \to \infty$ (this fact can be explicitly verified for the example discussed in section 4.2, see (4.2.4)). Hence, such integrals can be computed efficiently using the technique introduced in [45].

The solution of the arbitrary Dirichlet problem is derived in section 4.3. In particular, it is proven that this problem can be decomposed in three problems each of which is similar with the symmetric problem. Thus, the solution of the arbitrary Dirichlet problem does not present any new complications.

4.1 The Symmetric Dirichlet Problem.

Let the real function $q(x_1, x_2, t)$ satisfy the forced heat equation in the interior of an equilateral triangle

$$q_t - q_{x_1 x_1} - q_{x_2 x_2} = f, \ (x_1, x_2) \in D, \ 0 < t < T, \tag{4.1.1}$$

where T is a positive constant, $f(x_1, x_2, t)$ is a given function with sufficient smoothness and $D \subset \mathbb{R}^2$ denotes the interior of the equilateral triangle, defined in Chapter 2 in equation (2.1.1) with the parametrization (2.1.2), see Figure 2.1.

Equation (4.1.1) can be rewritten in the following divergence form:

$$\left(e^{-ikx+w(k)t}q\right)_t + \left(e^{-ikx+w(k)t}X^{(1)}\right)_{x_1} + \left(e^{-ikx+w(k)t}X^{(2)}\right)_{x_2} = e^{-ikx+w(k)t}f, \qquad (4.1.2)$$

where

$$kx = k_1x_1 + k_2x_2,\ w(k) = k_1^2 + k_2^2 \qquad (4.1.3)$$

and the functions $X^{(m)}$ are defined by

$$X^{(m)} = -q_{x_m} - ik_mq,\ m = 1, 2. \qquad (4.1.4)$$

Let $\hat{T}_j$ and $\hat{N}_j$, $j = 1, 2, 3$, denote the unit vectors along and normal to the sides (j), with the directions indicated in Figure 4.1. A unit vector from z_2 to z_3 makes an angle of $\pi + \frac{\pi}{6} = \frac{\pi}{2} + \frac{2\pi}{3}$ with the x_1-axis, thus it is characterized by the following complex number:

$$e^{i(\frac{\pi}{2}+\frac{2\pi}{3})} = ia = -\sin\frac{2\pi}{3} + i\cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - i\frac{1}{2},\ a = e^{\frac{2i\pi}{3}}. \qquad (4.1.5)$$

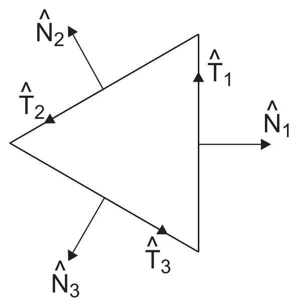

Figure 4.1: *The unit vectors on the Equilateral Triangle.*

Similarly a unit vector from z_3 to z_2 is characterized by the complex number

$$e^{i(\frac{\pi}{2}-\frac{2\pi}{3})} = i\bar{a} = \frac{\sqrt{3}}{2} - i\frac{1}{2}. \qquad (4.1.6)$$

Hence,

$$\begin{aligned}&\hat{T}_1=(0,1),\ \hat{N}_1=(1,0);\ \hat{T}_2=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\ \hat{N}_2=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right);\\&\hat{T}_3=\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\ \hat{N}_3=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right).\end{aligned}\tag{4.1.7}$$

Let ξ and η denote the components of the vector (x_1,x_2) along $\hat{T}$ and $\hat{N}$, see Figure 4.2. Then, if (x_1,x_2) is on any of the sides of the triangle,

$$\xi=s,\quad \eta=\frac{l}{2\sqrt{3}}.\tag{4.1.8}$$

Indeed,

on (1): $\xi=\left(\frac{l}{2\sqrt{3}},s\right)\cdot(0,1)=s,\quad \eta=\left(\frac{l}{2\sqrt{3}},s\right)\cdot(1,0)=\frac{l}{2\sqrt{3}}.$

on (2): $\xi=\left(\frac{l}{2\sqrt{3}}\cos\theta-s\sin\theta,\frac{l}{2\sqrt{3}}\sin\theta+s\cos\theta\right)\cdot(-\sin\theta,\cos\theta)=s,$

$$\eta=\left(\frac{l}{2\sqrt{3}}\cos\theta-s\sin\theta,\frac{l}{2\sqrt{3}}\sin\theta+s\cos\theta\right)\cdot(\cos\theta,\sin\theta)=\frac{l}{2\sqrt{3}},\quad \theta=\frac{2\pi}{3}.$$

Similarly for the side (3). Moreover, since the equilateral triangle is convex we find that

$$\xi_j\in\left[-\frac{l}{2},\frac{l}{2}\right]\quad\text{and}\quad \eta_j\in\left[-\frac{l}{\sqrt{3}},\frac{l}{2\sqrt{3}}\right],\quad j=1,2,3,\tag{4.1.9}$$

where ξ_j and η_j are the components of (x_1,x_2) along $\hat{T}_j$ and $\hat{N}_j$, respectively.

We will use the notations μ and λ for the component of (k_1,k_2) along $\hat{T}$ and $\hat{N}$ respectively. Also, we will use the notation $\frac{\partial}{\partial N_j}$ for the derivative along $\hat{N}_j$, i.e.

$$\mu_j=(k_1,k_2)\cdot\hat{T}_j,\quad \lambda_j=(k_1,k_2)\cdot\hat{N}_j,\quad \hat{N}_j\cdot\nabla q=\frac{\partial q}{\partial N_j},\quad j=1,2,3.\tag{4.1.10}$$

Let $\mathcal{T}$ denote the domain of validity of (4.1.1), i.e.

$$\mathcal{T}=\{x_1,x_2\in D,\ 0<t<T\}.\tag{4.1.11}$$

Proposition 4.1. ***(The Global Relation and the Integral Representation)*** *Suppose that there exists a solution of the forced heat equation (4.1.1) in the interior of the*

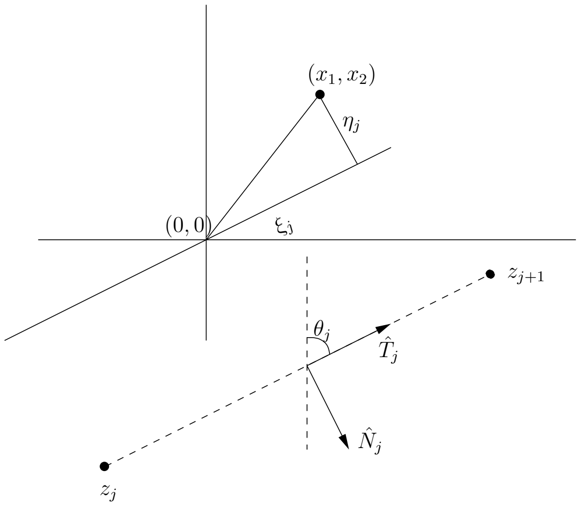

Figure 4.2: *The ξ_j and η_j.*

equilateral triangle and suppose that this solution has sufficient smoothness all the way to the boundary of the triangle. Then this solution can be expressed in the form

$$\begin{aligned} q(x_1, x_2, t) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} dk_1 \int_{\mathbb{R}} dk_2 e^{ikx - w(k)t} Q(k_1, k_2, t) \\ &- \frac{1}{(2\pi)^2} \sum_{j=1}^{3} \int_{\mathbb{R}} d\mu_j \int_{\partial D^-} d\lambda_j e^{i(\mu_j \xi_j + \lambda_j \eta_j) - w(k)t} \tilde{g}^{(j)}(\mu_j, \lambda_j, t), \ (x_1, x_2, t) \in \mathcal{T}, \end{aligned} \tag{4.1.12}$$

where the quantities appearing in (4.1.12) *are defined as follows:*

- *μ_j and λ_j are the components of (k_1, k_2) along $\hat{T}_j$ and $\hat{N}_j$, whereas ξ_j and η_j are the components of (x_1, x_2) along $\hat{T}_j$ and $\hat{N}_j$, hence*

$$kx = \mu_j \xi_j + \lambda_j \eta_j; \tag{4.1.13}$$

- *$w(k)$ is defined in the second of equations (4.1.3); furthermore the definitions of μ_j and λ_j* (4.1.10) *implies that $\mu_j^2 + \lambda_j^2 = w(k)$, $j = 1, 2, 3$;*

- *∂D^- denotes the same contours in the complex λ_j-planes, $j = 1, 2, 3$, namely the union of the rays $\arg \lambda_j = -\frac{\pi}{4}, -\frac{3\pi}{4}$, see Figure 3;*

- *the functions Q and $\left\{\tilde{g}^{(j)}\right\}_1^3$ are defined by*

$$
\begin{aligned}
Q(k_1, k_2, t) = & \iint_D e^{-ikx} q_0(x_1, x_2) dx_1 dx_2 \\
& + \iint_D \int_0^t e^{-ikx+w(k)\tau} f(x_1, x_2, \tau) dx_1 dx_2 d\tau
\end{aligned} \tag{4.1.14a}
$$

and

$$
\begin{aligned}
\tilde{g}^{(j)}(\mu_j, \lambda_j, t) = e^{-i\lambda_j \frac{l}{2\sqrt{3}}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^t e^{-i\mu_j s + w(k)\tau} \left[\frac{\partial q^{(j)}}{\partial N_j}(s, \tau) + i\lambda_j q^{(j)}(s, \tau) \right] ds d\tau, \\
j = 1, 2, 3, \ 0 < t < T, \ k_1, k_2 \in \mathbb{C}.
\end{aligned} \tag{4.1.14b}
$$

Furthermore the following relation, called the global relation, is valid:

$$
e^{w(k)t} \hat{q}(k_1, k_2, t) = Q(k_1, k_2, t) + \sum_{j=1}^{3} \tilde{g}^{(j)}(\mu_j, \lambda_j, t), \ k_1, k_2 \in \mathbb{C}, \ 0 < t < T, \tag{4.1.15}
$$

where $\hat{q}$ is defined by

$$
\hat{q}(k_1, k_2, t) = \iint_D e^{-ikx} q(x_1, x_2, t) dx_1 dx_2, \ k_1, k_2 \in \mathbb{C}, \ 0 < t < T. \tag{4.1.16}
$$

Proof. Integrating (4.1.2) over D we find

$$
\begin{aligned}
\left(e^{w(k)t} \hat{q} \right)_t = & \iint_D e^{-ikx+w(k)t} f(x_1, x_2, t) dx_1 dx_2 \\
& + \iint_D \left[\underline{\nabla} \cdot e^{-ikx+w(k)t} (\underline{\nabla} q + i\underline{k} q) \right] dx_1 dx_2,
\end{aligned} \tag{4.1.17}
$$

where $\underline{k}$ denotes the vector (k_1, k_2). Green's theorem implies that the second integral in the RHS of (4.1.17) equals

$$
\sum_{j=1}^{3} \int_{-\frac{l}{2}}^{\frac{l}{2}} \hat{N}_j \cdot e^{-ikx+w(k)t} \left(\underline{\nabla} q + i\underline{k} q \right) ds.
$$

Using this expression in (4.1.17), employing equations (4.1.9) and (4.1.10) and then integrating the resulting equation over $(0, t)$, we find (4.1.15).

Taking the inverse Fourier transform of (4.1.15) we find that q is given by the RHS of (4.1.12), where the integrals in the summation of the RHS of (4.1.12) involve integrals along the real line instead of integrals along $-\partial D^-$. The deformation of the contours of integration from the real line of the complex λ_j-plane to the curve $-\partial D^-$ can be justified as follows: The relevant integrand in the complex λ_j-plane is given by

$$\frac{1}{(2\pi)^2}e^{i(\mu_j\xi_j+\lambda_j\eta_j)-(\mu_j^2+\lambda_j^2)t}$$
$$\cdot e^{-i\lambda_j\frac{l}{2\sqrt{3}}}\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_0^t e^{-i\mu_j s+(\mu_j^2+\lambda_j^2)\tau}\left[\frac{\partial q^{(j)}}{\partial N_j}(s,\tau)+i\lambda_j q^{(j)}(s,\tau)\right]dsd\tau,\ \ j=1,2,3. \tag{4.1.18}$$

Equation (4.1.9) implies that

$$\eta_j-\frac{l}{2\sqrt{3}}\leq 0,\ \ j=1,2,3,$$

thus $\exp\left[i\lambda_j\left(\eta_j-\frac{l}{2\sqrt{3}}\right)\right]$ $\exp\left[-\lambda_j^2(t-\tau)\right]$ is bounded and analytic in the lower-half of the non-shaded domain of Figure 4.3 and hence Jordan's lemma implies the desired result.

$\square$

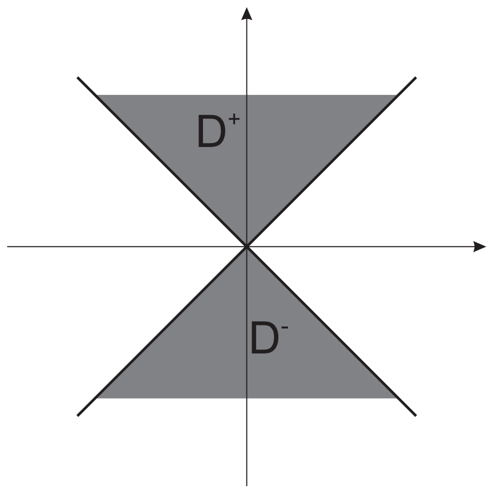

Figure 4.3: *The domains of boundedness and analyticity in the λ_j-plane. The boundaries of D^+ and D^-, denoted by ∂D^+ and ∂D^- are defined by the union of the rays* $\arg\lambda_j=\pi/4,3\pi/4$ *and* $\arg\lambda_j=-\pi/4,-3\pi/4$, *respectively.*

Remark 4.1. *It is straightforward to verify that under the transformations*

$$k_1\to -k_1 \text{ and } k_2\to k_2, \tag{4.1.19}$$

$\{\lambda_j,\mu_j\}_1^3$ *transform as follows:*

$$\lambda_1\to-\lambda_1,\ \ \mu_1\to\mu_1;\ \ \lambda_2\to-\lambda_3,\ \ \mu_2\to\mu_3;\ \ \lambda_3\to-\lambda_2,\ \ \mu_3\to\mu_2. \tag{4.1.20}$$

Indeed, relations (4.1.20) follows from (4.1.19) and the definition of λ_j and μ_j:

$$\begin{aligned}&\lambda_1=(k_1,k_2)\cdot(1,0)=k_1,\ \ \mu_1=(k_1,k_2)\cdot(0,1)=k_2;\\&\lambda_2=-\frac{k_1}{2}+\frac{\sqrt{3}}{2}k_2,\ \ \mu_2=-\frac{\sqrt{3}}{2}k_1-\frac{k_2}{2};\ \ \lambda_3=-\frac{k_1}{2}-\frac{\sqrt{3}}{2}k_2,\ \ \mu_3=\frac{\sqrt{3}}{2}k_1-\frac{k_2}{2}.\end{aligned}\tag{4.1.21}$$

Remark 4.2. *The transformation* (4.1.19)*, which leads to* (4.1.20)*, is the analogue of taking the Schwarz conjugate in the global relation (in Chapter 2), for the elliptic PDEs.*

Proposition 4.2. ***(The Symmetric Dirichlet problem)***

Let $q(x_1,x_2,t)$ satisfy (4.1.1) in the domain $\mathcal{T}$ defined in (4.1.11) with the same *Dirichlet boundary conditions on each side, i.e.*

$$\begin{aligned}&q(x_1,x_2,0)=q_0(x_1,x_2),\quad x_1,x_2\in D;\\&q\left(\frac{l}{2\sqrt{3}},s,t\right)=g_0(s,t),\ q\left(-\frac{l}{4\sqrt{3}}-\frac{s\sqrt{3}}{2},\frac{l}{4}-\frac{s}{2},t\right)=g_0(s,t),\\&q\left(-\frac{l}{4\sqrt{3}}+\frac{s\sqrt{3}}{2},-\frac{l}{4}-\frac{s}{2},t\right)=g_0(s,t),\ \ s\in\left(-\frac{l}{2},\frac{l}{2}\right),\end{aligned}\tag{4.1.22}$$

where q_0 and g_0 are sufficiently smooth and g_0 is compatible at the corners of the triangle and is also compatible with q_0.

Define Q in terms of q_0 and f by (4.1.14a) and define G_0 and G_1 in terms of Q and g_0 by the following equations:

$$G_0(\mu_j,\lambda_j^2,t)=\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_0^t e^{-i\mu_j s+(\mu_j^2+\lambda_j^2)\tau}g_0(s,\tau)dsd\tau,\quad \mu_j,\lambda_j\in\mathbb{C},\ \ j=1,2,3\tag{4.1.23}$$

and

$$\begin{aligned}G_1(k_1,k_2,t)&=e^{-i\lambda_3\frac{l}{2\sqrt{3}}}Q(-k_1,k_2,t)-e^{i\lambda_3\frac{l}{2\sqrt{3}}}Q(k_1,k_2,t)\\&-2i\lambda_1\cos\left(\mu_2\frac{l}{2}\right)G_0(\mu_1,\lambda_1^2,t)-2i\lambda_2\cos\left(\mu_1\frac{l}{2}\right)G_0(\mu_2,\lambda_2^2,t)-2i\lambda_3G_0(\mu_3,\lambda_3^2,t).\end{aligned}\tag{4.1.24}$$

The solution is given by

$$
\begin{aligned}
q(x_1,x_2,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dk_1 \int_{\mathbb{R}} dk_2 e^{ikx-w(k)t} Q(k_1,k_2,t) \\
&- \frac{i}{(2\pi)^2}\int_{\mathbb{R}} dk_2 \int_{\partial D^-} dk_1\ k_1 e^{-ik_1\frac{l}{2\sqrt{3}}} e^{-w(k)t} G_0(k_2,k_1^2,t)\mathcal{P}(k_1,k_2,x_1,x_2) \\
&+ \frac{i}{2(2\pi)^2}\int_{\mathbb{R}} dk_2 \int_{\partial D^-} dk_1 \frac{e^{-ik_1\frac{l}{2\sqrt{3}}} e^{-w(k)t}}{\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}\right]} G_1(k_1,k_2,t)\mathcal{P}(k_1,k_2,x_1,x_2) \\
&- \frac{1}{4(2\pi)^2}\int_{\partial D^+} dk_2 \int_{\partial D^-} dk_1 \frac{e^{-ik_1\frac{l}{2\sqrt{3}}} e^{ik_2\frac{l}{2}} e^{-w(k)t}}{\sin\left(k_2\frac{l}{2}\right)\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}\right]} G_1(k_1,k_2,t)\mathcal{P}(k_1,k_2,x_1,x_2),
\end{aligned}
\tag{4.1.25}
$$

where

$$
\mathcal{P}(k_1,k_2,x_1,x_2) = \sum_{j=1}^{3} e^{i(k_2\xi_j + k_1\eta_j)}. \tag{4.1.26}
$$

Proof. Let the unknown function $U(\mu_j,\lambda_j^2,t)$ be defined by

$$
U(\mu_j,\lambda_j^2,t) = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^t e^{-i\mu_j s + (\mu_j^2+\lambda_j^2)\tau} \frac{\partial q}{\partial N_j}(s,\tau) ds d\tau, \quad \mu_j,\lambda_j \in \mathbb{C}. \tag{4.1.27}
$$

For convenience, we introduce the following notations:

$$
G_0^{(j)} = G_0(\mu_j,\lambda_j^2,t), \quad U^{(j)} = U(\mu_j,\lambda_j^2,t). \tag{4.1.28}
$$

Using the definitions of $\tilde{g}^{(j)}$, $G_0^{(j)}$ and $U^{(j)}$, the integral representation (4.1.12) can be rewritten in the form

$$
\begin{aligned}
q(x_1,x_2,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dk_1 \int_{\mathbb{R}} dk_2\ e^{ikx-w(k)t} Q(k_1,k_2,t) \\
&- \frac{1}{(2\pi)^2}\sum_{j=1}^{3}\int_{\mathbb{R}} d\mu_j \int_{\partial D^-} d\lambda_j e^{i(\mu_j\xi_j+\lambda_j\eta_j)-w(k)t} i\lambda_j e^{-i\lambda_j\frac{l}{2\sqrt{3}}} G_0^{(j)} \\
&- \frac{1}{(2\pi)^2}\sum_{j=1}^{3}\int_{\mathbb{R}} d\mu_j \int_{\partial D^-} d\lambda_j e^{i(\mu_j\xi_j+\lambda_j\eta_j)-w(k)t} e^{-i\lambda_j\frac{l}{2\sqrt{3}}} U^{(j)}.
\end{aligned}
\tag{4.1.29}
$$

Furthermore the global relation can be written in the form

$$
e^{w(k)t}\hat{q}(k_1,k_2,t) = \sum_{j=1}^{3} e^{-i\lambda_j\frac{l}{2\sqrt{3}}} U^{(j)} + N(k_1,k_2,t), \ \ k_1,k_2 \in \mathbb{C}, \tag{4.1.30}
$$

where the known function N is defined by

$$N(k_1, k_2, t) = Q(k_1, k_2, t) + i\sum_{j=1}^{3} \lambda_j e^{-i\lambda_j \frac{l}{2\sqrt{3}}} G_0^{(j)}. \tag{4.1.31}$$

We will now implement the steps (i)-(iv) summarized at the beginning of this Chapter.

Step(i)

The transformations (4.1.19) and (4.1.20) imply the following relations:

$$\begin{aligned}
U^{(1)} &= U(\mu_1, \lambda_1^2, t) \to U(\mu_1, \lambda_1^2, t) = U^{(1)},\\
U^{(2)} &= U(\mu_2, \lambda_2^2, t) \to U(\mu_3, \lambda_3^2, t) = U^{(3)},\\
U^{(3)} &= U(\mu_3, \lambda_3^2, t) \to U(\mu_2, \lambda_2^2, t) = U^{(2)}.
\end{aligned}$$

Hence, under the transformations (4.1.19) and (4.1.20) the global relation (4.1.30) yields

$$e^{w(k)t}\hat{q}(-k_1, k_2, t) = \sum_{j=1}^{3} e^{i\lambda_j \frac{l}{2\sqrt{3}}} U^{(j)} + N(-k_1, k_2, t), \;\; k_1, k_2 \in \mathbb{C}. \tag{4.1.32}$$

Multiplying (4.1.30) by $e^{i\lambda_3 \frac{l}{2\sqrt{3}}}$, (4.1.32) by $e^{-i\lambda_3 \frac{l}{2\sqrt{3}}}$ and subtracting the resulting equations, we can eliminate $U^{(3)}$ from equations (4.1.30) and (4.1.32):

$$\begin{aligned}
&2i\sin\left(\mu_2 \frac{l}{2}\right) U^{(1)} - 2i\sin\left(\mu_1 \frac{l}{2}\right) U^{(2)} =\\
&e^{w(k)t}\left[e^{i\lambda_3 \frac{l}{2\sqrt{3}}}\hat{q}(k_1, k_2, t) - e^{-i\lambda_3 \frac{l}{2\sqrt{3}}}\hat{q}(-k_1, k_2, t)\right] + G_1(k_1, k_2, t),
\end{aligned} \tag{4.1.33}$$

where the known function $G_1(k_1, k_2, t)$ is defined in (4.1.24); for the derivation of this expression we have used the following identities which are a direct consequence of equation (4.1.21):

$$\begin{aligned}
\lambda_2 - \lambda_3 &= k_2\sqrt{3} = \mu_1\sqrt{3}\\
\lambda_1 - \lambda_3 &= k_1\frac{3}{2} + k_2\frac{\sqrt{3}}{2} = -\mu_2\sqrt{3}.
\end{aligned}$$

Step(ii)

We will first compute the contribution in (4.1.29) of the term involving $U^{(1)}$. In this respect we solve (4.1.33) for $U^{(1)}$ and then replace $U^{(1)}$ in (4.1.29); this yields the following expression:

$$
\begin{aligned}
&-\frac{1}{(2\pi)^2}\int_{\mathbb{R}} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left(\mu_2\frac{l}{2}\right)} \\
&\left\{G_1 + 2i\sin\left(\mu_1\frac{l}{2}\right)U^{(2)} + e^{w(k)t}\left[e^{i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(k_1,k_2,t) - e^{-i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(-k_1,k_2,t)\right]\right\}.
\end{aligned} \tag{4.1.35}
$$

The integral involving the terms $\hat{q}(k_1,k_2,t)$ and $\hat{q}(-k_1,k_2,t)$ vanishes. Indeed let us consider the part of this integral involving $\hat{q}(k_1,k_2,t)$:

$$
-\int_{\mathbb{R}} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}e^{i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(k_1,k_2,t). \tag{4.1.36}
$$

Using

$$
k_1=\lambda_1,\ k_2=\mu_1,\ \mu_2=-\frac{1}{2}(\mu_1+\sqrt{3}\lambda_1),\ \lambda_3=-\frac{1}{2}(\lambda_1+\sqrt{3}\mu_1),
$$

the above integral becomes

$$
\int_{\mathbb{R}} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left[(\mu_1+\sqrt{3}\lambda_1)\frac{l}{4}\right]}e^{-i(\lambda_1+\sqrt{3}\mu_1)\frac{l}{4\sqrt{3}}}\hat{q}(\lambda_1,\mu_1,t). \tag{4.1.37}
$$

For λ_1 in the lower half λ_1-plane the term $\sin\left[(\mu_1+\sqrt{3}\lambda_1)\frac{l}{4}\right]$ is dominated by $e^{i\sqrt{3}\lambda_1\frac{l}{4}}$. Furthermore, the definition of $\hat{q}(\lambda_1,\mu_1,t)$ in (4.1.16), i.e.

$$
\hat{q}(\lambda_1,\mu_1,t) = \iint_D e^{-i\lambda_1\eta_1-i\mu_1\xi_1}q(\eta_1,\xi_1,t)d\eta_1 d\xi_1,
$$

implies that this term behaves like $e^{-i\lambda_1\eta_1'}$ with $-\frac{l}{\sqrt{3}}<\eta_1'<\frac{l}{2\sqrt{3}}$. Hence the integrand of (4.1.37) with respect to λ_1 behaves like

$$
e^{i\lambda_1\left(\eta_1-\frac{l}{2\sqrt{3}}\right)}\frac{e^{-\frac{1}{2}i\lambda_1\frac{l}{2\sqrt{3}}}e^{-i\lambda_1\eta_1'}}{e^{\frac{3}{2}i\lambda_1\frac{l}{2\sqrt{3}}}} = e^{i\lambda_1\left(\eta_1-\frac{l}{2\sqrt{3}}\right)}e^{-i\lambda_1\left(\eta_1'+\frac{l}{\sqrt{3}}\right)}. \tag{4.1.38}
$$

Using

$$\left(\eta_1 - \frac{l}{2\sqrt{3}}\right) < 0, \ \left(\eta_1' + \frac{l}{\sqrt{3}}\right) > 0, \ Im\lambda_1 < 0,$$

it follows that the exponential in (4.1.38) is bounded and analytic in λ_1 for $\lambda_1 \in D^-$. Thus the integrand of (4.1.37) is bounded and analytic in the lower half λ_1-plane except for the points where $\sin\left[(\mu_1 + \sqrt{3}\lambda_1)\frac{l}{4}\right] = 0$; these points are characterized by

$$\sin\left[(\mu_1 + \sqrt{3}\lambda_1)\frac{l}{4}\right] = 0 \Leftrightarrow \mu_1 + \sqrt{3}\lambda_1 = \frac{4n\pi}{l}, \ n \in \mathbb{Z} \Leftrightarrow \lambda_1 = 0 \text{ and } \mu_1 = \frac{4n\pi}{l}, n \in \mathbb{Z}.$$

Hence the contribution of the term (4.1.36) equals

$$\sum_{n=-\infty}^{\infty} e^{i\frac{4n\pi}{l}\xi_1} e^{-in\pi} \hat{q}\left(0, \frac{4n\pi}{l}, t\right). \tag{4.1.39}$$

Similar considerations imply that the part of this integral involving $\hat{q}(-k_1, k_2, t)$ yields the contribution

$$-\sum_{n=-\infty}^{\infty} e^{i\frac{4n\pi}{l}\xi_1} e^{in\pi} \hat{q}\left(0, \frac{4n\pi}{l}, t\right). \tag{4.1.40}$$

Thus the integral in (4.1.35) involving $\hat{q}(k_1, k_2, t)$ and $\hat{q}(-k_1, k_2, t)$ vanishes. Hence the term in (4.1.35) yields

$$-\frac{1}{(2\pi)^2}\int_{\mathbb{R}} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1 + \lambda_1\eta_1) - w(k)t} \frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}\left[G_1 + 2i\sin\left(\mu_1\frac{l}{2}\right)U^{(2)}\right]. \tag{4.1.41}$$

In order to compute this term we rewrite it in the following form:

$$\begin{aligned}
&-\frac{1}{(2\pi)^2}\int_{\mathbb{R}} d\mu_1 \int_{\partial\widetilde{D}^-} d\lambda_1 e^{i(\mu_1\xi_1 + \lambda_1\eta_1) - w(k)t} \frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left(\mu_2\frac{l}{2}\right)} G_1 \\
&-\frac{1}{(2\pi)^2}\int_{\widetilde{\mathbb{R}}} d\mu_1 \int_{\partial\widetilde{D}^-} d\lambda_1 e^{i(\mu_1\xi_1 + \lambda_1\eta_1) - w(k)t} \frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}} e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_2\frac{l}{2}\right)} U^{(2)} \\
&-\frac{1}{(2\pi)^2}\int_{\partial\widetilde{D}^-} d\mu_1 \int_{\partial\widetilde{D}^-} d\lambda_1 e^{i(\mu_1\xi_1 + \lambda_1\eta_1) - w(k)t} \frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}} e^{-i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_2\frac{l}{2}\right)} U^{(2)},
\end{aligned} \tag{4.1.42}$$

where the contour $\widetilde{\mathbb{R}}$ denotes the deformation of $\mathbb{R}$ to $\mathbb{R} + i\epsilon_2$ and the contour $\partial\widetilde{D}^-$ denotes the deformation of ∂D^- in theλ_1-complex plane to a curve so that $Im\{\lambda_1\} \leq -\epsilon_1 < 0$, where $\epsilon_2 > \sqrt{3}\epsilon_1$; the deformation of ∂D^- to $\partial\widetilde{D}^-$ in the μ_j-complex plane is defined in a similar way. The reason of this deformation is to avoid the zeros of $\sin\left(\mu_2\frac{l}{2}\right)=0$, which are characterized by

$$\sin\left(\mu_2\frac{l}{2}\right)=0 \Leftrightarrow \mu_1+\sqrt{3}\lambda_1=\frac{4n\pi}{l},\ n\in\mathbb{Z}.$$

The deformation from the real line to the curve $-\partial D^-$ of the complex μ_1-plane can be justified by using the definition of $U^{(2)}$ given in (4.1.27). In particular:

(i) The term $e^{i\mu_1\xi_1}e^{i\mu_1\frac{l}{2}}$ is bounded and analytic in the upper half μ_1-plane;

(ii) the term $e^{i\mu_1\xi_1}e^{-i\mu_1\frac{l}{2}}$ is bounded and analytic in the lower half μ_1-plane;

(iii) the term $e^{-w(k)t}\frac{U^{(2)}}{2i\sin\left(\mu_2\frac{l}{2}\right)}$ gives rise to the term $\exp\left[-\mu_1^2(t-\tau)\right]$ which is bounded in the shaded area in Figure 3, as well as the term $\frac{e^{i\mu_2 s}}{2i\sin\left(\mu_2\frac{l}{2}\right)}$, which is bounded and analytic for every $\mu_1\in\mathbb{C}$ and $\lambda_1\in\mathbb{C}$ except from the points where $\sin\left(\mu_2\frac{l}{2}\right)=0$.

Step(iii)

Employing (4.1.33) in the second integral of (4.1.42), we find that the second term in (4.1.42) yields the following contribution:

$$\begin{aligned}
&-\frac{1}{(2\pi)^2}\int_{\partial D^+}d\mu_1\int_{\partial D^-}d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)}U^{(1)}\\
&+\frac{1}{(2\pi)^2}\int_{\partial D^+}d\mu_1\int_{\partial D^-}d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)2i\sin\left(\mu_2\frac{l}{2}\right)}G_1\\
&+\frac{1}{(2\pi)^2}\int_{\partial D^+}d\mu_1\int_{\partial D^-}d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)2i\sin\left(\mu_2\frac{l}{2}\right)}\\
&\qquad\left[e^{i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(k_1,k_2,t)-e^{-i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(-k_1,k_2,t)\right].
\end{aligned}\tag{4.1.43}$$

The last integral in the above expression vanishes: Indeed, observe that the relevant integrand is bounded and analytic in the lower half λ_1-plane, as well as in the upper half μ_1-plane except for the points where $\sin\left(\mu_2\frac{l}{2}\right)=0$, i.e. except from the points satisfying $\mu_1+\sqrt{3}\lambda_1=\frac{4n\pi}{l}$.

Performing the μ_1 integration and calculating the contribution from the residues at $\mu_1=-\sqrt{3}\lambda_1+\frac{4n\pi}{l}$, $n\in\mathbb{Z}$, we find

$$\sum_{n=-\infty}^{\infty}\int_{\partial D^-} d\lambda_1 e^{i\frac{4n\pi}{l}\xi_1 - i\sqrt{3}\lambda_1\xi_1 + i\lambda_1\eta_1} \frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{-i\sqrt{3}\lambda_1\frac{l}{2}}}{2i\sin\left(\sqrt{3}\lambda_1\frac{l}{2}\right)} e^{in\pi}$$
$$\left[e^{i\lambda_1\frac{l}{2\sqrt{3}}}\hat{q}\left(\lambda_1, -\sqrt{3}\lambda_1 + \frac{4n\pi}{l}, t\right) - e^{i\lambda_1\frac{l}{2\sqrt{3}}}\hat{q}\left(-\lambda_1, -\sqrt{3}\lambda_1 + \frac{4n\pi}{l}, t\right)\right]. \tag{4.1.44}$$

The contribution of each one of the above integrals equals the residue at $\lambda_1 = 0$, hence the above expression equals

$$\sum_{n=-\infty}^{\infty} e^{i\frac{4n\pi}{l}\xi_1} e^{in\pi}\left[\hat{q}\left(0, \frac{4n\pi}{l}, t\right) - \hat{q}\left(0, \frac{4n\pi}{l}, t\right)\right] = 0. \tag{4.1.45}$$

Using (4.1.42) and (4.1.43) it follows that the contribution in (4.1.29) of the term involving $U^{(1)}$ is given by

$$\begin{aligned}
C_1(x_1, x_2, t) = &-\frac{1}{(2\pi)^2}\int_{\mathbb{R}} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}G_1\\
&+\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)2i\sin\left(\mu_2\frac{l}{2}\right)}G_1\\
&-\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)}U^{(1)}\\
&-\frac{1}{(2\pi)^2}\int_{\partial D^-} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{-i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}U^{(2)}.
\end{aligned} \tag{4.1.46}$$

The first two integrals of the above expression are known functions and will be denoted by $F_1(x_1, x_2, t)$, whereas the last two integrals are unknown and will be denoted by $\widetilde{U}_1(x_1, x_2, t)$. Replacing in $F_1(x_1, x_2, t)$ the variables λ_1 and μ_1 in terms of k_1 and k_2, we find

$$F_1(x_1,x_2,t)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}dk_2\int_{\partial D^-}dk_1e^{i(k_2\xi_1+k_1\eta_1)-w(k)t}\frac{e^{-ik_1\frac{l}{2\sqrt{3}}}}{2i\sin\left((\sqrt{3}k_1+k_2)\frac{l}{4}\right)}G_1(k_1,k_2,t)$$
$$-\frac{1}{(2\pi)^2}\int_{\partial D^+}dk_2\int_{\partial D^-}dk_1e^{i(k_2\xi_1+k_1\eta_1)-w(k)t}\frac{e^{-ik_1\frac{l}{2\sqrt{3}}}e^{ik_2\frac{l}{2}}}{2i\sin\left(k_2\frac{l}{2}\right)2i\sin\left((\sqrt{3}k_1+k_2)\frac{l}{4}\right)}G_1(k_1,k_2,t). \tag{4.1.47}$$

Making in the second integral of $\widetilde{U}_1(x_1,x_2,t)$ the change of variables

$$\lambda_1=-\frac{\lambda_2}{2}-\frac{\sqrt{3}}{2}\mu_2,\ \ \mu_1=\frac{\sqrt{3}}{2}\lambda_2-\frac{\mu_2}{2},$$

we find

$$\begin{aligned}\widetilde{U}_1(x_1,x_2,t)=&-\frac{1}{(2\pi)^2}\int_{\partial D^+}d\mu_1\int_{\partial D^-}d\lambda_1e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)}U^{(1)}\\&+\frac{1}{(2\pi)^2}\int_{\partial D^+}d\mu_2\int_{\partial D^-}d\lambda_2e^{i(\mu_2\xi_2+\lambda_2\eta_2)-w(k)t}\frac{e^{-i\lambda_2\frac{l}{2\sqrt{3}}}e^{i\mu_2\frac{l}{2}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}U^{(2)}.\end{aligned} \tag{4.1.48}$$

Step(iv)

The contributions $\{F_j\}_2^3$ and $\{\widetilde{U}_j\}_2^3$ of $\{U^{(j)}\}_2^3$ are obtained in the same way, making the appropriate rotations on the subscripts of the relevant variables. Hence, the terms $U^{(2)}$ and $U^{(3)}$ in (4.1.29) give the contributions $F_2(x_1,x_2,t)$ and $F_3(x_1,x_2,t)$, which can be obtained from $F_1(x_1,x_2,t)$ via the substitutions

$$(\xi_1,\eta_1)\to(\xi_2,\eta_2)\text{ and }(\xi_1,\eta_1)\to(\xi_3,\eta_3). \tag{4.1.49}$$

Similarly, the terms $U^{(2)}$ and $U^{(3)}$ in (4.1.29) give the contributions $\widetilde{U}_2(x_1,x_2,t)$ and $\widetilde{U}_3(x_1,x_2,t)$, which can be obtained from $\widetilde{U}_1(x_1,x_2,t)$ via the substitutions

$$(\xi_1,\eta_1)\to(\xi_2,\eta_2),\ (\xi_2,\eta_2)\to(\xi_3,\eta_3)\text{ and }(\lambda_1,\mu_1)\to(\lambda_2,\mu_2),\ (\lambda_2,\mu_2)\to(\lambda_3,\mu_3)$$

and

$$(\xi_1,\eta_1)\to(\xi_3,\eta_3),\ (\xi_2,\eta_2)\to(\xi_1,\eta_1)\text{ and }(\lambda_1,\mu_1)\to(\lambda_3,\mu_3),\ (\lambda_2,\mu_2)\to(\lambda_1,\mu_1),$$

respectively.

Hence,

$$\begin{aligned}\widetilde{U}_2(x_1,x_2,t) = &-\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_2 \int_{\partial D^-} d\lambda_2 e^{i(\mu_2\xi_2+\lambda_2\eta_2)-w(k)t}\frac{e^{-i\lambda_2\frac{l}{2\sqrt{3}}}e^{i\mu_2\frac{l}{2}}}{2i\sin\left(\mu_2\frac{l}{2}\right)}U^{(2)} \\ &+\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_3 \int_{\partial D^-} d\lambda_3 e^{i(\mu_3\xi_3+\lambda_3\eta_3)-w(k)t}\frac{e^{-i\lambda_3\frac{l}{2\sqrt{3}}}e^{i\mu_3\frac{l}{2}}}{2i\sin\left(\mu_3\frac{l}{2}\right)}U^{(3)}\end{aligned} \tag{4.1.50}$$

and

$$\begin{aligned}\widetilde{U}_3(x_1,x_2,t) = &-\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_3 \int_{\partial D^-} d\lambda_3 e^{i(\mu_3\xi_3+\lambda_3\eta_3)-w(k)t}\frac{e^{-i\lambda_3\frac{l}{2\sqrt{3}}}e^{i\mu_3\frac{l}{3}}}{2i\sin\left(\mu_3\frac{l}{2}\right)}U^{(3)} \\ &+\frac{1}{(2\pi)^2}\int_{\partial D^+} d\mu_1 \int_{\partial D^-} d\lambda_1 e^{i(\mu_1\xi_1+\lambda_1\eta_1)-w(k)t}\frac{e^{-i\lambda_1\frac{l}{2\sqrt{3}}}e^{i\mu_1\frac{l}{2}}}{2i\sin\left(\mu_1\frac{l}{2}\right)}U^{(1)}.\end{aligned} \tag{4.1.51}$$

But

$$\widetilde{U}_1(x_1,x_2,t)+\widetilde{U}_2(x_1,x_2,t)+\widetilde{U}_3(x_1,x_2,t)=0,$$

thus the only contribution of the unknown terms $U^{(j)},\ j=1,2,3$ in (4.1.29), is given by

$$F_1(x_1,x_2,t)+F_2(x_1,x_2,t)+F_3(x_1,x_2,t),$$

where $F_1(x_1,x_2,t)$ is given by the expression (4.1.47) and $F_2(x_1,x_2,t)$ and $F_3(x_1,x_2,t)$ are obtained from $F_1(x_1,x_2,t)$ via the transformations (4.1.49). This yields the solution (4.1.25).

□

4.2 An example.

Let

$$l=\pi,\ g_0(s,t)=te^{-t}\cos s,\ f(x_1,x_2,t)=0,\ q_0(x_1,x_2)=0. \tag{4.2.1}$$

The definitions of G_0 and G_1 imply

$$G_0(k_1,k_2^2,t)=\frac{2\cos\left(k_1\frac{\pi}{2}\right)}{1-k_1^2}\frac{e^{(w(k)-1)t}\left[t\left(w(k)-1\right)-1\right]+1}{\left(w(k)-1\right)^2}, \tag{4.2.2}$$

and

$$G_1(k_1,k_2,t) = -2i\lambda_1 \cos\left(\mu_2\frac{\pi}{2}\right) G_0(\mu_1,\lambda_1^2,t) - 2i\lambda_2 \cos\left(\mu_1\frac{\pi}{2}\right) G_0(\mu_2,\lambda_2^2,t) - 2i\lambda_3 G_0(\mu_3,\lambda_3^2,t), \tag{4.2.3}$$

where $w(k) = k_1^2 + k_2^2$.

Thus, the solution (4.1.25) becomes:

$$\begin{aligned} q(x_1,x_2,t) &= -\frac{i}{(2\pi)^2}\int_{\mathbb{R}} dk_2 \int_{\partial D^-} dk_1\ k_1 e^{-ik_1\frac{\pi}{2\sqrt{3}}} e^{-w(k)t} G_0(k_2,k_1^2,t)\mathcal{P}(k_1,k_2,x_1,x_2) \\ &+ \frac{i}{2(2\pi)^2}\int_{\mathbb{R}} dk_2 \int_{\partial D^-} dk_1 \frac{e^{-ik_1\frac{\pi}{2\sqrt{3}}} e^{-w(k)t}}{\sin\left((\sqrt{3}k_1+k_2)\frac{\pi}{4}\right)} G_1(k_1,k_2,t)\mathcal{P}(k_1,k_2,x_1,x_2) \\ &- \frac{1}{4(2\pi)^2}\int_{\partial D^+} dk_2 \int_{\partial D^-} dk_1 \frac{e^{-ik_1\frac{\pi}{2\sqrt{3}}} e^{ik_2\frac{\pi}{2}} e^{-w(k)t}}{\sin\left(k_2\frac{\pi}{2}\right)\sin\left((\sqrt{3}k_1+k_2)\frac{\pi}{4}\right)} G_1(k_1,k_2,t)\mathcal{P}(k_1,k_2,x_1,x_2). \end{aligned} \tag{4.2.4}$$

- For the first integral, taking into consideration that $Im\{k_1\} < 0$, $Re\{k_1^2\} > 0$ and $Im\{k_2\} = 0$, it follows that:

(i) The function $e^{-w(k)t}G_0(k_2,k_1^2,t)$ is decaying exponentially;

(ii) each of the three terms of $\mathcal{P}(k_1,k_2,x_1,x_2)E(-ik_1)$ behaves like $e^{ik_2\xi_j}e^{ik_1\left(\eta_j-\frac{\pi}{2\sqrt{3}}\right)}$, thus these three terms decay exponentially.

- For the second integral, taking into consideration that $Im\{k_1\} < 0$, $Re\{k_1^2\} > 0$ and $Im\{k_2\} = 0$, it follows that:

(i) The function $e^{-w(k)t}\frac{G_1(k_1,k_2,t)}{\sin\left((\sqrt{3}k_1+k_2)\frac{\pi}{4}\right)}$ decays exponentially;

(ii) each of the three terms of $\mathcal{P}(k_1,k_2,x_1,x_2)e^{-ik_1\frac{\pi}{2\sqrt{3}}}$ behaves like $e^{ik_2\xi_j}e^{ik_1\left(\lambda_j-\frac{\pi}{2\sqrt{3}}\right)}$, thus these three terms decay exponentially.

- For the last integral, taking into consideration that $Im\{k_1\} < 0$, $Re\{k_1^2\} > 0$ and $Im\{k_2\} > 0$, $Re\{k_2^2\} > 0$, it follows that:

(i) The function $e^{-w(k)t}\frac{G_1(k_1,k_2,t)}{\sin\left(k_2\frac{\pi}{2}\right)\sin\left((\sqrt{3}k_1+k_2)\frac{\pi}{4}\right)}$ decays exponentially;

(ii) each of the three terms of $\mathcal{P}(k_1,k_2,x_1,x_2)e^{-ik_1\frac{\pi}{2\sqrt{3}}}e^{ik_2\frac{\pi}{2}}$ behaves like $e^{ik_2\left(\xi_j+\frac{\pi}{2}\right)}e^{ik_1\left(\lambda_j-\frac{\pi}{2\sqrt{3}}\right)}$, thus these three terms decay exponentially.

4.3 The General Dirichlet Problem.

Folowing the same ideas as those used for the solution of the arbitrary Dirichlet problem for the Laplace equation, it follows that the general Dirichlet problem can be decomposed into the following three problems:

1. Let $q(x_1,x_2,t)$ satisfy the symmetric the Dirichlet problem for (4.1.1) in the domain $\mathcal{T}$ defined in (4.1.11), i.e.

$$\begin{aligned}
&q(x_1,x_2,0)=q_1(x_1,x_2),\ \ x_1,x_2\in D;\\
&q\left(\frac{l}{2\sqrt{3}},s,t\right)=g_1(s,t),\ q\left(-\frac{l}{4\sqrt{3}}-\frac{s\sqrt{3}}{2},\frac{l}{4}-\frac{s}{2},t\right)=g_1(s,t),\\
&q\left(-\frac{l}{4\sqrt{3}}+\frac{s\sqrt{3}}{2},-\frac{l}{4}-\frac{s}{2},t\right)=g_1(s,t),\ \ s\in\left(-\frac{l}{2},\frac{l}{2}\right),
\end{aligned}\tag{4.3.1}$$

where q_1 and g_1 are sufficiently smooth and g_1 is compatible at the corners of the triangle and is also compatible with q_1.

2. Let $q(x_1,x_2,t)$ satisfy (4.1.1) in the domain $\mathcal{T}$ defined in (4.1.11) with the following Dirichlet boundary conditions on each side:

$$\begin{aligned}
&q(x_1,x_2,0)=q_2(x_1,x_2),\ \ x_1,x_2\in D;\\
&q\left(\frac{l}{2\sqrt{3}},s,t\right)=g_2(s,t),\ q\left(-\frac{l}{4\sqrt{3}}-\frac{s\sqrt{3}}{2},\frac{l}{4}-\frac{s}{2},t\right)=ag_2(s,t),\\
&q\left(-\frac{l}{4\sqrt{3}}+\frac{s\sqrt{3}}{2},-\frac{l}{4}-\frac{s}{2},t\right)=\bar{a}g_2(s,t),\ \ s\in\left(-\frac{l}{2},\frac{l}{2}\right),
\end{aligned}\tag{4.3.2}$$

where $a = e^{\frac{2i\pi}{3}}$, q_2 and g_2 are sufficiently smooth and g_2 is compatible at the corners of the triangle and is also compatible with q_2.

3. Let $q(x_1, x_2, t)$ satisfy (4.1.1) in the domain $\mathcal{T}$ defined in (4.1.11) with the following Dirichlet boundary conditions on each side:

$$\begin{aligned}
&q(x_1, x_2, 0) = q_3(x_1, x_2),\ x_1, x_2 \in D;\\
&q\left(\frac{l}{2\sqrt{3}}, s, t\right) = g_3(s,t),\ q\left(-\frac{l}{4\sqrt{3}} - \frac{s\sqrt{3}}{2}, \frac{l}{4} - \frac{s}{2}, t\right) = \bar{a}g_3(s,t),\\
&q\left(-\frac{l}{4\sqrt{3}} + \frac{s\sqrt{3}}{2}, -\frac{l}{4} - \frac{s}{2}, t\right) = ag_3(s,t),\ s \in \left(-\frac{l}{2}, \frac{l}{2}\right),
\end{aligned} \tag{4.3.3}$$

where $a = e^{\frac{2i\pi}{3}}$, q_3 and g_3 are sufficiently smooth and g_3 is compatible at the corners of the triangle and is also compatible with q_3.

It turns out that the boundary conditions of an arbitrary Dirichlet problem can be written as the sum of the Dirichlet conditions of these three problems. Indeed, consider the arbitrary Dirichlet problem with the following conditions:

$$\begin{aligned}
&q(x_1, x_2, 0) = q_0(x_1, x_2), x_1, x_2 \in D;\\
&q\left(\frac{l}{2\sqrt{3}}, s, t\right) = f_1(s,t),\ q\left(-\frac{l}{4\sqrt{3}} - \frac{s\sqrt{3}}{2}, \frac{l}{4} - \frac{s}{2}, t\right) = f_2(s,t),\\
&q\left(-\frac{l}{4\sqrt{3}} + \frac{s\sqrt{3}}{2}, -\frac{l}{4} - \frac{s}{2}, t\right) = f_3(s,t),\ s \in \left(-\frac{l}{2}, \frac{l}{2}\right),
\end{aligned} \tag{4.3.4}$$

where $q_0,\ f_1,\ f_2$ and f_3 are sufficiently smooth and $\{f_j\}_1^3$ are compatible at the corners of the triangle and are also compatible with q_0. The matrix of the following 3×3 algebraic system is non-singular:

$$\begin{pmatrix} f_1(s,t) \\ f_2(s,t) \\ f_3(s,t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix} \begin{pmatrix} g_1(s,t) \\ g_2(s,t) \\ g_3(s,t) \end{pmatrix}, \quad Det[\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix}] = i3\sqrt{3}. \tag{4.3.5}$$

Also, we choose $\{q_j(x_1,x_2)\}_1^3$ such that

$$q_0(x_1,x_2) = q_1(x_1,x_2) + q_2(x_1,x_2) + q_3(x_1,x_2).$$

In order for q_0 and $\{f_j\}_1^3$ to be compatible we make the following choice for $\{q_j\}_1^3$:

$$\begin{pmatrix} q_1(x_1,x_2) \\ q_2(x_1,x_2) \\ q_3(x_1,x_2) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{a} & a \\ 1 & a & \bar{a} \end{pmatrix} \begin{pmatrix} q_0(x_1,x_2) \\ q_0\left(-\frac{x_1}{2} - \frac{x_2\sqrt{3}}{2}, \frac{x_1\sqrt{3}}{2} - \frac{x_2}{2}\right) \\ q_0\left(-\frac{x_1}{2} + \frac{x_2\sqrt{3}}{2}, -\frac{x_1\sqrt{3}}{2} - \frac{x_2}{2}\right) \end{pmatrix}. \tag{4.3.6}$$

Due to uniqueness, the solution of the general Dirichlet problem is given by the sum of these three problems.

The solution of the problems (4.3.2) and (4.3.3) can be derived from Proposition 4.1 using similar steps with those used for the derivation of Proposition 4.2. In this respect, we make the following substitutions:

- for the problem (4.3.2):

$$U^{(j)} \longrightarrow a^{j-1}U^{(j)} \text{ and } G_0^{(j)} \longrightarrow a^{j-1}G_2^{(j)}, \quad j=1,2,3.$$

- for the problem (4.3.3):

$$U^{(j)} \longrightarrow \bar{a}^{j-1}U^{(j)} \text{ and } G_0^{(j)} \longrightarrow \bar{a}^{j-1}G_3^{(j)}, \quad j=1,2,3.$$

Hence, the analogue of the relation (4.1.33) for the problems (4.3.2) and (4.3.3) are now the following relations:

$$\begin{gathered} 2i\sin\left(\mu_2\frac{l}{2} + \frac{2\pi}{3}\right)U^{(1)} - 2i\sin\left(\mu_1\frac{l}{2} + \frac{2\pi}{3}\right)U^{(2)} = \\ e^{w(k)t}\left[ae^{i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(k_1,k_2,t) - \bar{a}e^{-i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(-k_1,k_2,t)\right] + \widetilde{G}_2(k_1,k_2,t) \end{gathered} \tag{4.3.7}$$

and

$$\begin{aligned}&2i\sin\left(\mu_2\frac{l}{2}-\frac{2\pi}{3}\right)U^{(1)}-2i\sin\left(\mu_1\frac{l}{2}-\frac{2\pi}{3}\right)U^{(2)}=\\&e^{w(k)t}\left[\bar{a}e^{i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(k_1,k_2,t)-ae^{-i\lambda_3\frac{l}{2\sqrt{3}}}\hat{q}(-k_1,k_2,t)\right]+\widetilde{G}_3(k_1,k_2,t),\end{aligned}\tag{4.3.8}$$

where $\widetilde{G}_2$ and $\widetilde{G}_3$ are known functions. Thus, the only difference in comparison with the proof of Proposition 4.2, is that the points of non-analyticity for the problems (4.3.2) and (4.3.3) are the points $\mu_2=\frac{4n\pi}{l}\pm\frac{4\pi}{3l}$, $n\in\mathbb{Z}$ instead of $\mu_2=\frac{4n\pi}{l}$, $n\in\mathbb{Z}$. However, the new points remain on the real line thus the contribution of the unknown functions $U^{(j)}$ can be analyzed *mutatis mutandis* as in the Proposition 4.2. Hence, the solution of the problem defined in (4.3.2) is given by the following expression:

$$\begin{aligned}q_2(x_1,x_2,t)=&\frac{1}{(2\pi)^2}\int_{\mathbb{R}}dk_1\int_{\mathbb{R}}dk_2e^{ikx-w(k)t}Q_2(k_1,k_2,t)\\&-\frac{i}{(2\pi)^2}\int_{\mathbb{R}}dk_2\int_{\partial D^-}dk_1\ k_1e^{-ik_1\frac{l}{2\sqrt{3}}}e^{-w(k)t}G_2(k_2,k_1^2,t)\mathcal{P}_2(k_1,k_2,x_1,x_2)\\&+\frac{i}{2(2\pi)^2}\int_{\mathbb{R}}dk_2\int_{\partial D^-}dk_1e^{-ik_1\frac{l}{2\sqrt{3}}}e^{-w(k)t}\frac{\widetilde{G}_2(k_1,k_2,t)\mathcal{P}_2(k_1,k_2,x_1,x_2)}{\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}+\frac{2\pi}{3}\right]}\\&-\frac{1}{4(2\pi)^2}\int_{\partial D^+}dk_2\int_{\partial D^-}dk_1e^{-ik_1\frac{l}{2\sqrt{3}}}e^{ik_2\frac{l}{2}}e^{-w(k)t}\\&\qquad\frac{\widetilde{G}_2(k_1,k_2,t)\mathcal{P}_2(k_1,k_2,x_1,x_2)}{\sin\left(k_2\frac{l}{2}+\frac{2\pi}{3}\right)\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}+\frac{2\pi}{3}\right]},\end{aligned}\tag{4.3.9}$$

where

$$\mathcal{P}_2(k_1,k_2,x_1,x_2)=e^{i(k_2\xi_1+k_1\eta_1)}+ae^{i(k_2\xi_2+k_1\eta_2)}+\bar{a}e^{i(k_2\xi_3+k_1\eta_3)},\tag{4.3.10}$$

$$\begin{aligned}Q_j(k_1,k_2,t)=&\frac{1}{3}\iint_D\int_0^t e^{-ikx+w(k)\tau}f(x_1,x_2,\tau)dx_1dx_2d\tau\\&+\iint_D e^{-ikx}q_j(x_1,x_2)dx_1dx_2,\quad k_1,k_2\in\mathbb{C},\ j=1,2,3,\end{aligned}\tag{4.3.11}$$

$$G_j(k_2,k_1^2,t)=\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_0^t e^{-ik_2s+(k_2^2+k_1^2)\tau}g_j(s,\tau)dsd\tau,\quad k_1,k_2\in\mathbb{C},\ j=1,2,3\tag{4.3.12}$$

and

$$\begin{aligned}\widetilde{G}_2(k_1,k_2,t) &= \bar{a}e^{-i\lambda_3\frac{l}{2\sqrt{3}}}Q_2(-k_1,k_2,t) - ae^{i\lambda_3\frac{l}{2\sqrt{3}}}Q_2(k_1,k_2,t)\\ &- 2i\lambda_1\cos\left(\mu_2\frac{l}{2}+\frac{2\pi}{3}\right)G_2(\mu_1,\lambda_1^2,t)\\ &- 2i\lambda_2\cos\left(\mu_1\frac{l}{2}+\frac{2\pi}{3}\right)G_2(\mu_2,\lambda_2^2,t) - 2i\lambda_3 G_2(\mu_3,\lambda_3^2,t).\end{aligned} \tag{4.3.13}$$

In a similar way, we obtain the solution of problem (4.3.3). Hence, the solutions of these 3 problems (4.3.1)-(4.3.3) yield the following proposition.

Proposition 4.3. ***(The Dirichlet problem)***

Let $q(x_1,x_2,t)$ satisfy (4.1.1) in the domain $\mathcal{T}$ defined in (4.1.11) with Dirichlet boundary conditions, i.e.

$$\begin{aligned}&q(x_1,x_2,0)=q_0(x_1,x_2), x_1,x_2\in D;\\ &q\left(\frac{l}{2\sqrt{3}},s,t\right)=f_1(s,t),\ q\left(-\frac{l}{4\sqrt{3}}-\frac{s\sqrt{3}}{2},\frac{l}{4}-\frac{s}{2},t\right)=f_2(s,t),\\ &q\left(-\frac{l}{4\sqrt{3}}+\frac{s\sqrt{3}}{2},-\frac{l}{4}-\frac{s}{2},t\right)=f_3(s,t),\ s\in\left(-\frac{l}{2},\frac{l}{2}\right),\end{aligned} \tag{4.3.14}$$

where q_0, f_1, f_2 and f_3 are sufficiently smooth and $\{f_j\}_1^3$ are compatible at the corners of the triangle and are also compatible with q_0. Define $\{g_j\}_1^3$ in terms of $\{f_j\}_1^3$ by

$$\begin{pmatrix} g_1(s,t)\\ g_2(s,t)\\ g_3(s,t)\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & \bar{a} & a\\ 1 & a & \bar{a}\end{pmatrix}\begin{pmatrix} f_1(s,t)\\ f_2(s,t)\\ f_2(s,t)\end{pmatrix} \tag{4.3.15}$$

and $\{q_j\}_1^3$ in terms of q_0 by equation (4.3.6).

Define Q in terms of q_0 and f by (4.1.14a), *$\{Q_j\}_1^3$ in terms of $\{q_j\}_1^3$ and f by* (4.3.11), *$\{G_j\}_1^3$ in terms of $\{g_j\}_1^3$ by equation* (4.3.12) *and $\{\widetilde{G}_j\}_1^3$ by the following equations:*

$$\begin{aligned}\widetilde{G}_1(k_1,k_2,t) &= e^{-i\lambda_3\frac{l}{2\sqrt{3}}}Q_1(-k_1,k_2,t) - e^{i\lambda_3\frac{l}{2\sqrt{3}}}Q_1(k_1,k_2,t)\\ &- 2i\lambda_1\cos\left(\mu_2\frac{l}{2}\right)G_1(\mu_1,\lambda_1^2,t)\\ &- 2i\lambda_2\cos\left(\mu_1\frac{l}{2}\right)G_1(\mu_2,\lambda_2^2,t) - 2i\lambda_3 G_1(\mu_3,\lambda_3^2,t),\end{aligned}\tag{4.3.16}$$

$$\begin{aligned}\widetilde{G}_2(k_1,k_2,t) &= \bar{a}e^{-i\lambda_3\frac{l}{2\sqrt{3}}}Q_2(-k_1,k_2,t) - ae^{i\lambda_3\frac{l}{2\sqrt{3}}}Q_2(k_1,k_2,t)\\ &- 2i\lambda_1\cos\left(\mu_2\frac{l}{2}+\frac{2\pi}{3}\right)G_2(\mu_1,\lambda_1^2,t)\\ &- 2i\lambda_2\cos\left(\mu_1\frac{l}{2}+\frac{2\pi}{3}\right)G_2(\mu_2,\lambda_2^2,t) - 2i\lambda_3 G_2(\mu_3,\lambda_3^2,t),\end{aligned}\tag{4.3.17}$$

$$\begin{aligned}\widetilde{G}_3(k_1,k_2,t) &= ae^{-i\lambda_3\frac{l}{2\sqrt{3}}}Q_3(-k_1,k_2,t) - \bar{a}e^{i\lambda_3\frac{l}{2\sqrt{3}}}Q_3(k_1,k_2,t)\\ &- 2i\lambda_1\cos\left(\mu_2\frac{l}{2}-\frac{2\pi}{3}\right)G_3(\mu_1,\lambda_1^2,t)\\ &- 2i\lambda_2\cos\left(\mu_1\frac{l}{2}-\frac{2\pi}{3}\right)G_3(\mu_2,\lambda_2^2,t) - 2i\lambda_3 G_3(\mu_3,\lambda_3^2,t).\end{aligned}\tag{4.3.18}$$

The solution is given by

$$\begin{aligned}q(x_1,x_2,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}dk_1\int_{\mathbb{R}}dk_2 e^{ikx-w(k)t}Q(k_1,k_2,t)\\ &-\frac{i}{(2\pi)^2}\int_{\mathbb{R}}dk_2\int_{\partial D^-}dk_1\ k_1 e^{-ik_1\frac{l}{2\sqrt{3}}}e^{-w(k)t}\sum_{j=1}^{3}G_j(k_2,k_1^2,t)\mathcal{P}_j(k_1,k_2,x_1,x_2)\\ &+\frac{i}{2(2\pi)^2}\int_{\mathbb{R}}dk_2\int_{\partial D^-}dk_1 e^{-ik_1\frac{l}{2\sqrt{3}}}e^{-w(k)t}\sum_{j=1}^{3}\frac{\widetilde{G}_j(k_2,k_1,t)\mathcal{P}_j(k_1,k_2,x_1,x_2)}{\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}+(j-1)\frac{2\pi}{3}\right]}\\ &-\frac{1}{4(2\pi)^2}\int_{\partial D^+}dk_2\int_{\partial D^-}dk_1 e^{-ik_1\frac{l}{2\sqrt{3}}}e^{ik_2\frac{l}{2}}e^{-w(k)t}\\ &\qquad\sum_{j=1}^{3}\frac{\widetilde{G}_j(k_2,k_1,t)\mathcal{P}_j(k_1,k_2,x_1,x_2)}{\sin\left(k_2\frac{l}{2}+(j-1)\frac{2\pi}{3}\right)\sin\left[(\sqrt{3}k_1+k_2)\frac{l}{4}+(j-1)\frac{2\pi}{3}\right]},\end{aligned}\tag{4.3.19}$$

where

$$
\begin{aligned}
\mathcal{P}_1(k_1,k_2,x_1,x_2) &= \sum_{j=1}^{3} e^{i(k_2\xi_j+k_1\eta_j)},\\
\mathcal{P}_2(k_1,k_2,x_1,x_2) &= \sum_{j=1}^{3} a^{j-1}e^{i(k_2\xi_j+k_1\eta_j)},\\
\mathcal{P}_3(k_1,k_2,x_1,x_2) &= \sum_{j=1}^{3} \bar{a}^{j-1}e^{i(k_2\xi_j+k_1\eta_j)}.
\end{aligned}
\tag{4.3.20}
$$

Proof. The solution of problem (4.3.14) is given by the sum of the solutions of the problems (4.3.1)-(4.3.3), where $\{g_j\}_1^3$ are defined in terms of $\{f_j\}_1^3$ in (4.3.15) and $\{q_j\}_1^3$ are defined in terms of q_0 in (4.3.6). The first term of (4.3.19) is obtained from the fact that $Q = Q_1 + Q_2 + Q_3$. □

It is straightforward to make the relevant results rigorous. In order to prove Proposition 4.1 we have *assumed* the a priori existence of the solution. However, this assumption can be eliminated. Indeed, equation (4.1.25) shows that if $f(x_1,x_2),\ x_1,x_2 \in \mathbb{R}$, $q_0(x_1,x_2),\ x_1,x_2 \in \mathbb{R}$ and $g_0(s,\tau),\ s \in \left(-\frac{l}{2},\frac{l}{2}\right),\ \tau \in [0,T]$, are in appropriate function spaces, then functions Q and G_1 appearing in the *definition* of q are well defined. Hence, q is also well defined. It is then straightforward to show that this function q solves the heat equation and satisfies the given initial and boundary conditions (for evolution equations in one spacial dimension this is implemented in [46] and [47]). It is important to emphasize that the relevant integrals are uniformly convergent at the boundary, thus it is straightforward to prove that q satisfies the given boundary condition; this is to be contrasted with the classical approaches when q is expressed in terms of an infinite series which is *not* uniformly convergent at the boundary.

Chapter 5

Explicit soliton asymptotics for the nonlinear Schrödinger equation on the half-line.

The Fokas method was further developed for the analysis of initial-boundary value problems for nonlinear integrable evolution equations by several authors, see for example [3], [2], [6], [9]. This method is based on the following ideas: (a) The derivation of an integral representation for the solution which involves the formulation of a Riemann-Hilbert problem. This derivation employs the *simultaneous* spectral analysis of both parts of the associated Lax pair(this is to be contrasted with the inverse scattering transform method which employs the spectral analysis of only the t-dependent part of the Lax pair). This integral representation involves the nonlinear Fourier transforms of the boundary values. (b) The characterization of the unknown boundary values in terms of the given boundary conditions. This involves the analysis of the global relation [7], [3]. In general the global relation yields a *nonlinear* Volterra integral equation. However, for a particular class of boundary conditions, called linearizable, this "nonlinearity" can be bypassed, and one can characterize the unknown boundary conditions using a linear procedure. In this case, the nonlinear Fourier transforms of both the initial and boundary conditions can be obtained via the spectral analysis of the x-dependent part of the Lax pair, as well as via certain algebraic manipulations. Here, we will analyze certain linearizable boundary value problems for the nonlinear Schrödinger equation(NLS).

5.1 Formulation of the problems.

The problems that we will discuss are certain initial-boundary problems on the half line $0 < x < \infty,\ t > 0$, applied to the NLS, i.e.

$$iq_t + q_{xx} - 2\lambda|q|^2 q = 0,\ \ \lambda = \pm 1. \tag{5.1.1}$$

This equation admits the following types of linearizable boundary conditions:

$$q(0,t) = 0; \quad q_x(0,t) = 0; \quad q_x(0,t) - \chi q(0,t) = 0, \quad \chi \in \mathbb{R}^*. \tag{5.1.2}$$

We will analyze three classes of Initial Boundary Value (IBV) problems. These problems involve one of the boundary conditions (5.1.2), as well as initial conditions characterized by the following three functions: (a) a soliton evaluated at $t = 0$; (b) a function describing a hump; and (c) an exponential function.

Regarding (a) we note that the focusing NLS, i.e. (5.1.1) with $\lambda = -1$, formulated on the line admits solitons. Thus, we can construct a solution of the IBV problem by simply restricting a soliton solution, denoted by $q_s(x,t)$; $\{q(x,0) = q_s(x,0),\ q(0,t) = q_s(0,t)\}$.

The IBV problem associated with a hump-shaped initial condition is defined as follows

$$q_0(x) = \begin{cases} 0, & 0 \le x < x_1, \\ h, & x_1 \le x \le x_2,\ h > 0, \\ 0, & x_2 < x < \infty, \end{cases} \tag{5.1.3}$$

and

$$\text{either} \quad q(0,t) = 0 \quad \text{or} \quad q_x(0,t) = 0, \quad t > 0. \tag{5.1.4}$$

The eigenfunctions associated with the function $q_0(x)$ can be computed explicitly in terms of trigonometric functions. This leads to an explicit formula for the functions $a(k)$ and $\Delta(k)$ defined in the Definition 5.1 and equation (5.3.2), respectively; the zeros of these functions characterize the asymptotic behavior of the solution. Although the explicit formulae of $a(k)$ and $\Delta(k)$ are complicated, the relevant zeros can be computed numerically. In this way we find that as $t \to \infty$, $q_0(x)$ generates, as expected, a finite number of solitons, whose number depends on the area under the graph of $q_0(x)$.

The IBV problem associated with an initial condition of an exponential function is defined as follows

$$q(x,0) = \begin{cases} e^{rx}, & 0 \leq x < s, \\ 0, & s < x < \infty, \end{cases} \tag{5.1.5}$$

$$q_x(0,t) - rq(0,t) = 0, \quad t > 0 \tag{5.1.6}$$

and we will consider two subcases, namely either $r < 0,\ s = \infty$ or $r > 0,\ s < \infty$.

Before analyzing the particular examples, we review the general theory of the IST for equation (5.1.1) on the half line and the main results of [9] and [2] regarding linearizable IBV problems. In Sections 5.4 - 5.6 we consider the three main classes of examples mentioned earlier, namely: Solitons; IBV problems with hump-shaped initial profiles; and IBV problems with exponential initial profiles.

5.2 Spectral Theory.

In this section we review the spectral theory of equation (5.1.1) on the half line. We will define three eigenfunctions $\{\mu_j\}_1^3$ of the Lax pair associated with (5.1.1)(see [2] and [9]) and then we will express the solution of equation (5.1.1) in terms of the solution of a 2×2 Riemann-Hilbert problem.

5.2.1 Lax pair.

Equation (5.1.1) admits the following Lax pair formulation

$$\begin{aligned} \mu_x + ik[\sigma_3, \mu] &= Q(x,t)\mu, \\ \mu_t + 2ik^2[\sigma_3, \mu] &= \widetilde{Q}(x,t,k)\mu, \end{aligned} \tag{5.2.1}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$,

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix}, \quad \widetilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 - i\lambda|q|^2\sigma_3. \tag{5.2.2}$$

The Lax pair (5.2.1) can be rewritten in the following differential form

$$d\left(e^{i(kx+k^2t)\hat{\sigma}_3}\mu(x,t,k)\right) = W(x,t,k), \tag{5.2.3}$$

where the exact 1-form W is defined by

$$W(x,t,k) = e^{i(kx+k^2t)\hat{\sigma}_3}(Q\mu dx + \widetilde{Q}\mu dt), \tag{5.2.4}$$

and $\hat{\sigma}_3$ denotes the commutator with respect to σ_3; if A is 2×2 matrix, the expression $(\exp\hat{\sigma}_3)A$ takes a simple form:

$$\hat{\sigma}_3 A = [\sigma_3, A], \qquad e^{\hat{\sigma}_3}A = e^{\sigma_3}Ae^{-\sigma_3}.$$

5.2.2 Bounded and Analytic Eigenfunctions.

Let equation (5.2.1) be valid for $0 < t < T$ and $0 < x < \infty$, where $T \leq \infty$. Assuming that the function $q(x,t)$ has sufficient smoothness and decay, we introduce three solutions $\mu_j,\ j = 1,2,3$ of (5.2.3) by

$$\mu_j(x,t,k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-i(kx+k^2t)\hat{\sigma}_3} W(\xi,\tau,k), \tag{5.2.5}$$

where I is the 2×2 identity matrix, $(x_1,t_1) = (0,T)$, $(x_2,t_2) = (0,0)$ and $(x_3,t_3) = (\infty,t)$. If $T = \infty$, the function μ_1 is only defined if $q(0,t)$ decays to zero as $t \to \infty$. Also we choose the contours shown in Figure 5.1. This choice implies the following inequalities on the contours,

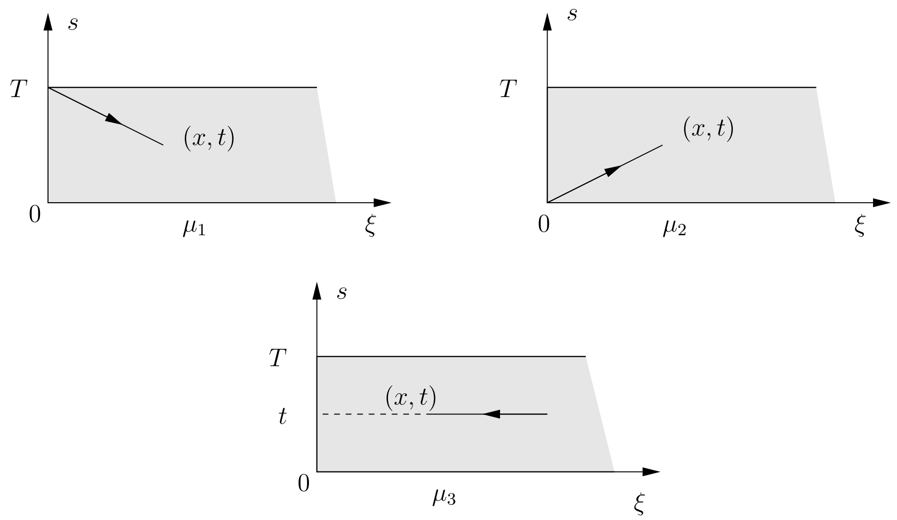

Figure 5.1: *The contours of integration of the spectral functions.*

$$\begin{aligned}
\mu_1 &: \ \xi - x \leq 0, \quad \tau - t \geq 0,\\
\mu_2 &: \ \xi - x \leq 0, \quad \tau - t \leq 0,\\
\mu_3 &: \ \xi - x \geq 0.
\end{aligned}$$

The second column of the matrix equation (5.2.5) involves $\exp[2ik(\xi - x) + 4ik^2(\tau - t)]$.

Using the above inequalities it follows that this exponential is bounded in the following regions of the complex plane

$$\begin{aligned}
\mu_1 &: \{Im\ k \leq 0 \cap Im\ k^2 \geq 0\}, \\
\mu_2 &: \{Im\ k \leq 0 \cap Im\ k^2 \leq 0\}, \\
\mu_3 &: \{Im\ k \geq 0\}.
\end{aligned}$$

Thus the second column vectors of μ_1, μ_2 and μ_3 are bounded and analytic for $\arg k$ in $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$ and $(0,\pi)$ respectively. We will denote these vectors with superscripts (3), (4) and (12) to indicate that they are bounded and analytic in the third quadrant, fourth quadrant and the upper half plane respectively. Similar conditions are valid for the first column vectors, thus

$$\mu_1(x,t,k) = (\mu_1^{(2)}, \mu_1^{(3)}),\ \mu_2(x,t,k) = (\mu_2^{(1)}, \mu_2^{(4)}),\ \mu_3(x,t,k) = (\mu_3^{(34)}, \mu_3^{(12)}).$$

Equation (5.2.5) and integration by parts imply that in the domains where $\{\mu_j\}_1^3$ are bounded, the following estimate is valid

$$\mu_j(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad j = 1,2,3.$$

The μ_j's are the fundamental eigenfunctions needed for the formulation of a Riemann-Hilbert problem in the complex k-plane.

5.2.3 Spectral functions.

We define $s(k)$ and $S(k)$ by the relations

$$\mu_3(x,t,k) = \mu_2(x,t,k) e^{-i(kx+2k^2t)\hat{\sigma}_3} s(k), \tag{5.2.6}$$

$$\mu_1(x,t,k) = \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}_3}S(k). \tag{5.2.7}$$

Evaluation of (5.2.6) and (5.2.7) at $(x,t) = (0,0)$ and $(x,t) = (0,T)$ implies

$$s(k) = \mu_3(0,0,k), \quad S(k) = \mu_1(0,0,k) = \left(\mu_2(0,T,k)e^{2ik^2T\hat{\sigma}_3}\right)^{-1}, \tag{5.2.8}$$

where the final equation is valid only when $T < \infty$. We use the following notation for $s(k)$ and $S(k)$:

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \lambda\overline{b(\bar{k})} & a(k) \end{pmatrix}, \quad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \lambda\overline{B(\bar{k})} & A(k) \end{pmatrix}. \tag{5.2.9}$$

5.2.4 The global relation.

Applying Stokes' theorem to the domain $\{0 < x < \infty,\ 0 < t < T\}$ for the closed one-form W with $\mu = \mu_3$, we find the following global relation:

$$B(k)a(k) - A(k)b(k) = \begin{cases} e^{4ik^2T}c^+(k) & \text{for } \arg k \in [0,\pi],\ T < \infty, \\ 0 & \text{for } \arg k \in [0,\pi/2], T = \infty, \end{cases} \tag{5.2.10}$$

where

$$c^+(k) = \int_0^\infty e^{2ik\xi}q(\xi,T)(Q\mu_3)_{22}(\xi,T,k)d\xi.$$

5.2.5 The Riemann-Hilbert problem.

Equations (5.2.6) and (5.2.7) can be rewritten in the following form, expressing the jump condition of a 2×2 RH problem:

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \tag{5.2.11}$$

where the matrices M_-, M_+ and J are defined by

$$\begin{aligned} M_+ &= \left(\frac{\mu_2^{(1)}}{a(k)}, \mu_3^{(12)}\right), \arg k \in [0, \frac{\pi}{2}]; \qquad M_- = \left(\frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(12)}\right), \arg k \in [\frac{\pi}{2}, \pi]; \\ M_+ &= \left(\mu_3^{(34)}, \frac{\mu_1^{(3)}}{\overline{d(\bar{k})}}\right), \arg k \in [\pi, \frac{3\pi}{2}]; \quad M_- = \left(\mu_3^{(34)}, \frac{\mu_2^{(4)}}{\overline{a(\bar{k})}}\right), \arg k \in [\frac{3\pi}{2}, 2\pi]; \end{aligned} \tag{5.2.12}$$

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}; \tag{5.2.13}$$

$$J(x,t,k) = \begin{cases} J_4, & \arg k = 0, \\ J_1, & \arg k = \frac{\pi}{2}, \\ J_2 = J_3 J_4^{-1} J_1, & \arg k = \pi, \\ J_3, & \arg k = \frac{3\pi}{2}; \end{cases} \tag{5.2.14}$$

with

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \lambda\bar{\gamma}(k)e^{2i\theta} & 1-\lambda|\gamma(k)|^2 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix}; \tag{5.2.15}$$

$$\theta(x,t,k) = kx + 2k^2t; \;\; \gamma(k) = \frac{b(k)}{\bar{a}(k)}, k \in \mathbb{R}; \;\; \Gamma(k) = \frac{\lambda\overline{B(\bar{k})}}{a(k)d(k)}, k \in \mathbb{R}^- \cup i\mathbb{R}^+. \tag{5.2.16}$$

The matrix $M(x,t,k)$ defined by equations (5.2.12) is, in general, a meromorphic function of k in $\mathbb{C}\backslash\{\mathbb{R} \cup i\mathbb{R}\}$. The possible poles of M are generated by the zeros of $a(k)$ and $d(k)$, and by the conjugate of these zeros.

Assumption 5.1. *We will make the following assumptions regarding the zeros:*

1. *If* $\lambda = -1$*,* $a(k)$ *has* n *simple zeros* $\{k_j\}_1^n$*,* $n = n_1 + n_2$*, where* $\arg k_j \in (0, \frac{\pi}{2})$*,* $j = 1, \ldots, n_1$*;* $\arg k_j \in (\frac{\pi}{2}, \pi)$*,* $j = n_1 + 1, \ldots, n_1 + n_2$*.*

2. *If* $\lambda = -1$*,* $d(k)$ *has* Λ *simple zeros* $\{\lambda_j\}_1^\Lambda$*, where* $\arg \lambda_j \in (\frac{\pi}{2}, \pi)$*,* $j = 1, \ldots, \Lambda$*.*
If $\lambda = 1$*,* $d(k)$ *has no zeros in the second quadrant.*

3. *None of the zeros of* $a(k)$ *for* $\arg k \in (\frac{\pi}{2}, \pi)$*, coincide with a zero of* $d(k)$*.*

Theorem 5.1. *Given* $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ *define the spectral functions* $a(k), b(k), A(k)$ *and* $B(k)$ *according to* (5.2.8) *and* (5.2.9)*, where* $\mu_1(0,t,k)$ *and* $\mu_3(x,0,k)$ *are obtained as the unique solutions of the Volterra linear integral equations*

$$\mu_1(0,t,k) = I + \int_0^t e^{2ik^2(\tau - t)\hat{\sigma}_3}(\widetilde{Q}\mu_2)(0,\tau,k)d\tau, \tag{5.2.17}$$

$$\mu_3(x,0,k) = I + \int_x^\infty e^{ik(\xi - x)\hat{\sigma}_3}(Q\mu_3)(\xi,0,k)d\xi, \tag{5.2.18}$$

and $Q(x,0)$*,* $\widetilde{Q}(0,t,k)$ *are given by equations* (5.2.2) *in terms of the initial and boundary values*

$$q_0(x) = q(x,0), \quad g_0(t) = q(0,t), \quad g_1(t) = q_x(0,t).$$

Suppose that the initial and boundary values are compatible in the sense that

- *they are compatible with the equation* (5.1.1) *at* $x = t = 0$*.*

- *the spectral functions satisfy the global relation* (3.1.7a)

Assume that the possible zeros $\{k_j\}_1^n$ *of* $a(k)$ *and* $\{\lambda_j\}_1^\Lambda$ *of* $d(k)$ *are as in Assumption 5.1. Define* $M(x,t,k)$ *as the solution of the following* 2×2 *matrix RH problem:*

- M *is sectionally meromorphic in* $k \in \mathbb{C} \backslash \{\mathbb{R} \cup i\mathbb{R}\}$*.*

- *M satisfies the jump condition*

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathbb{R} \cup i\mathbb{R},$$

where M is M_- for $\arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$, *$M$ is M_+ for* $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ *and J is defined in terms a, b, A and B by equations* (5.2.13)-(5.2.16).

- *The first column of M can have simple zeros at k_j, $j = 1, \dots, n_1$ and λ_j, $j = 1, \dots, \Lambda$; the second column of M can have simple zeros at $\bar{k}_j$, $j = 1, \dots, n_1$ and $\bar{\lambda}_j$, $j = 1, \dots, \Lambda$. The associated residues satisfy the following relations:*

$$\underset{k_j}{Res}[M(x,t,k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)}[M(x,t,k_j)]_2, \qquad j = 1, \dots, n_1, \tag{5.2.19a}$$

$$\underset{\bar{k}_j}{Res}[M(x,t,k)]_2 = \frac{1}{\dot{\bar{a}}(k_j)\bar{b}(k_j)} e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1, \qquad j = 1, \dots, n_1, \tag{5.2.19b}$$

$$\underset{\lambda_j}{Res}[M(x,t,k)]_1 = \underset{\lambda_j}{Res}\Gamma(k) e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2, \qquad j = 1, \dots, \Lambda, \tag{5.2.19c}$$

$$\underset{\bar{\lambda}_j}{Res}[M(x,t,k)]_2 = \underset{\bar{\lambda}_j}{Res}\overline{\Gamma(\bar{k})} e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \qquad j = 1, \dots, \Lambda, \tag{5.2.19d}$$

where $\theta(k_j) = k_j x + 2k_j^2 t$ *and* $\Gamma(k)$ *defined in* (5.2.16).

- $M(x,t,k) = I + O(\frac{1}{k}), \quad k \to \infty.$

Then $M(x,t,k)$ exists and is unique.

Define $q(x,t)$ in terms of $M(x,t,k)$ by

$$q(x,t) = 2i \lim_{k\to\infty}(kM(x,t,k))_{12}. \tag{5.2.20}$$

Then $q(x,t)$ solves equation (5.1.1). *Furthermore,*

$$q_0(x) = q(x,0), \quad g_0(t) = q(0,t), \quad g_1(t) = q_x(0,t).$$

Note: The Volterra equation (5.2.18) along with the definition of the spectral function $s(k)$ by the equations (5.2.6)-(5.2.9) suggest the following definition for the functions $a(k)$ and $b(k)$:

Definition 5.1. *(The spectral functions $a(k),\ b(k)$) The map*

$$\mathbb{S} : \{q_0(x)\} \Longrightarrow \{a(k), b(k)\}$$

is defined as follows:

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0,k) \tag{5.2.21}$$

where the vector-valued function $\varphi(x,k)$ is defined in terms of $q_0(x)$ by

$$\partial_x \varphi(x,k) + 2ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(x,k) = Q(x,0)\varphi(x,k),\ \ 0 < x < \infty,\ \ Imk \geq 0, \tag{5.2.22}$$

$$\lim_{x\to\infty} \varphi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.2.23}$$

where $Q(x,0)$ is given by:

$$Q(x,0) = \begin{pmatrix} 0 & q_0(x) \\ \lambda \bar{q}_0(x) & 0 \end{pmatrix}. \tag{5.2.24}$$

5.2.6 Asymptotic behavior of the solutions.

Here we review the main result -associated with this work- obtained via the asymptotic analysis of the relevant Riemman-Hilbert problem at Chapter 19 in [9].

If, for the focusing NLS, i.e. $\lambda = -1$, the discrete spectrum is not empty then solitons which are moving away from the boundary are generated. In particular, if $\{\kappa_j\}_1^N$ are roots of $a(k)$ or $d(k)$ then the asymptotics is given by a one-soliton in each of the N directions on the (x,t)-plane, namely

$$t \to \infty, \quad -\frac{x}{4t} = Re\{\kappa_j\} + O\left(\frac{1}{t}\right), \quad j = 1, \cdots, N. \tag{5.2.25}$$

Hence, solitons are generated only if $Re\{\kappa_j\} \leq 0$ because otherwise these solitons are moving to the left and after a finite time they disappear from the first quadrant. Note, also, that if $Re\{\kappa_j\} = 0$ then κ_j corresponds to a stationary soliton.

5.3 Linearizable Conditions.

It was shown in Theorem 5.1 that $q(x,t)$ can be expressed in terms of the solution of a 2×2 RH problem, which is uniquely defined in terms of the spectral functions $a(k),\ b(k),\ A(k)$ and $B(k)$. The functions $a(k)$ and $b(k)$ are defined in terms of $q_0(x)$ through the solution of the linear Volterra integral equation (5.2.18). However, the spectral functions $A(k)$ and $B(k)$ are defined in terms of both the known and unknown boundary conditions through the solution of the linear Volterra integral equation (5.2.17). The additional condition needed to determine the unknown boundary value is the requirement that they satisfy the global relation (5.2.10), which, in general, involves solving a nonlinear Volterra integral equation.

However, for a particular class of boundary value problems it is possible to compute $A(k)$ and $B(k)$, making only algebraic manipulation of the global relation.

Theorem 5.2. *Let $q(x,t)$ satisfy* (5.1.1), *the initial condition*

$$q(x,0) = q_0(x), \quad 0 < x < \infty$$

and the boundary condition

$$q_x(0,t) - \chi q(0,t) = 0, \; \chi \in \overline{\mathbb{R}}, \; t > 0. \tag{5.3.1}$$

We define

$$\Delta_\chi(k) = a(k)\overline{a(-\bar{k})} + \lambda \frac{2k - i\chi}{2k + i\chi} b(k)\overline{b(-\bar{k})}, \quad \chi \in \overline{\mathbb{R}}, \; \arg k \in [0, \pi] \tag{5.3.2}$$

and

$$\Gamma_\chi(k) = -\lambda \frac{2k - i\chi}{2k + i\chi} \frac{\overline{b(-\bar{k})}}{a(k)\Delta_\chi(k)}, \quad \chi \in \overline{\mathbb{R}}, \; k \in \mathbb{R}^- \cup i\mathbb{R}^+, \tag{5.3.3}$$

where $a(k)$ *and* $b(k)$ *are defined in Definition 5.1. Assume that the initial and boundary conditions are compatible at* $x = t = 0$*. Furthermore, if* $\lambda = -1$*, assume that:*

1. $a(k)$ *has a finite number of simple zeros for* $Imk > 0$.

2. $\Delta_\chi(k)$ *has a finite number of simple zeros in the second quadrant which do not coincide with any zero of* $a(k)$*.*

The solution $q(x,t)$ *can be constructed through equation* (5.2.20)*, where* M *satisfies the RH problem defined in Theorem 5.1, with jump matrices and residues conditions defined by replacing* $\Gamma(k)$ *with* $\Gamma_\chi(k)$ *in* (5.2.16)*.*

Proof. Recall that $A(k)$ and $B(k)$ are defined in terms of $\mu_2(0,t,k)$. Let $M(t,k) = \mu_2(0,t,k)e^{-ik^2t\hat{\sigma}_3}$, then $M(t,k)$ satisfies

$$M_t + 2ik^2 M = \widetilde{Q}(0,t,k), \quad M(0,k) = I. \tag{5.3.4}$$

The function $M(t,-k)$ satisfies a similar equation where $\widetilde{Q}(0,t,k)$ is replaced by $\widetilde{Q}(0,t,-k)$. Suppose that there exists a t-independent, nonsingular matrix $N(k)$ such that

$$(2ik^2\sigma_3 - \widetilde{Q}(0,t,-k))N(k) = N(k)(2ik^2\sigma_3 - \widetilde{Q}(0,t,k)). \tag{5.3.5}$$

Then

$$M(t,-k) = N(k)M(t,k)N(k)^{-1}. \tag{5.3.6}$$

The evaluation of this equation at $t = 0$ yields a relation between the spectral functions at k and $-k$. We note that a necessary condition for the existence of $N(k)$ is that the determinant of the matrix $2ik^2\sigma_3 - \widetilde{Q}(0,t,k)$ depends on k in the form of k^2. This condition implies

$$q(0,t)\bar{q}_x(0,t) - \bar{q}(0,t)q_x(0,t) = 0, \tag{5.3.7}$$

which is equivalent to (5.3.1). Given this condition, define the entries of the matrix $N(k)$ as follows:

$$N_{12} = N_{21} = 0, \quad (2k - i\chi)N_{22} + (2k + i\chi)N_{11} = 0.$$

Then equation (5.3.5) is satisfied and the second column of equation (5.3.6) evaluated at $t = T$ yields

$$A(k) = A(-k), \quad B(k) = -\frac{2k + i\chi}{2k - i\chi}B(-k), \quad k \in \mathbb{C}. \tag{5.3.8}$$

For convenience we assume that $T = \infty$. It can be shown that a similar analysis is valid if $T < \infty$. If $T = \infty$, the global relation becomes

$$a(k)B(k) - b(k)A(k) = 0, \quad \arg k \in \left[0, \frac{\pi}{2}\right]. \tag{5.3.9}$$

Letting $k \to -k$ in the definition of $\overline{d(\bar{k})}$ and using the symmetry relation (5.3.8) we find

$$A(k)\overline{a(-\bar{k})} + \lambda\frac{2k - i\chi}{2k + i\chi}B(k)\overline{b(-\bar{k})} = \overline{d(-\bar{k})}, \quad \arg k \in \left[0, \frac{\pi}{2}\right], \tag{5.3.10}$$

which along with the global relation (5.3.9) yield the following solution

$$A(k) = \frac{a(k)\overline{d(-\bar{k})}}{\Delta_\chi(k)}, \quad B(k) = \frac{b(k)\overline{d(-\bar{k})}}{\Delta_\chi(k)}, \quad \arg k \in \left[0, \frac{\pi}{2}\right]. \tag{5.3.11}$$

The function $\overline{d(\bar{k})}$ cannot be computed explicitly in terms of $a(k)$ and $b(k)$. However, this does not affect the solution of the RH problem of Theorem 2.1. Indeed, this RH problem is defined in terms of $\gamma(k) = \frac{b(k)}{\bar{a}(k)}$, $k \in \mathbb{R}$ and of $\Gamma(k)$ which involves $a(k), b(k)$ and $\frac{B(k)}{A(k)}$,

$$\Gamma(k) = \frac{\lambda \overline{\frac{B(\bar{k})}{A(\bar{k})}}}{a(k)\left(a(k) - \lambda b(k)\overline{\frac{B(\bar{k})}{A(\bar{k})}}\right)} = \Gamma_\chi(k), \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \tag{5.3.12}$$

The function $\Delta_\chi(k)$ is an analytic function in the upper half k-plane, and it satisfies the symmetry equation,

$$\Delta_\chi(k) = \overline{\Delta_\chi(-\bar{k})}. \tag{5.3.13}$$

It can be shown that the zero set of $\Delta_\chi(k)$ is the union

$$\{\lambda_j\}_{j=1}^{\Lambda} \cup \{-\bar{\lambda}_j\}_{j=1}^{\Lambda}. \tag{5.3.14}$$

Indeed, the global relation (5.3.9) implies that the zero sets of $A(k)$ and $a(k)$ coincide in the first quadrant. It also implies that if the zeros of $a(k)$ are simple, then the zeros of $A(k)$ have the same property. This and equation (5.3.11) imply that the zero sets of $\overline{d(-\bar{k})}$ and $\Delta_\chi(k)$ coincide in the first quadrant as well. Equation (5.3.13) implies that the zero set of $\Delta_\chi(k)$ is the given set given in (5.3.14).

Since the zeros λ_j of $d(k)$ coincide with the second quadrant zeros of $\Delta_\chi(k)$, equations (5.3.11) and (5.3.8) imply the relevant modifications on the residue conditions.

□

5.4 Solitons.

The one-soliton solution of the focusing NLS is given by

$$q_s(x,t) = \frac{1}{L}\frac{e^{i\left[\frac{v}{2}x - \left(\frac{v^2}{4} - \frac{1}{L^2}\right)t\right]}}{\cosh\frac{x - vt - x_0}{L}}, \tag{5.4.1}$$

where v, x_0, L are positive constants. The functions $q_s(0,t)$ and $(q_s)_x(0,t)$ satisfy the third of the linearizable boundary conditions (5.1.2) provided that

$$v = 0 \quad \text{and} \quad \chi = \frac{1}{L}\tanh\frac{x_0}{L}. \tag{5.4.2}$$

The fact that v vanishes, indicates that the relevant soliton is a stationary soliton. In this case

$$q_0(x) = \frac{1}{L\cosh\frac{x-x_0}{L}}. \tag{5.4.3}$$

Hence, the definitions of $a(k)$ and $\Delta(k)$ imply

$$a(k) = \frac{k - \frac{i}{2L}\tanh\frac{x_0}{L}}{k + \frac{i}{2L}}, \quad \Delta(k) = \frac{\left(k - i\frac{\chi}{2}\right)\left(k - \frac{i}{2L}\right)}{\left(k + i\frac{\chi}{2}\right)\left(k + \frac{i}{2L}\right)}. \tag{5.4.4}$$

Thus the zeros of $a(k)$ and $\Delta(k)$ are given by $k = \frac{i}{2L}\tanh\frac{x_0}{L}$ and $k = \frac{i}{2L}$, which confirms that the relevant solitons are stationary.

5.5 Hump-shaped initial profiles.

In this section we consider the IBV problem for equation (5.1.1) with initial and boundary conditions given by (5.1.3) and (5.1.4), respectively. Since the boundary conditions satisfy the equation (5.3.1) of Theorem 5.2, they are linearizable.

The definition of $a(k)$ for the initial value $q_0(x)$ given in (5.1.3) yields

$$a(k) = \frac{e^{ikl}}{\sqrt{\lambda h^2 - k^2}}\left[-ik\sinh\left(l\sqrt{\lambda h^2 - k^2}\right) + \sqrt{\lambda h^2 - k^2}\cosh\left(l\sqrt{\lambda h^2 - k^2}\right)\right], \tag{5.5.1}$$

where $l = x_2 - x_1$.

Now we investigate separately the following two cases:

(i)$\lambda = -1$. Using the transformation

$$k = ih\sin\theta,\ \theta \in \mathbb{C},\quad Re\{\sin\theta\} > 0, \tag{5.5.2}$$

we find that $a(k) = 0$ is equivalent to the equation

$$A\cos\theta - \theta = n\pi + \frac{\pi}{2},\ n \in \mathbb{Z},\quad \theta \neq n\pi + \frac{\pi}{2},\ A = hl. \tag{5.5.3}$$

Writing $\theta = \gamma + i\delta,\ \gamma, \delta \in \mathbb{R}$, it is straightforward to show that the solitons of (5.5.3) which satisfy the condition of the transformation (5.5.2), i.e. $Re\{\sin\theta\} > 0$, exist only when $\sin\theta > 0$. Hence, with no loss of generality, we can solve numerically equation (5.5.3) with $0 < \theta < \frac{\pi}{2}$. The graph at Figure 5.2 indicates that there exist finite many zeros(the intersections of the two graphs). The number of these zeros depends on the value of A and particularly if $A \in \left(m\pi + \frac{\pi}{2}, (m+1)\pi + \frac{\pi}{2}\right)$, then there exist exactly m solutions θ_i, which satisfy

$$A\cos\theta_i - \theta_i = n\pi + \frac{\pi}{2},\quad n \in \mathbb{Z}. \tag{5.5.4}$$

Hence, the set of the roots of $a(k)$ is $\{k_i,\ k_i = ih\sin\theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy (5.5.4).

Using the definition of $\Delta_\chi(k)$ in Theorem 5.2 for $\chi = \infty$ and 0, i.e. for $q(0,t) = 0$ and $q_x(0,t) = 0,\ t > 0$ we obtain the following expression

$$\Delta_\pm(k) = a(k)\overline{a(-\bar{k})} \pm b(k)\overline{b(-\bar{k})},\ \arg k \in \left[\frac{\pi}{2}, \pi\right],$$

respectively. Using the same transformation used earlier, i.e. $k = ih\sin\theta,\ \theta \in \mathbb{C}$ with

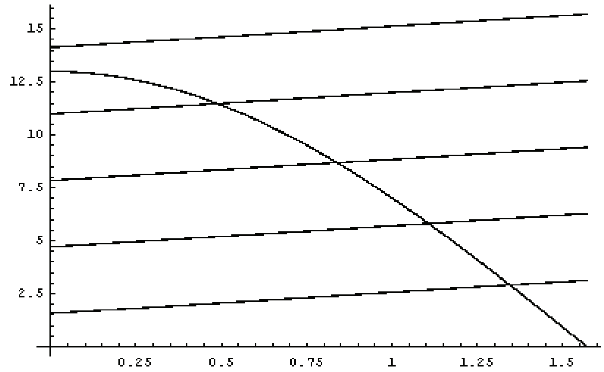

Figure 5.2: *The intersections of these plots are corresponding to the roots of* $a(k) = 0$ *for* $A = 13$.

$Re\{\sin\theta\} > 0$, we conclude that $\Delta_\pm(k) = 0$ is equivalent to the following equation

$$\sin(2A\cos\theta - \theta)\sin\theta \pm 1 = 0. \tag{5.5.5}$$

Writing again $\theta = \gamma + i\delta,\ \gamma, \delta \in \mathbb{R}$ in the first of the two equations (5.5.5) and making numerically the plots of $Re\{\sin(2A\cos\theta - \theta)\sin\theta\} = -1$ and $Im\{\sin(2A\cos\theta - \theta)\sin\theta\} = 0$ as shown in the Figure 5.3, we find again finite many solutions (the intersections of the two graphs) of the equation depending on the value of A. In particular, if $A \in \left(\left(m - \frac{1}{2}\right)\frac{\pi\sqrt{2}}{2}, \left(m + \frac{1}{2}\right)\frac{\pi\sqrt{2}}{2}\right)$ then there exist exactly m solutions θ_i, which satisfy

$$\sin(2A\cos\theta_i - \theta_i)\sin\theta_i + 1 = 0. \tag{5.5.6}$$

Hence the set of the roots of $d(k)$ is $\{\lambda_i,\ \lambda_i = ih\sin\theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy the equation (5.5.6).

Using similar arguments we can show that the second equation in (5.5.5) also has finite many solutions (the intersections of the two graphs) of the equation depending on the value of A. In particular, if $A \in \left((m-1)\frac{\pi\sqrt{2}}{2}, m\frac{\pi\sqrt{2}}{2}\right)$ then there exist exactly m solutions θ_i, which satisfy

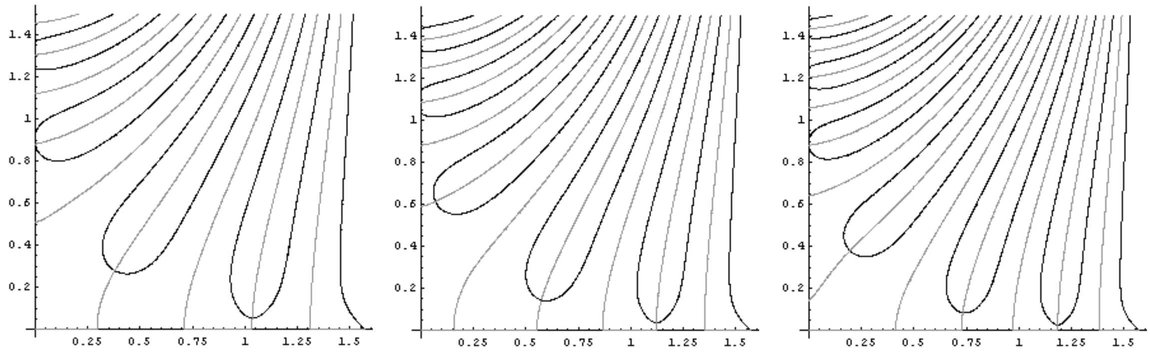

Figure 5.3: *The intersections of these plots are corresponding to the roots of* $d(k) = 0$ *for* $A = \frac{5}{2}\frac{\pi\sqrt{2}}{2},\ \frac{6}{2}\frac{\pi\sqrt{2}}{2},\ \frac{7}{2}\frac{\pi\sqrt{2}}{2}$, *respectively.*

$$\sin(2A\cos\theta_i - \theta_i)\sin\theta_i - 1 = 0. \tag{5.5.7}$$

Hence the set of the roots of $d(k)$ is $\{\lambda_i,\ \lambda_i = ih\sin\theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy (5.5.7).

(ii) $\lambda = 1$. Putting $k = h\sin\theta$, $\theta \in \mathbb{C}$, with $Im\{\sin\theta\} > 0$ makes $a(k) = 0$ equivalent to $A\cos\theta - i\theta = i(n\pi + \frac{\pi}{2})$, $n \in \mathbb{Z}$, with $\theta \neq n\pi + \frac{\pi}{2}$ where $A = hl$ =Area of the hump. Using similar arguments as before and in particular writing again $\theta = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}$ we conclude that there is no solution of this equation satisfying the restriction that $Im\{\sin\theta\} > 0$. This is in accordance with what was proven in [2], about the non-existence of soliton solutions of equation (5.1.1) when $\lambda = 1$.

The above results imply the following conclusions for the asymptotic behavior of the solution of equation (5.1.1) for large t:

- The real part of the zeros of $a(k)$ is zero, i.e. *Re* k_j=0. Hence, these zeros produce *only* stationary solitons.
- The real part of the zeros of $d(k)$ is *not* zero, i.e. *Re* $\lambda_j \neq 0$. Hence, these zeros produce *only* non-stationary solitons.
- Both the zeros of $a(k)$ and $d(k)$ (i) are finitely many, (ii) does not coincide with

each other and (iii) are dependent on the area that the initial condition(the hump) has; the number of zeros and hence the number of solitons increases as the area increases.

5.6 Exponential initial profiles.

In this section we consider the IBV problem for equation (5.1.1) with $\lambda = -1$ and initial and boundary conditions given by (5.1.5). Since the boundary conditions are of type (b) of Theorem 5.2, they are linearizable.

In what follows, we first consider the case $q_0(x) = e^{rx},\ r < 0,\ x > 0$. The definition of $a(k)$ for this initial condition yields the following expression

$$a(k) = \frac{(2a)^{-\frac{1}{2}+i\frac{k}{r}}}{\Gamma(\frac{1}{2} - i\frac{k}{r}) \cosh \frac{k\pi}{r}} I_{-\frac{1}{2}+i\frac{k}{r}}(-\frac{1}{r}), \tag{5.6.1}$$

where $I_a(x)$ denotes the modified Bessel function of first kind and $\Gamma(z)$ is the Euler gamma function. Making the transformation $k = -ir\nu$ with $Re\ \nu > 0$ we conclude that the zeros of $a(k)$ come from the zeros of $I_{\nu-\frac{1}{2}}(-\frac{1}{r})$. Arguments similar with those used in Section 4, imply that the roots of this Bessel function exist only when $\nu > 0$. Figure 5.4 implies that there exist finite many zeros depending on the value of r. In particular, if $-\frac{1}{r} \in \left(m\pi - \frac{\pi}{2}, m\pi + \frac{\pi}{2}\right)$, then there exist exactly m solutions ν_i. Note that the area below the graph of the initial data $q_0(x) = e^{rx}$ is given by $A(r) = -\frac{1}{r}$.

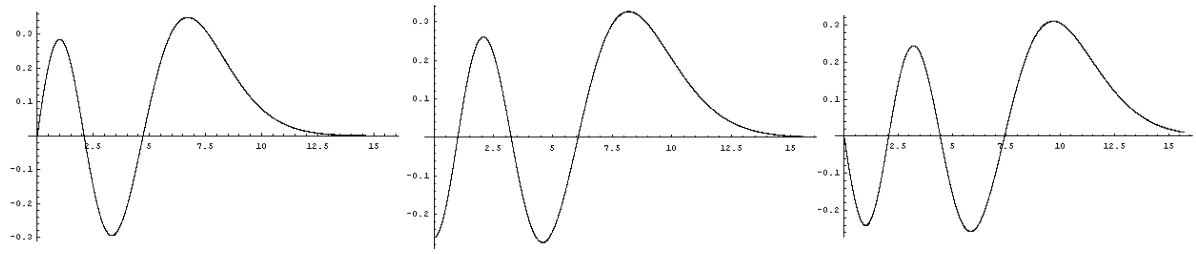

Figure 5.4: *The plot of $I_{\nu-\frac{1}{2}}(-\frac{1}{r})$ with $\nu > 0$ and $-\frac{1}{r} = \frac{5\pi}{2},\ \frac{6\pi}{2},\ \frac{7\pi}{2}$, respectively.*

The computation of $\Delta_\chi(k)$ with $\chi = r$ shows that the roots of $d(k)$ have the same distribution on the imaginary axis, as the roots of $a(k)$. Hence, asymptotically, there exist finitely many stationary solitons and the number of these solitons depends only on the area under the graph of the initial condition.

We now discuss the subcase $\{r > 0,\ s < \infty\}$. In this case the formulae of $a(k)$ and $\Delta(k)$ are more complicated. Actually, $a(k)$ is given by

$$a(k) = -\frac{e^{rs\left(\frac{1}{2}+i\frac{k}{r}\right)}\pi}{2r}\left[I_{\frac{1}{2}+i\frac{k}{r}}\left(\frac{e^{rs}}{r}\right)I_{-\frac{1}{2}-i\frac{k}{r}}\left(\frac{1}{r}\right) - I_{\frac{1}{2}+i\frac{k}{r}}\left(\frac{1}{r}\right)I_{-\frac{1}{2}-i\frac{k}{r}}\left(\frac{e^{rs}}{r}\right)\right]. \qquad (5.6.2)$$

Using arguments similar with those used above, it can be shown that the zeros of the functions $a(k)$ and $\Delta(k)$ are on the imaginary axis and depend again on the area below the graph of the initial condition,

$$A(r,s) = \frac{e^{rs}}{r} - \frac{1}{r}.$$

Chapter 6

Future work.

Some of the problems to be investigated using the Fokas method applied to (1.1.1) and (1.1.2) are the following:

- **Solution in the interior of more complicated domains.** The regular hexagon is an illustrative example of such domains. Some special problems have already been solved in this domain and yield the solutions of the corresponding problems for the equilateral triangle. The approach used for the solution of elliptic equations in the interior of an equilateral triangle suggest that we have to exploit further the invariances of the global relation, as well as the symmetries appearing in the integral representation.

- **Eigenvalues in the interior of more complicated domains.** Referring again to the regular hexagon, we note that the eigenvalues for this domain which coincide with those of the corresponding equilateral triangle can be rederived independently. These eigenvalues still correspond to trigonometric eigenfunctions, thus by postulating other types of eigenfunctions, perhaps we could obtain novel eigenvalues.

- **Solution in the exterior of convex polygons.** Since these domains are not convex, the analysis presented here can not be implemented directly. However, this approach can be used if the exterior domain is subdivided into convex subdomains.

Preliminary results are presented in [48].

- **Solution in three spatial dimensions.** Several problems for elliptic PDEs in these domains have been solved in the interior of a sphere and of a sprherical sector in [49]. The application of the Fokas method to other three dimensional domains is under investigation.

Bibliography

[1] A.S. Fokas. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond. A*, 453:1411–1443, 1997.

[2] A.S. Fokas, A.R. Its, L.Y. Sung. The nonlinear schrödinger equation on the half-line. *Nonlinearity*, 18:17711822, 2005.

[3] A.S. Fokas. Integrable nonlinear evolution equations on the half-line. *Commun. Math. Phys.*, 230(1):1–39, 2002.

[4] A.S. Fokas and B. Pelloni. A transform method for evolution pdes on a finite interval. *IMA J. Appl. Math.*, 70:564–587, 2005.

[5] B. Pelloni. The spectral representation of two-point boundary value problems for linear evolution equations. *Proc. Roy. Soc. Lond. A*, (461):2965–2984, 2005.

[6] A. Boutet de Monvel, D. Shepelsky. Initial boundary value problem for the mKdV equation on a finite interval. *Annales de l'institut Fourier*, 54(5):1477–1495, 2004.

[7] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky. Analysis of the Global Relation for the Nonlinear Schroedinger Equation on the Half-line. *Lett. Math. Phys.*, 65(3):199–212, 2003.

[8] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky. The Modified KdV Equation on the Half-Line. *J. of the Inst. of Math. Jussieu*, 3:139–164, 2004.

[9] A.S. Fokas. *A Unified Approach to Boundary Value Problems.* CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 2008.

[10] J.B. Keller. Progress and Prospects in the Theory of Linear Wave Propagation. *SIAM Review*, 21:229, 1979.

[11] I. Stakgold. *Boundary value problems of mathematical physics, Volumes I and II.* New York: The Macmillan Company; London: Collier-Macmillan Ltd, 1967.

[12] P.M.C. Morse and H. Feshbach. *Methods of Theoretical Physics.* McGraw-Hill Science/Engineering/Math, 1953.

[13] B. Friedman. *Principles and Techniques of Applied Mathematics.* John Wiley & Sons Inc, 1956.

[14] J.P. Keener. *Principles of Applied Mathematics.* Perseus Books, 1995.

[15] J. Ockendon, S. Howison, A. Lacey, and A. Movchan. *Applied Partial Differential Equations.* Oxford University Press, USA, 2003.

[16] E.C. Titchmarsh. *Eigenfunction Expansions Associated with Second-order Differential Equtions.* Clarendon Press, 2nd edition, 1962.

[17] L.C. Evans. *Partial differential equations.* American Mathematical Society Providence, RI, 1998.

[18] J. Keller. The scope of the image method. *Comm Pure Appl. Math*, 6:505–512, 1953.

[19] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, 2001.

[20] J.B. Keller. Oblique Derivative Boundary Conditions and the Image Method. *SIAM Journal on Applied Mathematics*, 41:294, 1981.

[21] A.K. Gautesen. Oblique derivative boundary conditions and the image method for wedges. *SIAM Journal on Applied Mathematics*, 48(6):1487–1492, 1988.

[22] R. Terras and R.A. Swanson. Electrostatic image problems with plane boundaries. *American Journal of Physics*, 48:526, 1980.

[23] R. Terras and R. Swanson. Image methods for constructing Green's functions and eigenfunctions for domains with plane boundaries. *Journal of Mathematical Physics*, 21:2140, 1980.

[24] M.A. Pinsky. The Eigenvalues of an Equilateral Triangle. *SIAM Journal on Mathematical Analysis*, 11:819, 1980.

[25] M. Práger. Eigenvalues and eigenfunctions of the Laplace operator on an equilateral triangle. *Applications of Mathematics*, 43(4):311–320, 1998.

[26] A Sommerfeld. *Optics: lecture on theoretical physics volume IV.* Academic Press New York, 1964.

[27] G. Dassios and RE Kleinman. On Kelvin Inversion and Low-Frequency Scattering. *SIAM Review*, 31(4):565–585, 1989.

[28] D. ben Avraham and A.S. Fokas. The Solution of the modified Helmholtz equation in a wedge and an application to diffusion-limited coalescence. *Physics Letters A*, 263(4-6):355–359, 1999.

[29] D. ben Avraham and A.S. Fokas. Solution of the modified Helmholtz equation in a triangular domain and an application to diffusion-limited coalescence. *Phys. Rev. E*, 64(016114), 2001.

[30] YA Antipov and AS Fokas. The modified Helmholtz equation in a semi-strip. *Mathematical Proceedings of the Cambridge Philosophical Society*, 138(02):339–365, 2005.

[31] A.S. Fokas and D. Pinotsis. The Dbar formalism for certain linear non-homogenous elliptic PDEs in two dimensions. *Euro. J. Appl. Math*, 17(3):323–346, 2006.

[32] G. Dassios and A.S. Fokas. The basic elliptic equations in an equilateral triangle. *Proc. R. Soc. Lond. A*, 461:2721–2748, 2005.

[33] A.S. Fokas and A.A. Kapaev. On a transform method for the Laplace equation in a polygon. *IMA J. Appl. Math.*, 68:355–408, 2003.

[34] G. Lamé. Mémoire sur la propagation de la chaluer dans les polyèdres. *Journal de lÉcole Polytechnique*, 22:194–251, 1833.

[35] M.A. Pinsky. Completeness of the Eigenfunctions of the Equilateral Triangle. *SIAM Journal on Mathematical Analysis*, 16:848, 1985.

[36] B.J. McCartin. Eigenstructure of the equilateral triangle. Part III. The Robin problem. *International Journal of Mathematics and Mathematical Sciences*, 2004(16):807–825, 2004.

[37] B.J. McCartin. Eigenstructure of the Equilateral Triangle, Part I: The Dirichlet Problem. *SIAM Review*, 45:267, 2003.

[38] B.J. McCartin. Eigenstructure of the Equilateral Triangle, Part II: The Neumann Problem. *Mathematical Problems in Engineering*, 8(6):517–539, 2002.

[39] A.S. Fokas. A new transform method for evolution partial differential equations. *IMA J. Appl. Math.*, 67(6):559–590, 2002.

[40] A.S. Fokas. The Davey-Stewartson equation on the half-plane. *Commun. Math. Phys.*, 289(3):957–993, 2009.

[41] G. Deift and RE Zhou. A Steepest Descent Method For Oscillatory Riemann-Hilbert Problems. *Bulletin of the American Mathematical Society*, 26(1):119–124, 1992.

[42] A.S. Fokas and A.A. Kapaev. A Riemann-Hilbert approach to the Laplace equation. *J. Math. Anal. Appl.*, 251:770–804, 2000.

[43] A.S. Fokas. Two-dimensional linear partial differential equations in a convex polygon. *Proc. R. Soc. Lond. A*, 457:371–393, 2001.

[44] A.S. Fokas and M. Zyskin. The fundamental differential form and boundary-value problems. *Q.J. Mech. Appl. Math*, 55(3):457–479, 2002.

[45] N. Flyer and A.S. Fokas. A hybrid analytical-numerical method for solving evolution partial differential equations. I. The half-line. *Proc. Roy. Soc. A*, 464(2095):1823–1849, 2008.

[46] G. M. Dujardin. Long time asymptotics of linear initial-boundary value problems with periodic boundary data. *Proc. Roy. Soc.*, 465(2111):3341–3360, 2009.

[47] A.S. Fokas and L.Y. Sung. Initial-boundary value problems for linear disperisve evolution equations on the half-line. *IMI Research Report 1999:11*.

[48] A. Charalambopoulos, G. Dassios, and A.S. Fokas. Laplace's equation in the exterior of a convex polygon: the equilateral triangle. *Q. Appl. Math.*, (to appear), 2009.

[49] G. Dassios and AS Fokas. Methods for Solving Elliptic PDEs in Spherical Coordinates. *SIAM Journal on Applied Mathematics*, 68:1080, 2008.